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EFFICIENT SYNCHRONIZATION WITH CHAOTIC QUADRATIC MAPS

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Abstract. We present a systematic way to design unidirectional and bidirectional coupling schemes for synchronizing arbitrary pairs of (identical or different) discrete dynamical systems. If the coupled chaotic systems are very similar or identical, using the singular value decomposition, it is possible to suppress the exponential divergence of the dynamics of the synchronization error, and exploit the existing contraction properties of the given systems. When non-identical systems are coupled, in order to achieve synchronization it is necessary to employ some other techniques from linear algebra, stability theory and control. We use two methods to study the stability of synchronous state: the linear stability and the Lyapunov functional analysis. In order to illustrate these methods, we use a system of two coupled chaotic quadratic maps. The map obtained by coupling exhibits a richer dynamics that the single quadratic map, but is still possible to study its behaviour.

1 Introduction

Many deterministic nonlinear systems exhibit, apart from fixed-point solutions and limit cycles, more complex invariant sets which act as attractors for their dynamics. Among them we can find chaotic attractors. One of the main cha-racteristic associated with the chaotic behaviour is the sensitive dependence on initial conditions, that is, any infinitesimal perturbations of the initial conditions lead to the divergence of nearby starting orbits. The possibility of two (or more) chaotic systems oscillate in a coherent and synchronized way is not an obvious phenomenon, since it is not possible to reproduce exactly the initial conditions. However, when ensembles of chaotic oscillators are coupled, the attractive effect of a suitable coupling can counterbalance the trend of the trajectories to approximate due to chaotic dynamics. As a result, it is possible to reach full synchronization in chaotic systems.

Coupled dynamical systems are constructed from simple, low-dimensional maps and form new and more complex organizations, with the belief that do-minant features of the underlying components will be retained. This building up approach can also be used to create a novel system whose behaviour is more flexible or richer than that of the components, but whose analysis and control remains tractable.

The first observations related to synchronization were reported by Huygens in 1665. In that case, the synchronization was indicated by the equal periods of coupled clocks. Synchronization of periodic signals is a well-know phenomenon in physics, nature, economics, engineering and many other scientific areas. Today, synchronization is used in a more generalized sense: occurring in periodic and chaotic coupled systems.

The seminal paper of Fujisaka and Yamada [2] review several results related to synchronization. Later, in 1990, Ott, Grebogi and Yorke [6], and Pecora and Carroll [7] lead to the establishment of two new areas of research: synchronization and control of chaotic dynamical systems. These papers immediately received a great deal of attention, and opened up a wide range of applications outside the traditional scope of chaos and nonlinear dynamics research. Since then, various synchronization methods and several news concepts necessary for analyzing synchronization have been developed. Synchronization and controle of chaotic motions have common roots based on driving the nonlinear system to restrict its motion – in each case one selects parameter regions or external forcing to achieve the collapse of the full state space to a selected subspace. In synchronization we seek subspaces of the coupled system space – the synchronization set – in which a special kind of motion, which relates the coupled system, takes place.

The most well-known regimes of synchronization are the identical (or complete) synchronization, the phase synchronization and the generalized synchronization. Identical synchronization means that the periodic or the chaotic oscillations of the coupled identical systems coincide exactly in time due to the strong interaction between them. Generalized synchronization is a kind of synchronization where exists a one-to-one smooth mapping between oscillations of each subsystem. Hence, knowing the state of one subsystem enables us to know the state of the other subsystem. Phase synchronization is defined as the appearance of a certain relationship between the phases of the coupled systems while the amplitudes can remain uncorrelated. For each regime, one can separate cases of full and partial synchronization based on the coupling degree. We can also consider two coupling mechanisms: unidirectional (or one-way) and bidirectional (or mutual) coupling.

In what follows we present a systematic way to design unidirectional and bidirectional coupling schemes for synchronizing arbitrary pairs of (identical or different) discrete dynamical systems. If the coupled chaotic systems are very similar or identical, their state vectors x and y converge to the same trajectory due to synchronization. A very short transition is observed in this case. Junge and Parlitz [5] show that the synchronization of chaotic systems can be explained by the suppression of expanding dynamics in the state space transversal to the synchronization manifold $x = y$. Using the singular value decomposition, it is possible to suppress the exponential divergence of the dynamics of the error $x - y$, and exploit the existing contraction properties of the given systems. With this method, systems can be synchronized using a minimum of transmitted information and is guarantied linear stability of the synchronized state in all points of the state space. When non-identical systems are coupled, in order to achieve synchronization it is necessary to employ several other techniques from linear algebra, stability theory and control. We use two types of analysis to determine the stability of synchronous state: the linear stability analysis [2], [1] and the Lyapunov functional analysis [9], [4].

The one-dimensional quadratic map $x_{n+1} = a - x_n^2$, where a is a real cons-tant, is a discrete dynamical system whose behaviour has been intensively stu-died. In order to illustrate these synchronization methods, we use a system of two coupled chaotic quadratic maps. The system obtained by coupling exhibits a much richer dynamics that the single quadratic map, but is still simple enough to allow the study of its behaviour. The most interesting and nontrivial phenomenon here is the synchronization transition. We start with the construction of the linear coupled system and explore its topological dynamics.

2 Coupled dynamical systems in discrete time

Let consider the following system

$$\begin{cases} x_{n+1} = f_a(x_n) \\ y_{n+1} = g_b(y_n) \end{cases}, \tag{1}$$

where x_n and y_n are real dynamical variables, $n = 0, 1, 2, ...,$ f_a and g_b are chaotic maps and the constants a and b represent the control parameters. Both maps in (1) satisfy a global dissipative condition and hence has global attractor.

As the dynamics of each map is chaotic, in the case of uncorrelated systems we can observe two independent random-like processes without any mutual correlation. Now let us introduce an interaction between the two systems. There are several ways to couple mathematically two maps: any additional term containing both x and y on the right-hand side of the equation will provide some coupling. We want, however, the coupling to have some relevant properties:

(i) the coupling is dissipative, that is, it tends to make the states x and y closer to each other;

(ii) the coupling does not affect the symmetric synchronous state $x = y$.

The proper way to apply a coupling operator to the nonlinear maps (1) is like follows:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \begin{bmatrix} f_a(x_n) \\ g_b(y_n) \end{bmatrix}, \tag{2}$$

where α and β are the coupling parameters. The bidirectionally coupled systems in (2) are synchronized if $|x_n - y_n| \to 0$ as $n \to \infty$, for a certain range of the control parameters. The main method of research on this problem is analytical: for given parameter values a and b, we want to find coupling values of α and β so that for close initial values x_0 and y_0, synchronization must occurs.

The simplest case is the identical synchronized regime, where $f_a = g_b$ and $\alpha = \beta = c$. In this case, the system (2) takes the form

$$\begin{cases} x_{n+1} = (1-c) f_a(x_n) + c f_a(y_n) \\ y_{n+1} = c f_a(x_n) + (1-c) f_a(y_n) \end{cases}, \tag{3}$$

which means that is completely symmetric with respect to changes of the varia-bles $x \longleftrightarrow y$, due to the choice of symmetrically coupled identical subsystems.

This birectional coupling scheme can be clearly interpreted in light dynamics of populations. One can think of $f_a(x_n)$ and $f_a(y_n)$ as simulating the population dynamics of a particular species at two adjacent locations. If the species can migrate in both directions within the time intervals between the stages of their reproduction and death, then c represents the fraction of these species, which migrate to the neighbouring location (c is a measure of the diffusion of individuals between the two locations). For $c \in [0, 1]$, the phase volume suffers additional contraction in comparison with the magnitude it would have without coupling. This fact gives grounds to call this type of coupling scheme as dissipative coupling.

There are other simple ways to couple two maps; for instance we could have linear coupling,

$$\begin{cases} x_{n+1} = f_a(x_n) + c(x_n - y_n) \\ y_{n+1} = f_a(y_n) + c(y_n - x_n) \end{cases} . \tag{4}$$

and bilinear coupling, if the linear terms in (4) is replaced by $\pm c x_n y_n$. However, such coupling schemes are not biologically realistic as they involve mixing of generations: some of the individuals have been allowed to reproduce and die and have also been allowed to move into the other location.

With this coupling scheme, large enough coupling strength c should eventually bring about the synchronization of the considered system (3), for any value of a. In particular, when $c = 0$, the two variables x and y are completely independent and uncorrelated, that is, the two systems act independently. For $c = 0.5$, after few iterations the two variables x and y become identical and we immediately observe the synchronous state $x_n = y_n$ for all n. When the synchronous state is reached, the dynamics of both systems corresponds to that of the single map f_a. As the coupling does not affect this state, the dynamics of x and y are the same as in the uncoupled systems, that is, chaotic. Such a regime, where each of the maps show chaos and their states are identical at each moment in time, is called full (or total) synchronization. So, when the synchronous (chaotic) state $x = y$ is reached, the chaotic dynamics of x and y are restricted to the one-dimensional invariant attracting subspace $x = y$, called the synchronization set. Thus, the problem of synchronization can be understood as a problem of stability of an one-dimensional chaotic attractor embedded in the two-dimensional phase space. If we consider the coupling parameter c as a bifurcation parameter that increases gradually from 0, a complex bifurcation structure is generally observed, but one clearly sees a tendency to closer correlation between the variables x and y. The goal is to keep the systems as loosely coupled as possible, but still have them synchronize. One can find a critical coupling $c_{crit} < 0.5$ – the coupling threshold – such that for $c > c_{crit}$ the synchronous state $x = y$ is established. The synchronization near this value of c appears to be highly sensitive. The points outside the diagonal $x = y$ represent the nonsynchronous state. With increasing of c, the distribution of the points tends towards the diagonal, and beyond the critical coupling c_{crit} all points satisfy $x = y$. The critical coupling value c_{crit} is obtained from the Lyapunov exponent of the uncoupled chaotic system.

Unidirectional coupling is described by the interaction matrix

$$\begin{bmatrix} 1 & 0 \\ c & 1-c \end{bmatrix},$$

and since we obtain

$$\begin{cases} x_{n+1} = f_a(x_n) \\ y_{n+1} = c f_a(x_n) + (1-c)\, f_a(y_n) \end{cases} ,$$

where x_n and y_n are the real dynamical variables of the drive (or master) and the response (or slave) systems, respectively. The expression $c\,[f_a\,(x_n) - f_a\,(y_n)]$ is called the coupling term. As we see, in unidirectional coupling only the dynamics of the response system is affected by the drive system through the coupling; the reverse does not hold. If the coupling term assume the expression $c\,(x_n - y_n)$ we have linear coupling; otherwise, we have dissipative coupling.

3 Stability analysis of the synchronous state

3.1 Linear stability analysis

The coupling threshold in which we obtain stable synchronous state in (3) can be computed by linearizing around the synchronous state, where $x_n = y_n$. To characterize the synchronization transition at $c = c_{crit}$, it is convenient to define two new variables

$$U_n = \frac{x_n + y_n}{2} \quad \text{and} \quad V_n = \frac{x_n - y_n}{2}.$$

Geometrically, the variable U_n is directed along the diagonal $x_n = y_n$, while the variable V_n corresponds to the direction tranverse to this diagonal. In the synchronous state, $V_n \equiv 0$ and $x_n = y_n = U_n$. Close to the synchronous state, the variable V_n is small and its evolution will determine the stability of the synchronous state. Note that, since system (3) remains invariant under the transformation $x \longleftrightarrow y$, the synchronous state $x_n = y_n$ is a solution of (3) for all values of c. So that, if the initial conditions are symmetric, $x_0 = y_0$, the symmetry is preserved in time. If we want the synchronous state to be observed not only for specific, but also for general initial states, we must impose the stability condition: the (full) synchronous state $V_n \equiv 0$ should be an attractor, that is, synchronization should establish even from nonsymmetric initial states. This stability condition will give us the critical coupling c_{crit} for the onset of synchronization. The rate of growth of a small difference along two trajectories on the chaotic attractor is measured through the Lyapunov exponent λ_U and the evolution of perturbations in the perpendicular direction, which determines the stability of the synchronous attractor, is characterized by the transversal Lyapunov exponent λ_V.

The system (3) can be written in the form

$$\begin{cases} U_{n+1} = \dfrac{1}{2}\left[f_a(U_n + V_n) + f_a(U_n - V_n)\right] \\[2ex] V_{n+1} = \dfrac{1-2c}{2}\left[f_a(U_n + V_n) - f_a(U_n - V_n)\right] \end{cases}.$$

We linearize this system near the synchronous state U_n, where the variable V_n is small, and obtain a linear maps for small perturbations of U_n and V_n given by

$$U_{n+1} = \left[Df_a(U_n)\right] U_n \quad \text{and} \quad V_{n+1} = (1-2c)\left[Df_a(U_n)\right] V_n, \tag{5}$$

where the derivative $Df_a(U_n)$ is evaluated along the synchronous state. Since in the linear approximation the perturbations of U_n and V_n do not interact, the perturbations can be treated separately. For $V_n \equiv 0$, the Lyapunov exponent λ_U is nothing else than the Lyapunov exponent λ for uncoupled map

$$\lambda_U = \lambda = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \ln\left|Df_a(U_i)\right|,$$

and the transversal Lyapunov exponent λ_V can be written in terms of λ in the following simple way

$$\lambda_V = \lambda + \ln\left|1 - 2c\right|.$$

Because of this simple dependence on λ, the boundary of stability of the synchronous state can be immediately deduced: the synchronous state is stable if the Lyapunov exponent corresponding to the difference variable V_n is negative, that is $\lambda_V < 0$ (and unstable if $\lambda_V > 0$), and the range of stability is given by

$$\frac{1-e^{-\lambda}}{2} < c < \frac{1+e^{-\lambda}}{2}.$$

The coupling threshold is then defined from the condition $\lambda_V = 0$, that is

$$\ln|1-2c_{crit}| = -\lambda \Rightarrow c_{crit} = \frac{1-e^{-\lambda}}{2}.$$

3.2 Global stability analysis

Global stability in a neighborhood of an equilibrium point is confirmed if there exist a positive definite function defined in that neighborhood, whose derivative is negative semi-definite [9]. To get conditions for the global stability of synchronization of two systems in variables x and y, we define the associated Lyapunov function by

$$L(x,y) = (x-y)^2.$$

Since $L(x,y) \geq 0$ the equality holds only when the systems are exactly synchronized. For the asymptotic global stability of the synchronous state, the Lyapunov function should satisfy the following condition in the region of stabi-lity,

$$\frac{L_{n+1}}{L_n} < 1. \tag{6}$$

If we consider the coupling scheme (3), the Lyapunov function is written as

$$L_{n+1} = (x_{n+1}-y_{n+1})^2 = (1-2c)^2\left[f_a(x_n)-f_a(y_n)\right]^2,$$

and, using the Taylor expansion of $f_a(x_n)$ about y_n, we obtain

$$\frac{L_{n+1}}{L_n} = (1-2c)^2\left[f_a'(y_n) + f_a''(y_n)\frac{x_n-y_n}{2} + \mathcal{O}(x_n-y_n)^2\right]^2. \tag{7}$$

If the expression in the square bracket on the right hand-side is bounded then always exist some values of c around $1/2$ for which the synchronous state will be stable.

Example. Consider the bidirectional coupling scheme (3) with the one-dimensional quadratic map $f_a(x) = a - x^2$,

$$\begin{cases} x_{n+1} = (1-c)\left(a-x_n^2\right) + c(a-y_n^2) \\ y_{n+1} = c(a-x_n^2) + (1-c)\left(a-y_n^2\right) \end{cases},$$

where $a \in \mathbb{R}$ is the control parameter. The system takes the equivalent form

$$\begin{cases} x_{n+1} = a - x_n^2 + c(x_n^2-y_n^2) \\ y_{n+1} = a - y_n^2 + c(y_n^2-x_n^2) \end{cases},$$

and, by (7), the corresponding Lyapunov function gives that

$$\begin{aligned}\frac{L_{n+1}}{L_n} &= (1-2c)^2\left[-2y_n + \frac{x_n - y_n}{2}(-2) + \mathcal{O}(x_n - y_n)^2\right]^2 \\ &= (1-2c)^2\left[-x_n - y_n + \mathcal{O}(x_n - y_n)^2\right]^2.\end{aligned}$$

Hence, it follows that we have the approximation

$$\frac{L_{n+1}}{L_n} \simeq (1-2c)^2 (x_n + y_n)^2,$$

and by considering $0 \leq x_n + y_n \leq 2$, we obtain

$$0 \leq \frac{L_{n+1}}{L_n} \leq 4(1-2c)^2$$

and finally by the synchronization condition (6) we have

$$-\frac{1}{2} < 1 - 2c < \frac{1}{2} \Leftrightarrow \frac{1}{4} < c < \frac{3}{4}.$$

Moreover, if we consider a more realistic bound for $x_n + y_n$ as $S = sup_n(x_n + y_n)$, a better range for c can be obtained

$$-\frac{1}{S} < 1 - 2c < \frac{1}{S} \Leftrightarrow \frac{1}{2}\left(1 - \frac{1}{S}\right) < c < \frac{1}{2}\left(1 + \frac{1}{S}\right).$$

In order to ilustrate the efficiency of this synchronization method we consider some numerical simulations. For example, if $a = 1.57 > a_M$, where a_M denotes the Misiurewicz point, $c = 0.26 < c_{crit}$ and different initial conditions $x_0 = 1.1$, $y_0 = 1.8$, we obtain that the synchronization takes place after few steps (8 iterations). Figure 1 illustrates the stabilization of the dynamics of the coupled system on the synchronization set $x = y$. The transition point (8 iterations) are the points outside the attractor. We also represent the time series of the x_n and y_n variables and the error $x_n - y_n$. For any value of c closer to the critical threshold the synchronization is attained in the shorter time interval.

4 Unidirectional coupling based on singular value decomposition

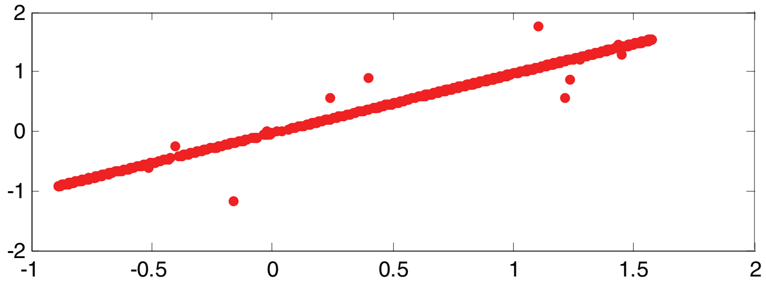
Consider two identical chaotic dynamical systems

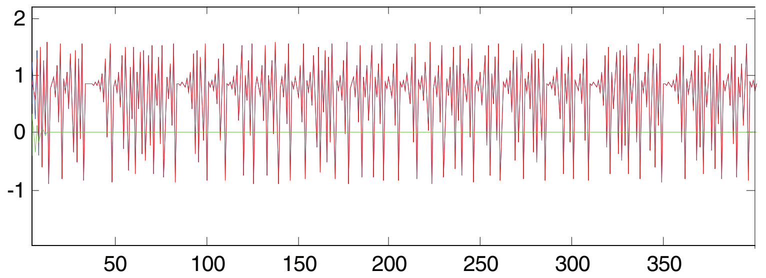
$$\begin{cases} x_{n+1} = f_a(x_n) \\ y_{n+1} = f_a(y_n) \end{cases},$$

where $a \in \mathbb{R}$ is the control parameter. We want to synchronize these systems by using a dissipative coupling, that is

$$y_{n+1} = f_a(y_n) + C(x_n - y_n) \qquad (8)$$

where $C = C(x_n)$ is the coupling matrix that suppresses the local expansion of the flow along the noncontracting directions.





Following Junge and Parlitz [5], we consider the singular value decomposition of the Jacobian matrix,

$$Df_a = U \cdot \Sigma \cdot V^T,$$

where U and V are orthogonal matrices and $\Sigma = diag\,(\sigma_i)$ is a diagonal matrix with positive elements σ_i, represented by the singular values of Df_a. Thus, we can use one of the most powerful coupling scheme defined by

$$\begin{cases} x_{n+1} = f_a(x_n) \\ y_{n+1} = f_a(y_n) + \sum\limits_{i=1}^{k} \sigma_i \left[\langle x_n, v_i\,(x_n) \rangle - \langle y_n, v_i\,(y_n) \rangle \right] u_i \end{cases},$$

where we assume that exist k local noncontracting directions at y_n and $v_1, v_2, ..., v_k$ are the column vectors of V with corresponding singular values $\sigma_1, \sigma_2, ..., \sigma_k \geq 1$. Note that, choosing

$$C = U \cdot diag\,(\sigma_1, \sigma_2, ..., \sigma_k, 0, ..., 0) \cdot V^T,$$

the matrix $Df_a - C$ governing the synchronization error dynamics $e_n = y_n - x_n$ yielding by (8) as

$$e_{n+1} = \left[Df_a\,(y_n) - C \right] \cdot e_n,$$

is given by the singular value decomposition

$$Df_a - C = U \cdot diag\,(0, ..., 0, \sigma_{k+1}, \sigma_{k+2}, ..., \sigma_m) \cdot V^T,$$

where appear only singular values that are smaller than 1. So that, the choice of this matrix C guarantees the linear stability of the synchronous state[1].

[1] Junge and Parlitz has showed that this scheme coupling allows for desactivating the coupling from time to time in order to reduce the information flow from the drive to the response system. They discuss two ways to exploit this feature: by sporadic coupling (where T iterations are performed before the next coupling signal is computed from the current state and transmitted to the response system where it is applied in the coupling) and in partitioned state space (the singular values σ_i de $Df_a\,(x_n)$ depend, in general, on the state x_n, so one may restrict the coupling to the regions in the state space where strong expansion has to be suppressed).

Example: To show the efficiency of the proposed coupling, in deficit to linear coupling, we consider again the one-dimensional quadratic map $f_a(x) = a - x^2, a \in \mathbb{R}$. The singular value of the Jacobian matrix of f_a is given by the square root of the eigenvalue of the matrix $Df_a^T \cdot Df_a$. Thus, we have the characteristic equation $4a^2x_n^2 - \lambda = 0$ and the singular value is $\sigma = -2a|x_n| = -2ax_n$.

For $a \in [1.545, 2]$ we observe chaotic motion for the quadratic map. Then we find a positive Lyapunov exponent and the singular value satisfy $\sigma = -2ax_n \geq 1$, when $x_n < 0.25$, indicating expansion.

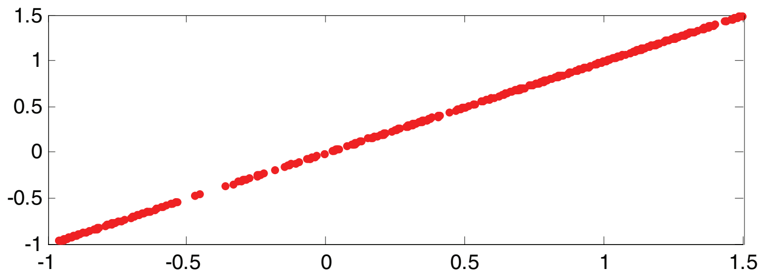
Choosing the coupling constant $C = C(x_n) = I_1 \cdot \Sigma \cdot I_1^T = -2ax_n$, the coupling term can be written as

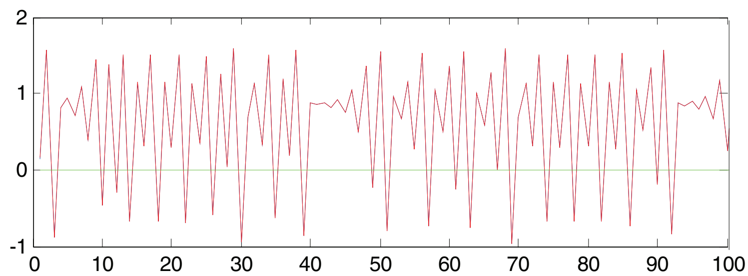
$$C(x_n - y_n) = -2ax_n [\langle x_n, v_1 \rangle - \langle y_n, v_1 \rangle] u = -2ax_n (x_n - y_n),$$

and the coupling scheme is given by

$$\begin{cases} x_{n+1} = a - x_n^2 \\ y_{n+1} = a - y_n^2 - 2ax_n(x_n - y_n) \end{cases}.$$

In practical simulation the synchronization is achieve in very short time. With $a = 1.6 > a_M$ and the initial conditions $x_0 = 0.15,\ y_0 = 0.16$ we obtain the Figure above.





The first question we want to address with this example is how much the approximation of the local singular values and vectors at the response system degrades the performance of this coupling. For this purpose we have compared the coupling given below with a corresponding coupling given by linear and quadratic coupling, as illustrated in Figure.

We can observe that for the similar initial conditions and parameter settings, the synchronization is achieved after more than 20 iterations. This is very important since in economical or biological systems the expenses to apply these synchronization techniques are very elevated and not practical if the time to achieve synchronization is very large.

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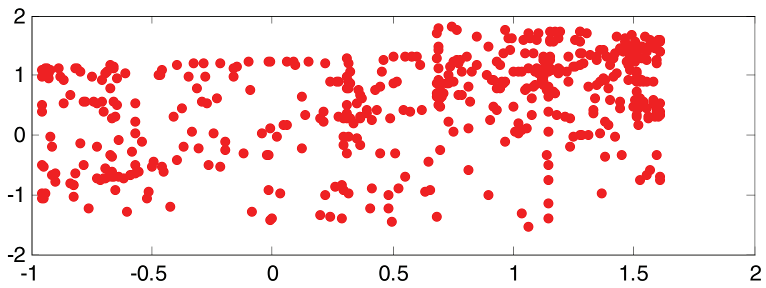

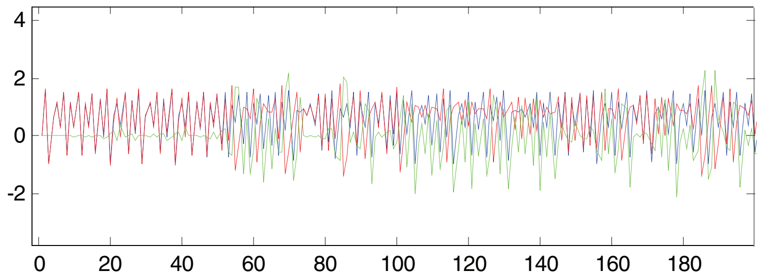

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