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A METHOD TO APPROXIMATE FIRST PASSAGE TIMES DISTRIBUTIONS IN DIRECT TIME MARKOV PROCESSES

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Abstract. A numerical method to approximate first passage times distributions in direct Markov processes will be presented. It is useful to compute sojourn times in queue systems, namely in Jackson queuing networks. Using this method (Kiessler *et al*., 1988) achieved to clear a problem that arises in the Jackson three node acyclic networks sojourn times.

Key words. randomisation procedure, sojourn time, Jackson three node acyclic networks

1 Introduction

In this work it will be described a general method, which key is the proceeding called, in the English language literature, "randomisation procedure" to approximate "first passage times" distributions in direct time Markov Processes, being the sojourn times in queue systems a particular case.

Call $\aleph = \{X(t): t \geq 0\}$ a regular Markov Process, in continuous time with a countable states space E and a bounded matrix infinitesimal generator Q.
The elements of Q are designated by

$$Q(x,y), x,y \in E \text{ and } Q(x) = \sum_{y \in E - \{x\}} Q(x,y).$$

$\psi(t)$ designates the $X(t)$ state probability vector:

$$\psi_t(x) = P\{X(t) = x\}, x \in E \qquad (1).$$

X models, for instance, the evolution of a queue system during the sojourn of a given, "marked", customer in it.

The states of E have two main components:

i) The queue system state,
ii) The "marked" customer position.

Call

- A the states subset that describes the system till the departure of the "marked" customer, and
- B the state subset that describes the system after the departure of that customer.

Evidently

- $\{A, B\}$ is a partition of E,
- If T is the time that the process $\aleph$ spends in A till attaining B, for the first time, T is precisely the sojourn time of the "marked" customer in the queue system.

It is supposed that $\aleph$ will remain in B, with probability 1 after having attained it for the first time. In fact, as the evolution of the system after the departure of the "marked" customer is irrelevant, it may be supposed that B is a closed set. That is, the process $\aleph$ cannot come back to A after reaching B. The quantity of interest is the T distribution function, $\tau(t)$. Note that

$$\tau(t) = P\{T \leq t\} = P\{X(t) \in B\} = 1 - P\{X(t) \in A\}, t \geq 0 \qquad (2)$$

since the presented hypotheses guarantee that $\{T \leq t\} = \{X(t) \in B\}$.

After (2) it is concluded that

- The problem of computing $\tau(t)$ is equivalent to the one of the computation of the transient distribution of $X(t)$ in A.

2 The Randomisation Procedure

From section 1 it follows that it is necessary to compute the vector $\psi_t, t \geq 0$. Being $P_t, t \geq 0$, the $\aleph$ n transition matrix,

$$\psi_t = \psi_0 P_t, t \geq 0 \qquad (3)$$

and

$$P_t = exp(Qt) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n, t \geq 0 \qquad (4).$$

The "randomisation procedure" consists in using in (4) an equivalent representation, see (Çinlar, 1975):

$$P_t = exp(-\alpha t) exp\left(\alpha t \left(I + \frac{1}{\alpha} Q\right)\right) = exp(-\alpha t) \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} R^n \qquad (5)$$

where

$$R = I + \frac{1}{\alpha} Q \qquad (6)$$

is called the "randomised matrix" in English language literature,

- I is the identity matrix, and
- α is a positive majorant of the whole $Q(x), x \in E$.

Note that, see (Melamed and Yadin, 1984, 1984a),

- Although the equation (5) seems more complex than (4), fulfils in fact more favourable computational properties. The most important is that R is a stochastic matrix while Q is not. Consequently, the computation using (5) is stable and using (4) is not,
- The "randomisation procedure" has an interesting probabilistic meaning, useful to determine bounds for $\tau(t)$. In fact, being R a stochastic matrix, it defines a discrete time Markov Process

$$\mathfrak{I} = \{Y_n : n = 0, 1, \dots\} \qquad (7)$$

if it is assumed $Y_0 = X(0)$. With this procedure, the relation between the processes ℵ and $\mathfrak{I}$ is quite simple as it will be seen next.

Extending the discrete time process $\mathfrak{I}$ to a continuous time Markov Process such that

i) The time intervals between jumps are exponential random variables i.i.d. with mean
ii) The jumps are commanded by *R*.

In (Melamed and Yadin, 1984) it is shown that the resulting process is precisely the original process ℵ; but when there is a sequence of jumps in $\mathfrak{I}$ from the state $x \in E$, this will be noticed in ℵ as a long sojourn in state *x*.

So, the "randomisation procedure" may be interpreted as a sowing in the process ℵ with "fake" random jumps between the true jumps. The resulting process, designated by $\overline{\aleph}$, at which the "fake" jumps are visible, has the same probabilistic structure than ℵ but with an advantage:

- The sequence of the jump instants in $\overline{\aleph}$, "fake" and "true", is now a Poisson Process. This is not, in general, the case of ℵ.

Note that Y_n is the state of $\overline{\aleph}$ in the instant of the n^{th} jump, "fake" or "true".

Suppose that $\overline{\aleph}$ reaches the set *B* in its n^{th} jump. Consequently the $\overline{\aleph}$ sojourn time, and so also the ℵ, in *A* is the sum of *n* exponential independent random variables with mean . That is the sojourn time has a *n* order Erlang distribution with parameter α. Its distribution function will be designated $E_{n,\alpha}(t)$.

Be $h(n)$ the probability that $\overline{\aleph}$ reaches *B* in its n^{th} jump. Call ϕ_n the state probability vector of Y_n:

$$\phi_n = \psi_0 R^n \qquad (8).$$

The quantities $h(n)$ are given by the equivalent formulae:

$$h(n) = \begin{cases} \sum_{x \in B} \phi_0(x), n = 0 \\ \sum_{x \in A} \sum_{y \in B} \phi_{n-1}(x) R(x, y), n > 0 \end{cases} \qquad (9)$$

or

$$h(n) = \begin{cases} 1 - \sum_{x \in A} \phi_0(x), n = 0 \\ \sum_{x \in A} \phi_{n-1}(x) - \sum_{x \in A} \phi_n(x), n > 0 \end{cases} \qquad (10).$$

Given the probabilities $h(n)$ and, noting that $\sum_{n=0}^{\infty} h(n) = 1$, it is obtained

$$\tau(t) = \sum_{n=0}^{\infty} h(n) E_{n,\alpha}(t), t \geq 0 \qquad (11),$$

$$E[T^m] = \frac{1}{\alpha^m} \sum_{n=0}^{\infty} n(n+1) \dots (n+m-1) h(n), m = 1, 2, \dots \qquad (12).$$

The formula (12) for $m = 1$ is

$$E[T] = \frac{1}{\alpha} E[H] \qquad (13)$$

being H the number of ℵ jumps till reaching B. Expression (13) is the Little's Formula in this context.

Equation (11) allows obtaining simple bounds for $\tau(t)$ that may, in principle, to become arbitrarily close. Equation (12) allows to obtain $E[T^k]$, in principle, so close of $E[T^k]$ as wished. So, given any integer $k \geq 0$

$$L_k(t) \leq \tau(t) \leq U_k(t) \qquad (14)$$

where

$$L_k(t) = \sum_{n=0}^{k} h(n) E_{n,\alpha}(t), t \geq 0 \qquad (15),$$

$$U_k(t) = 1 - \sum_{n=0}^{k} h(n) \bar{E}_{n,\alpha}(t), t \geq 0 \qquad (16)$$

and

$$E[T^m]_{L,k} \leq E[T^m], \;\; m = 1, 2, \dots . \qquad (17)$$

where

$$E[T^m]_{L,k} = \frac{1}{\alpha^m} \sum_{n=0}^{k} n(n+1) \dots (n+m-1) h(n), m = 1, 2, \dots \qquad (18).$$

It is easy to prove that

Proposition

If, for any $\varepsilon > 0$, k is chosen in accordance with the rule

$$k = min\left\{n \geq 0: \sum_{i=0}^{n} h(i) \geq 1 - \varepsilon\right\} = k(\varepsilon), \qquad (19)$$

or equivalently

$$J = min\left\{n \geq 0: \sum_{x \in A} \phi_n(x) \leq \varepsilon\right\} = J(\varepsilon), \qquad (20)$$

$$\left|\tau(t) - L_{J(\varepsilon)}\right| \leq \varepsilon \text{ and } \left|\tau(t) - U_{J(\varepsilon)}\right| \leq \varepsilon, \text{uniformely in } t \geq 0. \blacksquare$$

3 Concluding Remarks

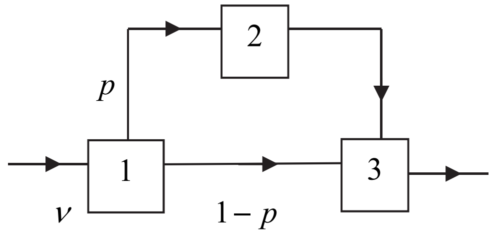
The main problem in the application of the method presented, that in principle would solve any computation problems related to the distribution of sojourn times, stays in the difficulty of the $h(n)$ computation. In fact, for it, it is necessary to compute the vectors ϕ_n but only in the subset *A* of the states space. When states space *E* is finite, has it happens, for instance in the case of closed queue networks, both $h(n)$ and ϕ_n can, at first glance, be computed exactly, apart the mistakes brought by the approximations.

In practice the states space is often infinite or, although finite prohibitively great. In this situations it is mandatory to truncate *E*. So, it must be considered a new level of approximation since the $h(n)$, ϕ_n, etc. must also be approximated now.

In fact, what is viable to obtain is $h(n)$ minorants because the *E* truncation is translated in probability loss (Melamed and Yadin, 1984a). So, with these $h(n)$ approximate values, (14) and (17) go on being valid but

- The uniform convergence property seen above is lost,
- The rules analogous to (19) and (20) are not equivalent. The one generated by (19) may be even unviable and in practice it is used only the one generated by (20) (Melamed and Yadin, 1984a).

Using this method (Kiessler *et al.*, 1988) achieved to show that, in a Jackson three node acyclic network, see Figure 1, the total sojourn time distribution function for a customer that follows



ν – *exogenous arrival rate*

p – *probability of a customer goes to* 2 *after* 1

Figure 1: **Jackson Three Node Acyclic Network**

the path integrated by the nodes 1, 2, and 3 is not the same obtained considering that S_1, S_2 and S_3 (the sojourn times at nodes 1, 2 and 3 respectively) are independent although this one, designated by $F(t)$, is a "good" approximation of that one. They show that in some particular cases it was not true that

$$F^L(t) \leq F(t) \leq F^U(t), t \geq 0 \qquad (21)$$

being $F^L(t)$ and $F^U(t)$ the minorant and the majorant, respectively, of that customer sojourn time distribution function, obtained through the described method.

This conclusion is important because, in spite of the dependence between S_1 and S_3, see for instance (Ferreira, 2010), $F(t)$ could be the $S = S_1 + S_2 + S_3$ distribution function. In fact, (Feller, 1966) presents an example of dependent random variables which sum has the same distribution as if the random variables were independent.

Finally note that the formula (12), apparently new, seems to be of great efficiency, although only allows to obtain moments minorants, because its field of application is a broad one.

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