SYNCHRONIZATION OF CHAOTIC DYNAMICAL SYSTEMS: A BRIEF REVIEW

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ABSTRACT

There are several reasons for the approach to chaos synchronization. This phenomenon is immediately interesting because of its high potential for applications. But, first of all, it is particularly interesting the study of a phenomenon that requires the adjustment of dynamic behaviors in order to obtain a coincident chaotic motion, being this possible even in chaotic dynamical systems in which sensitive dependence on initial conditions is one of the features. The possibility of applying techniques of chaos control in order to optimize the results of synchronization is alsoa motivating factor for the study of this phenomenon.

It is presented a brief review of preliminary notions on nonlinear dynamics and then is considered in detail the synchronization of chaotic dynamical systems, both in continuous and discrete time.

Key words: chaos synchronization, chaos control, local and global asymptotical stability

1. INTRODUCTION

The first observation of synchronization was reported by the Dutch physicist Christian Huygens in 1665 (1). In that case, the phenomenon was evidenced by the equal periods of mutually coupled clocks. Subsequently, the synchronization of periodic signals was being detected in many dynamical processes, becoming common in other scientific areas and the subject of various applications in engineering. Given this seminal discovery of Huygens, the synchronization phenomenon is, in the classical sense, associated with the ability of self-oscillatory dissipative systems to adjust their behaviors in order to follow a global periodic motion.

Today the term synchronization is used in a more generalized sense including discrete and continuous chaotic coupled systems. The numerical analysis of chaotic populations found in nature, even heterogeneous, reveals that the synchronization is possible and frequent.

2. CHAOTIC BEHAVIOR

The study of dynamical systems was based, for a long time, on examples of differential equations with regular solutions. If these solutions remained in a bounded region of the phase space, then they correspond to one of two types of behavior: a stable equilibrium point or a periodic (or quasi-periodic) oscillation.

Many deterministic nonlinear systems exhibit, apart from fixed-point solutions and limit cycles, more complex invariant sets which act as attractors for their dynamics. Among them we can find chaotic attractors. Chaos belongs to the wide field of nonlinear oscillations theory which significant development was initiated in the last century. In the 20s and 30s, Andronov proposed a classification of nonlinear behaviors (1933) and Van der Pol detected experimentally type-noise oscillations in electronic circuits (1927). However the experiment that boosted the consideration of chaotic behavior was due to Lorenz (2). In 1961, working in a simplified model of atmospheric transfer with three nonlinear differential equations, he observed numerically that making very small changes in the initial conditions he got a huge effect on their solutions. It was the evidenceof one of the main properties of chaotic dynamics which was later known as sensitive dependence on initial conditions. This property had already been investigated from the topological point of view by Poincare who described it in his monograph "Science and Method" (1903).

Due to this sensitivity, an infinitesimal perturbation of the initial conditions of a dynamical system, in discrete or continuous time, leads to exponential divergence of initially nearby starting orbits over time. For many years this property became undesirable chaos, since it reduces the predictability of the chaotic system over long time periods. However, the scientific community was gradually becoming aware of a third type of dynamical behavior: the chaotic behavior. Some experiments, which abnormal results had been previously explained in terms of experimental error or additional noise, were evaluated for an explanation in terms of chaos and it became the subject of a rigorous mathematical study. A measure of sensitivity to initial conditions is currently quantified by the Lyapunov exponents. Topological entropy and metric entropy are quantities that, such as the Lyapunov exponents, also measure the dynamical complexity of a system.

In the mid-70s, the term deterministic chaos was introduced by Li and Yorke(3) in a famous paper entitled "Period three implies chaos". In that paper they presented the study of the possible periods of periodic points for continuous real maps defined in an interval$I$of real numbers. A continuous map $f: I \to \mathbb{R}$ is chaotic if **(i)**f has periodic points of period n ,$\forall n \in \mathbb{N}$, **(ii)** there is an uncountable set $J \subset I$ such that

$$\lim_{n\to\infty} sup\,|f^n(x) - f^n(y)| > 0 \text{and} \lim_{n\to\infty} inf\,|f^n(x) - f^n(y)| = 0,$$

$\forall x, y \in J$, and **(iii)** for any $x \in J$ and x^* a periodic point of f is valid

$$\lim_{n\to\infty} sup\,|f^n(x) - f^n(x^*)| > 0.$$

By condition **(ii)**, any two orbits in J successively move away and approach over time and, by condition **(iii)**, the periodic points are not asymptotic to any point of J.

Also, these authors provided that a map$f$which has a *3*-periodic orbitis chaotic.However, this result can be considered as a corollary of a strong Schakowsky's result presented in a paper written in Russian (4) in 1964. That result about the existence of periodic points for continuous maps is based on a certain order of natural numbers designated by Schakowsky's sequence,

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd 2\cdot 7 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^2\cdot 5 \lhd 2^2\cdot 7 \lhd \cdots \lhd 2^n\cdot 3 \lhd 2^n\cdot 5 \lhd 2^n\cdot 7 \lhd \cdots \lhd 2^\infty \lhd \cdots \lhd 2^{n+1} \lhd 2^n \lhd \cdots \lhd 2 \lhd 1.$$

According to the Schakowsky's theorem, if a continuous map $f: I \to \mathbb{R}$ has a periodic point of period p and $p \lhd q$ in theSchakowsky's sequence, then f also has a periodic point of period q. Although there is no single formal definition of the term deterministic chaos, the chaotic behavior can be defined as an observable pattern that appears irregular and unpredictable in large time scales.

Geometrically, an attractor can be a single point, a curve, a manifold, or even a set with fractal structure known as strange attractor.An invariant subset $E \subseteq X$ of a dynamical system $S = (X, T, \phi^t)$ is chaotic if satisfies the properties **(P1)**ϕ^t has sensitive dependence to initial conditions, **(P2)**ϕ^t is topologically transitive in E, and **(P3)** the periodic points of ϕ^t are dense in E. The presence of a chaotic attractor in the phase space -which typically has embedded an infinite dense set of unstable periodic orbits - leads that it is impossible to determine the position of the system in the attractor over time, even knowing its position on that attractor at earlier time. The unstable periodic orbits embedded in the chaotic attractor are a dynamical invariant of the system and its systematic research allows the characterization and estimation of many other dynamical invariants such as natural invariant measure, spectrum of Lyapunov exponents and fractal dimensions. The relationship between the trajectories of the chaotic attractor and the unstable periodic orbits embedded in it has been explored in detail for hyperbolic dynamical systems, for which the separation into stable and unstable invariant subspaces is consistent (robust) under the dynamics evolution (5), and for non-hyperbolic systems with homoclinic tangencies. The infinite set of unstable periodic orbits embedded in a chaotic attractor located in some symmetrical invariant manifold plays a fundamental role in the destabilization mechanism of that attractor, since it is responsible by the dynamics of phenomena such as riddling of attraction basin and bubbling of chaotic attractor (6,7).

Many nonlinear dynamical systems depend on control parameters in a certain parameter space. For purposes of applications it is crucial to understand the evolution of the qualitative behavior of elements of a dynamical systems family when the parameters vary. The qualitative change of behavior induced by a variation of one of their parameters is known as bifurcation. The Poincare's work for describing the separation of the equilibrium solutions in a family of differential equations was pioneering about the importance given to qualitative/topological changes in the dynamical behavior and about the use of the term bifurcation. However, there is still no agreement with the rigorous meaning of this term. Note that the bifurcation theory is a theme with classical mathematical origins, for example in the Euler's works in the eighteenth century. To clarify the possible "routes to chaos", had great relevance a paper (8) of May, published in 1976 in Nature, with a description of the period-doubling bifurcation in the logistic map.

It may happen that, for different regions of parameter space of the system, there are different attractors (multiple attractors); so, also the qualitative asymptotic behavior of the system depends on the initial conditions taken. Even in systems with symmetry, the phenomenon of multi-stability may occurs, where take place the coexistence of more than one attractor for a given set of parameters. In this case, the qualitative asymptotic behavior of the system cannot be predicted because for each initial condition it is not known, in principle, in which of the attractors finally stabilizes the system dynamics.

3. CHAOS SYNCHRONIZATION

The chaotic behavior can be observed in natural systems as in experiments and computational models of many scientific areas. It appears to be a robust phenomenon. The ability of the chaotic dynamics for enlarge small perturbations potentiates its use in order to obtain specific desired states in a chaotic system, without substantially changing their main dynamical properties and with a small spent of energy. The developments in theory of chaos control and synchronization are a consequence of this. Synchronization and control of chaotic motions have common roots based on driving the nonlinear chaotic system to restrict its motion -- in each case one selects parameter regions to perturb or external forcing to achieve the collapse of the full state space to a selected accessible subspace. In chaos control we seek to move the system to an accessible invariant set where the movements are more regular, while in chaos synchronization we seek an invariant set in the coupled system space - the subspace of synchronization - in which the synchronous motion takes place. Although the concept of chaos synchronization has generally evolved independently, there is a recent tendency to unify the study of chaos control and synchronization under the same rubric.

The dynamics of a system displays chaotic behavior when it never repeats itself and even if initial conditions are correlated by proximity, the corresponding trajectories quickly become uncorrelated. As such, the possibility of two (or more) chaotic systems oscillates in a coherent and synchronized way is not an obvious phenomenon, since it is not possible to reproduce exactly the initial conditions. Even following the evolution of two identical chaotic systems which start from nearby points in phase space, but not exactly the same, we find that they diverge from each other in their behaviors, although both retain the same pattern of chaotic attractor. Two chaotic processes are observed without any mutual correlation (independent). Intuitively, the terms chaos and synchronization may seem mutually exclusive, even if it is chosen an appropriate coupling.

On the other hand, the possibility of qualitative/topological changes in the dynamic behavior of elements of a dynamical systems family indicated by the bifurcation points, difficult the synchronization of no identical systems, differing even by small parametric mismatches. An infinitesimal deviation in any of the parameters can leads to qualitatively different dynamics.

Studies in the last three decades have shown that it is false the intuitive idea of impossibility of chaotic systems behave synchronously. Although it is impossible to reproduce exactly the same initial conditions and parameters, there are sets of coupled chaotic oscillators in which the attractive effect of a sufficiently strong coupling can counterbalance the trend of the trajectories to diverge due to chaotic dynamics. As a result, it is possible to reach full synchronization in chaotic systems since they are coupled by a suitable dissipative coupling. Synchronizing chaotic systems is then to couple them in order to "force" that they follow the same trajectory in the chaotic attractor, by applying small perturbations between the systems. Still, two synchronized trajectories may lose the stability of synchronization due to the influence of external noise. However, due to ergodic property of chaotic trajectories, after a finite transient time, they will be coming again and can synchronize again. In this sense, the synchronization is robust against small external noise.

The possibility to synchronize two chaotic dynamical systems depends on several conditions, such as the parametric region of each one, the coupling strength used between them and the degree of differentiation of systems themselves. The goal is to synchronize de systems with a minimum coupling strength.

Although there is no single way to couple chaotic systems, the coupling must possess certain relevant properties. It is intended that the coupling **(i)** is dissipative, that is, that tends to make the state vectors of the systems coming together, and **(ii)** does not affect the synchronous chaotic state. It is possible to consider two coupling mechanisms: unidirectional (one-way or directional) and bidirectional (mutual or global) coupling. In unidirectional coupling, only the dynamics of a system - the response or slave system - is affected by the dynamic of the other - the drive, transport or master system. The bidirectional coupling implies mutual interaction between the systems.

Coupled chaotic dynamical systems are constructed from simple, low-dimensional elements to form new and more complex organizations, with the guarantee that dominant features of the underlying individual components will be retained, that is, the two systems are able to act mostly independent, with no one dominates the other. This building up approach ("cumulative" construction) can also be used to create a new system whose behavior is more flexible and/or richer than their components, but whose analysis and control are remains tractable. The concept that several systems nonlinearly interacting, collectively give rise to new dynamics that are not attributable to individual component parts is designated for emergency.

It is not possible to anticipate the consequences of a coupling between chaotic systems. It can drastically change the qualitative properties of dynamics. It can stabilize in periodic behavior, can occasionally produce hidden correlations between the elements (although the dynamics are apparently turbulent), or can induce synchronization of a subset of dynamical variables. The effectiveness of a coupling between systems of equal dimension is given, firstly, by the analysis of the difference between the coordinates of the respective variables of the systems involved - the synchronization error. In optimal case, a coupling between chaotic systems leads to its asymptotic synchronization, in which the synchronization error converges to zero over time. Two dynamical systems $S_1 = (X, T, \phi^t)$ and $S_2 = (X, T, \varphi^t)$ are inasymptotic synchronization if

$$\lim_{t\to+\infty} \|\phi^t(x) - \varphi^t(x)\| = 0$$

for a certain coupling strength between them. However, it was recently introduced by Stefański and Kapitaniak (9) a less demanding synchronization form - the practical synchronization in Kapitaniak sense - in which it is only expected to stabilize the synchronization error$\phi^t(x) - \varphi^t(x)$ below a constant less than one. The systems $S_1 = (X, T, \phi^t)$ and $S_2 = (X, T, \varphi^t)$ are in practical synchronization if

$$\|\phi^t(x) - \varphi^t(x)\| < K$$

for a positive constant $K < 1$.

When practical synchronization is not achieved, but the difference between the dynamical variables of the systems is bounded, we still can apply a chaos control technique. Note that, the chaotic dynamics introduces new freedom degrees in sets of coupled systems. However, when two (or more) chaotic oscillators are coupled and synchronization is achieved, in general the number of dynamical freedom degrees for the coupled system actually decreases. However, the specificities of chaotic behavior make impossible to apply, directly to the synchronization of chaotic systems, the methods developed for synchronization of periodic oscillations.

4. MECHANISMS OF COUPLING

4.1. Identical synchronization

Chaos synchronization began in the mid-80s with the studies of Kaneko (10) and Afraimovich*et al.*(11) about coupling of discrete and continuous identical systems, respectively, evolving from different initial conditions.

Also important, in the same decade, the research of Fujisaka and Yamada (12,13) who introduced the study of transversal Lyapunov exponents in the coupled system, emphasizing the idea of how the dynamics can change with the coupling, and the research of Pikovsky (14,15) and Afraimovich*et al.*(11) that presented many of the fundamental concepts in chaos synchronization. It was only in the 90s, with the research of Pecora and Carroll (16-18), that the chaos synchronization has been consolidated as a topic of independent search, together with the rigorous establishment of control chaos theory by Ott, Grebogy and Yorke(19). These papers immediately received a great deal of attention from the scientific community, and opened up a wide range of applications outside the traditional scope of chaos and nonlinear dynamics research. Since then, various synchronization methods and several news concepts necessary for analyzing synchronization have been developed.

In communication systems, especially those that involve signals which future behavior is difficult (or impossible) to predict during transmission, some type of synchronization between transmitter and receiver is obviously helpful. This fact motivated the research of Pecora and Carroll (16-18), where it was established a synchronization method by coupling two identical chaotic dynamical systems through transmission of a driver subsystem which acts as a common chaotic signal between them. In this way, the method of Pecora and Carroll, also known as complete replacement, suggests how a synchronous chaotic state can be used as a driver in communication. Thus, given a chaotic system, this synchronization method requires the decomposition of the system in order to obtain an appropriate driver subsystem. So they are usually tested several combinations of a subset of state variables in order to identify a stable driver subsystem. It seems counterintuitive that a non-dissipative system can leads to synchronization, but in a multidimensional volume preserving dynamical system, there must be at least one contractor direction so that volumes in phase space are preserved, which allows choosing a stable subsystem. Given the possibility of synchronizing chaotic systems, it is necessary to determine conditions under which the synchronization is stable. In these papers, beyond to establish the coupling mechanism, which is relatively straightforward but deceptively simple, Pecora and Carroll (16, 17) presented the first response to this question.

Although in complete replacement there is only a finite number of possible decompositions of a chaotic system, the basic idea of Pecora and Carroll of decompose chaotic systems in order to obtain a driver subsystem, led to another more general synchronization methods, which do not require the decomposition of original chaotic system into two stable subsystems. Kocarev and Parlitz (20,21) proposed the active-passive decomposition method based on decomposition of the transport system into two components - active and passive - and transmission of a scalar signal from the transport system to the system response which may be function of an information signal. Different replicas of the passive component synchronize when driven by the same active component. According Stefański and Kapitaniak(22) a trajectory of a chaotic system can synchronize with a chaotic trajectory of a similar system by adding a linear damper term proportional to the difference between the corresponding state variables of both systems. This term acts as a control signal applied only to the response system as negative feedback and, as such, the coupling is said to be by negative feedback control. Applying this method to different experimental models, by authors such as Lai and Grebogy(23), Kapitaniak(24), John and Amritkar(25), Anishchenko*et al.*(26), Ding and Ott(27), Pyragas(28) and Wu and Chua (29), has shown that it is effective when the coupled system has a single attractor.

According Fujisaka and Yamada (12,13), a natural way to introduce a bidirectional coupling between two identical chaotic systems is to add symmetric linear coupling terms to the expressions that define them. This coupling mechanism, which may be total or partial, is called linear diffusive coupling. A study of Stefański(30) showed that the properties of exponential divergence and convergence in total coupling allow to estimate the largest Lyapunov exponent of any chaotic dynamical system, a possibility which is especially useful in non-smooth systems, where the estimation of Lyapunov exponents is not straightforward.

As the above, there are various coupling mechanisms between identical chaotic systems so that they become synchronized. In this regime, known as identical synchronization, the chaotic trajectories of the identical chaotic systems in coupling coincide exactly in time due to the strong interaction between them. Each system retains its chaotic behavior, but when it is reached the symmetrical synchronous state, that is, when the evolution of their state vectors is coincident, the dynamics of coupled system is restricted to a synchronization hyperplane in the phase space.

In coupling of discrete chaotic systems are considered parameters through which is controlled the coupling strength between the systems. The same happens in linear diffusive coupling and coupling by negative feedback control of continuous systems. Based on the structure of coupling terms involving these parameters, we can distinguish several coupling schemes in discrete time: internal or external dissipative coupling, linear diffusive coupling, coupling by quadratic terms or bilinear coupling. The results of stability of synchronous chaotic state depend on the coupling parameters considered.

4.2. Generalized synchronization

Although the phenomenon of chaos synchronization has been detected from examples with identical chaotic systems, the behavior of these synchronized systems is just a sample of the abundance of different types of interdependent behavior that can be detected in coupled chaotic systems. Although the identical synchronization regime is the most current and with a large number of theoretical results, many studies have shown that it is also possible to obtain synchronization of nonidentical systems, defined by evolution laws that either differ only by small parametric mismatches or are even distinct (and may even differ in dimension), called generalized synchronization, in bidirectional or unidirectional configuration. Many of the coupling schemes between nonidentical chaotic systems are an extension of the well-known among identical systems.

The first mathematical definition of synchronized chaos in a generalized sense is due to Afraimovich*et al.*(11), in 1986, and is based on the existence of a homeomorphism between systems with parametric mismatches which relates the projections of the synchronized chaotic trajectories on subspaces of the phase spaces of transport and response systems. However, the term generalized synchronization was only introduced by Rulkov*et al.* (31), in 1995. Referring to the synchronization of periodic systems unidirectionally coupled, the central idea in that paper, as well as in the paper of Kocarev and Parlitz(32) in 1996, is to take the ability for predict the current state of the response system from the chaotic information measured in the transport system as a definition of generalized synchronization. Predictability points to the existence and stability of a chaotic attractor of the coupled system.

Thus, most methods for detecting considers that the generalized synchronization is reflected by the existence of a functional relationship between the systems, as regular as possible, whose graph contains an invariant stable manifold - the synchronization manifold - in which is embedded the attractor chaotic on the synchronization. This synchronization regime is weaker than the identical synchronization since the persistent (robust) and stable dependence between the state vectors of each system is not necessarily the identity of states. On the other hand, the absence, in general, of inherent symmetry in the coupled system makes it difficult to obtain a stable synchronous state. Note that the functional relationship need not be valid in whole phase space of coupled systems, but only in the invariant manifold.

The original definition of Afraimovich*et al.*(11), although it allows many analytical results about stability of synchronous states, it is not particularly satisfactory because it makes no reference to the attractive nature of the synchronization set and require verification of conditions whose validity in real experiments cannot always be displayed. On the other hand, the definition of Rulkov*et al.*(31), covers situations in physics, biology and economy, in which was detected chaos synchronization, but in which the requirement of a homeomorphism between the projections was not satisfy. However, for systems with invertible dynamics, the definition of Rulkov*et al.*(31) is equivalent to the existence of a continuous function between the states of the systems when they are in the synchronous chaotic attractor.

Studies have shown that the response system is asymptotically stable if there is a function - the synchronization function - that transforms each trajectory in the attractor of the transport system into a trajectory in the attractor of the response system. In this case, the synchronized chaotic trajectories are located in a stable synchronization manifold. Based on the equivalence of generalized synchronization in the coupled system and asymptotic stability of the response system, Abarbanel*et al.* (33) established a criterion for detecting generalized synchronization called the auxiliary system approach.

The generalized synchronization includes the identical synchronization as a particular case, in which the functional relationship is the identity function and the synchronization manifold is a hyperplane. However, while the identical synchronization is easily seen by representing the difference between the coordinates of the two systems in coupling, to detect generalized synchronization does not follow a simple method, especially when we analyze information obtained experimentally. Except in special cases of coupling between systems with small parametric mismatches, it is rarely possible to exhibit explicit formulas of the synchronization function or have a trivial synchronization manifold in the phase space. In general, the synchronized chaotic oscillations are different from those generated by the uncoupled chaotic systems. Therefore, the analogy between the synchronous chaotic attractor and the chaotic attractors of the uncoupled systems cannot be considered as a requirement to define generalized synchronization.

The asymptotic stability of the response system does not guarantee that the synchronization function is continuous, not even the existence of a synchronization function between the systems. It have been observed experimental situations in which the response system is asymptotically stable but the chaotic attractor for the coupled system has a complex structure and the synchronization function is not differentiable. The dynamics in the synchronization manifold is in general quite complex, due to the absence of symmetry in the coupled system or non-invertibility of transport system. Contrary to identical synchronization, where the trajectories are attracted to the symmetry plane and synchronization manifold is trivial, many real systems exhibit synchronization subspaces with non-trivial geometric structures inherent to the coupled system: wrinkles, cusps or bands, which can coexist in the same system and have a adverse effect on the synchronization detection. Some existing methods for detecting generalized synchronization, presented in (31), (34) and(35), are hampered by the presence of such structures. Wrinkles are caused in general by the existence of invariant sets embedded in the synchronization subspace in which the synchronization function has different Hölder degrees (of regularity). The different Hölder exponents, given by the modulus of the ratio between the Lyapunov exponent on the transversal contracting direction and the smaller negative Lyapunov exponent of the transport, depend on the intensity of the contraction rate in the transversal direction to the synchronization subspace (36). The presence of cusps typically arises from the existence of critical points in the attractor of the transport system defined by a smooth noninvertible map. In neighborhood of a critical point, where the Jacobian matrix is singular, there can be orbits of the transport system along which the contraction is arbitrarily large and the synchronization subspace is typically non-differentiable next to him. If the transport system is not invertible, the synchronization function cannot be continuous or not even exist as a function, since there are typical states of the transport that have more than one pre-image. In the synchronization subspace can then occur several bands, so the synchronization function is usually replaced by a "multiple" relationship between the coupled systems, although the response is still asymptotically stable. In this case we cannot predict a state of the response by the state of the transport.

The detection of features of generalized synchronization based on experimental data relies strongly on the continuity of synchronization function and, in general, also requires a certain degree of smoothness. In this case the functions on the coupled system are unknown and, when the generalized synchronization is stable, we can try in relatively simple cases the approximate construction of the synchronization function by means of numerical methods. However, if occur deviations to the functional dependence between the systems, it is never clear whether they are the result of loss of synchronization in the coupled system or the inaccuracy of the function considered. Thus, alternative definitions have emerged which differ on the properties of regularity imposed on the synchronization function and have shown different results in the detection of generalized synchronization in experimental data. Recently diffeomorphic properties have been required by authors as Abarbanel*et al.*(33), Pyragas(37) and Hunt *et al*. (36). Pyragas(37) also distinguishes between two types of generalized synchronization: strong synchronization for a smooth synchronization map and weak synchronization otherwise.

The generalized synchronization has practical interest (application) if it persists under arbitrarily small perturbations either the coupling or the dynamics of the systems components. As in identical synchronization, the stability of synchronization manifold can be local, guaranteed by the negativity of Lyapunov exponents which characterize the transversal perturbations to synchronization manifold (transversal or conditional Lyapunov exponents) (37) and/or by the study of eigenvalues of linearization of the coupled system; or global, guaranteed by the existence of a Lyapunov function (direct method of Lyapunov) (32). The results of local stability of the synchronization do not guarantee that it occurs when the coupled system is initiated from another initial condition. In order to investigate the possibility of stable synchronization in the coupled system, it is not indifferent the choice of the initial condition when it has more than one attractor in phase space. In contrast to the Lyapunov's direct method, the study of transversal Lyapunov exponents is very direct and can be easily applied even in very complicated systems. However, it has been suggested by Stefański and Kapitaniak(9) that, in practice, the negativity of Lyapunov exponents does not always guarantee that there are no unstable invariant sets in the synchronization manifold which could cause loss of synchronization when there is noise or small parametric mismatches. Note that, unlike what happens in unidirectional coupling, in bidirectional the Lyapunov exponents of one of the systems are not the same as the exponents that characterize the transversal perturbations. In unidirectional coupling, the behavior of the coupled system on the synchronization manifold is controlled only by the dynamics of the transport system. epequenasperturbações no sistemarespostacrescerão. When there is loss of synchronization, the transport system no longer has complete control over the behavior of the response system and small perturbations in the response system will grow. Although the process of synchronization loss is similar to that seen in identical synchronization, the identification of bubbling-type or blowout-type bifurcations can be hampered by the complexity of synchronization subspace. By continuous differentiation between the transport and the response when the strength coupling decreases, Barreto*et al.*(38) have proposed a method which can treat the problem using a decomposition based on the identification of unstable periodic orbits of transport system. It is described the creation and evolution of a complicated set of orbits that are developed outside the synchronization manifold, called the emerging set. It is also identified a critical transition point in this process.

5. CONCLUSION

The study of chaos synchronization phenomenonis quite recent in the theory of nonlinear dynamical systems and still continue to raise a high interest in the scientific community. However, despite the effort developed in investigating this phenomenon, many questions remain open.The existence of some analogies between chaos synchronization and chaos control and the possibility of applying techniques of chaos control in order to optimize the results of synchronization are factors to consider in future research. Also, the high potential of the chaos synchronization for applications is transversal to knowledge areas as distinct as physics, biology or economics.

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