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A FURTHER NOTE ABOUT THE RICATTI EQUATION APPLICATION TO THE M|G|∞ SYSTEM BUSY PERIOD STUDY

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Abstract. The M|G|∞ queue system can be applied in the modelation of many social problems: sickness, unemployment, emigration, ...(see, for instance, Ferreira (1995 and 1996)). In these situations it is very important to study the busy period length distribution of that system. We show, in this work, that the solution of the problem may be in the resolution of a Ricatti equation generalizing the work of Ferreira (1998).

Key Words. M|G|∞, Busy Period, Ricatti Equation

The M|G|∞ emptiness probability at time t study, as time function, being the initial instant the one of the beginning of a busy period (at which a customer arrives at the system finding it empty) is determined by the sign of $\frac{g(t)}{1-G(t)}-\lambda G(t),\ t\geq 0$ (Ferreira (1996)) where λ is the Poisson process arrival rate, $g(.)$ and $G(.)$ are, respectively, the service time p.d.f. and d.f..
Putting $\frac{g(t)}{1-G(t)}-\lambda G(t)=\beta(t)$ ($\beta(.)$ is any time function) we get

$$\frac{dG(t)}{dt}=-\lambda G^2(t)-(\beta(t)-\lambda)G(t)+\beta(t) \qquad (1)$$

that is a Ricatti equation about $G(.)$.

Solving it, after noting that $G(t)=1, t\geq 0$ is a solution, we get

$$G(t)=1-\frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw-\left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw},$$

$$,t\geq 0,-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{2}$$

Putting (2) in

$$\overline{B}(s)=1+\lambda^{-1}\left(s-\frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t[1-G(v)]dv}dt}\right) \tag{3}$$

that is the M|G|∞ busy period length Laplace transform (Stadje (1985)) we get

$$\overline{B}(s)=\frac{1-(s+\lambda)(1-G(0))L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]}{1-\lambda(1-G(0))L\left[e^{-\lambda t-\int_0^t \beta(u)du}\right]},-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{4}$$

where L means Laplace transform and

$$G(0)=\frac{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw+e^{-\rho}-1}{\lambda\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw} \tag{5}$$

After (4) we can compute $\frac{1}{s}\overline{B}(s)$ whose inversion gives

$$B(t)=\left((1-(1-G(0))(e^{-\lambda t-\int_0^t \beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw)\right)*$$

$$*\sum_{n=0}^{\infty}\lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t \beta(u)du}\right)^{*n},-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{6}$$

for the M|G|∞ busy period d.f., where $*$ is the convolution operator.
If $\beta(t)=\beta$ (constant), see Ferreira (1998),

$$G^{\beta}(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{-(\lambda+\beta)t}-1\right)+\lambda}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{7}$$

and

$$B^{\beta}(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{8}$$

So, if the service time d.f. is given by (7) the M|G|∞ busy period d.f. is the a mixture of a degenerate distribution at the origin and an exponential distribution.

Finally note that, for $\beta=\frac{\lambda}{e^{\rho}-1}$, $B^{\beta}(t)=1-e^{-\frac{\lambda}{e^{\rho}-1}t}$, $t\geq 0$ (purely exponential). And $B(t)$, given by (6) satisfies

$$B(t)\geq 1-e^{-\frac{\lambda}{e^{\rho}-1}t},\ t\geq 0,\ -\lambda\leq\frac{\int_0^t\beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{9}$$

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