Instituto Universitário de Lisboa

ISCTE-Lisbon University Institute

Business School

Department of Finance

# Three Essays on Option Pricing

A Thesis presented in partial fulfillment of the Requirements for the Degree of Doctor in Finance

by

Aricson César Jesus da Cruz

Supervisor:

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December, 2018

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# Resume

This thesis addresses option pricing problem in three separate and self-contained papers:

A. **The Binomial CEV Model and the Greeks**

   This article compares alternative binomial approximation schemes for computing the option hedge ratios studied by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011) under the lognormal assumption, but now considering the constant elasticity of variance (CEV) process proposed by Cox (1975) and using the continuous-time analytical Greeks recently offered by Larguinho et al. (2013) as the benchmarks. Among all the binomial models considered in this study, we conclude that an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011).

B. **Valuing American-Style Options under the CEV Model: An Integral Representation Based Method**

   This article derives a new integral representation of the early exercise boundary for valuing American-style options under the constant elasticity of variance (CEV) model. An important feature of this novel early exercise boundary characterization

is that it does not involve the usual (time) recursive procedure that is commonly employed in the so-called integral representation approach well known in the literature. Our non-time recursive pricing method is shown to be analytically tractable under the local volatility CEV process and the numerical experiments demonstrate its robustness and accuracy.

C. **A Note on Options and Bubbles under the CEV Model: Implications for Pricing and Hedging**

The discounted price process under the constant elasticity of variance (CEV) model is not a martingale for options markets with upward sloping implied volatility smiles. The loss of the martingale property implies the existence of (at least) two option prices for the call option, that is the price for which the put-call parity holds and the price representing the lowest cost of replicating the payoff of the call. This article derives closed-form solutions for the Greeks of the risk-neutral call option pricing solution that are valid for any CEV process exhibiting forward skew volatility smile patterns. Using an extensive numerical analysis, we conclude that the differences between the call prices and Greeks of both solutions are substantial, which might yield significant errors of analysis for pricing and hedging purposes.

*JEL Classification:* G13

*Keywords:* CEV model; Greeks; Binomial schemes; Numerical differentiation; Extended tree; Option pricing; American-style options; Early exercise boundary; Iterative method; Bubbles; Put-call parity; Local martingales.

# Resumo


Esta tese aborda a avaliação de opções em três artigos distintos:

A. **The Binomial CEV Model and the Greeks**

   Este artigo compara diferentes aproximações binomiais para o cálculo dos Greeks das opções estudadas por Pelsser and Vorst (1994), Chung and Shackleton (2002), e Chung et al. (2011), no âmbito da distribuição lognormal, mas agora considerando o processo *constant elasticity of variance* (CEV) proposto por Cox (1975), utilizando os Greeks analíticos em tempo continuo, recentemente propostos por Larguinho et al. (2013) como referência. Entre os modelos binomiais considerados neste estudo, concluimos que um modelo extended tree binomial CEV com uma aproximação convergente e monotona é o método mais eficiente para o cálculo dos Greeks no âmbito do processo de difusão CEV porque podemos aplicar a fórmula de extrapolação de dois pontos, sugerido por Chung et al. (2011).

B. **Valuing American-Style Options under the CEV Model: An Integral Representation Based Method**

   Este artigo deriva uma nova representação integral da barreira de exercício antecipado para a avaliação das opções Americanas no âmbito do modelo *constant elasticity of variance* (CEV), um importante aspeto desta nova caracterização da barreira de exercício antecipado é que este não envolve o usual processo recur-

sivo que é habitualmente aplicado e conhecido na literatura como a abordagem de representação integral. O nosso método de avaliação não recursivo é de fácil tratamento analítico sob o processo de difusão CEV e os resultados numéricos demonstram a sua rubustez e precisão.

C. **A Note on Options and Bubbles under the CEV Model: Implications for Pricing and Hedging**

O processo de desconto de preço no âmbito do modelo *constant elasticity of variance* (CEV) não é um martingale para os mercados de opções com uma *volatility smile* de inclinação ascendente. A perda da propriedade martingale implica a existência de (pelo menos) dois preços de opção para a opção de compra, que é o preço para qual se verifica a paridade *put-call* e este preço representa o menor custo de replicação do *payoff* da *call*. Este artigo deriva as soluções em fórmula fechada para os *Greeks* da opção call no risco neutral que são válidas para qualquer processo CEV que possui padrões de enviesamento ascencentes. Tendo por base uma analise numerica extensiva, concluímos que a diferença entre os preços da *call* e os *Greeks* de ambas as soluções são substanciais, o que pode gerar erros significativos de análises no calculo do preço da *call* e dos *Greeks*.

*JEL Classification:* G13

*Keywords:* Modelo CEV; Greeks; Árvores binomiais; Diferenciação numérica; Árvore estendida; avaliação de opções; opções Americanas; Barreira de exercício antecipado; méthodo iterativo; Bolhas; Paridade *put-call*; Martingales.

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# Contents

# 1. Introduction

This thesis addresses option pricing problem in three separate and self-contained papers.

The main purpose of the first article is to revisit the analysis performed by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011) for choosing appropriate methods when calculating option hedge ratios under the GBM assumption, but now using the constant elasticity of variance (henceforth, CEV) diffusion process proposed by Cox (1975). We note that while there are several papers in the literature comparing the convergence behavior of alternative binomial schemes under the GBM assumption, such extension to the CEV model is still missing mainly due to the absence of analytical solutions for Greeks of European-style options under this modeling setup. Such extension to the CEV model is now possible given the closed-form solutions of Greeks recently offered by Larguinho et al. (2013).

Option traders need to repeatedly and accurately calculate options sensitivity measures (usually known as Greeks) to successfully implement hedging strategies in their risk management activities, especially in the case of naked short options positions. This is so mainly because the option's risk characteristics change dynamically as the underlying stock price and the remaining time to maturity change.

We review the argument of Chung and Shackleton (2002), who demonstrated that the

Binomial Black-Scholes (henceforth, BBS) model advocated by Broadie and Detemple (1996) outperforms either a straight extended tree or a BBS extended tree. While such argument is true under the GBM setup considered in Chung and Shackleton (2002), we find that the use of a straight extended tree design is preferable for calculating Greeks under the state-dependent volatility CEV process, since it is better able to efficiently capture the leverage and volatility smile effects frequently found in the options markets. However, this is true only when we avoid the use of a Richardson extrapolation technique. Overall, we conclude that the use of an extended tree binomial CEV model possessing the smooth and monotonic convergence property substantially enhances the accuracy of Greeks because we can apply the extrapolation formula suggested by Chung et al. (2011).

The main aim of the second paper is to propose a simple iterative method to determine the optimal exercise boundary for valuing American-style options under the CEV diffusion process following the insights offered by Little et al. (2000) and Kim et al. (2013) in the context of the log-normal assumption. It is well known that the early exercise feature attached to American-style contingent claims turns the option pricing problem much more complex than its European-style counterpart, mainly because the early exercise boundary is not known *ex-ante* (i.e., before the solution of the pricing problem) and, therefore, it must be determined simultaneously as the solution of the same boundary value problem. In other words, the valuation of such claims requires the identification of the set of prices and times at which it is optimal to exercise the contract. To overcome this challenging difficulty, several alternative valuation methodologies have been proposed in the literature.

A common feature of the option pricing methodologies based on the integral representation approach is that they use a discretization scheme of a given number of implicit integral equations defining the optimal exercise points of the early exercise boundary.

The numerical procedure is initiated at the maturity date with appropriate boundary conditions and then the optimal stopping boundary is computed through a *time recursive iterative method* using the whole set of integral equations (i.e., the boundary is computed recursively via backward induction). Once such optimal exercise boundary is obtained, calculations of the early exercise premiums and option prices are then performed. Even though our method relies also on the integral representation approach, it uses instead a *non-time recursive iterative method* similar to the one employed by Kim et al. (2013) in a GBM modeling setup. Our numerical results show that the proposed method is accurate and efficient under the CEV model, thus being a viable alternative to the aforementioned option pricing methodologies under such local volatility model.

In the third article, we will focus our analysis on the so-called *constant elasticity of variance* (hereafter, CEV) model of Cox (1975), Cox and Ross (1976) and Emanuel and MacBeth (1982) to provide further insight on option pricing in markets with bubbles. This *local stochastic volatility* model is quite popular among researchers and practitioners because it offers several appealing features, namely: (i) the state-dependent volatility assumption of the CEV model allows volatility to be modeled using a simple and parsimonious specification, without the need of introducing an additional stochastic process as in the case of the Heston (1993) stochastic volatility model; (ii) it is known to be consistent with the existence of a negative correlation between stock returns and realized volatility (*leverage effect*) observed, for instance, in Black (1976), Beckers (1980), Christie (1982) and Bekaert and Wu (2000); (iii) it is able to accommodate the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*) documented, for example, in Dennis and Mayhew (2002) and Bakshi et al. (2003).

Even though the martingale property under the CEV model is preserved in the case of options markets exhibiting *volatility smirk patterns* (i.e., with downward sloping implied volatility smiles), the discounted price process under the CEV model is not a martingale for options markets exhibiting *forward skew patterns* (i.e., with upward sloping implied volatility smiles), as was first documented in Emanuel and MacBeth (1982), Lewis (2000) and Delbaen and Shirakawa (2002). Cox and Hobson (2005) and Heston et al. (2007) offered an economic interpretation for this technical irregularity of the CEV model as evidence for the presence of a stock price bubble. Heston et al. (2007) further show that this loss of the martingale property implies the existence of (at least) two option prices for the call option: the price for which the put-call parity holds and the price representing the lowest cost of replicating the payoff of the call.

Since the CEV process is widely used in many option pricing applications, the main aim of this article is to shed further light on the implications for option pricing and hedging purposes of the existence of multiple option prices under such state-dependent volatility setup. To accomplish this purpose, we offer novel closed-form solutions of Greeks for the risk-neutral call option pricing formula proposed by Heston et al. (2007) and for *any* elasticity parameter of a CEV process. This is achieved by combining the new sensitivity measures derived in this paper for the bubble formula—which can be simply expressed as the difference between the solution given by Emanuel and MacBeth (1982) and the cheapest solution of Heston et al. (2007)—and the analytical formulae of Greeks provided by Larguinho et al. (2013) for the CEV model and expressed in terms of the noncentral chi-square distribution function. Hence, our formulas can be applied to *any* CEV process possessing upward sloping implied volatility smiles, thus making the formulas recently presented in Veestraeten (2017) a special case of our general analytical solutions. This should be important for both academics and practitioners since such implied volatility behaviour is a characteristic that is often observed in some commodity spot prices and futures options—see, for example, Choi and Longstaff (1985), Geman and Shih (2009) and Dias and Nunes (2011).

This thesis proceeds as follows. Chapter 2 presents the first paper. Chapter 3 presents the second paper. Chapter 4 presents the third paper. Finally, Chapter 5 concludes.

# 2. The Binomial CEV Model and the Greeks*

**Abstract:** This article compares alternative binomial approximation schemes for computing the option hedge ratios studied by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011) under the lognormal assumption, but now considering the constant elasticity of variance (CEV) process proposed by Cox (1975) and using the continuous-time analytical Greeks recently offered by Larguinho et al. (2013) as the benchmarks. Among all the binomial models considered in this study, we conclude that an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011).

*JEL Classification:* G13

*Keywords:* CEV model; Greeks; Binomial schemes; Numerical differentiation; Extended tree.

*This paper is a joint work with José Carlos Dias and was published in Journal of Futures Markets.

## 2.1 Introduction

Option traders need to repeatedly and accurately calculate options sensitivity measures (usually known as Greeks) to successfully implement hedging strategies in their risk management activities, especially in the case of naked short options positions. This is so mainly because the option's risk characteristics change dynamically as the underlying stock price and the remaining time to maturity change.

Given the absence of closed-form solutions for pricing and hedging many financial option contracts possessing early exercise features and/or exotic payoffs, binomial models—such as the one initially proposed by Cox et al. (1979)—are commonly used by both academics and practitioners to value and hedge such derivative products. The computation of the required Greek measures is then often performed through a numerical differentiation procedure. However, it is well known that the use of such scheme for computing Greeks (and prices) may be flawed by the nature of the binomial discretization behavior observed in tree methods. See, for instance, Pelsser and Vorst (1994), Chung and Shackleton (2002, 2005), and Chung et al. (2011) for details under the geometric Brownian motion (henceforth, GBM) setup.

The main purpose of this article is to revisit the analysis performed by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011) for choosing appropriate methods when calculating option hedge ratios under the GBM assumption, but now using the constant elasticity of variance (henceforth, CEV) diffusion process proposed by Cox (1975). We note that while there are several papers in the literature comparing the convergence behavior of alternative binomial schemes under the GBM assumption, such extension to the CEV model is still missing mainly due to the absence of analytical solutions for Greeks of European-style options under this modeling setup.

Such extension to the CEV model is now possible given the closed-form solutions of Greeks recently offered by Larguinho et al. (2013).

We recall that such state-dependent volatility modeling framework was introduced in the option pricing literature by Cox (1975) as a way to overcome the undesirable constant volatility assumption underlying the Black and Scholes (1973) and Merton (1973) setup. The importance of the CEV model for traders is justified by its ability to accommodate two empirical stylized facts commonly observed in options markets, namely: The existence of an inverse relation between stock returns and realized volatility (*leverage effect*), as highlighted, for instance, by Black (1976) and Bekaert and Wu (2000); and the negative correlation between the implied volatility and the strike price of an option contract (*implied volatility skew*), as documented, for example, in Dennis and Mayhew (2002). Therefore, it is with no surprise that the CEV model is still widely used nowadays in a variety of contexts, e.g. by Chung and Shih (2009), Nunes (2009), Ruas et al. (2013), and Ballestra and Cecere (2015) for pricing and hedging plain-vanilla American-style options, or by Chung et al. (2013a,b), Tsai (2014), Dias et al. (2015), and Nunes et al. (2015) in the case of barrier option contracts, just to mention a few.

In this study, we review the argument of Chung and Shackleton (2002), who demonstrated that the Binomial Black-Scholes (henceforth, BBS) model advocated by Broadie and Detemple (1996) outperforms either a straight extended tree or a BBS extended tree. While such argument is true under the GBM setup considered in Chung and Shackleton (2002), we find that the use of a straight extended tree design is preferable for calculating Greeks under the state-dependent volatility CEV process, since it is better able to efficiently capture the leverage and volatility smile effects frequently found in the options markets. However, this is true only when we avoid the use of a Richardson extrapolation technique. Overall, we conclude that the use of an extended tree binomial CEV model possessing the smooth and monotonic convergence property substantially enhances the accuracy of Greeks because we can apply the extrapolation formula suggested by Chung et al. (2011).

Even though we are examining only approximation methods of Greeks for European-style options against their closed-form continuous-time benchmarks borrowed from Larguinho et al. (2013), the results should still be important for other option contracts. Options with early exercise features were not analyzed here given the absence of analytical solutions for prices and Greeks. However, our numerical experiments and discussions seem to suggest that the results highlighted in this paper are a consequence of the method used for evaluation and not the option style itself. Hence, these results should be also of interest when pricing and hedging American-style option contracts.

The remainder of the paper is organized as follows. Section 2.2 presents the theoretical CEV modeling setup and the binomial tree schemes that will be used for approximating the CEV continuous-time process, which are then numerically tested in Section 2.3. Finally, Section 3.5 summarizes the concluding remarks.

## 2.2 Five methods for computing Greeks under the binomial CEV model

For the analysis to remain self-contained, the next three subsections provide, respectively, a brief summary of the CEV model setup, the adopted binomial tree method for approximating the CEV diffusion process, and the five competing methods considered for calculating Greeks under the binomial CEV model.

### 2.2.1 CEV model setup

The CEV process proposed by Cox (1975) assumes that the asset price $\{S_t, t \geq 0\}$ is governed (under the risk-neutral probability measure $\mathbb{Q}$) by the stochastic differential equation

$$dS_t = (r - q)\, S_t\, dt + \delta\, S_t^{\beta/2}\, dW_t^{\mathbb{Q}}, \tag{2.1}$$

for $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, $q \geq 0$ represents the dividend yield for the underlying asset price, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.[2.1]

The stochastic differential equation (4.1) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$), as well as the absolute diffusion (when $\beta = 0$) and the square-root diffusion (for $\beta = 1$) models of Cox and Ross (1976), as special cases. We further notice that elasticity values of $\beta < 2$ (i.e. with a direct leverage effect) are observed for stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e. with an inverse leverage effect) are expected for some commodity spot prices and futures options with upward sloping implied volatility smiles, as documented, for instance, in Choi and Longstaff (1985), Davydov and Linetsky (2001), Geman and Shih (2009), and Dias and Nunes (2011).[2.2] In this paper, we will focus on equity options and, hence, we assume a CEV process with $\beta < 2$.

[2.1] We recall that the model parameter $\delta$ is a positive constant that can be interpreted as the scale parameter fixing the initial instantaneous volatility at the reference time $t_0 = 0$, i.e. $\sigma_0 = \sigma(S_0) = \delta\, S_0^{\beta/2-1}$. This calibration procedure is standard in the literature and it ensures that the differences found between CEV models with different $\beta$ values stem purely from the effect of the relationship between volatility and price levels.

[2.2] For additional background on the CEV process see, for instance, Cox (1975), Emanuel and MacBeth (1982), Schroder (1989), Davydov and Linetsky (2001), and Larguinho et al. (2013).

### 2.2.2 Approximating the CEV process through a lattice scheme

To approximate the CEV diffusion process with a binomial tree method we adopt the insights of Nelson and Ramaswamy (1990) and Chung and Shih (2009, Page 2145). First, we consider the $x$-transform $x(S) = S^{\alpha}/(\alpha\delta)$, with $\alpha = 1 - \beta/2$, and apply Itô's lemma to obtain

$$dx_t = \left[\frac{S_t^{\alpha-1}}{\delta}(r-q)\,S_t + \frac{\alpha-1}{2}\delta S_t^{-\alpha}\right]dt + dW_t^{\mathbb{Q}}. \tag{2.2}$$

Replacing the inverse transform $S = (x\alpha\delta)^{1/\alpha}$ in equation (2.2) results in a new process $x$ with a constant volatility equal to 1:

$$dx_t = \left[x_t\alpha\,(r-q) + \frac{\alpha-1}{2x_t\alpha}\right]dt + dW_t^{\mathbb{Q}}. \tag{2.3}$$

Figure 1 shows a simple two-period binomial $x$-tree with $x := x(S_{0,0})$, and where $S_{i,j}$ denotes the underlying asset value of an option contract in period $i$ and state $j$.[2.3] As usual, the time to maturity of the option contract is divided into $n$ evenly-spaced time points such that the time between intervals is $\Delta t := (T - t_0)/n$. Given the current value of $x$, the approximation of $x$ in the following time step is either $x^+ = x + \sqrt{\Delta t}$ for an up movement or $x^- = x - \sqrt{\Delta t}$ for a down movement. Repeating this procedure, we construct a recombined binomial tree for the $x$ process.

[Please insert Figure 1 about here.]

Then, using the inverse transform $S^{\pm} = f(x^{\pm}) = (x^{\pm}\alpha\delta)^{1/\alpha}$ if $x^{\pm} > 0$, or $S^{\pm} = 0$ if $x^{\pm} \leq 0$, results in a recombined binomial grid for the underlying asset price as depicted

[2.3] We recall that the subscript $j$ indicates the number of up moves that the underlying asset has made from its initial price $S_{0,0}$.

in Figure 2.2.[2.4]

[Please insert Figure 2.2 about here.]

Following Chung and Shih (2009, Page 2145), the risk-neutral probability $p^+$ of an upward movement is then derived as

$$p^+ := \begin{cases} \frac{Se^{(r-q)\Delta t}-S^-}{S^+-S^-} & \Leftarrow \quad x > 0 \text{ and } 0 \leq \frac{Se^{(r-q)\Delta t}-S^-}{S^+-S^-} \leq 1 \\ 0 & \Leftarrow \quad x \leq 0 \text{ or } \frac{Se^{(r-q)\Delta t}-S^-}{S^+-S^-} < 0 \\ 1 & \Leftarrow \quad x > 0 \text{ and } \frac{Se^{(r-q)\Delta t}-S^-}{S^+-S^-} > 1 \end{cases} . \tag{2.4}$$

Finally, we compute the corresponding option values $V(S_{i,j})$ over all the nodes of the tree by applying the usual terminal condition $V(S_{n,j}) = \max(\phi K - \phi S_{n,j}, 0)$ of a call (if $\phi = -1$) or put (if $\phi = 1$), with $K$ being the option's strike price, and then using the standard backward recursive procedure to obtain the time-$t_0$ option price $V(S_{0,0})$.

### 2.2.3 Five methods for computing Greeks

The *numerical differentiation method* for calculating the hedge ratio delta ($\Delta$) relies on the introduction of a small perturbation parameter on the current asset value $S_{0,0}$. More specifically, one typically chooses a small positive number $h$ and constructs new trees with novel initial underlying asset values $S_{0,0} = S_{0,0} + h$ and $S_{0,0} = S_{0,0} - h$. Assuming that $V(S_{0,0} + h, T)$ and $V(S_{0,0} - h, T)$ are the initial theoretical option values obtained from the corresponding trees for contracts expiring at time $T$, then $\Delta$ is approximated

[2.4] As highlighted by Davydov and Linetsky (2001, pg. 955), zero is an exit boundary whenever $1 \leq \beta < 2$, while for $\beta < 1$ zero is a regular boundary point that is specified as a killing boundary by adjoining a killing boundary condition. Hence, the inverse transform condition $S^{\pm} = 0$ if $x^{\pm} \leq 0$ is imposed to ensure that the CEV process is killed at the zero boundary.

by

$$\Delta = \frac{V(S_{0,0}+h,T) - V(S_{0,0}-h,T)}{2h}, \tag{2.5}$$

whereas the second derivative of the option price with respect to the underlying asset, i.e. gamma ($\Gamma$), is approximated by

$$\Gamma = 2\left[\frac{V(S_{0,0}+h,T) - V(S_{0,0},T)}{h} - \frac{V(S_{0,0},T) - V(S_{0,0}-k,T)}{k}\right]/(h+k), \tag{2.6}$$

with $k$ being a second small (positive) perturbation parameter.[2.5]

Pelsser and Vorst (1994) consider also the *binomial extended tree method*, which extends the original lattice scheme to nodes where both $i$ and $j$ may now be negative for dates prior to time $t_0 = 0$, as shown in Figure 2.3.

[Please insert Figure 2.3 about here.]

After computing the option values over the whole set of nodes of the binomial extended tree, we can approximate $\Delta$ and $\Gamma$ by

$$\Delta = \frac{V(S_{0,1}) - V(S_{0,-1})}{S_{0,1} - S_{0,-1}}, \tag{2.7}$$

and

$$\Gamma = 2\left[\frac{V(S_{0,1}) - V(S_{0,0})}{S_{0,1} - S_{0,0}} - \frac{V(S_{0,0}) - V(S_{0,-1})}{S_{0,0} - S_{0,-1}}\right]/(S_{0,1} - S_{0,-1}). \tag{2.8}$$

Pelsser and Vorst (1994) compare both methods for calculating $\Delta$ and $\Gamma$ under the GBM framework and conclude that the binomial extended tree method is not only faster than the numerical differentiation method, but also more accurate. As argued by Chung and Shackleton (2002), the implementation of the numerical differentiation method fails for

[2.5] Typically $k$ is set equal to $h$, but they might be different.

$\Delta$ because the tree takes discrete payoffs, and, therefore, for a small perturbation parameter $h$, the option value is not convex in $S_{0,0}$. Hence, $\Delta$ is locally constant because the option price itself is a locally linear function of the underlying asset price. For the case of $\Gamma$, the numerical differentiation method might even result in approximations that make no sense at all.

While Pelsser and Vorst (1994) focus their analysis on $\Delta$ and $\Gamma$, Chung and Shackleton (2002) consider also the partial derivative with respect to time, i.e. theta ($\theta$), which is approximated as

$$\theta = \frac{V(S_{0,0}, T-\tau) - V(S_{0,0}, T+\tau)}{2\tau}, \tag{2.9}$$

with $\tau$ being another small (positive) perturbation parameter, and

$$\theta = \frac{V(S_{-2,-1}) - V(S_{2,1})}{4\tau}, \tag{2.10}$$

under the numerical differentiation method and the binomial extended tree method, respectively.

The binomial extended tree method not only provides a faster computation, but also results in Greek sensitivity measures that do not suffer from the perturbation discreteness problem associated with the numerical differentiation method. However, the binomial extended tree method still possesses a remaining error that depends on the magnitude of the tree intervals chosen in time. Even though the use of fine trees instead of sparse trees reduces the discreteness errors attached to the binomial extended tree method, it has the shortcoming of increasing the corresponding computational burden.

As an alternative for computing Greeks, Chung and Shackleton (2002) suggest the use of the BBS method offered by Broadie and Detemple (1996), which essentially introduces Black-Scholes analytical option prices at the time step just before maturity originating a more accurate tree for option pricing purposes. Chung and Shackleton (2002) show that this method not only yields more accurate prices, but it also allows accurate calculation of Greek sensitivity measures under the GBM assumption. This is so mainly because the BBS method circumvents the pitfall of the piecewise linearity attached to the standard binomial option pricing model by introducing nonlinear and smoothly differentiable Black-Scholes functions into the continuation value of the final pricing nodes before maturity.

Another valuation approach explored in this study, usually known as the *binomial CEV method* (henceforth, BCEV method), is inspired by the BBS pricing scheme by including CEV option prices into the holding value of the penultimate nodes of the tree. To calculate such option values we use the closed-form solutions offered by Schroder (1989) and the Benton and Krishnamoorthy (2003) algorithm for computing the required non-central chi-square distribution functions, as suggested in Larguinho et al. (2013). Then, both the numerical differentiation and the binomial extended tree schemes are implemented via the BCEV pricing methodology.

Chung et al. (2011) suggest that one can apply the standard Richardson extrapolation technique to enhance the accuracy of binomial Greeks if their convergence patterns are monotonic and smooth. The monotonic convergence is attractive because more time steps guarantee more accurate prices. Furthermore, smooth convergence is also desirable because an extrapolation formula can be used to enhance the accuracy. Therefore, our fifth and last method applies the two-point extrapolation formula provided by Chung et al. (2011) to the BCEV extended tree scheme, since this is a binomial model accommodating the monotonic and smooth convergence property.[2.6]

[2.6] As highlighted by Chung and Shih (2009, Footnote 11), the convergence pattern of the BCEV price to the accurate option price is the same as that of the BBS method, i.e. the BCEV price converges monotonically and smoothly to the accurate value at a rate of $O\left(1/n\right)$. Therefore, it is possible to apply an extrapolation formula to enhance the accuracy of the extended tree BCEV prices and Greeks.

## 2.3 Numerical results

Armed with the five aforementioned methods, it is now possible to compare the robustness of each one for computing Greeks of standard European-style option contracts under the CEV model against the analytical solutions of Greek measures borrowed from Larguinho et al. (2013). Hereafter, to provide a clear identification of each binomial method to be tested, we will name each one as follows:

i. **NumDiff** computes Greeks through a standard binomial scheme using the numerical differentiation method.

ii. **ExtTree** computes Greeks through an extended binomial tree scheme.

iii. **NumDiffBCEV** computes Greeks through a standard binomial scheme using the numerical differentiation method, though calculating the continuation value of the penultimate nodes of the tree via the closed-form solutions of Schroder (1989).

iv. **ExtTreeBCEV** computes Greeks through an extended binomial tree scheme, though calculating the continuation value of the penultimate nodes of the tree via the closed-form solutions of Schroder (1989).

v. **ExtTreeBCEVR** computes Greeks by applying the two-point extrapolation formula suggested by Chung et al. (2011) to the ExtTreeBCEV method.

vi. **Closed-form solution** stands for the analytical formulae of Greek measures recently offered by Larguinho et al. (2013).

We recall that the continuation values of the penultimate nodes of the NumDiffBCEV and ExtTreeBCEV trees are calculated using the Schroder (1989) option pricing solutions which require the computation of noncentral chi-square distribution functions.

Similarly, the delta and theta analytical formulas provided by Larguinho et al. (2013) contains also such distribution laws, while their gamma solutions involve only probability density functions of a noncentral chi-square distribution. As expected, the numerical efficiency of the NumDiffBCEV, ExtTreeBCEV and ExtTreeBCEVR methods depends on the accuracy and speed of computation of the Schroder (1989) closed-form solutions. To the best of our knowledge—see, for instance, the numerical analysis performed by Larguinho et al. (2013, Section 4)—, the iterative procedure of Benton and Krishnamoorthy (2003, Algorithm 7.3) clearly offers the best speed-accuracy trade-off for evaluating option prices and Greeks under the (unrestricted) CEV model. Based on these insights and in order to make fair comparisons between competing methods, the necessary noncentral chi-square distribution functions appearing in the benchmark values of Larguinho et al. (2013) and in the NumDiffBCEV, ExtTreeBCEV and ExtTreeBCEVR methods are all calculated via the Benton and Krishnamoorthy (2003) algorithm.[2.7]

For illustrative purposes, we consider the constellation of parameters used by Pelsser and Vorst (1994) and Chung and Shackleton (2002), but augmented by the $\beta$ parameter. More specifically, we assume that $\{K, T, \sigma_0, r, q\} = \{100, 1, 0.25, 0.09, 0\}$ over a small range of current asset values $S_{0,0}$ using $n = 50$. Moreover, we set the perturbation parameters $h = k = 0.01$ and $\tau = 0.001$. Even though we will concentrate our numerical analysis on calls the results for puts are similar.

Figure 2.4 plots the call delta of European-style options computed via the aforementioned binomial methods against the benchmark offered by Larguinho et al. (2013, Equation A7), with $\beta = 0$, for the two graphs at the top, and $\beta = 1$, for the two graphs at the bottom. The two left-hand side graphs show that, similarly to what was pointed out

[2.7]For completeness, we further note that the probability density functions of a noncentral chi-square distribution law appearing in the analytical solutions of Greeks borrowed from Larguinho et al. (2013) are computed through the *ncx2pdf* built-in function available in Matlab.

by Pelsser and Vorst (1994) under the GBM assumption, the ExtTree method provides a much better approximation to the true call delta under the CEV model than the NumDiff method, since the latter results in option deltas that are highly discrete in $S_{0,0}$. The two right-hand side graphs perform essentially the same calculations, but now comparing the three versions of the BCEV approach against the benchmark. Even though we observe that the NumDiffBCEV method does not yield discrete values of delta, it is still much less accurate than the ExtTreeBCEV and ExtTreeBCEVR methods. Thus, while the inclusion of nonlinear and smoothly differentiable Schroder (1989) CEV functions into the final pricing nodes of the tree before maturity removes the discreteness problem associated with numerical differentiation, the extension of the original lattice scheme to nodes with dates prior to time $t_0 = 0$ seems to be a relevant feature under the CEV model since it reduces significantly the $\Delta$ estimation error of the ExtTree, ExtTreeBCEV and ExtTreeBCEVR methods.

This implies that the conclusion pointed out by Chung and Shackleton (2002)—for the same set of parameters but under the GBM framework—that the addition of a smooth function in the BBS tree and numerical differentiation with $h = 0.01$ outperforms either a standard extended tree or a BBS extended tree is not valid under the CEV model. As expected, as we augment the number of $n$ evenly-spaced time points all the methods tend to converge to the true value. However, a straight extended tree or a BCEV extended tree both offer advantages in terms of computational burden since the NumDiff and NumDiffBCEV methods require the tree to be recalculated whereas the ExtTree and ExtTreeBCEV methods do not.

Figure 2.4 highlights also that as we move further away from the limiting GBM process (i.e. $\beta = 2$) both the NumDiff and NumDiffBCEV methods amplify the magnitude of the overestimation errors when approximating option deltas under CEV models with alternative $\beta$ values. This suggests that the ExtTree, ExtTreeBCEV and ExtTreeBCEVR methods accommodate better the leverage and volatility smile effects—both of which are commonly observed across options markets and are captured by the CEV model specification—when approximating $\Delta$ values.

Other numerical results not reported here, but available upon request, show that the same line of reasoning occurs also for the CEV model with an inverse leverage effect (i.e. with $\beta > 2$). More specifically, as we move further away to the right of $\beta = 2$ the NumDiff and NumDiffBCEV methods produce again a significant biases, but now with increasing underestimation errors, while the ExtTree, ExtTreeBCEV and ExtTree-BCEVR methods are still very accurate. Therefore, while the introduction of nonlinear and smoothly differentiable Schroder (1989) CEV functions into the final pricing nodes of the tree before maturity removes the wavy erratic behavior of the NumDiff method, the NumDiffBCEV method still contains substantial approximation errors due to the distribution and nonlinearity errors discussed in Figlewski and Gao (1999).

[Please insert Figure 2.4 about here.]

Figure 2.5 illustrates the call (and put) gamma of European-style options, for $\beta = 1$, calculated through the mentioned binomial methods against the true value provided by Larguinho et al. (2013, Equation A12). The left-hand side graph shows that the ExtTree method leads to very good approximations of the CEV gamma.[2.8] Even though the NumDiffBCEV method is exhibited in the right-hand side graph it reveals also very poor approximations for $\Gamma$. By contrast, the ExtTreeBCEV and ExtTreeBCEVR models originate very good approximations for $\Gamma$ calculations. Once again, the extension of the

[2.8] We note, however, that the NumDiff method gives approximations that do not make any sense: it provides gamma values equal to zero everywhere except near the point where the underlying spot price is 100. At this point, the gamma value is very large when compared with the benchmark gamma value. For this reason, the NumDiff method is not plotted in Figure 2.5. This is in line with the observation made by Pelsser and Vorst (1994) when calculating gamma values under the GBM assumption.

tree to nodes with dates prior to time $t_0 = 0$ seems to be a key feature for computing accurate gammas under binomial CEV schemes.

[Please insert Figure 2.5 about here.]

Figure 2.6 plots the call theta of a European-style in-the-money option, i.e. with $S_{0,0} = 105$ as in Chung and Shackleton (2002, Figures 3 and 4), and for $\beta = 1$, using the two extended tree methods against the benchmark offered by Larguinho et al. (2013, Equation A16). Following the recommendation of Chung and Shackleton (2002), the graphs plot the right- and left-hand derivatives and their average that can be calculated from the extended tree as follows:

$$\theta_- = \frac{V(S_{-2,-1}) - V(S_{0,0})}{2\tau}, \tag{2.11}$$

$$\theta_+ = \frac{V(S_{0,0}) - V(S_{2,1})}{2\tau}, \tag{2.12}$$

while the central (or symmetric) difference $\theta = \frac{\theta_+ + \theta_-}{2}$ is given by equation $(2.10)$.

The left-hand side graph of Figure 2.6 shows the ExtTree method with backward ($\theta_-$), forward ($\theta_+$), and central ($\theta$) approximating schemes compared to the theoretical $\theta$ values. We conclude that the average of forward and backward differences performs best when compared with the benchmark value, though some oscillatory behavior is observed as the the number of time steps rise. By contrast, the right-hand side graph of Figure 2.6 reveals that no oscillation is found as we increase the number of tree steps under the ExtTreeBCEV method due to the insertion of a smooth function before maturity. Moreover, the average of forward and backward methods is still the preferable scheme for computing thetas since it provides closer values to the continuous-time benchmark. The graph reveals also that the theta value obtained through the ExtTree-BCEV method seems to converge to the benchmark value monotonically and smoothly.

Therefore, the use of the extrapolation formula derived by Chung et al. (2011) is able to enhance significantly the accuracy of the theta value, as highlighted by the ExtTree-BCEVR method.

[Please insert Figure 2.6 about here.]

While a straight numerical differentiation scheme (i.e. the NumDiff method) is known to perform poorly for approximating $\theta$ values (even under the GBM setup), the left-hand side graph of Figure 2.7 shows that the use of the NumDiffBCEV method with central differences improves significantly the $\theta$ estimates.[2.9] A direct comparison between methods is provided in the right-hand side graph of Figure 2.7. Even though the NumDiffBCEV method requires the calculation of a second tree, the obtained $\theta$ estimates are significantly more accurate than the ExtTreeBCEV method for a limited number of time steps, at least for the specific contract under analysis. This finding is consistent with Chung and Shackleton (2002, Figure 5) who show that the central numerical differentiation of the BBS tree also improves significantly the accuracy of $\theta$ under the GBM assumption. However, the ExtTreeBCEVR method is preferable since it provides theta values that are closer to the benchmark value.

[Please insert Figure 2.7 about here.]

Figures 2.4 to 2.7 plot Greek measures that do not represent a sufficiently large enough sample to take more robust conclusions, thus giving only a preliminary flavor of the results. Hence, to better assess the speed-accuracy trade-off between the competing methods we follow the guidelines of Broadie and Detemple (1996) by conducting a

[2.9] For completeness, we recall that the forward and backward differences for the numerical differentiation scheme are calculated as $\theta_+ = \frac{V(S_{0,0},T-\tau)-V(S_{0,0},T)}{\tau}$ and $\theta_- = \frac{V(S_{0,0},T)-V(S_{0,0},T+\tau)}{\tau}$, respectively, while the central difference $\theta = \frac{\theta_+ + \theta_-}{2}$ results in equation (2.9).

careful large sample evaluation of 2500 randomly generated contracts. To accomplish this purpose, we fix the initial asset price at $S_{t_0} = 100$ and take the strike price $K$ to be uniform between 70 and 130. The $\beta$ parameter is distributed uniformly between $-4$ and 2. The volatility $\sigma_0$ is distributed uniformly between 0.10 and 0.60, and the scale parameter $\delta$ is then computed. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. The dividend yield $q$ is uniform between 0.0 and 0.1. The riskless rate $r$ is uniform between 0.0 and 0.1.

The graph on the left-hand side of Figure 2.8 compares the speed-accuracy trade-off between the alternative valuation methods—all binomial models in this graph are based on the standard binomial scheme of Cox et al. (1979)—for computing deltas under the GBM assumption using the whole set of 2500 contracts. The plot highlights that the speed-accuracy trade-off between the ExtTree, NumDiffBBS and ExtTreeBBS methods are almost indistinguishable. We also observe that the NumDiffBBS method requires less evenly-spaced time points to achieve the same root mean square (henceforth, RMS) relative error obtained by the ExtTree and ExtTreeBBS methods. These results confirm the argument stated by Chung and Shackleton (2002) that the inclusion of nonlinear and smoothly differentiable Black-Scholes functions into the holding value of the final pricing nodes before maturity improves significantly the straight NumDiff method under the GBM framework. Nevertheless, the ExtTreeBBS method is a binomial model with the smooth and monotonic convergence property and, therefore, we may apply the two-point extrapolation formula derived by Chung et al. (2011). Consequently, the ExtTreeBBSR method clearly offers the best speed-accuracy trade-off among the methods tested here under the GBM setup.[2.10]

[2.10]See Chung et al. (2011) for thorough discussions on other recent binomial models under the GBM assumption whose binomial option prices converge to the true value monotonically and smoothly.

The graph on the right-hand side of Figure 2.8 also compares the speed-accuracy trade-off between the alternative valuation methods for computing deltas, but now under the CEV model. Following the insight outlined by Larguinho et al. (2013, Page 911), we excluded option parameter configurations where the abscissa value and the noncentrality parameter of the noncentral chi-square distribution functions appearing in the analytical solutions provided by Schroder (1989) are both $\geq 5000$. Out of the 2500 randomly generated contracts, 2474 did not satisfied this criteria.[2.11] The plot reveals that both the NumDiff and NumDiffBCEV methods perform poorly. Hence, contrary to what it is observed under the GBM case, the inclusion of nonlinear and smoothly differentiable CEV option pricing analytical solutions into the penultimate pricing nodes of the tree does not reduce significantly the errors of the standard NumDiff method, though it removes the discreteness problem associated with the straight numerical differentiation.

The conclusions to be taken from this large sample confirm that the extension of the original lattice scheme to nodes prior to time $t_0 = 0$ is an important feature to obtain accurate delta values via binomial CEV models. We observe that the ExtTree and ExtTreeBCEV methods offer a similar accuracy for the same number of time steps though the primer scheme is faster than the latter. This means that without extrapolation the ExtTree provides a better speed-accuracy trade-off. However, we can apply the two-point extrapolation formula to the ExtTreeBCEV method to significantly enhance the accuracy of the delta estimation. For instance, the ExtTreeBCEVR method requires less than 100 time steps to achieve the same RMS relative error as that of the ExtTree and ExtTreeBCEV methods with 1000 time steps. Hence, the ExtTreeBCEVR method is clearly the most efficient one.

[2.11] The remaining 26 randomly generated contracts are excluded from the sample to be used for performing speed-accuracy tests under CEV because they are all associated to $\beta$ parameters that are almost indistinguishable from the limiting GBM value of $\beta = 2$ coupled with low volatility values.

[Please insert Figure 2.8 about here.]

Figure 2.9 compares the speed-accuracy trade-off between the alternative valuation methods for computing gammas—left-hand side graph—and thetas—right-hand side graph—under the CEV model. As shown before, both the NumDiff and NumDiffBCEV methods are inadequate for computing gamma sensitivity measures. Therefore, these methods are not plotted here. Similarly to the delta case, the ExtTree method offers a better speed-accuracy trade-off if we do not make use of the extrapolation scheme, though both the ExtTree and ExtTreeBCEV methods provide an identical accuracy for the same number of evenly-spaced time points. Nevertheless, the ExtTreeBCEVR method provides a superior accuracy for computing gammas when $n > 200$.

Regarding the theta case, the graph reveals that without extrapolation the ExtTree method is still the preferable one. However, the ExtTreeBCEV procedure requires significantly fewer time-steps to achieve the same RMS relative error. Moreover, while Figure 2.7 highlighted that the use of numerical differentiation can be reinstated for computing theta values by simply including nonlinear and smoothly differentiable CEV option pricing analytical solutions into the penultimate pricing nodes of the tree, the results in Figure 2.9 show that the NumDiffBCEV method is less efficient than the other methods. Overall, we may conclude that the ExtTreeBCEVR is the best choice for computing thetas under the CEV model.

[Please insert Figure 2.9 about here.]

## 2.4 Conclusions

This article examines the choice of method for computing the option hedge ratios studied by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011), but assumes the underlying stock price is governed by a CEV diffusion process. Contrary to what was found by Chung and Shackleton (2002) under the GBM assumption, we show that, under the CEV model, an extended tree design is the key feature for generating accurate and fast calculations of Greeks if one ignores the use of a Richardson extrapolation technique. However, an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011).

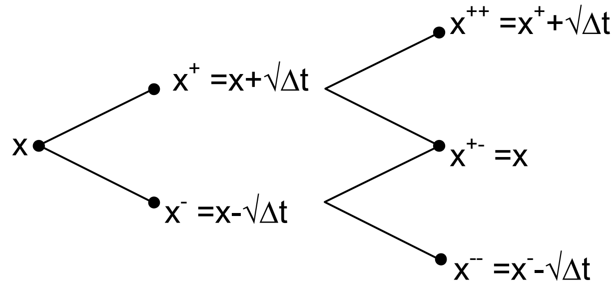


Figure 2.1: A simple two-period binomial tree for the $x$-transformed process.

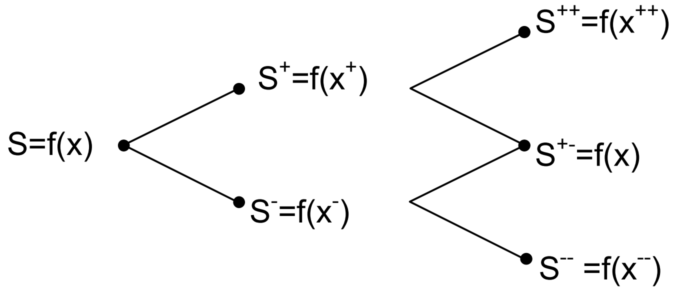


Figure 2.2: A simple two-period binomial tree for the inverse transform $S$.

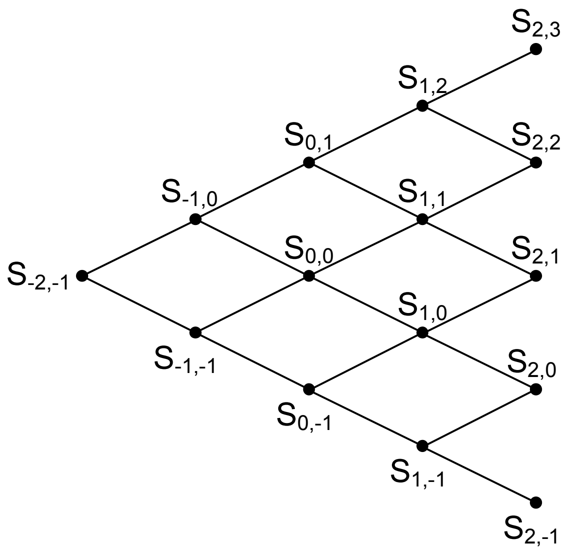


Figure 2.3: The extended binomial tree scheme.

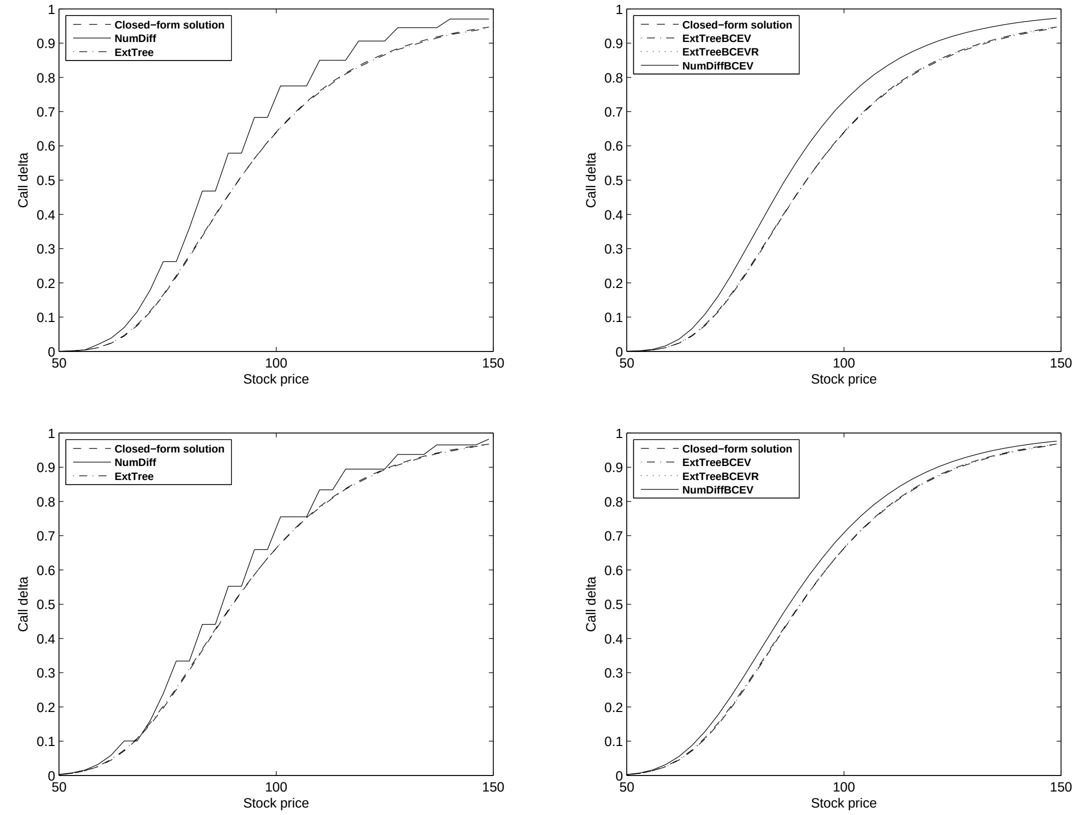


Figure 2.4: Call delta of European-style options computed via alternative binomial methods against the benchmark offered by Larguinho et al. (2013, Equation A7), with $\beta = 0$, for the two graphs at the top, and $\beta = 1$, for the two graphs at the bottom.

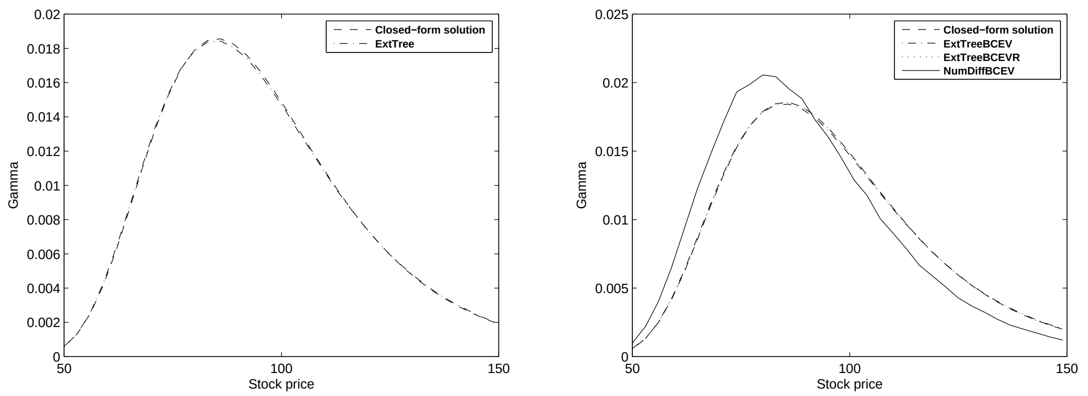


Figure 2.5: Call (and put) gamma of European-style options computed via alternative binomial methods against the benchmark offered by Larguinho et al. (2013, Equation A12), with $\beta = 1$.

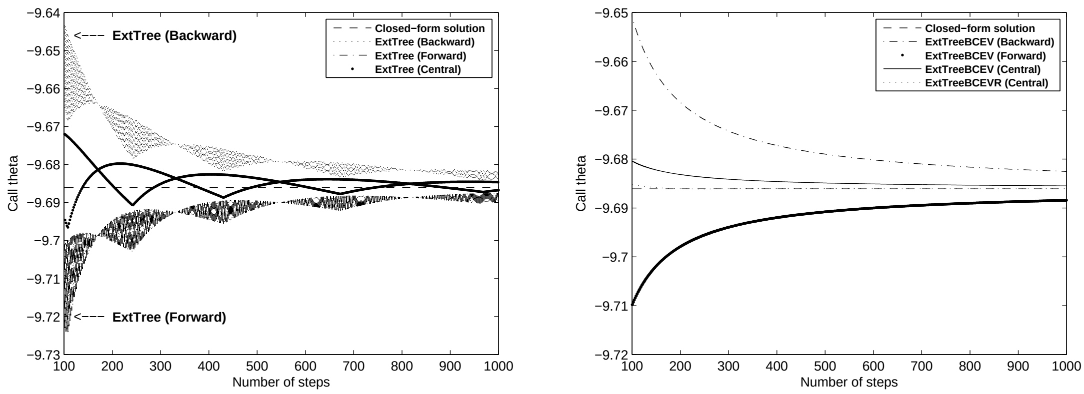


Figure 2.6: Call theta of European-style options computed via extended tree schemes against the benchmark offered by Larguinho et al. (2013, Equation A16), with $\beta = 1$.

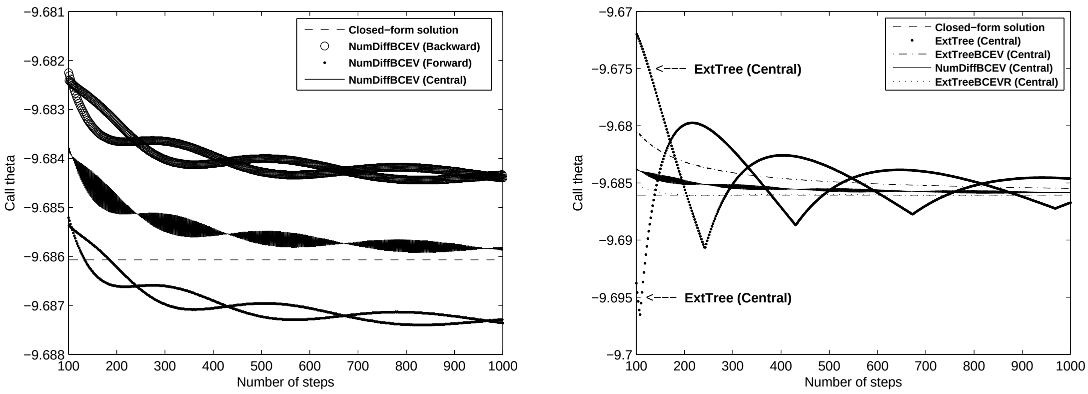


Figure 2.7: Call theta of European-style options computed via alternative binomial methods against the benchmark offered by Larguinho et al. (2013, Equation A16), with $\beta = 1$.

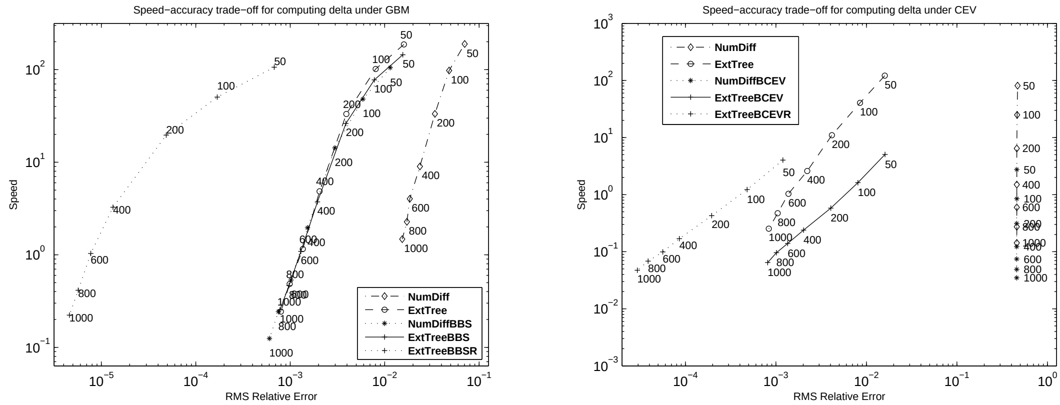


Figure 2.8: Speed-accuracy trade-off of alternative binomial methods for computing $\Delta$ under the GBM and CEV models using a random sample of 2500 and 2474 contracts, respectively. The RMS relative error is defined by $\sqrt{\frac{1}{m}\sum_{i=1}^{m}((\widehat{\Delta}_i - \Delta_i)/\Delta_i)^2}$, where $\Delta_i$ is the true delta value computed via the closed-form solutions of Black and Scholes (1973) and Larguinho et al. (2013, Equation A7), respectively, $\widehat{\Delta}_i$ is the approximate delta value estimated by the corresponding binomial method, and $m$ is the number of contracts. Speed is measured as the number of delta values calculated per second (on a 2.50GHz i3-3120M Toshiba Satellite). Preferred methods are in the upper-left corner.

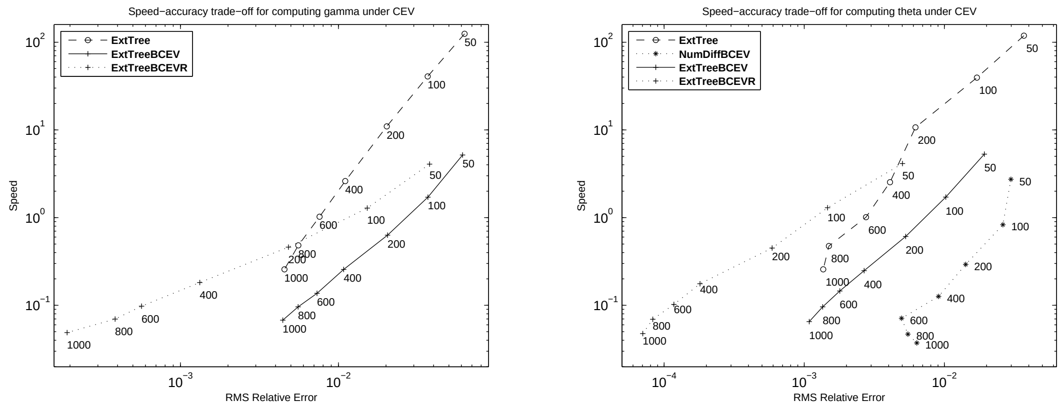


Figure 2.9: Speed-accuracy trade-off of alternative binomial methods for computing $\Gamma$ and $\theta$ under the CEV model using a random sample of 2474 contracts. The RMS relative error is defined by $\sqrt{\frac{1}{m}\sum_{i=1}^{m}((\widehat{f_i} - f_i)/f_i)^2}$, with $f \in \{\Gamma, \theta\}$ and where $f_i$ is the true $\Gamma$ or $\theta$ value computed via Larguinho et al. (2013, Equation A12 or A16), respectively, $\widehat{f_i}$ is the approximate $\Gamma$ or $\theta$ value estimated by the corresponding binomial method, and $m$ is the number of contracts. Speed is measured as the number of $\Gamma$ or $\theta$ values calculated per second (on a 2.50GHz i3-3120M Toshiba Satellite). Preferred methods are in the upper-left corner.

# 3. Valuing American-Style Options under the CEV Model: An Integral Representation Based Method*

**Abstract:** This article derives a new integral representation of the early exercise boundary for valuing American-style options under the constant elasticity of variance (CEV) model. An important feature of this novel early exercise boundary characterization is that it does not involve the usual (time) recursive procedure that is commonly employed in the so-called integral representation approach well known in the literature. Our non-time recursive pricing method is shown to be analytically tractable under the local volatility CEV process and the numerical experiments demonstrate its robustness and accuracy.

*JEL Classification:* G13

*Keywords:* CEV model; Option pricing; American-style options; Early exercise boundary; Iterative method.

*This paper is a joint work with José Carlos Dias and is under review in a peer-reviewed journal.

## 3.1 Introduction

The valuation and hedging of American-style options through analytical approximations and numerical schemes continues to receive much attention in the finance community given the impossibility of obtaining elegant closed-form solutions such as those offered by Black and Scholes (1973) and Merton (1973) under the *geometric Brownian motion* (hereafter, GBM) setup. The interest on these contracts is enhanced also by the fact that they are actively traded throughout the world on several options exchanges involving substantial amounts of trading volume. The main assumption underlying the GBM modeling framework is that asset prices follow a log-normal diffusion process with constant volatility, which permits a considerable amount of mathematical tractability and allows the valuation of European-style plain-vanilla options using simple analytical formulae. However, there is an abundant empirical evidence showing that the volatility of log-returns is far from being constant—especially after the stock market crash of October 1987, as documented, for example, in Jackwerth and Rubinstein (1996, 2012)—and, hence, several attempts have been developed in the literature to overcome such undesirable assumption.

In particular, Cox (1975), Cox and Ross (1976) and Emanuel and MacBeth (1982) propose the so-called *constant elasticity of variance* (hereafter, CEV) model, in which the volatility is specified as a function of the option's underlying asset price. Even though this *local stochastic volatility* model has been established more than forty years ago, it is still nowadays quite popular among researchers and practitioners because it offers several appealing features, namely: (i) the state-dependent volatility assumption of the CEV model allows volatility to be modeled using a simple and parsimonious specification, without the need of introducing an additional stochastic process as in the case of the Heston (1993) stochastic volatility model; (ii) it is known to be consistent with the existence of a negative correlation between stock returns and realized volatility (*leverage effect*) observed, for instance, in Black (1976), Beckers (1980), Christie (1982) and Bekaert and Wu (2000);[3.1] (iii) it is able to accommodate the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*) documented, for example, in Dennis and Mayhew (2002) and Bakshi et al. (2003).

The main aim of this paper is to propose a simple iterative method to determine the optimal exercise boundary for valuing American-style options under the CEV diffusion process following the insights offered by Little et al. (2000) and Kim et al. (2013) in the context of the log-normal assumption. It is well known that the early exercise feature attached to American-style contingent claims turns the option pricing problem much more complex than its European-style counterpart, mainly because the early exercise boundary is not known *ex-ante* (i.e., before the solution of the pricing problem) and, therefore, it must be determined simultaneously as the solution of the same boundary value problem. In other words, the valuation of such claims requires the identification of the set of prices and times at which it is optimal to exercise the contract. To overcome this challenging difficulty, several alternative valuation methodologies have been proposed in the literature.[3.2]

One of the first attempts to price American-style options under the CEV model is due to Kim and Yu (1996) and Detemple and Tian (2002), who extend the so-called *integral representation method*—initially established by Kim (1990), Jacka (1991), Carr et al. (1992) and Jamshidian (1992) in a GBM context—to the valuation of American-style options under alternative diffusion processes. More recently, Nunes (2009) pro-

[3.1] A common interpretation for this stylized fact is that when an asset price declines, the associated firm becomes more leveraged since its debt to equity ratio becomes larger. Therefore, the risk of the asset, namely its volatility, should become higher. Another possible economic rationale for this phenomenon is that the forecast of an increase in the volatility should be compensated by a higher rate of return, which can only be obtained via a decrease in the asset value.

[3.2] The valuation of American-style contingent claims has a long list of relevant contributions and an exhaustive literature review would be prohibitive. However, a general overview of the most important developments on this subject may be found, for example, in Myneni (1992), Broadie and Detemple (2004) and Barone-Adesi (2005).

poses an *optimal stopping approach* that is valid for any Markovian asset price process, Chung and Shih (2009) and Ruas et al. (2013) develop an efficient *static hedging portfolio approach* (hereafter, SHP), whereas Ballestra and Cecere (2015) derive elegant semi-analytical approximations (though expressed in terms of confluent hypergeometric functions and modified Bessel functions), by extending the pricing methodology previously proposed by Barone-Adesi and Whaley (1987) under the GBM assumption to a CEV diffusion setup.

A common feature of the option pricing methodologies based on the integral representation approach is that they use a discretization scheme of a given number of implicit integral equations defining the optimal exercise points of the early exercise boundary. The numerical procedure is initiated at the maturity date with appropriate boundary conditions and then the optimal stopping boundary is computed through a *time recursive iterative method* using the whole set of integral equations (i.e., the boundary is computed recursively via backward induction). Once such optimal exercise boundary is obtained, calculations of the early exercise premiums and option prices are then performed. Even though our method relies also on the integral representation approach, it uses instead a *non-time recursive iterative method* similar to the one employed by Kim et al. (2013) in a GBM modeling setup. Our numerical results show that the proposed method is accurate and efficient under the CEV model, thus being a viable alternative to the aforementioned option pricing methodologies under such local volatility model.

The rest of the paper proceeds as follows. Section 3.2 presents a short overview of the integral representation method under the CEV model. Section 3.3 highlights the main contribution of the paper: the extension of the early exercise premium representation proposed by Kim et al. (2013) under the GBM setup to a state-dependent volatility CEV process nesting, as a special case, the log-normal assumption. Section 3.4 tests the robustness of the proposed non-time recursive iterative method under CEV. Section

3.5 provides some concluding remarks and avenues for future research.

## 3.2 The American-style option pricing problem

This section provides a brief remainder of the CEV modeling setup and some background on the early exercise premium representation that has important implications for the iterative method to be developed in Section 3.3.

### 3.2.1 The CEV model

Let us assume an arbitrage-free and frictionless financial market with continuous trading on the time interval $[t_0, T]$, for some fixed time $T > t_0$. As usual, uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where the equivalent martingale probability measure $\mathbb{Q}$ associated to the numéraire money market account is taken as given.

The valuation of the American-style options considered in this paper is explored in the context of the CEV model initially developed by Cox (1975), Cox and Ross (1976) and Emanuel and MacBeth (1982). Under this state-dependent volatility framework, the asset price $\{S_t, t \geq 0\}$ is assumed to be governed (under the risk-neutral probability measure $\mathbb{Q}$) by the stochastic differential equation

$$dS_t = (r - q)\, S_t\, dt + \delta\, S_t^{\beta/2}\, dW_t^{\mathbb{Q}}, \tag{3.1}$$

for $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, $q \geq 0$ represents the dividend yield for the underlying asset price, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$ representing the history of the financial market.

As usual, the model parameter $\delta$ can be interpreted as the scale parameter fixing the initial instantaneous volatility at the reference time $t_0 = 0$, i.e. $\sigma_0 = \sigma(S_0) = \delta\, S_0^{\beta/2-1}$. This calibration procedure is standard in option pricing problems under the CEV diffusion process since it ensures that the differences found between CEV models with different $\beta$ values stem purely from the effect of the relationship between volatility and price levels. Moreover, elasticity values of $\beta < 2$ (i.e., with a direct leverage effect) are able to reproduce the so-called *reverse skew* or *volatility smirk* pattern that typically appears for individual stock options, stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e., with an inverse leverage effect) are normally expected for some commodity spot prices and futures options with upward sloping implied volatility smiles (also known as *forward skew* patterns), as discussed, for instance, in Choi and Longstaff (1985), Davydov and Linetsky (2001), Geman and Shih (2009), and Dias and Nunes (2011).

### 3.2.2 The early exercise premium representation

Since an American-style option contract can be exercised at any time until (and including) its expiry date, it is well established—see, for instance, Karatzas (1988, Theorem 5.4)—that the time-$t_0$ price of a put (if $\phi = 1$) or a call (if $\phi = -1$) on the stock price $S$, with strike price $K$, and maturity date $T$ ($\geq t_0$), can be represented by the following *Snell envelope*:

$$V_{t_0}(S, K, T; \phi) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left[ e^{-r[(T \wedge \tau) - t_0]} \left(\phi K - \phi S_{T \wedge \tau}\right)^+ \middle| \mathcal{F}_{t_0} \right], \tag{3.2}$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$ generated by the underlying asset price process and taking values in $[t_0, \infty[$.[3.3]

Denoting the first passage time of the underlying asset price to its time-dependent exercise boundary $\{B_t, t_0 \leq t \leq T\}$ by

$$\tau_e := \inf\{t > t_0 : S_t = B_t\} \tag{3.3}$$

and assuming that the American-style option is still alive at the valuation date $t_0$ (i.e., $\phi S_{t_0} > \phi B_{t_0}$), equation (3.2) can be restated as

$$\begin{aligned} & V_{t_0}(S, K, T; \phi) \qquad (3.4) \\ = \ & \mathbb{E}_{\mathbb{Q}}\left[ e^{-r[(T \wedge \tau_e) - t_0]} (\phi K - \phi S_{T \wedge \tau_e})^+ \middle| \mathcal{F}_{t_0} \right] \\ = \ & \mathbb{E}_{\mathbb{Q}}\left[ e^{-r(\tau_e - t_0)} (\phi K - \phi B_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] + e^{-r(T - t_0)} \mathbb{E}_{\mathbb{Q}}\left[ (\phi K - \phi S_T)^+ \mathbf{1}_{\{\tau_e \geq T\}} \middle| \mathcal{F}_{t_0} \right], \end{aligned}$$

where the first line of equation (3.4) follows from the first passage time (3.3), and $\mathbf{1}_{\{A\}}$ is the indicator function of the event $\{A\}$. We recall that $\phi K \geq \phi B_{\tau_e}$, because the early exercise boundary $t \mapsto B_t$ of the put (resp., call) is nondecreasing (resp., nonincreasing) on $[t_0, T]$ and is limited from above (resp., below) by $\phi(\phi K \wedge \phi r K / q)$—see, for instance, Kim (1990, Pages 558 and 560), Jacka (1991, Proposition 2.2), Huang et al. (1996, Footnote 5), Kim and Yu (1996, Pages 66 and 67), and Ruas et al. (2013, Equation 33).

Similarly to Kim and Yu (1996), Detemple and Tian (2002) and Nunes (2009), it is possible to express the American-style option price as the sum of two components: the corresponding European-style option and an early exercise premium. To accomplish

[3.3] As usual, $\mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_t]$ denotes the (time-$t$) expected value of the random variable $X$, conditional on $\mathcal{F}_t$, and computed under the equivalent martingale measure $\mathbb{Q}$. Moreover, for any two real numbers $x$ and $y$, we denote by $x \vee y$ and $x \wedge y$, respectively, their maximum and minimum.

this purpose, and since $\mathbf{1}_{\{\tau_e \geq T\}} = 1 - \mathbf{1}_{\{\tau_e < T\}}$, then equation (3.4) can be rewritten as

$$\begin{aligned}
& V_{t_0}(S,K,T;\phi) \\
= \ & \mathbb{E}_{\mathbb{Q}}\left[ e^{-r(\tau_e - t_0)} (\phi K - \phi B_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] + e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[ (\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] \\
& - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[ (\phi K - \phi S_T)^+ \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right], \\
= \ & v_{t_0}(S,K,T;\phi) + EEP_{t_0}(S,K,T;\phi), \qquad (3.5)
\end{aligned}$$

with

$$\begin{aligned}
v_{t_0}(S,K,T;\phi) & := e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[ (\phi K - \phi S_T)^+ \middle| \mathcal{F}_{t_0} \right] \\
& = \phi K e^{-r(T-t_0)} F_{\mathbb{Q}}(T,K;t_0,S_{t_0}) - \phi S_{t_0} e^{-q(T-t_0)} F_{\mathbb{Q}^S}(T,K;t_0,S_{t_0}) \qquad (3.6)
\end{aligned}$$

being understood as the European-style option component that can be efficiently computed via the option pricing solutions offered by Schroder (1989), and

$$\begin{aligned}
& EEP_{t_0}(S,K,T;\phi) \\
:= \ & \mathbb{E}_{\mathbb{Q}}\left[ e^{-r(\tau_e - t_0)} (\phi K - \phi B_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] - e^{-r(T-t_0)} \mathbb{E}_{\mathbb{Q}}\left[ (\phi K - \phi S_T)^+ \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\
= \ & \int_{t_0}^{T} \left[ \phi r K e^{-r(l-t_0)} F_{\mathbb{Q}}(l,B_l;t_0,S_{t_0}) - \phi q S_{t_0} e^{-q(l-t_0)} F_{\mathbb{Q}^S}(l,B_l;t_0,S_{t_0}) \right] dl \qquad (3.7)
\end{aligned}$$

denoting the early exercise premium representation considered in Kim and Yu (1996) and Detemple and Tian (2002),[3.4] whereas the required probability distribution func-

[3.4] For the sake of completeness, we notice that Nunes (2009, Proposition 1) also decomposes the price of an American-style option into the sum of the two aforementioned sources of value, as stated in equation (3.5). However, instead of using the integral representation (3.7), he proposes the use of an alternative characterization for the early exercise premium that requires an efficient valuation formula for the European-style counterpart and the knowledge of the underlying asset price transition density function.

tions are defined as[3.5]

$$F_{\mathbb{Q}}\left(l, B_l; u, S_u\right) := \begin{cases} F\left(2x(S_u); \frac{2}{2-\beta}, 2y(B_l)\right) & \Leftarrow \beta < 2, \phi = -1 \\ F\left(2y(B_l); 2 + \frac{2}{\beta-2}, 2x(S_u)\right) & \Leftarrow \beta > 2, \phi = -1 \\ Q\left(2x(S_u); \frac{2}{2-\beta}, 2y(B_l)\right) & \Leftarrow \beta < 2, \phi = 1 \\ Q\left(2y(B_l); 2 + \frac{2}{\beta-2}, 2x(S_u)\right) & \Leftarrow \beta > 2, \phi = 1 \end{cases} \tag{3.8}$$

and

$$F_{\mathbb{Q}^S}\left(l, B_l; u, S_u\right) := \begin{cases} Q\left(2y(B_l); 2 + \frac{2}{2-\beta}, 2x(S_u)\right) & \Leftarrow \beta < 2, \phi = -1 \\ Q\left(2x(S_u); \frac{2}{\beta-2}, 2y(B_l)\right) & \Leftarrow \beta > 2, \phi = -1 \\ F\left(2y(B_l); 2 + \frac{2}{2-\beta}, 2x(S_u)\right) & \Leftarrow \beta < 2, \phi = 1 \\ F\left(2x(S_u); \frac{2}{\beta-2}, 2y(B_l)\right) & \Leftarrow \beta > 2, \phi = 1 \end{cases}, \tag{3.9}$$

with $F\left(z; a, b\right)$ and $Q\left(z; a, b\right)$ representing, respectively, the distribution function and the complementary distribution function of a noncentral chi-square law with $a$ degrees of freedom and noncentrality parameter $b$ and

$$k = \frac{2(r-q)}{\delta^2(2-\beta)\left[e^{(r-q)(2-\beta)(l-u)} - 1\right]}, \tag{3.10}$$

$$\delta = \sigma\left(S_{t_0}\right) S_{t_0}^{1-\beta/2}, \tag{3.11}$$

$$x(S) = kS^{2-\beta}e^{(r-q)(2-\beta)(l-u)} \tag{3.12}$$

and

$$y(B) = kB^{2-\beta}. \tag{3.13}$$

The valuation of the American-style option pricing solution (3.5) requires the knowledge of the early exercise boundary $\{B_t, t_0 \leq t \leq T\}$. To accomplish this purpose, it is

[3.5] As usual, the risk-neutral measure $\mathbb{Q}$ is associated to the money market account numéraire, while the equivalent martingale measure $\mathbb{Q}^S$ takes as numéraire of the economy the underlying asset price.

necessary to solve recursively the equation

$$\begin{aligned} & \phi K - \phi B_t \qquad (3.14) \\ = \; & v_t(B_t, K, T; \phi) + \int_t^T \left[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}\left(l, B_l; t, B_t\right) - \phi q B_t e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l, B_l; t, B_t\right)\right] dl, \end{aligned}$$

by dividing the time to maturity of the option contract into $n$ evenly-spaced time points such that $\Delta t := (T - t_0)/n$, with each time $t_i := t_0 + i\Delta t$ (for $i = 0, 1, \ldots, n$). The use of this discretization scheme yields, via equation (3.14), $n$ implicit integral equations defining the optimal exercise points $\{B_{t_0}, B_{t_1}, \ldots, B_{t_{n-1}}\}$. Initializing the numerical procedure at the expiry date through the boundary condition $B_{t_n} = \phi\left(\phi K \wedge \phi r K/q\right)$, then the early exercise boundary $\{B_{t_i}, 0 \leq i \leq n-1\}$ can be computed recursively (via backward induction) using the $n$ integral equations. For $n$ sufficiently large, this integral representation approach produces a good approximation of the optimal stopping boundary. Once such optimal exercise boundary is obtained, calculations of the early exercise premiums and option prices are then straightforward.

## 3.3 The non-time recursive iterative method

In this section, we extend the early exercise premium representation derived by Kim et al. (2013) under the log-normal model to a more general CEV diffusion process that nests, as a special case, the GBM assumption.

It is well known that, at each time $t \in [t_0, T]$, there exists a (unique) *critical asset price* $B_t$ below (resp., above) which the American-style put (resp., call) price equals its intrinsic value and, therefore, early exercise should occur.[3.6] If this is the case, then the optimal policy should be to exercise the American-style option when the underlying asset price

[3.6] See, for example, Detemple and Tian (2002) and Dias and Nunes (2018) and the references therein.

first enters the *exercise* (or *stopping*) region $\mathcal{E} := \{(S,t) \in [0,\infty[ \times [t_0,T] : \phi S_t \le \phi B_t\}$.[3.7] In the exercise region $\mathcal{E}$, the value of an American-style option is equal to its intrinsic value, i.e. $V_t(S_t,K,T;\phi) = \phi K - \phi S_t$ and, hence, equations (3.5), (3.6) and (3.7) yield

$$\begin{aligned} & \phi K - \phi S_t \\ = \;& v_t(S_t,K,T;\phi) + \int_t^T \left[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}(l,B_l;t,S_t) - \phi q S_t e^{-q(l-t)} F_{\mathbb{Q}^S}(l,B_l;t,S_t)\right] dl. \end{aligned} \tag{3.15}$$

The next step to obtain the representation of the exercise boundary requires a judicious choice for $S_t \in \mathcal{E}$. Similarly to Little et al. (2000) and Kim et al. (2013), we let $S_t = \epsilon B_t$ with $\epsilon \in ]0,1]$ for $\phi = 1$ and $\epsilon \in [1,\infty[$ for $\phi = -1$.[3.8] With this substitution, equation (3.15) becomes

$$\begin{aligned} & \phi K - \phi \epsilon B_t \\ = \;& v_t(\epsilon B_t,K,T;\phi) + \int_t^T \left[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}(l,B_l;t,\epsilon B_t) - \phi q \epsilon B_t e^{-q(l-t)} F_{\mathbb{Q}^S}(l,B_l;t,\epsilon B_t)\right] dl. \end{aligned} \tag{3.16}$$

Now it is necessary to differentiate both sides of equation (3.16) with respect to $\epsilon$. The derivative of the expression on left-hand side of the equation follows easily as

$$\frac{\partial}{\partial \epsilon}(\phi K - \phi \epsilon B_t) = -\phi B_t. \tag{3.17}$$

The differentiation of the expression on the right-hand side of equation (3.16) requires the derivatives of the noncentral chi-square distribution laws (3.8) and (3.9) with respect

[3.7] $\mathcal{C} := \{(S,t) \in [0,\infty[ \times [t_0,T] : \phi S_t > \phi B_t\}$ is defined as the corresponding *continuation* (or *holding*) region.

[3.8] Note that, for any $T > t$, equation (3.15) holds for every $S_t$ below (resp., above) the boundary value $B_t$ of the put (resp., call). While applications under the CEV model using the usual integral representation approach—e.g., Kim and Yu (1996) and Detemple and Tian (2002)—look at the integral representation (3.15) only for $S_t = B_t$, thus neglecting the region $\phi S_t < \phi B_t$ and leading to the convolution type integral equation representation (3.14), the early exercise representation exploited here—as well as in Little et al. (2000) and Kim et al. (2013)—considers all the exercise region $\phi S_t \le \phi B_t$.

to $\epsilon$. Exploiting the ideas of Larguinho et al. (2013, Appendix A) and taking, for example, the case with $\beta < 2$ and $\phi = 1$ contained in the third branch of equation (3.9), then

$$\begin{aligned} \frac{\partial}{\partial \epsilon} F_{\mathbb{Q}^S}\left(l, B_l; t, \epsilon B_t\right) &= \frac{\partial}{\partial \epsilon} F\left(2y(B_l); 2 + \frac{2}{2-\beta}, 2x(\epsilon B_t)\right) \\ &= \frac{\partial}{\partial 2x(\epsilon B_t)} F\left(2y(B_l); 2 + \frac{2}{2-\beta}, 2x(\epsilon B_t)\right) \frac{\partial}{\partial \epsilon} 2x(\epsilon B_t) \\ &= -p\left(2y(B_l); 4 + \frac{2}{2-\beta}, 2x(\epsilon B_t)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t), \end{aligned} \quad (3.18)$$

where the last line arises after straightforward calculus and because

$$\frac{\partial}{\partial b} F\left(z; a, b\right) = -\frac{\partial}{\partial b} Q\left(z; a, b\right) = -p\left(z; a+2, b\right), \quad (3.19)$$

as noticed in Larguinho et al. (2013, Equation A2b), and with $p\left(z; a, b\right)$ being the probability density function of a noncentral chi-square law with $a$ degrees of freedom and noncentrality parameter $b$.

Similarly, the case with $\beta > 2$ and $\phi = 1$ contained in the fourth branch of equation (3.9) can be computed as

$$\begin{aligned} \frac{\partial}{\partial \epsilon} F_{\mathbb{Q}^S}\left(l, B_l; t, \epsilon B_t\right) &= \frac{\partial}{\partial \epsilon} F\left(2x(\epsilon B_t); \frac{2}{\beta-2}, 2y(B_l)\right) \\ &= \frac{\partial}{\partial 2x(\epsilon B_t)} F\left(2x(\epsilon B_t); \frac{2}{\beta-2}, 2y(B_l)\right) \frac{\partial}{\partial \epsilon} 2x(\epsilon B_t) \\ &= p\left(2x(\epsilon B_t); \frac{2}{\beta-2}, 2y(B_l)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t), \end{aligned} \quad (3.20)$$

where the last line is obtained using simple calculations and because

$$\frac{\partial}{\partial z} F\left(z; a, b\right) = -\frac{\partial}{\partial z} Q\left(z; a, b\right) = p\left(z; a, b\right), \quad (3.21)$$

as presented in Larguinho et al. (2013, Equation A2a).

Following the same line of reasoning to calculate the remaining derivatives of the non-central chi-square distributions (3.8) and (3.9) with respect to $\epsilon$ yields

$$\begin{aligned} F_{\mathbb{Q}}^{\{\epsilon\}}\left(l, B_l; t, \epsilon B_t\right) &:= \frac{\partial}{\partial \epsilon} F_{\mathbb{Q}}\left(l, B_l; t, \epsilon B_t\right) \\ &= \begin{cases} -\phi p\left(2x(\epsilon B_t); \frac{2}{2-\beta}, 2y(B_l)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t) & \Leftarrow \beta < 2 \\ \phi p\left(2y(B_l); 4 + \frac{2}{\beta-2}, 2x(\epsilon B_t)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t) & \Leftarrow \beta > 2 \end{cases} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(l, B_l; t, \epsilon B_t\right) &:= \frac{\partial}{\partial \epsilon} F_{\mathbb{Q}^S}\left(l, B_l; t, \epsilon B_t\right) \\ &= \begin{cases} -\phi p\left(2y(B_l); 4 + \frac{2}{2-\beta}, 2x(\epsilon B_t)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t) & \Leftarrow \beta < 2 \\ \phi p\left(2x(\epsilon B_t); \frac{2}{\beta-2}, 2y(B_l)\right) \frac{2-\beta}{\epsilon} 2x(\epsilon B_t) & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \tag{3.23}$$

Armed with equations (3.22) and (3.23), the differentiation of the expression on the right-hand side of equation (3.16) follows straightforwardly. Using equation (3.6) with the substitution $S = \epsilon B$, then the derivative of the European-style option component with respect to $\epsilon$ is obtained as

$$\begin{aligned} \frac{\partial}{\partial \epsilon} v_t(\epsilon B_t, K, T; \phi) &= \phi K e^{-r(T-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(T, K; t, \epsilon B_t\right) - \phi B_t e^{-q(T-t)} F_{\mathbb{Q}^S}\left(T, K; t, \epsilon B_t\right) \\ &\quad - \phi \epsilon B_t e^{-q(T-t)} F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(T, K; t, \epsilon B_t\right), \end{aligned} \tag{3.24}$$

while the derivative of the early exercise premium with respect to $\epsilon$ is given by

$$\begin{aligned} & \frac{\partial}{\partial \epsilon}\left\{\int_t^T \left[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}\left(l, B_l; t, \epsilon B_t\right) - \phi q \epsilon B_t e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l, B_l; t, \epsilon B_t\right)\right] dl\right\} \\ = & \int_t^T \Big[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(l, B_l; t, \epsilon B_t\right) - \phi q B_t e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l, B_l; t, \epsilon B_t\right) \\ & - \phi q \epsilon B_t e^{-q(l-t)} F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(l, B_l; t, \epsilon B_t\right)\Big] dl. \end{aligned} \tag{3.25}$$

Combining equations (3.17), (3.24) and (3.25), the differentiation of both sides of equation (3.16) with respect to $\epsilon$ becomes

$$
\begin{aligned}
-\phi B_t \;=\; & \phi K e^{-r(T-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(T,K;t,\epsilon B_t\right) - \phi B_t e^{-q(T-t)} F_{\mathbb{Q}^S}\left(T,K;t,\epsilon B_t\right) \\
& -\phi \epsilon B_t e^{-q(T-t)} F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(T,K;t,\epsilon B_t\right) \\
& + \int_t^T \Big[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(l,B_l;t,\epsilon B_t\right) - \phi q B_t e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l,B_l;t,\epsilon B_t\right) \\
& - \phi q \epsilon B_t e^{-q(l-t)} F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(l,B_l;t,\epsilon B_t\right)\Big]\, dl. 
\end{aligned} \tag{3.26}
$$

Taking the limit $\epsilon \uparrow 1$, if $\phi = 1$, or $\epsilon \downarrow 1$, if $\phi = -1$, and rearranging terms yields the following implicit definition of the early exercise boundary $\{B_t, t_0 \leq t \leq T\}$

$$
\begin{aligned}
B_t \;=\; & \phi K \left[e^{-r(T-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(T,K;t,B_t\right) + r\int_t^T e^{-r(l-t)} F_{\mathbb{Q}}^{\{\epsilon\}}\left(l,B_l;t,B_t\right) dl\right] \\
& \times \Big[-\phi + \phi e^{-q(T-t)} \Big(F_{\mathbb{Q}^S}\left(T,K;t,B_t\right) + F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(T,K;t,B_t\right)\Big) \\
& + \phi q \int_t^T e^{-q(l-t)} \left[F_{\mathbb{Q}^S}\left(l,B_l;t,B_t\right) + F_{\mathbb{Q}^S}^{\{\epsilon\}}\left(l,B_l;t,B_t\right)\right] dl\Big]^{-1}.
\end{aligned} \tag{3.27}
$$

Even though the focus of our numerical experiments to be presented in the next section is on the valuation of American-style options, the computation of the corresponding *hedge ratios*, $\Delta_t(.)$, follows straightforwardly once the optimal boundary is obtained. To accomplish this purpose, it is only necessary to differentiate equation (3.5) with respect to the underlying asset price $S$, that is

$$
\begin{aligned}
\Delta_t\left(S,K,T;\phi\right) \;:=\; & \frac{\partial}{\partial S} V_t\left(S,K,T;\phi\right) \\
=\; & \frac{\partial}{\partial S} v_t\left(S,K,T;\phi\right) + \frac{\partial}{\partial S} EEP_t\left(S,K,T;\phi\right),
\end{aligned} \tag{3.28}
$$

with the derivative of the European-style option component (3.6) with respect to $S$ being obtained as[3.9]

$$\begin{aligned}\frac{\partial}{\partial S}v_t(S,K,T;\phi) &= \phi K e^{-r(T-t)} F^{\{S\}}_{\mathbb{Q}}\left(T,K;t,S_t\right) - \phi e^{-q(T-t)} F_{\mathbb{Q}^S}\left(T,K;t,S_t\right) \\ &\quad -\phi S_t e^{-q(T-t)} F^{\{S\}}_{\mathbb{Q}^S}\left(T,K;t,S_t\right), \end{aligned} \tag{3.29}$$

while the derivative of the early exercise premium (3.7) with respect to $S$ is given by

$$\begin{aligned} &\frac{\partial}{\partial S}\left\{\int_t^T \left[\phi r K e^{-r(l-t)} F_{\mathbb{Q}}\left(l,B_l;t,S\right) - \phi q S e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l,B_l;t,S\right)\right] dl\right\} \\ = &\int_t^T \Big[\phi r K e^{-r(l-t)} F^{\{S\}}_{\mathbb{Q}}\left(l,B_l;t,S_t\right) - \phi q e^{-q(l-t)} F_{\mathbb{Q}^S}\left(l,B_l;t,S_t\right) \\ &- \phi q S_t e^{-q(l-t)} F^{\{S\}}_{\mathbb{Q}^S}\left(l,B_l;t,S_t\right)\Big]\, dl, \end{aligned} \tag{3.30}$$

and the required derivatives of the noncentral chi-square distributions (3.8) and (3.9) with respect to $S$ are

$$\begin{aligned} F^{\{S\}}_{\mathbb{Q}}\left(l,B_l;t,S\right) &:= \frac{\partial}{\partial S}F_{\mathbb{Q}}\left(l,B_l;t,S\right) \\ &= \begin{cases} -\phi p\left(2x(S_t); \frac{2}{2-\beta}, 2y(B_l)\right) \frac{2-\beta}{S} 2x(S_t) & \Leftarrow \beta < 2 \\ \phi p\left(2y(B_l); 4+\frac{2}{\beta-2}, 2x(S_t)\right) \frac{2-\beta}{S} 2x(S_t) & \Leftarrow \beta > 2 \end{cases} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} F^{\{S\}}_{\mathbb{Q}^S}\left(l,B_l;t,S\right) &:= \frac{\partial}{\partial S}F_{\mathbb{Q}^S}\left(l,B_l;t,S\right) \\ &= \begin{cases} -\phi p\left(2y(B_l); 4+\frac{2}{2-\beta}, 2x(S_t)\right) \frac{2-\beta}{S} 2x(S_t) & \Leftarrow \beta < 2 \\ \phi p\left(2x(S_t); \frac{2}{\beta-2}, 2y(B_l)\right) \frac{2-\beta}{S} 2x(S_t) & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \tag{3.32}$$

The implementation of our non-time recursive iterative method is similar to the one employed by Kim et al. (2013) in a GBM modelling setup. More specifically, it takes the

[3.9] Note that equation (3.29) is simply a compact formula of Larguinho et al. (2013, Equations A7-A10).

right-hand side of equation (3.27) as a functional form for calculating the early exercise boundary and begins the iteration scheme with the function

$$B_t^0 = \phi\left(\phi K \wedge \phi r K/q\right), \tag{3.33}$$

which can be used on the right-hand side of equation (3.27) to obtain the left-hand side as the first-round approximation denoted by $B_t^1$. This first-round approximation is then substituted on the right-hand side of equation (3.27) to obtain the second-round approximation $B_t^2$. This procedure is repeated until convergence is obtained. In each round $k$, the early exercise boundary at maturity, i.e. $B_T^k$, is set to be equal to the right-hand side of equation (3.33).

The proposed iterative method requires the use of a numerical integration scheme. In all numerical computations presented in this paper, we use the global adaptive quadrature built-in function “integral” that is available in *Matlab R2017a*, though any other numerical integration scheme might be used.[3.10] Following the insights of Kim et al. (2013), we also use a polynomial interpolation method to accelerate the computation of the optimal exercise boundary. More specifically, we approximate $B_t$ through a polynomial of degree $n$ that interpolates all points in the set of $\{B_{t_i}^k\}_{0\leq i\leq n}$ at each $k$th-round iteration.

In summary, the implementation of the iterative method is based on the following steps:

0. Set $n+1$ to be the number of nodes for time to maturity—i.e., the time to expiration of the option contract is divided into $n$ evenly-spaced time points with $\Delta t := (T - t_0)/n$ such that $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$—and $k$ the number of iterations.

[3.10] For completeness, we note also that the required noncentral chi-square distribution functions are computed through Benton and Krishnamoorthy (2003, Algorithm 7.3), which has been also used in many recent articles involving the CEV model, e.g. in Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015), and Cruz and Dias (2017).

1. The iteration is initialized at $k = 0$ with $B_t^0 = \phi\,(\phi K \wedge \phi r K/q)$. For each round $k \in \{0, 1, 2, \ldots\}$ the early exercise boundary at maturity is set to be $B_T^k = \phi\,(\phi K \wedge \phi r K/q)$.

2. Calculate $B_t^k$ (for $k = 1, 2, \ldots$).

   (a) Calculate the value $\{B_{t_i}^k\}_{0 \leq i \leq n}$ of the approximate optimal exercise boundary by substituting $B_t$ in the right-hand side of equation (3.27) with $B_t^{k-1}$. We use the mentioned global adaptive quadrature method to calculate the integrations in equation (3.27).

   (b) Construct the function $B_t^k$ by interpolating the values $\{B_{t_i}^k\}_{0 \leq i \leq n}$ with a polynomial of order $n$.

   (c) Repeat steps (2a) and (2b) until required accuracy is obtained.

3. Calculate the value of the American-style option (3.5) and/or the hedge ratio (3.28). The early exercise premium (3.7) and/or its derivative (3.30) are computed using the aforementioned global adaptive quadrature method.

## 3.4 Numerical results

The aim of this section is to test the robustness of the proposed non-time recursive iterative method by comparing our results with the ones reported in Nunes (2009) and Ruas et al. (2013).[3.11] Table 1 values standard American-style put options under the CEV model, adopting the parameters constellation of Nunes (2009, Table 2) and Ruas et al. (2013, Table 1). The third column presents the corresponding European-style put option prices, while the fourth column shows the *exact* American-style put option

[3.11] As expected, most of the computational burden of the proposed iterative method is due to the required numerical integrations. Even though other numerical integration schemes might be applied to diminish the computational effort, such efficiency considerations are outside the main scope of the present paper and, hence, speed-accuracy trade-off analysis are not exploited here.

values computed through the trinomial lattice scheme of Boyle and Tian (1999) with 10,000 time steps. Columns 5-9 (resp., 10-14) report the American-style put prices calculated through the proposed iterative method using six (resp., ten) iterations and five different polynomial interpolation schemes of order $n \in \{8, 16, 20, 24, 32\}$. The accuracy of the method is measured by the mean absolute percentage error (hereafter, MAPE) considering the whole set of 20 option contracts and with respect to the *exact* American-style put option price. Table 2 reports similar numerical experiments but now for the case of standard American-style call options under the CEV model and adopting the parameters configuration of Nunes (2009, Table 3) and Ruas et al. (2013, Table 2).

[Please insert Table 1 about here.]

[Please insert Table 2 about here.]

The results of both tables indicate that the proposed method is accurate under the local volatility CEV model: the MAPE in all tested cases is well below the typical bid-ask spread observed in the market. Similar results—not reported here but available upon request—are also obtained for the MAPE of the hedge ratios. As expected, for a given number of $k$ iterations the convergence of the method is improved when the order $n$ of the interpolation scheme increases. Moreover, the values reported in Ruas et al. (2013, Table 1) for American-style puts allow us to conclude that the iterative method with $n = 16$ (resp., $n = 24$) provides similar results in terms of accuracy to the optimal stopping approach of Nunes (2009) with a five degree polynomial specification for the early exercise boundary (resp., to the SHP scheme of Chung and Shih (2009) and Ruas et al. (2013) with $n = 24$).[3.12] Similar conclusions can also be taken for the case of American-style calls.

[3.12] We have also tested the benchmark considered in Nunes (2009, Table 2) and Ruas et al. (2013, Table 1)—i.e., the Crank-Nicolson finite difference scheme with 15,000 time intervals and 10,000 space steps—and, as expected, the corresponding MAPE values are similar to the ones reported in Tables 1 and 2.

Another salient feature of Tables 1 and 2 is that augmenting the number of iterations $k$ from 6 to 10 does not improve the accuracy significantly. To clarify this point, Figure 1 plots the early exercise boundary of an American-style put option under the CEV model using the parameters configuration borrowed from Chung and Shih (2009, Figure 4). The left-hand side plot shows the boundaries obtained by the proposed iterative method for different numbers of iterations $k \in \{0, 1, 2, 3, 4, 5, 6\}$ and $n = 252$. In this case, it seems that four, five or six round-iterations are enough to obtain a suitably accurate optimal exercise boundary. This is also true for others parameters combinations. The right-hand side plot presents the early exercise boundary calculated through the iterative method with $k = 6$ and $n = 252$ and the corresponding boundary computed by the trinomial grid of Boyle and Tian (1999) with 10,000 time steps.

[Please insert Figure 1 about here.]

To sum up, Figure 1 indicates that our non-time recursive iterative method is able to produce accurate values of the optimal boundary under the CEV model. It is also clear from Figure 1 that the early exercise boundary exhibits the expected pattern in the case of an American-style put: it is an increasing function of calendar time and the critical exercise price near the maturity date of the option rises very rapidly with an increasing slope, which is consistent with the asymptotic behavior of the early exercise boundary near expiration highlighted in Chevalier (2005, Theorem 3.1), Chung and Shih (2009, Equation 28) and Ruas et al. (2013, Equation 70).

## 3.5 Conclusions

This paper provides a simple non-time recursive iterative method to obtain the early exercise boundary of American-style options under the CEV model, which allows us to compute accurate option prices and hedge ratios under such local volatility diffusion process. The new early exercise premium representation nests, as a special case, the integral representation derived by Kim et al. (2013) in the context of the log-normal diffusion.

Although the non-time recursive iterative method is accurate and efficient for valuing and hedging options under both the GBM and CEV diffusions, it also has the potential to be applicable for more general processes beyond these two models. For example, it would be interesting to study the possibility of using both the time and non-time recursive iterative methods in the context of the jump to default extended CEV model of Carr and Linetsky (2006) and compare the results with the ones reported in Nunes (2009) and Ruas et al. (2013). To the best of our knowledge, however, such extension has not been performed yet even under the standard integral representation approach (i.e., under the time recursive iterative method) and, therefore, it is outside the scope of the present article and it is suggested as a possible avenue for future research.

Table 1: Prices of standard American-style puts under the CEV model ($S_{t_0} = 100$, $\beta = 3$ and $T - t_0 = 0.5$ years).

| | | | American-style put | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $k = 6$ | | | | | $k = 10$ | | | | |
| Parameters | Strike | European | Exact | 8 | 16 | 20 | 24 | 32 | 8 | 16 | 20 | 24 | 32 |
| | 80 | 0.159 | 0.162 | 0.163 | 0.162 | 0.162 | 0.162 | 0.162 | 0.163 | 0.162 | 0.162 | 0.162 | 0.162 |
| $r = 7\%$ | 90 | 1.255 | 1.297 | 1.299 | 1.297 | 1.297 | 1.297 | 1.297 | 1.299 | 1.297 | 1.297 | 1.297 | 1.297 |
| $q = 3\%$ | 100 | 4.579 | 4.792 | 4.795 | 4.793 | 4.792 | 4.792 | 4.792 | 4.795 | 4.793 | 4.792 | 4.792 | 4.792 |
| $\delta = 0.02$ | 110 | 10.542 | 11.215 | 11.218 | 11.216 | 11.215 | 11.215 | 11.215 | 11.218 | 11.216 | 11.215 | 11.215 | 11.215 |
| | 120 | 18.452 | 20.025 | 20.023 | 20.025 | 20.025 | 20.025 | 20.025 | 20.023 | 20.025 | 20.025 | 20.025 | 20.025 |
| | 80 | 2.293 | 2.331 | 2.334 | 2.332 | 2.332 | 2.332 | 2.332 | 2.334 | 2.332 | 2.332 | 2.332 | 2.332 |
| $r = 7\%$ | 90 | 5.385 | 5.491 | 5.495 | 5.492 | 5.492 | 5.491 | 5.491 | 5.495 | 5.492 | 5.492 | 5.491 | 5.491 |
| $q = 3\%$ | 100 | 10.030 | 10.262 | 10.267 | 10.264 | 10.263 | 10.263 | 10.263 | 10.267 | 10.264 | 10.263 | 10.263 | 10.263 |
| $\delta = 0.04$ | 110 | 16.043 | 16.474 | 16.479 | 16.475 | 16.475 | 16.474 | 16.474 | 16.479 | 16.475 | 16.475 | 16.474 | 16.474 |
| | 120 | 23.132 | 23.843 | 23.847 | 23.844 | 23.844 | 23.843 | 23.843 | 23.847 | 23.844 | 23.844 | 23.843 | 23.843 |
| | 80 | 0.822 | 0.852 | 0.854 | 0.852 | 0.852 | 0.852 | 0.852 | 0.854 | 0.852 | 0.852 | 0.852 | 0.852 |
| $r = 7\%$ | 90 | 2.843 | 2.969 | 2.973 | 2.970 | 2.970 | 2.969 | 2.969 | 2.973 | 2.970 | 2.970 | 2.969 | 2.969 |
| $q = 0\%$ | 100 | 6.698 | 7.060 | 7.066 | 7.062 | 7.062 | 7.061 | 7.061 | 7.066 | 7.062 | 7.062 | 7.061 | 7.061 |
| $\delta = 0.03$ | 110 | 12.371 | 13.175 | 13.181 | 13.177 | 13.176 | 13.176 | 13.176 | 13.181 | 13.177 | 13.176 | 13.176 | 13.176 |
| | 120 | 19.493 | 20.992 | 20.995 | 20.993 | 20.992 | 20.992 | 20.992 | 20.995 | 20.993 | 20.992 | 20.992 | 20.992 |
| | 80 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 | 1.419 |
| $r = 3\%$ | 90 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 | 4.311 |
| $q = 7\%$ | 100 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 | 9.254 |
| $\delta = 0.03$ | 110 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 | 15.980 |
| | 120 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 | 23.978 |
| Mean absolute percentage error | | | | 0.0698% | 0.0208% | 0.0141% | 0.0104% | 0.0064% | 0.0698% | 0.0207% | 0.0140% | 0.0103% | 0.0063% |

This table values standard American-style put options under the CEV model, adopting the parameters configuration of Nunes (2009, Table 2) and Ruas et al. (2013, Table 1). The third column presents the corresponding European-style put option prices, while the fourth column shows the *exact* American-style put option values computed through the trinomial lattice scheme of Boyle and Tian (1999) with 10,000 time steps. Columns 5-9 (resp., 10-14) report the American-style put prices calculated through the proposed iterative method using six (resp., ten) iterations and five different polynomial interpolation schemes of order $n \in \{8, 16, 20, 24, 32\}$.

Table 2: Prices of standard American-style calls under the CEV model ($S_{t_0} = 100$, $\beta = 1$ and $T - t_0 = 1$ year).

| | | | American-style call | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $k = 6$ | | | | | $k = 10$ | | | | |
| Parameters | Strike | European | Exact | 8 | 16 | 20 | 24 | 32 | 8 | 16 | 20 | 24 | 32 |
| | 80 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 | 23.370 |
| $r = 7\%$ | 90 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 | 15.735 |
| $q = 3\%$ | 100 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 | 9.635 |
| $\delta = 2$ | 110 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 | 5.315 |
| | 120 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 | 2.630 |
| | 80 | 28.249 | 28.254 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 | 28.253 |
| $r = 7\%$ | 90 | 22.203 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 | 22.205 |
| $q = 3\%$ | 100 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 | 17.083 |
| $\delta = 4$ | 110 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 | 12.870 |
| | 120 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 | 9.499 |
| | 80 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 | 28.022 |
| $r = 7\%$ | 90 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 | 21.061 |
| $q = 0\%$ | 100 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 | 15.221 |
| $\delta = 3$ | 110 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 | 10.567 |
| | 120 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 | 7.047 |
| | 80 | 20.301 | 21.882 | 21.886 | 21.883 | 21.883 | 21.883 | 21.882 | 21.887 | 21.883 | 21.883 | 21.883 | 21.882 |
| $r = 3\%$ | 90 | 14.257 | 15.187 | 15.194 | 15.189 | 15.189 | 15.188 | 15.188 | 15.194 | 15.189 | 15.189 | 15.188 | 15.188 |
| $q = 7\%$ | 100 | 9.552 | 10.084 | 10.091 | 10.086 | 10.085 | 10.085 | 10.085 | 10.091 | 10.086 | 10.085 | 10.085 | 10.085 |
| $\delta = 3$ | 110 | 6.106 | 6.401 | 6.408 | 6.403 | 6.402 | 6.402 | 6.402 | 6.408 | 6.403 | 6.402 | 6.402 | 6.402 |
| | 120 | 3.728 | 3.886 | 3.892 | 3.888 | 3.887 | 3.887 | 3.887 | 3.892 | 3.888 | 3.887 | 3.887 | 3.887 |
| Mean absolute percentage error | | | | 0.0187% | 0.0054% | 0.0037% | 0.0027% | 0.0018% | 0.0187% | 0.0054% | 0.0037% | 0.0027% | 0.0017% |

This table values standard American-style call options under the CEV model, adopting the parameters configuration of Nunes (2009, Table 3) and Ruas et al. (2013, Table 2). The third column presents the corresponding European-style call option prices, while the fourth column shows the *exact* American-style call option values computed through the trinomial lattice scheme of Boyle and Tian (1999) with 10,000 time steps. Columns 5-9 (resp., 10-14) report the American-style call prices calculated through the proposed iterative method using six (resp., ten) iterations and five different polynomial interpolation schemes of order $n \in \{8, 16, 20, 24, 32\}$.

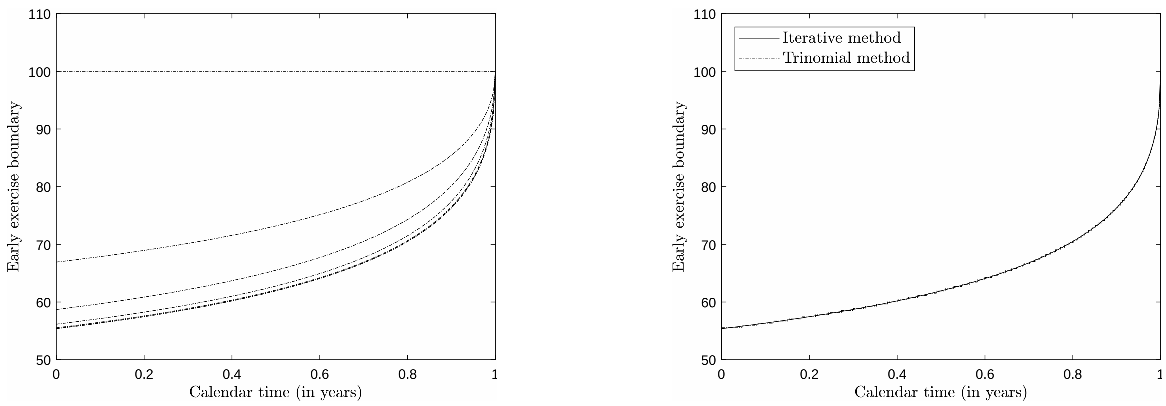


Figure 1: The early exercise boundary of an American-style put option under the CEV model with parameters configuration of Chung and Shih (2009, Figure 4): $K = 100$, $T = 1$, $\beta = 4/3$, $\delta = 2$, $r = 10\%$ and $q = 5\%$. The left-hand side plot shows the boundaries obtained by the proposed iterative method for different numbers of iterations $k \in \{0, 1, 2, 3, 4, 5, 6\}$ and $n = 252$. The right-hand side plot presents the early exercise boundary calculated through the iterative method with $k = 6$ and $n = 252$ and the corresponding boundary computed by the trinomial lattice scheme of Boyle and Tian (1999) with 10,000 time steps.

# 4. A Note on Options and Bubbles under the CEV Model: Implications for Pricing and Hedging*

**Abstract:** The discounted price process under the constant elasticity of variance (CEV) model is not a martingale for options markets with upward sloping implied volatility smiles. The loss of the martingale property implies the existence of (at least) two option prices for the call option, that is the price for which the put-call parity holds and the price representing the lowest cost of replicating the payoff of the call. This article derives closed-form solutions for the Greeks of the risk-neutral call option pricing solution that are valid for any CEV process exhibiting forward skew volatility smile patterns. Using an extensive numerical analysis, we conclude that the differences between the call prices and Greeks of both solutions are substantial, which might yield significant errors of analysis for pricing and hedging purposes.

*JEL Classification:* G13

*Keywords:* Bubbles; CEV model; Greeks; Option pricing; Put-call parity; Local martingales.

*This paper is a joint work with José Carlos Dias, and will be submitted to a peer-reviewed journal.

## 4.1 Introduction

A *bubble* is characterized by the existence of an underlying asset whose discounted price process is a *strict local martingale* under the risk-neutral probability measure but not a *martingale*. The presence of bubbles in spot and option prices imply that many standard results from option pricing theory do not hold. Hence, it is with no surprise that this issue has attracted much attention in the literature—see, for example, Loewenstein and Willard (2000), Cox and Hobson (2005), Heston et al. (2007), Ekström and Tysk (2009), Pal and Protter (2010), Guasoni and Rásonyi (2015) and Veestraeten (2017), just to name a few.

In particular, Cox and Hobson (2005), Heston et al. (2007) and Pal and Protter (2010) show that unusual properties in option values arise in the presence of bubbles. For instance, put-call parity fails—consequently, one can choose either put-call parity or risk-neutral option pricing but not both (i.e., these properties are mutually exclusive)—, the price of an American-style call on a non-dividend paying stock exceeds the price of a similar European-style option, American-style call options have no optimal exercise policy, the price of a European-style call is not convex as a function of the stock price, call prices do not tend to zero as strike increases to infinity and lookback call options have infinite value.

We recall that the equivalence of *no-arbitrage* with the existence of an *equivalent probability martingale measure* (and not simply a *strict local probability martingale measure*) is at the basis of the option pricing theory. For example, the option pricing methodology developed by Black and Scholes (1973) and Merton (1973) (henceforth, BSM) relies on delta-hedging arguments to value option contracts based on the assumption of no-arbitrage strategies that profit instantaneously and it establishes that option values must satisfy a particular partial differential equation (hereafter, PDE). The existence of a unique PDE solution for a given option pricing problem highlights the absence of arbitrage opportunities. As argued by Heston et al. (2007), however, multiple PDE solutions entail different strategies that exactly replicate identical option payoffs at different costs. This implies that two distinct replication strategies will produce different returns at the option's expiry date. The asset with dominated returns has an *asset pricing bubble* because its payoffs can be replicated by a cheaper investment strategy. Although the stock price process of the BSM model is a martingale under the risk-neutral probability measure, there are some important models in the option pricing literature for which the price process is a local martingale under the pricing measure, but not a true martingale.

In the present article, we will focus our analysis on the so-called *constant elasticity of variance* (hereafter, CEV) model of Cox (1975), Cox and Ross (1976) and Emanuel and MacBeth (1982) to provide further insight on option pricing in markets with bubbles. This *local stochastic volatility* model is quite popular among researchers and practitioners because it offers several appealing features, namely: (i) the state-dependent volatility assumption of the CEV model allows volatility to be modeled using a simple and parsimonious specification, without the need of introducing an additional stochastic process as in the case of the Heston (1993) stochastic volatility model; (ii) it is known to be consistent with the existence of a negative correlation between stock returns and realized volatility (*leverage effect*) observed, for instance, in Black (1976), Beckers (1980), Christie (1982) and Bekaert and Wu (2000); (iii) it is able to accommodate the inverse relation between the implied volatility and the strike price of an option contract (*implied volatility skew*) documented, for example, in Dennis and Mayhew (2002) and Bakshi et al. (2003).

Even though the martingale property under the CEV model is preserved in the case of options markets exhibiting *volatility smirk patterns* (i.e., with downward sloping implied volatility smiles), the discounted price process under the CEV model is not a martingale for options markets exhibiting *forward skew patterns* (i.e., with upward sloping implied volatility smiles), as was first documented in Emanuel and MacBeth (1982), Lewis (2000) and Delbaen and Shirakawa (2002). Cox and Hobson (2005) and Heston et al. (2007) offered an economic interpretation for this technical irregularity of the CEV model as evidence for the presence of a stock price bubble. Heston et al. (2007) further show that this loss of the martingale property implies the existence of (at least) two option prices for the call option: the price for which the put-call parity holds and the price representing the lowest cost of replicating the payoff of the call.

Since the CEV process is widely used in many option pricing applications, the main aim of this article is to shed further light on the implications for option pricing and hedging purposes of the existence of multiple option prices under such state-dependent volatility setup. To accomplish this purpose, we offer novel closed-form solutions of Greeks for the risk-neutral call option pricing formula proposed by Heston et al. (2007) and for *any* elasticity parameter of a CEV process. This is achieved by combining the new sensitivity measures derived in this paper for the bubble formula—which can be simply expressed as the difference between the solution given by Emanuel and MacBeth (1982) and the cheapest solution of Heston et al. (2007)—and the analytical formulae of Greeks provided by Larguinho et al. (2013) for the CEV model and expressed in terms of the noncentral chi-square distribution function. Hence, our formulas can be applied to *any* CEV process possessing upward sloping implied volatility smiles, thus making the formulas recently presented in Veestraeten (2017) a special case of our general analytical solutions. This should be important for both academics and practitioners since such implied volatility behaviour is a characteristic that is often observed in some commodity spot prices, energy markets and futures options—see, for example, Choi and Longstaff (1985), Adland et al. (2008), Geman and Shih (2009), Dias and Nunes (2011) and Lindström and Regland (2012).

The remainder of the article is organized as follows. Section 4.2 provides a short overview of the CEV model and its boundary conditions and the call options pricing solutions of Emanuel and MacBeth (1982) and Heston et al. (2007). Section 4.3 derives closed-form solutions of Greeks for the risk-neutral call option pricing formula of Heston et al. (2007). Section 4.4 provides computational experiments with the aim of discussing the sensitivities of the option prices to their input parameters in the presence of bubbles. Section 4.5 gives some concluding remarks.

## 4.2 The CEV option pricing model

### 4.2.1 Model setup

The CEV process of Cox (1975) assumes that the asset price $\{S_t, t \geq 0\}$ is governed (under the risk-neutral probability measure $\mathbb{Q}$) by the stochastic differential equation

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma(S_t)\, dW_t^{\mathbb{Q}}, \tag{4.1}$$

with a local volatility function given by

$$\sigma(S_t) = \delta\, S_t^{\frac{\beta}{2}-1}, \tag{4.2}$$

for $\delta \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, $q \geq 0$ represents the dividend yield for the underlying asset price, $\sigma(S_t)$ corresponds to the instantaneous volatility per unit of time of asset returns, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.

We recall that the elasticity of return variance with respect to price is equal to $\beta - 2$ given that $dv(S_t)/v(S_t) = (\beta - 2)\, dS_t/S_t$, where $v(S_t) = \delta^2 S_t^{\beta-2}$ is the instantaneous variance of asset returns. Moreover, the stochastic differential equation (4.1) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$), as well as the absolute diffusion (when $\beta = 0$) and the square-root diffusion (for $\beta = 1$) models of Cox and Ross (1976), as special cases. Whenever $\beta < 2$ (resp., $\beta > 2$) the local volatility function (4.2) is a decreasing (resp., increasing) function of the asset price, thus being able to generate downward-sloping (resp., upward-sloping) volatility skews. The model parameter $\delta$ is a positive constant that can be interpreted as the scale parameter fixing the initial instantaneous volatility at the reference time $t = 0$, i.e. $\sigma_0 = \sigma(S_0) = \delta\, S_0^{\beta/2-1}$. This calibration procedure ensures that the differences found between CEV models with different $\beta$ values stem purely from the effect of the relationship between volatility and price levels, which is captured by the CEV volatility specification (4.2).

### 4.2.2 Boundary characterization of the CEV diffusion

From Andersen and Andreasen (2000, Lemma 2) and Davydov and Linetsky (2001, Page 955), we recall that the CEV diffusion has the following boundary characterization:[4.1] infinity is a *natural boundary* for $\beta < 2$; zero is an *exit boundary* for $1 \leq \beta < 2$; and, for $\beta < 1$, zero is a *regular boundary point* and is specified as a killing boundary by adjoining a killing boundary condition. In technical terms, the imposition of such condition for the boundary behavior at $S = 0$ ensures that the stochastic differential

[4.1] For $\beta = 2$ (the lognormal case), both zero and infinity are natural boundaries.

equation $(4.1)$ has a unique solution and satisfies the *local Lipschitz continuity condition* enunciated in Andersen and Andreasen (2000, Theorem 1). Consequently, the origin becomes the absorbing state for the stock price process $\{S_t, t \geq 0\}$ whenever $\beta < 2$. Therefore, $\tau_0 := \inf\{t > t_0 : S_t = 0\} < \infty$ for these cases. This implies that the CEV process with $\beta < 2$ hits zero with positive probability, though such killing probability is generally quite small.

For $\beta > 2$, however, zero is a *natural boundary* and $\tau_0 = \infty$ (i.e. the probability that the stock price hits the point 0 is zero). Moreover, and still for $\beta > 2$, the CEV local volatility $(4.2)$ is unbounded as $S \to \infty$, and thus $+\infty$ is an *entrance boundary* for the CEV diffusion. Furthermore, and as initially observed by Emanuel and MacBeth (1982), the mean of the CEV density function for the case $\beta > 2$ is less than $S_t e^{(r-q)\tau}$ for all $S > 0$ and $\tau > 0$. However, one would expect a mean of $S_t e^{(r-q)\tau}$ given the (risk-neutral) dynamics $(4.1)$.

Notwithstanding Emanuel and MacBeth (1982) were the first to observe the aforementioned paradox for the case $\beta > 2$—and offered a call option pricing solution that avoids such problem by integrating the corresponding density function over domains that do not include $+\infty$—, they did not mention the possible existence of arbitrage opportunities under the CEV model. By exploring the powerful linkage between the CEV diffusion and squared Bessel processes, Delbaen and Shirakawa (2002) show the existence of a unique equivalent martingale measure and derive the Cox (1975) arbitrage free call option pricing formula (for $\beta < 2$) through the properties of squared Bessel processes. However, Delbaen and Shirakawa (2002, Theorem 4.2) highlight that the CEV model admits arbitrage opportunities when it is conditioned to be strictly positive, as it happens whenever $\beta > 2$.

We note that such problem is caused by the explosive nature of the stochastic process when $\beta > 2$, which implies that the stock price process under CEV has a bubble, as shown in Heston et al. (2007, Example 2.3). Technically speaking, the CEV process with $\beta > 2$ does not satisfy the so-called *linear growth condition* enunciated by Andersen and Andreasen (2000, Theorem 1). Clearly, this is not a desirable feature for financial applications since there is no equivalent martingale measure for the CEV specification when $\beta > 2$. As shown in Andersen and Andreasen (2000, Section 4.2), Davydov and Linetsky (2001, Footnote 8 and Appendix B), and Windcliff et al. (2001, Page 224), however, this problem can be easily circumvented (from a practical point of view) through a regularizing scheme for large values of $S$, which modifies the volatility specification (4.2) to a regularized version of the CEV model—known as *limited CEV* (henceforth, LCEV) process—whose volatility is bounded as

$$\sigma_\varepsilon(S_t) = \delta \, \min\left\{ S_t^{\frac{\beta}{2}-1}, \varepsilon^{\frac{\beta}{2}-1} \right\}, \tag{4.3}$$

where $\varepsilon$ is a fixed large number for which the LCEV process becomes a geometric Brownian motion diffusion when the asset price crosses over such switching level.[4.2] Consequently, $+\infty$ is a natural boundary, the mean of $S_T$ is equal to $S_t e^{(r-q)\tau}$, and the process $\{S_t e^{(q-r)\tau}; t \geq 0\}$ is a martingale on any finite time interval.

In summary, such modification of the CEV process exhibits more appealing growth and boundary characteristics. Even though the technical irregularities of the CEV process can be surpassed through a regularized version of the CEV model, it is still interesting to understand the effective impact (for practical applications) of the existence of bubbles under CEV when pricing and hedging options with upward sloping implied volatility smiles, i.e. CEV models with $\beta > 2$. For completeness, the next three subsections summarize the call option pricing solutions offered by Emanuel and MacBeth (1982) and Heston et al. (2007) and the implications of multiple solutions for the put-call parity

[4.2] A similar regularization scheme can be applied at small price levels for the CEV process with $\beta < 2$ to avoid absorption at zero.

property.

### 4.2.3 The Emanuel and MacBeth (1982) call option pricing solution

The CEV option pricing solutions for standard European-style contracts have been initially expressed in terms of the standard complementary gamma distribution function by Cox (1975) for $\beta < 2$ and by Emanuel and MacBeth (1982) for $\beta > 2$, whereas Schroder (1989) has subsequently extended such formulae in terms of the noncentral chi-square distribution.

Focusing on the case with $\beta > 2$ and following Schroder (1989, Footnote 2), the time-$t$ value of a European-style call option on the asset price $S$, with strike $K$, and maturity at time $T$ ($\geq t$) is given by

$$c_t^1(S_t, K, T) = S_t\, e^{-q\tau}\, Q\left(2x; 2v, 2y\right) - K\, e^{-r\tau}\, \left[1 - Q\left(2y; 2 + 2v, 2x\right)\right], \tag{4.4}$$

with $Q(.; v, \lambda)$ being the complementary distribution function of a noncentral chi-square law with $v \geq 0$ degrees of freedom and noncentrality parameter $\lambda \geq 0$, and where

$$k := \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau} - 1]}, \tag{4.5}$$

$$x := k\, S_t^{2-\beta}\, e^{(r-q)(2-\beta)\tau}, \tag{4.6}$$

$$y := kK^{2-\beta}, \tag{4.7}$$

$$v := \frac{1}{\beta - 2}, \tag{4.8}$$

and

$$\tau := T - t. \tag{4.9}$$

### 4.2.4 The Heston et al. (2007) call option pricing solution

The call option formula (4.4) is often (incorrectly) assumed to be the risk-neutral expected discounted value of the payoff of a European-style call option. Heston et al. (2007) show that this is not the case and offer the novel (corrected) solution expressed as[4.3]

$$c_t^2(S_t, K, T) = S_t\, e^{-q\tau} \left[ Q\left(2x; 2v, 2y\right) - \frac{\Gamma(v,x)}{\Gamma(v)} \right] - K\, e^{-r\tau} \left[ 1 - Q\left(2y; 2+2v, 2x\right)\right], \quad (4.10)$$

with $\Gamma\left(a, z\right)$ and $\Gamma\left(a\right)$ representing the upper incomplete gamma function and the Euler gamma function given in Abramowitz and Stegun (1972, Equations 6.5.3 and 6.1.1), respectively, for $a, z \in \mathbb{R}_+$.

The option pricing formula (4.10) is the risk-neutral expected discounted payoff of the call and is also the cheapest nonnegative solution subject to the same boundary conditions as the solution (4.4). One notes that the solutions' difference is

$$c_t^1(S_t, K, T) - c_t^2(S_t, K, T) = \Pi_t(S_t, K, T), \quad (4.11)$$

where the bubble value is simply given by

$$\Pi_t(S_t, K, T) = S_t\, e^{-q\tau}\, \frac{\Gamma(v,x)}{\Gamma(v)}. \quad (4.12)$$

The minimum quantity needed to replicate the option payoff at maturity is given by the risk-neutral solution (4.10). This implies that there is an arbitrage opportunity (even though an equivalent local martingale measure exists) and there are bubbles on option

[4.3] We note that equation (4.10) corrects the misprint error of Heston et al. (2007, Page 367) highlighted in Veestraeten (2017, Footnote 1).

values and on the stock price.

### 4.2.5 The put-call parity property

Although solution (4.4) does not satisfy risk-neutral pricing, put-call parity will hold for $c_t^1(.)$ if the put is risk-neutral priced, that is

$$c_t^1(S_t, K, T) = p_t(S_t, K, T) + S_t\, e^{-q\tau} - K\, e^{-r\tau}, \tag{4.13}$$

with

$$p_t(S_t, K, T) = K\, e^{-r\tau}\, Q\,(2y; 2 + 2v, 2x) - S_t\, e^{-q\tau}\; [1 - Q\,(2x; 2v, 2y)] \tag{4.14}$$

being the risk-neutral put price shown, for instance, in Larguinho et al. (2013) and Hull (2018, Chapter 27). However, since $c_t^2(.) < c_t^1(.)$ the risk-neutral pricing solution (4.10) does not satisfy the put-call parity property, that is put-call parity cannot hold if the put and call are both risk-neutral priced. In summary, as argued by Heston et al. (2007) one must choose between risk-neutral pricing and put-call parity since choosing both is not possible.

## 4.3 Sensitivity measures of the bubble formula

To better understand the importance and magnitude of the bubble values for pricing and hedging European-style plain-vanilla options under the CEV model, next proposition expresses the Greeks of the bubble formula (4.12) analytically and for any $\beta > 2$. Hence, we are able to offer novel closed-form solutions for the Greeks of the risk-neutral call option formula (4.10) using the results borrowed from Larguinho et al.

(2013).

**Proposition 4.1** *Let $k$, $x$, $v$, and $\tau$ be defined as in equations* (4.5), (4.6), (4.8), *and* (4.9), *respectively, and take $S_t > 0$.*

*i. The delta of the bubble formula* (4.12) *is given by*

$$\Delta_\Pi = \frac{\partial \Pi_t\,(.)}{\partial S} = e^{-q\tau}\,\frac{\Gamma(v,x)}{\Gamma(v)} + e^{-q\tau - x}\,\frac{x^v}{\Gamma(v+1)}. \tag{4.15}$$

*ii. The gamma of the bubble formula* (4.12) *is given by*

$$\Gamma_\Pi = \frac{\partial \Pi_t^2\,(.)}{\partial S^2} = \frac{\partial \Delta_\Pi}{\partial S} = e^{-q\tau - x}\,\frac{x^{v+1}}{v S_t\,\Gamma(v+1)}. \tag{4.16}$$

*iii. The theta of the bubble formula* (4.12) *is given by*

$$\Theta_\Pi = -\frac{\partial \Pi_t\,(.)}{\partial \tau} = q\,S_t\,e^{-q\tau}\,\frac{\Gamma(v,x)}{\Gamma(v)} + S_t\,e^{-q\tau - x}\,\frac{x^v}{\Gamma(v+1)}\,\frac{r-q}{e^{(r-q)(2-\beta)\tau}-1}. \tag{4.17}$$

*iv. The vega of the bubble formula* (4.12) *is given by*

$$\begin{aligned} \mathcal{V}_\Pi = \frac{\partial \Pi_t\,(.)}{\partial \sigma} &= 2S_t^{2-\beta/2}\,e^{-q\tau - x}\,\frac{x^v}{\delta\,\Gamma(v)} \\ &= 2S_t\,e^{-q\tau - x}\,\frac{x^v}{\sigma\,\Gamma(v)}. \end{aligned} \tag{4.18}$$

*v. The rho of the bubble formula* (4.12) *is given by*

$$\rho_\Pi = \frac{\partial \Pi_t\,(.)}{\partial r} = -S_t\,e^{-q\tau - x}\,\frac{x^v}{\Gamma(v)}\left[\frac{1}{r-q} - \frac{(2-\beta)\tau}{e^{(r-q)(2-\beta)\tau}-1}\right]. \tag{4.19}$$

**Proof.** The proof follows from straightforward calculations using the result in Abramowitz and Stegun (1972, Equation 6.5.25) and is available upon request. ■

**Corollary 1** *The Greeks of the risk-neutral call option pricing solution* (4.10) *arise immediately after subtracting the bubble Greeks shown in Proposition 4.1 from the corresponding sensitivity measures offered by Larguinho et al. (2013) for the pricing formula* (4.4)*.*

As discussed in Schroder (1989, Section V), the complementary noncentral chi-square distribution function with odd degrees of freedom can be represented by the sum of normal distributions and elementary functions. Relying on this idea, Veestraeten (2017) offer analytical solutions only for the case with $\beta = 4$—i.e., for $\alpha = \beta/2 = 2$ in their formulation, which implies the special cases with degrees of freedom of 1 and 3—with the argument that Greeks expressed in terms of the noncentral chi-square distributions would "become too elaborate for a meaningful analysis". The sensitivity measures of the bubble formula (4.12) offered in Proposition 4.1 and Corollary 1 clearly show that this is not the case since all the required functions—namely, the upper incomplete gamma function, Euler gamma function and the probability density function and the complementary distribution function of a noncentral chi-square law—can be easily and efficiently computed and are available as built-in functions in any computer language.[4.4]

In summary, the proposed novel Greeks are valid for any CEV model with $\beta > 2$ and nest the sensitivity measures presented in Veestraeten (2017) as a special case of our general solutions. Moreover, the use of our unrestricted Greeks is better able to provide a more detailed analysis of the presence of bubbles under the CEV model and its impact for hedging purposes.

**Remark 4.1** *Similarly to Larguinho et al. (2013, Equation A15), the bubble vega shown in equation* (4.18) *is derived with respect to the initial volatility* $\sigma \equiv \sigma_0$*. By contrast,*

[4.4] For instance, the unrestricted Greeks solutions of Larguinho et al. (2013) have been proved to be crucial in a wide variety of option pricing applications under the CEV model—see, for example, Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015) and Cruz and Dias (2017).

*Veestraeten (2017, Section 4.3) derives vegas with respect to the scale parameter $\delta$ (denoted by $\sigma$ in his notation). By multiplying the formulas of his vegas by $S^{1-\alpha}$—representing $\partial\delta/\partial\sigma_0$, with $\delta = \sigma_0 S^{1-\alpha}$—, then the vega values for $\alpha = 2$ are equal to the ones obtained through our solutions with $\beta = 4$.*

One notes that the bubble formula (4.12) does not depend on the strike price $K$. Thus, the so-called *eta* (sometimes also known as *strike delta*) is the same for both equations (4.4) and (4.10). This implies that the second derivative with respect to an option's strike price—often used to imply out state-contingent prices as highlighted in Breeden and Litzenberger (1978)—is also equal.

**Proposition 4.2** *Let $k$, $x$, $v$, $y$, and $\tau$ be defined as in equations (4.5), (4.6), (4.7), (4.8), and (4.9), respectively, and take $K > 0$. Then, the eta of the option pricing solutions (4.4) and (4.10) is given by*

$$\eta_{c^1} = \frac{\partial c_t^1\,(.)}{\partial K} = \eta_{c^2} = \frac{\partial c_t^2\,(.)}{\partial K} = -e^{-r\tau}\,\left[1 - Q\left(2y; 2+2v, 2x\right)\right] \tag{4.20}$$

*and the derivative of eta with respect to an option's strike price is obtained, for $i \in \{1, 2\}$, as*

$$\frac{\partial \eta_{c^i}\,(.)}{\partial K} = -e^{-r\tau}\,\frac{2y(2-\beta)}{K}\,p(2y; 2+2v, 2x), \tag{4.21}$$

*with*

$$p(x; v, \lambda) = \frac{1}{2}\,e^{-(\lambda+x)/2}\left(\frac{x}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda x}), \quad x > 0, \tag{4.22}$$

*being the probability density function of a noncentral chi-square law with $v \geq 0$ degrees of freedom and noncentrality parameter $\lambda \geq 0$, as given in Johnson et al. (1995, Equation 29.4), while $I_q(\cdot)$ is the modified Bessel function of the first kind of order $q$, as defined by Abramowitz and Stegun (1972, Equation 9.6.10).*

**Proof.** We first note that $\eta_{c^1} = \eta_{c^2}$, because $\frac{\partial \Pi_t(.)}{\partial K} = 0$. Hence, for $i \in \{1, 2\}$,

$$\begin{aligned} \eta_{c^i} &= \frac{\partial c_t^i(.)}{\partial K} \qquad (4.23) \\ &= S_t\, e^{-q\tau} \frac{\partial Q\,(2x; 2v, 2y)}{\partial K} - e^{-r\tau}\, [1 - Q\,(2y; 2+2v, 2x)] + K e^{-r\tau} \frac{\partial Q\,(2y; 2+2v, 2x)}{\partial K}. \end{aligned}$$

Using the derivatives shown in Larguinho et al. (2013, Equations A2a and A2b) and since

$$\frac{\partial 2x}{\partial K} = 0 \qquad (4.24)$$

and

$$\frac{\partial 2y}{\partial K} = \frac{2y(2-\beta)}{K}, \qquad (4.25)$$

then it is straightforward to compute the following partial derivatives:

$$\frac{\partial Q(2y; v, 2x)}{\partial K} = \frac{\partial Q(2y; v, 2x)}{\partial 2y} \frac{\partial 2y}{\partial K} = -\frac{2y(2-\beta)}{K}\, p(2y; v, 2x) \qquad (4.26)$$

and

$$\frac{\partial Q(2x; v, 2y)}{\partial K} = \frac{\partial Q(2x; v, 2y)}{\partial 2y} \frac{\partial 2y}{\partial K} = \frac{2y(2-\beta)}{K}\, p(2x; v+2, 2y). \qquad (4.27)$$

Replacing equations (4.26) and (4.27) into equation (4.23) and considering the relation

$$p\,(\lambda; v, x) = \left(\frac{\lambda}{x}\right)^{\frac{v-2}{2}} p\,(x; v, \lambda)\,, \qquad (4.28)$$

equation (4.23) can be rewritten as

$$\begin{aligned} \eta_{c^i} &= -e^{-r\tau}\, [1 - Q\,(2y; 2+2v, 2x)] \\ &\quad + \frac{2y(2-\beta)}{K}\, p\,(2x; 2+2v, 2y) \left[ S_t\, e^{-q\tau} - \left(\frac{x}{y}\right)^{-v} K e^{-r\tau} \right]. \qquad (4.29) \end{aligned}$$

Finally, equation (4.20) arises immediately because $(x/y)^{-v} K e^{-r\tau} = S_t e^{-q\tau}$. Equation (4.21) is obtained by using the partial derivative (4.26). ■

As expected, Equation (4.20) shows that $\eta_{c^i}$, for $i \in \{1, 2\}$, can be interpreted as the discounted risk-neutral probability of the call ending up in-the-money (assuming one takes the absolute value of the call strike delta). Alternatively, Equation (4.20) implies that $\eta_{c^i}$ can be interpreted also as a short position in a European-style cash-or-nothing call option that pays nothing if the underlying asset price $S_t$ ends up below or equal to the strike price $K$ at the maturity date $T$ and pays a fixed cash amount of \$1 if it ends up above the strike price (that is, if $S_T > K$). Since the time-$T$ price of a European-style cash-or-nothing call on the asset price $S$, with strike $K$, predetermined fixed cash amount of \$1 and maturity at time $T$ ($\geq t$) is equal to $\mathbf{1}_{\{S_T > K\}}$, then the presence of bubbles will not imply different pricing solutions for European-style digital and range digital options.[4.5]

## 4.4 Numerical applications

Panels A and B of Table 1 report the prices of European-style calls and the corresponding sensitivity measures adopting the parameters configuration of Larguinho et al. (2013, Table 3), that is $S_0 = 100$, $K \in \{95, 100, 105\}$, $\sigma_0 = \sigma(S_0) = 0.25$, $r = 0.10$, $q = 0$ and $\tau = 0.50$, but with $\beta \in \{3, 4, 5, 6, 7, 8, 9\}$. As usual, the scale parameter $\delta$ is computed through the local volatility function (4.2). The call prices shown in the third column of panels A and B of the table are obtained using equations (4.4) and (4.10), respectively. The Greeks reported in Panel A are calculated using the solutions offered by Larguinho et al. (2013), while the Greeks highlighted in Panel B are computed

[4.5] However, similarly to the case of plain-vanilla calls, the presence of bubbles will imply at least two different solutions for European-style asset-or-nothing calls, range asset options, gap call options and (single and double) barrier call options whenever $\beta > 2$.

through Corollary 1. All the required noncentral chi-square distribution functions are computed via Benton and Krishnamoorthy (2003) algorithm.

[Please insert Table 1 about here.]

Panels A and B of Table 2 show the bubble values and the absolute percentage relative errors defined by $(\hat{f}_i - f_i)/f_i$, respectively, where $f_i$ is the risk-neutral value obtained via Heston et al. (2007) formula and $\hat{f}_i$ denotes the corresponding value estimated by Emanuel and MacBeth (1982) solution, using the same parameters configuration of Table 1.

[Please insert Table 2 about here.]

There are several points that are noteworthy to highlight from these two tables. Call prices, deltas, gammas, vegas and rhos obtained via Emanuel and MacBeth (1982) formula are higher than the corresponding ones computed through the risk-neutral solutions based on Heston et al. (2007), thus originating positive bubble values for this constellation of parameters. For the theta case, however, we observe the oppositive behavior, which results in a negative bubble value for theta. These are the most common relations between both pricing solutions and the corresponding Greeks. We note also that, as expected, the bubble values shown in Panel A of Table 2 are the same for different moneyness levels since they do not depend on the strike price $K$.

We observe also that bubble values are almost insignificant for $\beta \in \{3, 4\}$, at least for the set of parameters under analysis. However, the case with $\beta = 5$ produces a bubble value in the call that is about 1 penny, as shown in Panel A of Table 2. Hence, the bubble values of call prices and Greeks for $\beta = 5$ seem to be not negligible anymore. Much more pronounced differences are revealed for the cases with $\beta \in \{6, 7, 8, 9\}$.

Moreover, we document that both the risk-neutral gamma and vega can be negative for the three moneyness levels, as revealed in Panel B of Table 1, and not only for the limiting case of $K = 0$ highlighted in Veestraeten (2017). It is well known that this limiting case produces a concave relation between the risk-neutral call price and the stock price, as noted in Cox and Hobson (2005) and Ekström and Tysk (2009). However, larger values for the strike price $K$ may create also unusual patterns in which the familiar convex relation between the risk-neutral call price and the stock price turns into a concave relationship for larger stock prices. As argued by Veestraeten (2017), such concavity behavior in the call option price might suggest the presence of a bubble in the underlying stock price.

We note also that the gammas and vegas presented in Panel A of Table 2 are equal to the risk-neutral sensitivity measures of the corresponding puts, as shown in Larguinho et al. (2013, Equations A13 and A15), respectively. Hence, observing larger gammas and vegas of puts than those of identical risk-neutral calls might be used as an option-based test for the potential presence of a bubble in the underlying stock price. Interestingly, Panel B of Table 2 reveals that the absolute percentage relative errors are equal for gammas and vegas, that is

$$\frac{\Gamma_{\Pi}}{\Gamma_{c^2}} = \frac{\mathcal{V}_{\Pi}}{\mathcal{V}_{c^2}}. \tag{4.30}$$

The rationale for this result is explained by the existence of a direct link between gamma and vega that is valid not only for the bubble values and the (risk-neutral) gamma and vegas appearing in equation (4.30), but also for the gamma and vega derived in Larguinho et al. (2013, Equations A13 and A15), that is

$$\frac{\Gamma_{\Pi}}{\mathcal{V}_{\Pi}} = \frac{\Gamma_{c^2}}{\mathcal{V}_{c^2}} = \frac{\Gamma_{c^1}}{\mathcal{V}_{c^1}}. \tag{4.31}$$

Using equations (4.16) and (4.18) we are able to express a general relation between gammas and vegas of a CEV process as

$$\frac{\Gamma_\Pi}{\mathcal{V}_\Pi} = \frac{\sigma x}{2v^2 S^2}. \tag{4.32}$$

Clearly, the use of this link allows us to compute the value of any vega under the CEV process (with $\beta > 2$) through the corresponding gamma value multiplied by the positive term $2v^2S^2(\sigma x)^{-1}$. This should have the potential to reduce the hedging costs when implementing a delta-gamma-vega neutral strategy, because such objective might be accomplished by simply using only one traded option instead of two traded derivatives on the underlying asset that are usually required for a portfolio to be both gamma and vega neutral.

The risk-neutral theta can assume positive values, as shown in Panel B of Table 1. Such possibility have been also discussed in Pal and Protter (2010) in the context of an inverse Bessel process and in Veestraeten (2017) under the nested CEV process. Finally, the risk-neutral rho typically returns a positive value. However, it can be negative for the case of calls that are deep in-the-money.

As expected, Tables 1 and 2 report results that do not represent a sufficiently large enough sample to take more robust conclusions, thus giving only a preliminary flavor of the results. Hence, to better assess the impact of different call and Greeks solutions when $\beta > 2$ we follow the guidelines of Broadie and Detemple (1996) by conducting a careful large sample evaluation of 17,500 randomly generated contracts. To accomplish this purpose, we fix the initial asset price at $S_0 = 100$ and take the strike price $K$ to be uniform between 70 and 130. The volatility $\sigma_0$ is distributed uniformly between 0.10 and 0.60 and the scale parameter $\delta$ is then computed. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. The dividend yield $q$ is uniform between 0.0 and 0.1. The riskless rate $r$ is uniform between 0.0 and 0.1. The $\beta$ parameter is distributed uniformly between each of the seven considered intervals with $\beta \in ]2, 3], ..., ]8, 9]$.

Table 3 reports the mean, maximum, minimum, median and the 75th and 95th percentile statistic measures of the bubble values for 2,500 random samples of prices and Greeks of European-style call options under the CEV model and for different intervals of $\beta$ values, thus resulting in a large sample of 17,500 contracts. Bubbles in call prices are computed via equation $(4.12)$, while bubbles in Greeks are calculated through Proposition 4.1.

[Please insert Table 3 about here.]

While the bubble values for $\beta \in \{3, 4\}$ reported in Table 2 seem to be negligible from a practical point of view, the results of Table 3 clearly show that significant bubble values might be obtained even in the range of $\beta \in ]2, 5]$. This suggests that for some CEV applications with $\beta > 2$ the use of call prices and Greeks based on the Emanuel and MacBeth (1982) formula might originate expensive option prices and inadequate sensitivity measures for hedging purposes.[4.6] To sum up, it is better to replace the usual solutions based on Emanuel and MacBeth (1982) formula by the risk-neutral solutions for future theoretical and empirical applications of the CEV process with $\beta > 2$.

## 4.5 Conclusions

The discounted price process under the CEV is not a martingale for options markets with upward sloping implied volatility smiles, which implies the existence of (at least)

[4.6] For example, Choi and Longstaff (1985, Page 252) reported $\beta$ values ranging from 4.38 to 5.02 in their application of the CEV model for pricing options on agricultural futures, but using the Emanuel and MacBeth (1982) solution.

two option prices for the call option: the price for which the put-call parity holds and the price representing the lowest cost of replicating the payoff of the call. This article derives closed-form solutions for the Greeks of the risk-neutral call option pricing solution that are valid for any CEV process exhibiting forward skew volatility smile patterns. Overall, we find that the Greeks of the risk-neutral call offered by Heston et al. (2007) can differ significantly from the ones calculated for the Emanuel and MacBeth (1982) solution, which may lead to much more expensive hedging strategies when using the latter formulae.

Table 1: Prices and Greeks of European-style call options under the CEV model

| $K$ | $\beta$ | Call | Delta | Gamma | Vega | Theta | Rho |
|---|---|---|---|---|---|---|---|
| Panel A: Emanuel and MacBeth (1982) | | | | | | | |
| 95 | 9 | 12.1710 | 0.8453 | 0.0141 | 21.1237 | -11.6464 | 31.8274 |
| 95 | 8 | 12.2195 | 0.8322 | 0.0145 | 21.1837 | -11.6410 | 31.7254 |
| 95 | 7 | 12.2720 | 0.8189 | 0.0150 | 21.3085 | -11.6505 | 31.6167 |
| 95 | 6 | 12.3281 | 0.8053 | 0.0155 | 21.4905 | -11.6731 | 31.5023 |
| 95 | 5 | 12.3877 | 0.7912 | 0.0161 | 21.7198 | -11.7066 | 31.3830 |
| 95 | 4 | 12.4507 | 0.7767 | 0.0167 | 21.9923 | -11.7499 | 31.2590 |
| 95 | 3 | 12.5174 | 0.7616 | 0.0174 | 22.3075 | -11.8030 | 31.1304 |
| 100 | 9 | 9.7146 | 0.7625 | 0.0187 | 28.0625 | -12.5133 | 27.4885 |
| 100 | 8 | 9.6747 | 0.7444 | 0.0189 | 27.5108 | -12.3736 | 27.4796 |
| 100 | 7 | 9.6439 | 0.7270 | 0.0191 | 27.0913 | -12.2670 | 27.4709 |
| 100 | 6 | 9.6206 | 0.7102 | 0.0194 | 26.7924 | -12.1907 | 27.4628 |
| 100 | 5 | 9.6034 | 0.6938 | 0.0197 | 26.5816 | -12.1366 | 27.4562 |
| 100 | 4 | 9.5915 | 0.6774 | 0.0201 | 26.4399 | -12.1002 | 27.4513 |
| 100 | 3 | 9.5845 | 0.6611 | 0.0206 | 26.3580 | -12.0791 | 27.4483 |
| 105 | 9 | 7.9209 | 0.6859 | 0.0220 | 32.9772 | -12.9528 | 23.5422 |
| 105 | 8 | 7.7639 | 0.6602 | 0.0218 | 31.7991 | -12.6424 | 23.4632 |
| 105 | 7 | 7.6250 | 0.6366 | 0.0217 | 30.8169 | -12.3851 | 23.4041 |
| 105 | 6 | 7.5009 | 0.6149 | 0.0217 | 30.0438 | -12.1835 | 23.3629 |
| 105 | 5 | 7.3880 | 0.5944 | 0.0218 | 29.4169 | -12.0217 | 23.3376 |
| 105 | 4 | 7.2843 | 0.5749 | 0.0220 | 28.8940 | -11.8889 | 23.3270 |
| 105 | 3 | 7.1884 | 0.5561 | 0.0222 | 28.4533 | -11.7794 | 23.3304 |
| Panel B: Heston et al. (2007) | | | | | | | |
| 95 | 9 | 9.0658 | 0.4506 | -0.0137 | -20.4309 | 0.6661 | 22.2082 |
| 95 | 8 | 10.2474 | 0.5607 | -0.0085 | -12.3893 | -1.9259 | 25.1162 |
| 95 | 7 | 11.3351 | 0.6749 | -0.0002 | -0.2272 | -5.5655 | 28.1117 |
| 95 | 6 | 12.0860 | 0.7618 | 0.0096 | 13.2881 | -9.4106 | 30.4429 |
| 95 | 5 | 12.3743 | 0.7882 | 0.0155 | 20.9513 | -11.4997 | 31.3092 |
| 95 | 4 | 12.4507 | 0.7767 | 0.0167 | 21.9919 | -11.7498 | 31.2590 |
| 95 | 3 | 12.5174 | 0.7616 | 0.0174 | 22.3075 | -11.8030 | 31.1304 |
| 100 | 9 | 6.6094 | 0.3679 | -0.0090 | -13.4921 | -0.2008 | 17.8692 |
| 100 | 8 | 7.7026 | 0.4728 | -0.0042 | -6.0622 | -2.6585 | 20.8704 |
| 100 | 7 | 8.7070 | 0.5830 | 0.0039 | 5.5556 | -6.1821 | 23.9658 |
| 100 | 6 | 9.3786 | 0.6668 | 0.0134 | 18.5900 | -9.9282 | 26.4034 |
| 100 | 5 | 9.5900 | 0.6907 | 0.0191 | 25.8131 | -11.9297 | 27.3823 |
| 100 | 4 | 9.5915 | 0.6774 | 0.0201 | 26.4395 | -12.1001 | 27.4512 |
| 100 | 3 | 9.5845 | 0.6611 | 0.0206 | 26.3580 | -12.0791 | 27.4483 |
| 105 | 9 | 4.8157 | 0.2913 | -0.0057 | -8.5774 | -0.6402 | 13.9230 |
| 105 | 8 | 5.7919 | 0.3887 | -0.0012 | -1.7740 | -2.9273 | 16.8540 |
| 105 | 7 | 6.6881 | 0.4927 | 0.0065 | 9.2813 | -6.3002 | 19.8991 |
| 105 | 6 | 7.2588 | 0.5714 | 0.0158 | 21.8415 | -9.9210 | 22.3034 |
| 105 | 5 | 7.3746 | 0.5914 | 0.0212 | 28.6484 | -11.8148 | 23.2637 |
| 105 | 4 | 7.2843 | 0.5749 | 0.0220 | 28.8936 | -11.8888 | 23.3270 |
| 105 | 3 | 7.1884 | 0.5561 | 0.0222 | 28.4533 | -11.7794 | 23.3304 |

Panels A and B of this table value European-style call options and the corresponding sensitivity measures based on the option pricing solutions of Emanuel and MacBeth (1982) and Heston et al. (2007), respectively, adopting the parameter configurations of Larguinho et al. (2013, Table 3), that is $S_0 = 100$, $K \in \{95, 100, 105\}$, $\sigma_0 = \sigma(S_0) = 0.25$, $r = 0.10$, $q = 0$ and $\tau = 0.50$, but with $\beta \in \{3, 4, 5, 6, 7, 8, 9\}$. The call prices shown in the third column of the table are obtained using equations (4.4) and (4.10), respectively. The Greeks reported in Panel A are calculated using the solutions offered by Larguinho et al. (2013), while the Greeks highlighted in Panel B are computed through Corollary 1.

Table 2: Bubble values and absolute percentage relative errors of prices and Greeks of European-style call options under the CEV model

| $K$ | $\beta$ | Call | Delta | Gamma | Vega | Theta | Rho |
|---|---|---|---|---|---|---|---|
| Panel A: Bubble values | | | | | | | |
| 95 | 9 | 3.1052 | 0.3947 | 0.0278 | 41.5546 | -12.3125 | 9.6192 |
| 95 | 8 | 1.9721 | 0.2715 | 0.0230 | 33.5730 | -9.7151 | 6.6092 |
| 95 | 7 | 0.9369 | 0.1440 | 0.0152 | 21.5356 | -6.0849 | 3.5050 |
| 95 | 6 | 0.2420 | 0.0434 | 0.0059 | 8.2024 | -2.2625 | 1.0594 |
| 95 | 5 | 0.0134 | 0.0030 | 0.0006 | 0.7685 | -0.2069 | 0.0738 |
| 95 | 4 | 3.47E-06 | 1.12E-06 | 3.31E-07 | 4.35E-04 | -1.14E-04 | 2.76E-05 |
| 95 | 3 | 0.00E+00 | 4.89E-26 | 3.05E-26 | 3.91E-23 | -1.00E-23 | 1.23E-24 |
| 100 | 9 | 3.1052 | 0.3947 | 0.0278 | 41.5546 | -12.3125 | 9.6192 |
| 100 | 8 | 1.9721 | 0.2715 | 0.0230 | 33.5730 | -9.7151 | 6.6092 |
| 100 | 7 | 0.9369 | 0.1440 | 0.0152 | 21.5356 | -6.0849 | 3.5050 |
| 100 | 6 | 0.2420 | 0.0434 | 0.0059 | 8.2024 | -2.2625 | 1.0594 |
| 100 | 5 | 0.0134 | 0.0030 | 0.0006 | 0.7685 | -0.2069 | 0.0738 |
| 100 | 4 | 3.47E-06 | 1.12E-06 | 3.31E-07 | 4.35E-04 | -1.14E-04 | 2.76E-05 |
| 100 | 3 | 0.00E+00 | 4.89E-26 | 3.05E-26 | 3.91E-23 | -1.00E-23 | 1.23E-24 |
| 105 | 9 | 3.1052 | 0.3947 | 0.0278 | 41.5546 | -12.3125 | 9.6192 |
| 105 | 8 | 1.9721 | 0.2715 | 0.0230 | 33.5730 | -9.7151 | 6.6092 |
| 105 | 7 | 0.9369 | 0.1440 | 0.0152 | 21.5356 | -6.0849 | 3.5050 |
| 105 | 6 | 0.2420 | 0.0434 | 0.0059 | 8.2024 | -2.2625 | 1.0594 |
| 105 | 5 | 0.0134 | 0.0030 | 0.0006 | 0.7685 | -0.2069 | 0.0738 |
| 105 | 4 | 3.47E-06 | 1.12E-06 | 3.31E-07 | 4.35E-04 | -1.14E-04 | 2.76E-05 |
| 105 | 3 | 0.00E+00 | 4.89E-26 | 3.05E-26 | 3.91E-23 | -1.00E-23 | 1.23E-24 |
| Panel B: Absolute percentage relative errors | | | | | | | |
| 95 | 9 | 34.25 | 87.58 | 203.39 | 203.39 | 1,848.46 | 43.31 |
| 95 | 8 | 19.24 | 48.43 | 270.98 | 270.98 | 504.44 | 26.31 |
| 95 | 7 | 8.27 | 21.33 | 9,479.10 | 9,479.10 | 109.33 | 12.47 |
| 95 | 6 | 2.00 | 5.70 | 61.73 | 61.73 | 24.04 | 3.48 |
| 95 | 5 | 0.11 | 0.38 | 3.67 | 3.67 | 1.80 | 0.24 |
| 95 | 4 | 2.78E-05 | 1.44E-04 | 1.98E-03 | 1.98E-03 | 9.73E-04 | 8.84E-05 |
| 95 | 3 | 0.00E+00 | 6.42E-24 | 1.75E-22 | 1.75E-22 | 8.50E-23 | 3.96E-24 |
| 100 | 9 | 46.98 | 107.28 | 307.99 | 307.99 | 6,130.87 | 53.83 |
| 100 | 8 | 25.60 | 57.42 | 553.81 | 553.81 | 365.43 | 31.67 |
| 100 | 7 | 10.76 | 24.69 | 387.64 | 387.64 | 98.43 | 14.62 |
| 100 | 6 | 2.58 | 6.51 | 44.12 | 44.12 | 22.79 | 4.01 |
| 100 | 5 | 0.14 | 0.44 | 2.98 | 2.98 | 1.73 | 0.27 |
| 100 | 4 | 3.61E-05 | 1.66E-04 | 1.65E-03 | 1.65E-03 | 9.45E-04 | 1.01E-04 |
| 100 | 3 | 0.00E+00 | 7.40E-24 | 1.48E-22 | 1.48E-22 | 8.30E-23 | 4.49E-24 |
| 105 | 9 | 64.48 | 135.49 | 484.47 | 484.47 | 1,923.08 | 69.09 |
| 105 | 8 | 34.05 | 69.86 | 1,892.53 | 1,892.53 | 331.88 | 39.21 |
| 105 | 7 | 14.01 | 29.22 | 232.03 | 232.03 | 96.58 | 17.61 |
| 105 | 6 | 3.33 | 7.60 | 37.55 | 37.55 | 22.80 | 4.75 |
| 105 | 5 | 0.18 | 0.51 | 2.68 | 2.68 | 1.75 | 0.32 |
| 105 | 4 | 4.76E-05 | 1.95E-04 | 1.51E-03 | 1.51E-03 | 9.61E-04 | 1.19E-04 |
| 105 | 3 | 0.00E+00 | 8.80E-24 | 1.38E-22 | 1.38E-22 | 8.52E-23 | 5.29E-24 |

Panels A and B of this table report the bubble values and the absolute percentage relative errors defined by $(\hat{f}_i - f_i)/f_i$, respectively, where $f_i$ is the risk-neutral value obtained via Heston et al. (2007) formula and $\hat{f}_i$ denotes the corresponding value estimated by Emanuel and MacBeth (1982) solution using the parameters configuration of Table 1.

Table 3: Bubble values for large samples of randomly generated prices and Greeks of European-style call options under the CEV model

| | Call | Delta | Gamma | Vega | Theta | Rho |
|---|---|---|---|---|---|---|
| | | | $\beta \in ]8,9]$ | | | |
| Mean | 10.6002 | 0.5220 | 0.0113 | 32.9920 | -6.9980 | 29.8515 |
| Maximum | 52.9749 | 0.9887 | 0.0275 | 93.6827 | 1.3621 | 163.8070 |
| Minimum | 0.00E+00 | 3.26E-22 | 9.88E-22 | 1.06E-19 | -18.0448 | 1.70E-21 |
| Median | 9.1934 | 0.6420 | 0.0102 | 37.6620 | -7.6218 | 20.4989 |
| 75th Percentile | 17.5038 | 0.7971 | 0.0186 | 43.4777 | -1.2841 | 31.5320 |
| 95th Percentile | 27.9577 | 0.8938 | 0.0248 | 63.2290 | -2.39E-16 | 109.6067 |
| | | | $\beta \in ]7,8]$ | | | |
| Mean | 10.0141 | 0.4691 | 0.0102 | 33.4435 | -6.9786 | 27.6693 |
| Maximum | 54.0942 | 0.9846 | 0.0236 | 98.8360 | 1.3207 | 156.0623 |
| Minimum | 0.00E+00 | 3.69E-33 | 1.41E-32 | 1.45E-30 | -18.9103 | 1.88E-32 |
| Median | 7.7978 | 0.5699 | 0.0097 | 39.9060 | -6.9546 | 17.9209 |
| 75th Percentile | 17.0282 | 0.7543 | 0.0169 | 45.7373 | -0.8539 | 29.8269 |
| 95th Percentile | 28.8182 | 0.8720 | 0.0215 | 66.7960 | -1.81E-19 | 106.2715 |
| | | | $\beta \in ]6,7]$ | | | |
| Mean | 9.0903 | 0.4021 | 0.0087 | 33.1418 | -6.7642 | 24.8640 |
| Maximum | 54.6910 | 0.9829 | 0.0201 | 104.3462 | 1.1735 | 143.9497 |
| Minimum | 0.00E+00 | 3.81E-49 | 1.74E-48 | 1.85E-46 | -19.0374 | 1.97E-48 |
| Median | 5.8005 | 0.4476 | 0.0092 | 40.8787 | -5.1307 | 14.1560 |
| 75th Percentile | 15.5639 | 0.6945 | 0.0145 | 48.7142 | -0.4761 | 27.4921 |
| 95th Percentile | 29.6413 | 0.8335 | 0.0181 | 69.2197 | -4.32E-23 | 101.9081 |
| | | | $\beta \in ]5,6]$ | | | |
| Mean | 7.6999 | 0.3177 | 0.0069 | 31.4264 | -6.1151 | 21.1408 |
| Maximum | 55.6142 | 0.9768 | 0.0163 | 105.8785 | 1.0194 | 131.9322 |
| Minimum | 0.00E+00 | 5.24E-89 | 3.24E-88 | 3.42E-86 | -19.9382 | 2.70E-88 |
| Median | 3.1676 | 0.2695 | 0.0078 | 35.9988 | -3.5548 | 8.6280 |
| 75th Percentile | 12.7450 | 0.5961 | 0.0120 | 51.3724 | -0.0809 | 23.8891 |
| 95th Percentile | 29.7843 | 0.7791 | 0.0146 | 75.7301 | -1.81E-32 | 95.1721 |
| | | | $\beta \in ]4,5]$ | | | |
| Mean | 5.5542 | 0.2084 | 0.0045 | 26.5558 | -4.6379 | 15.9319 |
| Maximum | 55.6434 | 0.9625 | 0.0127 | 116.3126 | 5.37E-01 | 120.1965 |
| Minimum | 0.00E+00 | 2.99E-187 | 2.72E-186 | 2.81E-184 | -20.3148 | 1.53E-186 |
| Median | 0.8610 | 0.0842 | 0.0033 | 16.2364 | -2.0230 | 2.6981 |
| 75th Percentile | 7.4501 | 0.3884 | 0.0090 | 50.3903 | -6.29E-04 | 16.1825 |
| 95th Percentile | 27.6970 | 0.6883 | 0.0112 | 80.6022 | -3.22E-53 | 85.5039 |
| | | | $\beta \in ]3,4]$ | | | |
| Mean | 2.8168 | 0.0872 | 0.0016 | 15.6629 | -1.8069 | 9.0263 |
| Maximum | 53.0964 | 0.9258 | 0.0091 | 125.2368 | 0.00E+00 | 109.8012 |
| Minimum | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | -17.8788 | 0.00E+00 |
| Median | 3.10E-03 | 4.42E-04 | 5.47E-05 | 1.62E-01 | -4.01E-02 | 1.65E-02 |
| 75th Percentile | 9.94E-01 | 6.99E-02 | 2.90E-03 | 1.73E+01 | -1.42E-13 | 3.15E+00 |
| 95th Percentile | 21.0235 | 0.5425 | 0.0067 | 86.5274 | -1.12E-134 | 69.2600 |
| | | | $\beta \in ]2,3]$ | | | |
| Mean | 0.3754 | 0.0107 | 0.0002 | 3.3857 | -0.1945 | 1.5211 |
| Maximum | 32.6016 | 0.6606 | 0.0048 | 132.5431 | 0.00E+00 | 87.8428 |
| Minimum | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | -9.2470 | 0.00E+00 |
| Median | 0.00E+00 | 4.77E-55 | 3.80E-55 | 5.65E-52 | -1.80E-52 | 1.68E-53 |
| 75th Percentile | 8.44E-10 | 1.64E-10 | 2.59E-11 | 1.32E-07 | 0.00E+00 | 8.73E-09 |
| 95th Percentile | 0.7734 | 0.0324 | 9.67E-04 | 17.8168 | 0.00E+00 | 5.2441 |

This table reports some statistics of the bubble values for 2,500 random samples of prices and Greeks of European-style call options under the CEV model and for different intervals of $\beta$ values. Bubbles in call prices are computed via equation (4.12), while bubbles in Greeks are calculated through Proposition 4.1.

# 5. Conclusion

This thesis provides important results concerning the valuation of European and American options in three separate articles.

The first paper examines the choice of method for computing the option hedge ratios studied by Pelsser and Vorst (1994), Chung and Shackleton (2002), and Chung et al. (2011), but assumes the underlying stock price is governed by a CEV diffusion process. Contrary to what was found by Chung and Shackleton (2002) under the GBM assumption, we show that, under the CEV model, an extended tree design is the key feature for generating accurate and fast calculations of Greeks if one ignores the use of a Richardson extrapolation technique. However, an extended tree binomial CEV model with the smooth and monotonic convergence property is the most efficient method for computing Greeks under the CEV diffusion process because one can apply the two-point extrapolation formula suggested by Chung et al. (2011).

The second paper provides a simple non-time recursive iterative method to obtain the early exercise boundary of American-style options under the CEV model, which allows us to compute accurate option prices and hedge ratios under such local volatility diffusion process. The new early exercise premium representation nests, as a special case, the integral representation derived by Kim et al. (2013) in the context of the log-normal diffusion.

Although the non-time recursive iterative method is accurate and efficient for valuing and hedging options under both the GBM and CEV diffusions, it also has the potential to be applicable for more general processes beyond these two models.

The third article derives closed-form solutions for the Greeks of the risk-neutral call option pricing solution that are valid for any CEV process exhibiting forward skew volatility smile patterns. Overall, we find that the Greeks of the risk-neutral call offered by Heston et al. (2007) can differ significantly from the ones calculated for the Emanuel and MacBeth (1982) solution, which may lead to much more expensive hedging strategies when using the latter formulae.

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