WORKING PAPERS

N° 1290

January 2022

“Best-response dynamics in directed network games”

Peter Bayer, György Kozics and Nora Gabriella Szöke

Toulouse School of Economics

Best-response dynamics in directed network games*

Péter Bayer[†] György Kozics[‡] Nóra Gabriella Szőke[§]

September 27, 2021

Abstract

We study public goods games played on networks with possibly non-reciprocal relationships between players. Examples for this type of interactions include one-sided relationships, mutual but unequal relationships, and parasitism. It is well known that many simple learning processes converge to a Nash equilibrium if interactions are reciprocal, but this is not true in general for directed networks. However, by a simple tool of rescaling the strategy space, we generalize the convergence result for a class of directed networks and show that it is characterized by transitive weight matrices and quadratic best-response potentials. Additionally, we show convergence in a second class of networks; those rescalable into networks with weak externalities. We characterize the latter class by the spectral properties of the absolute value of the network's weight matrix and by another best-response potential structure.

Keywords: Networks, externalities, local public goods, potential games, non-reciprocal relations.

JEL classification: C72, D62, D85.

*We thank Yann Bramoullé, Péter Csóka, P. Jean-Jacques Herings, Botond Kőszegi, Mánuel László Mágó, Ronald Peeters, Miklós Pintér, Ádám Szeidl, Frank Thuijsman, Yannick Viossat, and Mark Voorneveld for valuable feedback and suggestions.

[†]Corresponding author. Toulouse School of Economics, 1 Esplanade de l'université 31080 Toulouse. E-mail: peter.bayer@tse-fr.eu

[‡]Central European University, Department of Economics and Business, Nádor utca 9, 1051 Budapest, Hungary. E-mail: kozics_gyorgy@phd.ceu.edu.

[§]Université Grenoble-Alpes, Institut Fourier, CS 40700, 38058 Grenoble cedex 09, France. E-mail: nora.gabriella.szoke@gmail.com. Research supported by the Swiss National Science Foundation, Early Postdoc.Mobility fellowship no. P2ELP2_184531.

1 Introduction

All social and economic networks feature relationships which cannot be described as a partnership of equals. There are relationships between pairs of agents in which only one party is interested and the other is not. Some relationships are beneficial for one party but harmful for the other. Even when both parties benefit or both are harmed, the extent by which they are affected by their counterpart's decisions is not necessarily equal. Nevertheless, most scientific works, both theoretical and applied, are using models in which the reciprocity of interactions is a fundamental property. These frameworks, simple graphs and weighted networks, have received a lot of recent interest from economic theorists due to them providing a highly accurate, rich, and efficient description of real-life interactions. However for reasons relating to either convenience or convention, non-reciprocal relations, and their relevance to economic theory, are relatively unexplored in network literature.

In particular, most models featuring externalities in networks such as Ballester et al. (2006) and Bramoullé and Kranton (2007) assume reciprocal interactions. These highly influential theoretical papers opened the way for a number of applications, such as R&D expenditure between interlinked firms (König et al., 2019), peer effects (Blume et al., 2010), defense expenditures (Sandler and Hartley, 1995, 2007), and crime (Ballester et al., 2010). Most of the applied literature continues to assume reciprocity of relations and performs equilibrium analysis. However, as it will be apparent from our results, the basis behind using the Nash equilibrium as a prediction in networks with non-reciprocal interactions is much weaker than with only reciprocal ones. Thus, predictions made by models using simple graphs or weighted networks are only justified for applications where the underlying interaction network is shown to consist only of reciprocal relations.

In this paper we extend the theoretical literature of learning in networks to include non-reciprocal relations by the use of directed networks. By doing so we intend to fill a gap in the theory literature and provide a jumping off point for a stronger connection between subsequent applied literature and real-life interaction networks. Our setting generalizes games played on weighted networks: instead of one weight, each link is defined by two distinct weights, one for each direction. The three important types of non-reciprocal relations we highlight are (1) one-directional links with one of the weights of the link being zero and the other being non-zero, e.g., an upstream city along a river affects a neighboring downstream city by polluting the river but not vice versa, (2) parasitic links with one weight being positive and the other being negative, e.g., a criminal organization engaging in the extortion of a small business gains benefits from the interaction but it comes with losses for the business, and (3) mutual but unequal benefit or harm, with both weights being positive or both being negative but they are not equal, e.g., a monopolistic seller and a competitive buyer both benefit from their economic relationship but the gains from the interaction tend to be greater for the seller than for the buyer.

One of the focuses of theoretical literature of public goods and networks is the convergence of learning processes to the Nash equilibrium. This is to provide a behavioral and an evolutionary motivation to use equilibrium as a prediction in applied settings. Under symmetric weight matrices a number of powerful convergence results are known: Stability of Nash equilibria with respect to the continuous best-response dynamic has been established by Bramoullé et al. (2014). Convergence of the continuous best-response dynamic to some Nash equilibrium in such games has been shown by Bervoets and Faure (2019). Bayer et al. (2019) shows convergence of a class of one-sided learning processes. Bervoets et al. (2020) constructs a convergent learning process not requiring the sophistication of the best response. Bayer et al. (2021) studies the impact of a farsighted agent on the process in a population of myopic players. Together, these results allow for an interpretation of the game's Nash equilibria as the results of a sequence of improvements made separately by the players.

All of the above papers assume reciprocal network interactions. Thus, they make use of more general results of games of reciprocal interactions (Dubey et al., 2006; Kukushkin, 2005), as well as that of generalized aggregative games (Jensen, 2010). The latter structure allows for the use of the theory of potential games (Monderer and Shapley, 1996), specifically, best-response potential games (Voorneveld, 2000). The main intuition behind the existence of a differentiable best-response potential function for reciprocal interaction networks is that the potential's Hessian, a symmetric matrix, must correspond to the Jacobian of the system of best-responses, which, in this case, is equal to the network's weight matrix. In this paper, however, we show that the best-response potential structure can be exploited in important classes of networks even when the relations, as expressed by the interaction matrix, are not reciprocal.

We consider convergence to Nash equilibria under one-sided improvement dynamics taking place in discrete time. Starting from a profile of production decisions, in every time period one player receives an opportunity to change her production, while every other player remains on the previous period's level. In the next period, another player receives a revision opportunity, and so on. We consider three versions of the best-response dynamic. In increasing order of generality, these are the pure best-response dynamic, in which every revision takes the player to her current best choice given the actions of others, the best-response-approaching dynamic, in which every revision moves the player into the interval between her current action and her current best choice, and the best-response-centered dynamic, in which every revision reduces the distance between her action and her current best choice. Pure best-response dynamics as described above are widely studied. In directed network games, best-response-approaching dynamics include the naive learning dynamics introduced by Bervoets et al. (2020), while best-response-centered dynamics are studied in Bayer et al. (2019). These two dynamics are similar to the directional learning model (Selten and Stoecker, 1986; Selten and Buchta, 1998) in which players are making attempts to find their targets by adjusting towards the direction they believe the target is located. Such qualitative learning models are known to explain experimental behavior in various settings (Cachon and Camerer, 1996; Cason and Friedman, 1997; Kagel and Levin, 1999; Nagel and Vriend, 1999).

While none of these dynamics can cycle under reciprocal interactions, in directed network games cycles can emerge. Cycles indicate that convergence to the Nash equilibrium is not a universal property of learning processes. We discuss two examples: (1) directed 'net' cycle networks allow for best-response cycles as economic activity of the players flows in the opposite direction as the external 'net' effects of the network, and (2) parasite-host networks lead to best-response cycles as the parasite's economic activity increases with the host's activity level, while the host's economic activity decreases with that of the parasite.

Nevertheless, classes of networks exist without cycles where convergence to a Nash equilibrium can be shown, given some mild assumptions on the order of updates. In this paper we identify two such classes, networks with transitive relative importance, and games rescalable to exhibit weak influences or weak externalities.

The former class captures networks that can be transformed into symmetric networks with some appropriate rescaling of the action space, an idea raised by Bramoullé et al. (2014). Rescaling can be understood as changing the measurement of one player's production from, e.g., euros to dollars. Rescaling does not affect the equilibrium structure or the convergence properties of the game, but it does change its nominal interaction structure as expressed by the network's weight matrix. Thus, a network with reciprocal interactions can be rescaled into non-reciprocal ones, which thus inherit its convergence properties. A network can be rescaled in such a way if and only if it satisfies the property of transitive relative importance and it does not have one-way or parasitic interactions.

The relative importance of a link for a player is measured by the relative payoff-effects between the two players. If a link is reciprocal, then the relative importance of that link for both players is unity. A link with a larger weight for one player and a small weight for the other is relatively more important to the former and, inversely, not so important for the latter. Transitive relative importance restricts the network through these values. For instance, if the link $\{i, j\}$ is more important to i than to j, and if the link $\{j, k\}$ has equal importance to both participants, then the link $\{i, k\}$ must be more important to i than to k. This property presumes a common hierarchy of players with important players whose production matters greatly in relative terms for all individuals and less important players whose production matters little.

This property is closely connected to Kolmogorov's reversibility criterion for Markov chains (see e.g. Kelly (2011)), as well as transitive matrices (Farkas et al., 1999) a property applied in pairwise comparison matrices (Bozóki et al., 2010) and Analytic Hierarchy Processes (Saaty, 1988). We show that these networks and only these can be rescaled into symmetric ones, and these are the only ones that have a quadratic best-response potential function.

The second class of networks with convergent best-response dynamics are those that are rescalable to exhibit weak influences or weak externalities. A player is influenced weakly by her opponents if the total effects of a unit change in all of her opponents' actions on her are smaller than the effect of a unit change in her own action. These networks are characterized by row diagonally dominant weight matrices. In social networks, this property can be interpreted as a form of individualism. On the other hand, weak external effects are characterized by column diagonally dominant weight matrices, meaning a unit change in any player's action has a larger effect on herself than on all the other players combined. In economics, small level of externalities is a characteristic of efficient markets.

In non-cooperative game theory, weakness of externalities is known be sufficient for uniqueness of the Nash equilibrium as well as convergence to the unique Nash equilibrium under best-response dynamics (Gabay and Moulin, 1980; Moulin, 1986). In networks, weak influences are studied by Parise and Ozdaglar (2019) while both types of diagonal dominance are studied by Scutari et al. (2014). Since in our model, network weights are allowed to be both negative and positive, weak externalities is a separate condition from small network effects (Bramoullé et al., 2014; Belhaj et al., 2014). Under small network effects, given an equilibrium production profile, a tremor in a player's production decision is dampened by the network such that the system returns to the original equilibrium under best-response dynamics. We show that small network effects in absolute value, that is, the spectral radius of the absolute value of the weight matrix being less than one is sufficient and necessary for rescalability into networks with weak influences and those with weak externalities. Moreover, we show that this class of games is also characterized by a potential structure. We thus fully characterize the class of networks for which uniqueness of equilibrium and global convergence of best-response dynamics can be shown by either type of diagonal dominance. In addition, this class includes the set of directed acyclic networks.

Overall, our results have a number of implications with respect to convergence in directed network games. A negative finding is that, in networks with directed 'net' cycles and parasite-host interactions, the interpretation of the Nash equilibrium as an outcome of a series of decentralized improvements by the players is questionable as the convergence of simple learning processes is not assured. We complement this with a pair of positive results by identifying and characterizing two interesting classes of networks where convergence is assured. Our two sets of positive results uncover insights into the relationship between reciprocity of network interactions, the spectral properties of the network, and the games' potential structure as well as generalize the powerful results achieved for the case of reciprocal interactions.

Our paper is organized as follows: Section 2 presents our setting, introducing directed network games, best-response dynamics, and best-response potential games. Section 3 contains our characterization and convergence results for networks that can be rescaled into symmetric networks. Section 4 contains the same sets of results for networks rescalable to games with weak influences or weak externalities. Section 5 concludes.

2 The model

Let $I = \{1, \ldots, n\}$ be the set of players. For $i \in I$ and upper bounds $\overline{x}_i > 0$ the set $X_i = [0, \overline{x}_i]$ is called the action set of player i, $X = \prod_{i \in I} X_i$ is called the set of action profiles. We let $x_i \in X_i$ denote the action taken by player i while $x_{-i} \in \prod_{j \in I \setminus \{i\}} X_j$ denotes the truncated action profile of all players except player i, and $x = (x_i)_{i \in I}$ denotes the action profile of all players.

The formal definition of directed network games used in this paper is as follows.

Definition 2.1. The tuple $G = (I, X, (\pi_i)_{i \in I})$ is called a *directed network game* with payoff functions $\pi_i \colon X \to \mathbb{R}$ given by

$$\pi_i(x) = f_i\left(\sum_{j \in I} w_{ij} x_j\right) - c_i x_i, \tag{1}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f_i' > 0$, $f_i'' < 0$, $w_{ij} \in \mathbb{R}$, $w_{ii} = 1$, and $c_i > 0$ for every $i, j \in I$.

Assumption 2.2. We assume that f_i are given such that for every $i \in I$ we have $t_i \in (0, \overline{x}_i)$ such that $f'(t_i) = c_i$.

The interpretation is the following. Each player produces a specialized good with linear production technology, incurring costs c_i for every unit of the good produced. Players derive benefits from the consumption of their own goods and they are affected by their opponents' production decisions. Player i's enjoyment of player j's good is represented by the weight $w_{ij} \in \mathbb{R}$. Without loss of generality, we normalize the interaction parameter of each player i with herself, w_{ii}, to 1. The overall benefits of player i are given by the benefit function f_i over the weighted sum of her and her opponents' goods. Crucially, we do not impose reciprocal relations, meaning that $w_{ij} \neq w_{ji}$ may hold, so the weight matrix $(w_{ij})_{i,j \in I} = W$ might not be symmetric.

Since the benefit functions f_i are strictly concave, the sets X_i are compact, and the cost parameters c_i are positive, for every x_{-i} there is a unique value of x_i

that maximizes $\pi_i(x)$. The target values t_i are implicitly defined by $f_i'(t_i) = c_i$, i.e. the value player i would produce if all others produce 0. We make the simplifying assumption that every player is able to produce her target amount of the good. Let $t = (t_i)_{i \in I}$ denote the vector of targets.

Note that if $w_{ij} > 0$ and $w_{ji} > 0$, then the goods of players i and j are strategic substitutes. If $w_{ij} < 0$ and $w_{ji} < 0$ then their goods are strategic complements. If $w_{ij} > 0$ and $w_{ji} < 0$, then we say that players i and j share a parasitic link. If $w_{ij} = 0$, then player i is not directly affected by player j's production decision.

For player $i \in I$ and $x \in X$, $b_i(x) = \operatorname{argmax}_{x_i \in X_i} \pi_i(x)$ denotes player i's best-response function. It is easy to see that the best response functions are the following:

$$b_i(x) = \begin{cases} 0 & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j < 0, \\ t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j \in [0, \overline{x}_i], \\ \overline{x}_i & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j > \overline{x}_i. \end{cases} \tag{2}$$

Let $\tilde{b}_i(x) = t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j$ denote player i's unconstrained best response.

Let the set of Nash equilibria be denoted by X^*. It is also easy to see that every directed network game has at least one Nash equilibrium, thus X^* is non-empty. Bramoullé et al. (2014)'s analogue result using Brouwer's fixed-point theorem for a positive and symmetric weight matrix is applicable in the directed network case. While the functional form of the best response and the existence of Nash equilibria follow along the same lines in both undirected and directed networks, the convergence results of the undirected case do not generalize to the directed case.

We now formally introduce the learning processes of our paper, the best-response dynamic and two extensions.

Definition 2.3. The sequence of action profiles $(x^k)_{k \in \mathbb{N}}$ is a *one-sided dynamic* if for every $k \in \mathbb{N}$ there exists an $i^k \in I$ such that $x^k_{-i^k} = x^{k+1}_{-i^k}$. We say that players *update infinitely many times* if for every $i \in I$ the set $\{k \in \mathbb{N} \colon i^k = i\}$ is infinite. We say that players *update regularly* if there exists an $K > 0$ such that for every $i \in I$ and $k \in \mathbb{N}$ there exists $k' \in \{k, \ldots, k+K\}$ with $i^{k'} = i$.

Definition 2.4 (Best-response dynamics)**.** The one-sided dynamic $(x^k)_{k \in \mathbb{N}}$ is a

- *best-response dynamic* (BRD), if we have $x^{k+1}_{i^k} = b_{i^k}(x^k)$.

- *best-response-approaching dynamic* (BRAD) with approach parameter $0 \leq \beta < 1$, if $|x_{i^k}^{k+1} - b_{i^k}(x^k)| \leq \beta |x_{i^k}^k - b_{i^k}(x^k)|$ and if $x_{i^k}^{k+1} \neq b_{i^k}(x^k)$, then $\text{sgn}(x_{i^k}^{k+1} - b_{i^k}(x^k)) = \text{sgn}(x_{i^k}^k - b_{i^k}(x^k))$.

- *best-response-centered dynamic* (BRCD) with centering parameter $0 \leq \alpha < 1$, if $|x_{i^k}^{k+1} - b_{i^k}(x^k)| \leq \alpha |x_{i^k}^k - b_{i^k}(x^k)|$,

for every $k \in \mathbb{N}$.

In a one-sided dynamic exactly one player changes her action in every time period. In a BRD, every revision takes the updating player to her best response, in a BRAD, players move closer to their best responses without overshooting it, while in a BRCD, players move closer to their best responses and are allowed to overshoot. The approach and centering parameters of a BRAD and a BRCD, respectively, indicate the maximum fraction to which the distances are allowed to decrease. These processes allow payoff-maximizing players to make mistakes, their ability of reaching the best-response is captured by the two parameters with lower values indicating a higher level of accuracy. It is clear that every BRD is a BRAD and every BRAD is a BRCD.

Players revising infinitely many times and regularly are mild technical restrictions that ensure that (1) a convergent dynamic will converge to a Nash equilibrium as no player can get stuck playing a suboptimal action indefinitely by not having the opportunity to revise, (2) we can iterate the best-responses of all players. It is clear that the latter implies the former.

Bramoullé et al. (2014) and Bervoets and Faure (2019) consider the BRD in continuous time. Parise and Ozdaglar (2019), in addition to the continuous-time dynamic, also considers discrete-time BRD, both simultaneous and sequential updating in a fixed order of players. Our BRD process is more general than the latter as revision opportunities may arrive in any order. Bayer et al. (2019) considers both the BRD and the BRCD as above.

Definition 2.5 (Cycles)**.** A sequence $(x^k)_{k \in \mathbb{N}}$ has a *cycle* if there exist three time periods, $k < k' < k''$ such that $x^k = x^{k''}$, but $x^k \neq x^{k'}$.

In words, a process has a cycle if it non-trivially revisits an action profile in two different time periods. The absence of best-response cycles is a necessary but not sufficient condition of the convergence of best-response dynamics (Kukushkin, 2015), which holds in networks with reciprocal interactions. The following two examples show that directed 'net' cycle networks and parasitic links lead to cycling and thus hinder any general convergence results.

Example 2.6 (Directed 'net' cycle network)**.** Let $w \geq 0$ be given. Consider a three-player directed network game with $I = \{1, 2, 3\}$, $X_i = [0, 1]$ for $i \in I$, and weight matrix

$$W = \begin{pmatrix} 1 & w & w+1 \\ w+1 & 1 & w \\ w & w+1 & 1 \end{pmatrix},$$

Let $f_i(x) = \log(1 + x)$ and $c_i = 1/(1 + t_i)$ for $i \in I$, so player i's payoff is

$$\pi_i(x) = \log\left(1 + \sum_{j \in I} w_{ij} x_j\right) - \frac{1}{1 + t_i} x_i.$$

As $f_i'(t_i) = c_i$ for all i, t is indeed the vector of targets. Fix $t = (1, 1, 1)^\top$. We call W a directed 'net' cycle as it comprises two directed cycles of opposite directions, one with weight w, and one with a larger weight, $w + 1$.

Consider the BRD with the initial action profile $x = (1/(w + 1), 0, 0)^\top$. Let player 3 receive the first revision opportunity, followed by player 1. Then, player 3 will set her production to $1/(w + 1)$, while player 1 will set hers to 0. Production shits in opposite direction of the 'net' cycle. Thus, if the next revision is made by player 2, followed by player 1, production shifts again, this time to player 2. Player 1 revising next followed by player 3 completes the cycle. Figure 1 shows two revisions of the 'pure' directed cycle network, with $w = 0$.

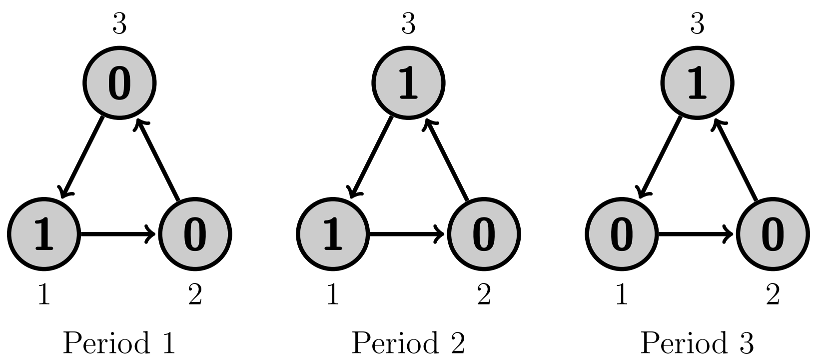
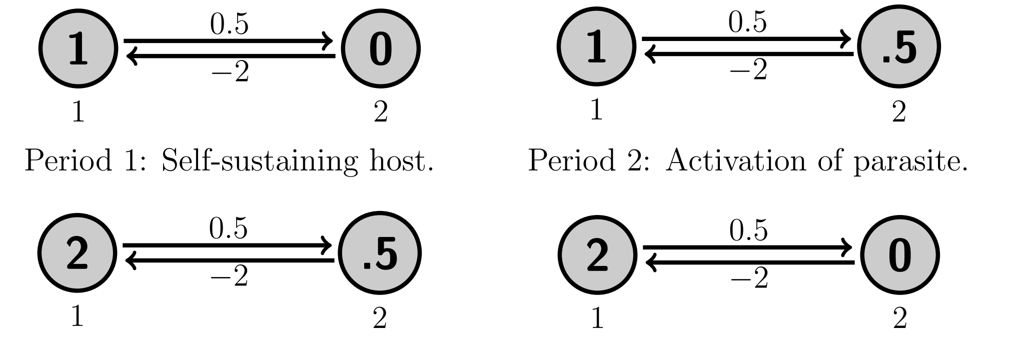


Figure 1: For $w = 0$, Example 2.6 produces a pure directed cycle network. Production shifts in the opposite direction of the cycle under the BRD.

Example 2.7 (Parasitism)**.** Let $I = \{1, 2\}$, $f_i(x) = \log(1 + x)$, $c_i = 1/(1 + t_i)$ for $i \in I$, $t_1, t_2 = 1$, and let

$$W = \begin{pmatrix} 1 & -2 \\ 0.5 & 1 \end{pmatrix}.$$

Under BRD, starting from the action profile $x = (1, 0)^\top$, if both players revise in turns, the game has a best-response cycle of length four.



Period 1: Self-sustaining host. Period 2: Activation of parasite.

Period 3: Increased host activity. Period 4: Passive, free-riding parasite.

Figure 2: The best-response cycle of Example 2.7.

In this example, player 1 is called the host and player 2 is called the parasite. A self-sustaining host is engaged by a parasite. The host responds by increasing her activity to offset the parasite's negative effects. The parasite's benefits from the host are large enough that it ceases production entirely and free-rides on the host. The host is then able to return to the self-sustaining stage, completing the cycle. The full process is shown in Figure 2. It is easy to see that the same cyclic pattern of parasite-host interaction can be replicated by a broad range of parameters and larger networks. Such cycles necessitate that the parasitic link is *amplifying*, i.e. $|w_{12}w_{21}| \geq 1$.

Examples 2.6 and 2.7 together indicate that directed cycles and parasitic relations lead to the cycling of BRDs, and hence those of BRADs and BRCDs. In such networks, convergence to the Nash equilibrium is not guaranteed.

We define one of the main concepts used in this paper, best-response potential games.

Definition 2.8 (Voorneveld (2000))**.** A game $G = (I, X, (\pi_i)_{i \in I})$ is a *best-response potential game*, if there exists a *best-response potential function* $\phi : X \to \mathbb{R}$ such that for every $i \in I$, and every $x_{-i} \in X_{-i}$ it holds that

$$\underset{x_i \in X_i}{\operatorname{argmax}}\ \pi_i(x) = \underset{x_i \in X_i}{\operatorname{argmax}}\ \phi(x).$$

G is a best-response potential game if the best-response behavior of all players can be characterized by a single real-valued function ϕ, called the best-response potential. As per Voorneveld (2000)'s Theorem 3.1, for general classes of strategic games a best-response potential if and only if the game admits no best-response cycles, and a technical condition is met.[1] This condition is empty for games with countable action spaces, hence, the paper's second main result, Theorem 3.2, shows that a best-response potential exists if and only if the game has no best-response cycles. Furthermore, if X is finite (as in a discretized setting à la Bayer et al. (2021)), if there are no best-response cycles, then any BRD where the players update regularly converges to a Nash equilibrium.

As we explore classes of networks with convergent best-response dynamics, we thus also characterize them by their potential structure.

3 Transitive relative importance

As shown in the previous section, allowing for non-reciprocal interactions in network games changes their convergence properties under BRDs. However, there are interesting classes of directed network games where convergence to the game's Nash equilibrium can still be shown.

It is well known in the literature that convergence of BRDs in networks of reciprocal interactions can be shown by exploiting the game's potential structure. The main intuition this is that in the case $w_{ij} = w_{ji}$ for all i, j, the function

$$x^\top t - \frac{1}{2} x^\top W x$$

is a best-response potential. Every player's update will weakly increase the value of the best-response potential, which is bounded as the function is continuous and the action space is compact. Assuming that there is a finite amount of Nash equilibria, regular updating ensures that, in time, every player will be close to her

[1]The condition requires that the tuple $(X_\sim, \prec)$ be properly ordered, where $\prec$ is the binary relation defined as follows: $x \prec y$ if there exists a BRD from x to y, $\sim$ is the equivalence relation $x \sim y$ if $x \prec y$ and $y \prec x$; and $X_\sim$ is the set of equivalence classes on X generated by $\sim$.

current best-response. This means that the process will converge to an isolated peak of the potential value landscape, corresponding to a Nash equilibrium.

In section VI, Bramoullé et al. (2014) raises the idea that, by an appropriate rescaling of the action space, this method may be extended for some class of directed networks as well. In this section we identify and characterize the class of directed networks for which this can be done.

Take a vector $a \in \mathbb{R}^n$, called a scaling vector, such that $a > 0$, i.e., for every $i \in I$ we have $a_i > 0$ and let $y_i = a_i x_i$ and $\overline{y}_i = a_i \overline{x}_i$. Furthermore, let $Y_i = [0, \overline{y}_i]$ and $Y = \prod_{i \in I} Y_i$. It is clear that any BRD, BRAD, or BRCD in the game with strategy space X is a BRD, BRAD, or BRCD, respectively, in the game with Y with the same approach or centering parameters in the cases of BRAD and BRCD. Similarly, the convergence properties of the processes in the game with strategy space Y are identical to those with X.

The unconstrained best-response function of player i in the rescaled game is given by

$$t_i a_i - \sum_{j \in I \setminus \{i\}} w_{ij} \frac{a_i}{a_j} y_j.$$

Let $w_{ij} a_i / a_j = v_{ij}$ and let $V = (v_{ij})_{i,j \in I}$ denote the matrix of rescaled weights. Our goal in this section is to characterize the class of networks that are rescalable into a symmetric network, i.e., the set of networks W for which there exists a vector $a > 0$ such that for every $i, j \in I$ we have

$$\frac{a_i}{a_j} w_{ij} = v_{ij} = v_{ji} = \frac{a_j}{a_i} w_{ji}. \tag{3}$$

Note that rescaling a network means that we conjugate its weight matrix by the diagonal matrix of scaling weights. That is, consider $S = \operatorname{diag}(a_1, \ldots, a_n)$. Then, we have $V = SWS^{-1}$. We continue to talk about rescalability instead of conjugate matrices as S being a diagonal matrix is crucial for our results, as it will be apparent in Theorem 3.4. A similar notion appears in Golub and Jackson (2012) who rely on this special case of matrix similarity to symmetric matrices.

We now define transitive relative importance of links.

Definition 3.1. We say that the weight matrix W shows *transitive relative importance* if for every $3 \leq m \leq n$ and for all pairwise distinct $i_1, i_2, \ldots, i_m \in I$ we have

$$w_{i_1 i_2} w_{i_2 i_3} \ldots w_{i_{m-1} i_m} w_{i_m i_1} = w_{i_1 i_m} w_{i_m i_{m-1}} \ldots w_{i_3 i_2} w_{i_2 i_1}.$$

For simplicity we write that W is transitive if it satisfies Definition 3.1. Notice that if W is symmetric, then it is also transitive. Also notice that if for every $i, j \in I$ it holds that $w_{ij} \neq 0$, then $w_{ij}w_{jk}w_{ki} = w_{ik}w_{kj}w_{ji}$ for every pairwise distinct $i, j, k \in I$ implies transitivity.

The interpretation is as follows: for $w_{ij}, w_{ji} \neq 0$ define the value $r_{ij} = w_{ij}/w_{ji}$ as player i's relative importance on the link between i and j. It is clear that, if well-defined, the matrix $R = (r_{ij})_{i,j\in I}$ is symmetrically reciprocal. In this case the transitivity property reduces to having

$$r_{ij}r_{jk} = r_{ik}, \tag{4}$$

for every $i, j, k \in I$. Thus, qualitatively, if the link $\{i, j\}$ is more important to i than to j and if the link $\{j, k\}$ is more important to j than to k, then the link $\{i, k\}$ must be more important to i than to k.

Remark 3.2. Transitivity as defined in Definition 3.1 is identical as Kolmogorov's characterization of reversible Markov chains with the exception that network weights are allowed to be negative. If R is defined, then transitivity of W means that R is a consistent pairwise comparison matrix of an Analytic Hierarchical Process (Saaty, 1988). Here, the literature seems to use the terms consistent (Bozóki et al., 2010) and transitive (Farkas et al., 1999) interchangeably when describing the symmetrically reciprocal comparison matrix R. We use transitivity of W to indicate the intuition imposed on links.

The final definition we require is that of sign-symmetry.

Definition 3.3. We say that the weight matrix W is sign-symmetric if for every $\{i, j\} \subseteq I$ we have $\operatorname{sgn}(w_{ij}) = \operatorname{sgn}(w_{ji})$.

Sign-symmetry of networks rules out one-way interactions and parasitic interactions between players.

We are ready to present the main result of this section.

Theorem 3.4. *The following statements are equivalent:*

1. *the network W is rescalable into a symmetric matrix,*

2. *the network W is transitive and sign-symmetric,*

3. *there exists a game G on W that has a quadratic best-response potential function,*

4. *every game G on W has a quadratic best-response potential function.*

The proof is shown in the appendix.

Theorem 3.4 shows that transitivity of a network combined with sign-symmetry is equivalent to it being rescalable into a symmetric network. Additionally, no other network has a quadratic potential function, which shows that this property cannot be exploited further. Our result thus gives a full characterization for which types of directed networks satisfy the requirements put forward in Bramoullé et al. (2014) section VI.

Theorem 3.4 combined with Bayer et al. (2019)'s Theorem 5.3 give rise to the following corollary.

Corollary 3.5. *Let W be a transitive and sign-symmetric network and let t be given such that $|X^*| < \infty$. Then, every BRD and BRCD in which players update regularly converges to a Nash equilibrium.*

For a fixed symmetric network, since the number of Nash equilibria is finite for almost every target vector (Bayer et al., 2019), Corollary 3.5 also applies to every network and almost every target vector, thus convergence is generically established for this class.

4 Weak influences and weak externalities

In this section we characterize another class of networks with convergent dynamics. A key concept in describing this class is the players' influence and the externalities they produce. A player i's decisions are influenced by her opponents through her incoming weights, measured by their total magnitude: $\sum_{j \in I \setminus \{i\}} |w_{ij}|$. Similarly, a player i's external effects on her opponents is measured by the total magnitude of her outgoing weights: $\sum_{j \in I \setminus \{i\}} |w_{ji}|$. In this section we consider cases where such influences/externalities are smaller than the players' own weight with themselves, $w_{ii} = 1$. As a motivating example, consider a parametric version of Example 2.6's 'pure' directed cycle network.

k	x_1^k	x_2^k	x_3^k	i^k	$\sum_{j\in I} w_{i^k j} x_j$	$b_{i^k}(x_{i^k})$
0	1	0	0	3	0	1
1	1	0	1	1	δ	$1-\delta$
2	$1-\delta$	0	1	2	$\delta-\delta^2$	$1-\delta+\delta^2$
3	$1-\delta$	$1-\delta+\delta^2$	1	3	$\delta-\delta^2+\delta^3$	$1-\delta+\delta^2-\delta^3$

Table 1: The best-response dynamic of Example 4.1.

Example 4.1 (Directed cycle with weights)**.** Consider the weight matrix

$$W = \begin{pmatrix} 1 & 0 & \delta \\ \delta & 1 & 0 \\ 0 & \delta & 1 \end{pmatrix},$$

with $\delta \in (0,1)$. Let $f_i(x) = \log(1+x)$, $c_i = 1/(1+t_i)$ and fix $t_i = 1$ for $i \in \{1,2,3\}$ as previously. The best-response functions are $b_1(x) = 1 - \delta x_3$, $b_2(x) = 1 - \delta x_1$, $b_3(x) = 1-\delta x_2$. The only Nash equilibrium is $x^* = (1/(1+\delta), 1/(1+\delta), 1/(1+\delta))^\top$.

In Table 1 we that show the sequence of action profiles in the BRD where players receive revision opportunities in the same, repeating order $(3,1,2)$, starting, from the action profile $(1,0,0)^\top$. In this order of revisions, the player holding the revision opportunity in period k will revise to $\sum_{\ell=0}^{k}(-\delta)^\ell = (1-(-\delta)^{k+1})/(1+\delta)$. Playing on in this order will produce no cycles, and lead to convergence to the Nash equilibrium.

Example 4.1 suggests that if influences or externalities are even marginally weaker than the players' own weight, best-response cycles disappear and we get convergence to a unique Nash equilibrium. As we will show in this section, this turns out to be a general property. We first introduce these games formally.

Definition 4.2. A network W has

- *weak influences* if for every $i \in I$ it holds that $\sum_{j\in I\setminus\{i\}} |w_{ij}| < 1$,
- *weak externalities* if for every $i \in I$ it holds that $\sum_{j\in I\setminus\{i\}} |w_{ji}| < 1$.

Networks with weak influences are characterized by *row diagonally dominant* weight matrices, while those with weak externalities have *column diagonally dominant* weight matrices. An equivalent characterization of these two classes in terms of matrix norms is as follows: A network $W \in \mathbb{R}^{n\times n}$ has

- weak influences if $\|W - I_n\|_\infty < 1$,

- weak externalities if $\|W - I_n\|_1 < 1$,

where I_n denotes the $n \times n$ identity matrix. Games with weak influences satisfy Assumption 2b of Parise and Ozdaglar (2019), while both classes are covered by Scutari et al. (2014)'s Proposition 7.

It turns out that both classes of games have a unique Nash equilibrium and every BRD and BRAD converges to it. The main intuition for this is that the iterated BRD is contracting. Once again we can make use of rescaling: any network which is rescalable into one of the two classes inherits the uniqueness of the Nash equilibrium as well as the convergence properties. As before, for $a \in \mathbb{R}^n$, $a > 0$, define $V = (v_{ij})_{i,j \in I}$ as $v_{ij} = w_{ij} a_i / a_j$. We show that the two classes are rescalable into each other. Furthermore, a network is rescalable to either class if and only if the spectral radius of $|W| - I_n$ is less than one, where $|W| = (|w_{ij}|)_{i,j \in I}$. Finally, these classes are also characterized by a best-response structure.

Recall that the *spectral radius* $\rho(M)$ of a square matrix $M \in \mathbb{C}^{n \times n}$ is the largest absolute value of its eigenvalues, i.e.,

$$\rho(M) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } M\}.$$

The next result characterizes rescalability to these classes.

Theorem 4.3. *The following statements are equivalent for W.*

1. *There exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled network V has weak externalities.*

2. *There exists a scaling vector $a' \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix V has weak influences.*

3. $\lim_{k\to\infty}(|W| - I_n)^k = 0$.

4. $\rho(|W| - I_n) < 1$.

5. *There exists a vector $a \in \mathbb{R}^n$, $a > 0$ such that*

$$\phi'(x) = -\sum_{i \in I} a_i |x_i - \tilde{b}_i(x)|$$

is a best-response potential function of every game played on W.

The fact that a diagonally dominant W satisfies points 3 and 4 of Theorem 4.3, as well as the equivalence of 3 and 4 are well-known in linear algebra. Our characterization adds the notion of rescalability, as well the existence of a best-response potential function for all games played on W. Berman and Plemmons (1994) prove the equivalence of points 1 to 4 of this theorem for matrices with negative off-diagonal elements in Theorem 2.3 while we prove it in the general case. We highlight the difference between Section 3's similar result, Theorem 3.4. In the present case there may some be games played on a network W that is not rescalable to weak influences/externalities, yet admits the potential $\phi'(x)$.

Proposition 4.4. *If $\rho(|W| - I_n) < 1$, then every game played on the network W has a unique Nash equilibrium.*

By Ui (2016), every positive definite W has a unique equilibrium. If W is symmetric, then the condition $\rho(|W| - I_n) < 1$ implies positive definiteness, but for an asymmetric W neither condition is implied by the other.

Proposition 4.4 follows from the fact that such networks are rescalable to diagonally dominant matrices (Theorem 4.3), allowing us to use Moulin (1986), Chapter 6, Theorems 2 and 3. As a result, games played on such networks are *dominance solvable* Moulin (1984), that is, iterated elimination of dominated strategies leads to a unique solution. Thus, they have a unique Nash equilibrium and convergent BRD. We now show a stronger convergence property: the convergence of BRAD.

Theorem 4.5. *If $\rho(|W| - I_n) < 1$, then in every game played on network W, every BRAD (and hence every BRD) in which players revise infinitely many times converges to the unique Nash equilibrium.*

The proofs of Theorems 4.3, 4.5 and Proposition 4.4 are found in the appendix.

Theorem 4.5 is related to Theorem 4 of Moulin (1986), Chapter 6, which shows local stability of equilibria with respect to BRD for which the Jacobian's spectral radius is less than one. Our condition is global due to the linear best response functions of the network environment hence we have global convergence. Another related result is Theorem 4.1 of Gabay and Moulin (1980) that shows convergence of BRAD where revisions arrive in a fixed order for diagonally dominant Jacobians. Our result is more general in directed network games as we cover a wider class of networks and in that revision opportunities may arrive in any order.

As demonstrated by Example 2.6, a spectral radius equal to 1 leads to best-response cycles, hence these results are tight. Additionally, notice that the BRCD may lead to cycles in this gameclass. For instance, the BRD shown in Example 2.6 is a BRCD in Example 4.1 for $\delta = 0.9$.

We end this section with two remarks. First, it is easy to show that every network that is rescalable to one with weak externalities lacks any amplifying links, i.e. for all i, j, $|w_{ij} w_{ji}| < 1$ must hold. To show this, just suppose that $|w_{ij} w_{ji}| \geq 1$. Then, for any rescaling parameters a_i, a_j we have

$$|v_{ij} v_{ji}| = |\frac{a_i}{a_j} w_{ij} \frac{a_j}{a_i} w_{ji}| \geq 1,$$

hence one of v_{ij} or v_{ji} is at least 1 in absolute value, and thus the rescaled network V cannot be one of weak externalities. Thus, Definition 4.2 rules out the best-response cycles along parasitic links shown in Example 2.7 as these rely on amplifying links.

As our second remark we mention a special class of networks that are contained in this class are directed acyclic networks (DANs). W is a DAN if it is lower- or upper triangular (possibly after a suitable relabeling of the players). If W is a DAN, every eigenvector of $W - I_n$ is 0, hence all statements in this section hold for this class. DANs describe a hierarchy of players in which externalities flow downstream, e.g. a supply chain, pollution along a river, military chains of command, or a trophic network with apex predators on the highest level and prey animals on lower levels.

5 Conclusion

In this paper we analyze directed network games, a generalization of the private provision of public goods games model to include possibly non-reciprocal relationships. These cover one-way interactions, unequal interactions, and parasitism. While weighted networks and simple graphs are very useful frameworks, more nuanced models of social and economic networks should include non-reciprocal interactions.

While best-response dynamics on games played on symmetric networks are known to converge to a Nash equilibrium due to the games' potential structure, this is not true in general for networks with asymmetric weight matrices. In this paper we show that both one-way interactions and parasitic interactions can create best-response cycles. This questions the interpretation of the Nash equilibrium as the result of individual improvements by the players. Together with other known problems of the Nash equilibrium both conceptual and behavioral, equilibrium analysis of such games may be of questionable value in settings with possibly non-reciprocal interactions.

There are classes of asymmetric networks, however, where the predictive power of the Nash equilibrium is retained. In this paper we highlight two such classes; those that can be rescaled into symmetric networks and those that are rescalable to networks with weak influences or weak externalities. We characterize the former type by transitive relative importance of players and sign-symmetry of the weight matrix. Additionally, this class captures all networks with quadratic best-response potential functions, indicating that other network types with convergence require different approaches to identify.

The latter class captures individualistic social networks as well as situations where the economic externalities have been internalized. We show that these types are equivalent with respect to rescaling and any network with a spectral radius less than one is rescalable to either. Such games are best-response potential games, have a unique Nash equilibrium, and all BRDs and BRADs converge to it. A necessary condition for a network to be rescalable to one with weak externalities is the absence of amplifying links.

Our results unlock a number of insights into network games. The most apparent general result is a negative one: the convergence properties of games played on symmetric networks do not generalize well for the asymmetric case. For directed ('net') cycles and parasitic interactions best-response cycles may appear, thus identifying convergent classes of networks that include any of these types of interactions are likely to require different methodologies than the best-response potentials, rescaling, and spectral properties used in this paper.

Our positive contribution consists of the full exploration of the idea of rescalability into symmetric matrices as well as the identification of weak influences/externalities as networks with convergent dynamics and the full characterization of the latter class. We thus broaden the set of sufficient conditions that guarantee convergence in network games. Finding broader sets of sufficient conditions necessitates finding additional classes of best-response potentials that characterize them, while identifying the set of all networks with convergent best-response dynamics necessitates a sufficient and necessary condition for the existence of a best-response potential function. This is a highly interesting problem left for future research. Our results constitute two important steps in this direction: Section 3 identifies all classes of networks with quadratic potential functions. As the best-response functions are linear, this is a key step. Section 4 identifies a different, non-polynomial class of potential games.

A Appendix

Proofs for Section 3

Theorem 3.4

It is clear that statement 4 implies statement 3. We show the remaining three implications.

Proof of 1 ⇒ 4. Suppose that the matrix W can be rescaled into a symmetric matrix $(v_{ij})_{i,j\in I}$ with the vector $a \in \mathbb{R}^n$, $a > 0$, i.e., we have $v_{ij} = w_{ij}a_i/a_j$. Let G be any game on W. We now show that the following quadratic function is a best-response potential of the game.

$$\phi^Q(x) = \sum_{i\in I} a_i^2 x_i t_i - \frac{1}{2}\sum_{i\in I}\sum_{j\in I} x_i x_j a_i a_j v_{ij}.$$

For every $i \in I$, the partial derivative is as follows:

$$\begin{aligned}
\frac{\partial \phi^Q}{\partial x_i}(x) &= a_i^2 t_i - \sum_{j\in I} x_j a_j a_i v_{ij} \\
&= a_i^2 t_i - \sum_{j\in I} x_j a_i^2 w_{ij} \\
&= a_i^2 \left(t_i - \sum_{j\in I} x_j w_{ij} \right) \\
&= a_i^2 \cdot (\tilde{b}_i(x) - x_i)
\end{aligned}$$

We used that $v_{ij} = v_{ji}$ and that $a_j a_i v_{ij} = w_{ij} a_i^2$.

Also, we have

$$\frac{\partial^2 \phi^Q}{\partial x_i^2}(x) = -a_i^2 w_{ii} = -a_i^2 < 0.$$

Therefore, if $b_i(x) = \tilde{b}_i(x)$, then $\phi^Q(x)$ is maximal when $\tilde{b}_i(x) = b_i(x) = x_i$. If $\tilde{b}_i(x) < 0$ for some x_{-i}, then the derivative of ϕ^Q with respect to x_i is uniformly negative on $[0, \overline{x}_i]$, so the maximum is achieved when $x_i = 0$. On the other hand, when $\overline{x}_i < \tilde{b}_i(x)$, then the derivative of ϕ^Q with respect to x_i is positive on $[0, \overline{x}_i]$, and hence it takes its maximum in $\overline{x}_i$.

Therefore, for a fixed x_{-i} vector the function ϕ^Q is maximal when player i is in her best response, so ϕ^Q is a best-response potential function. ∎

Proof of 3 ⇒ 2. Suppose that there exists a quadratic best-response potential function, ϕ^Q, for a game G on W. We show that the matrix W is sign symmetric and transitive.

Let the potential function be given as follows.

$$\phi^Q(x) = \sum_{i\in I} p_i x_i - \frac{1}{2}\sum_{i\in I} q_{ii} x_i^2 - \sum_{i,j\in I, i>j} q_{ij} x_i x_j,$$

where $p_i \in \mathbb{R}$ for every $i \in I$ and $q_{ij} \in \mathbb{R}$ for every $i, j \in I$, $i \geq j$. For $j > i$, we set $q_{ij} = q_{ji}$ for convenience. With this notation, the partial derivative of ϕ^Q is:

$$\frac{\partial \phi^Q}{\partial x_i}(x) = p_i - q_{ii} x_i - \sum_{j\in I\setminus\{i\}} q_{ij} x_j. \tag{5}$$

The function ϕ^Q is a best-response potential of the weighted network game G, so for every $x \in X$ and for every $i \in I$, the partial derivative of ϕ^Q is zero exactly when player i is in her best response. Note that we have $0 < t_i < \overline{x}_i$ for every $i \in I$, this means that $t_i = b_i(0) = \tilde{b}_i(0) \in (0, \overline{x}_i)$ for every $i \in I$. Therefore, there exists a neighborhood of 0 where each player's unconstrained best response is equal to her best response. Let $\varepsilon > 0$ be so that for every $x \in [0, \varepsilon]^n$ and for all $i \in I$ we have $b_i(x) = \tilde{b}_i(x)$. For a fixed i, the functions $\frac{\partial \phi^Q}{\partial x_i}(x)$ and $\tilde{b}_i(x) - x_i$ are both linear in x and they have the same zero set when $x \in [0, \varepsilon]^n$. Hence, they must be equal up to a constant factor: there exists $d_i \neq 0$ such that we have

$$t_i - \sum_{j\in I} w_{ij} x_j = \tilde{b}_i(x) - x_i = d_i \frac{\partial \phi^Q}{\partial x_i}(x) = d_i \left(p_i - \sum_{j\in I\setminus\{i\}} q_{ij} x_j - q_{ii} x_i \right) \tag{6}$$

Additionally, as we are maximizing ϕ^Q, the second derivative of ϕ^Q with respect to x_i has to be negative, so $q_{ii} > 0$ for every $i \in I$. From (6), we get the following for every $i, j \in I$.

$$t_i = d_i \cdot p_i$$
$$w_{ii} = 1 = d_i \cdot q_{ii}$$
$$w_{ij} = d_i \cdot q_{ij} \tag{7}$$

Since $q_{ii} > 0$, we must have $d_i > 0$ as well for all $i \in I$.

The first two equations give no constraints for W. We have to show that if (7) holds for all $i, j \in I$, that implies the transitivity and the sign-symmetry of the weight matrix W.

Since the d_i's are positive and $q_{ij} = q_{ji}$, we have

$$\operatorname{sgn}(w_{ij}) = \operatorname{sgn}(d_i q_{ij}) = \operatorname{sgn}(q_{ij}) = \operatorname{sgn}(q_{ji}) = \operatorname{sgn}(d_j q_{ji}) = \operatorname{sgn}(w_{ji}),$$

so the matrix W is sign-symmetric.

Now let $3 \leq s \leq n$, then for all $i_1, i_2, ..., i_s \in I$ pairwise distinct, we have

$$\begin{aligned} w_{i_1 i_2} w_{i_2 i_3} \dots w_{i_{s-1} i_s} w_{i_s i_1} &= (d_{i_1} q_{i_1 i_2})(d_{i_2} q_{i_2 i_3}) \dots (d_{i_{s-1}} q_{i_{s-1} i_s})(d_{i_s} q_{i_s i_1}) \\ &= (d_{i_1} d_{i_2} \dots d_{i_s})(q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_{s-1} i_s} q_{i_s i_1}) \\ &= (d_{i_1} d_{i_2} \dots d_{i_s})(q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_s i_{s-1}} q_{i_1 i_s}) \\ &= (d_{i_2} q_{i_2 i_1})(d_{i_3} q_{i_3 i_2}) \dots (d_{i_s} q_{i_s i_{s-1}})(d_{i_1} q_{i_1 i_s}) \\ &= w_{i_2 i_1} w_{i_3 i_2} \dots w_{i_s i_{s-1}} w_{i_1 i_s} \end{aligned}$$

using $w_{ij} = d_i q_{ij}$ for every $i, j \in I$. Therefore, the matrix W is transitive. ■

Proof of 2 ⇒ 1. Suppose that the matrix W is transitive and sign-symmetric. We would like to find a scaling vector $a \in \mathbb{R}^n$, $a > 0$, such that the rescaled matrix is symmetric, i.e., for every pair $i, j \in I$ we have $w_{ij} a_i / a_j = w_{ji} a_j / a_i$.

First, let us assume that the graph of W is connected, so there exists a path between any two players. If the graph is not connected, we follow the described algorithm for every connected component of the graph separately in order to define the vector a.

We start by ordering the players in the following way. Choose player 1 arbitrarily. Let the neighbors of player 1 be $2, \dots, n_1$. Players $n_1 + 1, \dots, n_2$ are those neighbors of 2 that are not neighbors of 1, and so on: players $n_k+1, n_k+2, \dots, n_{k+1}$ are those neighbors of player $k + 1$ that are not neighbors of players $1, 2, \dots, k$.

(If there are no such players, then we have $n_k = n_{k+1}$.) Due to the sign-symmetry of W, if two players are neighbors, there is a directed edge between them in both directions. Hence, since W is connected, there exists a directed path from player 1 to every player, so after at most n steps we reach all players.

Let $a_1 = 1$. We define all a_j's for $j \geq 2$ recursively in the following way. Suppose we have already defined $a_1, a_2, ..., a_{j-1}$, and $n_{i-1} < j \leq n_i$, so player j is a neighbor of i but not of players $1, ..., i-1$. In other words, i is the neighbor of j with the smallest index. Clearly $i < j$, so a_i is already defined. Also, we have $w_{ji} \neq 0 \neq w_{ij}$, since i and j are neighbors. Let

$$a_j = a_i \frac{\sqrt{|w_{ij}|}}{\sqrt{|w_{ji}|}}. \tag{8}$$

Now we show that with this scaling vector a, for any $k, \ell \in I$ we have $v_{k\ell} = v_{\ell k}$. Take any $k, \ell \in I$. If $k = \ell$, then $w_{k\ell} = v_{k\ell} = v_{\ell k} = w_{\ell k} = 1$, so we can assume that $k \neq \ell$. Let $i_1 \in I$ be so that k is a neighbor of i_1, but not of players $1, \ldots, i_1 - 1$, so $n_{i_1 - 1} < k \leq n_{i_1}$. Similarly, let $j_1 \in I$ be so that $n_{j_1 - 1} < \ell \leq n_{j_1}$. For every $m \in \mathbb{N}$, we define i_m and j_m recursively:

$$\begin{aligned} i_{m+1} &= \min\{i \in I : w_{i_m i} \neq 0\}, \\ j_{m+1} &= \min\{j \in I : w_{j_m j} \neq 0\}. \end{aligned}$$

Both sequences are decreasing, and they both stabilize when they reach 1. Let $r, s \in \mathbb{N}$ be minimal so that $i_r = j_s$. Note that this is going to happen, the latest when they are both equal to 1, and also note that from this point on, the two sequences coincide. Therefore, we have $k > i_1 > \cdots > i_r$ and $\ell > j_1 > \cdots > j_s$, and the numbers $k, i_1, \ldots, i_{r-1}, \ell, j_1, \ldots, j_s$ are distinct.

We can calculate a_k as follows.

$$\begin{aligned} a_k &= a_{i_1} \frac{\sqrt{|w_{i_1 k}|}}{\sqrt{|w_{k i_1}|}} \\ &= a_{i_2} \frac{\sqrt{|w_{i_2 i_1}|}}{\sqrt{|w_{i_1 i_2}|}} \frac{\sqrt{|w_{i_1 k}|}}{\sqrt{|w_{k i_1}|}} \\ &\vdots \\ &= a_{i_r} \frac{\sqrt{|w_{i_r i_{r-1}}|}}{\sqrt{|w_{i_{r-1} i_r}|}} \cdots \frac{\sqrt{|w_{i_2 i_1}|}}{\sqrt{|w_{i_1 i_2}|}} \frac{\sqrt{|w_{i_1 k}|}}{\sqrt{|w_{k i_1}|}} \\ &= a_{i_r} \left| \frac{w_{i_r i_{r-1}} \ldots w_{i_2 i_1} w_{i_1 k}}{w_{i_{r-1} i_r} \ldots w_{i_1 i_2} w_{k i_1}} \right|^{\frac{1}{2}}. \end{aligned}$$

Similarly, we have

$$a_\ell = a_{j_s} \left| \frac{w_{j_s j_{s-1}} \dots w_{j_2 j_1} w_{j_1 \ell}}{w_{j_{s-1} j_s} \dots w_{j_1 j_2} w_{\ell j_1}} \right|^{\frac{1}{2}}.$$

Now we can compute $v_{k\ell}$ using that $i_r = j_s$.

$$\begin{aligned}
v_{k\ell} &= \frac{a_k}{a_\ell} w_{k\ell} = a_k (a_\ell)^{-1} w_{k\ell} \\
&= \left(a_{i_r} \left| \frac{w_{i_r i_{r-1}} \dots w_{i_2 i_1} w_{i_1 k}}{w_{i_{r-1} i_r} \dots w_{i_1 i_2} w_{k i_1}} \right|^{\frac{1}{2}} \right) \left(a_{j_s} \left| \frac{w_{j_s j_{s-1}} \dots w_{j_2 j_1} w_{j_1 \ell}}{w_{j_{s-1} j_s} \dots w_{j_1 j_2} w_{\ell j_1}} \right|^{\frac{1}{2}} \right)^{-1} w_{k\ell} \\
&= \left| \frac{(w_{i_r i_{r-1}} \dots w_{i_2 i_1} w_{i_1 k})(w_{j_{s-1} j_s} \dots w_{j_1 j_2} w_{\ell j_1})}{(w_{i_{r-1} i_r} \dots w_{i_1 i_2} w_{k i_1})(w_{j_s j_{s-1}} \dots w_{j_2 j_1} w_{j_1 \ell})} \right|^{\frac{1}{2}} w_{k\ell} \\
&= \left| \frac{w_{i_1 k} w_{i_2 i_1} \dots w_{i_r i_{r-1}} w_{j_{s-1} j_s} \dots w_{j_1 j_2} w_{\ell j_1}}{w_{k i_1} w_{i_1 i_2} \dots w_{i_{r-1} i_r} w_{j_s j_{s-1}} \dots w_{j_2 j_1} w_{j_1 \ell}} \right|^{\frac{1}{2}} \left| \frac{w_{k\ell}}{w_{\ell k}} \right|^{\frac{1}{2}} \cdot \operatorname{sgn}(w_{k\ell}) \sqrt{|w_{k\ell} w_{\ell k}|} \\
&= \left| \frac{w_{i_1 k} w_{i_2 i_1} \dots w_{j_s i_{r-1}} w_{j_{s-1} j_s} \dots w_{j_1 j_2} w_{\ell j_1} w_{k\ell}}{w_{k i_1} w_{i_1 i_2} \dots w_{i_{r-1} j_s} w_{j_s j_{s-1}} \dots w_{j_2 j_1} w_{j_1 \ell} w_{\ell k}} \right|^{\frac{1}{2}} \operatorname{sgn}(w_{k\ell}) \sqrt{|w_{k\ell} w_{\ell k}|}
\end{aligned}$$

By the transitivity assumption for $k, i_1, \dots i_{r-1}, j_s, \dots j_1, \ell \in I$, we have

$$w_{k i_1} w_{i_1 i_2} \dots w_{i_{r-1} j_s} w_{j_s j_{s-1}} \dots w_{j_1 \ell} w_{\ell k} = w_{i_1 k} w_{i_2 i_1} \dots w_{j_s i_{r-1}} w_{j_{s-1} j_s} \dots w_{\ell j_1} w_{k\ell},$$

and hence $v_{k\ell} = \operatorname{sgn}(w_{k\ell}) \sqrt{|w_{k\ell} w_{\ell k}|}$. Similarly, we have $v_{\ell k} = sgn(w_{\ell k}) \sqrt{|w_{\ell k} w_{k\ell}|}$. Since W is sign-symmetric, these two numbers are equal, so $v_{k\ell} = v_{\ell k}$ for all $k, \ell \in I$.

Finally, we show that the statement holds even if the graph of W is not connected. In this case we order the players in each component and define the a_i's for the components separately as described in the beginning of the proof. Now take any $i, j \in I$. If i and j are the same connected component, we have already shown that $v_{ij} = v_{ji}$. If they are in different components, we know that $w_{ij} = w_{ji} = 0$, therefore $v_{ij} = 0 = v_{ji}$. This concludes the proof. ■

Proofs for Section 4

Let us introduce some notations and terminology that will be used in the proofs of this section.

Recall that a matrix norm $\|\cdot\|\colon \mathbb{C}^{m\times n} \to \mathbb{R}_+$ is an *induced norm* if it is induced by vector norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, i.e., there exist norms $\|\cdot\|_{\mathbb{C}^m}\colon \mathbb{C}^m \to \mathbb{R}_+$ and $\|\cdot\|_{\mathbb{C}^n}\colon \mathbb{C}^n \to \mathbb{R}_+$ such that for $M \in \mathbb{C}^{m\times n}$ we have

$$\|M\| = \sup\{\|Mx\|_{\mathbb{C}^m} : x \in \mathbb{C}^n, \|x\|_{\mathbb{C}^n} = 1\}.$$

This definition implies that we have

$$\|Mx\|_{\mathbb{C}^m} \leq \|M\| \cdot \|x\|_{\mathbb{C}^n} \tag{9}$$

for every $M \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^n$.

We will use a variation of the ∞-norm: the weighted maximum norm. Fix a weight vector $u \in \mathbb{R}^n$, $u > 0$. For $x \in \mathbb{C}^n$, let

$$\|x\|_\infty^u = \max\{|x_i|/u_i : 1 \leq i \leq n\}.$$

The induced matrix norm is the following: for $M = (m_{ij}) \in \mathbb{C}^{n\times n}$, we have

$$\|M\|_\infty^u = \sup\{\|Mx\|_\infty^u : \|x\|_\infty^u = 1\} = \max\left\{\frac{1}{u_i}\sum_{j=1}^n u_j|m_{ij}| : 1 \leq i \leq n\right\}. \tag{10}$$

For a vector $u \in \mathbb{R}^n$, $u > 0$, let $u^{-1} \in \mathbb{R}^n$ denote the vector with entries u_i^{-1}. Let $D_u = \operatorname{diag}(u_1, u_2, \ldots, u_n) \in \mathbb{R}^{n\times n}$, by (10), we have

$$\|M\|_\infty^u = \|D_u^{-1}MD_u\|_\infty.$$

If W is a network and $a \in \mathbb{R}^n$, $a > 0$ is a scaling vector, then for the rescaled matrix V we have $v_{ij} = w_{ij}a_i/a_j$. Equivalently, $V = D_aWD_a^{-1}$. Therefore, we have

$$\|W\|_\infty^{a^{-1}} = \|D_{a^{-1}}^{-1}WD_{a^{-1}}\|_\infty = \|D_aWD_a^{-1}\|_\infty = \|V\|_\infty. \tag{11}$$

For a matrix $M = (m_{ij})_{1\leq i,j\leq n} \in \mathbb{C}^{n\times n}$, we will denote by $|M| \in \mathbb{R}_+^{n\times n}$ the matrix $(|m_{ij}|)_{1\leq i,j\leq n}$.
We will use the following statement.

Proposition A.1 (Perron-Frobenius Theorem)**.** *Let $M \in \mathbb{R}_+^{n\times n}$. Then, there exists a vector $z \geq 0$, $z \neq 0$ such that $Mz = \rho(M)z$.*

Furthermore, for any $\varepsilon > 0$ there exists a vector $u > 0$ such that $\rho(M) < \|M\|_\infty^u < \rho(M) + \varepsilon$.

For a proof see Chapter 2, Proposition 6.6 of Bertsekas and Tsitsiklis (1989).

Theorem 4.3

Let us start by proving the equivalence of 2, 3, and 4:

Proof of 4 ⇒ 2. Assume that $\rho(|W| - I_n) < 1$. Then, by Proposition A.1 for $0 < \varepsilon < 1-\rho(|W|-I_n)$, there exists a vector $u \in \mathbb{R}^n$, $u > 0$ such that $\||W|-I_n\|_\infty^u < 1$. Let us use $u^{-1} = a$ as a scaling vector, i.e., let $v_{ij} = w_{ij}a_i/a_j$. Then, we have

$$1 > \||W| - I_n\|_\infty^u = \||W| - I_n\|_\infty^{a^{-1}} = \||V| - I_n\|_\infty = \|V - I_n\|_\infty,$$

so the matrix V is row diagonally dominant. In other words, V has weak influences.

■

Proof of 2 ⇒ 3. Let $a \in \mathbb{R}^n$, $a > 0$ be a scaling vector so that the rescaled matrix V has weak influences, i.e., V is a row diagonally dominant matrix. Therefore, $\||V| - I_n\|_\infty = \|V - I_n\|_\infty < 1$, so by (11), we have

$$\||W| - I_n\|_\infty^{a^{-1}} = \|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty = \||V| - I_n\|_\infty < 1.$$

For any induced matrix norm $\|\cdot\|$, we have $\|MN\| \leq \|M\|\|N\|$ for any matrices M, N. Therefore, $\|M^k\| \leq \|M\|^k$ for any $k \in \mathbb{N}$ and any matrix M. Hence,

$$\lim_{k\to\infty} \|(|W| - I_n)^k\|_\infty^{a^{-1}} \leq \lim_{k\to\infty} \left(\||W| - I_n\|_\infty^{a^{-1}}\right)^k = 0,$$

since $\||W| - I_n\|_\infty^{a^{-1}} < 1$. The norm of $(|W| - I_n)^k$ converges to 0, this is only possible if the matrices converge to the 0 matrix, so we have $\lim_{k\to\infty}(|W| - I_n)^k = 0$. ■

Proof of 3 ⇒ 4. Assume that the limit is 0. Let λ be any eigenvalue of $|W| - I_n$, and $z \neq 0$ the corresponding eigenvector. Note that z is also an eigenvector of $(|W| - I_n)^k$ with eigenvalue λ^k. We have

$$0 = \left(\lim_{k\to\infty} (|W| - I_n)^k\right) z = \lim_{k\to\infty} (|W| - I_n)^k z = \lim_{k\to\infty} \lambda^k z = \left(\lim_{k\to\infty} \lambda^k\right) z.$$

Since $z \neq 0$, we must have $\lim_{k\to\infty} \lambda^k = 0$, hence $|\lambda| < 1$. This is true for any eigenvalue, so $\rho(|W| - I_n) < 1$. ■

Hence, conditions 2, 3, and 4 are equivalent. Now notice that for any matrix M, we have $\lim_{k\to\infty} M^k = 0$ if and only if $\lim_{k\to\infty}(M^\top)^k = 0$. Therefore, we have the following equivalences: the network W can be rescaled into a row diagonally dominant matrix ⇔ $\lim_{k\to\infty}(|W| - I_n)^k = 0$ ⇔ $\lim_{k\to\infty}((|W| - I_n)^\top)^k = 0$ ⇔ $W^\top$ can be rescaled into a row diagonally dominant matrix ⇔ W can be rescaled into a column diagonally dominant matrix.

Column diagonal dominance means exactly that the network has weak externalities, so we proved the equivalence of 1 with the other three statements. Finally, we add 5. The direction 1 $\Rightarrow$ 5 will need two lemmas.

Lemma A.2. *If the network W can be rescaled into a network with weak externalities using the vector $a \in \mathbb{R}^n$, $a > 0$, then for every $j \in I$, we have $\sum_{i \in I \setminus \{j\}} a_i |w_{ij}| < a_j$.*

Proof. Take a vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled network V is with weak externalities. By definition, this means that for every $j \in I$, we have

$$\sum_{i \in I \setminus \{j\}} |v_{ij}| < |v_{jj}| = 1$$
$$\sum_{i \in I \setminus \{j\}} \frac{a_i}{a_j} |w_{ij}| < |w_{jj}| = 1$$
$$\sum_{i \in I \setminus \{j\}} a_i |w_{ij}| < a_j.$$

■

Lemma A.3. *Let $i, j \in I$ and let $(x^k)_{k \in \mathbb{N}}$ be a best-response dynamic. If player i's action changes by Δ, player j's unconstrained best response changes by $|\Delta \cdot w_{ji}|$, i.e., we have $|\tilde{b}_j(x^k) - \tilde{b}_j(x^{k+1})| = |w_{ji}| \cdot |x_i^k - x_i^{k+1}|$ for any $k \in \mathbb{N}$.*

Proof. By (2), the unconstrained best response function of player j is $\tilde{b}_j(x) = t_j - \sum_{i \in I \setminus \{j\}} w_{ji} x_i$. Therefore, if player i's action changes by Δ, j's unconstrained best response changes by $|\Delta \cdot w_{ji}|$. ■

Proof of 1 $\Rightarrow$ 5. Suppose that W can be rescaled into a network with weak externalities using the scaling vector $a \in \mathbb{R}^n$, $a > 0$. We will show that for the same vector a the function

$$\phi'(x) = -\sum_{i \in I} a_i |x_i - \tilde{b}_i(x)|$$

is a best-response potential function.

Let $i \in I$ and fix $x \in X$. We need to prove that only $b_i(x)$ maximizes $\phi'(\cdot, x_{-i})$. Assume that we have $x_i^1 \in \operatorname{argmax}_{x_i \in X_i} \phi'(x_i, x_{-i})$. Let $x^1 = (x_i^1, x_{-i})$ and $i^1 = i$. Then $x^2 = (b_i(x^1), x_{-i}) = (x_j^2)_{j \in I}$. We have that $\phi'(x^1) \geq \phi'(x^2)$ since $x_i^1 \in$

$\operatorname{argmax}_{x_i \in X_i} \phi'(x_i, x_{-i}^1)$. Therefore, we have

$$\begin{aligned}
0 &\geq \phi'(x^2) - \phi'(x^1) \\
&= -\left(\sum_{j \in I \setminus \{i\}} a_j |x_j^2 - \tilde{b}_j(x^2)|\right) - \left(-\sum_{j \in I} a_j |x_j^1 - \tilde{b}_j(x^1)|\right) \\
&= a_i |x_i^1 - b_i(x^1)| + \sum_{j \in I \setminus \{i\}} a_j \left(|x_j^1 - \tilde{b}_j(x^1)| - |x_j^1 - \tilde{b}_j(x^2)|\right) \\
&\geq a_i |x_i^1 - b_i(x^1)| - \sum_{j \in I \setminus \{i\}} a_j |\tilde{b}_j(x^1) - \tilde{b}_j(x^2)| \\
&= a_i |x_i^1 - b_i(x^1)| - \sum_{j \in I \setminus \{i\}} a_j |w_{ji}||x_i^1 - x_i^2| \qquad \text{by Lemma A.3} \\
&= \left(a_i - \sum_{j \in I \setminus \{i\}} a_j |w_{ji}|\right) |x_i^1 - b_i(x^1)| \\
&\geq 0 \qquad \text{by Lemma A.2.}
\end{aligned}$$

Hence, we must have equality everywhere. Since $a_i - \sum_{j \in I \setminus \{i\}} a_j |w_{ji}| > 0$, equality holds in the last line if and only if $|x_i^1 - b_i(x^1)| = 0$, i.e., iff $x_i^1 = b_i(x^1) = b_i(x)$. Thus, we have $\operatorname{argmax}_{x_i \in X_i} \phi'(x_i, x_{-i}) = \{b_i(x)\}$, so ϕ' is a best-response potential function. ∎

Proof of 5 ⇒ 1. Let $a \in \mathbb{R}^n$, $a > 0$ be such that

$$\phi'(x) = -\sum_{i \in I} a_i |x_i - \tilde{b}_i(x)|$$

is a best-response potential function for every game played on W. We show that W is rescalable to a network with weak externalities using the scaling vector a.

Suppose for contradiction that there exists $k \in I$ such that the kth column of the rescaled matrix does not satisfy the diagonally dominant property, i.e., we have

$$\sum_{i \neq k} \frac{a_i}{a_k} |w_{ik}| > 1. \tag{12}$$

Let $t_k > 0$ and for every $i \neq k$ choose t_i as follows:

- if $w_{ik} > 0$, then let $t_i < w_{ik} t_k$,
- if $w_{ik} \leq 0$, then t_i can be arbitrary.

Let us define the action profile $x \in \mathbb{R}^n$ as follows. $x_j = 0$ for all $j \neq k$ and

$$x_k = \max\left(\{\frac{t_i}{w_{ik}} : i \in I \setminus \{k\}, w_{ik} > 0\} \cup \{0\}\right).$$

Note that we have $x_k < t_k$ by the choice of t.

We will show that if we are in action profile x and it is player k's turn, the value of ϕ' decreases as player k moves to her best response. Let $\hat{x}$ denote the next step, i.e., $\hat{x}_i = 0$ for $i \neq k$ and $\hat{x}_k = t_k$. We have

$$\begin{aligned}
\phi'(x) - \phi'(\hat{x}) &= -\sum_{i \in I} a_i |x_i - \tilde{b}_i(x)| + \sum_{i \in I} a_i |\hat{x}_i - \tilde{b}_i(\hat{x})| \\
&= -\sum_{i \neq k} a_i |\tilde{b}_i(x)| - a_k |x_k - t_k| + \sum_{i \neq k} a_i |\tilde{b}_i(\hat{x})| + a_k |\hat{x}_k - \tilde{b}_k(\hat{x})| \\
&= -\sum_{i \neq k} a_i |t_i - w_{ik} x_k| - a_k |x_k - t_k| + \sum_{i \neq k} a_i |t_i - w_{ik} t_k| - a_k |t_k - t_k| \\
&= \sum_{i \neq k} a_i \left(|t_i - w_{ik} t_k| - |t_i - w_{ik} x_k| \right) - a_k (t_k - x_k) \\
&= \sum_{\substack{i \neq k, \\ w_{ik} \leq 0}} a_i (w_{ik} x_k - w_{ik} t_k) - \sum_{\substack{i \neq k, \\ w_{ik} > 0}} a_i (w_{ik} t_k - w_{ik} x_k) - a_k (t_k - x_k),
\end{aligned}$$

since for $w_{ik} > 0$, both $t_i - w_{ik} t_k$ and $t_i - w_{ik} x_k$ are non-positive by the choice of t and x. Hence, we have

$$\begin{aligned}
\phi'(x) - \phi'(\hat{x}) &= \sum_{i \neq k} a_i |w_{ik}| (t_k - x_k) - a_k (t_k - x_k) \\
&= (t_k - x_k) \left(\sum_{i \neq k} a_i |w_{ik}| - a_k \right) \\
&\geq 0,
\end{aligned}$$

since $t_k - x_k > 0$ and $\sum_{i \neq k} a_i |w_{ik}| > a_k$ by (12). Therefore, ϕ' decreases as player k moves to her best response, and hence it cannot be a best-response potential, which is a contradiction. This finishes the proof of the statement. ■

Proposition 4.4

Lemma A.4. *Recall that $b \colon X \to X$ denotes the best response mapping. For any vector $u \in \mathbb{R}^n$, $u > 0$ and for all $x, x' \in X$, we have*

$$\|b(x) - b(x')\|_\infty^u \leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u.$$

Proof. First, consider the unconstrained best responses $\tilde{b}(x)$ and $\tilde{b}(x')$. We have

$$\begin{aligned}
\|\tilde{b}(x) - \tilde{b}(x')\|_\infty^u &= \|(t - (W - I_n)x) - (t - (W - I_n)x')\|_\infty^u \\
&= \|(W - I_n)(x' - x)\|_\infty^u \\
&\leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u \qquad \text{by (9).}
\end{aligned}$$

Notice that for every $i \in I$, we have $|b_i(x) - b_i(x')| \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$. Indeed, without loss of generality, we can assume that $\tilde{b}_i(x) \leq \tilde{b}_i(x')$ and we can verify the inequality in all cases:

- If $\tilde{b}_i(x) \leq \tilde{b}_i(x') \leq 0$, then $b_i(x) = b_i(x') = 0$, so $|b_i(x) - b_i(x')| = 0 \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $\overline{x}_i \leq \tilde{b}_i(x) \leq \tilde{b}_i(x')$, then $b_i(x) = b_i(x') = \overline{x}_i$, so $|b_i(x) - b_i(x')| = 0 \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $\tilde{b}_i(x) \leq 0 \leq \tilde{b}_i(x')$ or if $\tilde{b}_i(x) \leq \overline{x}_i \leq \tilde{b}_i(x')$, then $\tilde{b}_i(x) \leq b_i(x) \leq b_i(x') \leq \tilde{b}_i(x')$, and hence $|b_i(x) - b_i(x')| \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $0 \leq \tilde{b}_i(x) \leq \tilde{b}_i(x') \leq \overline{x}_i$, then $b_i(x) = \tilde{b}_i(x)$ and $b_i(x') = \tilde{b}_i(x')$, so $|b_i(x) - b_i(x')| = |\tilde{b}_i(x) - \tilde{b}_i(x')|$.

Therefore, we have

$$\begin{aligned}
\|b_i(x) - b_i(x')\|_\infty^u &= \sum_{i \in I} |b_i(x) - b_i(x')|/u_i \\
&\leq \sum_{i \in I} |\tilde{b}_i(x) - \tilde{b}_i(x')|/u_i \\
&= \|\tilde{b}(x) - \tilde{b}(x')\|_\infty^u \\
&\leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u .
\end{aligned}$$

This concludes the proof of the lemma. ■

Proof of Proposition 4.4. By Theorem 4.3, there exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix is row diagonally dominant, i.e., we have $\|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty < 1$.

Assume that x^* and x^{**} are two different Nash equilibria, i.e., we have $b(x^*) = x^*$, $b(x^{**}) = x^{**}$ and $x^* \neq x^{**}$. By the assumption that $\|W - I_n\|_\infty^{a^{-1}} < 1$ and by Lemma A.4, we get

$$\|x^* - x^{**}\|_\infty^{a^{-1}} = \|b(x^*) - b(x^{**})\|_\infty^{a^{-1}} \leq \|W - I_n\|_\infty^{a^{-1}} \|x^* - x^{**}\|_\infty^{a^{-1}} < \|x^* - x^{**}\|_\infty^{a^{-1}}$$

which is a contradiction. Hence, there exists a unique Nash equilibrium. ■

Theorem 4.5

Proof of Theorem 4.5. By Theorem 4.3, there exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix is row diagonally dominant, i.e., we have $\|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty < 1$. Let $\gamma = \|W - I_n\|_\infty^{a^{-1}}$, then $0 \leq \gamma < 1$.

By Proposition 4.4, there is a unique Nash equilibrium, let us denote it by x^*. Notice that a BRD is a BRAD with parameter 0, so it is enough to prove the statement for BRAD's. Consider any BRAD $(x^k)_{k\in\mathbb{N}}$ with approach parameter $0 \leq \beta < 1$.

By the definition of the weighted maximum norm, for every $x, y \in X$ we have

$$a_i|x_i - y_i| \leq \|x - y\|_\infty^{a^{-1}}. \tag{13}$$

Hence, for an arbitrary $k \in \mathbb{N}$, we have

$$\begin{aligned} a_{i^k}|b_{i^k}(x^k) - x^*_{i^k}| &= a_{i^k}|b_{i^k}(x^k) - b_{i^k}(x^*)| \\ &\leq \|b(x^k) - b(x^*)\|_\infty^{a^{-1}} \qquad \text{by (13)} \\ &\leq \|W - I_n\|_\infty^{a^{-1}} \|x^k - x^*\|_\infty^{a^{-1}} \qquad \text{by Lemma A.4} \\ &= \gamma \cdot \|x^k - x^*\|_\infty^{a^{-1}}. \end{aligned} \tag{14}$$

For the next claim, note that since $\gamma + \beta - \gamma\beta = 1 - (1-\gamma)(1-\beta)$, we have $0 \leq \gamma + \beta - \gamma\beta < 1$.

Claim A.5. *For every $k \in \mathbb{N}$, we have*

$$a_{i^k}|x^{k+1}_{i^k} - x^*_{i^k}| \leq (\gamma + \beta - \gamma\beta)\|x^k - x^*\|_\infty^{a^{-1}}.$$

Proof. Let us use the notation $D = a_{i^k}^{-1}\|x^k - x^*\|_\infty^{a^{-1}} \in \mathbb{R}_+$. By (13), we have $|x_{i^k} - x^*_{i^k}| \leq a_{i^k}^{-1}\|x^k - x^*\|_\infty^{a^{-1}} = D$, and hence $x^k_{i^k}$ is contained in the interval of lenght $2D$ with midpoint $x^*_{i^k}$. By (14), $b_{i^k}(x^k)$ is in the interval of length $\gamma 2D$ centered at $x^*_{i^k}$.

For $p, q \in \mathbb{R}$, let us use the notation $[p, q] = [\min\{p, q\},\ \max\{p, q\}]$ for the interval between p and q.

From the definition of BRAD, we have $x^{k+1}_{i^k} \in [b_{i^k}(x^k),\ (1-\beta)b_{i^k}(x^k) + \beta x^k_{i^k}]$, since this is the β-contracted image of $[b_{i^k}(x^k),\ x^k_{i^k}]$ towards $b_{i^k}(x^k)$. This implies that $x^{k+1}_{i^k}$ is contained in the β-contracted image of $[x^*_{i^k} - D,\ x^*_{i^k} + D]$ around $b_{i^k}(x^k)$, which is a point of $[x^*_{i^k} - \gamma D,\ x^*_{i^k} + \gamma D]$. Hence, we get the worst upper bound for $x^{k+1}_{i^k}$ if $b_{i^k}(x^k)$ takes the maximal value in $[x^*_{i^k} - \gamma D,\ x^*_{i^k} + \gamma D]$, and the worst lower bound if $b_{i^k}(x^k)$ takes the minimal value in the interval. We can compute these bounds: if $b_{i^k}(x^k) = x^*_{i^k} + \gamma D$, then the contracted image of $x^*_{i^k} + D$ is

$$(1-\beta)(x^*_{i^k} + \gamma D) + \beta(x^*_{i^k} + D) = x^*_{i^k} + (\gamma + \beta - \gamma\beta)D.$$

Similarly, for $b_{i^k}(x^k) = x^*_{i^k} - \gamma D$ we get the lower bound

$$(1-\beta)(x^*_{i^k} - \gamma D) + \beta(x^*_{i^k} - D) = x^*_{i^k} - (\gamma + \beta - \gamma\beta)D.$$

Therefore, $x^{k+1}_{i^k} \in [x^*_{i^k} - (\gamma + \beta - \gamma\beta)D,\ x^*_{i^k} + (\gamma + \beta - \gamma\beta)D]$, and hence

$$a_{i^k}|x^{k+1}_{i^k} - x^*_{i^k}| \leq (\gamma + \beta - \gamma\beta)\|x^k - x^*\|^{a^{-1}}_{\infty},$$

as desired. ■

Claim A.6. *For every $m \in \mathbb{N}$ there exists $K(m) \in \mathbb{N}$ such that for all $k \geq K(m)$ we have*

$$\|x^k - x^*\|^{a^{-1}}_{\infty} \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|^{a^{-1}}_{\infty}.$$

Proof. We prove the statement by induction on m. For $m = 0$, it clearly holds with $K(0) = 0$. Assume that $K(m-1)$ exists, and we would like to find $K(m)$.

Take an arbitrary $k \geq K(m-1)$ and let player i be the one who moves at time k. Then, we have

$$\begin{aligned}
a_i|x^{k+1}_i - x^*_i| &\leq (\gamma + \beta - \gamma\beta)\|x^k - x^*\|^{a^{-1}}_{\infty} && \text{by Claim A.5} \\
&\leq (\gamma + \beta - \gamma\beta)(\gamma + \beta - \gamma\beta)^{m-1}\|x^0 - x^*\|^{a^{-1}}_{\infty} && \text{by ind. hypothesis} \\
&= (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|^{a^{-1}}_{\infty}.
\end{aligned}$$

Therefore, we can see that $a_i|x^{\ell}_i - x^*_i| \leq \gamma^m \|x^0 - x^*\|^{a^{-1}}_{\infty}$ for all $\ell > k$, since it is true after every move of player i, and it remains true in all other players' turns because that does not change the action of player i.

For every $i \in I$, let k_i be the first time player i moves after $K(m-1)$. Let

$$K(m) = \max\{k_i : i \in I\} + 1.$$

By time $K(m)$, every player moved at least once since $K(m-1)$, so for every $i \in I$ and all $k \geq K(m)$, we have $a_i|x^k_i - x^*_i| \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|^{a^{-1}}_{\infty}$. Therefore, we also have

$$\|x^k - x^*\|^{a^{-1}}_{\infty} = \max\{a_i|x^k_i - x^*_i| : i \in I\} \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|^{a^{-1}}_{\infty}.$$

This proves the statement for every $m \in \mathbb{N}$. ■

Now we can show the convergence of the BRAD $(x^k)_{k\in\mathbb{N}}$ to the Nash equilibrium x^*. Take any $\varepsilon > 0$, then there exists $m \in \mathbb{N}$ such that $(\gamma+\beta-\gamma\beta)^m\|x^0 - x^*\|_\infty^{a^{-1}} < \varepsilon$, since $\gamma+\beta-\gamma\beta < 1$. Therefore, if $k \geq K(m)$ from Claim A.6, then we have

$$\|x^k - x^*\|_\infty^{a^{-1}} \leq (\gamma+\beta-\gamma\beta)^m\|x^0 - x^*\|_\infty^{a^{-1}} < \varepsilon.$$

Hence, $(x^k)_{k\in\mathbb{N}}$ converges to the Nash equilibrium x^*. ■

References

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417.

Ballester, C., Zenou, Y., and Calvó-Armengol, A. (2010). Delinquent networks. *Journal of the European Economic Association*, 8(1):34–61.

Bayer, P., Herings, P. J.-J., and Peeters, R. (2021). Farsighted manipulation and exploitation in networks. *Journal of Economic Theory*, 196:105311.

Bayer, P., Herings, P. J.-J., Peeters, R., and Thuijsman, F. (2019). Adaptive learning in weighted network games. *Journal of Economic Dynamics and Control*, 105:250–264.

Belhaj, M., Bramoullé, Y., and Deroïan, F. (2014). Network games under strategic complementarities. *Games and Economic Behavior*, 88(C):310–319.

Berman, A. and Plemmons, R. J. (1994). M-Matrices. In *Nonnegative Matrices in the Mathematical Sciences*, chapter 6, pages 132 – 164. Academic Press.

Bertsekas, D. P. and Tsitsiklis, J. N. (1989). *Parallel and Distributed Computation: Numerical Methods*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA.

Bervoets, S., Bravo, M., and Faure, M. (2020). Learning with minimal information in continuous games. *Theoretical Economics*, 15(4):1471–1508.

Bervoets, S. and Faure, M. (2019). Stability in games with continua of equilibria. *Journal of Economic Theory*, 179(C):131–162.

Blume, L. E., Brock, W. A., Durlauf, S. N., and Ioannides, Y. M. (2010). Identification of Social Interactions. Economics Series 260, Institute for Advanced Studies.

Bozóki, S., Fülöp, J., and Rónyai, L. (2010). On optimal completion of incomplete pairwise comparison matrices. *Mathematical and Computer Modelling*, 52(1-2):318–333.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic Interaction and Networks. *American Economic Review*, 104(3):898–930.

Cachon, G. P. and Camerer, C. (1996). The sunk cost fallacy, forward induction and behavior in coordination games. *Quarterly Journal of Economics*, 111:165–194.

Cason, T. and Friedman, D. (1997). Price formation in single call markets. *Econometrica*, 65(2):311–346.

Dubey, P., Haimanko, O., and Zapechelnyuk, A. (2006). Strategic complements and substitutes, and potential games. *Games and Economic Behavior*, 54(1):77–94.

Farkas, A., Rózsa, P., and Stubnya, E. (1999). Transitive matrices and their applications. *Linear Algebra and its Applications*, 302-303:423–433.

Gabay, D. and Moulin, H. (1980). On the uniqueness and stability of Nash equilibria in noncooperative games. In Bensoussan, A., Kleindorfer, P. R., and Tapiero, C. S., editors, *Applied Stochastic Control in Econometrics and Management Science*, pages 271–294. North-Holland.

Golub, B. and Jackson, M. O. (2012). How homophily affects the speed of learning and best-response dynamics. *The Quarterly Journal of Economics*, 127(3):1287–1338.

Jensen, M. (2010). Aggregative games and best-reply potentials. *Economic Theory*, 43(1):45–66.

Kagel, J. and Levin, D. (1999). Common value auctions with insider information. *Econometrica*, 67(5):1219–1238.

Kelly, F. (2011). Reversibility and stochastic networks. *Cambridge University Press.*

König, M. D., Liu, X., and Zenou, Y. (2019). R&d networks: Theory, empirics, and policy implications. *The Review of Economics and Statistics*, 101(3):476–491.

Kukushkin, N. S. (2005). Strategic supplements in games with polylinear interactions. *EconWPA Paper*, 411008.

Kukushkin, N. S. (2015). Cournot tatonnement and potentials. *Journal of Mathematical Economics*, 59:117 – 127.

Monderer, D. and Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, 14(1):124–143.

Moulin, H. (1984). Dominance solvability and cournot stability. *Mathematical Social Sciences*, 7:83–102.

Moulin, H. (1986). *Game Theory for the Social Sciences.* New York University Press, Washington Square, New York, USA.

Nagel, R. and Vriend, N. J. (1999). An experimental study of adaptive behavior in an oligopolistic market game. *Journal of Evolutionary Economics*, 9(1):27–65.

Parise, F. and Ozdaglar, A. (2019). A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis. *Games and Economic Behavior*, 114(C):47–82.

Saaty, T. L. (1988). What is the analytic hierarchy process? In Mitra, G., Greenberg, H. J., Lootsma, F. A., Rijkaert, M. J., and Zimmermann, H. J., editors, *Mathematical Models for Decision Support*, pages 109–121. Springer, Berlin, Heidelberg.

Sandler, T. and Hartley, K. (1995). *The Economics of Defense.* Cambridge University Press.

Sandler, T. and Hartley, K. (2007). Defense in a Globalized World: An Introduction. In Hartley, K. and Sandler, T., editors, *Handbook of Defense Economics*, volume 2, chapter 20, pages 607–621. Elsevier.

Scutari, G., Facchinei, F., Pang, J.-S., and Palomar, D. P. (2014). Real and complex monotone communication games. *IEEE Transactions on Information Theory*, 60(7):4197–4231.

Selten, R. and Buchta, J. (1998). Experimental sealed bid first price auctions with directly observed bid functions. Games and Human Behavior: Essays in Honor of Amnon Rapoport.

Selten, R. and Stoecker, R. (1986). End behavior in sequences of finite prisoner's dilemma supergames a learning theory approach. *Journal of Economic Behavior & Organization*, 7(1):47 – 70.

Ui, T. (2016). Bayesian nash equilibrium and variational inequalities. *Journal of Mathematical Economics*, 63:139–146.

Voorneveld, M. (2000). Best-response potential games. *Economics Letters*, 66(3):289–295.