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Invariant Set of Weight of Perceptron Trained by Perceptron Training Algorithm

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ABSTRACT

In this paper, an invariant set of the weight of the perceptron trained by the perceptron training algorithm is defined and characterized. The dynamic range of the steady state values of the weight of the perceptron can be evaluated via finding the dynamic range of the weight of the perceptron inside the largest invariant set. Also, the necessary and sufficient condition for the forward dynamics of the weight of the perceptron to be injective as well as the condition for the invariant set of the weight of the perceptron to be attractive is derived.

Index Terms—Perceptron training algorithm, neurodynamics, symbolic dynamics, chaos, invariant set.

I. INTRODUCTION

Pattern recognitions, such as speech recognitions [3], infra red imagery military vehicle detections [18], English letter recognitions [19] and facial recognitions [20], play an important role in our daily life. The existing pattern recognition methods could be mainly categorized into three different approaches, namely, the statistical approaches, the neural network approaches and the structural approaches. The structural approaches are problem dependent and these approaches are difficult for solving general pattern recognition problems. The statistical approaches require information on the prior probabilities of each class and the condition probabilities of the feature vectors in which this information is usually not available in many practical pattern recognition problems. Hence, the neural network approaches are the most practical approaches for solving general pattern recognition problems. The simplest neural network is a perceptron. A perceptron is a single neuron that applies the single bit quantization function to the inner product of its weight and its input [1], [2], [23]. As the output of the perceptron is either 1 or -1 [1], [2], the output of the perceptron is used for representing two different classes of objects of pattern recognition systems. Hence, perceptrons are widely employed for solving general pattern recognition problems [17].

To implement the perceptron, the weight of the perceptron is required to be known a prior and it is usually trained by perceptron training algorithms [1], [2], [4]-[7], [23]. There are many different perceptron training algorithms [1], [2], [4]-[7], [23], in which the one proposed in [1] and [2] is the commonest perceptron training algorithm employed in industries (First, an arbitrary weight is initialized. Then the new weight is obtained by adding the old weight to the product of its input and the half difference between the desirable output and the perceptron output. By computing the new weight again and again, if the new weight converges, then the converged weight is employed as the weight of the perceptron [1], [2].). Many efficient hardware and software packages [8] have been developed for the implementation of the perceptron training algorithm [1], [2].

It is well known from the perceptron training algorithm [1], [2] that the weight of the perceptron would converge if the set of input vectors is linearly separable. When the set of input vectors is nonlinearly separable, the weight of the perceptron could exhibit chaotic behaviors (Chaotic behavior is a kind of nonlinear system behaviors in which the system is sensitive to its initial condition, topological transitive and with dense periodic orbits [25]. It is worth noting that in general non-converging behaviors may not be chaotic behaviors. For an example, an impulse response of an unstable linear system is diverging, but this diverging behavior is not a chaotic behavior. Also, a limit cycle behavior is not a chaotic behavior too because the system response consists of a finite number of periodic orbits. Hence, in this paper chaotic behaviors are not referring to the non-converging behaviors.). Recent researches [9]-[11] show that the exhibition of chaotic behaviors of the weight of the perceptron could be applied for the recognition of chaotic attractors [9], nonlinear dynamical systems [10], [21], [22] and cardiovascular time series [11].

However, there are some fundamental questions remained unaddressed when the weight of the perceptron exhibits chaotic behaviors. For examples, what is the dynamic range of the steady state values of the weight of the perceptron when it exhibits chaotic behaviors? Are there any attractive regions that the weight of the perceptron will eventually move to and stay inside once the weight of the perceptron enters these regions? These two fundamental questions are important from a practical point of view because the dynamic range of the steady state values of the weight of the perceptron has to be within a certain range for an implementation and safety reason. Also, as the existence of the attractive regions implies that the weights of the perceptron will be stayed inside these attractive regions if the initial weight of the perceptron is inside these attractive regions, and the existence of these attractive regions implies the weights of the perceptron will move to these attractive regions, the existence of these attractive regions would guarantee the robust local stability of the perceptron. The objective of this paper is to address these two issues.

To investigate the dynamic range of the steady state values of the weight of the perceptron, an invariant set approach [12]-[16] is proposed. The dynamic range of the steady state values of the weight of the perceptron could be evaluated by characterizing the largest invariant set and finding the dynamic range of the weight of the perceptron inside the largest invariant set. To investigate whether there exist attractive regions that the weight of the perceptron will eventually move to, it is equivalent to investigate whether the invariant set is attractive or not.

However, it is very challenging to characterize an invariant set of the weight of the perceptron. There are mainly two reasons. First, no existing result has been reported on the characterization of an invariant set of the weight of the perceptron. Since conventional perceptrons are usually operated with a set of linearly separable input vectors, existing results are not applicable for the characterization of an invariant set of the weight of the perceptron when the weight of the perceptron exhibits chaotic behaviors. Second, as the forward dynamics of the weight of the perceptron depends on the output of the perceptron, in which it is obtained by applying the single bit quantization function on the inner product of the weight and the input of the perceptron, the forward dynamics of the weight of the perceptron is governed by a nonlinear map. Moreover, as the input vectors keep multiplying to the weight of the perceptron, the input of the perceptron is periodically time varying with the period equal to the total number of the input vectors. Hence, the forward dynamics of the weight of the perceptron is governed by a time varying map. Overall, the forward dynamics of the weight of the perceptron is governed by a nonlinear time varying map. This results to a very difficult characterization of an invariant set and the corresponding invariant map of the weight of the perceptron.

To address these difficulties, this paper proposes to downsample the weight of the perceptron with the downsampling rate equal to the total number of the input vectors. Here, the set of the downsampled weights of the perceptron refers to the set of the weights of the perceptron with the time indices equal to an integer multiple of the total number of the input vectors. Since the next weight depends on the current weight, the current input vector and the current desirable output, the system map relating the current weight and the next weight is time variant. However, as all input vectors and desirable outputs are sum up for the calculation of the next downsampled weight, the next downsampled weight only depends on the current weight. As a result, the system map relating the current downsampled weight and the next downsampled weight is time invariant. Hence, the forward dynamics of the downsampled weight of the perceptron is now governed by a nonlinear time invariant map. An invariant set of the weight of the perceptron is defined as a set of the downsampled weights that maps to itself.

Besides, it is also challenging to investigate whether an invariant set of the weight of the perceptron is attractive or not. Since if the invariant set is attractive, then some weights outside the invariant set will map to a weight inside the invariant set. As the weight inside the invariant set will also map to a weight inside the invariant set, there exist at least two different weights, one inside the invariant set and another one outside the invariant set, that will map to the same weight inside the invariant set. In other words, there exist at least two different backward dynamics of the weight of the perceptron that will map the weight inside the invariant set to the weights both inside and outside the invariant set. As the backward dynamics of the weight of the perceptron is not uniquely defined, the analysis of the attractive property of the invariant set of the weight of the perceptron is very challenging. To address this difficulty, first it is required to define a backward dynamics of the weight of the perceptron so that the weight inside the invariant set will map (based on the defined backward dynamics of the weight of the perceptron) to a weight inside the invariant set. The obtained result will be discussed in Lemma 1. Second, it is required to investigate the injective property of the forward dynamics of the weight of the perceptron. Here, the injective property of the forward dynamics of the weight of the perceptron refers to whether the forward dynamics is one to one or many to one. This result will determine whether the backward dynamics of the weight of the perceptron is uniquely defined or not. The result derived in Lemma 1 will be applied for this investigation and the obtained result will be discussed in Lemma 2 and Corollary 1. Third, it is required to define an invariant set and the corresponding invariant map of the weight of the perceptron so that the corresponding invariant map is bijective. Hence, the weights of the perceptron will be stayed within the invariant set if the initial weight is inside the invariant set. The result derived in Lemma 2 and Corollary 1 will be applied for this investigation and the obtained result will be discussed in Theorem 1. By the way, it is worth investigating whether the invariant set is empty or not. Lemma 3 is addressing this issue. Now, it is ready to evaluate the dynamic range of the steady state values of the weight of the perceptron by finding the dynamic range of the weight of the perceptron inside the largest invariant set. The corresponding result will be discussed in Corollary 2. Fourth, it is required to investigate the dynamics of the weights of the perceptron outside the invariant set. The obtained result will be discussed in Lemma 4 and Theorem 2. Fifth, it is required to investigate the surjective property of the forward dynamics of the weight of the perceptron. The obtained result will be discussed in Theorem 3. Based on the obtained results, it can be concluded whether there exist some weights outside the invariant set that will eventually move to the invariant set or not. In other words, it can be concluded whether the invariant set is attractive or not. Finally, all possible output sequences of the perceptron in which the initial weights outside the invariant set will eventually move to the invariant set will be identified. The obtained result will be discussed in Lemma 5. An interesting property of the phase diagram will be discussed in Lemma 6.

The outline of this paper is as follows. Notations used throughout this paper are introduced in Section II. In Section III, an invariant set of the weight of the perceptron is defined and characterized. Some numerical computer simulation results are illustrated. Finally, a conclusion is drawn in Section IV.

II. NOTATIONS

Denote N as the total number of bounded training feature vectors and d as the dimension of these training feature vectors. Denote the elements of these training feature vectors as $x_i(k)$ for $i=1,2,\cdots,d$ and for $k=0,1,\cdots,N-1$. Define the input vectors as $\mathbf{x}(k)\equiv[1, \quad x_1(k), \quad \cdots, \quad x_d(k)]^T$ for $k=0,1,\cdots,N-1$, in which the superscript ^{T} denotes the transposition operator. In this paper, we assume that $\mathbf{x}(k)\neq\mathbf{0}$ for $k=0,1,\cdots,N-1$. Define $\mathbf{x}(Nn+k)\equiv\mathbf{x}(k)$ $\forall n\in Z\setminus\{0\}$ and for $k=0,1,\cdots,N-1$ so that $\mathbf{x}(k)$ is periodic with period N. Denote the weights of the perceptron as $w_i(n)$ for $i=1,2,\cdots,d$ and $\forall n\in Z$. Denote the threshold of the perceptron as $w_0(n)$ $\forall n\in Z$ and the activation function of the perceptron as $Q(z)\equiv\begin{cases}1 & z\geq 0\\ -1 & z<0\end{cases}$. Define $\mathbf{w}(n)\equiv[w_0(n), \quad w_1(n), \quad \cdots, \quad w_d(n)]^T$ $\forall n\in Z$ and denote the output of the perceptron as $y(n)$ $\forall n\in Z$, then $y(n)=Q(\mathbf{w}^T(n)\mathbf{x}(n))$ $\forall n\in Z$. Denote the desirable output of the perceptron corresponding to $\mathbf{x}(n)$ as $t(n)$ $\forall n\in Z$. Assume that the perceptron training algorithm proposed in [1], [2] and [23] is employed for the training, then the forward dynamics of the weight of the perceptron is governed by the following equation:

$$\mathbf{w}(n+1)=\mathbf{w}(n)+\frac{t(n)-Q(\mathbf{w}^T(n)\mathbf{x}(n))}{2}\mathbf{x}(n) \quad \forall n\in Z, \tag{1}$$

and denoted as $\tilde{\Im}_k^F : \Re^{d+1} \to \Re^{d+1}$, that is $\tilde{\Im}_k^F(\mathbf{w}(k)) \equiv \mathbf{w}(k+1)$ $\forall k \in Z$. In order to investigate a backward dynamics of the weight of the perceptron, the most direct approach is to characterize a system map such that the current weight $\mathbf{w}(k)$ will be moved to the previous weight $\mathbf{w}(k-1)$. That means, it is required to find an equation expressing $\mathbf{w}(k-1)$ in terms of $\mathbf{w}(k)$. Define $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1}$ such that

$$\tilde{\Im}_k^B(\mathbf{w}(k)) \equiv \mathbf{w}(k) - \frac{t(k-1) - Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right)}{2}\mathbf{x}(k-1) \quad \forall k \in Z . \tag{2}$$

It is worth noting that the time index of the weight in the activation function in $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1}$ $\forall k \in Z$ is not equal to $k-1$. It will be shown in Section III that $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1}$ is the backward dynamics of the weight of the perceptron that will map the weight inside the invariant set to the weight inside the invariant set.

It will be shown in Section III that there exist at least two different initial weights (one inside the invariant set and another one outside the invariant set) that will map to the same weight inside the invariant set. Denote $\mathbf{w}(0)$ and $\mathbf{w}'(0)$ as these two initial weights, respectively, and $\mathbf{w}'(j)$ $\forall j \in Z$ as the weight of the perceptron at the time index j based on the initial weight $\mathbf{w}'(0)$, that is $\mathbf{w}'(j+1) \equiv \tilde{\Im}_j^F(\mathbf{w}'(j))$ $\forall j \in Z$. Denote $y'(j)$ as the corresponding output of the perceptron, that is $y'(j) \equiv Q\left(\mathbf{w}'^T(j)\mathbf{x}(j)\right)$ $\forall j \in Z$. Suppose that $\exists k \in Z$ such that these two initial weights of the perceptron will map to the same weight at the time index k, that is $\mathbf{w}'(k) = \mathbf{w}(k)$.

A set S is called an invariant set under an invariant map T if $T(S) = S$. Denote the absolute value of a real number as $|\cdot|$ and the 2-norm of a vector as $\|\mathbf{v}\| \equiv \sqrt{\sum_{i=0}^{d} v_i^2}$, where $\mathbf{v} \equiv [v_0, \ \cdots, \ v_d]^T$.

III. DEFINITION AND CHARACTERIZATION OF AN INVARIANT SET OF THE WEIGHT OF THE PERCEPTRON

It has been discussed in Section II that $\tilde{\Im}_k^F : \Re^{d+1} \to \Re^{d+1} \quad \forall k \in Z$ is the forward dynamics of the weight of the perceptron. The following lemma reveals that $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1} \quad \forall k \in Z$ is one of the possible backward dynamics of the weight of the perceptron.

Lemma 1

$$\tilde{\Im}_{k-1}^F\left(\tilde{\Im}_k^B(\mathbf{w}(k))\right) = \mathbf{w}(k) \quad \forall k \in Z .$$

Proof:

$$\begin{aligned}
&\tilde{\Im}_{k-1}^F\left(\tilde{\Im}_k^B(\mathbf{w}(k))\right) \\
&= \tilde{\Im}_{k-1}^F\left(\mathbf{w}(k) - \frac{t(k-1) - Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right)}{2}\mathbf{x}(k-1)\right) \\
&= \begin{cases} \tilde{\Im}_{k-1}^F(\mathbf{w}(k)) & t(k-1) = Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) \\ \tilde{\Im}_{k-1}^F(\mathbf{w}(k) - \mathbf{x}(k-1)) & t(k-1) = 1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = -1 \\ \tilde{\Im}_{k-1}^F(\mathbf{w}(k) + \mathbf{x}(k-1)) & t(k-1) = -1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = 1 \end{cases} \\
&= \begin{cases} \mathbf{w}(k) + \frac{t(k-1) - Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right)}{2}\mathbf{x}(k-1) & t(k-1) = Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) \\ \mathbf{w}(k) - \mathbf{x}(k-1) + \frac{t(k-1) - Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1) - \|\mathbf{x}(k-1)\|^2\right)}{2}\mathbf{x}(k-1) & t(k-1) = 1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = -1 \\ \mathbf{w}(k) + \mathbf{x}(k-1) + \frac{t(k-1) - Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1) + \|\mathbf{x}(k-1)\|^2\right)}{2}\mathbf{x}(k-1) & t(k-1) = -1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = 1 \end{cases} \\
&= \begin{cases} \mathbf{w}(k) & t(k-1) = Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) \\ \mathbf{w}(k) - \mathbf{x}(k-1) + \mathbf{x}(k-1) & t(k-1) = 1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = -1 \\ \mathbf{w}(k) + \mathbf{x}(k-1) - \mathbf{x}(k-1) & t(k-1) = -1 \text{ and } Q\left(\mathbf{w}^T(k)\mathbf{x}(k-1)\right) = 1 \end{cases} \\
&= \mathbf{w}(k)
\end{aligned} \tag{3}$$

$\forall k \in Z$. This completes the proof. ■

Lemma 1 states that the weight of the perceptron will map to itself if it is first mapped according to $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1} \quad \forall k \in Z$ and then mapped according to $\tilde{\Im}_{k-1}^F : \Re^{d+1} \to \Re^{d+1}$ $\forall k \in Z$. This implies that $\tilde{\Im}_k^B : \Re^{d+1} \to \Re^{d+1} \quad \forall k \in Z$ is one of the possible backward dynamics of the weight of the perceptron.

It is worth noting that although $\tilde{\Im}_{k-1}^F\left(\tilde{\Im}_k^B(\mathbf{w}(k))\right) = \mathbf{w}(k) \quad \forall k \in Z$, the inverse of $\tilde{\Im}_{k-1}^F : \Re^{d+1} \to \Re^{d+1} \quad \forall k \in Z$ may not exist. In other words, the backward dynamics of the weight of the perceptron may not be uniquely defined, and the forward dynamics of the weight of the perceptron may be many to one. Hence, it is required to investigate the injective property of the forward dynamics of the weight of the perceptron and the result is summarized below:

Lemma 2

Assume that $\|\mathbf{x}(k)\|^2 \neq |\mathbf{w}^T(k)\mathbf{x}(k)|$. Then $\widetilde{\Im}_k^F$ is not injective if and only if $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$.

Proof:

For the necessity, $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$ implies that $\mathbf{x}^T(k)\mathbf{x}(k) > \mathbf{w}^T(k)\mathbf{x}(k)$ for $\mathbf{w}^T(k)\mathbf{x}(k) \geq 0$ and $\mathbf{x}^T(k)\mathbf{x}(k) > -\mathbf{w}^T(k)\mathbf{x}(k)$ for $\mathbf{w}^T(k)\mathbf{x}(k) < 0$. This implies that

$$Q(\mathbf{w}^T(k)\mathbf{x}(k)) = -Q(\mathbf{w}^T(k)\mathbf{x}(k) - \mathbf{x}^T(k)\mathbf{x}(k)) \text{ for } \mathbf{w}^T(k)\mathbf{x}(k) \geq 0 \tag{4}$$

and

$$Q(\mathbf{w}^T(k)\mathbf{x}(k)) = -Q(\mathbf{w}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k)\mathbf{x}(k)) \text{ for } \mathbf{w}^T(k)\mathbf{x}(k) < 0. \tag{5}$$

This further implies that

$$Q(\mathbf{w}^T(k)\mathbf{x}(k)) = -Q(\mathbf{w}^T(k)\mathbf{x}(k) - \mathbf{x}^T(k)Q(\mathbf{w}^T(k)\mathbf{x}(k))\mathbf{x}(k)). \tag{6}$$

As $y(k) = Q(\mathbf{w}^T(k)\mathbf{x}(k))$, we have

$$y(k) = -Q(\mathbf{w}^T(k)\mathbf{x}(k) - \mathbf{x}^T(k)y(k)\mathbf{x}(k)) = -Q((\mathbf{w}(k) - y(k)\mathbf{x}(k))^T\mathbf{x}(k)). \tag{7}$$

Define $\mathbf{w}''(k) \equiv \mathbf{w}(k) - y(k)\mathbf{x}(k)$ and $y''(k) \equiv Q(\mathbf{w}''^T(k)\mathbf{x}(k))$. Then $y(k) = -Q(\mathbf{w}''^T(k)\mathbf{x}(k)) = -y''(k)$ and

$$\begin{aligned}
&\tilde{\mathfrak{I}}_k^F(\mathbf{w}''(k)) \\
&= \mathbf{w}''(k) + \frac{t(k) - Q(\mathbf{w}''^T(k)\mathbf{x}(k))}{2}\mathbf{x}(k) \\
&= \mathbf{w}''(k) + \frac{t(k) - y''(k)}{2}\mathbf{x}(k) \\
&= \mathbf{w}(k) - y(k)\mathbf{x}(k) + \frac{t(k) + y(k)}{2}\mathbf{x}(k). \\
&= \mathbf{w}(k) + \frac{t(k) - y(k)}{2}\mathbf{x}(k) \\
&= \mathbf{w}(k) + \frac{t(k) - Q(\mathbf{w}^T(k)\mathbf{x}(k))}{2}\mathbf{x}(k) \\
&= \tilde{\mathfrak{I}}_k^F(\mathbf{w}(k))
\end{aligned} \quad (8)$$

Obviously, $\mathbf{w}''(k) \neq \mathbf{w}(k)$ because $y(k) \neq 0$ and $\mathbf{x}(k) \neq \mathbf{0}$. Hence, $\tilde{\mathfrak{I}}_k^F$ is not injective. This proves the necessity.

To prove the sufficiency, if $\tilde{\mathfrak{I}}_k^F$ is not injective, then there exists $\mathbf{w}(k), \mathbf{w}''(k) \in \Re^{d+1}$ such that $\mathbf{w}(k) \neq \mathbf{w}''(k)$ and $\mathbf{w}(k) + \frac{t(k) - Q(\mathbf{w}^T(k)\mathbf{x}(k))}{2}\mathbf{x}(k) = \mathbf{w}''(k) + \frac{t(k) - Q(\mathbf{w}''^T(k)\mathbf{x}(k))}{2}\mathbf{x}(k)$. This implies that $\mathbf{w}''(k) - \mathbf{w}(k) = \frac{Q(\mathbf{w}''^T(k)\mathbf{x}(k)) - Q(\mathbf{w}^T(k)\mathbf{x}(k))}{2}\mathbf{x}(k)$. This further implies that

$$Q(\mathbf{w}''^T(k)\mathbf{x}(k)) = -Q(\mathbf{w}^T(k)\mathbf{x}(k)) \text{ and } \mathbf{w}''(k) = \mathbf{w}(k) - Q(\mathbf{w}^T(k)\mathbf{x}(k))\mathbf{x}(k). \quad (9)$$

Consequently, we have

$$Q(\mathbf{w}''^T(k)\mathbf{x}(k)) = Q\left((\mathbf{w}(k) - Q(\mathbf{w}^T(k)\mathbf{x}(k))\mathbf{x}(k))^T \mathbf{x}(k)\right) = Q(\mathbf{w}^T(k)\mathbf{x}(k) - \mathbf{x}^T(k)Q(\mathbf{w}^T(k)\mathbf{x}(k))\mathbf{x}(k)). \quad (10)$$

As $Q(\mathbf{w}''^T(k)\mathbf{x}(k)) = -Q(\mathbf{w}^T(k)\mathbf{x}(k))$, if $\mathbf{w}^T(k)\mathbf{x}(k) > 0$, then $\mathbf{w}^T(k)\mathbf{x}(k) - \mathbf{x}^T(k)\mathbf{x}(k) < 0$. This implies that $\|\mathbf{x}(k)\|^2 > \mathbf{w}^T(k)\mathbf{x}(k)$. This further implies that $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$. If $\mathbf{w}^T(k)\mathbf{x}(k) < 0$, then $\mathbf{w}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k)\mathbf{x}(k) \geq 0$. This implies that $\|\mathbf{x}(k)\|^2 \geq -\mathbf{w}^T(k)\mathbf{x}(k)$. This further implies that $\|\mathbf{x}(k)\|^2 \geq |\mathbf{w}^T(k)\mathbf{x}(k)|$. Since $\|\mathbf{x}(k)\|^2 \neq |\mathbf{w}^T(k)\mathbf{x}(k)|$, we have $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$. If $\mathbf{w}^T(k)\mathbf{x}(k) = 0$, since we assume that $\mathbf{x}(k) \neq \mathbf{0}$, then we have $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$. Hence, this proves the sufficiency and it completes the proof. ■

Lemma 2 states that the necessary and sufficient condition for the forward dynamics of the weight of the perceptron being not injective is the square of the 2-norm of the input vectors being larger than the absolute value of the inner product of the weight and the input of the perceptron. When $\|\mathbf{x}(k)\|^2 > |\mathbf{w}^T(k)\mathbf{x}(k)|$, the forward dynamics of the weight of the perceptron is not injective. Hence, $\Im_k^F$ is not invertible and the backward dynamics of the weight of the perceptron is not uniquely defined.

This lemma also implies that the weight of the perceptron has to be within some neighborhood around the origin in order for the forward dynamics of the weight of the perceptron being not injective, and the sizes of the neighborhood depend on the magnitudes of the input vectors. If an invariant set exists and is attractive, then the invariant set has to be located within some neighborhood around the origin.

Corollary 1

Assume that $\|\mathbf{x}(k-1)\|^2 \neq |\mathbf{w}^T(k-1)\mathbf{x}(k-1)|$ and $\Im_k^F$ is not injective, then

$$\|\mathbf{x}(k-1)\|^2 > |\mathbf{w}^T(k-1)\mathbf{x}(k-1)|, \tag{11}$$

$$Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) = -Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \tag{12}$$

and

$$\mathbf{w}'(k-1) = \mathbf{w}(k-1) - Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1))\mathbf{x}(k-1). \tag{13}$$

Proof:

The result follows directly from Lemma 2, so the proof is omitted here. ■

Corollary 1 states that if there exist two weights $\mathbf{w}'(k-1)$ and $\mathbf{w}(k-1)$ that will map to the same weight $\mathbf{w}'(k) = \mathbf{w}(k)$, then the relationship between these two weights is governed by $\mathbf{w}'(k-1) = \mathbf{w}(k-1) - Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1))\mathbf{x}(k-1)$. Also, as the output of the perceptron corresponding to these two weights $\mathbf{w}'(k-1)$ and $\mathbf{w}(k-1)$ are $Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1))$ and $Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1))$, respectively, and the output of the perceptron is either "1" or "-1", Corollary 1 implies that the outputs of the perceptron corresponding to these two weights $\mathbf{w}'(k-1)$ and $\mathbf{w}(k-1)$ are different. Moreover, as there exist two weights $\mathbf{w}'(k-1)$ and $\mathbf{w}(k-1)$ that will map to the same weight $\mathbf{w}'(k) = \mathbf{w}(k)$, this implies that the forward dynamics of the weight of the perceptron is not injective.

According to Lemma 2 and Corollary 1, the square of the 2-norm of the input vectors is larger than the absolute value of the inner product of the weight and the input of the perceptron.

Now, it is ready to define an invariant set of the weight of the perceptron. It has been discussed in Section I that an invariant set of the weight of the perceptron is defined as the set of the downsampled weights that will map to itself. Define $\wp \equiv \{\mathbf{w}(qN) \quad \forall q \in Z$ such that $\forall j \in Z$ and $\forall n \in Z$

$$\mathbf{w}(jN) \neq \mathbf{w}(nN) + \sum_{p=0}^{N-1} \frac{Q\left(\left(\mathbf{w}(p+jN)\right)^T \mathbf{x}(p)\right) - Q\left(\left(\mathbf{w}(p+nN)\right)^T \mathbf{x}(p)\right)}{2} \mathbf{x}(p) \Bigg\}. \tag{14}$$

Define $\Im^F : \wp \to \wp$ such that $\Im^F(\mathbf{w}(qN)) \equiv \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}(qN)) \quad \forall \mathbf{w}(qN) \in \wp$. The following theorem reveals that the above definitions on $\wp$ and $\Im^F : \wp \to \wp$ actually correspond to an invariant set and an invariant map of the weight of the perceptron, respectively.

Theorem 1

$\Im^F$ is bijective and $\wp$ is an invariant set under the map $\Im^F$.

Proof:

As $\forall \mathbf{w}(qN) \in \wp$, $\Im^F(\mathbf{w}(qN)) = \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}(qN)) = \mathbf{w}((q+1)N) \in \wp \quad \forall q \in Z$, we have $\Im^F(\wp) \subseteq \wp$. As $\forall \mathbf{w}(qN) \in \wp$, $\exists \mathbf{w}((q-1)N) \in \wp$ such that

$$\Im^F(\mathbf{w}((q-1)N)) = \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}((q-1)N)) = \mathbf{w}(qN) \in \wp, \tag{15}$$

we have $\Im^F(\wp) \supseteq \wp$ and $\Im^F$ is surjective. Consequently, $\Im^F(\wp) = \wp$ and $\wp$ is an invariant set under the map $\Im^F$.

Assume that $\mathbf{w}(jN) \neq \mathbf{w}(nN)$ such that $\Im^F(\mathbf{w}(nN)) = \Im^F(\mathbf{w}(jN))$. This implies that

$$\mathbf{w}(jN) + \sum_{p=0}^{N-1} \frac{t(p) - Q\left(\left(\mathbf{w}(p+jN)\right)^T \mathbf{x}(p)\right)}{2} \mathbf{x}(p) = \mathbf{w}(nN) + \sum_{p=0}^{N-1} \frac{t(p) - Q\left(\left(\mathbf{w}(p+nN)\right)^T \mathbf{x}(p)\right)}{2} \mathbf{x}(p). \tag{16}$$

This further implies that

$$\mathbf{w}(jN) = \mathbf{w}(nN) + \sum_{p=0}^{N-1} \frac{Q\left(\left(\mathbf{w}(p+jN)\right)^T \mathbf{x}(p)\right) - Q\left(\left(\mathbf{w}(p+nN)\right)^T \mathbf{x}(p)\right)}{2} \mathbf{x}(p). \tag{17}$$

However, there is a contradiction. Consequently, $\Im^F$ is injective. As a result, $\Im^F$ is bijective and this completes the proof. ■

Theorem 1 states that the above definitions on $\wp$ and $\Im^F : \wp \to \wp$ actually correspond to an invariant set and an invariant map of the weight of the perceptron, respectively. This implies that the weight of the perceptron inside the invariant set will map to a weight inside the invariant set. In other words, the weights of the perceptrons are stayed within the invariant set if the initial weight is inside the invariant set. Hence, the local stability of the perceptron is guaranteed even though the set of the input vectors is nonlinearly separable. Besides, any weights inside the invariant set are guaranteed to be mapped by some weights inside the invariant set.

Although an invariant set is defined and proved in (14) and Theorem 1, respectively, it is worth to see if this invariant set would be empty or not. The following lemma addresses this issue.

Lemma 3

$\wp$ is nonempty.

Proof:

$\forall \mathbf{w}(0) \in \Re^{d+1}$, there always exists a sequence of vectors $\{\mathbf{w}(qN) \quad \forall q \in Z\}$ and this sequence of vectors $\{\mathbf{w}(qN) \quad \forall q \in Z\}$ consists of an infinite number of vectors. As an invariant set is a set defined as $\wp \equiv \{\mathbf{w}(qN) \quad \forall q \in Z$ such that $\forall j \in Z$ and $\forall n \in Z$ $\mathbf{w}(jN) \neq \mathbf{w}(nN) + \sum_{p=0}^{N-1} \frac{Q\left(\left(\mathbf{w}(p+jN)\right)^T \mathbf{x}(p)\right) - Q\left(\left(\mathbf{w}(p+nN)\right)^T \mathbf{x}(p)\right)}{2} \mathbf{x}(p)\Big\}$, if $\wp = \emptyset$, then this implies that there exists different two time indices j and n such that $j \neq n$, $\mathbf{w}(jN) \neq \mathbf{w}((j+1)N) \neq \mathbf{w}(nN)$, $\tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}(jN)) = \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}(nN)) = \mathbf{w}((j+1)N)$, $j+1 < n$ and $\tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F \circ \cdots \circ \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F(\mathbf{w}((j+1)N)) = \mathbf{w}(jN)$. Otherwise, all the vectors in the subsequence of vectors $\{\mathbf{w}(qN) \quad \forall q \leq n\}$ or in the subsequence of vectors $\{\mathbf{w}(qN) \quad \forall q \leq j\}$ could be removed from the original sequence of vectors and the new sequence of vectors forms a non-empty invariant set. However, as the forward dynamics of the weight of the perceptron is well defined, it is impossible that there exists two different vectors $\mathbf{w}((j+1)N)$ with one leads to the vector $\mathbf{w}(jN)$ and the other one leads to the vector $\mathbf{w}(nN)$. Hence, $\wp$ is nonempty and this completes the proof. ■

Lemma 3 clearly states that the invariant set defined by (14) is nonempty.

Now, it is ready to evaluate the dynamic range of the steady state values of the weight of the perceptron. The following corollary addresses this issue.

Corollary 2

The dynamic range of the steady state values of the weight of the perceptron is bounded by $\max_{\mathbf{w}(nN)\in\wp}\min_{\mathbf{w}(jN)\in\wp}\|\mathbf{w}(jN)-\mathbf{w}(nN)\|$.

Proof:

This result is trivial, so the proof is omitted here. ■

Corollary 2 gives the bound on the dynamic range of the steady state values of the weight of the perceptron, so it can be checked easily whether the perceptron satisfies the implementation and safety constraints or not.

The next question is whether the weight of the perceptron outside the invariant set will eventually move to the invariant set or not. In other words, is the invariant set attractive? The following lemma and theorem reveal that the invariant set is actually attractive.

Lemma 4

$\forall jN<k$ and $\forall nN<k$,

$$\mathbf{w}(nN)+\sum_{p=0}^{k-1-nN}\frac{t(p)-Q\left(\mathbf{w}^T(p+nN)\mathbf{x}(p)\right)}{2}\mathbf{x}(p)-\sum_{p=0}^{k-1-jN}\frac{t(p)-Q\left(\mathbf{w}'^T(p+jN)\mathbf{x}(p)\right)}{2}\mathbf{x}(p)\notin\wp. \quad (18)$$

Proof:

Since $\exists k\in Z$ such that $\mathbf{w}'(k)=\mathbf{w}(k)$, we have $\forall jN<k$ and $\forall nN<k$,

$$\mathbf{w}'(jN)=\mathbf{w}(nN)+\sum_{p=0}^{k-1-nN}\frac{t(p)-Q\left(\mathbf{w}^T(p+nN)\mathbf{x}(p)\right)}{2}\mathbf{x}(p)-\sum_{p=0}^{k-1-jN}\frac{t(p)-Q\left(\mathbf{w}'^T(p+jN)\mathbf{x}(p)\right)}{2}\mathbf{x}(p). \quad (19)$$

Suppose that $\mathbf{w}'(jN)\in\wp$, then $\exists p,q\in Z^+\cup\{0\}$ and $\exists m\in\{0,1,\cdots,N-1\}$ such that $\mathbf{w}'((j+p)N)\in\wp$, $\mathbf{w}((n+q)N)\in\wp$ and $\mathbf{w}'((j+p)N+m)=\mathbf{w}((n+q)N+m)=\mathbf{w}(k)$. This implies that $\mathbf{w}'((j+p+1)N)=\mathbf{w}((n+q+1)N)\in\wp$. However, it contradicts to Theorem 1. Hence,

$\mathbf{w}'(jN) \notin \wp$ and this completes the proof. ■

Lemma 4 states that if there exist two weights $\mathbf{w}'(jN)$ and $\mathbf{w}(nN)$ that will eventually map to the same weight at the time index k, that is $\mathbf{w}'(k) = \mathbf{w}(k)$, and if $\mathbf{w}(nN)$ is inside the invariant set of the weight of the perceptron, then $\mathbf{w}'(jN)$ is outside the invariant set. This lemma is important because it excludes some weights of the perceptron outside the invariant set so that it guarantees that the invariant map is bijective. The weights outside the invariant set will eventually move to the invariant set.

Define $\tilde{\Im}^F : \Re^{d+1} \to \Re^{d+1}$ such that $\tilde{\Im}^F(\mathbf{w}''(qN)) \equiv \tilde{\Im}^F_{N-1} \circ \cdots \tilde{\Im}^F_0(\mathbf{w}''(qN))$ $\forall \mathbf{w}''(qN) \in \Re^{d+1}$.

Theorem 2

$\tilde{\Im}^F$ is not injective.

Proof:

As $\mathbf{w}'(k) = \mathbf{w}(k)$, $\exists m \in \{0,1,\cdots,N-1\}$, $\exists \mathbf{w}'(jN) \notin \wp$ and $\exists \mathbf{w}(nN) \in \wp$ such that $\mathbf{w}'(jN+m) = \mathbf{w}(nN+m) = \mathbf{w}(k)$. Obviously, $\mathbf{w}'(jN) \neq \mathbf{w}(nN)$ and

$$\tilde{\Im}^F(\mathbf{w}'(jN)) = \mathbf{w}'((j+1)N) = \mathbf{w}((n+1)N) = \tilde{\Im}^F(\mathbf{w}(nN)). \quad (20)$$

Hence, $\tilde{\Im}^F$ is not injective and this completes the proof. ■

Theorem 2 states that $\tilde{\Im}^F$ is not injective. This implies that some initial weights outside the invariant set of the weight of the perceptron will eventually move to the invariant set. Hence, the invariant set is attractive. As if the weights are inside an invariant set, then they will stay inside the invariant set forever. If the weights are outside an invariant set, then these weights will move to the invariant set after certain iterations. Hence, a logic diagram can be used to represent the dynamics of the weight of the perceptron and the logic diagram is shown in Figure 1.

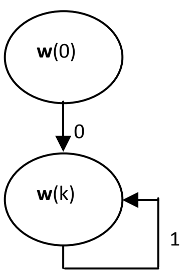

Figure 1. Logic diagram for the representation of the weight of the perceptron. The symbol "0" and

the symbol "1" represent whether the weights are outside and inside the invariant set, respectively.

Theorem 3

$\tilde{\Im}^F$ is surjective.

Proof:

$\forall \mathbf{w} \in \Re^{d+1}$, define $\mathbf{v} = \tilde{\Im}_1^B \circ \cdots \circ \tilde{\Im}_N^B(\mathbf{w})$. Obviously, $\mathbf{v} \in \Re^{d+1}$. By Lemma 1, we have $\tilde{\Im}^F(\mathbf{v}) = \tilde{\Im}_{N-1}^F \circ \cdots \circ \tilde{\Im}_0^F \circ \tilde{\Im}_1^B \circ \cdots \circ \tilde{\Im}_N^B(\mathbf{w}) = \mathbf{w}$. Hence, $\tilde{\Im}^F$ is surjective and this completes the proof. ■

Theorem 3 states that $\tilde{\Im}^F$ is surjective. This implies that for any arbitrary weight of the perceptron in the $d+1$ dimensional real-valued space, there always exist some weights in the same space that will map to that weight.

Since there exist initial weights outside the invariant set of the weight of the perceptron that will eventually move to the invariant set, it is important to identify these initial weights. The following lemma is to identify all possible output sequences of the perceptron that the initial weight is outside the invariant set but will eventually move to the invariant set.

Lemma 5

$$y'(0) = Q\left(\mathbf{w}^T(0)\mathbf{x}(0) + \frac{1}{2}\left(\begin{bmatrix} Q(\mathbf{w}'^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix} - \begin{bmatrix} Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}\right)^T \begin{bmatrix} \mathbf{x}^T(0)\mathbf{x}(0) \\ \vdots \\ \mathbf{x}^T(k-1)\mathbf{x}(0) \end{bmatrix}\right) \quad (21)$$

and

$$y'(j) = Q\left(\mathbf{w}^T(0)\mathbf{x}(j) + \frac{1}{2}\left(\begin{bmatrix} t(0) \\ \vdots \\ t(j-1) \\ Q(\mathbf{w}'^T(j)\mathbf{x}(j)) \\ \vdots \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix} - \begin{bmatrix} Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}\right)^T \begin{bmatrix} \mathbf{x}^T(0)\mathbf{x}(j) \\ \vdots \\ \mathbf{x}^T(k-1)\mathbf{x}(j) \end{bmatrix}\right) \quad \text{for } j = 1,2,\cdots,k-1. \quad (22)$$

Proof:

Since $y'(j) \equiv Q(\mathbf{w}'^T(j)\mathbf{x}(j))$ and $\mathbf{w}'(j+1) \equiv \tilde{\Im}_j^F(\mathbf{w}'(j))$ $\forall j \in Z$ as well as $\exists k \in Z$ such that $\mathbf{w}'(k) = \mathbf{w}(k)$, we have

$$\begin{aligned}&\mathbf{w}'(0)+\frac{1}{2}[\mathbf{x}(0),\quad \cdots,\quad \mathbf{x}(k-1)]\begin{bmatrix} t(0)-Q(\mathbf{w}'^T(0)\mathbf{x}(0)) \\ \vdots \\ t(k-1)-Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix} \\ &=\mathbf{w}(0)+\frac{1}{2}[\mathbf{x}(0),\quad \cdots,\quad \mathbf{x}(k-1)]\begin{bmatrix} t(0)-Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ t(k-1)-Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}.\end{aligned} \tag{23}$$

This further implies that

$$\mathbf{w}'(0)=\mathbf{w}(0)+\frac{1}{2}[\mathbf{x}(0),\quad \cdots,\quad \mathbf{x}(k-1)]\begin{bmatrix} Q(\mathbf{w}'^T(0)\mathbf{x}(0))-Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1))-Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix} \tag{24}$$

and

$$\mathbf{w}'(j)\equiv\mathbf{w}(0)+\frac{1}{2}[\mathbf{x}(0),\quad \cdots,\quad \mathbf{x}(k-1)]\left(\begin{bmatrix} t(0) \\ \vdots \\ t(j-1) \\ Q(\mathbf{w}'^T(j)\mathbf{x}(j)) \\ \vdots \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}-\begin{bmatrix} Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}\right) \quad \text{for} \quad j=1,2,\cdots,k-1. \tag{25}$$

As $y'(j)\equiv Q(\mathbf{w}'^T(j)\mathbf{x}(j))$ $\forall j\in Z$, the result follows directly and this completes the proof. ■

To evaluate $y'(k-1)$, as

$$y'(k-1)=Q\left(\mathbf{w}^T(0)\mathbf{x}(k-1)+\frac{1}{2}\left(\begin{bmatrix} t(0) \\ \vdots \\ t(k-2) \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}-\begin{bmatrix} Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}\right)^T\begin{bmatrix} \mathbf{x}^T(0)\mathbf{x}(k-1) \\ \vdots \\ \mathbf{x}^T(k-1)\mathbf{x}(k-1) \end{bmatrix}\right), \tag{26}$$

it can be seen easily that the above equation is satisfied if $y'(k-1)=y(k-1)$. However, the above equation may also be satisfied when $y'(k-1)=-y(k-1)$. Once all the possible values of $y'(k-1)$ are determined, then $y'(k-2)$ can also be determined as follows. As

$$y'(k-2)=Q\left(\mathbf{w}^T(0)\mathbf{x}(k-2)+\frac{1}{2}\left(\begin{bmatrix} t(0) \\ \vdots \\ t(k-3) \\ Q(\mathbf{w}'^T(k-2)\mathbf{x}(k-2)) \\ Q(\mathbf{w}'^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}-\begin{bmatrix} Q(\mathbf{w}^T(0)\mathbf{x}(0)) \\ \vdots \\ Q(\mathbf{w}^T(k-1)\mathbf{x}(k-1)) \end{bmatrix}\right)^T\begin{bmatrix} \mathbf{x}^T(0)\mathbf{x}(k-2) \\ \vdots \\ \mathbf{x}^T(k-1)\mathbf{x}(k-2) \end{bmatrix}\right), \tag{27}$$

all possible values of $y'(k-1)$ have already been determined and $y'(k-2)\in\{-1,1\}$, all possible values of $y'(k-2)$ could be determined accordingly. Similarly, all possible values of $y'(j)$ for $j=0,1,\cdots,k-1$ could be determined accordingly. Hence, all possible output sequences of the perceptron that the initial weight is outside the invariant set but will eventually move to the invariant set could be identified.

Plotting the state trajectory on the phase diagram is a very important technique for the understanding of the dynamics of nonlinear systems. The following lemma describes an interesting property of the state trajectory of the weight of the perceptron.

Lemma 6

$$w_0(n)-w_0(0)\in Z \quad \forall n\in Z .$$

Proof:

Since $\mathbf{w}(n)=\mathbf{w}(0)+\sum_{k=0}^{n-1}\frac{t(k)-y(k)}{2}\mathbf{x}(k)$ $\forall n\in Z$, the first element of $\mathbf{x}(n)$ is 1 $\forall n\in Z$ and $\frac{t(n)-y(n)}{2}\in\{1,0,-1\}$ $\forall n\in Z$, the result follows directly and this completes the proof. ■

Lemma 6 states that the difference of the thresholds of the weight between any time indices and the initial time index is always an integer. This implies that the weight occurs only at certain hyperplanes and no weight can be found between these hyperplanes.

To illustrate the developed theory, three different types of examples are shown below. The first type of examples illustrates the exhibition of the fixed point behavior, the second type of examples illustrates the exhibition of the limit cycle behavior, while the last type of examples illustrates the exhibition of the chaotic behavior. For the first type of examples, in order for the weights to exhibit the fixed point behavior, the necessary and sufficient condition is that the sets of the input vectors are linearly separable. Actually, this necessary and sufficient condition does not directly relate to the values of the input vectors (on the condition that the sets of the input vectors are linearly separable). However, in terms of the illustration purpose, simple input vectors, such as the elements of the input vectors are either 1 or -1, are employed for the illustration. Consider the following set of the input vectors $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$. Assume that the corresponding set of the desirable outputs is $\{1,-1,1,-1\}$. Also, assume that $\mathbf{w}(0)=[0,\quad 2,\quad 0]^T$. It can be verified that $\mathbf{w}(k)=[0,\quad 2,\quad 0]^T\quad \forall k\in Z$. Hence, the set of the weights of the perceptron only contains a single weight, that is $\{[0,\quad 2,\quad 0]^T\}$, and the dynamics of the weight of the perceptron exhibits a fixed point behavior. The invariant set of the weight of the perceptron also consists of a single weight, that is $\wp=\{[0,\quad 2,\quad 0]^T\}$. It is trivial to see that the invariant map $\Im^F:\{[0,\quad 2,\quad 0]^T\}\to\{[0,\quad 2,\quad 0]^T\}$ is bijective because the invariant set only contains a single element and the mapping is just a one to one mapping. However, the map $\tilde{\Im}^F:\Re^3\to\Re^3$ is not injective because $\|\mathbf{x}(k)\|^2>|\mathbf{w}^T(k)\mathbf{x}(k)|\quad \forall k\in Z$. Hence, some weights outside the invariant set, such as $[1,\quad 1,\quad -1]^T$, would converge to the invariant set. In other words, the invariant set is attractive.

Now, consider another example that the weight of the perceptron also exhibits a fixed point behavior. Suppose that the set of the input vectors is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$ and the corresponding set of the desirable outputs is $\{1,-1,1,-1\}$. Also, assume that $\mathbf{w}(0)=[-1,\quad 1,\quad 1]^T$. It can be verified that $\mathbf{w}(0)=\mathbf{w}(1)=\mathbf{w}(2)=[-1,\quad 1,\quad 1]^T$ and $\mathbf{w}(k)=[0,\quad 2,\quad 0]^T\quad \forall k\geq 3$. Hence, the set of the downsampled weights of the perceptron is $\{[-1,\quad 1,\quad 1]^T,[0,\quad 2,\quad 0]^T\}$. As both the weights $[-1,\quad 1,\quad 1]^T$ and $[0,\quad 2,\quad 0]^T$ map to the same weight $[0,\quad 2,\quad 0]^T$, according to (14), the weight $[-1,\quad 1,\quad 1]^T$ is removed from the set $\{[-1,\quad 1,\quad 1]^T,[0,\quad 2,\quad 0]^T\}$ and the new set $\{[0,\quad 2,\quad 0]^T\}$ forms an invariant set of the weight of the perceptron. As this invariant set is the same as that in the previous example, it can be seen easily that this invariant set is attractive.

For the second type of examples, in order for the weights to exhibit the limit cycle behavior, the most common well known example is the XOR example. Hence, the elements of the input vectors are selected as either 1 or -1 and the corresponding desirable outputs are chosen in such a way that the input vectors and the corresponding desirable outputs correspond to the XOR truth table. Consider the following example. Suppose that the set of the input vectors is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$. Assume that the corresponding set of the desirable outputs is $\{1,-1,1,-1\}$.

Also, assume that $\mathbf{w}(0)=[-1,\ \ -1,\ \ -1]^T$. It can be verified that the set of the weights of the perceptron is $\left\{\begin{bmatrix}-1\\-1\\-1\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}-1\\-1\\1\end{bmatrix},\begin{bmatrix}0\\-2\\0\end{bmatrix}\right\}$, that is $\mathbf{w}(4k)=[-1,\ \ -1,\ \ -1]^T$, $\mathbf{w}(4k+1)=[0,\ \ 0,\ \ 0]^T$, $\mathbf{w}(4k+2)=[-1,\ \ -1,\ \ 1]^T$ and $\mathbf{w}(4k+3)=[0,\ \ -2,\ \ 0]^T$ $\forall k\in Z$. The dynamics of the weight of the perceptron exhibits a limit cycle behavior with period 4. The set of the downsampled weights of the perceptron consists of a single weight, which is $\left\{\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}$. As the invariant set of the weight of the perceptron is defined as the set of the downsampled weights that maps to itself, the invariant set of the weight of the perceptron also consists of a single weight, that is $\wp=\left\{\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}$. It is trivial to see that the invariant map $\Im^F:\left\{\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}\to\left\{\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}$ is bijective because the invariant set only contains a single element and the map is just a one to one mapping. However, the map $\tilde{\Im}^F:\Re^3\to\Re^3$ is not injective because $\|\mathbf{x}(k)\|^2\geq\left|\mathbf{w}^T(k)\mathbf{x}(k)\right|$ $\forall k\in Z$. Hence, some weights outside the invariant set, such as $[0, \ 0, \ 0]^T$, would converge to the invariant set. In other words, the invariant set is attractive.

Now, consider another example that the weight of the perceptron also exhibits a limit cycle behavior. Suppose that the set of the input vectors is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the corresponding set of the desirable outputs is $\{1,-1,1,-1\}$. Also, assume that $\mathbf{w}(0)=[0, \ 0, \ 0]^T$. It can be verified that $\mathbf{w}(0)=\mathbf{w}(1)=[0, \ 0, \ 0]^T$, $\mathbf{w}(2)=[-1, \ -1, \ 1]^T$, $\mathbf{w}(3)=[0, \ -2, \ 0]^T$, $\mathbf{w}(4k)=[-1, \ -1, \ -1]^T \quad \forall k \geq 1$, $\mathbf{w}(4k+1)=[0, \ 0, \ 0]^T \quad \forall k \geq 1$, $\mathbf{w}(4k+2)=[-1, \ -1, \ 1]^T \quad \forall k \geq 1$ and $\mathbf{w}(4k+3)=[0, \ -2, \ 0]^T \quad \forall k \geq 1$. Hence, the set of the downsampled weights of the perceptron is $\{[0, \ 0, \ 0]^T, [-1, \ -1, \ -1]^T\}$. As both the weights $[0, \ 0, \ 0]^T$ and $[-1, \ -1, \ -1]^T$ map to the same weight $[-1, \ -1, \ -1]^T$, according to (14), the weight $[0, \ 0, \ 0]^T$ is removed from the set $\{[0, \ 0, \ 0]^T, [-1, \ -1, \ -1]^T\}$ and the new set $\{[-1, \ -1, \ -1]^T\}$ forms an invariant set of the weight of the perceptron. As this invariant set is the same as that in the previous example, it can be seen easily that this invariant set is attractive.

Finally, the last example is to illustrate the exhibition of the chaotic behavior of the weight of the perceptron. As $\mathbf{w}(n+1)=\mathbf{w}(n)+\frac{t(n)-Q\left((\mathbf{w}(n))^T\mathbf{x}(n)\right)}{2}\mathbf{x}(n)$ and the values of $\frac{t(n)-Q\left((\mathbf{w}(n))^T\mathbf{x}(n)\right)}{2}$ are in the set $\{-1,0,1\}$, $\mathbf{w}(n)$ is the sum of $\mathbf{w}(0)$ and the integer combinations of $\mathbf{x}(n)$. In order for the weights to exhibit the chaotic behavior, the weights could not exhibit a periodic behavior. One way to achieve this condition is that the elements of $\mathbf{w}(0)$ and $\mathbf{x}(n)$ are irrational numbers and relatively prime. In this case, $\mathbf{w}(n)$ could not exhibit the limit cycle behavior. As it is shown in [24] that the weights of the perceptron are bounded, so in this case the weights will most likely exhibit the chaotic behavior. Hence, the elements of $\mathbf{w}(0)$ and $\mathbf{x}(n)$

are chosen as irrational numbers rounded by certain numbers of significant figures. Assume that the set of the input vectors is $\left\{\begin{bmatrix}1\\0.1746\\-0.1867\end{bmatrix},\begin{bmatrix}1\\0.7258\\-0.5883\end{bmatrix},\begin{bmatrix}1\\2.1832\\-0.1364\end{bmatrix},\begin{bmatrix}1\\0.1139\\1.0668\end{bmatrix}\right\}$ and the corresponding set of the desirable outputs is $\{1,-1,1,-1\}$. Also, assume that $\mathbf{w}(0)=[-1, \quad 0.7923, \quad -0.2133]^T$. It can be verified that the set of the weights of the perceptron consists of three hyperplanes as shown in Figure 2. The dynamics of the weight of the perceptron exhibits a chaotic behavior. The invariant set of the weight of the perceptron also consists of these three hyperplanes. It can be checked easily that the map from the invariant set to itself is bijective but the map $\tilde{\Im}^F:\Re^3\to\Re^3$ is not injective because $\exists k\in Z$ such that $\|\mathbf{x}(k)\|^2>|\mathbf{w}^T(k)\mathbf{x}(k)|$. Hence, some weights outside the invariant set, such as $[-1, \quad -1, \quad 0]^T$, would converge to the invariant set. In other words, the invariant set is attractive.

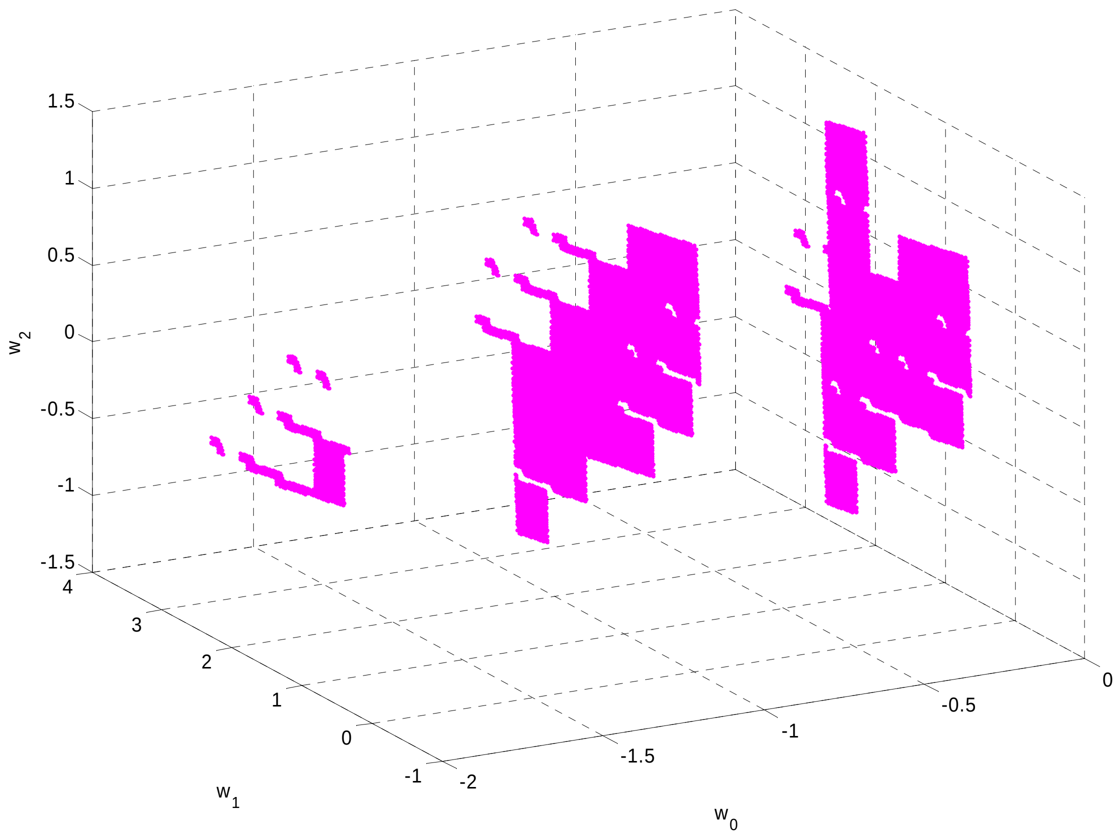

Figure 2. Phase diagram of the weights of the perceptron when

$$\mathbf{x}(k) \in \left\{ \begin{bmatrix} 1 \\ 0.1746 \\ -0.1867 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.7258 \\ -0.5883 \end{bmatrix}, \begin{bmatrix} 1 \\ 2.1832 \\ -0.1364 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.1139 \\ 1.0668 \end{bmatrix} \right\}, \quad \mathbf{w}(0) = [-1, \quad 0.7923, \quad -0.2133]^T \quad \text{and}$$

$$t(k) \in \{1, -1, 1, -1\}.$$

IV. CONCLUSIONS

In this paper, an invariant set of the weight of the perceptron is defined as a set of the downsampled weights that maps to itself. In order to investigate the dynamic range of the steady state values of the weight of the perceptron, first a backward dynamics of the weights of the perceptron is defined. Based on the definition of the backward dynamics of the weights of the perceptron, it is shown in this paper that the forward dynamics of the weight of the perceptron is in general not injective and the necessary and sufficient condition for the forward dynamics of the weight of the perceptron to be injective is characterized. As a result, the set of the weight of the perceptron that the forward dynamics is injective is characterized and it is shown that this set of the weight of the perceptron is actually a nonempty invariant set in which the map that maps this invariant set to itself is a bijective map. Consequently, the dynamic range of the steady state values of the weight of the perceptron can be evaluated via finding the dynamic range of the weight of the perceptron inside the largest invariant set of the weight of the perceptron. Finally, all possible output sequences of the perceptron in which the initial weights outside the invariant set will eventually move to the invariant set are identified.

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