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# Description of Vortical Flows of Incompressible Fluid in Terms of Quasi-Potential Function

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### Abstract

It has been shown [1, 2] that a wide class of 3D motions of incompressible viscous fluid in Cartesian coordinates can be described by only one scalar function dubbed the quasi-potential. This class of fluid flows is characterized by three-component velocity field having two-component vorticity field; both these fields can depend of all three spatial variables and time, in general. Governing equations for the quasi-potential have been derived and simple illustrative examples of 3D flows have been presented. In this paper the concept of quasi-potential is further developed for fluid flows in cylindrical coordinates. It is shown that the introduction of a quasi-potential in curvilinear coordinates is non-trivial and may be a subject of additional restrictions. In the cases when it is possible, we construct illustrative examples which can be of interest for some practical applications.

### Introduction

A great success in the solution of fluid dynamic problems is associated with the reduction of a primitive set of hydrodynamic equations to only one equation for any scalar function, e.g., velocity potential or stream function [3, 4, 5, 6, 7]. It has been shown in [1, 2] that the class of exactly solvable hydrodynamic problems can be widened by introduction one more scalar function, the quasi-potential. The starting point for the introduction of the quasi-potential is the condition of incompressibility of a fluid: $\operatorname{div}\mathbf{U} = 0$, where $\mathbf{U}$ is the fluid velocity. This allows us to introduce a vector-potential $\mathbf{A}$ such that $\mathbf{U} = \operatorname{curl}\mathbf{A}$ automatically satisfies this equation. However, there is a gauge invariance in the choice of the vector-potential as it is defined up to the gradient of any scalar function $f(t,x,y,z)$, because $\operatorname{curl}(\nabla f) \equiv 0$. Therefore, if we use another vector-potential $\mathbf{A}' = \mathbf{A} + \nabla f$, this does not affect the velocity vector $\mathbf{U}$.

Due to the freedom of choice of an arbitrary function $f(t,x,y,z)$, we can eliminate one of the components of the vector-potential. Therefore without loss of generality the vector potential $\mathbf{A}$ can be chosen consisting of two components only. In the Cartesian rectilinear coordinates it does not matter which component is eliminated, because vector differential operations are symmetrical with respect to all spatial variables $x$, $y$ and $z$. However, it is not the case in curvilinear coordinates, in particular, in cylindrical coordinates. In any case, an arbitrary 3D velocity field can be described, in general, by two-component vector-potential, i.e. by two scalar functions – the corresponding components of the vector-potential. If there is any additional link between these two components, then the description of a fluid flow can be done in terms of only one scalar function, equation for which follows from the primitive Navier–Stokes equation. This approach has been exploited in [1, 2] in Cartesian coordinates and nontrivial examples of fluid motions have been found.

Below we consider fluid flow in cylindrical coordinates and show how the quasi-potential can be introduced when one of the components of the vector-potential $\mathbf{A}$ is eliminated. An illustrative example is constructed.

### Governing equations and quasi-potential in cylindrical coordinates

Case 1 – Derivation of basic equations when $A_r = 0$

Consider first the case when the vector-potential does not contain the radial component $A_r$ and reads: $\mathbf{A} = (F_1/r)\mathbf{e}_\varphi - F_2\mathbf{e}_z$, where $F_1$ and $F_2$ are functions of time and all spatial variables.

The velocity and vorticity fields for such vector-potential read:

$$\mathbf{U} = F_3\mathbf{e}_r + \frac{\partial F_2}{\partial r}\mathbf{e}_\varphi + \frac{1}{r}\frac{\partial F_1}{\partial r}\mathbf{e}_z, \quad F_3 = \frac{-1}{r}\left(\frac{\partial F_2}{\partial \varphi} + \frac{\partial F_1}{\partial z}\right), \quad (1)$$

$$\omega = \left(\frac{1}{r^2}\frac{\partial^2 F_1}{\partial r \partial \varphi} - \frac{\partial^2 F_2}{\partial r \partial z}\right)\mathbf{e}_r +$$
$$\left[\frac{\partial F_3}{\partial z} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_1}{\partial r}\right)\right]\mathbf{e}_\varphi + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial F_2}{\partial r}\right) - \frac{\partial F_3}{\partial \varphi}\right]\mathbf{e}_z. \quad (2)$$

Consider a particular class of fluid flows having only two components of the vorticity. To eliminate the first component of the vorticity let us introduce such function $P$ that

$$\frac{1}{r^2}\frac{\partial F_1}{\partial r} = \frac{\partial P}{\partial z}, \quad \frac{\partial F_2}{\partial r} = \frac{\partial P}{\partial \varphi}. \quad (3)$$

Thus, the assumption that the $r$-component of vorticity is zero implies that $P$ is the potential function of $\varphi$ and $z$.

It is convenient to introduce further a quasi-potential $\Phi$ such that $P = \Phi_r$ (here and below the indices of function $\Phi$ stand for partial derivatives with respect to the corresponding variables). Substitute then expressions for $\mathbf{U}$ and $\omega$ into the Navier–Stokes equation in the Helmholtz form [3]:

$$\frac{\partial \omega}{\partial t} + \operatorname{curl}[\omega \times \mathbf{U}] = \nu\Delta\omega. \quad (4)$$

Bearing in mind that thankful to the condition (3) the $r$-component of the vorticity is zero, in the case of perfect fluid ($\nu = 0$) the $r$-component of this vector equation reduces to:

$$\Delta\Phi + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(H - \frac{\partial \Phi}{\partial r}\right)\right] = 0. \quad (5)$$

where $H(\Phi_r)$ is an arbitrary function of $\Phi_r$.

Two other components ($\varphi$ and $z$) of Eq. (4) after simple manipulations reduce to one equation:

$$\left(H' - 1\right)\frac{\partial}{\partial r}\left[\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2}\right] +$$
$$\left(H' - 1\right)\left[\left(H - \frac{\partial \Phi}{\partial r}\right)\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^3}\left(\frac{\partial \Phi}{\partial \varphi}\right)^2\right] = Q(t,r), \quad (6)$$

where $Q(t,r)$ is an arbitrary function of its arguments, and $H'$ stands for a derivative of function $H$ with respect to its argument.

In terms of function $\Phi$ the velocity and vorticity fields read:

$$\mathbf{U} = \nabla\Phi + [H(\Phi_r) - \Phi_r]\,\mathbf{e}_r\,, \tag{7}$$

$$\omega = (H' - 1)\left(\frac{\partial^2\Phi}{\partial r\partial z}\,\mathbf{e}_\varphi - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\varphi}\,\mathbf{e}_z\right). \tag{8}$$

In the particular case of $H(\Phi_r) \equiv \Phi_r$, the quasi-potential $\Phi$ becomes the conventional hydrodynamic velocity potential. Equation (5) reduces to the Laplace equation, Eq. (6) disappears, and vorticity vanishes.

In general the main equations to be solved simultaneously are Eqs. (5) and (6) with given functions $H(\Phi_r)$ and $Q(t, r)$. As there is a freedom in the choice of these function, we obtain a good perspective to construct exact solutions to hydrodynamic equations and accommodate them to practical needs.

Unfortunately, in the case of a viscous fluid ($\nu \neq 0$) the introduction of a quasi-potential does not help to simplify the basic equation (4).

### Case 2 – Derivation of basic equations when $A_\varphi = 0$

Consider now the case when the vector potential has only $r$- and $z$-components: $\mathbf{A} = -rF_1\,\mathbf{e}_r + rF_2\,\mathbf{e}_z$. In this case the velocity and vorticity fields become:

$$\mathbf{U} = \frac{\partial F_2}{\partial\varphi}\,\mathbf{e}_r + \frac{F_3}{r}\,\mathbf{e}_\varphi + \frac{\partial F_1}{\partial\varphi}\mathbf{e}_z\,, \tag{9}$$

where $F_3 = -r\left[\frac{\partial(rF_1)}{\partial z} + \frac{\partial(rF_2)}{\partial r}\right]$,

$$\omega = \frac{1}{r}\left(\frac{\partial^2 F_1}{\partial\varphi^2} - \frac{\partial F_3}{\partial z}\right)\mathbf{e}_r +$$

$$\frac{\partial}{\partial\varphi}\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial r}\right)\mathbf{e}_\varphi + \frac{1}{r}\left(\frac{\partial F_3}{\partial r} - \frac{\partial^2 F_2}{\partial\varphi^2}\right)\mathbf{e}_z. \tag{10}$$

Let us eliminate now the second component of the vorticity. To this end introduce such function $P$ that

$$\frac{\partial^2 F_2}{\partial\varphi^2} - \frac{\partial F_3}{\partial r} = \frac{\partial P}{\partial r}, \quad \frac{\partial^2 F_1}{\partial\varphi^2} - \frac{\partial F_3}{\partial z} = \frac{\partial P}{\partial z}. \tag{11}$$

In the case of a perfect fluid the second $\varphi$-component of vector equation (4) reduces to:

$$\Delta\left[P + r^2 H(P)\right] = \frac{1}{r^2}\,\frac{\partial^2 P}{\partial\varphi^2}, \tag{12}$$

where $H(P)$ is an arbitrary function of $P$.

This equation can be presented in terms of a quasi-potential $\Phi$ which is defined as $P + r^2 H(P) = \partial\Phi/\partial\varphi$. Alternatively, one can think that $P$ is an arbitrary function $G$ of $\Phi_\varphi$ and $r$, viz. $P = G(\Phi_\varphi, r)$. Then Eq. (12) reads:

$$\Delta\Phi - \frac{1}{r^2}\frac{\partial G(\Phi_\varphi, r)}{\partial\varphi} = R(t, r, z), \tag{13}$$

where $R(t, r, z)$ is an arbitrary function of its arguments.

Two other components ($r$ and $z$) of Eq. (4) after simple manipulations can be reduced to the following one equation:

$$\frac{\partial G}{\partial t} + \nabla\Phi\cdot\nabla G - \frac{G}{r^2}\frac{\partial G}{\partial\varphi} = Q(t, \varphi), \tag{14}$$

where $Q(t, \varphi)$ is an arbitrary function of its arguments.

In terms of quasi-potential the velocity and vorticity fields read:

$$\mathbf{U} = \nabla\Phi - \frac{1}{r}G(\Phi_\varphi, r)\,\mathbf{e}_\varphi\,, \tag{15}$$

$$\omega = \frac{1}{r}\left(\frac{\partial G}{\partial z}\,\mathbf{e}_r - \frac{\partial G}{\partial r}\,\mathbf{e}_z\right). \tag{16}$$

In the particular case of $G(\Phi_\varphi, r) \equiv 0$, the quasi-potential $\Phi$ becomes $\varphi$-independent conventional hydrodynamic velocity potential. Equation (13) reduces to the Laplace equation with $R(t, r, z) \equiv 0$, Eq. (14) disappears, provided that $Q(t, \varphi) \equiv 0$, and vorticity vanishes. The velocity field becomes plane and depends only of $r$ and $z$.

In general the main equations to be solved for $\Phi$ simultaneously are Eqs. (13) and (14) with given functions $G(\Phi_\varphi, r)$, $Q(t, \varphi)$, and $R(t, r, z)$. Despite of the more complex character of these equations, the freedom of choice of arbitrary functions allows us to obtain again a good perspective to construct exact solutions to hydrodynamic equations and accommodate them to the practical needs.

Unfortunately, in the case of a viscous fluid ($\nu \neq 0$) the introduction of a quasi-potential does not help to simplify the basic equation (4).

### Case 3 – Derivation of basic equations when $A_z = 0$

Consider at last the case when the vector potential has only $r$- and $\varphi$-components: $\mathbf{A} = (F_1/r)\,\mathbf{e}_r - F_2\,\mathbf{e}_\varphi$.

In this case the velocity and vorticity fields become:

$$\mathbf{U} = \frac{\partial F_2}{\partial z}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial F_1}{\partial z}\,\mathbf{e}_\varphi + F_3\,\mathbf{e}_z\,, \tag{17}$$

where $F_3 = -\frac{1}{r}\left(\frac{\partial rF_2}{\partial r} + \frac{1}{r}\frac{\partial F_1}{\partial\varphi}\right)$,

$$\omega = \frac{1}{r}\left(\frac{\partial F_3}{\partial\varphi} - \frac{\partial^2 F_1}{\partial z^2}\right)\mathbf{e}_r + \left(\frac{\partial^2 F_2}{\partial z^2} - \frac{\partial F_3}{\partial r}\right)\mathbf{e}_\varphi\,, \tag{18}$$

where we have eliminated already the third component of vorticity assuming that there is a relationship between functions $F_1$ and $F_2$: $\partial F_1/\partial r = \partial F_2/\partial\varphi$. This relationship is satisfied automatically if we introduce a quasi-potential $\Phi$ such that

$$\frac{\partial F_1}{\partial z} = \frac{\partial\Phi}{\partial\varphi}\,, \quad \frac{\partial F_2}{\partial z} = \frac{\partial\Phi}{\partial r}.$$

Substituting the expressions for $\mathbf{U}$ and $\omega$ into Eq. (4), we obtain from the third, $z$-component, of this vector equation even with the viscous term accounted for:

$$\Delta\Phi + \frac{\partial}{\partial z}\left(H - \frac{\partial\Phi}{\partial z}\right) = R(t, r, \varphi). \tag{19}$$

where $H(\Phi_z)$ is an arbitrary function of $\Phi_z$ and $R(t, r, \varphi)$ is an arbitrary function of its arguments.

Two other components ($r$ and $\varphi$) of Eq. (4) with $\nu \neq 0$ after simple manipulations can be reduced to the following one equation:

$$\frac{\partial}{\partial t}\left(H - \frac{\partial\Phi}{\partial z}\right) + \nabla\Phi\cdot\nabla\left(H - \frac{\partial\Phi}{\partial z}\right) +$$

$$\frac{1}{2}\frac{\partial}{\partial z}\left(H - \frac{\partial\Phi}{\partial z}\right)^2 - \nu\Delta\left(H - \frac{\partial\Phi}{\partial z}\right) = Q(t, z), \tag{20}$$

where $Q(t, z)$ is an arbitrary function of its arguments.

In terms of quasi-potential the velocity and vorticity fields read:

$$\mathbf{U} = \nabla\Phi + \left(H - \frac{\partial\Phi}{\partial z}\right)\mathbf{e}_z, \tag{21}$$

$$\omega = \frac{1}{r}\frac{\partial}{\partial\varphi}\left(H - \frac{\partial\Phi}{\partial z}\right)\mathbf{e}_r - \frac{\partial}{\partial r}\left(H - \frac{\partial\Phi}{\partial z}\right)\mathbf{e}_\varphi. \tag{22}$$

In the particular case of $H(\Phi_z) \equiv \Phi_z$, the quasi-potential $\Phi$ becomes the conventional hydrodynamic velocity potential. Equation (19) reduces to the Laplace equation with $R(t, r, \varphi) \equiv 0$, Eq. (20) disappears, and vorticity vanishes. In the case of viscous fluid there is no additional restrictions.

Below we present an example of non-trivial fluid flow described by these Eqs. (19) and (20).

### Examples

We managed to construct two examples illustrating the developed theory.

#### Example of a vertical flow in Case 2

Consider first an example for the case when the vector potential has only $r$- and $z$-components in cylindrical coordinates. Let us assume that $G(\Phi_\varphi, r) \equiv (1-\lambda^2)\Phi_\varphi$, where $\lambda \neq 1$ is a constant.

One can readily prove that the following quasi-potential

$$\Phi = \left(\frac{\sqrt{r^2+z^2}+z}{r}\right)^\lambda \sin\varphi \tag{23}$$

satisfies Eqs. (13) and (14). Then the velocity and vorticity fields read:

$$\mathbf{U} = \left(\frac{\sqrt{r^2+z^2}+z}{r}\right)^\lambda \frac{\lambda}{r\sqrt{r^2+z^2}}\Big[-z\sin\varphi\,\mathbf{e}_r + \lambda\sqrt{r^2+z^2}\cos\varphi\,\mathbf{e}_\varphi + r\sin\varphi\,\mathbf{e}_z\Big], \tag{24}$$

$$\omega = \left(\frac{\sqrt{r^2+z^2}+z}{r}\right)^\lambda \frac{\lambda(1-\lambda^2)\cos\varphi}{r^2\sqrt{r^2+z^2}}\left(r\,\mathbf{e}_r + z\,\mathbf{e}_z\right). \tag{25}$$

This solution describes stationary vortical flow periodic in azimuthal coordinate $\varphi$ and having two components of vorticity. When $\lambda = 0$, the velocity and vorticity fields vanish, whereas when $\lambda = 1$, the flow becomes potential with the zero vorticity.

In Cartesian coordinates both the velocity and vorticity fields are three-component and read:

$$\mathbf{U} = \left(\frac{R+z}{\sqrt{x^2+y^2}}\right)^\lambda \frac{\lambda}{R\left(x^2+y^2\right)^{3/2}} \times \Big[-xy\left(\lambda R+z\right)\mathbf{i} + \left(\lambda x^2 R - y^2 z\right)\mathbf{j} + y\left(x^2+y^2\right)\mathbf{k}\Big], \tag{26}$$

where $R = \sqrt{x^2+y^2+z^2}$;

$$\omega = \left(\frac{R+z}{\sqrt{x^2+y^2}}\right)^\lambda \frac{\lambda\left(1-\lambda^2\right)x}{R\left(x^2+y^2\right)^{3/2}}\left(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\right). \tag{27}$$

Figures 1 and 2 illustrate the velocity and vorticity fields as per Eqs. (26) and (27).

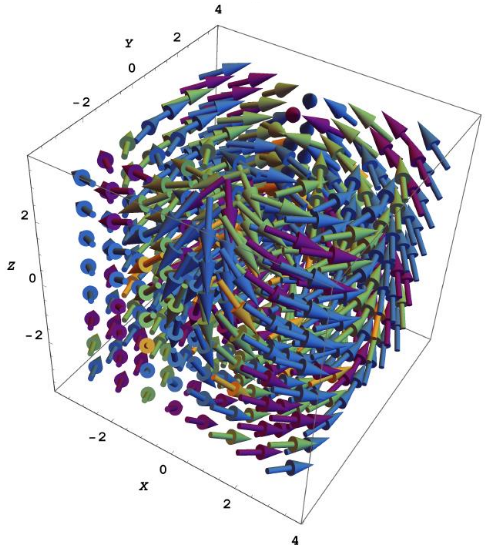


Figure 1: A fragment of velocity field as per Eq. (26).

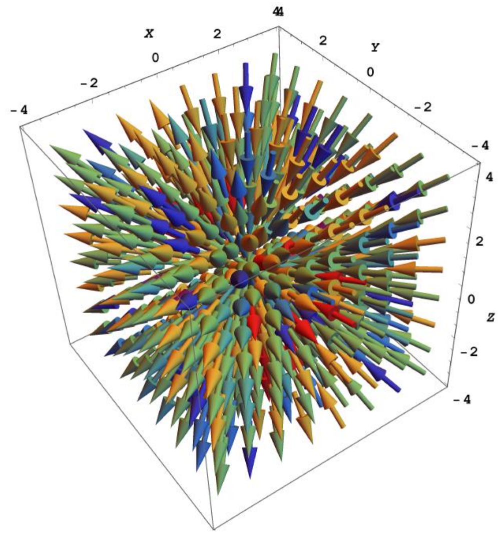


Figure 2: A fragment of vorticity field as per Eq. (27).

The modulus of the velocity field is:

$$|\mathbf{U}| = \lambda\left(\frac{R+z}{\sqrt{x^2+y^2}}\right)^\lambda \frac{\sqrt{\lambda^2x^2+y^2}}{\left(x^2+y^2\right)}. \tag{28}$$

And the modulus of the vorticity field is:

$$|\omega| = \lambda\left(1-\lambda^2\right)\left(\frac{R+z}{\sqrt{x^2+y^2}}\right)^\lambda \frac{x}{\left(x^2+y^2\right)^{3/2}}. \tag{29}$$

#### Example of a vertical flow in Case 3

Consider now an example of the vortical flow for the third case when the vector potential has only $r$- and $\varphi$-components in cylindrical coordinates. Let us assume that $H(\Phi_z) \equiv -\lambda^2\Phi_z$, where $\lambda \neq 1$ is a constant.

One can readily prove that the following quasi-potential

$$\Phi = \frac{T(t)}{\lambda}\cos\left(\lambda r\sin\varphi\right)\cos z, \tag{30}$$

where $T(t) = C\exp\left[-\left(1+\lambda^2\right)\nu t\right]$, satisfies Eqs. (19) and (20). Then the velocity and vorticity fields read:

$$\mathbf{U} = -T(t)\left[\cos z\,\sin\left(r\lambda\sin\varphi\right)\left(\sin\varphi\,\mathbf{e}_r + \cos\varphi\,\mathbf{e}_\varphi\right) - \right.$$

$$\lambda \sin z \cos(r\lambda \sin\varphi)\,\mathbf{e}_z], \tag{31}$$

$$\omega = \left(1+\lambda^2\right) T(t) \sin z \cos\varphi \sin(r\lambda \sin\varphi)\left(-\mathbf{e}_r + r\,\mathbf{e}_\varphi\right). \tag{32}$$

This solution describes exponentially decaying with time velocity and vorticity fields. The vector fields are periodic in $\varphi$ and $z$. In Cartesian coordinates the velocity field becomes two-component:

$$\mathbf{U} = -\lambda T(t)\left(\sin\lambda y \cos z\,\mathbf{j} - \lambda \cos\lambda y \sin z\,\mathbf{k}\right), \tag{33}$$

whereas the vorticity field has only one component:

$$\omega = -\lambda\left(1+\lambda^2\right) T(t) \sin(\lambda y)\, \sin z\,\mathbf{i}. \tag{34}$$

Both these fields do not depend on $x$ and therefore can be described by the conventional stream function $\psi = -\lambda T(t) \sin\lambda y \sin z$, so that the velocity components are:

$$U_y = \frac{\partial\psi}{\partial z}, \quad U_z = -\frac{\partial\psi}{\partial y}.$$

For purely imaginary $\lambda = i$, the quasi-potential reduces to the conventional hydrodynamic potential, and the fluid field becomes potential with the zero vorticity.

Thus, this example describes a vortical flow double periodic in the $(y, z)$ plane. Figure 3 illustrates the velocity field for $t = 0$ and $\lambda = 1$ when $y$ and $z$ vary in the range of $[0, \pi]$. There is a mirror symmetry with respect to planes $y = 0$ and $z = 0$. Figure 4 illustrates the corresponding vorticity field for $t = 0$ and $\lambda = 1$.

As one can see from these two examples, the introduction of quasi-potentials allows us to construct exact solutions for fairly complex vortical flows in cylindrical coordinates. The transformation of a two-component vorticity field in cylindrical coordinates can lead to either on-component or three-component vorticity field in Cartesian coordinates. There is no regular method to construct three-component vortical flows in Cartesian coordinates, whereas the developed theory allows us to find exact solutions in terms of quasi-potential.

## Conclusion

It has been demonstrated that introduction of quasi-potential is possible in cylindrical geometry, whereas it is not a trivial generalisation of quasi-potential theory developed in [1, 2]. Quasi-potential approach helps us to construct exact solutions describing fairly complicated three-dimensional fluid field with a multi-component vorticity field. In the particular cases the quasi-potential theory reduces to the conventional potential theory and to the theory based on introduction of a stream-function. It is believed that the quasi-potential approach can be used also in other curvilinear coordinates, in particular, in a spherical geometry. The results for spherical coordinates will be published elsewhere.

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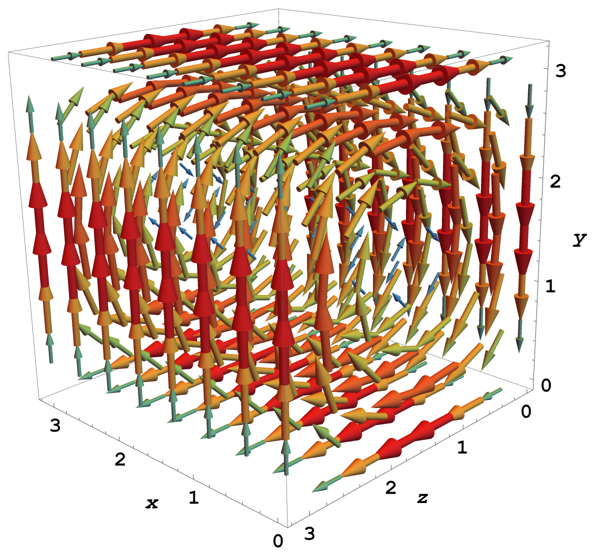


Figure 3: A fragment of velocity field as per Eq. (33).

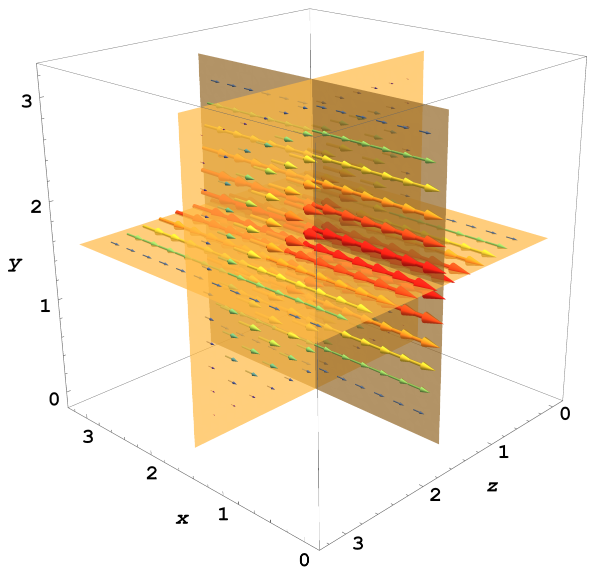


Figure 4: A fragment of vorticity field as per Eq. (34).