# Optimum Design of Discrete-time Differentiators via Semi-infinite Programming Approach

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## ABSTRACT

In this paper, a general optimum full band high order discrete-time differentiator design problem is formulated as a peak constrained least square optimization problem. That is, the objective of the optimization problem is to minimize the total weighted square error of the magnitude response subject to the peak constraint of the weighted error function. This problem formulation provides a great flexibility for the tradeoff between the ripple energy and the ripple magnitude of the discrete-time differentiator. The optimization problem is actually a semi-infinite programming problem. Our recently developed dual parametrization algorithm is applied for solving the problem. The main advantage of employing the dual parameterization algorithm for solving the problem is the guarantee of the convergence of the algorithm and the obtained solution being the global optimal solution that satisfies the corresponding continuous constraints. Moreover, the computational cost of the algorithm is lower than that of algorithms implementing the semi-definite programming approach.



*Index Terms*—Discrete-time differentiators, semi-infinite programming, dual parameterization algorithm, peak constrained least square approach, eigen approach, Remez approach, semi-definite programming approach.


## I. INTRODUCTION

Discrete-time differentiators have many important applications in physics and engineering [1]. In particular, they are used to obtain a set of data relating to the rate of change of some physical quantities, such as the estimation of heating rates from temperature data, net flow rates of fluid from measurements of volume level, and velocity from position data, etc.

Two common methods for the design of discrete-time differentiators are based on the eigen approach [2] and the Remez algorithm approach [3]. However, the eigen approach does not guarantee the obtained solution satisfying the required specifications. On the other hand, the Remez algorithm approach would result to a discrete-time differentiator with large ripple energy. To address this problem, the discrete-time differentiator design problem is formulated as a peak constraint least square optimization problem. That is, the total weighted square error of the magnitude response is minimized subject to the peak constraint of the weighted error function [4]-[6]. Although this problem formulation provides a great flexibility for the tradeoff between the ripple energy and the ripple magnitude of the discrete-time differentiator, this optimization problem is actually a semi-infinite programming problem. The common method for solving semi-infinite programming problems is via the semi-definite programming approach [4]. That is, the continuous constraints are discretized into finite number of discrete constraints. However, this approach does not guarantee that the continuous constraints are satisfied among the discretization points. Although the deviation between the continuous constraints and the discrete constraints can be reduced by increasing the number of discretization points, the exact number of discretization points required for the optimization problem is unknown and the increase in the number of discrete constraints will result to the increase of the computational complexity. Although new primal quadratic programming approach was proposed for solving the problem [5], the convergence of the algorithm is not guaranteed. In this paper, the dual parameterization algorithm is employed for solving the problem [6]. The semi-infinite programming problem is reduced to a sequence of approximating sub-problems followed by a nonlinear finite programming problem. Each of the approximating sub-problems can be readily solved by quadratic programming. The global solution of the finite nonlinear program can then be obtained from the approximated solution. If the feasible set is nonempty, then an exact optimal solution is guaranteed. Also, the convergence of the algorithm is proved. Moreover, since the total number of the finite constraints in the approximating sub-problems is smaller than that of the corresponding semi-definite programming problems, the computational complexity is low.

The outline of this paper is as follows. In Section II, the optimum discrete-time differentiator design problem is formulated as a semi-infinite programming problem. The dual parameterization algorithm is summarized in Section III. The computer numerical simulation results are presented in Section IV. Finally, a conclusion is drawn in Section V.

## II. PROBLEM FORMULATION

Let $h(n)$ be the impulse response of the discrete-time differentiator. For $N$ is odd, we assume

$$\begin{cases} h(k) = -h(N-1-k), \quad k = 0,1,2,\cdots,\dfrac{N-3}{2} \\ h\left(\dfrac{N-1}{2}\right) = 0 \end{cases}. \tag{1}$$

For $N$ is even, we assume

$$h(k) = -h(N-1-k), \text{ for } k = 0,1,2,\cdots,\frac{N}{2}-1. \tag{2}$$

Define

$$\mathbf{x} \equiv \begin{cases} \begin{bmatrix} a_1, & a_2, & \cdots, & a_{\frac{N-1}{2}} \end{bmatrix}^T, & N \text{ is odd} \\ \begin{bmatrix} a_1, & a_2, & \cdots, & a_{\frac{N}{2}} \end{bmatrix}^T, & N \text{ is even} \end{cases}, \tag{3}$$

$$\boldsymbol{\eta}(\omega) \equiv \begin{cases} \left[\sin\omega, \quad \sin 2\omega, \quad \cdots, \quad \sin\left(\left(\frac{N-1}{2}\right)\omega\right)\right]^T, & N \text{ is odd} \\ \left[\sin\frac{\omega}{2}, \quad \sin\frac{3\omega}{2}, \quad \cdots, \quad \sin\left(\left(\frac{N-1}{2}\right)\omega\right)\right]^T, & N \text{ is even} \end{cases}, \tag{4}$$

and

$$H_0(\omega) \equiv (\boldsymbol{\eta}(\omega))^T \mathbf{x}, \tag{5}$$

in which

$$a_n \equiv \begin{cases} 2h\left(\frac{N-1}{2} - n\right), & N \text{ is odd and } n = 1,2,\cdots,\frac{N-1}{2} \\ 2h\left(\frac{N}{2} - n\right), & N \text{ is even and } n = 1,2,\cdots,\frac{N}{2} \end{cases}, \tag{6}$$

then the frequency response of the discrete-time differentiator can be expressed as

$$H(\omega) = je^{-j\omega\left(\frac{N-1}{2}\right)} H_0(\omega), \tag{7}$$

where $j \equiv \sqrt{-1}$.

Define $B_d \equiv \left[\frac{d}{2} - \pi, \pi - \frac{d}{2}\right]$, where $d$ is the width of the transition band. Then the total weighted square error of the magnitude response of the discrete-time differentiator can be represented as

$$J(\mathbf{x}) \equiv \int_{B_d} W(\omega) \left| H_0(\omega) - D(\omega) \right|^2 d\omega = \frac{1}{2} \mathbf{x}^T \mathbf{Q}\, \mathbf{x} + \mathbf{b}^T \mathbf{x} + p, \tag{8}$$

where $D(\omega)$ is the desired magnitude response, $W(\omega)$ is the weighted function with $W(\omega) > 0$ for $\omega \in B_d$,

$$\mathbf{Q} = 2\int_{B_d} W(\omega) \boldsymbol{\eta}(\omega) (\boldsymbol{\eta}(\omega))^T d\omega, \tag{9}$$

$$\mathbf{b} = -2\int_{B_d} W(\omega) D(\omega) \boldsymbol{\eta}(\omega) d\omega, \tag{10}$$

and

$$p = \int_{B_d} W(\omega) (D(\omega))^2 d\omega. \tag{11}$$

It can be checked easily that matrix $\mathbf{Q}$ is positive definite. To specify the constraints, let $\delta$ be the peak constraint of the weighted error function. Then, the constraint can be expressed as:

$$W(\omega)\left| H_0(\omega) - D(\omega) \right| \leq \delta \text{, for } \omega \in B_d \text{,} \tag{12}$$

which implies that

$$\mathbf{A}(\omega)\,\mathbf{x} \leq \mathbf{c}(\omega) \text{, for } \omega \in B_d \text{,} \tag{13}$$

where

$$\mathbf{A}(\omega) = W(\omega)\left[\boldsymbol{\eta}(\omega), \quad -\boldsymbol{\eta}(\omega)\right]^T \text{, for } \omega \in B_d \text{,} \tag{14}$$

and

$$\mathbf{c}(\omega) = \left[D(\omega)W(\omega) + \delta, \quad \delta - D(\omega)W(\omega)\right]^T \text{, for } \omega \in B_d \text{.} \tag{15}$$

Clearly, $\mathbf{A}(\omega)$ and $\mathbf{c}(\omega)$ are continuously differentiable with respect to $\omega \in B_d$. Consequently, the optimum discrete-time differentiator design problem can be formulated as the following semi-infinite programming problem:

**Problem** (P)

$$\min_{\mathbf{x}} \qquad J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{Q}\,\mathbf{x} + \mathbf{b}^T \mathbf{x} + p \text{,} \tag{16a}$$

$$\text{subject to} \qquad \mathbf{g}(\mathbf{x}, \omega) = \mathbf{A}(\omega)\,\mathbf{x} - \mathbf{c}(\omega) \leq \mathbf{0} \text{, for } \omega \in B_d \text{.} \tag{16b}$$

## III. DUAL PARAMETERIZATION ALGORITHM

The above problem can be solved using the dual parameterization algorithm [6]. We first consider the Dorn's dual of problem (**P**) as

**Problem** (D)

$$\min_{(\mathbf{x}, \boldsymbol{\Lambda})} \qquad L(\mathbf{x}, \boldsymbol{\Lambda}) \text{,} \tag{17a}$$

$$\text{subject to} \qquad \mathbf{Q}\,\mathbf{x} + \mathbf{b} + \mathbf{A}^* \boldsymbol{\Lambda} = \mathbf{0} \text{,} \tag{17b}$$

$$\boldsymbol{\Lambda} \geq \mathbf{0} \text{,} \tag{17c}$$

where

$$L(\mathbf{x}, \boldsymbol{\Lambda}) = \frac{1}{2}\mathbf{x}^T \mathbf{Q}\,\mathbf{x} + \int_{B_d} \mathbf{c}(\omega) d\boldsymbol{\Lambda}, \tag{18}$$

$\mathbf{A}$ is the operator from $\Re^{N'}$ to $C\left(B_d, \Re^{N'}\right)$ defined by $\mathbf{A}(\omega)$ according to $\left(\mathbf{A}\,\mathbf{x}\right)(\omega) = \mathbf{A}(\omega)\,\mathbf{x}$ for $\omega \in B_d$, where $\mathbf{A}^*$ is the dual operator of $\mathbf{A}$, $N'$ is the length of the vector $\mathbf{x}$ and $C\left(B_d, \Re^{N'}\right)$ is the Banach space of all continuous real functions on $B_d$.

Assume the Slater's qualification holds, that is, there exists $\mathbf{x}_0 \in \Re^{N'}$ satisfying $\mathbf{g}\left(\mathbf{x}_0, \omega\right) < \mathbf{0}$ $\forall \omega \in B_d$. Since

(i) $J$ and $\mathbf{g}\left(\mathbf{x}, \omega\right)$ are convex in $\mathbf{x}$, $\forall \omega \in B_d$,

(ii) $J$ is differentiable on $\Re^{N'}$,

(iii) $\mathbf{g}(\mathbf{x}, \omega)$ is continuous in $\omega$, $\forall \mathbf{x} \in \Re^{N'}$, and continuously differentiable in $\mathbf{x}$ on $\Re^{N'} \times B_d$,

the strong duality theorem holds. That is, if the minimum of the primal problem (**P**) is achieved by some $\mathbf{x}^* \in \Re^{N'}$, then there exists a solution $\boldsymbol{\Lambda}^*$ of the dual problem (**D**), such that

$$J(\mathbf{x}^*) = \min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\Lambda}^*), \tag{19a}$$

$$\text{subject to} \quad \int_{B_d} \left(\mathbf{A}(\omega)\,\mathbf{x}^* - \mathbf{c}(\omega)\right) d\boldsymbol{\Lambda}^* = \mathbf{0}, \tag{19b}$$

$$\boldsymbol{\Lambda}^* \geq \mathbf{0}. \tag{19c}$$

Since $\mathbf{g}\left(\mathbf{x}, \omega\right)$ is continuously Fréchet differentiable, the Karush-Kuhn-Tucker (KKT) conditions for problem (**P**) are also satisfied. That is, the minimum of problem (**P**) can be achieved at $\mathbf{x}^* \in \Re^{N'}$ if and only if $\mathbf{x}^*$ is feasible and there exists a $\boldsymbol{\Lambda}^*$ such that

$$\mathbf{Q}\,\mathbf{x}^* + \mathbf{b} + \mathbf{A}^*\mathbf{\Lambda}^* = \mathbf{0}\,, \tag{20a}$$

$$\int_{B_d} \left(\mathbf{A}(\omega)\,\mathbf{x}^* - \mathbf{c}(\omega)\right) d\mathbf{\Lambda}^*(\omega) = \mathbf{0}\,, \tag{20b}$$

$$\mathbf{\Lambda}^* \geq \mathbf{0}\,. \tag{20c}$$

In general, the multiplier $\mathbf{\Lambda}^*$ satisfying the KKT conditions is not unique. However, as we assume that the Slater constraint qualification is satisfied, and the optimal solution of the primal problem **(P)** is achieved at $\mathbf{x}^* \in \Re^{N'}$. So the set of multipliers satisfying the KKT conditions of problem (**P**) will necessarily include a measure with finite support at no more than $N'$ points unless it is empty. This can be proved by the Carathéodory's theorem. Hence, there exists a solution pair $(\mathbf{x}^*, \mathbf{\Lambda}^*)$ of the dual problem **(D)** where the measure $\mathbf{\Lambda}^*$ has a finite support of no more than $N'$ points.

The dual semi-infinite problem (**D**) can be reduced to the finite dimensional optimization problem **(PD)**, called the parameterized dual of problem (**P**), as the following:

**Problem** (PD)

$$\min_{(\mathbf{x},\mathbf{t},\boldsymbol{\lambda})} L_k(\mathbf{x},\mathbf{t},\boldsymbol{\lambda})\,, \tag{21a}$$

$$\text{subject to} \qquad \boldsymbol{\lambda}_i \geq \mathbf{0}\,, \qquad i = 1,2,\cdots,N'\,, \tag{21b}$$

$$\omega_i \in B_d\,, \qquad i = 1,2,\cdots,k \leq N'\,, \tag{21c}$$

where the integer $k$ is the parameterization number, $\mathbf{t} = [\omega_1, \;\; \omega_2, \;\; \cdots, \;\; \omega_k]^T$ and $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1, \;\; \boldsymbol{\lambda}_2, \;\; \cdots, \;\; \boldsymbol{\lambda}_k]$, in which $\boldsymbol{\lambda}_i = [\lambda_{i,1}, \;\; \lambda_{i,2}, \;\; \cdots, \;\; \lambda_{i,m}]^T \in \Re^m$ and $m$ is the number of rows in matrix $\mathbf{A}$. The cost function $L_k(\mathbf{x},\mathbf{t},\boldsymbol{\lambda})$ is given by

$$L_k(\mathbf{x},\mathbf{t},\boldsymbol{\lambda}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\,\mathbf{x} + \sum_{i=1}^{k} \mathbf{c}^T(\omega_i)\,\boldsymbol{\lambda}_i\,. \tag{22}$$

According to the dual parameterization theory, once a solution $(\mathbf{x}^*, \mathbf{t}^*, \boldsymbol{\lambda}^*)$ is obtained from solving the problem (**PD**), the optimal solution of the primal problem (**P**) will also be $\mathbf{x}^*$. To state the algorithm for solving the problem (**P**), denote the problem obtained from problem (**PD**) by fixing $\mathbf{t}$ as problem **(PD(t))**. It can be shown easily that problem **(PD(t))** is the dual problem of the following problem **(P(t))** for fixed $\omega_i \in B_d$, $i = 1, 2, \cdots, k$.

**Problem** (P(t))

$$\min_{\mathbf{x}} \qquad J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{Q}\, \mathbf{x} + \mathbf{b}^T \mathbf{x} + p, \tag{23a}$$

$$\text{subject to} \qquad \mathbf{g}(\mathbf{x}, \omega_i) \le \mathbf{0}, \quad \text{for } i = 1, 2, \cdots, k. \tag{23b}$$

Hence, we have the following theorem:

*Theorem 1*

Consider problems **(P)**, **(P(t))** and **(PD)**. The following statements hold:

(i) Let $\bar{\mathbf{x}}$ be an optimum solution of problem **(P(t))**. If $\bar{\mathbf{x}}$ satisfies the infinite constraint (16b), then $\bar{\mathbf{x}}$ is the optimal solution of the primal problem (**P**).

(ii) Let $\mathbf{v}_k$ be the optimal value of problem (**PD**) with parameterization number $k$, then sequence $\{\mathbf{v}_k\}$ is decreasing, and there exists $k^*$ such that $\mathbf{v}_{k^*} = \mathbf{v}_k$, for all $k \ge k^*$. Furthermore, if $k^* \ge 1$, then $\mathbf{v}_{k^*-1} > \mathbf{v}_{k^*}$.

(iii) The $k^*$ in (ii) is the minimum integer such that for $k \ge k^*$, a global solution of the finite problem (**PD**) provides the solution for the primal problem (**P**) in the sense that if $(\mathbf{x}^*, \mathbf{t}^*, \boldsymbol{\lambda}^*)$ is a global optimizer of problem (**PD**), then $\mathbf{x}^*$ is the global optimizer of the primal problem (**P**).

(iv) If $0 \le k < k^*$, then $\mathbf{v}_k > \mathbf{v}_{k+1}$.

The proof of Theorem 1 can be found in [6]. The number $k^*$ in Theorem 1 is called the minimum parameterization number. If the optimal primal solution is an interior point of the feasible region, then $k^* = 0$.

Let $\{k_i\}$ be the given sequence of the parameterization numbers satisfying $k_i \le k_{i+1}$. For each $i$, let $\Omega_i = \{\omega_j^i : j = 1,2,\cdots,k_i\}$ be a given subset of $B_d$ and let $\mathbf{t}^i = [\omega_0^i \quad \omega_1^i \quad \cdots \quad \omega_{k_i}^i]^T$. Define the density distance between $\Omega_i$ and $B_d$ as

$$d(\Omega_i, B_d) \equiv \max_{\omega\in B_d} \min_{1\le j\le l_i} \left|\omega - \omega_j^i\right|. \tag{24}$$

We have the following theorem:

*Theorem 2*

Let $\{\mathbf{t}^i\}$ be the sequence given as above. Suppose $(\bar{\mathbf{x}}^i, \bar{\boldsymbol{\lambda}}^i)$ is a solution of problem (PD($\mathbf{t}^i$)). If $d(\Omega_i, B_d) \to 0$ as $i \to +\infty$, then it holds that

(i) $\{\bar{\mathbf{x}}^i\}$ converges to the solution of the primal problem (**P**).

(ii) $\mathbf{v}(\mathbf{PD}(\mathbf{t}^i)) \to \mathbf{v}(\mathbf{D})$, where $\mathbf{v}(\mathbf{S})$ denotes the optimal value of a given problem (**S**).

The proof of Theorem 2 can be found in [6].

We finally obtain the following optimization algorithm:

*Algorithm:*

*Step 0* (*Initialization*): Select a small number $\varepsilon > 0$. Choose a sequence of index sets $\Omega_i$. Set $i = 1$.

*Step 1* (*Compute a local optimum*): Solve the finite problem $(\mathbf{PD}(\mathbf{t}^i))$. Denote the local optimal solution as $(\mathbf{x}^i, \boldsymbol{\lambda}^i)$.

*Step 2* (*Test improvement of the objective*): If $i \ge 2$ and $\left|\mathbf{v}(\mathbf{PD}(\mathbf{t}^i)) - \mathbf{v}(\mathbf{D})\right| < \varepsilon$, then go to *step 3*, else $i = i+1$ and go to *step 1*.

*Step 3* (*Compute the global optimum*): Implement a local search for the finite dual problem (**PD**) with $k = k_i$. The solution is denoted as $(\mathbf{x}^*, \mathbf{t}^*, \boldsymbol{\lambda}^*)$, and $\mathbf{x}^*$ is taken as the optimizer of problem (**P**).

## IV. COMPUTER NUMERICAL SIMULATION RESULTS

To demonstrate the applicability of our proposed algorithm, a full band high order discrete-time differentiator is preferred. However, the magnitude response of the discrete-time differentiator would rise very fast if its order is high. Hence, it requires many filter coefficients for the implementation. To tradeoff between these two factors, a full band fifth order discrete-time differentiator is illustrated in this paper. That is, $D(\omega) = \omega^5 \quad \forall \omega \in B_d$. To design a full band fifth order discrete-time differentiator, small ripple magnitude and small transition bandwidth of the differentiator are usually preferred. However, it requires many filter coefficients for the implementation. To tradeoff among these factors, $N = 32$, $\delta = 0.0064 \times \pi^5$ and $d = 0.06\pi$ are chosen as the specifications. To demonstrate the performance of the full band fifth order differentiator, the effect of the weighted function should be removed and a uniform weighted function is employed, that is $W(\omega) = 1 \quad \forall \omega \in B_d$. In our proposed dual parameterization algorithm, a small value of $\varepsilon$ is usually preferred. However, too small value of $\varepsilon$ would increase the number of iterations and so the computational complexity is increased. To tradeoff between these two factors, $\varepsilon = 1 \times 10^{-6}$ is chosen. A large number of discrete frequencies in the index sets are usually preferred. However, too many discrete frequencies would increase the computational complexity. Since the number of extrema of the magnitude response of the full band fifth order differentiator designed via the Remez algorithm is equal to $N + 2$, the number of discrete frequencies in the first index set is $N + 2$. For the simplicity, a uniform sampling scheme is employed. So the first index set is initialized as $\Omega_1 = \left\{ \omega_k : \omega_k = \frac{d}{2} - \pi + k\left(\frac{2\pi - d}{N+1}\right) \text{for } k = 0,1,\cdots,N+1 \right\}$. The other index sets are constructed based on the previous index set by adding all violated index points of a refined set of grid points to the previous index set while dropping all unnecessary points from $\Omega_{i-1}$ for $i > 1$.

Our computer numerical simulation results are compared to that designed based on the eigen approach [2], the Remez approach [3] and the semi-definite programming approach [4]. These approaches are compared because these approaches are the most common approaches for the design of full band high order differentiators. The magnitude response of the full band fifth order differentiators designed via various approaches are shown in Figure 1, while the corresponding weighted error functions are shown in Figure 2. We can see from Figure 2 that the maximum ripple magnitude of the full band fifth order differentiator designed via the eigen approach is very large that it fails to satisfy the specification. Although the full band fifth order differentiator designed via the Remez approach achieves the smallest ripple magnitude among these approaches, the total weighted square error is the largest among these approaches. The full band fifth order differentiator designed via the semi-definite programming approach also fails the specification because the number of discrete frequencies is not large enough. On the other hand, the full band fifth order differentiator designed via the peak constraint least square approach satisfies the required specification and minimizes the total weighted square error.

Actually it is difficult to have a fair comparison on the computational complexity of our proposed method to other existing methods because almost none of them solve the design problem via the semi-infinite programming approach with the guarantee of the convergence of the algorithms. In our proposed method, it is found that the algorithm terminated after three iterations, and the number of discrete frequencies in the last index set is 1345. If the design problem is formulated via the semi-definite programming approach with the same number of discrete frequencies, it is shown in Figure 2 that the solution obtained does not satisfy the specification. Hence, more discrete frequencies are required for the semi-definite programming approach. It implies that the computational complexity of the semi-definite programming approach is much higher than our proposed algorithm [4].

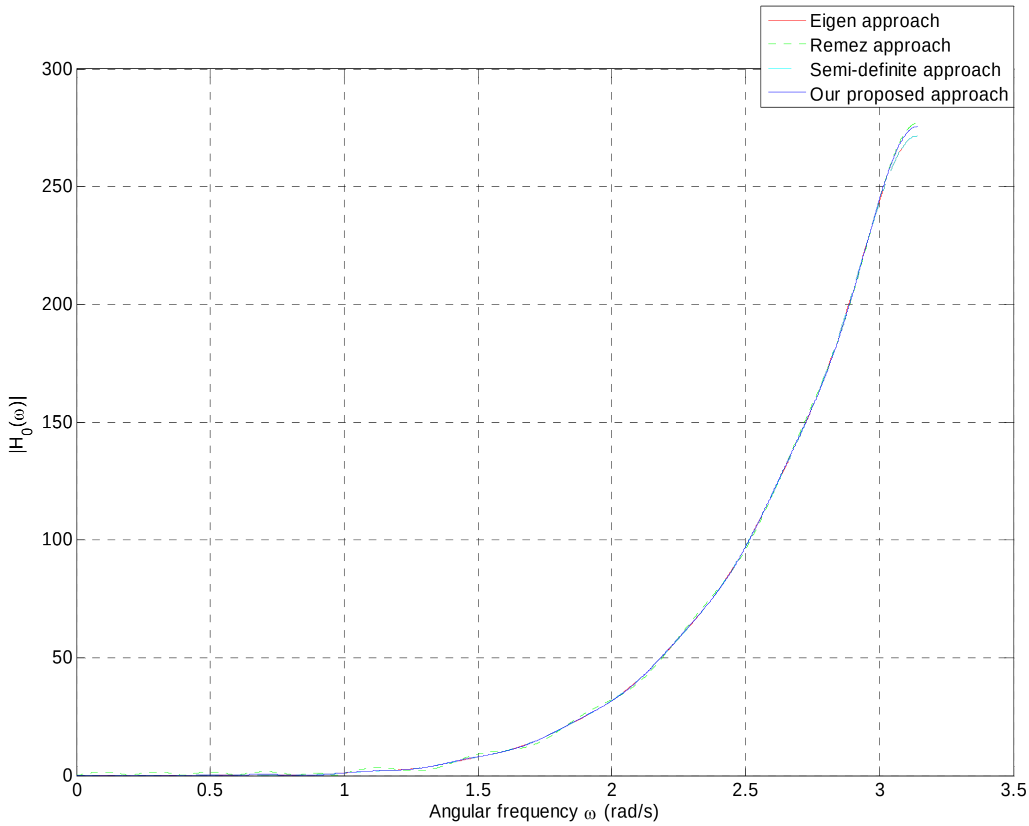


Figure 1. Magnitude response of the full band fifth order differentiators.

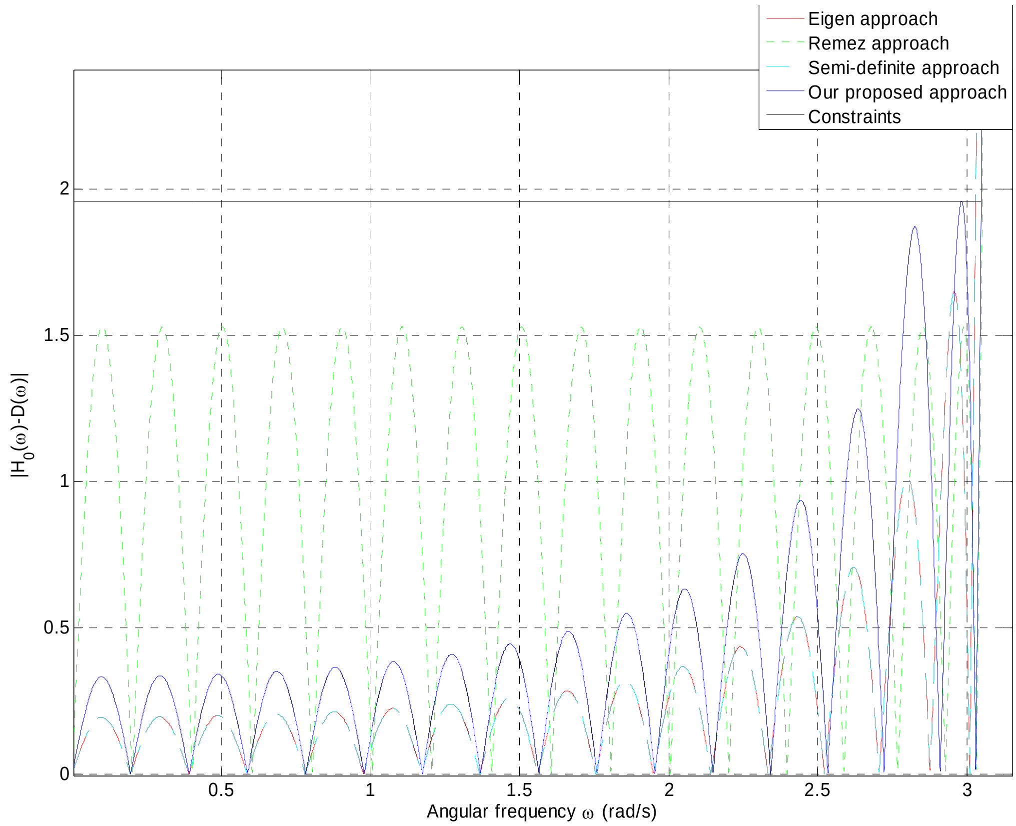


Figure 2. Weighted error function of the full band fifth order differentiators.

## V. CONCLUSION

The main contribution of this paper is to formulate the optimum discrete-time differentiator design problem as a peak constrained least square optimization problem. Actually, the problem formulation can also be applied to non-full band arbitrary order discrete-time differentiator design problems. The formulated problem is a semi-infinite programming problem and our proposed dual parameterization algorithm is employed for solving the problem. The main advantages of our proposed algorithm are i) the guarantee of the solutions converging to the optimum one that satisfies the continuous constraints if the solution exists; and ii) low computational complexity because the semi-infinite programming problem is transformed to a finite dimensional optimization problem and just a few active points are sufficient to give enough information for searching the optimal solution.

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## REFERENCES

[1] San L. Lin and Samiha Mourad, "On-chip rise-time measurement," *IEEE Transactions on Instrumentation and Measurement*, vol. 53, no. 6, pp. 1510-1516, 2004.

[2] Soo-Chang Pei and Jong-Jy Shyu, "Eigenfilter design of higher-order digital differentiators," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 4, pp. 505-511, 1989.

[3] Emmanouil Z. Psarakis and George V. Moustakides, "A robust initialization scheme for the Remez exchange algorithm," *IEEE Signal Processing Letters*, vol. 10, no. 1, pp. 1-3, 2003.

[4] W.-S. Lu, "A unified approach for the design of 2-D digital filters via semidefinite programming," *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, vol. 49, no. 6, pp. 814-826, 2002.

[5] J. W. Adams, P. Kruethong, R. Hashemi, J. L. Sullivan and D. Gleeson, "New quadratic programming algorithms for designing FIR digital filters," *The Twenty-Seventh Asilomar Conference on Signals, Systems and Computers*, vol. 2, pp. 1206-1210, 1-3 November, 1993.

[6] Y. Liu, C. H. Tseng and K. L. Teo, "A unified quadratic semi-infinite programming approach to time and frequency domain constrained digital filter

design," *Communications in Information and Systems*, vol. 2, no. 4, pp. 399-410, 2002.