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Some null submanifolds of indefinite nearly Sasakian manifolds

This Dissertation is Submitted in fulfillment of the academic requirements for the degree of Masters in Mathematics in the School of Mathematics, Statistics and Computer Sciences, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, South Africa

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This Dissertation is submitted in fulfillment of the Academic requirement for the degree of Master of Philosophy in Mathematics to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and science, University of KwaZulu-Natal.

As the candidate's supervisor, I have approved this dissertation for submission.

[redacted]

Prof. Fortuné Massamba

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Islam F. M. Osman
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Signed ..[redacted].......

Dedication

I dedicated this work to
the family of Mr. Fawzi and Mrs. Amal
my husband Mohamed Hamed
my sister Safia and my brothers

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First and foremost, I thank the Almighty God. Sincere gratitude goes to my advisor, Professor Fortuné Massamba who introduced me to differential geometry, for his inspiring guidance, great enthusiasm for mathematics and invaluably constructive motivation during the dissertation work. I would like thank to my husband Mohamed Hamed his support and immense contribution to the success of this work. To my family and friends for their great encouragement and unending love. God bless you all.

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Abstract

We investigate some geometric aspects of indefinite nearly Sasakian manifolds. We study specific cases of null submanifolds, namely, invariant, GCR, Screen transversal and Radical screem transversal null submanifolds. Under some conditions there exist leaves that are nearly Sasakian, immersed in the ambient manifold as submanifolds. Furthermore, we give some geometric configuration of the existence of totally geodesic foliations in some distributions.

Contents

Chapter One

Introduction

1.1 Introduction

Let $\overline{M}$ be a semi-Riemannian manifold. A submanifold M of $\overline{M}$ is said to be a null(degenerate) submanifold if there exists a metric on M which is degenerate. The null submanifolds of a semi-Riemannian manifold have been introduced by Duggal and Bejancu and Kupeli in [6] and [14], respectively. Kupeli's approach is intrinsic while Duggal-Bejancu's approach is extrinsic. The geometry of null submanifolds of semi-Riemannian manifolds was studied by Duggal and Bejancu [6]. The null submanifolds have their applications in mathematical physics, more precisely in general relativity. Indeed, null submanifolds appear in general relativity as some smooth parts of event horizons of the Kruskal and Kerr black holes [11]. The geometry of null submanifolds of indefinite nearly Sasakian manifolds was first studied in [17], in which several aspects of such submanifolds were investigated. However, the invariant null submanifolds were not investigated.

The null submanifolds of indefinite nearly Sasakian manifolds are defined according to the behavior of the contact structure of indefinite nearly Sasakian manifolds. The indefinite Sasakian version was studied by Duggal and Sahin in [7]. They defined and studied, null submanifolds, invariant submanifolds, screen transversal submanifolds, contact CR-null submanifolds of indefinite Sasakian manifolds and contact screen CR-null submanifolds. Later on, Duggal and Sahin studied contact generalized CR-null submanifolds of indefinite nearly Sasakian manifolds [7, 8, 9].

From these studies, it appears that invariant, screen real, CR-null, screen CR-null and transversal null submanifolds of indefinite nearly Sasakian manifolds do not include real null curves. In spite of that null real curves have many applications in mathematical physics for more details see ([6], [7]), a null (lightlike) geodesic curve represents the path of a photon in general relativity.

To this end, in this dissertation, we introduce invariant null submanifolds of indefinite nearly Sasakian manifolds. We prove several new results concerning such submanifolds and give some examples to illustrate the main concepts.

We introduce the general notions of contact Cauchy-Riemann(CR)-null submanifolds, a first attempt towards the general theory of null submanifolds of nearly Sasakian manifolds, and study its properties. We study the integrability conditions of their distributions and investigate the geometry of leaves of the distributions involved in the induced contact CR-structure. We find geometric conditions for contact CR-submanifolds M to be irrotational. It is important to mention that contrary to the Riemannian case [22], but, similar to the Duggal-Bejancu's concept of null CR-submanifolds of Kählerian manifolds [6], the contact CR-null submanifolds are always non-trivial, that is, they do not include the invariant and the real subcases. We introduce contact screen Cauchy-Riemann(SCR)-null submanifolds, which include invariant and screen real submanifolds and study some of their properties.

1.2 Main results

Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be a null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. In this dissertation, we prove the following results:

1. If M is invariant, then it is nearly Sasakian. Moreover, M cannot be proper totally umbilical.

2. Assume that M is a contact GCR-null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. If the distribution $D \oplus \{\xi\}$ is integrable, then its leaves are nearly Sasakian, immersed in $\overline{M}$ as submanifolds if and only if
$$th(X, Y) = 0, \;\; \forall X,\, Y \in \Gamma(D \oplus \{\xi\}).$$

3. If M is a radical screen transversal null submanifold, tangent to the structure vector field ξ, then the radical distribution $\mathrm{Rad}(TM)$ is integrable if and only if
$$g(A_{\overline{\phi}E_1}E_2 - A_{\overline{\phi}E_2}E_1 - 2(\overline{\nabla}_{E_1}\phi)E_2, \overline{\phi}X) = 0,$$
for $X \in \Gamma\left(S(TM) - \{\xi\}\right)$ and $E_1, E_2 \in \Gamma\left(\mathrm{Rad}(TM)\right)$.
Moreover, if the radical distribution $\mathrm{Rad}(TM)$ is integrable, then
$$HX + \overline{\phi}X \in \Gamma(S(TM) \perp \mathrm{ltr}(TM)), \tag{1.1}$$
for any $X \in \Gamma(\mathrm{Rad}\, TM))$.

4. If M is a radical screen transversal null submanifold and the screen distribution $S(TM)$ defines a totally geodesic foliation, then
$$h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X)$$
has no components in $\overline{\phi}\,(\mathrm{Rad}(TM))$, for any $X,\, Y \in \Gamma\,(S(TM))$.

5. If M is a contact SCR-null submanifold and the distribution $D^{\perp}$ defines a totally geodesic foliation on M, then

   (i) $A_{\overline{\phi}X}Y + A_{\overline{\phi}Y}X$ has no components in $\overline{D}$.

   (ii) For any X, $Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(D)$ and $N \in \Gamma(\mathrm{ltr}(TM))$,

$$\overline{g}(B'h^{s}(X,\overline{\phi}Z),Y) + \overline{g}(B'h^{s}(Y,\overline{\phi}Z),X) = 0,$$
$$\text{and } \overline{g}(B'D^{s}(X,\overline{\phi}N),Y) + \overline{g}(B'D^{s}(Y,\overline{\phi}N),X) = 0.$$

1.3 Dissertation organization

The dissertation is organized as follows. In Chapter 2, we introduce the preliminaries and present the basic information on the geometry of semi-Riemannian manifolds and geometry of null submanifolds. In Chapter 3, we define and study the indefinite nearly Sasakian manifolds and invariant submanifolds. We present the concept of GCR-null submanifolds of indefinite contact manifolds. In the same chapter, we introduce screen transversal of null submanifolds and radical screen transversal null submanifolds and we give examples of such submanifolds. Chapter 4 deals with the concept of contact CR-submanifolds of indefinite nearly Sasakian manifolds supported by an example as well as the contact screen CR-null submanifolds. Finally, we end the dissertation in Chapter 5 by some concluding remarks and perspectives.

Chapter Two

Preliminaries

In this chapter, we introduce the basic notions on the geometry of semi-Riemannian manifolds, the null geometry of submanifolds in general needed for this dissertation. The chosen approach is that of Duggal-Bejancu [6] and Duggal-Sahin [7].

2.1 Geometry of semi-Riemannian manifolds

In this section, we introduce the basic notions of a semi-Riemannian manifold from [6] and [7], required in dissertation.

Let $\mathcal{V}$ be a real m-dimensional vector space with a symmetric bilinear mapping $\overline{g} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$. We say that $\overline{g}$ is degenerate [6] on $\mathcal{V}$ if there exists a vector $E \neq 0$, of $\mathcal{V}$, such that

$$\overline{g}(E, v) = 0, \quad \forall v \in \mathcal{V},$$

otherwise $\overline{g}$ is called non-degenerate if and only if

$$\overline{g}(u, v) = 0, \quad \forall v \in \mathcal{V} \quad \text{implies} \quad u = 0.$$

Definition 2.1.1. [6] The radical or the null space of $\mathcal{V}$, with respect to the symmetric bilinear form $\overline{g}$, is a subspace $Rad\,\mathcal{V}$ of $\mathcal{V}$ defined by

$$Rad\,\mathcal{V} = \{E \in \mathcal{V};\, \overline{g}(E, v) = 0,\, v \in \mathcal{V}\}.$$

The dimension of $Rad\,\mathcal{V}$ is called the nullity degree of $\overline{g}$, denoted by $\text{null}\,\mathcal{V}$. Clearly, $\overline{g}$ is degenerate or non-degenereate on $\mathcal{V}$ if and only if $\text{null}\,\mathcal{V} > 0$ or $\text{null}\,\mathcal{V} = 0$, respectively.

Definition 2.1.2. [6] Let $\overline{M}$ be a real m-dimensional smooth manifold and $\overline{g}$ be a symmetric tensor field of type (0,2) on $\overline{M}$. We suppose $\overline{g}$ is non-degenerate on $T\overline{M}$. Hence each $T\overline{M}$ becomes an $2m$-dimensional semi-Euclidean space. Then, the tensor field $\overline{g}$ is called a semi-Riemannian metric (metric tensor field) and $\overline{M}$ endowed with $\overline{g}$ is called a semi-Riemannian manifold.

As $T\overline{M}$ is a semi-Euclidean space, any tangent vector $u \in T\overline{M}$ is said to be

- spacelike, if $\overline{g}(u,u) > 0$ or $u = 0$,
- timelike, if $\overline{g}(u,u) < 0$,
- null (lightlike), if $\overline{g}(u,u) = 0$ and $u \neq 0$.

The class into which a tangent vector falls is called its causal character.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold. Suppose q is the index of $\overline{M}$, that is, q is the common value of the index of $\overline{g}$. In case $q = 0$ and $q = 1$, $m \geq 2$, $\overline{M}$ is called Riemannian manifold and a Lorentz manifold, respectively. In case $0 < q < m$, we say that $\overline{M}$ is a proper semi-Riemannian manifold.

Let denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ.

Definition 2.1.3. [6] A vector field Y on $\overline{M}$ is said to be *parallel* with respect to a linear connection $\overline{\nabla}$ if

$$\overline{\nabla}_X Y = 0,$$

for any $X \in \Gamma(T\overline{M})$.

Definition 2.1.4. [6] A linear connection $\overline{\nabla}$ on $(\overline{M}, \overline{g})$ is said to be a metric connection if the metric tensor field $\overline{g}$ is parallel with respect to $\overline{\nabla}$, i.e., for any $X \in \Gamma(T\overline{M})$,

$$(\overline{\nabla}_X \overline{g})(Y, Z) = 0. \tag{2.1}$$

That is,

$$X\left(\overline{g}(Y, Z)\right) = \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z), \tag{2.2}$$

for any X, $Y \in \Gamma(T\overline{M})$.

The metric connection $\overline{\nabla}$ defined in the Definition 2.1.4 is called the Levi-Civita connection and it is given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_X Y, Z) = {} & X\left(\overline{g}(Y, Z)\right) + Y\left(\overline{g}(X, Z)\right) - Z\left(\overline{g}(X, Y)\right) \\ & + \overline{g}([X, Y], Z) + \overline{g}([Z, X], Y) - \overline{g}([Y, Z], X), \end{aligned} \tag{2.3}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. In particular, on $\mathbb{R}^m_q$ the Levi-Civita connection is defined by

$$\overline{\nabla}_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k. \tag{2.4}$$

Definition 2.1.5. [7] The semi-Riemannian curvature tensor, denoted by $\overline{R}$, of $\overline{M}$ is a (1,3) tensor field defined by

$$\overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, \tag{2.5}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

The torsion tensor, denoted by $\overline{T}$, of $\overline{\nabla}$ is a (1,2) tensor defined by

$$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y].$$

$\overline{R}$ is skew-symmetric in the first two slots. In case $\overline{T}$ vanishes on $\overline{M}$ we say that $\overline{\nabla}$ is torsion-free or symmetric metric connection on $\overline{M}$.

The curvature tensor $\overline{R}$ of the Levi-Civita connection satisfies

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0, \tag{2.6}$$

$$(\overline{\nabla}_X \overline{R})(Y,Z) + (\overline{\nabla}_Y \overline{R})(Z,X) + (\overline{\nabla}_Z \overline{R})(X,Y) = 0, \tag{2.7}$$

respectively, for any $X, Y, Z \in \Gamma(T\overline{M})$.

Proposition 2.1.1. *[7] Let $(\mathcal{V}, g)$ be a real n-dimensional null vector space such that $\dim(Rad\,\mathcal{V}) = r < n$. Then any complementary subspace to $Rad\,\mathcal{V}$ is non-degenerate.*

Proof. Let $S\mathcal{V}$ be a complementary subspace to $Rad\,\mathcal{V}$ in $\mathcal{V}$, we have the decomposition

$$\mathcal{V} = Rad\,\mathcal{V} \oplus_{orth} S\mathcal{V}, \tag{2.8}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Suppose there exists a non-zero $u \in S\mathcal{V}$ such that $g(u,v) = 0$ for any $v \in S\mathcal{V}$. From (2.8) implies $g(u,E) = 0$, $\forall E \in Rad\,\mathcal{V}$. Therefore, $u \in Rad\,\mathcal{V}$. But $Rad\,\mathcal{V}$ and $S\mathcal{V}$ being complementary subspaces, $S\mathcal{V}$ is non-degenerate, which completes the proof. □

Definition 2.1.6. [6] Let $\overline{M}$ be a real m-dimensional smooth manifold. A distribution D of rank r on $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an r-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ is said to belong to D if $X(x) \in D_x$ for each $x \in \overline{M}$.

The distribution D is said to be smooth, if for any $x \in \overline{M}$ there exist r smooth linearly independent vector fields $X_\alpha \in \Gamma(D)$, $\alpha \in \{1, ..., r\}$, in a coordinate neighbourhood of x.

Definition 2.1.7. [6] A distribution D is said to be integrable if

$$[X,Y] \in \Gamma(D),$$

for all $X, Y \in \Gamma(D)$.

2.2 Null submanifolds

In this section, we present a few aspects of null submanifolds, needed for this dissertation.

A submanifold M^n of a semi-Riemannian manifold $(\overline{M}^{n+k}, \overline{g})$ is said to be a null submanifold if the induced metric $g = \overline{g}_{|_M}$ on M is degenerate and its radical distribution $\mathrm{Rad}(TM)$ is such that $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, where [8]

$$TM^{\perp} = \bigcup_{x\in M} \{u \in T_x\overline{M} : \overline{g}(u, s) = 0, \forall s \in T_xM\}.$$

The submanifold M of $\overline{M}$ is said to be r-null submanifold if the mapping

$$Rad\,TM : x \in M \longrightarrow Rad\,T_xM$$

defines a smooth distribution on M of rank $r > 0$, where $Rad\,TM$ is called the radical (or null) distribution on M [12]. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = \mathrm{Rad}(TM) \perp S(TM).$$

Choose a screen transversal bundle $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\mathrm{Rad}(TM)$ in $TM^{\perp}$ such that

$$TM^{\perp} = \mathrm{Rad}(TM) \perp S(TM^{\perp}).$$

As $S(TM^{\perp})$ is a vector subbundle of $S(TM)^{\perp}$ and since both are non-degenerate we have the following orthogonal direct decomposition [6]:

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}.$$

Since, for any local basis $\{E_i\}$ of $\mathrm{Rad}(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that [10]

$$\overline{g}(E_i, N_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0,$$

it follows that there exists a null transversal vector bundle $\mathrm{ltr}(TM)$ locally spanned by $\{N_i\}$. Consider $tr(TM)$ be complementary vector bundle to TM in $T\overline{M}$. Then [8]

$$tr(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}), \tag{2.9}$$

$$T\overline{M} = TM \oplus tr(TM) = S(TM) \perp (\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)) \perp S(TM^{\perp}).$$

Proposition 2.2.1. *[6] The null second fundamental forms of a null submanifolds* $\big(M, g, S(TM), S(TM^{\perp})\big)$ *do not depend on* $S(TM)$, $S(TM^{\perp})$ *and* $\mathrm{ltr}(TM)$.
A submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$ *of* $\overline{M}$ *is called*

(i) r-null if $r < min\{n, k\}$;

(ii) co-isotropic if $r = k < n$, $S(TM^{\perp}) = \{0\}$;

(iii) isotropic if $r = n < k$, $S(TM) = \{0\}$;

(iv) totally null if $r = n = k$, $S(TM) = S(TM^{\perp}) = \{0\}$,

where r *is a rank of radical distribution* $\mathrm{Rad}(TM)$, *and* $1 \le r \le n$.

Let $\overline{\nabla}$, ∇ and ∇^{t} denote the linear connections on $\overline{M}$, M and vector bundle $tr(TM)$, respectively. Then the Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.10}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^{t}_X U, \quad \forall U \in \Gamma(tr(TM)), \tag{2.11}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{t}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM)$, respectively and A_U is the shape operator of M with respect to U. Moreover, according to the decomposition (2.9), h^{l} and h^{s} are $\Gamma\left(\mathrm{ltr}(TM)\right)$ valued and $\Gamma\left(S(TM^{\perp})\right)$ valued null second fundamental form and screen second fundamental form of M, respectively [18], then

$$\overline{\nabla}_X Y = \nabla_X Y + h^{l}(X, Y) + h^{s}(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.12}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{l}_X N + D^{s}(X, N), \quad N \in \Gamma\left(\mathrm{ltr}(TM)\right), \tag{2.13}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^{s}_X W + D^{l}(X, W), \quad W \in \Gamma\left(S(TM^{\perp})\right), \tag{2.14}$$

where $D^{l}(X, W)$, $D^{s}(X, N)$ are the projections of ∇^{t} on $\Gamma\left(\mathrm{ltr}(TM)\right)$ and $\Gamma\left(S(TM^{\perp})\right)$, respectively and ∇^{l}, ∇^{s} are linear connections on $\Gamma\left(\mathrm{ltr}(TM)\right)$ and $\Gamma\left(S(TM^{\perp})\right)$, respectively. We call ∇^{l}, ∇^{s} the null and screen transversal connections on M, and A_N, A_W are shape operators on M with respect to N and W, respectively. Then from (2.10), (2.12)-(2.14), we obtain [10]

$$\overline{g}\left(h^{s}(X, Y), W\right) + \overline{g}\left(Y, D^{l}(X, W)\right) = g(A_W X, Y), \tag{2.15}$$

and

$$\overline{g}\left(D^{s}(X, N), W\right) = \overline{g}\left(N, A_W X\right). \tag{2.16}$$

Let P denote the projection of TM on $S(TM)$ and let ∇^{*} and ∇^{*t} be the linear connections on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. Then from the decomposition of tangent bundle of null submanifold we have

$$\nabla_X PY = \nabla^{*}_X PY + h^{*}(X, PY), \tag{2.17}$$

$$\text{and } \nabla_X E = -A^{*}_E X + \nabla^{*^t}_X E, \tag{2.18}$$

for all $X, Y \in \Gamma(TM)$ and $E \in \Gamma(\mathrm{Rad}(TM))$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. From (2.17) and 2.18, we obtain

$$\overline{g}\left(h^l(X, PY), E\right) = g(A^*_E X, PY), \tag{2.19}$$

$$\overline{g}\left(h^*(X, PY), N\right) = g(A_N X, PY), \tag{2.20}$$

$$\overline{g}\left(h^l(X, E), E\right) = 0, \quad A^*_E E = 0. \tag{2.21}$$

The linear connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, from (2.12), we get

$$(\nabla_X g)(Y, Z) = \overline{g}\left(h^l(X, Y), Z\right) + \overline{g}\left(h^l(X, Z), Y\right). \tag{2.22}$$

Consider a local quasi-orthonormal fields of frames of $\overline{M}$ along M, on a neighbourhood $\mathcal{U}$ as

$$\{E_1, ..., E_r, N_1, ..., N_r, Z_{r+1}, ..., Z_m, W_{r+1}, ..., W_n\},$$

where $\{Z_{r+1}, ..., Z_m\}$ and $\{W_{r+1}, ..., W_n\}$ are respectively orthogonal basis of $\Gamma\left(S(TM)|_{\mathcal{U}}\right)$ and $\Gamma\left(S(TM^{\perp})|_{\mathcal{U}}\right)$ and that $\epsilon_a = g(Z_a, Z_a)$ and $\epsilon_\alpha = \overline{g}(W_\alpha, W_\alpha)$ are the signatures of $\{Z_a\}$ and $\{W_\alpha\}$ respectively. The following range of indices will be used. $i, j, k \in \{1, ..., r\}$; $\alpha, \beta, \eta \in \{r+1, ..., n\}$; $a, b, c \in \{r+1, ..., m\}$. [12]

Let P be the projection morphism of TM onto $S(TM)$. Then,

$$X = PX + \sum_{i=1}^{r} \eta_i(X) E_i, \tag{2.23}$$

for any $X \in \Gamma(TM)$, where the 1-forms η_i are given by

$$\eta_i(X) = \overline{g}(X, N_i), \quad \forall X \in \Gamma(TM), \tag{2.24}$$

the Gauss-Weingartein equations [12] of an r-null submanifold M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h^l_i(X, Y) N_i + \sum_{\alpha=r+1}^{n} h^s_\alpha(X, Y) W_\alpha, \tag{2.25}$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_\alpha, \tag{2.26}$$

$$\overline{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^{r} \varphi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X) W_\beta, \tag{2.27}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h^*_i(X, PY) E_i, \tag{2.28}$$

$$\nabla_X E_i = -A^*_{E_i} X - \sum_{j=1}^{r} \tau_{ji}(X) E_j, \quad \forall X, Y \in \Gamma(TM), \tag{2.29}$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$ respectively, h_i^l and h_α^s are symmetric bilinear forms known as local null and screen fundamental forms of TM respectively. Also h_i^* are the second fundamental forms of $S(TM)$. A_{N_i}, $A_{E_i}^*$ and A_{W_α} are linear operators on TM while τ_{ij}, $\rho_{i\alpha}$, $\varphi_{\alpha i}$ and $\theta_{\alpha\beta}$ are 1-forms on TM given by [19]

$$\begin{aligned} &\tau_{ij}(X) = \overline{g}(\overline{\nabla}_X N_i, E_j), \quad \epsilon_\alpha \rho_{i\alpha}(X) = \overline{g}(\overline{\nabla}_X N_i, W_\alpha), \quad \varphi_{\alpha_l}(X) = \overline{g}(\overline{\nabla}_X W_\alpha, E_i) \\ \text{and} \quad &\epsilon_\beta \theta_{\alpha\beta}(X) = \overline{g}(\overline{\nabla}_X W_\alpha, W_\beta), \quad \forall X \in \Gamma(TM). \end{aligned} \tag{2.30}$$

It is easy to see from (2.25) that

$$h_i^l(X,Y) = \overline{g}(\overline{\nabla}_X Y, E_i), \quad \forall X, Y \in \Gamma(TM), \tag{2.31}$$

from which we deduce the independence of h_i^l on the choice of $S(TM)$, and that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection and satisfies the relation

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r} \{h_i^l(X,Y)\eta_i(Z) + h_i^l(X,Z)\eta_i(Y)\}, \tag{2.32}$$

for any $X, Y \in \Gamma(TM)$ and η_i are differential 1-forms given by [19]

$$\eta_i(X) = \overline{g}(X, N_i), \quad \forall X \in \Gamma(TM). \tag{2.33}$$

The above three local second fundamental forms are related to their shape operators by the following set of equations

$$g(A_{E_i}^* X, Y) = h_i^l(X,Y) + \sum_{j=1}^{r} h_j^l(X, E_i)\lambda_j(X), \quad \overline{g}(A_{E+_i}^* X, N_j) = 0, \tag{2.34}$$

$$g(A_{W\alpha} X, Y) = \epsilon_\alpha h_\alpha^s(X,Y) + \sum_{i=1}^{r} \nu_{\alpha i}(X)\lambda_i(Y), \tag{2.35}$$

$$\overline{g}(A_{W\alpha} X, N_i) = \epsilon_\alpha \tau_{i\alpha}(X), \quad g(A_{N_i} X, Y) = h_i^*(X, PY), \tag{2.36}$$

for any $X, Y \in \Gamma(TM)$.

Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional r-null submanifold of a $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Let $\overline{R}$, R, R^l and R^s denote the curvature tensors of $\overline{\nabla}$, ∇, ∇^l and ∇^s, respectively (see[6]). Then we have

$$\begin{aligned} \overline{R}(X,Y)Z = R(X,Y)Z &+ A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &- A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &+ D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &- (\nabla_Y h^s)(X,Z) + D^s\left(X, h^l(Y,Z)\right) - D^s\left(Y, h^l(X,Z)\right), \end{aligned} \tag{2.37}$$

$$
\begin{aligned}
\overline{R}(X,Y)N = {} & R^l(X,Y)N + h^l(Y,A_N X) - h^l(X,A_N Y) + D^l\left(X, D^s(Y,N)\right) \\
& - D^l\left(Y, D^s(X,N)\right) + (\nabla_Y A)(N,X) - (\nabla_X A)(N,Y) \\
& + A_{D^s(X,N)}Y - A_{D^s(Y,N)}X + (\nabla_X D^s)(Y,Z) \\
& - (\nabla_Y D^s)(X,N) + h^s(Y,A_N X) - h^s(X,A_N Y), \qquad (2.38)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{R}(X,Y)W = {} & R^s(X,Y)W + h^s(Y,A_W X) - h^s(X,A_W Y) + D^s\left(X, D^l(Y,W)\right) \\
& - D^s\left(Y, D^l(X,W)\right) + (\nabla_Y A)(W,X) - (\nabla_X A)(W,Y) \\
& + A_{D^l(X,W)}Y - A_{D^l(Y,W)}X + (\nabla_X D^l)(Y,W) \\
& - (\nabla_Y D^l)(X,W) + h^l(Y,A_W X) - h^l(X,A_W Y), \qquad (2.39)
\end{aligned}
$$

for any $X, Y, Z \in \Gamma(TM)$, $N \in \Gamma\left(\mathrm{ltr}(TM)\right)$ and $W \in \Gamma\left(S(TM^\perp)\right)$.

Definition 2.2.1. [7] A null submanifold M of a semi-Riemannian manifold is said to be an irrotational submanifold if $\overline{\nabla}_X E \in \Gamma(TM)$, for all $X \in \Gamma(TM)$ and $E \in \Gamma\left(\mathrm{Rad}(TM)\right)$.

From (2.12), we conclude that M is an irrotational null submanifold if and only if the following are satisfied:

$$h^s(X,E) = 0, \quad h^l(X,E) = 0. \qquad (2.40)$$

Definition 2.2.2. [7] A null submanifold (M,g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma\left(tr(TM)\right)$ on M, called the transversal curvature vector field of M such that

$$h(X,Y) = Hg(X,Y), \qquad (2.41)$$

for all $X, Y \in \Gamma(TM)$. Furthermore, M is said to be totally geodesic if $H = 0$. Moreover, it is easy to see that M is totally umbilical in $\overline{M}$ if and only if on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $H^l \in \Gamma\left(\mathrm{ltr}(TM)\right)$ and $H^s \in \Gamma\left(S(TM^\perp)\right)$ and smooth functions $H^l_i \in F\left(\mathrm{ltr}(TM)\right)$ and $H^s_\alpha \in F\left(S(TM^\perp)\right)$ such that

$$h^l(X,Y) = H^l g(X,Y), \; h^s(X,Y) = H^s g(X,Y), \; D^l(X,W) = 0 \qquad (2.42)$$

$$h^l_i(X,Y) = H^l_i g(X,Y), \; h^s_\alpha(X,Y) = H^s_\alpha g(X,Y), \qquad (2.43)$$

for all $X, Y \in \Gamma(TM)$, $W \in \Gamma\left(S(TM^\perp)\right)$.

Definition 2.2.3. [8] If the second fundamental form h of a submanifold M, tangent to the structure vector field ξ, of an indefinite nearly Sasakian manifold $\overline{M}$ is of the form

$$h(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \qquad (2.44)$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called totally contact umbilical and totally contact geodesic if $\alpha = 0$.
Also for a totally contact umbilical M, we have

$$h^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_l + \eta(X)h^l(Y,\xi) + \eta(Y)h^l(X,\xi),$$
$$h^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_s + \eta(X)h^s(Y,\xi) + \eta(Y)h^s(X,\xi), \tag{2.45}$$

where $\alpha_s \in \Gamma(S(TM^{\perp}))$ and $\alpha_l \in \Gamma\,(\mathrm{ltr}(TM))$.

Chapter Three

Screen transversal null submanifolds of indefinite nearly Sasakian manifolds

In this chapter, we define and study the indefinite nearly Sasakian manifolds and invariant submanifolds. We present the contact of GCR-null submanifolds of indefinite contact manifolds. We introce screen transversal of null submanifolds and radical screen transversal submanifolds, and we give examples of submanifolds.

3.1 Indefinite nearly Sasakian manifolds

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with a contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying [1]

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \text{ and } \overline{\phi}\xi = 0. \tag{3.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is a contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [16], for any vector field X, Y on $\overline{M}$,

$$\overline{g}(\overline{\phi} X, \overline{\phi} Y) = \overline{g}(X, Y) - \eta(X)\,\eta(Y). \tag{3.2}$$

It follows that, for any vector X on $\overline{M}$,

$$\eta(X) = \overline{g}(\xi, X).$$

If, moreover,

$$(\overline{\nabla}_X \overline{\phi})Y + (\overline{\nabla}_Y \overline{\phi})X = 2\overline{g}(X, Y)\xi - \eta(Y)X - \eta(X)Y, \tag{3.3}$$

for any vector fields X, Y on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an *indefinite nearly Sasakian manifold.*

More precisely, $\overline{M}$ is called nearly cosymplectic manifold if

$$(\overline{\nabla}_X\overline{\phi})Y + (\overline{\nabla}_Y\overline{\overline{\phi}})X = 0, \tag{3.4}$$

for any vector fields X, Y on $\overline{M}$.

Let Ω be the fundamental 2-form of $\overline{M}$ defined by

$$\Omega(X,Y) = \overline{g}(X,\overline{\phi}Y), \quad X,Y \in \Gamma(T\overline{M}).$$

Replacing Y by ξ in (3.3), we obtain

$$\overline{\nabla}_X\xi + \overline{\phi}(\overline{\nabla}_\xi\overline{\phi})X = -\overline{\phi}X, \quad \forall X \in \Gamma(T\overline{M}). \tag{3.5}$$

Introduce a (1,1)-tensor H on $\overline{M}$ by taking

$$HX = \overline{\phi}(\overline{\nabla}_\xi\overline{\phi})X, \quad \forall X \in \Gamma(T\overline{M}),$$

such that (3.5)reduces to

$$\overline{\nabla}_X\xi = -HX - \overline{\phi}X. \tag{3.6}$$

The linear operator H has the properties

$$H\overline{\phi} + \overline{\phi}H = 0, \quad H\xi = 0, \quad \eta \circ H = 0, \tag{3.7}$$

and

$$\overline{g}(HX,Y) = -\overline{g}(X,HY). \tag{3.8}$$

That is, H is skew-symmetric.

Lemma 3.1.1 ([16])**.** *Let $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$ be an indefinite nearly Sasakian manifold. Then,*

$$\Omega(X,Y) = d\eta(X,Y) + \overline{g}(HX,Y), \tag{3.9}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Proof. The relation (3.9) follows from a straightforward calculation. The second assertion follows from Theorem 3.2 in [2]. □

Proposition 3.1.1 ([2])**.** *Let $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$ be a nearly Sasakian manifold. Then, the vector field ξ is a Killing vector field.*

Proof. For any X and $Y \in \Gamma(T\overline{M})$, one has

$$\begin{aligned}(\mathcal{L}_\xi\overline{g})(X,Y) &= \overline{g}(\overline{\nabla}_X\xi,Y) + \overline{g}(X,\overline{\nabla}_Y\xi)\\ &= \overline{g}(-HX-\overline{\phi}X,Y) + \overline{g}(X,-HY-\overline{\phi}Y)\\ &= -\overline{g}(HX,Y) - \overline{g}(\overline{\phi}X,Y) - \overline{g}(X,HY) - \overline{g}(X,\overline{\phi}Y)\\ &= 0,\end{aligned}$$

which shows that ξ is Killing. □

Definition 3.1.1 ([1]). A contact metric manifold is said to be a normal [3] if the torsion tensor $N^{(1)}$ vanishes, i.e.,

$$N^{(1)} = [\overline{\phi}, \overline{\phi}] + 2d\eta \otimes \xi = 0, \tag{3.10}$$

where

$$[\overline{\phi}, \overline{\phi}](X, Y) = \overline{\phi}^2[X, Y] + [\overline{\phi}X, \overline{\phi}Y] - \overline{\phi}[\overline{\phi}X, Y] - \overline{\phi}[X, \overline{\phi}Y], \tag{3.11}$$

is the Nijenhuis tensor of $\overline{\phi}$ and d denotes the exterior derivative operator.

The relation (3.11) can be rewritten as

$$[\overline{\phi}, \overline{\phi}](X, Y) = ((\overline{\nabla}_{\overline{\phi}X}\overline{\phi})Y - (\overline{\nabla}_{\overline{\phi}Y}\overline{\phi})X) - \overline{\phi}((\overline{\nabla}_X\overline{\phi})Y - (\overline{\nabla}_Y\overline{\phi})X). \tag{3.12}$$

Using (3.12) and Lemma 3.1.1, the component of (3.10) in the direction of ξ is given by

$$\begin{aligned} \eta\left(N^{(1)}(X, Y)\right) &= \eta([\overline{\phi}, \overline{\phi}](X, Y)) + 2d\eta(X, Y) \\ &= -2\{\overline{g}(HX, Y) + \Omega(X, Y)\} + 2d\eta(X, Y) \\ &= -2\{\overline{g}(HX, Y) + d\eta(X, Y) + \overline{g}(HX, Y)\} + 2d\eta(X, Y) \\ &= -4\overline{g}(HX, Y), \end{aligned} \tag{3.13}$$

for any X, $Y \in \Gamma(T\overline{M})$.

Note that, for any X, Y, $Z \in \Gamma(T\overline{M})$,

$$\overline{g}\left((\overline{\nabla}_Z\overline{\phi})X, Y\right) = -\overline{g}\left(X, (\overline{\nabla}_Z\overline{\phi})Y\right).$$

This means that the tensor $\overline{\nabla}\,\overline{\phi}$ is skew-symmetric.

Lemma 3.1.2. *Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite nearly Sasakian manifold. Then, $\overline{M}$ is Sasakian if and only if H vanishes identically on $\overline{M}$.*

Proof. If $\overline{M}$ is Sasakian, then it is normal, i.e., the tensor $N^{(1)}$ vanishes. Using (3.13), we have, for any $X, Y \in \Gamma(T\overline{M})$, $\overline{g}(HX, Y) = 0$. Thus $H = 0$. Conversely, if $H = 0$, by (3.13), $\eta\left(N^{(1)}(X, Y)\right) = 0$, for any $X, Y \in \Gamma(T\overline{M})$. This means that $\eta([\overline{\phi}, \overline{\phi}](X, Y)) = -2d\eta(X, Y)$. Since $\eta([\overline{\phi}, \overline{\phi}](X, Y)) = -2\Omega(X, Y)$, one has

$$\Omega(X, Y) = d\eta(X, Y).$$

This means that $\overline{M}$ is Sasakian and this completes the proof. □

3.2 Invariant null submanifolds

Definition 3.2.1. [8] Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be a null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \phi, \xi, \eta, \overline{g})$. For any vector field X tangent to M, we put

$$\overline{\phi}X = TX + FX, \tag{3.14}$$

where TX and FX are the tangential and the transversal parts of $\overline{\phi}X$, respectively. Moreover, P is skew-symmetric on $S(TM)$.

Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be a null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Assume that $\xi \in \Gamma(Rad\,TM)$. Then there exists $N \in \Gamma(\mathrm{ltr}(TM))$, such that $\overline{g}(N, \xi) = 1$. On the other hand,

$$0 = \overline{g}(\overline{\phi}N, \overline{\phi}\xi) = \overline{g}(N, \xi) - \eta(N)\eta(\xi), \tag{3.15}$$

which implies $\overline{g}(N, \xi) = 0$, a contradiction. Therefore, the characteristic vector field ξ does not belong $Rad\,TM$. This means that if M is tangent to the characteristic vector field ξ, then ξ belongs to $S(TM)$.

We say that M is *invariant* in $\overline{M}$ if M is tangent to the structure vector field ξ and

$$\overline{\phi}X = TX, \quad \forall X \in \Gamma(TM). \tag{3.16}$$

That is, $\overline{\phi}X \in \Gamma(TM)$. This definition is equivalent to

$$\overline{\phi}(Rad\,TM) = Rad\,TM \text{ and } \overline{\phi}(S(TM)) = S(TM). \tag{3.17}$$

Similarly, for any vector field $V \in \Gamma\,(tr(TM))$, let us put

$$\overline{\phi}V = tV + fV, \tag{3.18}$$

where tV and fV are tangential and transversal components of $\overline{\phi}V$, respectively. Clearly, the submanifold M which is tangent to the structure vector field ξ is invariant in $\overline{M}$ if $\overline{\phi}V = fV$.

Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be an invariant null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$.

Put $Y = \xi$ in (2.12), we get

$$\overline{\nabla}_X\xi = \nabla_X\xi + h^l(X, \xi) + h^s(X, \xi).$$

From (3.6) and (3.16) we have

$$-H^M X - H^l X - H^s X - TX = \nabla_X\xi + h^l(X, \xi) + h^s(X, \xi),$$

from transversal parts, we obtain

$$h^l(X,\xi) = -H^l X, \quad h^s(X,\xi) = -H^s X, \quad \nabla_X \xi = -H^M X - TX. \tag{3.19}$$

Then using (2.10), (2.11) and (3.16), we have

$$\begin{aligned}(\overline{\nabla}_X\overline{\phi})Y + (\overline{\nabla}_Y\overline{\phi})X &= \overline{\nabla}_X\overline{\phi}Y + \overline{\nabla}_Y\overline{\phi}X - \overline{\phi}(\overline{\nabla}_X Y) - \overline{\phi}(\overline{\nabla}_Y X)\\ &= \nabla_X TY + h(X,TY) + \nabla_Y TX + h(Y,TX) - \overline{\phi}(\nabla_X Y) - \overline{\phi}h(X,Y)\\ &\quad - \overline{\phi}(\nabla_Y X) - \overline{\phi}h(X,Y)\\ &= (\nabla_X T)Y + (\nabla_Y T)X + h(X,TY) + h(Y,TX) - 2\overline{\phi}h(X,Y), \end{aligned} \tag{3.20}$$

for all $X, Y \in \Gamma(TM)$.

Using (3.3), the relation (3.20) becomes

$$\begin{aligned}2\overline{g}(X,Y)\xi - \eta(Y)X - \eta(X)Y &= (\nabla_X T)Y + (\nabla_Y T)X\\ &\quad + h(X,TY) + h(Y,TX) - 2\overline{\phi}h(X,Y).\end{aligned} \tag{3.21}$$

Comparing the tangential and normal components, we have

$$(\nabla_X T)Y + (\nabla_Y T)X = 2\overline{g}(X,Y)\xi - \eta(Y)X - \eta(X)Y, \tag{3.22}$$

$$\text{and} \ \ h(X,TY) + h(Y,TX) = 2\overline{\phi}h(X,Y). \tag{3.23}$$

Therefore, we have the following.

Theorem 3.2.1. *Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be an invariant null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then M is nearly Sasakian. Moreover, M cannot be proper totally umbilical.*

Example 3.2.1. Let $\overline{M} = (\mathbb{R}^5_1, \overline{g})$ be a semi-Riemannian manifold of signature (-,+,+,+,+) with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5\}.$$

Let M be a submanifold given by

$$\begin{aligned}&x_1 = u_3, \quad x_2 = \sin\alpha u_1 + \cos\alpha u_3,\\ &x_3 = \cos\alpha u_1 + \sin\alpha u_3, \quad x_4 = u_2, \quad x_5 = 0.\end{aligned}$$

Then TM is spanned by $\{Z_1, Z_2, Z_3\}$, where

$$\begin{aligned}Z_1 &= -\sin\alpha\partial x_2 + \cos\alpha\partial x_3,\\ Z_2 &= \partial x_4,\\ Z_3 &= \partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial x_3.\end{aligned}$$

Hence M is a null submanifold of $\mathbb{R}^5_1$ with

$$\mathrm{Rad}(TM) = \mathrm{span}\{Z_3\},$$

and

$$S(TM) = \mathrm{span}\{Z_1, Z_2\}.$$

It is easy to see that

$$\overline{\phi}Z_3 = Z_3 \in \Gamma\left(\mathrm{Rad}(TM)\right), \quad \overline{\phi}Z_1 = Z_1 \in \Gamma\left(S(TM)\right), \\ \overline{\phi}Z_2 = Z_2 \in \Gamma\left(S(TM)\right),$$

which mean that $S(TM)$ and $\mathrm{Rad}(TM)$ is invariant with respect to $\overline{\phi}$. On the other hand by direct calculations, we get the null transversal bundle and screen transversal distribution are spanned by

$$N = \frac{1}{2}\{-\partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial x_3\}, \quad W = \partial x_5,$$

respectively. It is clear that $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ are invariant distributions. Then M is an invariant null submanifold of $\overline{M}$.

3.3 GCR-null submanifolds

In the section, we present the concept GCR-null submanifolds of indefinite contact manifolds. The discussion is mainly based on the references [6], [7] and [19].

Now, we introduce GCR-null submanifolds of indefinite contact manifold $(\overline{M}, \overline{g})$. Duggal and Sahin [7] (also see [9] and [17]) introduced a class of contact CR-null submanifold, called contact generalized Cauchy-Riemann (GCR)-null submanifold as follows.

Definition 3.3.1. [7] Let $(M, g, S(TM))$ be a real null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{g})$ such that ξ is tangent to M. Then, M is called contact generalized Cauchy-Riemann(GCR)-null submanifold if the following conditions are satisfied:

(1) There exist two subbundles D_1 and D_2 of $Rad\,TM$ such that

$$Rad\,TM = D_1 \oplus D_2, \quad \overline{\phi}(D_1) = D_1, \quad \overline{\phi}(D_2) \subset S(TM). \tag{3.24}$$

(2) There exist two subbundles D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \{\overline{\phi}(D_2) \oplus \overline{D}\}\perp D_0 \perp \{\xi\}, \quad \overline{\phi}(\overline{D}) = \mathcal{L}_1 \perp \mathcal{L}_2, \tag{3.25}$$

where D_0 is an invariant non-degenerate distribution on M, $\{\xi\}$ is the 1-dimensional distribution spanned by ξ, $\mathcal{L}_1$ and $\mathcal{L}_2$ are vector subbundles of $\mathrm{ltr}(TM)$ and $S(TM^\perp)$ respectively.

Then, from the above definition, the tangent bundle TM of M decomposed as follows

$$TM = \{D \oplus \overline{D}\} \perp \{\xi\}, \quad D = \mathrm{Rad}(TM) \oplus D_0 \oplus \overline{\phi}(D_2). \tag{3.26}$$

A contact GCR-null submanifold $(M, g, S(TM))$ is called a proper if $D_2 \neq \{0\}$, $D_1 \neq \{0\}$, $D_0 \neq \{0\}$ and $\mathcal{L}_2 \neq \{0\}$.

Proposition 3.3.1. *There exist no coisotropic, isotropic or totally null proper contact GCR-null submanifold of an indefinite nearly Sasakian manifold.*

Proof. Suppose M is isotropic or totally null submanifold of $\overline{M}$. Then we have $S(TM) = \{0\}$, which is a contradiction with definition of proper contact GCR-null submanifold. Similarly, if M is coisotropic or totally null submanifolds of $\overline{M}$, we have $S(TM^\perp) = \{0\}$, there is a contradiction. This completes the proof. □

Next, we use the techniques and notations used by Duggal and Sahin in [7]. Let $\big(M, g, S(TM), S(TM^\perp)\big)$ be a contact GCR-null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$

Let Q, P_1, P_2 and P_3 be the morphisms on D, $\overline{\phi}\mathcal{L}_1$ and $\overline{\phi}\mathcal{L}_2$, respectively. Then we have, for any $X \in \Gamma(TM)$,

$$X = QX + P_1X + P_2X + \eta(X)\xi. \tag{3.27}$$

Also, we write

$$\overline{\phi}X = TX + FX, \tag{3.28}$$

where TX and FX are the tangential and transversal parts of $\overline{\phi}X$. Applying $\overline{\phi}$ to (3.27), we have

$$\overline{\phi}X = TX + FP_1X + FP_2X, \tag{3.29}$$

where $TX = \overline{\phi}QX \in \Gamma(D)$, $FP_1X \in \Gamma(\mathcal{L}_1)$ and $FP_2X \in \Gamma(\mathcal{L}_2)$. Similarly,

$$\overline{\phi}V = tV + fV, \ \ \forall\, V \in \Gamma(tr(TM)), \tag{3.30}$$

where tV and fV are sections of TM and $tr(TM)$, respectively.

Comparing tangential and normal components (3.29) and (3.30), we have the following Lemma.

Lemma 3.3.1. *For a contact GCR-null submanifold $(M, g, S(TM))$ of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$, the following equalities holds:*

$$T^2 + tFP_1 + tFP_2 = -\mathbb{I} + \eta \otimes \xi, \tag{3.31}$$

$$FP_1T + FP_2T + fFP_1 + fFP_2 = 0, \tag{3.32}$$

$$f^2 - Ft = -\mathbb{I}, \tag{3.33}$$

$$Tt + tf = 0. \tag{3.34}$$

Differentiating (3.29), and using (2.10) and (2.14), we have

$$\begin{aligned}(\overline{\nabla}_X\overline{\phi})Y &= (\nabla_X T)Y + h(X, TY) - A_{FP_1Y}X - A_{FP_2Y}X + \nabla^l_X FP_1Y \\ &+ D^s(X, FP_1Y) + \nabla^s_X FP_2Y + D^l(X, FP_2Y) - FP_1\nabla_X Y \\ &- FP_2\nabla_X Y - th^l(X, Y) - th^s(X, Y) - fh^l(X, Y) - fh^s(X, Y),\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}(\overline{\nabla}_Y\overline{\phi})X &= (\nabla_Y T)X + h(Y, TX) - A_{FP_1X}Y - A_{FP_2X}Y + \nabla^l_Y FP_1X \\ &+ D^s(Y, FP_1X) + \nabla^s_Y FP_2X + D^l(Y, FP_2X) - FP_1\nabla_Y X \\ &- FP_2\nabla_Y X - th^l(Y, X) - th^s(Y, X) - fh^l(Y, X) - fh^s(Y, X).\end{aligned} \tag{3.36}$$

From (3.3), and putting the pieces (3.35) and (3.36), for any X and $Y \in \Gamma(TM)$,

$$\begin{aligned}2g(X, Y)\xi - \eta(Y)X - \eta(X)Y &= (\nabla_X T)Y + (\nabla_Y T)X + h(X, TY) + h(Y, TX) \\ &- A_{FP_1Y}X - A_{FP_2Y}X - A_{FP_1X}Y - A_{FP_2X}Y + \nabla^l_X FP_1Y + \nabla^l_Y FP_1X \\ &+ D^s(X, FP_1Y) + D^s(Y, FP_1X) + \nabla^s_X FP_2Y + \nabla^s_Y FP_2X + D^l(X, FP_2Y) \\ &+ D^l(Y, FP_2X) - FP_1\nabla_X Y - FP_1\nabla_Y X - FP_2\nabla_X Y - FP_2\nabla_Y X \\ &- 2th^l(X, Y) - 2th^s(X, Y) - 2fh^l(X, Y) - 2fh^s(X, Y).\end{aligned} \tag{3.37}$$

Comparing the tangential and the transversal components, we have the following.

Proposition 3.3.2. *Let $(M, g, S(TM))$ be a contact GCR-null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. Then, we have*

$$\begin{aligned}(\nabla_X T)Y + (\nabla_Y T)X &= 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y + A_{FY}X + A_{FX}Y \\ &+ 2th^l(X, Y) + 2th^s(X, Y),\end{aligned} \tag{3.38}$$

$$\begin{aligned}h^l(X, TY) + h^l(Y, TX) &= -\nabla^l_X FP_1Y - \nabla^l_Y FP_1X - D^l(X, FP_2Y) - D^l(Y, FP_2X) \\ &+ FP_1\nabla_X Y + FP_1\nabla_Y X + 2fh^l(X, Y),\end{aligned} \tag{3.39}$$

$$\begin{aligned}h^s(X, TY) + h^s(Y, TX) &= -\nabla^s_X FP_2Y - \nabla^s_Y FP_2X - D^s(X, FP_1Y) - D^s(Y, FP_1X) \\ &+ FP_2\nabla_X Y + FP_2\nabla_Y X + 2fh^s(X, Y),\end{aligned} \tag{3.40}$$

for any vector fields X and Y on M.

Now, for any X, $Y \in \Gamma(D \oplus \{\xi\})$, from (3.38), one has

$$(\nabla_X T)Y + (\nabla_Y T)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y + 2th^l(X,Y) + 2th^s(X,Y). \tag{3.41}$$

Theorem 3.3.1. *Let $(M, g, S(TM))$ be a contact GCR-null submanifold of an indefinite nearly Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$. If the distribution $D \oplus \{\xi\}$ is integrable, then its leaves are nearly Sasakian, immersed in $\overline{M}$ as submanifolds if and only if*

$$th(X,Y) = 0, \quad \forall X,\, Y \in \Gamma(D \oplus \{\xi\}).$$

Using Definition 3.3.1 and the relation (3.17), we have the following.

Theorem 3.3.2. *There exist no invariant contact GCR-null submanifolds of indefinite nearly Sasakian manifolds.*

3.4 Screen transversal submanifolds

In this section, we introduce screen transversal null submanifold of an indefinite nearly Sasakian manifold.

Definition 3.4.1. [23] Let M be an r-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, we say that M is a screen transversal null submanifold of $\overline{M}$ if there exists a screen transversal bundle $S(TM^\perp)$ such that

$$\overline{\phi}\,(\mathrm{Rad}(TM)) \subset S(TM^\perp).$$

Similarly, as in [23], we have the following lemma

Lemma 3.4.1. *Let M be an r-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Suppose that $\overline{\phi}\,(\mathrm{Rad}(TM))$ is a vector subbundle of $S(TM^\perp)$. Then, $\overline{\phi}\,(\mathrm{ltr}(TM))$ is also vector subbundle of the screen transversal bundle $S(TM^\perp)$. Furthermore,*

$$\overline{\phi}\,(\mathrm{Rad}(TM)) \cap \overline{\phi}\,(\mathrm{ltr}(TM)) = \{0\}.$$

Proof. Suppose that $\mathrm{ltr}(TM)$ is invariant with respect to $\overline{\phi}$, i.e.,

$$\overline{\phi}\,(\mathrm{ltr}(TM)) = \mathrm{ltr}(TM).$$

From definition of a null submanifold, there exist vector fields $E \in \Gamma\,(\mathrm{Rad}(TM))$ and $N \in \Gamma\,(\mathrm{ltr}(TM))$ such that $\overline{g}(E, N) = 1$. Also, from (3.2) we get

$$\overline{g}(E, N) = \overline{g}(\overline{\phi}E, \overline{\phi}N) = 1.$$

However, if $\overline{\phi}N \in \Gamma(\mathrm{ltr}(TM))$ then by hypoDissertation, we get $\overline{g}(\overline{\phi}E, \overline{\phi}N) = 0$. Hence, we obtain a contradiction which implies that $\overline{\phi}N \notin \mathrm{ltr}(TM)$. Now, let $\overline{\phi}N \in \Gamma(S(TM))$. Then, from (3.2) we obtain

$$1 = \overline{g}(E, N) = \overline{g}(\overline{\phi}E, \overline{\phi}N) = 0,$$

since $\overline{\phi}E \in \Gamma\left(S(TM^{\perp})\right)$ and $\overline{\phi}N \in \Gamma(S(TM))$. Thus, $\overline{\phi}N \notin S(TM)$. We can also obtain that $\overline{\phi}N \notin \mathrm{Rad}(TM)$. Then, we can say that $\overline{\phi}N \in \Gamma\left(S(TM^{\perp})\right)$. Suppose that there exist a vector field $X \in \Gamma\big(\overline{\phi}(\mathrm{Rad}(TM)) \cap \overline{\phi}(\mathrm{ltr}(TM))\big)$. Then, from (3.2) we get

$$0 \neq \overline{g}(\overline{\phi}X, N) = -\overline{g}(X, \overline{\phi}N) = 0,$$

which is a contradiction. Then, we get

$$\overline{\phi}(\mathrm{Rad}(TM)) \cap \overline{\phi}(\mathrm{ltr}(TM)) = \{0\},$$

which completes the proof. □

From the above definition and Lemma 3.4.1 it follows that $\overline{\phi}(\mathrm{ltr}(TM)) \subset S(TM^{\perp})$.

Definition 3.4.2. [23] Let M be screen transversal null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, we say that M is a screen transversal anti-invariant null submanifold of $\overline{M}$ if $S(TM)$ is screen transversal with respect to $\overline{\phi}$, that is,

$$\overline{\phi}(S(TM)) \subset S(TM^{\perp}).$$

Remark 3.4.1. If M is screen transversal anti-invariant null submanifold, from definition 3.4.2, we have

$$S(TM^{\perp}) = \overline{\phi}(\mathrm{Rad}(TM)) \oplus \overline{\phi}(\mathrm{ltr}(TM)) \perp \overline{\phi}(S(TM)) \perp D_0,$$

where D_0 is a non-degenerate orthogonal complementary distribution to

$$\overline{\phi}(\mathrm{Rad}(TM)) \oplus \overline{\phi}(\mathrm{ltr}(TM)) \perp \overline{\phi}(S(TM)),$$

in $S(TM^{\perp})$. For the distribution D_0, we have the following results that were proved by C. Yildirim and B. Sahin in [23].

Proposition 3.4.1. *Let M be a screen transversal anti-invariant null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, the distribution D_0 is invariant with respect to $\overline{\phi}$.*

Proof. For $X \in \Gamma(D_0)$, $E \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(ltr(TM))$, we have

$$\overline{g}(\overline{\phi}X, \overline{\phi}E) = \overline{g}(X, E) = 0, \text{ and } \overline{g}(\overline{\phi}X, N) = -\overline{g}(X, \overline{\phi}N) = 0,$$

which implies that $\overline{\phi}(D_0)\cap\overline{\phi}(\mathrm{Rad}(TM)) = \{0\}$ and $\overline{\phi}(D_0)\cap\overline{\phi}(\mathrm{ltr}(TM)) = \{0\}$. Moreover, for $Z \in \Gamma(S(TM))$, since $\overline{\phi}Z \in \Gamma\overline{\phi}(S(TM))$, $\overline{\phi}(S(TM))$ and D_0 are orthogonal, we obtain $\overline{g}(\overline{\phi}X, Z) = -\overline{g}(X, \overline{\phi}Z) = 0$, which shows that $\overline{\phi}(D_0) \cap S(TM) = \{0\}$. Hence, we also have $\overline{\phi}(D_0) \cap \overline{\phi}(S(TM)) = \{0\}$. Thus, we get

$$\overline{\phi}(D_0) \cap (TM) = \{0\}, \qquad \overline{\phi}(D_0) \cap \mathrm{ltr}(TM) = \{0\},$$

and

$$\overline{\phi}(D_0) \cap \{\overline{\phi}(S(TM)) \perp \overline{\phi}(\mathrm{Rad}(TM)) \oplus \overline{\phi}(\mathrm{ltr}(TM))\} = \{0\},$$

which prove that D_0 is invariant. □

Proposition 3.4.2. *Suppose that M is a real null curve submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then M is an isotropic screen transversal null submanifold.*

Proof. Let M is a real null curve, we obtain

$$TM = \mathrm{Rad}(TM) = \mathrm{span}\{E\},$$

where E is the tangent vector to M. Then, from (3.2) we have $\overline{g}(\overline{\phi}E, E) = 0$ which implies that $\overline{\phi}E \notin \mathrm{ltr}(TM)$. Since $\dim(TM) = 1$, $\overline{\phi}E$ and E are linearly independent $\overline{\phi}E \notin TM$. Hence, we can say that $\overline{\phi}E \in \Gamma\left(S(TM^{\perp})\right)$. Similarly, from (3.2) we obtain $\overline{g}(\overline{\phi}N, N) = 0$, which shows that $\overline{\phi}N \notin \mathrm{Rad}(TM)$. Also, we have

$$\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0,$$

because $\overline{\phi}E \in \Gamma\left(S(TM^{\perp})\right)$. Then we obtain $\overline{\phi}N \notin \mathrm{ltr}(TM)$, since, $\overline{\phi}N \in \Gamma\left(S(TM^{\perp})\right)$. Furthermore, from (3.2) we get

$$\overline{g}(\overline{\phi}N, \overline{\phi}E) = \overline{g}(N, E) = 1.$$

Thus, we conclude

$$S(TM^{\perp}) = \overline{\phi}(\mathrm{Rad}(TM)) \oplus \overline{\phi}(\mathrm{ltr}(TM)) \perp D_0,$$

where D_0 is a non-degenerate distribution and this completes the proof. □

3.5 Radical screen transversal submanifolds

Definition 3.5.1. [23] Let M be a screen transversal null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, we say that M is a radical screen transversal null submanifold if $S(TM)$ is invariant with respect to $\overline{\phi}$.

Theorem 3.5.1. *Let M be a radical screen transversal null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$, tangent to the characteristic vector field ξ. Then the radical distribution* $\mathrm{Rad}(TM)$ *is integrable if and only if*

$$g(A_{\overline{\phi}E_1}E_2 - A_{\overline{\phi}E_2}E_1 - 2(\overline{\nabla}_{E_1}\phi)E_2, \overline{\phi}X) = 0,$$

for $X \in \Gamma\left(S(TM) - \{\xi\}\right)$ and $E_1, E_2 \in \Gamma\left(\mathrm{Rad}(TM)\right)$.

Moreover, if the radical distribution $\mathrm{Rad}(TM)$ *is integrable, then*

$$HX + \overline{\phi}X \in \Gamma(S(TM) \perp \mathrm{ltr}(TM)), \tag{3.42}$$

for any $X \in \Gamma(\mathrm{Rad}\, TM))$.

Proof. By the definition of a radical screen transversal null submanifold, $\mathrm{Rad}(TM)$ is integrable if and only if $g([E_1, E_2], X) = 0$, for any $X \in \Gamma(S(TM))$ and E_1, $E_2 \in \Gamma(\mathrm{Rad}(TM))$. Let R be the projection on $S(TM) - \{\xi\}$. Then, for any $X \in \Gamma(S(TM))$,

$$X = RX + \eta(X)\xi,$$

From (3.2), (3.3) and (3.6), we have

$$\begin{aligned}
\overline{g}([E_1, E_2], X) &= \overline{g}([E_1, E_2], RX) + \eta(X)\overline{g}([E_1, E_2], \xi) \\
&= \overline{g}([E_1, E_2], RX) + \eta(X)\left\{\overline{g}(\overline{\nabla}_{E_1}E_2, \xi) - \overline{g}(\overline{\nabla}_{E_2}E_1, \xi)\right\} \\
&= \overline{g}([E_1, E_2], RX) + \eta(X)\left\{-\overline{g}(E_2, \overline{\nabla}_{E_1}\xi) + \overline{g}(E_1, \overline{\nabla}_{E_2}\xi)\right\} \\
&= \overline{g}([E_1, E_2], RX) + \eta(X)\left\{\overline{g}(E_2, HE_1 + \overline{\phi}E_1) - \overline{g}(E_1, HE_2 + \overline{\phi}E_2)\right\} \\
&= \overline{g}([E_1, E_2], RX) + 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\} \\
&= \overline{g}(\overline{\phi}[E_1, E_2], \overline{\phi}X) + 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\} \\
&= \overline{g}(\overline{\phi}\,\overline{\nabla}_{E_1}E_2, \overline{\phi}X) - \overline{g}(\overline{\phi}\,\overline{\nabla}_{E_2}E_1, \overline{\phi}X) + 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\} \\
&= \overline{g}(\overline{\nabla}_{E_2}\overline{\phi}E_1 - (\overline{\nabla}_{E_1}\overline{\phi})E_2, \overline{\phi}X) - \overline{g}(\overline{\nabla}_{E_1}\overline{\phi}E_2 - (\overline{\nabla}_{E_2}\overline{\phi})E_1, \overline{\phi}X) \\
&+ 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\} \\
&= \overline{g}(\overline{\nabla}_{E_2}\overline{\phi}E_1, \overline{\phi}X) - g(\overline{\nabla}_{E_1}\overline{\phi}E_2, \overline{\phi}X) - 2g((\overline{\nabla}_{E_1}\overline{\phi})E_2, \overline{\phi}X) \\
&+ 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\}.
\end{aligned}$$

Using (2.14), we obtain

$$\begin{aligned}
\overline{g}([E_1, E_2], X) &= \overline{g}\left(-A_{\overline{\phi}E_1}E_2 + \nabla^s_{E_2}\overline{\phi}E_1 + D^l(E_2, \overline{\phi}E_1), \overline{\phi}X\right) \\
&- \overline{g}\left(-A_{\overline{\phi}E_2}E_1 + \nabla^s_{E_1}\overline{\phi}E_2 + D^l(E_1, \overline{\phi}E_2), \overline{\phi}X\right) \\
&- 2g((\overline{\nabla}_{E_1}\overline{\phi})E_2, \overline{\phi}X) + 2\eta(X)\left\{\overline{g}(E_2, HE_1) + \overline{g}(E_2, \overline{\phi}E_1)\right\},
\end{aligned}$$

and this completes the proof. □

Theorem 3.5.2. *Let M be a radical screen transversal null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, if the screen distribution $S(TM)$ defines a totally geodesic foliation, then*

$$h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X)$$

has no components in $\overline{\phi}\,(\mathrm{Rad}(TM))$, for any $X,\ Y \in \Gamma\,(S(TM))$.

Proof. By the definition of radical screen transversal null submanifold $S(TM)$ is a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, N) = 0$, for any $X,\ Y \in \Gamma\,(S(TM))$, $N \in \Gamma\,(\mathrm{ltr}(TM))$. From (2.12) and (3.2), we have

$$\begin{aligned}\overline{g}(\nabla_X Y, N) &= \overline{g}(\overline{\nabla}_X Y, N)\\ &= \overline{g}(\overline{\phi}\ \overline{\nabla}_X Y, \overline{\phi}N).\end{aligned}$$

Also, from (3.3) we get

$$\overline{g}(\nabla_X Y, N) + \overline{g}(\nabla_Y X, N) = \overline{g}(\overline{\nabla}_X \overline{\phi}Y, \overline{\phi}N) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X, \overline{\phi}N).$$

Using (2.12), we obtain

$$\begin{aligned}\overline{g}(\nabla_X Y, N) + \overline{g}(\nabla_Y X, N) &= \overline{g}(\nabla_X \overline{\phi}Y + h^l(X, \overline{\phi}Y) + h^s(X, \overline{\phi}Y), \overline{\phi}N)\\ &\quad + \overline{g}(\nabla_Y \overline{\phi}X + h^l(Y, \overline{\phi}X) + h^s(Y, \overline{\phi}X)), \overline{\phi}N)\\ &= \overline{g}(h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X), \overline{\phi}N),\end{aligned}$$

and the assertion follows. □

Example 3.5.1. Let $\overline{M} = (\mathbb{R}^9_2, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbb{R}^9_2$ defined by

$$x^1 = 0, x^2 = u_1, x^3 = u_2, x^4 = u_3,$$

$$y^1 = u_1, y^2 = 0, y^3 = -u_3, y^4 = u_2.$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned}Z_1 &= 2(\partial x_2 + \partial y_1 + y^2 \partial z),\\ Z_2 &= 2(\partial x_3 + \partial y_4 + y^3 \partial z),\\ Z_3 &= 2(\partial x_4 - \partial y_3 + y^4 \partial z),\\ Z_4 &= \xi = 2\partial z.\end{aligned}$$

Thus, M is a 1-null submanifold with $\mathrm{Rad}(TM) = \mathrm{span}\{Z_1\}$. Also, $\overline{\phi}Z_2 = Z_3$ implies that $\overline{\phi}S(TM) = S(TM)$. Null transversal bundle $\mathrm{ltr}(TM)$ is spanned by

$$N = 2(-\partial x_2 - 2\partial y_1 - y^2\partial z).$$

Also, screen transversal bundle $S(TM^{\perp})$ is spanned by

$$\begin{aligned}
W_1 &= 2(\partial x_1 - \partial y_2 + y^1\partial z),\\
W_2 &= 2(-2\partial x_1 + \partial y_2 - 2y^1\partial z),\\
W_3 &= 2(\partial x_2 - \partial x_4 + \partial y_1 - \partial y_3 + (y^2 - y^4)\partial z),\\
W_4 &= 2(\partial x_1 - \partial x_3 - \partial y_2 + \partial y_4 + (y^1 - y^3)\partial z).
\end{aligned}$$

It follows that $\overline{\phi}Z_1 = W_1$, $\overline{\phi}N = W_2$ and $\overline{\phi}W_3 = W_4$. Hence M is radical screen transversal 1-null submanifold.

Contact CR-submanifolds of indefinite nearly Sasakian manifolds

In this chapter, we present the general concept of contact CR-submanifolds of indefinite nearly Sasakian manifolds, supported by an example. We also introduce contact screen CR-null submanifold of an indefinite nearly Sasakian manifold and give the example of contact SCR-null submanifolds of indefinite nearly Sasakian manifolds.

4.1 Contact CR-null submanifolds

In this section we use background definitions and techniques given in [7] in the context of indefinite nearly Sasakian manifolds.

Definition 4.1.1. [7] Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be a null submanifold, tangent to the structure vector field ξ, immersed in an indefinite nearly Sasakian manifold $(\overline{M}, \phi, \xi, \eta, \overline{g})$. We say that M is contact CR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$ if

(A) There exists a distribution $\mathrm{Rad}(TM)$ on M such that
$\mathrm{Rad}(TM) \cap \overline{\phi}\,(\mathrm{Rad}(TM)) = \{0\}$.

(B) There exist vector bundles D_0 and $D^{'}$ over M such that

$$S(TM) = \{\overline{\phi}\,(\mathrm{Rad}(TM)) \oplus D^{'}\} \perp D_0 \perp \{\xi\},$$
$$\overline{\phi}(D_0) = D_0,\ \overline{\phi}(D^{'}) = K_1 \perp \mathrm{ltr}(TM),$$

where D_0 is non-degenerate and K_1 is a vector subbundle of $S(TM^{\perp})$, we have the following decomposition:

$$TM = \{D \oplus D^{'}\} \perp \{\xi\}, \quad D = \mathrm{Rad}(TM) \perp \overline{\phi}\,(\mathrm{Rad}(TM)) \perp D_0. \tag{4.1}$$

Contact CR-null submanifold is said to be proper if $D_0 \neq \{0\}$ and $K_1 \neq \{0\}$. We see that any contact CR-null of 3-dimensional submanifold is 1-null.
Denote the orthogonal complement subbundle to the vector subbundle K_1 in $S(TM^\perp)$ by $K_1^\perp$. For contact CR-null submanifold M, we put

$$\overline{\phi}X = fX + \omega X, \quad \forall X \in \Gamma(TM), \tag{4.2}$$

where $fX \in \Gamma(D)$ and $\omega X \in \Gamma(K_1 \perp \mathrm{ltr}(TM))$. Similarly, we write

$$\overline{\phi}W = BW + CW, \quad \forall W \in \Gamma\big(S(TM^\perp)\big), \tag{4.3}$$

where $BW \in \Gamma(\overline{\phi}K_1)$ and $CW \in \Gamma(K_1^\perp)$.

Example 4.1.1. Let $\overline{M} = (\mathbb{R}^9_2, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Consider a submanifold M of $\mathbb{R}^9_2$ defined by

$$x^1 = y^4, \quad x^2 = \sqrt{1-(y^2)^2}, \quad y^2 \neq \pm 1.$$

Then a local frame of TM is given by

$$\begin{aligned}
Z_1 &= 2(\partial x_1 + \partial y_4 + y^1 \partial z),\\
Z_2 &= 2(\partial x_4 - \partial y_1 + y^4 \partial z),\\
Z_3 &= \partial x_3 + y^3 \partial z,\\
Z_4 &= \partial y_3,\\
Z_5 &= -\frac{y^2}{x^2}\partial x_2 + \partial y_2 - \frac{(y^2)^2}{x^2}\partial z,\\
Z_6 &= \partial x_4 + \partial y_1 + y^4 \partial z,\\
Z_7 &= \xi = 2\partial z.
\end{aligned}$$

Hence, $\mathrm{Rad}(TM) = \mathrm{span}\{Z_1\}$, $\overline{\phi}\,(\mathrm{Rad}(TM)) = \mathrm{span}\{Z_2\}$, and $\mathrm{Rad}(TM) \cap \overline{\phi}\,(\mathrm{Rad}(TM)) = \{0\}$. Hence (A) holds. Next, $\overline{\phi}(Z_3) = -Z_4$ implies that $D_0 = \{Z_3, Z_4\}$ is invariant with respect to $\overline{\phi}$. By direct calculations, we get

$$S(TM^\perp) = \mathrm{span}\{W = \partial x_2 + \frac{y^2}{x^2}\partial y_2 + y^2 \partial z\}$$

such that $\overline{\phi}(W) = -Z_5$ and $\mathrm{ltr}(TM)$ is spanned by

$$N = -\partial x_1 + \partial y_4 - y^1 \partial z$$

such that $\overline{\phi}(N) = Z_6$. Hence, M is contact CR-null submanifold.

Proposition 4.1.1. *Let M be a contact CR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. Then, D and $D^{'} \oplus D$ are not integrable.*

Proof. Suppose that D is integrable. Then, we have $g([X,Y],\xi)=0$, for any $X,Y \in \Gamma(D)$. Also, we derive

$$\begin{aligned} g([X,Y],\xi) &= \overline{g}(\overline{\nabla}_X Y,\xi) - \overline{g}(\overline{\nabla}_Y X,\xi) \\ &= -\overline{g}(Y,\overline{\nabla}_X \xi) + \overline{g}(X,\overline{\nabla}_Y \xi). \end{aligned}$$

Since $\overline{\nabla}$ is a metric connection and from (3.6), we get

$$\begin{aligned} g([X,Y],\xi) &= -\overline{g}(Y,-HX-\overline{\phi}X) + \overline{g}(X,-HY-\overline{\phi}Y) \\ &= \overline{g}(Y,HX) + \overline{g}(Y,\overline{\phi}X) - \overline{g}(X,HY) - \overline{g}(X,\overline{\phi}Y). \end{aligned}$$

Hence

$$g([X,Y],\xi) = 2\overline{g}(Y,HX) + 2\overline{g}(Y,\overline{\phi}X).$$

As M being proper and D_0 non-degenerate, then, we can choose non-null vector fields $X,Y \in \Gamma(D)$ such that $\overline{g}(Y,HX)+\overline{g}(Y,\overline{\phi}X) \neq 0$, which is a contradiction. Hence, D is not integrable. By a similar way it is easy to see that $D^{'} \oplus D$ is not integrable. □

Similarly, as in [7], we have the following proposition in case contact CR-null submanifold of an indefinite nearly Sasakian manifold.

Proposition 4.1.2. *Let M be a contact CR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$, tangent to the vector field ξ. If $D\perp\{\xi\}$ is a totally geodesic foliation, then*

$$h^l(X,\overline{\phi}Y) + h^l(\overline{\phi}X,Y) = 0, \tag{4.4}$$

and

$$h^s(X,\overline{\phi}Y) + h^s(\overline{\phi}X,Y),$$

has no components in K_1.

Proof. From Definition 4.1.1, $D\perp\{\xi\}$ defines a totally geodesic foliation if

$$\overline{g}(\nabla_X Y,\overline{\phi}E) = \overline{g}(\nabla_X Y,\overline{\phi}W) = 0,$$

for $X,Y \in \Gamma(D\perp\{\xi\})$ and $W \in \Gamma(K_1)$. Then, we have

$$\overline{g}(\nabla_X Y,\overline{\phi}E) = -\overline{g}(\overline{\phi}\,\overline{\nabla}_X Y,E) = -\overline{g}(\nabla_X \overline{\phi}Y,E) + \overline{g}((\nabla_X \overline{\phi})Y,E).$$

Now, from (3.3) and (2.12), we get

$$\begin{aligned} \overline{g}(\nabla_X Y,\overline{\phi}E) + \overline{g}(\nabla_Y X,\overline{\phi}E) &= -\overline{g}(\nabla_X \overline{\phi}Y + \nabla_Y \overline{\phi}X,E) + \overline{g}((\nabla_X \overline{\phi})Y + (\nabla_Y \overline{\phi})X,E) \\ &= -\overline{g}\left(h^l(X,\overline{\phi}Y) + h^l(Y,\overline{\phi}X),E\right). \end{aligned} \tag{4.5}$$

Similarly, from (3.3) and (2.12), we get

$$\begin{aligned}\overline{g}(\nabla_X Y, \overline{\phi}W) + \overline{g}(\nabla_Y X, \overline{\phi}W) &= -\overline{g}(\nabla_X \overline{\phi}Y + \nabla_Y \overline{\phi}X, W) + \overline{g}((\nabla_X \overline{\phi})Y + (\nabla_Y \overline{\phi})X, W) \\ &= -\overline{g}\left(h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X), W\right). \qquad (4.6)\end{aligned}$$

Then, from (4.5) and (4.6), we complete the proof. □

Proposition 4.1.3. *Let M be a contact CR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$, tangent to the vector field ξ. If D' is a totally geodesic foliation, then*

(i) HZ and $A_N Z$ has no components in $\overline{\phi}K_1 \perp \overline{\phi}(\mathrm{Rad}TM)$.

(ii) $A_{\overline{\phi}W}Z + A_{\overline{\phi}Z}W$ has no components in $D_0 \perp \mathrm{Rad}TM$,

for any Z, $W \in \Gamma(D')$.

Proof. Note that D' defines a totally geodesic foliation if and only if

$$g(\nabla_Z W, N) = g(\nabla_Z W, \overline{\phi}N) = g(\nabla_Z W, X) = g(\nabla_Z W, \xi) = 0,$$

for any Z, $W \in \Gamma(D')$, $N \in \Gamma(\mathrm{ltr}(TM))$ and $X \in \Gamma(D_0)$. From (3.3) and (3.6), one obtains

$$\begin{aligned}g(\nabla_Z W, \xi) = \overline{g}(\overline{\nabla}_Z W, \xi) &= -\overline{g}(W, \overline{\nabla}_Z \xi) \\ &= \overline{g}(W, HZ + \overline{\phi}Z) \\ &= \overline{g}(W, HZ). \qquad (4.7)\end{aligned}$$

On the other hand, $\overline{\nabla}$ is a metric connection and (2.13) implies

$$\overline{g}(\nabla_Z W, N) = g(W, A_N Z). \qquad (4.8)$$

Using (2.15), (3.3) and (3.6), we obtain

$$\begin{aligned}g(\nabla_Z W, \overline{\phi}N) + g(\nabla_W Z, \overline{\phi}N) &= \overline{g}(\overline{\nabla}_Z W, \overline{\phi}N) + \overline{g}(\overline{\nabla}_W Z, \overline{\phi}N) \\ &= -\overline{g}(\overline{\phi}\,\overline{\nabla}_Z W, N) - \overline{g}(\overline{\phi}\,\overline{\nabla}_W Z, N) \\ &= -\overline{g}(\overline{\nabla}_Z \overline{\phi}W, N) - \overline{g}(\overline{\nabla}_W \overline{\phi}Z, N) + \overline{g}((\overline{\nabla}_Z \overline{\phi})W, N) \\ &+ \overline{g}((\overline{\nabla}_W \overline{\phi})Z, N) \\ &= \overline{g}(A_{\overline{\phi}W}Z + A_{\overline{\phi}Z}W, N). \qquad (4.9)\end{aligned}$$

Likewise,

$$\begin{aligned}g(\nabla_Z W, \overline{\phi}N) + g(\nabla_W Z, \overline{\phi}X) &= \overline{g}(\overline{\nabla}_Z W, \overline{\phi}X) + \overline{g}(\overline{\nabla}_W Z, \overline{\phi}X) \\ &= -\overline{g}(\overline{\phi}\,\overline{\nabla}_Z W, X) - \overline{g}(\overline{\phi}\,\overline{\nabla}_W Z, X) \\ &= -\overline{g}(\overline{\nabla}_Z \overline{\phi}W, X) - \overline{g}(\overline{\nabla}_W \overline{\phi}Z, X) + \overline{g}((\overline{\nabla}_Z \overline{\phi})W, X) \\ &+ \overline{g}((\overline{\nabla}_W \overline{\phi})Z, X) \\ &= \overline{g}(A_{\overline{\phi}W}Z + A_{\overline{\phi}Z}W, X). \qquad (4.10)\end{aligned}$$

The proof of the assertions follows from the relations (4.7), (4.8), (4.9) and (4.10). □

Lemma 4.1.1. *Let M be a totally contact umbilical proper contact CR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$, tangent to the characteristic vector field ξ. Then $\alpha_l = 0$.*

Proof. The proof is similar to one given in [7, Lemma 7.4.8]. □

4.2 Contact SCR-null submanifolds

Definition 4.2.1. [7] Let $\big(M, g, S(TM), S(TM^{\perp})\big)$ be a null submanifold, tangent to the structure vector field ξ of an indefinite nearly Sasakian manifold $\overline{M}$. Then M is contact SCR-null submanifold of $\overline{M}$ if the following holds:

(1) There exist real non-null distributions D, $D^{\perp}$ such that

$$S(TM) = \big(D \oplus D^{\perp}\big) \perp \{\xi\}, \quad \overline{\phi}(D^{\perp}) \subset S(TM^{\perp}), \quad D \cap D^{\perp} = \{0\}, \quad (4.11)$$

where $D^{\perp}$ is orthogonally complementary to $D \perp \{\xi\}$ in $S(TM)$.

(2) The distributions D and $\mathrm{Rad}(TM)$ are invariant with respect to $\overline{\phi}$. It follows that $\mathrm{ltr}(TM)$ is also invariant with respect to $\overline{\phi}$. Hence, we have

$$TM = \big(\overline{D} \oplus D^{\perp}\big) \perp \{\xi\}, \quad \overline{D} = D \perp \mathrm{Rad}(TM). \quad (4.12)$$

Denote the orthogonal complement to $\overline{\phi}(D^{\perp})$ in $S(TM^{\perp})$ by μ. We say that M is a proper contact SCR-null submanifold of $\overline{M}$ if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. We note the following properties of contact SCR-null submanifold:

(i) condition (2) implies that $\dim(\mathrm{Rad}(TM)) = r = 2p \geq 2$;

(ii) for proper M, (2) implies that $\dim(D) = 2s \geq 2$, $\dim(D^{\perp}) \geq 1$. Thus $\dim(M) \geq 5$, $\dim(\overline{M}) \geq 9$.

For any $X \in \Gamma(TM)$ and $W \in \Gamma\big(S(TM^{\perp})\big)$, now we can put

$$\overline{\phi}X = P'X + F'X, \quad \overline{\phi}W = B'W + C'W, \quad (4.13)$$

where $P'X \in \Gamma(\overline{D})$, $F'X \in \Gamma(\overline{\phi}D^{\perp})$, $B'W \in \Gamma(D^{\perp})$ and $C'W \in \Gamma(\mu)$.

Example 4.2.1. [8] Let M be a submanifold of $\overline{M} = (\mathbb{R}^9_2, \overline{g})$ defined by

$$x^1 = x^2, \quad y^1 = y^2, \quad x^4 = \sqrt{1 - (y^4)^2}, \quad y^4 \neq \pm 1. \quad (4.14)$$

There exist a local frame of TM given by

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_2 + (y^1 + y^2)\partial z, \\ Z_2 &= \partial y_1 + \partial y_2, \\ Z_3 &= \partial x_3 + y^3 \partial z, \\ Z_4 &= \partial y_3, \\ Z_5 &= -y^4 \partial x_4 + x^4 \partial y_4 - (y^4)^2 \partial z, \\ \xi &= 2\partial z. \end{aligned} \tag{4.15}$$

Then, $\mathrm{Rad}(TM) = \mathrm{span}\{Z_1, Z_2\}$ and $\overline{\phi}(Z_1) = -Z_2$. Since $\mathrm{Rad}(TM)$ is invariant with respect to $\overline{\phi}$. Also, $\overline{\phi}(Z_3) = -Z_4$ implies that $D = \mathrm{span}\{Z_3, Z_4\}$. Then, we get $S(TM^\perp) = \mathrm{span}\{W = x^4\partial x_4 + y^4 \partial y_4 + x^4 y^4 \partial z\}$ such that $\overline{\phi}(W) = -Z_5$ and null transversal bundle $\mathrm{ltr}(TM)$ is spanned by

$$\begin{aligned} N_1 &= 2\left(-\partial x_1 + \partial x_2 + (-y^1 + y^2)\partial z\right), \\ N_2 &= 2(-\partial y_1 + \partial y_2). \end{aligned}$$

It follows that $\overline{\phi}(N_2) = N_1$. Thus, $\mathrm{ltr}(TM)$ is also invariant. Hence, M is contact SCR-null submanifold.

Then we can prove the following results by direct use of Definition 4.2.1.

(1) Contact SCR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$ is invariant if and only if $D^\perp = 0$ (resp., $D = \{0\}$).

(2) Any contact SCR-coisotropic, isotropic, and totally null submanifold of $\overline{M}$ is an invariant null submanifold. Consequently, there exist no proper contact SCR or screen real coisotropic or isotropic or totally null submanifold of $\overline{M}$.

The following result is well known.

Proposition 4.2.1. *[7]. A contact GCR-null submanifold of an indefinite Sasakian manifold* $(\overline{M}, \phi, \xi, \eta, \overline{g})$*, is a contact CR (respectively, contact SCR-null submanifold) if and only if* $D_1 = \{0\}$ *(respectively,* $D_2 = \{0\}$*).*

Proof. Let M be a contact CR-null submanifold. Then $\phi(\mathrm{Rad}\,TM)$ is a distribution on M such that $\phi(\mathrm{Rad}\,TM) \cap \mathrm{Rad}\,TM = \{0\}$. Therefore, $D_2 = \mathrm{Rad}\,TM$ and $D_1 = \{0\}$. Hence, $\phi(\mathrm{ltr}(TM)) \cap \mathrm{ltr}(TM) = \{0\}$. Then it follows that $\phi(\mathrm{ltr}(TM)) \subset S(TM)$. Conversely, suppose M is a contact GCR-null submanifold such that $D_1 = \{0\}$. Then, $D_2 = \mathrm{Rad}\,TM$. Hence, $\phi(\mathrm{Rad}\,TM) \cap \mathrm{Rad}\,TM = \{0\}$, that is, $\phi(\mathrm{Rad}\,TM)$ is a vector sub-bundle of $S(TM)$. Thus, M is contact CR-null. □

Theorem 4.2.1. *Let M be a contact SCR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. If $\overline{D}\perp\{\xi\}$ defines a totally geodesic foliation in M, then*

$$h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X),$$

has no components in $\overline{\phi}(D^\perp)$, for any X, $Y \in \Gamma(\overline{D}\perp\{\xi\})$.

Proof. From equation (2.12), we have

$$g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z),$$

for any X, $Y \in \Gamma(\overline{D}\perp\{\xi\})$ and $Z \in \Gamma(D^\perp)$. Using (3.2) and (3.3), we obtain

$$\begin{aligned} g(\nabla_X Y + \nabla_Y X, Z) &= \overline{g}(\overline{\phi}\,\overline{\nabla}_X Y, \overline{\phi}Z) + \overline{g}(\overline{\phi}\,\overline{\nabla}_Y X, \overline{\phi}Z) \\ &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, \overline{\phi}Z) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X, \overline{\phi}Z) - \overline{g}((\overline{\nabla}_X \overline{\phi})Y, \overline{\phi}Z) \\ &- \overline{g}((\overline{\nabla}_Y \overline{\phi})X, \overline{\phi}Z) \\ &= \overline{g}(h^s(X, \overline{\phi}Y) + h^s(Y, \overline{\phi}X), \overline{\phi}Z), \end{aligned}$$

which completes the proof. □

Similarly, as in [7] we have the following theorem.

Theorem 4.2.2. *Let M be a contact SCR-null submanifold of an indefinite nearly Sasakian manifold $\overline{M}$. If the distribution $D^\perp$ defines a totally geodesic foliation on M, then*

(i) $A_{\overline{\phi}X}Y + A_{\overline{\phi}Y}X$ has no components in $\overline{D}$,

(ii) For any X, $Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D)$ and $N \in \Gamma(\mathrm{ltr}(TM))$,

$$\begin{aligned} &\overline{g}(B'h^s(X, \overline{\phi}Z), Y) + \overline{g}(B'h^s(Y, \overline{\phi}Z), X) = 0, \\ \textit{and }\ &\overline{g}(B'D^s(X, \overline{\phi}N), Y) + \overline{g}(B'D^s(Y, \overline{\phi}N), X) = 0. \end{aligned}$$

Proof. Suppose $D^\perp$ defines a totally geodesic foliation in M. Then $\nabla_X Y \in \Gamma(D^\perp)$. Using (3.3), we have, for any X, $Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$,

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, \overline{\phi}Z) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X, \overline{\phi}Z) \\ &- \overline{g}((\overline{\nabla}_X \overline{\phi})Y + (\overline{\nabla}_Y \overline{\phi})X, \overline{\phi}Z) \\ &= -\overline{g}(A_{\overline{\phi}X}Y, \overline{\phi}Z) - \overline{g}(A_{\overline{\phi}Y}X, \overline{\phi}Z). \end{aligned} \tag{4.16}$$

Similarly,

$$g(\nabla_X Y, N) + g(\nabla_Y X, N) = -\overline{g}(A_{\overline{\phi}X}Y, \overline{\phi}N) - \overline{g}(A_{\overline{\phi}Y}X, \overline{\phi}N), \tag{4.17}$$

for any $N \in \Gamma(\mathrm{ltr}(TM))$. Applying a similar method to (4.16) and (4.17), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= -\overline{g}(A_{\overline{\phi}X} Y, \overline{\phi}Z) - \overline{g}(A_{\overline{\phi}Y} X, \overline{\phi}Z) \\ &= -\overline{g}(h^s(X, \overline{\phi}Z), \overline{\phi}Y) - \overline{g}(h^s(Y, \overline{\phi}Z), \overline{\phi}X) \\ &- \overline{g}(\overline{\phi}Z, D^l(X, \overline{\phi}Y)) - \overline{g}(\overline{\phi}Z, D^l(Y, \overline{\phi}X)) \\ &= -\overline{g}(h^s(X, \overline{\phi}Z), \overline{\phi}Y) - \overline{g}(h^s(Y, \overline{\phi}Z), \overline{\phi}X), \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, N) &= -\overline{g}(A_{\overline{\phi}X} Y, \overline{\phi}N) - \overline{g}(A_{\overline{\phi}Y} X, \overline{\phi}N) \\ &= -\overline{g}(D^s(X, \overline{\phi}N), \overline{\phi}Y) - \overline{g}(D^s(Y, \overline{\phi}N), \overline{\phi}X) \end{aligned} \tag{4.19}$$

The two assertions (i) and (ii) follow from the relations (4.16), (4.17), (4.18) and (4.19). □

Proposition 4.2.2. *There exists a Levi-Civita connection on an irrotational screen real null submanifold of an indefinite nearly Sasakian manifold, tangent the structure vector field ξ such that*

$$h^l(X, \overline{\phi}Y = -h^l(Y, \overline{\phi}X), \tag{4.20}$$

for any X, $Y \in \Gamma(TM)$.

Proof. From (2.12), we have

$$\overline{g}\left(h^l(X,Y), E\right) = \overline{g}(\overline{\nabla}_X Y, E) = -\overline{g}(Y, \overline{\nabla}_X E),$$

for all X, $Y \in \Gamma(TM)$. By using (3.2) and (3.3), we obtain

$$\begin{aligned} 2\overline{g}\left(h^l(X,Y), E\right) &= \overline{g}\left(h^l(X,Y), E\right) + \overline{g}\left(h^l(Y,X), E\right) \\ &= \overline{g}(\overline{\phi}\,\overline{\nabla}_X Y, \overline{\phi}E) + \overline{g}(\overline{\phi}\,\overline{\nabla}_Y X, \overline{\phi}E) \\ &= \overline{g}\left(\overline{\nabla}_X \overline{\phi}Y + \overline{\nabla}_Y \overline{\phi}X, \overline{\phi}E\right) - \overline{g}\left((\overline{\nabla}_X \overline{\phi})Y + (\overline{\nabla}_Y \overline{\phi})X, \overline{\phi}E\right) \\ &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, \overline{\phi}E) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X, \overline{\phi}E) + \eta(X)\overline{g}(Y, \overline{\phi}E) + \eta(Y)\overline{g}(X, \overline{\phi}E) \\ &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, \overline{\phi}E) + \overline{g}(\overline{\nabla}_Y \overline{\phi}X, \overline{\phi}E). \end{aligned}$$

Using (2.12), we obtain

$$\begin{aligned} 2\overline{g}\left(h^l(X,Y), E\right) &= \overline{g}\left(h^l(X, \overline{\phi}Y) + h^l(Y, \overline{\phi}X), \overline{\phi}E\right) \\ &= -\overline{g}\left(\overline{\phi}h^l(X, \overline{\phi}Y) + \overline{\phi}h^l(Y, \overline{\phi}X), E\right). \end{aligned} \tag{4.21}$$

M being irrotational implies that $\overline{g}\left(h^l(X,Y), E\right) = 0$, that is, $h^l = 0$, since $\mathrm{ltr}(TM)$ is invariant under $\overline{\phi}$, we have, $\overline{\phi}h^l(X, \overline{\phi}Y) + \overline{\phi}h^l(Y, \overline{\phi}X) = 0$, that is,

$$h^l(X, \overline{\phi}Y) + h^l(Y, \overline{\phi}X) = 0,$$

which completes the proof. □

Chapter Five

Conclusions and Perspectives

In conclusion, we investigated the geometry of some null submanifolds of indefinite nearly Sasakian manifolds by paying attention to those of Generalized Cauchy-Riemmann (CR), contact CR, and contact Screen CR, and screen transversal null submanifolds.

The integrability of some distributions has been studied and the results have given some necessary conditions when the some of those distributions defines totally geodesic foliations. We have proven that there exists a Levi-Civita connection on an irrotational screen real null submanifold of an indefinite nearly Sasakian manifold, tangent the structure vector field ξ whose the second fundamental form in the direction the transversal bundle satisfies a certain condition.

In our future work, we would like to study the leaves of the foliations defined in some of the distributions generated by the CR-null submanifolds of indefinite nearly Sasakian manifolds. We will also focus on the screen contact CR manifolds developed in null submanifold in case of almost contact settings. From the idea of screen contact CR manifold, we shall investigate the integrability of screen distribution and a totally geodesic on thos settings.

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