

Existence and uniqueness of solution for Stieltjes differential equations with several derivators

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Abstract

In this paper, we study some existence and uniqueness results for systems of differential equations in which each of equations of the system involves a different Stieltjes derivative. Specifically, we show that this problems can only have one solution under the Osgood condition, or even, the Montel–Tonelli condition. We also explore some results guaranteeing the existence of solution under these conditions. Along the way, we obtain some interesting properties for the Lebesgue–Stieltjes integral associated to a finite sum of nondecreasing and left–continuous maps, as well as a characterization of the pseudometric topologies defined by this type of maps.

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1 Introduction

Stieltjes differential equations have gained popularity in the recent years. The main difference with respect to regular differential problems is the presence of the Stieltjes derivative, a modification of the usual derivative on the real line through a nondecreasing and left–continuous map. This change allows us to study impulsive differential equations and equations on time scales in a unified framework, see for example [6,14], or even [3] for the corresponding integral formulation counterpart.

The usual setting for Stieltjes differential equations in the literature involves a single derivator either in its theoretical –see for example [4,6,9,11,15,16,22]– or numerical studies [4,5]. This is also the case for other differential problems involving Stieltjes derivatives such as in [13,17,21], or even the corresponding integral counterparts. Nevertheless, it is the new setting of differential problems with Stieltjes derivatives that offers the possibility of a new type of problems: systems of differential equations in which each of the components is differentiated with respect to a different nondecreasing and left–continuous function. This was the case in papers such as [10,12], or even [7], where the authors considered differentiation with respect to functions that are not necessarily monotonous. Here, we aim to improve the work along this line regarding systems of equations. Specifically, we will consider maps $g_i : \mathbb{R} \rightarrow \mathbb{R}^n$, $i = 1, 2, \dots, n$, such that each g_i is nondecreasing and left–continuous, and we will discuss some existence and uniqueness results for the system

$$x'_{g_i}(t) = f_i(t, x(t)), \quad x_i(t_0) = x_{0,i}, \quad i = 1, 2, \dots, n, \quad (1.1)$$

where x'_{g_i} denotes the Stieltjes derivative of x with respect to g_i in the sense presented in [14]. In this setting, we build on the work in [10], adapting some of the results in [6,15] to the context of Stieltjes differential equations with several derivators.

The paper is structured as follows: first, in Section 2, we present the basic tools for the Lebesgue–Stieltjes integration on the real line defined in terms of a nondecreasing and left–continuous map. We also obtain some interesting results regarding the Lebesgue–Stieltjes outer measure, as well as a fundamental property regarding the Lebesgue–Stieltjes measure for the finite sum of nondecreasing and left–continuous functions –more information on the Lebesgue–Stieltjes measure can be found in [1,2,6,8,14,25] and for the more general Kurzweil–Stieltjes integral in [18]. Next, in Section 3, we introduce the Stieltjes derivative in the sense of [6] and we explore some concepts of continuity in

a similar fashion to [6, 10]. In particular, we discuss some of the limitations of the mentioned work. Furthermore, throughout this section we obtain important information regarding the pseudometric topology defined by a nondecreasing and left-continuous function, showing that it can be fully characterized in terms of some interesting sets related to such map. Finally, in Section 4, we turn our attention to the study of problems of the form (1.1). First, we continue the study of everywhere solutions started in [10] and, later, we follow the arguments in [15] to obtain some existence and uniqueness results involving Osgood and Montel–Tonelli conditions.

2 The Lebesgue–Stieltjes measure

Throughout this paper, we will make use of the Lebesgue–Stieltjes integral associated to nondecreasing and left-continuous functions. This integral is constructed as the integral with respect to a measure defined in terms of the mentioned map through the classical Carathéodory’s extension theorem, see for example [1, 2, 19, 20, 23]. Specifically, given a nondecreasing and left-continuous map $g : \mathbb{R} \rightarrow \mathbb{R}$, and denoting by $\mathcal{P}(\mathbb{R})$ the set of all subsets of \mathbb{R} , we define the map $\mu_g^* : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$ as

$$\mu_g^*(A) = \inf \left\{ \sum_{n=1}^{\infty} (g(b_n) - g(a_n)) : A \subset \bigcup_{n=1}^{\infty} [a_n, b_n), \{[a_n, b_n)\}_{n=1}^{\infty} \subset \mathcal{C} \right\}, \quad (2.2)$$

with $\mathcal{C} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. The map μ_g^* is an outer measure and, by considering its restriction to the following σ -algebra,

$$\mathcal{L}\mathcal{S}_g = \{A \in \mathcal{P}(\mathbb{R}) : \mu_g^*(E) = \mu_g^*(E \cap A) + \mu_g^*(E \setminus A) \text{ for all } E \in \mathcal{P}(\mathbb{R})\},$$

we obtain the Lebesgue–Stieltjes measure associated to g , which we denote by μ_g .

REMARK 2.1 *Every Borel set belongs to $\mathcal{L}\mathcal{S}_g$. In particular, this means that $\mathcal{C} \subset \mathcal{L}\mathcal{S}_g$. Furthermore, we have that $\mu_g([a, b)) = g(b) - g(a)$ for any $[a, b) \in \mathcal{C}$.*

For simplicity, in what follows we will use the term “ g -measurable” for a set or function to refer to μ_g -measurability in the corresponding sense; and we will denote the integration with respect to μ_g as

$$\int_X f(s) d g(s).$$

In a similar way, we will replace μ_g by g in other expressions such as “ P holds for μ_g -a.a. $x \in X$ ” or “ P holds μ_g -a.e. in X ”. Along these lines, it is important to note that the set

$$C_g := \{t \in \mathbb{R} : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\},$$

i.e. the set of points around which g is constant, has null g -measure, as pointed out in [14, Proposition 2.5]. Furthermore, observe that, by definition, C_g is open.

The aim of this section is to show that the expression used to compute μ_g , (2.2), can be simplified, as well as to prove some interesting properties regarding the Lebesgue–Stieltjes measure associated to the sum of a finite family of nondecreasing and left-continuous functions.

We begin by showing that (2.2) can be simplified. Specifically, we will show that the infimum in that expression can be considered over an smaller set, namely, assuming that the families in \mathcal{C} are pairwise disjoint. To that end, we introduce the following lemma.

LEMMA 2.2 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function. For every $\{[a_n, b_n)\}_{n \in \mathbb{N}} \subset \mathcal{C}$, there exists $\mathcal{V} = \{[c_n, d_n)\}_{n \in \mathbb{N}} \subset \mathcal{C}$ such that the sets in \mathcal{V} are pairwise disjoint and*

$$\bigcup_{n \in \mathbb{N}} [a_n, b_n) = \bigcup_{n \in \mathbb{N}} [c_n, d_n), \quad \sum_{n \in \mathbb{N}} (g(d_n) - g(c_n)) \leq \sum_{n \in \mathbb{N}} (g(b_n) - g(a_n)).$$

Proof. Let $U = \bigcup_{n \in \mathbb{N}} [a_n, b_n)$ and C_U be the set of all connected components of U . First, note that the set C_U is at most countable. Secondly, observe that all the elements of C_U are connected subsets of \mathbb{R} . Thus, we have that they are intervals (including the whole \mathbb{R}) or singletons. Nevertheless, observe that an element of C_U cannot be a singleton as each point of U belongs to $[a_n, b_n)$ for some $n_0 \in \mathbb{N}$. Furthermore, we claim that $\sup I \notin I$ for any $I \in C_U$ bounded from above. Indeed, let $I \in C_U$ be bounded from above and suppose that $\sup I \in I \subset U$. In this conditions, there exists $n_1 \in \mathbb{N}$ such that $\sup I \in [a_{n_1}, b_{n_1})$. Hence, we have that the set $I \cup [a_{n_1}, b_{n_1})$ is a connected set containing I , which is a contradiction with $I \in C_U$. Therefore, C_U is, by construction, an at most countable collection of pairwise disjoint sets of the form (a, b) , $[a, b)$, $[a, +\infty)$, $(-\infty, b)$, $a, b \in \mathbb{R}$; or $C_U = \{\mathbb{R}\}$.

For each $I \in C_U$ and define $\mathcal{F}_I \subset \mathcal{C}$ as follows:

- if $I = (a, b)$, $a, b \in \mathbb{R}$, then $\mathcal{F}_I = \{[a + (b - a)/(n + 1), a + (b - a)/n]\}_{n \in \mathbb{N}}$;
- if $I = [a, b)$, $a, b \in \mathbb{R}$, then $\mathcal{F}_I = \{[a + (n - 1)(b - a)/n, a + n(b - a)/(n + 1)]\}_{n \in \mathbb{N}}$;
- if $I = [a, +\infty)$, $a \in \mathbb{R}$, then $\mathcal{F}_I = \{[a + n - 1, a + n]\}_{n \in \mathbb{N}}$;
- if $I = (-\infty, b)$, $b \in \mathbb{R}$, then $\mathcal{F}_I = \{[b - n, b - n + 1]\}_{n \in \mathbb{N}}$;
- if $I = \mathbb{R}$, then $\mathcal{F}_I = \{[n, n + 1]\}_{n \in \mathbb{Z}}$.

Proceeding this way, for each $I \in C_U$ we find a countable pairwise disjoint family contained in \mathcal{C} , \mathcal{F}_I , such that $I = \bigcup_{J \in \mathcal{F}_I} J$. Furthermore, for each $I \in C_U$, it follows from Remark 2.1 and the fact that μ_g is a measure that

$$\sum_{J \in \mathcal{F}_I} (g(\sup J) - g(\inf J)) = \sum_{J \in \mathcal{F}_I} \mu_g(J) = \mu_g \left(\bigcup_{J \in \mathcal{F}_I} J \right) = \mu_g(I). \quad (2.3)$$

Define $\mathcal{V} = \bigcup_{I \in C_U} \mathcal{F}_I$. First, observe that \mathcal{V} is a countable set by definition. Furthermore, $\mathcal{V} \subset \mathcal{C}$. Hence, we can write $\mathcal{V} = \{[c_n, d_n]\}_{n \in \mathbb{N}}$ for some $c_n, d_n \in \mathbb{R}$. Let us show that \mathcal{V} satisfies the properties in the statement of the result.

First, note that the sets in \mathcal{V} are pairwise disjoint. Indeed, let $[c_n, d_n]$ and $[c_m, d_m]$ be two elements of \mathcal{V} . If they belong to the same connected component, $I \in C_U$, then, by construction of \mathcal{F}_I , we have that $[c_n, d_n] \cap [c_m, d_m] = \emptyset$. Otherwise, $[c_n, d_n] \in I$ and $[c_m, d_m] \in I'$ for some $I, I' \in C_U$, $I \neq I'$. Then, the definition of connected component guarantees that $I \cap I' = \emptyset$, which yields $[c_n, d_n] \cap [c_m, d_m] = \emptyset$. Hence, the family \mathcal{V} is pairwise disjoint. Furthermore,

$$U = \bigcup_{I \in C_U} I = \bigcup_{I \in C_U} \left(\bigcup_{J \in \mathcal{F}_I} J \right) = \bigcup_{V \in \mathcal{V}} V = \bigcup_{n \in \mathbb{N}} [c_n, d_n].$$

Finally, using (2.3) and Remark 2.1, we have that

$$\begin{aligned} \sum_{n \in \mathbb{N}} (g(d_n) - g(c_n)) &= \sum_{I \in C_U} \left(\sum_{J \in \mathcal{F}_I} (g(\sup J) - g(\inf J)) \right) \\ &= \sum_{I \in C_U} \mu_g(I) = \mu_g(U) \leq \sum_{n \in \mathbb{N}} (g(b_n) - g(a_n)), \end{aligned}$$

where the last inequality follows from (2.2). \square

The following result contains a characterization of the outer measure μ_g^* . The result follows from Lemma 2.2 by considering the relations between the infima involved.

THEOREM 2.3 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous functions and μ_g^* be as in (2.2). Then, for any $A \in \mathcal{P}(\mathbb{R})$,*

$$\mu_g^*(A) = \inf \left\{ \sum_{n=1}^{\infty} (g(b_n) - g(a_n)) : A \subset \bigcup_{n=1}^{\infty} [a_n, b_n), \{[a_n, b_n)\}_{n=1}^{\infty} \subset \mathcal{C} \text{ pairwise disjoint} \right\}. \quad (2.4)$$

Theorem 2.3 not only provides an easier way to compute the outer measure of sets, but it is also a fundamental tool for the proof of the following result which, to the best of our knowledge, is not available in the existing literature on the topic of Lebesgue–Stieltjes measures. Essentially, Proposition 2.4 guarantees that, given a finite family of nondecreasing and left-continuous functions, the measure defined as the sum of the corresponding Lebesgue–Stieltjes measures is, in fact, the Lebesgue–Stieltjes measure associated to the corresponding sum of nondecreasing and left-continuous functions.

PROPOSITION 2.4 *Let $g_1, g_2, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$ be a family of nondecreasing and left-continuous functions and define $\widehat{g} : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$\widehat{g}(t) = g_1(t) + g_2(t) + \dots + g_n(t), \quad t \in \mathbb{R}. \quad (2.5)$$

Then, for any $E \in \mathcal{P}(\mathbb{R})$,

$$\mu_{\widehat{g}}^*(E) = \mu_{g_1}^*(E) + \mu_{g_2}^*(E) + \dots + \mu_{g_n}^*(E). \quad (2.6)$$

Proof. We shall only prove the result for $n = 2$, as the general case can be deduced from this.

Let $E \in \mathcal{P}(\mathbb{R})$. Then, computing the corresponding outer measures as in (2.2), we have that

$$\begin{aligned} \mu_g^*(E) &= \inf \left\{ \sum_{n=1}^{\infty} (\widehat{g}(b_n) - \widehat{g}(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} [(g_1(b_n) - g_1(a_n)) + (g_2(b_n) - g_2(a_n))] : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} (g_1(b_n) - g_1(a_n)) + \sum_{n=1}^{\infty} (g_2(b_n) - g_2(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \right\} \\ &\geq \sum_{i=1}^2 \inf \left\{ \sum_{n=1}^{\infty} (g_i(b_n) - g_i(a_n)) : E \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \right\} = \mu_{g_1}^*(E) + \mu_{g_2}^*(E). \end{aligned}$$

For the reverse inequality, let $\varepsilon > 0$. It follows from (2.4) that there exist $\mathcal{R}_1 = \{[a_{1,n}, b_{1,n}]\}_{n=1}^{\infty}$ and $\mathcal{R}_2 = \{[a_{2,m}, b_{2,m}]\}_{m=1}^{\infty}$, each of them pairwise disjoint, such that

$$\begin{aligned} E \subset \bigcup_{n=1}^{\infty} [a_{1,n}, b_{1,n}], & \quad \sum_{n=1}^{\infty} (g_1(b_{1,n}) - g_1(a_{1,n})) \leq \mu_{g_1}^*(E) + \frac{\varepsilon}{2}, \\ E \subset \bigcup_{m=1}^{\infty} [a_{2,m}, b_{2,m}], & \quad \sum_{m=1}^{\infty} (g_2(b_{2,m}) - g_2(a_{2,m})) \leq \mu_{g_2}^*(E) + \frac{\varepsilon}{2}. \end{aligned}$$

Define $\mathcal{R} = \{[a_{1,n}, b_{1,n}] \cap [a_{2,m}, b_{2,m}] : n, m = 1, 2, \dots\} \setminus \{\emptyset\}$. Observe that the elements of \mathcal{R} are of the form $[c, d]$, $c, d \in \mathbb{R}$, $c < d$, since, by construction, we removed those intersections that might be empty. Specifically, denoting $a_n^m = \max\{a_{1,n}, a_{2,m}\}$, $b_n^m = \min\{b_{1,n}, b_{2,m}\}$, $n, m = 1, 2, \dots$,

$$\mathcal{R} = \{[a_n^m, b_n^m]\}_{(n,m) \in \mathcal{I}}, \quad \mathcal{I} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : a_n^m < b_n^m\}.$$

Observe that \mathcal{R} is a countable set. Furthermore, given $x \in E$, since \mathcal{R}_1 and \mathcal{R}_2 are covers of E , there exist $n_0, m_0 \in \mathbb{N}$ such that $x \in [a_{1,n_0}, b_{1,n_0}]$ and $x \in [a_{2,m_0}, b_{2,m_0}]$, which ensures that $x \in [a_{n_0}^{m_0}, b_{n_0}^{m_0}]$. This guarantees that \mathcal{R} is a countable cover of E . Therefore,

$$\mu_g^*(E) \leq \sum_{(n,m) \in \mathcal{I}} (g(b_n^m) - g(a_n^m)) = \sum_{(n,m) \in \mathcal{I}} (g_1(b_n^m) - g_1(a_n^m)) + \sum_{(n,m) \in \mathcal{I}} (g_2(b_n^m) - g_2(a_n^m)). \quad (2.7)$$

Hence, it is enough to show that

$$\sum_{(n,m) \in \mathcal{I}} (g_i(b_n^m) - g_i(a_n^m)) \leq \sum_{k=1}^{\infty} (g_i(b_{i,k}) - g_i(a_{i,k})), \quad i = 1, 2, \quad (2.8)$$

to conclude the result since, in that case, (2.7) yields

$$\mu_g^*(E) \leq \sum_{n=1}^{\infty} (g_1(b_{1,n}) - g_1(a_{1,n})) + \sum_{m=1}^{\infty} (g_2(b_{2,m}) - g_2(a_{2,m})) < \mu_{g_1}^*(E) + \mu_{g_2}^*(E) + \varepsilon,$$

which ensures that $\mu_g^*(E) \leq \mu_{g_1}^*(E) + \mu_{g_2}^*(E)$ as $\varepsilon > 0$ was arbitrarily fixed.

Let us show that (2.8) holds. For each $n \in \mathbb{N}$, define $\mathcal{I}_n = \{m \in \mathbb{N} : (n, m) \in \mathcal{I}\}$. Note that $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$, and so,

$$\sum_{(n,m) \in \mathcal{I}} (g_1(b_n^m) - g_1(a_n^m)) = \sum_{n=1}^{\infty} \left(\sum_{m \in \mathcal{I}_n} (g_1(b_n^m) - g_1(a_n^m)) \right). \quad (2.9)$$

On the other hand, by definition, we have that $\bigcup_{m \in \mathcal{I}_n} [a_n^m, b_n^m] \subset [a_{1,n}, b_{1,n}]$ for each $n \in \mathbb{N}$. Furthermore, since \mathcal{R}_2 is pairwise disjoint, it follows that $\{[a_n^m, b_n^m]\}_{m=1}^{\infty}$ is also pairwise disjoint for each $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$,

$$\sum_{m \in \mathcal{I}_n} (g_1(b_n^m) - g_1(a_n^m)) = \mu_{g_1} \left(\bigcup_{m \in \mathcal{I}_n} [a_n^m, b_n^m] \right) \leq \mu_{g_1}([a_{1,n}, b_{1,n}]) = g_1(b_{1,n}) - g_1(a_{1,n}).$$

Now (2.8) for $i = 1$ follows from (2.9). The case $i = 2$ is analogous and we omit it. \square

In particular, Proposition 2.4 ensures that every set which is g_i -measurable for all $i \in \{1, 2, \dots, n\}$ is also \widehat{g} -measurable and $\mu_{\widehat{g}}(E) = \mu_{g_1}(E) + \mu_{g_2}(E) + \dots + \mu_{g_n}(E)$ for all $E \in \bigcap_{i=1}^n \mathcal{L}_{S_{g_i}}$. The same can be said in regards to the measurability of maps. Furthermore, it follows that if a map $f : X \rightarrow \mathbb{R}$ is g_i -integrable for all $i \in \{1, 2, \dots, n\}$, then f is \widehat{g} -integrable and

$$\int_X f(s) d\widehat{g}(s) = \int_X f(s) dg_1(s) + \int_X f(s) dg_2(s) + \dots + \int_X f(s) dg_n(s).$$

This fact was implicitly used in [10] in the proof of the Theorem 4.3, but it was never discussed if such property was true. Here, we have shown that this is the case. This information will be fundamental for the work ahead. Furthermore, it is important to note that throughout this paper, the map \widehat{g} will denote the map defined as in (2.5).

3 The Stieltjes derivative and the concept of continuity

In this section we gather some information available in [6, 10, 14] regarding one of the fundamental tools for this paper: the Stieltjes derivative. Furthermore, we also include some information in those papers regarding different concept of continuity there presented which are also required for the study of differential equations with several Stieltjes derivatives. In particular, we revisit those in [10] as the results there present some limitations. For the aims of this section, as well as the rest of the paper, we shall assume that \mathbb{R}^n is endowed with the maximum norm, i.e.,

$$\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

We start by introducing the concept of Stieltjes derivator. From now on, we will refer to nondecreasing and left-continuous maps on \mathbb{R} as *derivators*. Given a derivator g , we will denote by D_g the set of all discontinuity points of g . Observe that, given that g is nondecreasing, we can write $D_g = \{t \in \mathbb{R} : \Delta g(t) > 0\}$ where $\Delta g(t) = g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right handside limit of g at t . With this remark, we now have all the information required to introduce the following definition in [14].

DEFINITION 3.1 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be derivator and $f : \mathbb{R} \rightarrow \mathbb{R}$. We define the Stieltjes derivative, or g -derivative, of f at a point $t \in \mathbb{R} \setminus C_g$ as*

$$f'_g(t) = \begin{cases} \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g, \\ \lim_{s \rightarrow t^+} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in D_g, \end{cases}$$

provided the corresponding limits exist. In that case, we say that f is g -differentiable at t .

REMARK 3.2 *Observe that the points of C_g are excluded from the definition of g -derivative. This is because the corresponding limit does not make sense in any neighborhood of these points. Nevertheless, as mentioned before, $\mu_g(C_g) = 0$ so, in most cases, this has no impact. Furthermore, note that for $t \in D_g$, $f'_g(t)$ exists if and only if $f(t^+)$ exists and, in that case,*

$$f'_g(t) = \frac{f(t^+) - f(t)}{\Delta g(t)}.$$

For more information on the Stieltjes derivatives, we refer the reader to [6, 14]. Here, we will restrict ourselves to the information strictly necessary for the contents of this paper. Along these lines, we include a reformulation of [14, Theorem 5.4] with [14, Definition 5.1] added to its statement.

THEOREM 3.3 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be derivator and $F : [a, b] \rightarrow \mathbb{R}$. The following conditions are equivalent:*

1. *The function F is g -absolutely continuous on $[a, b]$, which we write as $F \in \mathcal{AC}_g([a, b], \mathbb{R})$, according to the following definition: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$,*

$$\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta \implies \sum_{n=1}^m |F(b_n) - F(a_n)| < \varepsilon.$$

2. The function F satisfies the following conditions:

- (i) there exists $F'_g(t)$ for g -a.a. $t \in [a, b]$;
- (ii) $F'_g \in \mathcal{L}_g^1([a, b], \mathbb{R})$, the set of Lebesgue–Stieltjes integrable functions with respect to μ_g ;
- (iii) for each $t \in [a, b]$,

$$F(t) = F(a) + \int_{[a,t)} F'_g(s) \, d g(s).$$

REMARK 3.4 A particularly interesting case of g -absolutely continuous function can be found in Theorem 2.4 and Proposition 5.2 in [14]. Given $f \in \mathcal{L}_g^1([a, b], \mathbb{R})$, the map $F : [a, b] \rightarrow \mathbb{R}$ defined as

$$F(t) = \int_{[a,t)} f(s) \, d g(s),$$

is well-defined, $F \in \mathcal{AC}_g([a, b], \mathbb{R})$ and $F'_g(t) = f(t)$ for g -a.a. $t \in [a, b]$.

In the work ahead, we will consider systems of differential equations in \mathbb{R}^n where each component is differentiated with respect to a different derivator. Specifically, we will consider $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator, and we will be looking for solutions on the following set:

$$\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n) = \prod_{i=1}^n \mathcal{AC}_{g_i}([a, b], \mathbb{R}).$$

That is, we will look for \mathbf{g} -absolutely continuous functions (see [10, Definition 3.4]), or in other words, functions such that the i -th component is g_i -absolutely continuous, $i = 1, 2, \dots, n$.

REMARK 3.5 Observe that for the particular case in which $\mathbf{g} = (g, g, \dots, g)$ for some derivator g , we have that $\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n) = \mathcal{AC}_g([a, b], \mathbb{R}^n)$ in the sense presented in [6]. Note that, in particular, if $F \in \mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$, then $F \in \mathcal{AC}_{\hat{g}}([a, b], \mathbb{R}^n)$. To see that, it is enough to note that for each $i \in \{1, 2, \dots, n\}$ and each open pairwise disjoint family of subintervals of $[a, b]$, $\{(a_k, b_k)\}_{k=1}^m$,

$$\sum_{k=1}^m (g_i(b_k) - g_i(a_k)) \leq \sum_{k=1}^m (\hat{g}(b_k) - \hat{g}(a_k)).$$

Theorem 3.3 ensures that, given $F \in \mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$, $F = (F_1, F_2, \dots, F_n)$, $(F_i)'_{g_i}(t)$ exists for g_i -a.a. $t \in [a, b]$ and $(F_i)'_{g_i} \in \mathcal{L}_{g_i}^1([a, b], \mathbb{R})$, $i \in \{1, 2, \dots, n\}$. Moreover, we have that

$$F_i(t) = F_i(a) + \int_{[a,t)} (F_i)'_{g_i}(s) \, d g_i(s), \quad t \in [a, b], \quad i = 1, 2, \dots, n.$$

Throughout this paper, we will use “component-by-component” notation for the derivatives and integrals so that expressions such as the previous one can simply be reduced to

$$F(t) = F(a) + \int_{[a,t)} F'_g(s) \, d \mathbf{g}(s), \quad t \in [a, b].$$

The rest of this section is dedicated to the study of the concept of continuity with respect to a map $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator. Some research on this topic was carried out in [6] in the setting of a unique derivator, and in [10] in the general setting. Unfortunately, some of the results in the latter are incorrect. Here, we show their limitations and amend some of those errors. We start by introducing the following concept that contains [6, Definition 3.1].

DEFINITION 3.6 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that g, g_i , $i = 1, 2, \dots, n$, are derivators; and consider $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. We say that f is g -continuous at $t \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(s) - f(t)\| < \varepsilon, \quad \text{for all } s \in A \text{ such that } |g(s) - g(t)| < \delta.$$

We say that f is \mathbf{g} -continuous at $t \in A$ if f_i is g_i -continuous at t for each $i \in \{1, 2, \dots, n\}$. If f is \mathbf{g} -continuous at every $t \in A$, we say that f is \mathbf{g} -continuous on A .

REMARK 3.7 We denote by $\mathcal{C}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ the set of \mathbf{g} -continuous functions on $[a, b]$ with values in \mathbb{R}^n and by $\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ the set of functions in $\mathcal{C}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ which are also bounded. Note that, by definition, we have that

$$\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n) = \prod_{i=1}^n \mathcal{BC}_{g_i}([a, b], \mathbb{R}),$$

which, together with [6, Proposition 5.5], ensures that $\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n) \subset \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$.

The following result, which can be directly deduced from [6, Proposition 3.2], contains some basic properties regarding \mathbf{g} -continuous functions that will be useful in what follows.

PROPOSITION 3.8 Let $f : [a, b] \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a \mathbf{g} -continuous map on $[a, b]$. Then:

1. For each $i \in \{1, 2, \dots, n\}$, f_i is continuous from the left at every $t \in (a, b]$.
2. For each $i \in \{1, 2, \dots, n\}$, f_i is continuous at every $t \in (a, b)$ at which g_i is continuous.
3. For each $i \in \{1, 2, \dots, n\}$, f_i is constant on every $[c, d] \subset [a, b]$ at which g_i is constant.

Proposition 3.8 allows us to establish some relations between the continuity of a derivator with respect to another one and some of their characteristic sets, as presented in the next result.

PROPOSITION 3.9 Let $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ be derivators. Then g_1 is g_2 -continuous if and only if $C_{g_2} \subset C_{g_1}$ and $D_{g_1} \subset D_{g_2}$.

Proof. First, assume that g_1 is g_2 -continuous. If $t \in C_{g_2}$, then there exists $\delta > 0$ such that g_2 is constant in $(t - \delta, t + \delta)$. In particular, g_2 is constant on $[t - \delta/2, t + \delta/2]$. Proposition 3.8 guarantees that g_1 is constant on that same interval, so it is constant on $(t - \varepsilon, t + \varepsilon)$, $\varepsilon \in (0, \delta/2)$, i.e., $t \in C_{g_1}$.

On the other hand, if $t \in \mathbb{R} \setminus D_{g_2}$, we have that g_2 is continuous at t . In that case, Proposition 3.8 yields that g_1 is continuous at t , and so $t \in \mathbb{R} \setminus D_{g_1}$. Thus, $D_{g_1} \subset D_{g_2}$.

Conversely, assume that $C_{g_2} \subset C_{g_1}$ and $D_{g_1} \subset D_{g_2}$ and let $t \in \mathbb{R}$. Observe that showing that g_1 is g_2 -continuous at t is equivalent to showing that for each $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$0 \leq g_1(s) - g_1(t) < \varepsilon, \quad \text{for all } s \geq t \text{ such that } 0 \leq g_2(s) - g_2(t) < \delta_1, \quad (3.10)$$

$$0 \leq g_1(t) - g_1(s) < \varepsilon, \quad \text{for all } s < t \text{ such that } 0 \leq g_2(t) - g_2(s) < \delta_2. \quad (3.11)$$

Let $\varepsilon > 0$. In order to show that (3.10) holds, define $A_t = \{s \in [t, +\infty) : g_2(s) = g_2(t)\} \neq \emptyset$. If A_t is not bounded from above, we have that $g_2(s) = g_2(t)$ for all $s \in [t, +\infty)$. In that case, $(t, +\infty) \subset C_{g_2} \subset C_{g_1}$, which means that $g_1(s) = g_1(t)$ for all $s \in (t, +\infty)$, and so (3.10) holds for any $\delta_1 > 0$. Otherwise, A_t is bounded from above and we can define $a_t = \sup A_t \in [t, +\infty)$. Observe that, by definition, we have that $g_2(s) = g_2(t)$ for all $s \in [t, a_t)$ and $g_2(s) > g_2(t)$ for all $s \in (a_t, +\infty)$. In particular, we have that $(t, a_t) \subset C_{g_2} \subset C_{g_1}$, which implies that $g_1(s) = g_1(t)$ for all $s \in [t, a_t)$. Now, since g_1 and g_2 are left-continuous, it follows that $g_1(s) = g_1(t)$ and $g_2(s) = g_2(t)$ for all $s \in [t, a_t]$.

Now, if $a_t \in D_{g_1}$, we have that $a_t \in D_{g_2}$. In that case, taking $\delta_1 = \Delta g_2(a_t) > 0$, we have that $\{s \geq t : 0 \leq g_2(s) - g_2(t) < \delta_1\} = [t, a_t]$, and so (3.10) follows (even when $a_t = t$, as it becomes trivial). Otherwise, we have that $a_t \notin D_{g_1}$ which means that g_1 is continuous from the right at a_t , so there exists $\delta' > 0$ such that

$$0 \leq g_1(s) - g_1(a_t) < \varepsilon \quad \text{for all } s \geq a_t \text{ such that } 0 \leq s - a_t < \delta'. \quad (3.12)$$

Observe that it is enough to show that there exists $\delta_1 > 0$ such that

$$0 \leq s - a_t < \delta' \quad \text{for all } s \geq a_t \text{ such that } 0 \leq g_2(s) - g_2(a_t) < \delta_1, \quad (3.13)$$

to obtain that (3.10) holds. Indeed, if $a_t = t$, then (3.10) follows directly from (3.12) and (3.13). Otherwise, $a_t > t$ and, in that case, if $s \geq t$ is such that $0 \leq g_2(s) - g_2(t) < \delta_1$ we have that either $s \in [t, a_t]$, implying that $g_1(s) - g_1(t) = 0$, or $s > a_t$ and, in that case, it follows again from (3.12) and (3.13).

Reasoning by contradiction, if (3.13) does not hold, then for every $n \in \mathbb{N}$ there exists $s_n \geq a_t$ such that $s_n - a_t \geq \delta'$ and $0 \leq g_2(s_n) - g_2(a_t) < 1/n$. Let $s = \inf\{s_n\}_{n \in \mathbb{N}} \geq a_t + \delta' > a_t$. Since g_2 is nondecreasing, $0 \leq g_2(s) - g_2(a_t) \leq g_2(s_n) - g_2(a_t) < 1/n$ for every $n \in \mathbb{N}$. Therefore, $g_2(s) = g_2(a_t)$ which contradicts the fact that $g_2(s) > g_2(t) = g_2(a_t)$ for all $s \in [a_t, +\infty)$.

Finally, for (3.11) we proceed in a similar manner. Define $B_t = \{s \in (-\infty, t] : g_2(s) = g_2(t)\} \neq \emptyset$. If B_t is not bounded from below, an analogous reasoning to the one for A_t shows that (3.11) holds trivially. Otherwise, B_t is bounded from below and we can define $b_t = \inf B_t \in (-\infty, t]$. Hence, by

definition, we have that $g_2(s) = g_2(t)$ for all $s \in (b_t, t]$ and $g_2(s) < g_2(t)$ for all $s \in (-\infty, b_t]$. In particular, $(b_t, t) \subset C_{g_2} \subset C_{g_1}$, which guarantees that $g_1(s) = g_1(t)$ for all $s \in (b_t, t]$.

If $b_t \in D_{g_2}$, then taking $\delta_2 = \Delta g_2(b_t) > 0$, we have that $\{s < t : 0 \leq g_2(s) - g_2(t) < \delta_2\} = (b_t, t)$, and so (3.10) follows (even when $b_t = t$, as it becomes vacuous). Otherwise, $b_t \notin D_{g_2}$ which implies that $b_t \notin D_{g_1}$. Observe that this ensures that

$$g_1(s) = g_1(t) \quad \text{and} \quad g_2(s) = g_2(t), \quad \text{for all } s \in [b_t, t]. \quad (3.14)$$

Furthermore, since g_1 is left-continuous at b_t , there exists $\delta'' > 0$ such that

$$0 \leq g_1(b_t) - g_1(s) < \varepsilon \quad \text{for all } s < b_t \text{ such that } 0 < b_t - s < \delta''.$$

Now, an analogous reasoning to the one for (3.13) shows that, given that (3.14) holds, there exists $\delta_2 > 0$ such that $0 < b_t - s < \delta''$ for all $s < b_t$ satisfying that $0 < g_2(b_t) - g_2(s) < \delta_2$, which is enough to conclude that (3.11) holds in a similar fashion to (3.10). \square

REMARK 3.10 *Condition (3.10) can be interpreted as “being g_2 -continuous at t from the right” and, similarly, (3.11) as “being g_2 -continuous at t from the left”.*

As pointed out in [6], the concept of g -continuity can be understood as continuity in the topological sense. Specifically, a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is g -continuous if $f : (\mathbb{R}, \tau_g) \rightarrow (\mathbb{R}, \tau_u)$ is continuous, where τ_u is the usual topology of \mathbb{R} and τ_g is the topology generated by the sets

$$B_g(x, r) := \{y \in \mathbb{R} : |g(y) - g(x)| < r\}, \quad x \in \mathbb{R}, \quad r > 0. \quad (3.15)$$

With this characterization, observe that g_1 is g_2 -continuous if and only if $\tau_{g_1} \subset \tau_{g_2}$. Indeed, first observe that g_1 is trivially g_1 -continuous. This means that $g_1^{-1}(U) \in \tau_{g_1}$ for any $U \in \tau_u$. Hence, if $\tau_{g_1} \subset \tau_{g_2}$, we have that $g_1^{-1}(U) \in \tau_{g_2}$ for any $U \in \tau_u$, or equivalently, g_1 is g_2 -continuous. Conversely, if g_1 is g_2 -continuous, and given that the corresponding topologies are generated by the sets in (3.15), it is enough to show that for any $t \in \mathbb{R}$, $r > 0$ and $s \in B_{g_1}(t, r)$, there exists $r_s > 0$ such that $B_{g_2}(s, r_s) \subset B_{g_1}(t, r)$. Given $t \in \mathbb{R}$, $r > 0$ and $s \in B_{g_1}(t, r)$, define $\delta = r - |g_1(t) - g_1(s)| > 0$. Then, $B_{g_1}(s, \delta) \subset B_{g_1}(t, r)$ and, since g_1 is g_2 -continuous, there exists $r_s > 0$ such that

$$|g_1(z) - g_1(s)| < \delta, \quad \text{for all } z \in \mathbb{R} \text{ such that } |g_2(z) - g_2(s)| < r_s.$$

Hence, if $z \in B_{g_2}(s, r_s)$, we have that $z \in B_{g_1}(s, \delta) \subset B_{g_1}(t, r)$, which finishes the proof of the equivalence. Combining this with Proposition 3.9, we obtain the following result.

COROLLARY 3.11 *Let $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ be derivators. Then the following statements are equivalent:*

1. g_1 is g_2 -continuous and g_2 is g_1 -continuous.
2. $C_{g_2} = C_{g_1}$ and $D_{g_1} = D_{g_2}$.
3. $\tau_{g_1} = \tau_{g_2}$.

As we will see later, this result will allow us to relate the concept of g -continuity with a similar type of continuity in [10]. Furthermore, Corollary 3.11 implies that the topologies of derivators can be classified in terms of the sets C_g and D_g as the following theorem shows.

THEOREM 3.12 (CLASSIFICATION OF DERIVATOR TOPOLOGIES) *Let G be the family of all derivators and define the following equivalence relation on G :*

$$g_1 \sim g_2 : \iff \tau_{g_1} = \tau_{g_2}.$$

Let $\mathcal{G} := G|_{\sim}$ and $\mathcal{H} = \{(C, D) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) : C \text{ open, } D \text{ at most countable, } C \cap D = \emptyset\}$. Then, the map $\Phi : \mathcal{G} \rightarrow \mathcal{H}$, $\Phi([g]) = (C_g, D_g)$, is a bijection.

Proof. First, observe Corollary 3.11 ensures that Φ is well defined and injective. To see that it is surjective, let us denote by m the Lebesgue measure on \mathbb{R} and $\llbracket a, b \rrbracket := [\min\{a, b\}, \max\{a, b\}]$, $a, b \in \mathbb{R}$. Then, given $(C, D) \in \mathcal{H}$, let $D = \{d_n\}_{n \in \Lambda}$, $\Lambda \subset \mathbb{N}$, and define $g_{C,D} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$g_{C,D}(t) := \text{sgn}(t)m(\llbracket 0, t \rrbracket \setminus C) + \sum_{\substack{d_n \in D \\ d_n < t}} 2^{-n}.$$

Note that $\llbracket 0, t \rrbracket \setminus C$ is a closed set, thus Lebesgue-measurable. Hence, $g_{C,D}$ is well defined. Furthermore, $g_{C,D}$ is left-continuous and nondecreasing, $D_g = D$ and $C_g = C$, which shows that Φ is surjective. \square

EXAMPLE 3.13 To illustrate Theorem 3.12 we consider $[a, b] = [0, 1]$ and the class of the derivator

$$g(t) = t + \sum_{\substack{\frac{1}{n} < t \\ n \in \mathbb{N}}} 2^{-n}.$$

The class $[g]$ is the set of all derivators that generate the same topology. Observe that $C_g = \emptyset$ and $D_g = \{\frac{1}{n}\}_{n \in \mathbb{N}}$, so $[g]$ is characterized by $(\emptyset, \{\frac{1}{n}\}_{n \in \mathbb{N}}) \in \mathcal{H}$. Thus, $[g]$ can be explicitly expressed as

$$[g] = \left\{ h(t) = \phi(t) + \sum_{\substack{\frac{1}{n} < t \\ n \in \mathbb{N}}} \alpha_n : \alpha_n > 0, \sum_{n \in \mathbb{N}} \alpha_n < \infty, \phi \text{ strictly increasing and continuous} \right\}.$$

As mentioned before, in [10, Definition 3.1] we find a definition of continuity with respect to \mathbf{g} alternative to the one provided in Definition 3.6. For completeness, we include [10, Definition 3.1] before comparing the two concepts.

DEFINITION 3.14 Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator. A function $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, is $\vec{\mathbf{g}}$ -continuous at point $t \in A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(t) - f(s)\| < \varepsilon, \quad \text{for all } s \in A \text{ such that } \|\mathbf{g}(t) - \mathbf{g}(s)\| < \delta.$$

If it is $\vec{\mathbf{g}}$ -continuous at every point $t \in A$, we say that f is $\vec{\mathbf{g}}$ -continuous on A .

In [10], the authors claimed that Definitions 3.6 and 3.14 are equivalent. Nevertheless, a careful reader might notice that the proof only shows that \mathbf{g} -continuity implies $\vec{\mathbf{g}}$ -continuity. Furthermore, the reverse implication is not true, as shown in the next example.

EXAMPLE 3.15 Consider $\mathbf{g}, f : \mathbb{R} \rightarrow \mathbb{R}^2$, $\mathbf{g} = (g_1, g_2)$, $f = (f_1, f_2)$, defined as

$$\mathbf{g}(t) = \begin{cases} (0, t), & \text{if } t \leq 0, \\ (0, t+1), & \text{if } t > 0, \end{cases} \quad f(t) = \begin{cases} (t, 0), & \text{if } t \leq 0, \\ \left(t+1, \frac{\sin(1/t)}{t}\right), & \text{if } t > 0. \end{cases} \quad (3.16)$$

Note that f cannot be \mathbf{g} -continuous as f_1 is not constant, see Proposition 3.8. However, f is $\vec{\mathbf{g}}$ -continuous. Indeed, first note that $\|\mathbf{g}(t) - \mathbf{g}(s)\| = |g_2(t) - g_2(s)|$, $s, t \in \mathbb{R}$. Thus, showing that f is $\vec{\mathbf{g}}$ -continuous is equivalent to showing that f is g_2 -continuous. Since we are considering the max-norm in \mathbb{R}^n , it suffices to show that f_1 and f_2 are g_2 -continuous. Now, f_1 is trivially g_2 -continuous as $f_1 = g_2$, and [6, Example 3.3] shows that f_2 is g_2 -continuous.

It is important to note that the misinformation about the relations between Definitions 3.6 and 3.14 does not affect the existence and uniqueness results in [10]. However, it has some consequences when it comes to the study of solutions in the classical sense. Specifically, this affects Proposition 4.6 and Theorem 4.8 in [10]. In the next section we discuss the implications of these facts. Nevertheless, in some contexts, those results remain true as a consequence of the following result.

PROPOSITION 3.16 Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator. Then, the following are equivalent:

- (i) Every $\vec{\mathbf{g}}$ -continuous map is \mathbf{g} -continuous.
- (ii) For each $j, k \in \{1, 2, \dots, n\}$, the map $g_k : \mathbb{R} \rightarrow \mathbb{R}$ is g_j -continuous.
- (iii) For each $j, k \in \{1, 2, \dots, n\}$, $C_{g_j} = C_{g_k}$ and $D_{g_j} = D_{g_k}$.
- (iv) For each $j, k \in \{1, 2, \dots, n\}$, $\tau_{g_j} = \tau_{g_k}$.

Proof. Assume that (i) holds. Fix $k \in \{1, 2, \dots, n\}$ and consider the map

$$G(t) = (g_k(t), g_k(t), \dots, g_k(t)), \quad t \in \mathbb{R}.$$

Observe that it is enough to show that G is $\vec{\mathbf{g}}$ -continuous to prove (ii). This is straightforward. Indeed, let $t \in \mathbb{R}$, $\varepsilon > 0$ and take $\delta = \varepsilon$. Then, if $s \in \mathbb{R}$ is such that $\|\mathbf{g}(t) - \mathbf{g}(s)\| < \delta$, it follows that

$$\|G(t) - G(s)\| = |g_k(t) - g_k(s)| \leq \|\mathbf{g}(t) - \mathbf{g}(s)\| < \delta = \varepsilon.$$

Conversely, assume that (ii) holds and let $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a \vec{g} -continuous map. Fix $j \in \{1, 2, \dots, n\}$, $t \in A$ and $\varepsilon > 0$. Since f is \vec{g} -continuous, there exists $\gamma > 0$ such that

$$\|f(t) - f(s)\| < \varepsilon, \quad \text{for all } s \in A \text{ such that } \|\mathbf{g}(t) - \mathbf{g}(s)\| < \gamma.$$

On the other hand, for each $k \in \{1, 2, \dots, n\}$, g_k is g_j -continuous and so, there exists $\delta_k > 0$ such that

$$|g_k(t) - g_k(s)| < \gamma, \quad \text{for all } s \in \mathbb{R} \text{ such that } |g_j(t) - g_j(s)| < \delta_k.$$

Take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. In that case, if $s \in A$ such that $|g_j(t) - g_j(s)| < \delta$, we have that $\|\mathbf{g}(t) - \mathbf{g}(s)\| < \gamma$, which ensures that $|f_j(t) - f_j(s)| \leq \|f(t) - f(s)\| < \varepsilon$. Hence, f_j is g_j -continuous and, since $j \in \{1, 2, \dots, n\}$ was arbitrarily fixed, f is \mathbf{g} -continuous.

The rest of the result is a consequence of Corollary 3.11. \square

Interestingly enough, it is possible to establish some other connections between the continuity in the sense of Definition 3.14 and [6, Definition 3.1] for an adequate choice of a derivator.

PROPOSITION 3.17 *Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator. Then, $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is \vec{g} -continuous at $t \in A$ if and only if f is \widehat{g} -continuous at t .*

Proof. Observe that for all $s \in A$,

$$\|\mathbf{g}(t) - \mathbf{g}(s)\| \leq |\widehat{g}(t) - \widehat{g}(s)| \leq n\|\mathbf{g}(t) - \mathbf{g}(s)\|. \quad (3.17)$$

Indeed, it follows from the triangular inequality that $|\widehat{g}(t) - \widehat{g}(s)| \leq n\|\mathbf{g}(t) - \mathbf{g}(s)\|$ for any $s \in A$. For the other inequality we consider two cases. If $t \geq s$, since \widehat{g} and each g_i are nondecreasing, we have

$$|\widehat{g}(t) - \widehat{g}(s)| = \widehat{g}(t) - \widehat{g}(s) = \sum_{i=1}^n (g_i(t) - g_i(s)) \geq \max_{i=1, \dots, n} \{g_i(t) - g_i(s)\} = \|\mathbf{g}(t) - \mathbf{g}(s)\|. \quad (3.18)$$

On the other hand, if $t < s$, we proceed analogously to (3.18), interchanging the roles of t and s . Hence, (3.17) holds. Now, the equivalence between the two types of continuity follows. \square

Proposition 3.17 not only provides a simple condition to check if a map is \vec{g} -continuous, but we can also deduce some interesting properties for this type of maps through the results in [6]. In particular, [6, Corollary 3.5] yields the following result that is fundamental for the uniqueness results in the following section.

PROPOSITION 3.18 *Let $A \subset \mathbb{R}$ be a Borel set and $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be \vec{g} -continuous on A . Then, f is Borel measurable.*

REMARK 3.19 *Since every Borel set is Lebesgue–Stieltjes measurable, it follows that every Borel measurable map is Lebesgue–Stieltjes measurable. In particular, we have that if A is a Borel set and $f : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is \vec{g} -continuous on A , then f is g_j -measurable for all $j \in \{1, 2, \dots, n\}$.*

Another type of continuity defined in terms of \mathbf{g} that was introduced in [10] is what the authors called $(\vec{g} \times \text{Id})$ -continuity, which is a generalization of [6, Definition 7.7]. These concepts of continuity were introduced for the study of classical solutions in both papers. As mentioned earlier, the results in [10] are partially incorrect and we aim to correct them in this paper. To that end, we introduce the following definition, which is an alternative generalization of [6, Definition 7.7].

DEFINITION 3.20 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that g , g_i , $i = 1, 2, \dots, n$, are derivators; and consider $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. We say that f is $(\mathbf{g} \times \text{Id})$ -continuous at $(t, x) \in A \times B$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(s, y) - f(t, x)\| < \varepsilon \quad \text{for all } (s, y) \in A \times B \text{ such that } |g(s) - g(t)| < \delta \text{ and } \|y - x\| < \delta.$$

We say that f is $(\mathbf{g} \times \text{Id})$ -continuous at (t, x) if each f_i is $(g_i \times \text{Id})$ -continuous at (t, x) , $i \in \{1, 2, \dots, n\}$. We say that f is $(\mathbf{g} \times \text{Id})$ -continuous in $A \times B$ if it is $(\mathbf{g} \times \text{Id})$ -continuous at every $(t, x) \in A \times B$.

On the other hand, the corresponding definition in [10] reads as follows.

DEFINITION 3.21 Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, be such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator. A function $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$ is $(\bar{\mathbf{g}} \times \text{Id})$ -continuous at $(t, x) \in A \times B$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(s, y) - f(t, x)\| < \varepsilon \quad \text{for all } (s, y) \in A \times B \text{ such that } \|\mathbf{g}(s) - \mathbf{g}(t)\| < \delta \quad \text{and} \quad \|y - x\| < \delta.$$

We say that f is $(\bar{\mathbf{g}} \times \text{Id})$ -continuous in $A \times B$ if it is $(\bar{\mathbf{g}} \times \text{Id})$ -continuous at every $(t, x) \in A \times B$.

The relations between Definitions 3.20 and 3.21 are analogous to the ones between Definitions 3.6 and 3.14. In particular, we have the following result.

PROPOSITION 3.22 Let $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. If f is $(\mathbf{g} \times \text{Id})$ -continuous at $(t, x) \in A \times B$, then f is $(\bar{\mathbf{g}} \times \text{Id})$ -continuous at (t, x) .

Proof. Fix $\varepsilon > 0$. Given $i \in \{1, 2, \dots, n\}$, we have that f_i is $(g_i \times \text{Id})$ -continuous at (t, x) , so there exists $\delta_i > 0$ such that

$$|f_i(s, y) - f_i(t, x)| < \varepsilon \quad \text{for all } (s, y) \in A \times B \text{ such that } |g_i(s) - g_i(t)| < \delta_i \quad \text{and} \quad \|y - x\| < \delta_i.$$

Take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. Then, if $\|\mathbf{g}(t) - \mathbf{g}(s)\| < \delta$ and $\|x - y\| < \delta_i$, we have that $|g_i(t) - g_i(s)| < \delta_i$ and $\|x - y\| < \delta_i$ for all $i \in \{1, 2, \dots, n\}$, which implies that $\|f(t, x) - f(s, y)\| < \varepsilon$. \square

Once again, the reverse implication does not hold. To see that this is the case, it is enough to consider \mathbf{g} and f as in (3.16) and $F : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as $F(t, (x, y)) = f(t)$, and note F does not depend on (x, y) , which implies that $(\mathbf{g} \times \text{Id})$ and $(\bar{\mathbf{g}} \times \text{Id})$ -continuity reduce to \mathbf{g} and $\bar{\mathbf{g}}$ -continuity. Furthermore, using a similar reasoning, we can obtain analogous results to Propositions 3.16 and 3.17 in the context of Definitions 3.20 and 3.21.

Interestingly enough, within the proof of [10, Theorem 4.8] the authors proved correctly the following superposition result involving Definitions 3.14 and 3.21.

LEMMA 3.23 Let $f : A \times B \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $(\bar{\mathbf{g}} \times \text{Id})$ -continuous function on $A \times B$. If $x : A \rightarrow B$ is $\bar{\mathbf{g}}$ -continuous on A , then the composition $f(\cdot, x(\cdot))$ is $\bar{\mathbf{g}}$ -continuous on A .

REMARK 3.24 Lemma 3.23 guarantees that the composition of a \mathbf{g} -continuous map with a $(\mathbf{g} \times \text{Id})$ -continuous one is $\bar{\mathbf{g}}$ -continuous. Nevertheless, we cannot assure that the composition is \mathbf{g} -continuous. Indeed, consider $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^2$, $\mathbf{g} = (g_1, g_2)$, be such that g_1, g_2 are derivators satisfying $\Delta g_1(t_0) = 0$ and $\Delta g_2(t_0) > 0$ for some $t_0 \in \mathbb{R}$. Let I be a neighborhood of t_0 and consider the maps $x : I \rightarrow \mathbb{R}^2$, $f : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f = (f_1, f_2)$, defined as

$$x(t) = (g_1(t), g_2(t)), \quad f(t, (y, z)) = (g_1(t) - z, g_2(t) - y).$$

It is clear that x is \mathbf{g} -continuous at t_0 . Furthermore, observe that

$$\lim_{t \rightarrow t_0^+} f_1(t, x(t)) = \lim_{t \rightarrow t_0^+} g_1(t) - g_2(t) = g_1(t_0) - g_2(t_0^+) < g_1(t_0) - g_2(t_0) = f_1(t_0, x(t_0)).$$

This implies that the map $f(\cdot, x(\cdot))$ is not \mathbf{g} -continuous at t_0 , as $f_1(\cdot, x(\cdot))$ is not g_1 -continuous at t_0 , see Proposition 3.8. However, the map f is $(\mathbf{g} \times \text{Id})$ -continuous at $(t_0, (g_1(t_0), g_2(t_0)))$. Indeed, we shall only show that f_1 is $(g_1 \times \text{Id})$ -continuous at t_0 as the proof for f_2 being $(g_2 \times \text{Id})$ -continuous is analogous. Let $\varepsilon > 0$ and take $0 < \delta < \varepsilon/2$. Denote $u_0 = (g_1(t_0), g_2(t_0))$. If $(t, (x, y)) \in I \times \mathbb{R}^2$ is such that $|g_1(t_0) - g_1(t)| < \delta$ and $\|u_0 - (y, z)\| < \delta$, then

$$|f_1(t_0, u_0) - f_1(t, (y, z))| \leq |g_1(t_0) - g_1(t)| + |y - g_2(t_0)| \leq |g_1(t_0) - g_1(t)| + \|u_0 - (x, y)\| < 2\delta < \varepsilon.$$

4 The initial value problem with several derivators

We now turn our attention to the study of initial value problems in the context of \mathbf{g} -differential equations. That is, given $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator, we will study problems of the form

$$x'_{\mathbf{g}}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad (4.19)$$

with $t_0, T \in \mathbb{R}$, $T > 0$, $X \subset \mathbb{R}^n$, $x_0 \in X$ and $f : [t_0, t_0 + T) \times X \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$. To that end, we introduce the concept of solution that is fundamental for the aims of this section. In what follows we denote by $I_\sigma = [t_0, t_0 + \sigma)$, $\sigma \in (0, T]$, and $I = [t_0, t_0 + T)$ and by \bar{I}_σ and \bar{I} the corresponding closure sets with respect to the usual topology in \mathbb{R} .

DEFINITION 4.1 A solution of (4.19) on an interval I_σ , $\sigma \in (0, T]$, is a function $x \in \mathcal{AC}_g(\bar{I}_\sigma, \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, such that $x(t_0) = x_0$, $x(t) \in X$ for all $t \in I_\sigma$ and

$$x'_{g_i}(t) = f_i(t, x(t)), \quad g_i\text{-a.a. } t \in I_\sigma, \quad i \in \{1, 2, \dots, n\}. \quad (4.20)$$

If $\sigma = T$, we say that x is a global solution of (4.19); otherwise, i.e. if $\sigma \in (0, T)$, we say that x is a local solution of (4.19).

Before studying some existing and uniqueness results for (4.19) we will reflect on the concept of classical solution in [10]. When we talk about solutions in the classical sense, we mean solutions of (4.19) that solve the problem on their whole interval of definition and have continuous derivatives. Of course, given the nature of Definition 3.1, this is impossible unless $C_{g_i} = \emptyset$, $i = 1, 2, \dots, n$. Nevertheless, we can talk about “everywhere” solutions referring to solutions solving the problem on the biggest set possible, i.e., excluding the corresponding set C_{g_i} . With this idea in mind, we start exploring some basic properties of “everywhere” solutions that will culminate in Theorem 4.6. The first results concern the continuity of the derivative.

PROPOSITION 4.2 Let $\sigma \in (0, T]$, $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, be a $(\vec{g} \times \text{Id})$ -continuous function on $I_\sigma \times \mathbb{R}^n$ and $x \in \mathcal{AC}_g(\bar{I}_\sigma, \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, be such that $x(t_0) = x_0$ and

$$(x_i)'_{g_i}(t) = f_i(t, x(t)), \quad \text{for all } t \in I_\sigma \setminus C_{g_i}, \quad i = 1, 2, \dots, n. \quad (4.21)$$

Then, $(x_i)'_{g_i}$ is \vec{g} -continuous on $I_\sigma \setminus C_{g_i}$.

Proof. Given that x is g -absolutely continuous on \bar{I}_σ , we have that x is g -continuous on \bar{I}_σ , which implies that it is \vec{g} -continuous on \bar{I}_σ . Thus Lemma 3.23 yields that $f(\cdot, x(\cdot))$ is \vec{g} -continuous. Hence, it follows that, for each $i \in \{1, 2, \dots, n\}$, the map $f_i(\cdot, x(\cdot))$ is \vec{g} -continuous and, since $(x_i)'_{g_i}$ is defined on $I_\sigma \setminus C_{g_i}$, the result follows. \square

It follows from Proposition 3.17 that, under the hypotheses of Proposition 4.2, $(x_i)'_{g_i}$ is \widehat{g} -continuous on $I_\sigma \setminus C_{g_i}$. Furthermore, notice that if we replace the concept of \vec{g} -continuity for g -continuity in the hypotheses of Proposition 4.2 we still obtain \vec{g} -continuous solutions. Nevertheless, with a similar reasoning to the one used for Proposition 4.2, we can obtain the following result ensuring g -continuity.

PROPOSITION 4.3 Let $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$ and $x \in \mathcal{AC}_g(\bar{I}_\sigma, \mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$, $\sigma \in (0, T]$, be such that $x(t_0) = x_0$ and (4.21) is satisfied. If there exists $i \in \{1, 2, \dots, n\}$ such that $f_i(\cdot, x(\cdot))$ is g_i -continuous on I_σ , then $(x_i)'_{g_i}$ is g_i -continuous on $I_\sigma \setminus C_{g_i}$. In particular, if $f(\cdot, x(\cdot))$ is g -continuous on I_σ , then $(x_i)'_{g_i}$ is g_i -continuous on $I_\sigma \setminus C_{g_i}$ for all $i \in \{1, 2, \dots, n\}$.

Propositions 4.2 and 4.3 show that, under suitable conditions, the solutions of (4.19) have continuous derivatives in some sense. In particular, it is required that condition (4.21) is satisfied. Nevertheless, solutions in the sense of Definition 4.1 need not satisfy such condition. In the following results, we provide some conditions ensuring that (4.21) or similar conditions are satisfied.

PROPOSITION 4.4 Let $x = (x_1, x_2, \dots, x_n)$ be a solution of (4.19) on I_σ , $\sigma \in (0, T]$. Then:

(i) If $f(\cdot, x(\cdot))$ is \vec{g} -continuous on I_σ , then

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in (I_\sigma \setminus (D_{\vec{g}} \cup C_{g_i})) \cup D_{g_i}, \quad i = 1, 2, \dots, n.$$

(ii) If $f_i(\cdot, x(\cdot))$ is g_i -continuous for some $i \in \{1, 2, \dots, n\}$, then

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\sigma \setminus C_{g_i}.$$

In particular, if $f(\cdot, x(\cdot))$ is g -continuous on I_σ , then (4.21) holds.

Proof. Fix $i \in \{1, 2, \dots, n\}$. Since x is a solution of (4.19), we have that $x_i \in \mathcal{AC}_{g_i}(\bar{I}_\sigma, \mathbb{R}^n)$ and $(x_i)'_{g_i}(t) = f_i(t, x(t))$ for g_i -a.a. $t \in I_\sigma$. Hence, in particular, $(x_i)'_{g_i}(t) = f_i(t, x(t))$ for all $t \in D_{g_i}$, so it is enough to show that the corresponding equalities holds for all $t \in I_\sigma \setminus (C_{g_i} \cup D_{\vec{g}})$ for (i), and for all $t \in I_\sigma \setminus (C_{g_i} \cup D_{g_i})$ for (ii). We will first prove (ii) and then, by making small modifications to that proof, we will obtain (i).

Fix $t \in I_\sigma \setminus (C_{g_i} \cup D_{g_i})$. Since g_i is not constant on any neighbourhood of t , we may have $g_i(s) < g_i(t)$ for all $s < t$, $g_i(s) > g_i(t)$ for all $s > t$, or both. If $g_i(s) < g_i(t)$ for all $s < t$ and $t > t_0$, then, for all $s \in [t_0, t)$, the Fundamental Theorem of Calculus yields

$$\mu_{g_i}([s, t]) \inf_{s \leq r < t} f_i(r, x(r)) \leq \int_{[s, t]} f_i(r, x(r)) \, d g_i(r) = x_i(t) - x_i(s) \leq \mu_{g_i}([s, t]) \sup_{s \leq r < t} f_i(r, x(r)),$$

which implies that

$$\inf_{s \leq r < t} f_i(r, x(r)) \leq \frac{x_i(s) - x_i(t)}{g_i(s) - g_i(t)} \leq \sup_{s \leq r < t} f_i(r, x(r)). \quad (4.22)$$

On the other hand, if $g_i(s) > g_i(t)$ for all $s > t$, then, following an analogous reasoning, we deduce that

$$\inf_{t \leq r < s} f_i(r, x(r)) \leq \frac{x_i(s) - x_i(t)}{g_i(s) - g_i(t)} \leq \sup_{t \leq r < s} f_i(r, x(r)). \quad (4.23)$$

Now, for (ii), assume that $f_i(\cdot, x(\cdot))$ is g_i -continuous on \bar{I}_σ . In that case, since g_i is continuous at t , we have that $f_i(\cdot, x(\cdot))$ is continuous at t . Therefore, if t is such that $g_i(s) < g_i(t)$ for all $s < t$, (4.22) implies that the following limit exists and

$$\lim_{s \rightarrow t^-} \frac{x_i(s) - x_i(t)}{g_i(s) - g_i(t)} = f_i(t, x(t)). \quad (4.24)$$

If $g_i(s) = g_i(t)$ on some $[t, t + \delta]$, $\delta > 0$, then the limit in (4.24) is $(x_i)_{g_i}'(t)$ and the proof is complete. Similarly, if t is such that $g_i(s) > g_i(t)$ for all $s > t$, the continuity of $f_i(\cdot, x(\cdot))$ at t and (4.23) ensure that the following limit exists and

$$\lim_{s \rightarrow t^+} \frac{x_i(s) - x_i(t)}{g_i(s) - g_i(t)} = f_i(t, x(t)). \quad (4.25)$$

This covers all of the remaining cases, so the proof is finished for $f_i(\cdot, x(\cdot))$ g_i -continuous, and subsequently, for $f(\cdot, x(\cdot))$ \mathbf{g} -continuous.

On the other hand, if $t \in I_\sigma \setminus (C_{g_i} \cup D_{\hat{g}})$, we have that $t \in I_\sigma \setminus (C_{g_i} \cup D_{g_i})$. Hence, (4.22) holds if t is such that $g_i(s) < g_i(t)$ for all $s < t$, and (4.23), if $g_i(s) > g_i(t)$ for all $s > t$. Now, under the hypotheses of (i), we have that $f(\cdot, x(\cdot))$ is $\vec{\mathbf{g}}$ -continuous, so it is $\hat{\mathbf{g}}$ -continuous at t . As a consequence, $f(\cdot, x(\cdot))$ is continuous at t as $\hat{\mathbf{g}}$ is continuous at that point. Hence, by making analogous reasonings, we can obtain (4.24) and (4.25) to finish the proof. \square

Essentially, the proof of Proposition 4.4 is a revision of the proofs of [6, Proposition 7.6] and [10, Proposition 4.6] in the context of (4.19). Of course, in the latter, the authors worked on the same framework as in this paper. Nevertheless, [10, Proposition 4.6] is not correct due to Definitions 3.6 and 3.20 not being equivalent to Definitions 3.14 and 3.21, respectively. The following result serves as a proper reformulation of [10, Proposition 4.6] based on Proposition 4.4.

THEOREM 4.5 *Let $\sigma \in (0, T]$, $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $(\vec{\mathbf{g}} \times \text{Id})$ -continuous function on $I_\sigma \times \mathbb{R}^n$ and $x = (x_1, x_2, \dots, x_n)$ be a solution of (4.19) on I_σ . If there exists $i \in \{1, 2, \dots, n\}$ such that*

$$(x_i)_{g_i}'(t) = f_i(t, x(t)), \quad \text{for all } t \in D_{\hat{\mathbf{g}}} \setminus D_{g_i}, \quad (4.26)$$

then

$$(x_i)_{g_i}'(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\sigma \setminus C_{g_i}, \quad (4.27)$$

and $(x_i)_{g_i}'$ is $\vec{\mathbf{g}}$ -continuous on $I_\sigma \setminus C_{g_i}$. In particular, if (4.26) holds for all $i \in \{1, 2, \dots, n\}$, then (4.27) holds for all $i \in \{1, 2, \dots, n\}$ and each $(x_i)_{g_i}'$, $i \in \{1, 2, \dots, n\}$, is $\vec{\mathbf{g}}$ -continuous on $I_\sigma \setminus C_{g_i}$.

Proof. By definition, x is \mathbf{g} -absolutely continuous on \bar{I}_σ which implies that it is $\vec{\mathbf{g}}$ -continuous on that set. Hence, given that f is $(\vec{\mathbf{g}} \times \text{Id})$ -continuous on $\bar{I}_\sigma \times \mathbb{R}^n$, Proposition 4.4 ensures that

$$(x_i)_{g_i}'(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\sigma \setminus (D_{\hat{\mathbf{g}}} \cup C_{g_i}) \cup D_{g_i}, \quad i = 1, 2, \dots, n.$$

Hence, for each $i \in \{1, 2, \dots, n\}$ such that (4.26) holds, we have that $(x_i)_{g_i}'(t) = f_i(t, x(t))$, $t \in I_\sigma \setminus C_{g_i}$. Now, Proposition 4.2 ensures that $(x_i)_{g_i}'$ is $\vec{\mathbf{g}}$ -continuous in $I_\sigma \setminus C_{g_i}$, which finishes the proof. \square

Observe that (4.26) becomes vacuous when $D_{g_j} = D_{g_k}$ for all $j, k \in \{1, 2, \dots, n\}$ which, as stated by Proposition 3.16, is guaranteed to happen when Definitions 3.6 and 3.14 are equivalent. This justifies the statement of [10, Proposition 4.6], since the authors wrongly used both definitions equivalently. This, of course, affected [10, Theorem 4.8], where the authors guaranteed the existence of a local solution (and everything that follows from [10, Proposition 4.6]) under the assumption of an extra hypothesis: for each $i \in \{1, 2, \dots, n\}$, there exists $h_i \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ such that

$$|f_i(t, x)| \leq h_i(t), \quad g_i\text{-a.a. } t \in I, \quad x \in \overline{B(x_0, r)}. \quad (4.28)$$

It is possible to obtain a correct formulation of [10, Theorem 4.8] by noting that the $(\bar{\mathbf{g}} \times \text{Id})$ -continuity of f together with condition (4.28) are enough to obtain the existence of a local solution through [10, Theorem 4.5]. After that, all that is left to do is to consider Theorem 4.5 to obtain the right version of [10, Theorem 4.8].

As a final note, we obtain an analogous result to Theorem 4.5 yielding \mathbf{g} -continuity instead of $\bar{\mathbf{g}}$ -continuity. This result follows from Proposition 4.3 combined with statement (ii) in Proposition 4.4.

THEOREM 4.6 *Let $x = (x_1, x_2, \dots, x_n)$ be a solution of (4.19) on I_σ , $\sigma \in (0, T]$. If $f_i(\cdot, x(\cdot))$ is g_i -continuous on I_σ for some $i \in \{1, 2, \dots, n\}$, then*

$$(x_i)'_{g_i}(t) = f_i(t, x(t)) \quad \text{for all } t \in I_\sigma \setminus C_{g_i},$$

and $(x_i)'_{g_i}$ is g_i -continuous on $I_\sigma \setminus C_{g_i}$. In particular, if $f(\cdot, x(\cdot))$ is \mathbf{g} -continuous on I_σ , then (4.21) holds and $(x_i)'_{g_i}$ is g_i -continuous on $I_\sigma \setminus C_{g_i}$ for all $i \in \{1, 2, \dots, n\}$.

4.1 Uniqueness of solution

We now continue the work in [10] by researching some uniqueness conditions for (4.19). In particular, we shall adapt the results in [15] to the context of differential equations with several Stieltjes derivatives. For this endeavour, as well as for the question of existence of solution, we can assume that \mathbf{g} is continuous at t_0 , as pointed out in [10]. To see that this is the case, it is enough to use an analogous reasoning to that in [6, Section 5].

Our first uniqueness criterion for (4.19) is the analogous to [15, Theorem 3.9] and it is inspired by the ideas of [24, Theorem 4.8]. In order to obtain the mentioned result we need the following reformulation of [24, Theorem 1.40] in the context of the Lebesgue–Stieltjes integral, see [15, Lemma 3.8].

LEMMA 4.7 *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing left-continuous function, and let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous nondecreasing function such that $\omega(0) = 0$, $\omega(s) > 0$ for $s > 0$. For a fixed $u_0 > 0$, define*

$$\Omega(r) = \int_{u_0}^r \frac{1}{\omega(s)} \, d s, \quad r \in (0, +\infty).$$

and denote by $\alpha = \lim_{r \rightarrow 0^+} \Omega(r) \geq -\infty$, $\beta = \lim_{r \rightarrow +\infty} \Omega(r) \leq +\infty$. If $\psi : [a, b] \rightarrow [0, +\infty)$ is a bounded function, and there exists $\kappa > 0$ such that

$$\psi(s) \leq \kappa + \int_{[a, s)} \omega(\psi(\sigma)) \, d h(\sigma), \quad s \in [a, b],$$

and $\Omega(\kappa) + h(b) - h(a) < \beta$, then

$$\psi(s) \leq \Omega^{-1}(\Omega(\kappa) + h(s) - h(a)), \quad s \in [a, b],$$

where $\Omega^{-1} : (\alpha, \beta) \rightarrow \mathbb{R}$ is the inverse function of Ω .

We are now able to state and prove the following uniqueness result under an Osgood type condition.

THEOREM 4.8 *Let $X \subset \mathbb{R}^n$ be such that $x_0 \in X$, $f : [t_0, t_0 + T) \times X \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$, $\omega(s) > 0$ for all $s > 0$ and*

$$\lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{u_0} \frac{1}{\omega(s)} \, d s = +\infty, \tag{4.29}$$

for some $u_0 > 0$. If there exists $\sigma \in (0, T]$ such that for each $i \in \{1, 2, \dots, n\}$,

$$|f_i(t, x) - f_i(t, y)| \leq \omega(\|x - y\|), \quad g_i\text{-a.a. } t \in I_\sigma, \quad x, y \in X,$$

then (4.19) has at most one solution on I_σ .

Proof. Let x, y be solutions of (4.19) on I_σ . Define $\psi : \bar{I}_\sigma \rightarrow [0, +\infty)$, $\Omega : (0, +\infty) \rightarrow (0, +\infty)$ as

$$\psi(t) = \|x(t) - y(t)\|, \quad \Omega(r) = \int_{u_0}^r \frac{1}{\omega(s)} \, d s. \tag{4.30}$$

First, note that, given that $x, y \in \mathcal{AC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$, we have that each component of $x - y$ is Borel measurable (see Remark 3.19) and, since ψ is the pointwise maximum of Borel measurable maps,

we have that ψ is Borel measurable. Now, given that ω is continuous, it follows that $\omega \circ \psi$ is Borel measurable, which guarantees that it is \widehat{g} and g_i -measurable, $i = 1, 2, \dots, n$. Moreover, Remark 3.7 ensures that $x - y$ is bounded, which implies that so is ψ , yielding that $\omega \circ \psi$ is bounded as well. Hence, it follows that $\omega \circ \psi$ is integrable with respect to \widehat{g} and g_i , $i = 1, 2, \dots, n$.

Let $K > 0$ be an upper bound of $\omega \circ \psi$. Without loss of generality, we assume that \mathbf{g} is continuous, which ensures that \widehat{g} is continuous at t_0 . Then, for each $\gamma \in (0, \sigma)$,

$$\int_{[t_0, t_0 + \gamma)} \omega(\psi(s)) \, d\widehat{g}(s) \leq \int_{[t_0, t_0 + \gamma)} K \, d\widehat{g}(s) = K \mu_{\widehat{g}}([t_0, t_0 + \gamma)) < \widehat{\varepsilon}(\gamma),$$

where $\widehat{\varepsilon}(\gamma) = K \mu_{\widehat{g}}([t_0, t_0 + \gamma)) + \gamma > 0$. Noting that $\widehat{\varepsilon}$ and ω are in the same circumstances as ε and ω in the proof of [15, Theorem 3.9] with \widehat{g} in place of g , we can repeat the same arguments there to see that there exists $0 < R < \gamma$ such that

$$\Omega(\widehat{\varepsilon}(\delta)) + \widehat{g}(t_0 + \sigma) - \widehat{g}(t_0 + \delta) < \beta := \lim_{r \rightarrow \infty} \Omega(r) \quad \text{for } \delta \in (0, R).$$

By definition we have that, for each $t \in \bar{I}_\sigma$, there is $j_t \in \{1, 2, \dots, n\}$ such that $\psi(t) = |x_{j_t}(t) - y_{j_t}(t)|$. Therefore, Theorem 3.3 yields that for each $t \in \bar{I}_\sigma$,

$$\begin{aligned} \psi(t) &\leq \int_{[t_0, t)} |f_{j_t}(s, x(s)) - f_{j_t}(s, y(s))| \, dg_{j_t}(s) \leq \int_{[t_0, t)} \omega(\|x(s) - y(s)\|) \, dg_{j_t}(s) \\ &\leq \sum_{i=1}^n \int_{[t_0, t)} \omega(\|x(s) - y(s)\|) \, dg_i(s) = \int_{[t_0, t)} \omega(\|x(s) - y(s)\|) \, d\widehat{g}(s) \\ &= \int_{[t_0, t_0 + \delta)} \omega(\psi(s)) \, d\widehat{g}(s) + \int_{[t_0 + \delta, t)} \omega(\psi(s)) \, d\widehat{g}(s) < \widehat{\varepsilon}(\delta) + \int_{[t_0 + \delta, t)} \omega(\psi(s)) \, d\widehat{g}(s) \end{aligned}$$

for all $\delta \in (0, R)$. Therefore, the assumptions of Lemma 4.7 are satisfied, which guarantees that

$$\psi(t) \leq \Omega^{-1}(\Omega(\varepsilon(\delta)) + \widehat{g}(t) - \widehat{g}(t_0 + \delta)), \quad \delta \in (0, R), \quad t \in \bar{I}_\sigma.$$

Applying Ω on both sides of the inequality, we obtain

$$\Omega(\psi(t)) - \Omega(\varepsilon(\delta)) \leq \widehat{g}(t) - \widehat{g}(t_0 + \delta) \leq \widehat{g}(t) - \widehat{g}(t_0), \quad \delta \in (0, R), \quad t \in \bar{I}_\sigma.$$

Suppose $\psi \neq 0$ on \bar{I}_σ . In that case, there must exist $t^* \in \bar{I}_\sigma$ such that $\psi(t^*) > 0$. Then, for all $\delta \in (0, R)$ such that $\delta < t^* - t_0$,

$$\int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)} \, ds = \Omega(\psi(t^*)) - \Omega(\varepsilon(\delta)) < \widehat{g}(t^*) - \widehat{g}(t_0),$$

and, by taking the limit as $\delta \rightarrow 0^+$, we obtain

$$\lim_{\delta \rightarrow 0^+} \int_{\varepsilon(\delta)}^{\psi(t^*)} \frac{1}{\omega(s)} \, ds < \widehat{g}(t^*) - \widehat{g}(t_0) < +\infty,$$

which contradicts (ii). Hence, we must have $\psi = 0$ on \bar{I}_σ , i.e. $x = y$ on that interval. \square

Observe that Theorem 4.8 returns [15, Theorem 3.9] in the corresponding setting (namely, when $\mathbf{g} = (g, g, \dots, g)$ for some derivator g) as, for any $u \in (0, +\infty)$,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^u \frac{1}{\omega(s)} \, ds = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^{u_0} \frac{1}{\omega(s)} \, ds + \int_{u_0}^u \frac{1}{\omega(s)} \, ds. \quad (4.31)$$

Furthermore, it is possible to obtain the following more general result, which is the analog of the Montel–Tonelli uniqueness result in [15] in the setting of (4.19). In its proof, we make use of the Kurzweil–Stieltjes integral, which we denote by

$${}^{(KS)} \int_a^b F(s) \, dg(s).$$

For more information on this integral, we refer the reader to [18].

THEOREM 4.9 *Let $X \subset \mathbb{R}^n$ be such that $x_0 \in X$, $f : [t_0, t_0 + T) \times X \rightarrow \mathbb{R}^n$, $f = (f_1, f_2, \dots, f_n)$, and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$, $\omega(s) > 0$ for all $s > 0$ and (4.29) holds for some $u_0 > 0$. If there exists $\sigma \in (0, T]$ and $\varphi : I_\sigma \rightarrow [0, +\infty)$ such that for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}_{g_i}^1(I_\sigma, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \leq \varphi(t)\omega(\|x - y\|), \quad g_i\text{-a.a. } t \in I_\sigma, \quad x, y \in X, \quad (4.32)$$

then (4.19) has at most one solution on I_σ .

Proof. Let x, y be solutions of (4.19) on I_σ and define $\psi : \bar{I}_\sigma \rightarrow [0, +\infty)$ as in (4.30). We can show, using the same arguments as in Theorem 4.8, that $\omega \circ \psi$ is bounded and Borel measurable. Therefore, given $K > 0$ an upper bound of $\omega \circ \psi$, we have that $|\omega(\psi(t))\varphi(t)| \leq K\varphi(t)$, $t \in \bar{I}_\sigma$. This fact, together with Proposition 2.4, ensure that $\varphi \cdot \omega \circ \psi$ is \widehat{g} -integrable on I_σ . Define $\bar{g} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\bar{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \int_{[t_0, t)} \varphi(s) d\widehat{g}(s), & \text{if } t_0 < t \leq t_0 + \sigma, \\ \int_{[t_0, t_0 + \sigma)} \varphi(s) d\widehat{g}(s), & \text{if } t > t_0 + \sigma. \end{cases}$$

Recalling the relation between the Lebesgue–Stieltjes and Kurzweil–Stieltjes integrals (see [18, Chapter 6, Section 12]) and the substitution formula for Kurzweil–Stieltjes integral, for each $t \in \bar{I}_\sigma$ we have

$$\int_{[t_0, t)} \omega(\psi(s))\varphi(s) d\widehat{g}(s) = {}^{(KS)}\int_{t_0}^t \omega(\psi(s))\varphi(s) d\widehat{g}(s) = {}^{(KS)}\int_{t_0}^t \omega(\psi(s)) d\bar{g}(s) = \int_{[t_0, t)} \omega(\psi(s)) d\bar{g}(s). \quad (4.33)$$

Observe that the integrals in (4.33) are well-defined in the Lebesgue–Stieltjes sense as the map $\omega \circ \psi$ is Borel measurable. Once again, without loss of generality, we assume that g is continuous at t_0 , which implies that so is \widehat{g} and, as a consequence, so is \bar{g} . This, together with (4.33), yields

$$\int_{[t_0, t_0 + \gamma)} \omega(\psi(s)) d\bar{g}(s) \leq \int_{[t_0, t_0 + \gamma)} K d\bar{g}(s) < K\mu_{\bar{g}}([t_0, t_0 + \gamma)) + \gamma =: \bar{\varepsilon}(\gamma), \quad \gamma \in (0, \sigma).$$

Therefore, the result can be proved by reasoning as in the proof of Theorem 4.8 with the appropriate adjustments, i.e. replacing \widehat{g} by \bar{g} and $\widehat{\varepsilon}(\cdot)$ by $\bar{\varepsilon}(\cdot)$ accordingly. \square

4.2 Existence and uniqueness of solution

As a final step on the study of (4.19), we combine the uniqueness results in the previous section with some information available in the literature regarding the existence of solution. Specifically, we will use [10, Theorem 4.5], which requires the following definition, a slightly more general version of [6, Definition 4.7].

DEFINITION 4.10 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function, $J \subset \mathbb{R}$ and $X \subset \mathbb{R}^m$, $X \neq \emptyset$. A function $f : J \times X \rightarrow \mathbb{R}^n$ is said to be g -Carathéodory if the following properties are satisfied:*

- (i) *For each $x \in X$, the map $f(\cdot, x)$ is g -measurable.*
- (ii) *The map $f(t, \cdot)$ is continuous for g -a.a. $t \in J$.*
- (iii) *For each $r > 0$, there exists $h_r \in \mathcal{L}_g^1(J, [0, +\infty))$ such that*

$$\|f(t, x)\| \leq h_r(t), \quad g\text{-a.a. } t \in J, \quad x \in X, \quad \|x\| \leq r.$$

REMARK 4.11 *As proved in [15, Theorem 4.1], a sufficient condition for the f in (4.19) to be g -Carathéodory for some derivator g is that $f(\cdot, x_0) \in \mathcal{L}_g^1(I_\sigma, \mathbb{R})$ and conditions (i) in Definition 4.10 and (4.32) with ω continuous at 0, $\omega(0) = 0$, are satisfied.*

Based on this definition, we find the following Peano-type existence result in [10].

THEOREM 4.12 *Let $r > 0$ and $f : I \times \overline{B(x_0, r)} \rightarrow \mathbb{R}^n$ be such that for every $i = 1, 2, \dots, n$, the map f_i is g_i -Carathéodory. Then, there exists $\sigma \in (0, T]$ such that (4.19) has a solution on I_σ .*

As a direct consequence of Theorem 4.12 and Remark 4.11, we obtain the following Montel–Tonelli existence result in a similar fashion to [15, Theorem 4.2]. In fact, Theorem 4.13 reduces to the mentioned result in the corresponding setting, i.e., for $\mathbf{g} = (g, g, \dots, g)$ for a derivator g .

THEOREM 4.13 *Let $r > 0$ and $f : I \times \overline{B(x_0, r)} \rightarrow \mathbb{R}^n$ be such that, for each $i \in \{1, 2, \dots, n\}$, the following conditions are satisfied:*

- (i) *For each $x \in \overline{B(x_0, r)}$, $f_i(\cdot, x)$ is g_i -measurable.*
- (ii) *$f_i(\cdot, x_0) \in \mathcal{L}_{g_i}^1(I, \mathbb{R})$.*
- (iii) *There exist $\varphi_i \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ and $\omega_i : [0, +\infty) \rightarrow [0, +\infty)$ nondecreasing, continuous at 0 with $\omega_i(0) = 0$ and such that (4.32) holds for $\sigma = T$ and $X = \overline{B(x_0, r)}$.*

Then, there exists $\sigma \in (0, T]$ such that (4.19) has a solution on I_σ .

Naturally, combining the hypotheses of Theorems 4.9 and 4.13 we can obtain an Montel–Osgood–Tonelli type existence and uniqueness result of local solutions of (4.19). This result is a generalization of [15, Theorem 4.3]. To see that this is the case, it is enough to bear in mind expression (4.31).

THEOREM 4.14 *Let $r > 0$, $f : I \times \overline{B(x_0, r)} \rightarrow \mathbb{R}^n$ and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$, $s > 0$. Assume that the following conditions are satisfied:*

- (i) *For every $i = 1, 2, \dots, n$ and each $x \in \overline{B(x_0, r)}$, the map $f_i(\cdot, x)$ is g_i -measurable.*
- (ii) *For every $i = 1, 2, \dots, n$, $f_i(\cdot, x_0) \in \mathcal{L}_{g_i}^1(I, \mathbb{R})$.*
- (iii) *There exists $u_0 > 0$ such that (4.29) holds.*
- (iv) *There exists a map $\varphi : I \rightarrow [0, +\infty)$ such that, for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \leq \varphi(t)\omega(\|x - y\|), \quad g_i\text{-a.a. } t \in I, \quad x, y \in \overline{B(x_0, r)}.$$

Then, there exists $\sigma \in (0, T]$ such that (4.19) has a unique solution on I_σ .

Observe that for the particular choice of $\omega(r) = r$, $r \geq 0$, we obtain [10, Theorem 4.4]. As commented in Section 2, the proof of such result relies, implicitly, on the fact that if $\varphi \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ for all $i \in \{1, 2, \dots, n\}$, then $\varphi \in \mathcal{L}_{\mathbf{g}}^1(I, [0, +\infty))$, which was unclear in that setting. Here, and specifically, in the proof of Theorem 4.9 we make use of Proposition 2.4 to ensure that this is the case, hence improving the work in [10].

Note that Theorem 4.14 is stated in the context of a neighborhood of the x_0 . Following the steps of [15], we can obtain an Montel–Osgood–Tonelli type existence and uniqueness result with the map f in (4.19) defined on the whole $I \times \mathbb{R}^n$. To that end, we introduce the following lemma, which is a reformulation of [15, Theorem 4.4] in the context of (4.19).

LEMMA 4.15 *Let $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$, $s > 0$. Assume that \mathbf{g} is continuous at t_0 and that the following conditions are satisfied:*

- (i) *For every $i = 1, 2, \dots, n$ and $x \in \mathbb{R}^n$, the map $f_i(\cdot, x)$ is g_i -measurable.*
- (ii) *For each $i \in \{1, 2, \dots, n\}$, $f_i(\cdot, x_0) \in \mathcal{L}_{g_i}^1(I, \mathbb{R})$.*
- (iii) *There exists $u_0 > 0$ such that (4.29) holds.*
- (iv) *There exists a map $\varphi : I \rightarrow [0, +\infty)$ such that, for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ and*

$$|f_i(t, x) - f_i(t, y)| \leq \varphi(t)\omega(\|x - y\|), \quad g_i\text{-a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then there exist $t_1 \in (t_0, t_0 + T]$ and a nondecreasing function $h : [t_0, t_1] \rightarrow \mathbb{R}$ such that for every solution of (4.19), $x : \bar{I}_\sigma \rightarrow \mathbb{R}^n$, $\sigma \in (0, T]$, we have

$$\|x(t) - x_0\| \leq h(t), \quad t \in \bar{I}_\sigma \cap [t_0, t_1].$$

Proof. Define $\kappa : \bar{I} \rightarrow \mathbb{R}$ as

$$\kappa(t) = \int_{[t_0, t]} |f_i(s, x_0)| d\hat{g}(s), \quad t \in \bar{I}.$$

Note that hypothesis (ii) and Proposition 2.4 ensure that κ is well-defined. Let $x : \bar{I}_\sigma \rightarrow \mathbb{R}^n$ be a solution of (4.19). Remark 3.5 ensures that $x - x_0 \in \mathcal{AC}_{\hat{g}}(\bar{I}_\sigma, \mathbb{R}^n)$, which yields $\|x - x_0\| \in$

$\mathcal{AC}_{\widehat{g}}(\overline{I}_\sigma, \mathbb{R}^n)$. Hence, it is Borel measurable and bounded. Given that ω is continuous, it follows that $\omega(\|x - x_0\|)$ is Borel measurable and thus, \widehat{g} and g_i -measurable for each $i \in \{1, 2, \dots, n\}$. Furthermore, since ω is nondecreasing, it follows that it is bounded, which guarantees that the map $\varphi(t)\omega(\|x(t) - x_0\|)$ is \widehat{g} and g_i -integrable for each $i \in \{1, 2, \dots, n\}$.

Now, for each $t \in \overline{I}_\sigma$, there is $j_t \in \{1, 2, \dots, n\}$ such that $\|x(t) - x_0\| = |x_{j_t}(t) - x_{0,j_t}(t)|$. Hence, condition (iv) yields that, for each $t \in \overline{I}_\sigma$,

$$\begin{aligned} \|x(t) - x_0\| &\leq \int_{[t_0, t)} |f_{j_t}(s, x_0)| \, dg_{j_t}(s) + \int_{[t_0, t)} |f_{j_t}(s, x(s)) - f_{j_t}(s, x_0)| \, dg_{j_t}(s) \\ &\leq \sum_{i=1}^n \left(\int_{[t_0, t)} |f_i(s, x_0)| \, dg_i(s) + \int_{[t_0, t)} \varphi(s)\omega(\|x(s) - x_0\|) \, dg_i(s) \right) \\ &\leq \kappa(t) + \int_{[t_0, t)} \varphi(s)\omega(\|x(s) - x_0\|) \, d\widehat{g}(s). \end{aligned}$$

Define $\overline{g} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\overline{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \int_{[t_0, t)} \varphi(s) \, d\widehat{g}(s), & \text{if } t_0 < t \leq t_0 + \sigma, \\ \int_{[t_0, t_0 + \sigma)} \varphi(s) \, d\widehat{g}(s), & \text{if } t > t_0 + \sigma. \end{cases}$$

Recalling the relation (4.33), it follows that

$$\|x(t) - x_0\| \leq \kappa(t) + \int_{[t_0, t)} \omega(\|x(s) - x_0\|) \, d\overline{g}(s), \quad t \in \overline{I}_\sigma. \quad (4.34)$$

In order to apply [24, Theorem 1.40], consider the map $\Omega : (0, +\infty) \rightarrow (0, +\infty)$ as in (4.30). Since $\lim_{r \rightarrow 0^+} \Omega(r) = -\infty$, there exists $R > 0$ such that

$$\Omega(R) + \overline{g}(t_0 + T) - \overline{g}(t_0) < \beta := \lim_{r \rightarrow \infty} \Omega(r) \leq +\infty.$$

Since \mathbf{g} is continuous at t_0 , we have that \widehat{g} is continuous at t_0 as well. Then, we can find $t_1 \in (t_0, t_0 + T]$ such that $\kappa(t_1) \leq R$. The monotonicity of Ω then yields

$$\Omega(\kappa(t_1)) + \overline{g}(t_1) - \overline{g}(t_0) < \beta.$$

The inequality above together with (4.34) shows that the assumptions of Lemma 4.7 are satisfied on the interval $[t_0, t_1]$, therefore

$$\|x(t) - x_0\| \leq \Omega^{-1}(\Omega(\kappa(t_1)) + \overline{g}(t) - \overline{g}(t_0)) =: h(t), \quad t \in \overline{I}_\sigma \cap [t_0, t_1],$$

and $h : [t_0, t_1] \rightarrow \mathbb{R}$ is the desired monotone function. \square

Furthermore, for the proof of Theorem 4.17, we will also require the following result which is a reformulation of [10, Proposition 3.5] based on [6, Proposition 5.5] and Remark 3.7.

PROPOSITION 4.16 *Let \mathcal{S} be a subset of $\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$. Assume that for each $i \in \{1, 2, \dots, n\}$ the following conditions are satisfied:*

- (i) *The set $\{F_i(a) : F = (F_1, F_2, \dots, F_n) \in \mathcal{S}\}$ is bounded.*
- (ii) *There exists $h_i \in \mathcal{L}_{g_i}^1([a, b], [0, +\infty))$ such that*

$$|(F_i)'_{g_i}(t)| \leq h_i(t), \quad g_i\text{-a.a. } t \in [a, b], \quad \text{for all } F = (F_1, F_2, \dots, F_n) \in \mathcal{S}.$$

Then \mathcal{S} is a relatively compact subset of $\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$.

Now, we can state and prove the following Montel–Osgood–Tonelli existence and uniqueness of solution result for problem (4.19). Observe that, although we are imposing global conditions on \mathbb{R}^n , we can only ensure the existence of a unique local solution. In that sense, Theorem 4.17 is only a partial improvement with respect to [10, Theorem 4.3].

THEOREM 4.17 *Let $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function such that $\omega(0) = 0$ and $\omega(s) > 0$, $s > 0$. Assume that the following conditions are satisfied:*

- (i) For every $i = 1, 2, \dots, n$ and $x \in \mathbb{R}^n$, the map $f_i(\cdot, x)$ is g_i -measurable.
- (ii) For each $i \in \{1, 2, \dots, n\}$, $f_i(\cdot, x_0) \in \mathcal{L}_{g_i}^1(I, \mathbb{R})$.
- (iii) There exists $u_0 > 0$ such that (4.29) holds.
- (iv) There exists a map $\varphi : I \rightarrow [0, +\infty)$ such that, for each $i \in \{1, 2, \dots, n\}$, $\varphi \in \mathcal{L}_{g_i}^1(I, [0, +\infty))$ and

$$|f_i(t, x) - f_i(t, y)| \leq \varphi(t)\omega(\|x - y\|), \quad g_i\text{-a.a. } t \in I, \quad x, y \in \mathbb{R}^n.$$

Then there exists $\sigma \in (0, T]$ such that (4.19) has a unique solution on I_σ .

Proof. Without loss of generality, we assume that \mathbf{g} is continuous at t_0 . Let $h : [t_0, t_1] \rightarrow \mathbb{R}$ be the function whose existence is guaranteed by Lemma 4.15 and denote $R = h(t_1)$. Define $\bar{g} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\bar{g}(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \int_{[t_0, t)} \varphi(s) d\hat{g}(s), & \text{if } t_0 < t \leq t_0 + T, \\ \int_{[t_0, t_0+T)} \varphi(s) d\hat{g}(s), & \text{if } t > t_0 + T. \end{cases}$$

Since \mathbf{g} is continuous at t_0 , we have that so are g_i , $i = 1, 2, \dots, n$, and \hat{g} and, as a consequence, \bar{g} is continuous at t_0 as well. Thus, we can choose $\sigma \in (0, T]$ such that $t_0 + \sigma \leq t_1$ and

$$\omega(R)\mu_{\bar{g}}([t_0, t_0 + \sigma]) + \sum_{i=1}^n \int_{[t_0, \sigma)} |f_i(s, x_0)| dg_i(s) < R. \quad (4.35)$$

Consider $B = \{x \in \mathcal{BC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n) : \|x(t) - x_0\| \leq R, t \in \bar{I}_\sigma\}$. Clearly, B is a closed and convex subset of $\mathcal{BC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$. Now let us define $F : B \rightarrow \mathcal{BC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$ by

$$Fx(t) = x_0 + \int_{[t_0, t)} f(s, x(s)) d\mathbf{g}(s), \quad t \in I_\sigma.$$

It follows from Remark 4.11 that f_i is g_i -Carathéodory, $i = 1, 2, \dots, n$. Furthermore, given $x \in B$, we have that x is Borel measurable and therefore, g_i -measurable for each $i \in \{1, 2, \dots, n\}$. Thus, [6, Lemma 7.2] and Remark 3.4 ensure that F is well-defined. Moreover, the continuity of ω together with condition (iv) implies that F is continuous. Furthermore, following a similar argument as the one used to obtain (4.34), and using the relation (4.33), it follows from condition (iv) and (4.35) that for $x \in B$,

$$\|Fx(t) - x_0\| \leq \int_{[t_0, t)} \omega(\|x(s) - x_0\|) d\bar{g}(s) + \sum_{i=1}^n \int_{[t_0, \sigma)} |f_i(s, x_0)| dg_i(s) < R,$$

for every $t \in \bar{I}_\sigma$. That is, $F(B) \subset B$. It remains to verify that $F(B)$ is relatively compact in $\mathcal{BC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$. Firstly, note that for $x \in B$ and $i \in \{1, 2, \dots, n\}$,

$$|f_i(t, x(t))| \leq |f_i(t, x(t)) - f_i(t, x_0)| + |f_i(t, x_0)| \leq \varphi(t)\omega(R) + |f_i(t, x_0)| =: M_i(t),$$

for g_i -a.a. $t \in I_\sigma$. Observe that $F(B) \subset \mathcal{AC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$ and

$$((Fx)_i)'_{g_i}(t) = f_i(t, x(t)), \quad g_i\text{-a.a. } t \in I_\sigma, \quad x \in B, \quad i = 1, 2, \dots, n.$$

Since $M_i \in \mathcal{L}_{g_i}^1(I_\sigma, [0, +\infty))$ for each $i \in \{1, 2, \dots, n\}$, it follows from Proposition 4.16 that $F(B)$ is relatively compact in $\mathcal{BC}_{\mathbf{g}}(\bar{I}_\sigma, \mathbb{R}^n)$. Thus, Schauder's Fixed Point Theorem guarantees the existence of solution of (4.19) on I_σ , while the uniqueness is a consequence of Theorem 4.8. \square

As a final note, we provide an example where the results obtained in this paper can be applied while those in [10] cannot. To that end, we consider the maps $\exp^{[k]} : \mathbb{R} \rightarrow \mathbb{R}$, $k = 0, 1, 2, \dots$, defined

$$\exp^{[0]}(t) = t, \quad \exp^{[k]}(t) = \exp(\exp^{[k-1]}(t)), \quad k \in \mathbb{N},$$

the functions $\log^{[k]} : (\exp^{[k-1]}(0), +\infty) \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, defined as

$$\log^{[1]}(t) = \log(t), \quad \log^{[k]}(t) = \log(\log^{[k-1]}(t)), \quad k \geq 2,$$

and the family of functions $\omega_k : [0, +\infty) \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, defined as

$$\omega_k(t) = \begin{cases} 0 & \text{if } t = 0, \\ t \prod_{j=1}^k \log^{[j]} \left(\frac{1}{t} \right) & \text{if } 0 < t < \frac{1}{e_k}, \\ \frac{1}{e_k^2} \prod_{j=1}^k \exp^{[j]}(1) & \text{if } t \geq \frac{1}{e_k}, \end{cases} \quad (4.36)$$

where $e_k := \exp^{[k]}(1)$. In [15, Proposition 5.1], the authors proved some interesting properties of ω_k , $k \in \mathbb{N}$. In particular, they showed that ω_k is nondecreasing, differentiable in $(0, 1/e_k)$ and

$$\lim_{t \rightarrow 0^+} \omega_k'(t) = +\infty.$$

Furthermore, it is shown that ω_k , $k \in \mathbb{N}$, satisfies (4.29) for any $u_0 > 0$ and, in [15, Example 5.2], that for any $x, y \in \mathbb{R}$,

$$|\omega_k(|x|) - \omega_k(|y|)| \leq \omega_k(|x - y|). \quad (4.37)$$

With this, we have all the information necessary for the following example.

EXAMPLE 4.18 Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, such that each g_i , $i \in \{1, 2, \dots, n\}$, is a derivator; and $\varphi : [0, 1) \rightarrow [0, +\infty)$ be a Borel measurable map which is bounded. Observe that this implies that $\varphi \in \mathcal{L}_{g_i}^1([0, 1), [0, +\infty))$ for each $i \in \{1, 2, \dots, n\}$. For each $k \in \mathbb{N}$, consider the initial value problem

$$x_{\mathbf{g}}'(t) = f_k(t, x(t)), \quad g\text{-a.a. } t \in [0, 1), \quad x(0) = x_0, \quad (4.38)$$

where $f_k : [0, 1) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n})$, is given by

$$f_{k,i}(t, x) = \varphi(t) \omega_k(\|x\|), \quad (t, x) \in [0, 1) \times \mathbb{R}^n, \quad i \in \{1, 2, \dots, n\},$$

with ω_k as in (4.36). Note that each $f_{k,i}$, $i \in \{1, 2, \dots, n\}$, $k \in \mathbb{N}$, does not satisfied a Lipschitz condition on the whole \mathbb{R} as the derivative of ω_k is unbounded on any neighbourhood of 0. Therefore, the hypotheses of [10, Theorem 4.3] cannot be satisfied. Similarly, [10, Theorem 4.4] cannot be applied for any ball around x_0 containing 0. However, Theorem 4.17 yields that (4.38) has a unique solution. Indeed, it is clear that conditions (i) and (ii) are satisfied. Furthermore, as mentioned before, [15, Proposition 5.1] guarantees that condition (iii) is satisfied. Now, for condition (iv), for each $i \in \{1, 2, \dots, n\}$, the fact that ω_k is nondecreasing and (4.37) yield

$$|f_{k,i}(t, x) - f_{k,i}(t, y)| = \varphi(t) |\omega_k(\|x\|) - \omega_k(\|y\|)| \leq \varphi(t) \omega_k(\|\|x\| - \|y\|\|) \leq \varphi(t) \omega_k(\|x - y\|), \quad k \in \mathbb{N},$$

which shows that condition (iv) is satisfied. Hence, the hypotheses of Theorem 4.17 are satisfied. Using analogous reasonings, we can show that the hypotheses of Theorem 4.14 are satisfied if we consider f_k to be defined on $[0, 1) \times B(x_0, r)$, $r > 0$, instead of on $[0, 1) \times \mathbb{R}$.

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