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A sharp interpolation between the Hölder and Gaussian Young inequalities

Paolo Da Pelo* and Alberto Lanconelli†

Dipartimento di Matematica, Universita' Degli Studi di Bari,
Via E. Orabona, 4, 70125 Bari, Italia
**paolo.dapelo@uniba.it*
†*alberto.lanconelli@uniba.it*

Aurel I. Stan‡

Department of Mathematics, The Ohio State University at Marion,
1465 Mount Vernon Avenue, Marion, OH 43302, USA
stan.7@osu.edu

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We prove a very general sharp inequality of the Hölder–Young-type for functions defined on infinite dimensional Gaussian spaces. We begin by considering a family of commutative products for functions which interpolates between the pointwise and Wick products; this family arises naturally in the context of stochastic differential equations, through Wong–Zakai-type approximation theorems, and plays a key role in some generalizations of the Beckner-type Poincaré inequality. We then obtain a crucial integral representation for that family of products which is employed, together with a generalization of the classic Young inequality due to Lieb, to prove our main theorem. We stress that our main inequality contains as particular cases the Hölder inequality and Nelson's hypercontractive estimate, thus providing a unified framework for two fundamental results of the Gaussian analysis.

Keywords: Gaussian T-Wick product; second quantization operator; exponential functions; Hölder inequality; Lieb inequality; Minkowski inequality; Jensen inequality.

AMS Subject Classification: 60H40, 60H10

1. Introduction

The celebrated Wong–Zakai approximation theorem[28] establishes that if $\{W_t^\epsilon\}_{t\geq 0}$ denotes a "good" approximation of the white noise $\{W_t\}_{t\geq 0}$, then for any smooth

‡Corresponding author

functions $b, \sigma : \mathbb{R} \to \mathbb{R}$ the solution of the random differential equation

$$\dot{X}_t^\epsilon = b(X_t^\epsilon) + \sigma(X_t^\epsilon) \cdot W_t^\epsilon \tag{1.1}$$

converges in probability, as ϵ goes to zero, to the solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t) \circ dW_t, \tag{1.2}$$

where the symbol $\circ\, dW_t$ denotes Stratonovich stochastic integration (recently an analogous result in the context of stochastic partial differential equations has been obtained in the beautiful paper, Ref. 16). Replacing the pointwise product appearing between $\sigma(X_t^\epsilon)$ and W_t^ϵ in Eq. (1.1) with the Wick product, one gets,[17] under the assumption of a linear diffusion coefficient σ, the convergence, as ϵ goes to zero, to the Itô version of Eq. (1.2).

In Ref. 13 the authors introduced the following family of multiplications

$$f \diamond_\tau g := \tau^{-N}(\tau^N f \cdot \tau^N g), \quad \tau \in (0, 2], \tag{1.3}$$

where N denotes the number or Ornstein–Uhlenbeck operator (we refer the reader to the next section for a rigorous definition of this product in terms of second quantization operators). This family interpolates between the pointwise product (when $\tau = 1$) and the Wick product (in the limit as τ tends to zero); replacing Eq. (1.1) with

$$\dot{Y}_t^\epsilon = b(Y_t^\epsilon) + Y_t^\epsilon \diamond_\tau W_t^\epsilon,$$

one can prove[13] the convergence of Y_t^ϵ to the solution of

$$dY_t = b(Y_t)dt + Y_t \circ_\tau dW_t,$$

where $\circ_\tau\, dW_t$ denotes stochastic integration with evaluation point at

$$t_i^* := t_{i-1} + \frac{\tau}{2}(t_i - t_{i-1})$$

(this gives Stratonovich for $\tau = 1$ and Itô for $\tau = 0$). See Refs. 1 and 2 for a more general theory of stochastic integration.

The family of products defined in (1.3) turns out to be useful also in the study of Poincaré-type inequalities. In fact, an important generalization of the classic Poincaré inequality[10,25]

$$\int f^2(w)d\mu(w) - \left(\int f(w)d\mu(w)\right)^2 \leq \int \|Df(w)\|^2 d\mu(w) \tag{1.4}$$

(here μ is a Gaussian measure defined on a possibly infinite dimensional space and Df is a suitable notion of gradient of f) is the one proposed in Ref. 4 which reads for $\tau \in [0, 1]$ as

$$\int f^2(w)d\mu(w) - \int |\tau^N f(w)|^2 d\mu(w) \leq (1 - \tau) \int \|Df(w)\|^2 d\mu(w). \tag{1.5}$$

Inequality (1.5) coincides with (1.4) for $\tau = 0$ and with the logarithmic Sobolev inequality[15] in the limit as τ tends to one (after an application of the Nelson's hyper-contractive estimate). Observe that since for any g one has

$$\int \tau^{-N} g(w)d\mu(w) = \int g(w)d\mu(w)$$

it is possible to rewrite (1.5) as

$$\int f^2(w)d\mu(w) - \int \tau^{-N}|\tau^N f(w)|^2 d\mu(w) \leq (1-\tau)\int \|Df(w)\|^2 d\mu(w)$$

or equivalently as

$$\int f^2(w)d\mu(w) - \int (f \diamond_\tau f)(w)d\mu(w) \leq (1-\tau)\int \|Df(w)\|^2 d\mu(w). \qquad (1.6)$$

It has been proved in Ref. 14 the validity of inequality (1.6) for all the probability measures obtained convolving the Gaussian measure μ with a probability measure satisfying an exponential integrability condition.

It is then clear from the preceding discussion that the family of products defined in (1.3) connects intrinsically pointwise multiplication and Stratonovich integral on one side and Wick product and Itô integral on the other side. This connection is in addition related to the interplay between the Poincaré and logarithmic Sobolev inequalities. The aim of the paper is to obtain an inequality for the L^r-norm of $f \diamond_\tau g$ in terms of the L^p-norm of f and the L^q-norm of g for suitable $p, q, r \in [1, +\infty]$ and $\tau \in [0, 2]$. We obtain a very general and sharp inequality which coincides with the classic Hölder inequality for $\tau = 1$, as expected from the point of view of the interpolating nature of our family of products, and with the sharp Young-type inequality for the Wick product obtained in Ref. 11 for $\tau = 0$. (From a probabilistic point of view, the Wick product plays in Gaussian spaces the same role played by the convolution in spaces equipped with the Lebesgue measure; that is why we call the inequality for the Wick product of Young-type).

The main purpose of this paper is to find necessary and sufficient conditions to have inequalities of the form:

$$\|f \diamond_\tau g\|_r \leq \|\Gamma(C)f\|_\tau \cdot \|\Gamma(D)g\|_\tau,$$

for all $f \in L^p(\mu)$ and $g \in L^q(\mu)$.

In general, as also observed in Ref. 11, inequalities about the norms of Wick products are related to sharp inequalities (that means inequalities with best constants) from classic Harmonic Analysis like: Young and Lieb inequalities, see: Refs. 3, 7, 23 and 24. The sharp constant being 1 allows us to pass to the limit as the dimension d goes to infinity, having the same inequalities even in the infinite-dimensional case.

The paper is structured as follows. In Sec. 2 we give a minimal background on the construction of an infinite dimensional Gaussian probability space and second quantization operators. In Sec. 3, we review the definition of the t-Wick product and extend it to the definition of the T-Wick product, where T is an operator. We also review the definition of the exponential functions. In Sec. 4 we prove an important integral representation for the Gaussian T-Wick products for a specific class of operators T. In Sec. 5, we use the integral representation found in Sec. 4 and Lieb theorem from Ref. 23, to prove the main inequality from this paper in the dimension $d = 1$. We use Minkowski integral inequality, to extend the inequality from dimension $d = 1$, to every finite dimension $d \geq 2$. Finally, we extend the inequality to the infinite dimensional case.

2. Background

There are many ways to introduce the Gaussian Wick product and second quantization operators, all of them being equivalent. One can use an abstract Gel'fand triple and work with Hida White Noise Distribution Theory, see Refs. 19 or 27. Another way is to use Malliavin Calculus, see Ref. 5. There is also a third way, using the theory of Gaussian Hilbert spaces, see Ref. 18. We will use Hida White Noise Distribution Theory, to make the connection with the stochastic integral.

Let E be a real separable Hilbert space, and A a self-adjoint operator on E having a discrete spectrum $\{\lambda_n\}_{n\geq 0}$, such that:

(1) There exists an orthonormal basis $\{e_n\}_{n\geq 0}$ of E, such that for all $n \geq 0$,

$$Ae_n = \lambda_n e_n. \tag{2.1}$$

(2) $1 < \lambda_1 < \lambda_2 < \cdots$
(3) The operator A^{-1} is a Hilbert–Schmidt operator.

The inner product and norm of E are denoted by $(\cdot, \cdot)$ and $|\cdot|_0$, respectively. For each $p \geq 0$, we define the norm:

$$|f|_p^2 := |A^p f|_0^2 = \sum_{n=0}^{\infty} \lambda_n^{2p} (f, e_n)^2. \tag{2.2}$$

For each $p \geq 0$, we define the space:

$$\mathcal{E}_p := \{f \in E \,|\, |f|_p < \infty\}. \tag{2.3}$$

$\mathcal{E}_p$ is a Hilbert space with norm $|\cdot|_p$. If $0 \leq p < q$, then $\mathcal{E}_q \subset \mathcal{E}_p$.

We define the space:

$$\mathcal{E} = \bigcap_{p=0}^{\infty} \mathcal{E}_p \tag{2.4}$$

and equip it with the locally convex topology given by the family of norms $\{|\cdot|_p\}_{p\geq 0}$. The space $\mathcal{E}$ is a nuclear space.

For each $p \geq 0$, the dual of the space $\mathcal{E}_p$ is the space $\mathcal{E}_{-p}$, which is the completion of the space E, with respect to the norm $|\cdot|_{-p}$, defined as:

$$|f|_{-p}^2 := |A^{-p} f|_0^2 := \sum_{n=0}^{\infty} \lambda_n^{-2p} (f, e_n)^2. \tag{2.5}$$

Of course, if $0 \leq p < q$, we have:

$$E \subset \mathcal{E}_{-p} \subset \mathcal{E}_{-q}. \tag{2.6}$$

The dual of the space $\mathcal{E}$ is the space $\mathcal{E}'$, which can be written as:

$$\mathcal{E}' = \bigcup_{p=0}^{\infty} \mathcal{E}_{-p}. \tag{2.7}$$

The dual space $\mathcal{E}'$ is equipped with the inductive limit topology of the (locally convex) topologies given by the norms $\{|\cdot|_{-p}\}_{p\geq 0}$. We obtain in this way the following Gel'fand triple:

$$\mathcal{E} \subset E \subset \mathcal{E}'. \tag{2.8}$$

By Minlos theorem there exists a unique probability measure μ on the dual space $\mathcal{E}'$ of $\mathcal{E}$, such that, for all $\xi \in \mathcal{E}$, we have:

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-(1/2)|\xi|_0^2}, \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing of $\mathcal{E}'$ and $\mathcal{E}$, see p. 16 of Ref. 19. Formula (2.9) says that, as a random variable, the continuous function $\langle \cdot, \xi \rangle$ is normally distributed with mean 0 and variance $|\xi|_0^2$, for every $\xi \in \mathcal{E}$. This observation is very important, since by approximating in the norm $|\cdot|_0$ of E, every element f of E, by a sequence $\{\xi_n\}_{n\geq 1}$ of elements of $\mathcal{E}$, we obtain a Cauchy sequence $\{\langle \cdot, \xi_n \rangle\}_{n\geq 1}$ in $L^2(\mathcal{E}', \mu)$ of normally distributed random variables. The L^2-limit of this sequence is denoted by $\langle \cdot, f \rangle$ and is a normally distributed random variable with mean 0 and variance $|f|_0^2$.

For every real Hilbert space H, we denote by H_c its complexification. We define the *trace operator* τ as the following element of $(\mathcal{E}_c')^{\hat{\otimes}2}$, where $\hat{\otimes}$ denotes the symmetric tensor product:

$$\langle \tau, \xi \otimes \eta \rangle := \langle \xi, \eta \rangle, \tag{2.10}$$

for all ξ and η in $\mathcal{E}_c$. We define the *Wick tensor* $: x^{\otimes n} :$, for every $x \in \mathcal{E}'$, as:

$$: x^{\otimes n} := \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-1)^k x^{\otimes(n-2k)} \hat{\otimes} \tau^{\otimes k}.$$

If we denote by (L^2) the space of all complex valued square integrable functions defined on $(\mathcal{E}', \mu)$, then for every function φ in (L^2), there exists a unique sequence $\{f_n\}_{n\geq 0}$, where for all $n \geq 0$, $f_n \in E_c^{\hat{\otimes}n}$, such that:

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle. \tag{2.11}$$

Moreover, the square of the (L^2)-norm of φ is:

$$\|\varphi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty, \tag{2.12}$$

where $|f_n|_0$ denotes the norm of f_n computed in the space $E_c^{\hat{\otimes}n}$.

If B is a densely defined operator on E, and φ is given by (2.11), then we define the *second quantization operator* of B, as:

$$\Gamma(B)\varphi(x) := \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, B^{\otimes n} f_n \rangle. \tag{2.13}$$

It is not hard to see that if B is a bounded operator on E, of operatorial norm $\|B\| \le 1$, then $\Gamma(B)$ is a bounded operator on (L^2) of operatorial norm $\|\Gamma(B)\| = 1$.

In particular, if we take $B := A$, the unbounded operator used to define the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$, then the second quantization operator $\Gamma(A)$ has properties similar to those of the operator A:

- $\Gamma(A)$ has positive eigenvalues and a set of eigenfunctions that forms an orthogonal basis of (L^2).
- $\Gamma(A)^{-1}$ is a bounded operator.
- For every $p > 1$, $\Gamma(A)^{-p}$ is a Hilbert–Schmidt operator on (L^2).

Repeating the same constructions as before, with (L^2) and $\Gamma(A)$ replacing E and A, respectively, we obtain a new Gel'fand triple:

$$(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*. \tag{2.14}$$

$(\mathcal{E})$ is called the space of *test functions*, while $(\mathcal{E})^*$ is named the space of *generalized functions* (or *Hida distributions*).

The bilinear pairing between $(\mathcal{E})^*$ and $(\mathcal{E})$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. It must be mentioned that while the spaces involved in the first Gel'fand triple:

$$\mathcal{E} \subset E \subset \mathcal{E}'$$

are vector spaces over $\mathbb{R}$, the spaces used in the second Gel'fand triple:

$$(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$$

are vector spaces over $\mathbb{C}$.

The following two theorems can be found in Ref. 27, pp. 35–36.

Theorem 2.1. *Let $\phi \in (L^2)$ have the following Wiener–Itô expansion*:

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle,$$

where $x \in \mathcal{E}'$, and for each $n \ge 0$, $f_n \in E_c^{\hat{\otimes} n}$, such that:

$$\sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty.$$

Then $\phi \in (\mathcal{E})$ if and only if, for all $n \ge 0$, $f_n \in \mathcal{E}_c^{\hat{\otimes} n}$, and for all $p \ge 0$:

$$\|\phi\|_p^2 := \sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty. \tag{2.15}$$

Theorem 2.2. *For each $\phi \in (\mathcal{E})^*$ there exists a unique sequence $\{F_n\}_{n\geq 0}$, such that, for all $n \geq 0$, $F_n \in \mathcal{E}_c'^{\hat{\otimes} n}$:*

$$\langle\langle \phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \tag{2.16}$$

for all $\varphi \in (\mathcal{E})$, where φ and $\{f_n\}_{n=0}^{\infty}$ are related by the previous theorem.

Conversely, given a sequence $\{F_n\}_{n=0}^{\infty}$, such that, for each $n \geq 0$, $F_n \in \mathcal{E}_c'^{\hat{\otimes} n}$ and there exists $p \geq 0$, such that:

$$\sum_{n=0}^{\infty} n! |F_n|_{-p}^2 < \infty, \tag{2.17}$$

a generalized functional $\phi \in (E)^*$ is defined by (2.16). In this case, we write:

$$\phi(x) := \sum_{n=1}^{\infty} \langle : x^{\otimes n} :, F_n \rangle. \tag{2.18}$$

If $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, F_n \rangle$, and there exists $N \in \mathbb{N}$, such that, for all $n \geq N$, $F_n = 0$, then we call ϕ a *polynomial* generalized function.

As a particular example of the above general construction we present the following. Let k be a natural number and $E := L^2(\mathbb{R}^k, dx)$, where dx denotes the Lebesgue measure. We consider the following self-adjoint operator on E:

$$A := \left(x_1^2 - \frac{d^2}{dx_1^2} + 1 \right) \left(x_2^2 - \frac{d^2}{dx_2^2} + 1 \right) \cdots \left(x_k^2 - \frac{d^2}{dx_k^2} + 1 \right). \tag{2.19}$$

Then A satisfies the conditions required by our construction. In this case, the nuclear space $\mathcal{E}$ becomes the Schwartz space of rapidly decreasing smooth functions, and its dual $\mathcal{E}'$ the space of tempered distributions.

Let $\mathcal{B}_f$ denote the set of all Borel subsets of $\mathbb{R}^k$ of finite Lebesgue measure.

Since for every set X in $\mathcal{B}_f$, its characteristic function 1_X belongs to $L^2(\mathbb{R}^k, dx)$, we can define the (L^2) random variable:

$$B_X := \langle \cdot, 1_X \rangle. \tag{2.20}$$

Then the family of random variables $\{B_X\}_{X \in \mathcal{B}_f}$ is a Brownian sheet.

In particular, if $k = 1$, and for every $t \geq 0$, we define:

$$B_t := \langle \cdot, 1_{[0,t]} \rangle, \tag{2.21}$$

then $\{B_t\}_{t\geq 0}$ is a Brownian motion process.

The derivative of the Brownian motion is the following polynomial generalized function:

$$\dot{B}_t = \langle \cdot, \delta_t \rangle, \tag{2.22}$$

where δ_t denotes the Dirac delta measure, for all $t \in \mathbb{R}$.

3. Generalized Wick Products and Exponential Functions

For any non-negative integer n, let us denote by $\mathcal{G}_n$, the following closed subspace of (L^2):

$$\mathcal{G}_n := \{\langle : x^{\otimes n} :, f_n\rangle \mid f_n \in E_c^{\hat{\otimes} n}\}. \tag{3.1}$$

We call $\mathcal{G}_n$ the space of *homogenous polynomial random variables of degree* n. It is clear that the spaces $\{\mathcal{G}_n\}_{n\geq 0}$ are mutually orthogonal.

For all $n \geq 0$, let us define:

$$\mathcal{F}_n := \sum_{k=0}^{n} \mathcal{G}_k. \tag{3.2}$$

Let P_n and $P_{<n}$ denote the orthogonal projection of (L^2) onto the closed subspaces $\mathcal{G}_n$ and $\mathcal{F}_{n-1}$, respectively.

If $\varphi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, F_n\rangle$ and $\psi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, G_n\rangle$, with F_n and G_n in $\mathcal{E}_c'^{\hat{\otimes} n}$ for all $n \geq 0$, then we define the *classic Wick product*, $\varphi \diamond \psi$, of φ and ψ, as the following generalized function:

$$(\varphi \diamond \psi)(x) := \sum_{k=0}^{\infty}\langle : x^{\otimes k} :, h_k\rangle, \tag{3.3}$$

where

$$h_k := \sum_{u+v=k} F_u \hat{\otimes} G_v. \tag{3.4}$$

It is shown in Ref. 19, Theorem 8.12, p. 92, that the Wick product is continuous from $(\mathcal{E})^* \times (\mathcal{E})^*$ into $(\mathcal{E})^*$.

For every $t > 0$ (and later on we will restrict t to $(0, 2]$), the *generalized Wick product* or t-*Wick product*, introduced by Da Pelo and Lanconelli (see Ref. 12), can be defined, using the second quantization operator of $\sqrt{t}$ times the identity operator I as:

$$\varphi \diamond_t \psi := \Gamma\left(\frac{1}{\sqrt{t}}I\right)[\Gamma(\sqrt{t}I)\varphi \cdot \Gamma(\sqrt{t}I)\psi], \tag{3.5}$$

for every (φ, ψ) in a dense subspace V_t of $(L^2) \times (L^2)$, for which $\Gamma((1/\sqrt{t})I)[\Gamma(\sqrt{t}I)\varphi \cdot \Gamma(\sqrt{t}I)\psi]$ belongs to (L^2). Such a space V_t can be taken as the vector space spanned by exponential functions (which will be defined later).

We know for sure that for every two polynomial random variables φ and ψ in (L^2), $\varphi \diamond_t \psi$ is also a polynomial random variable. Moreover, $\varphi \diamond_t \psi$ can be viewed as a polynomial in the variable t with coefficients in the spaces $E_c^{\hat{\otimes} n}$, for $n \geq 0$, whose constant term (i.e. the term without t) is the classic Wick product $\varphi \diamond \psi$. To understand this, let us write: $\varphi = \sum_{p=0}^{m} f_p$ and $\psi = \sum_{q=0}^{n} g_q$, where for all $0 \leq p \leq m$, $f_p \in \mathcal{G}_p$, and for all $0 \leq q \leq n$, $g_q \in \mathcal{G}_q$. Since, for all $(p, q) \in \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$, $f_p \cdot g_q \in \mathcal{F}_{p+q}$, and the Gaussian probability measure is symmetric, we have:

$$\begin{aligned} f_p \cdot g_q &= P_{p+q}(f_p \cdot g_q) + P_{<(p+q)}(f_p \cdot g_q) \\ &= f_p \diamond g_q + \sum_{\substack{k<p+q \\ k\equiv(p+q)(\text{mod } 2)}} P_k(f_p \cdot g_q). \end{aligned} \tag{3.6}$$

Thus, for all $t \in (0, 2]$, we have:

$$\begin{aligned} \varphi \diamond_t \psi &= \Gamma\left(\frac{1}{\sqrt{t}}I\right)[\Gamma(\sqrt{t}I)\varphi \cdot \Gamma(\sqrt{t}I)\psi] \\ &= \Gamma\left(\frac{1}{\sqrt{t}}I\right)\left[\sum_{p=0}^{m} t^{p/2} f_p \cdot \sum_{q=0}^{n} t^{q/2} g_q\right] \\ &= \Gamma\left(\frac{1}{\sqrt{t}}I\right)\left[\sum_{k=0}^{m+n} t^{k/2} \sum_{p+q=k} f_p \cdot g_q\right] \\ &= \sum_{k=0}^{m+n} t^{k/2} \sum_{p+q=k} \Gamma\left(\frac{1}{\sqrt{t}}I\right)\left(f_p \diamond g_q + \sum_{\substack{r<k \\ r\equiv k(\text{mod } 2)}} P_r(f_p \cdot g_q)\right). \end{aligned}$$

Since for all $l \geq 0$ and $h \in \mathcal{G}_l$, we have $\Gamma(1/(\sqrt{t})I)h = t^{-l/2}h$, we obtain:

$$\begin{aligned} \varphi \diamond_t \psi &= \sum_{k=0}^{m+n} t^{k/2} \sum_{p+q=k} \left[t^{-k/2} f_p \diamond g_q + \sum_{\substack{r<k \\ r\equiv k(\text{mod } 2)}} t^{-r/2} P_r(f_p \cdot g_q)\right] \\ &= \sum_{k=0}^{m+n} \sum_{p+q=k} f_p \diamond g_q + \sum_{k=1}^{m+n} \sum_{p+q=k} \sum_{\substack{r<k \\ r\equiv k(\text{mod } 2)}} t^{(k-r)/2} P_r(f_p \cdot g_q) \\ &= \varphi \diamond g + \sum_{k=1}^{m+n} \sum_{p+q=k} \sum_{\substack{r<k \\ r\equiv k(\text{mod } 2)}} t^{(k-r)/2} P_r(f_p \cdot g_q). \end{aligned}$$

It follows from the last relation that, at least in the case of polynomial random variables, the classic Wick product can be understood as the 0-Wick product in the sense of Da Pelo and Lanconelli. That means:

$$\varphi \diamond \psi = \varphi \diamond_0 \psi := \lim_{t\to 0^+} \varphi \diamond_t \psi. \tag{3.7}$$

For this reason, from now on, we will take $t \geq 0$, when speaking about the family of t-Wick products.

We can generalize this product in the following way. For every bounded self-adjoint operator T on E, that commutes with the operator A used to define the

Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$, such that $T > 0$, we define the T-Wick product as:

$$\varphi \diamond_T \psi := \Gamma\left(\frac{1}{\sqrt{T}}\right)[\Gamma(\sqrt{T})\varphi \cdot \Gamma(\sqrt{T})\psi], \tag{3.8}$$

for every (φ, ψ) in a dense subspace V_T of $(L^2) \times (L^2)$, for which $\Gamma((1/\sqrt{T}))[\Gamma(\sqrt{T})$ $\varphi \cdot \Gamma(\sqrt{T})\psi]$ belongs to (L^2). Such a space V_T can be taken as the vector space spanned by exponential functions.

Here when we speak of $\sqrt{T}$ and $1/\sqrt{T}$, we understand the self-adjoint operators whose eigenvectors are $\{e_n\}_{n\geq 0}$ (the same as the eigenvectors of A) and whose eigenvalues are $\{\sqrt{\mu_n}\}_{n\geq 0}$ and $\{1/\sqrt{\mu_n}\}_{n\geq 0}$, respectively, where $\{\mu_n\}_{n\geq 0}$ are the eigenvalues of the operator T. It is clear that the T-Wick product is commutative and associative.

As before, if $T \geq 0$, by gently passing to the limit as $\epsilon \to 0^+$, we can define the T-Wick product as:

$$\varphi \diamond_T \psi := \lim_{\epsilon \to 0^+} \varphi \diamond_{T_\epsilon} \psi, \tag{3.9}$$

where

$$T_\epsilon := T + \epsilon P_{\mathrm{Ker}(T)}, \tag{3.10}$$

where

$$\mathrm{Ker}(T) := \{x \in E \,|\, Tx = 0_E\} \tag{3.11}$$

and $P_{\mathrm{Ker}(T)}$ denotes the orthogonal projection of E onto $\mathrm{Ker}(T)$.

We recall now an important family of random variables called the (*renormalized*) *exponential functions*.

For every $\xi \in E_c$, we define the *exponential function* φ_ξ *generated by* ξ as:

$$\begin{aligned} \varphi_\xi &:= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \cdot, \xi \rangle^{\diamond n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, \xi^{\otimes n} \rangle, \end{aligned} \tag{3.12}$$

where $\langle \cdot\, , \xi \rangle^{\diamond n} := \langle \cdot\, , \xi \rangle \diamond \langle \cdot\, , \xi \rangle \diamond \cdots \diamond \langle \cdot\, , \xi \rangle$ (n times). The pointwise formula for φ_ξ is:

$$\varphi_\xi(x) = e^{\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle}, \tag{3.13}$$

for almost all $x \in \mathcal{E}'$.

It is easy to see that φ_ξ belongs to $L^p(\mathcal{E}', \mu)$, for all $1 \leq p < \infty$, and all ξ in E_c. The family of exponential functions is closed with respect to the Wick product and second quantization operators. Moreover, the vector space spanned by the exponential functions is closed with respect to each T-Wick product, for every bounded self-adjoint and nonnegative operator T, and dense in every space

$L^p(\mathcal{E}', \mu)$, for $1 \le p < \infty$. We have the lemma.

Lemma 3.1. *For all* ξ *and* η *in* E_c, *and* T *a bounded self-adjoint operator on* E_c, *such that* $T \ge 0$, *we have:*

- $$\Gamma(T)\varphi_\xi = \varphi_{T\xi}. \tag{3.14}$$

- $$\varphi_\xi \diamond_T \varphi_\eta = e^{\langle T\xi,\eta\rangle}\varphi_{\xi+\eta}. \tag{3.15}$$

- $$\varphi_\xi \diamond \varphi_\eta = \varphi_{\xi+\eta}. \tag{3.16}$$

Proof. Since for all $n \ge 0$, $\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle \in \mathcal{G}_n$, we have:

$$\begin{aligned}\Gamma(T)\varphi_\xi &= \sum_{n=0}^{\infty} \frac{1}{n!}\langle :\cdot^{\otimes n}:, T^{\otimes n}\xi^{\otimes n}\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\langle :\cdot^{\otimes n}:, (T\xi)^{\otimes n}\rangle \\ &= \varphi_{T\xi}.\end{aligned}$$

For all $x \in \mathcal{E}'$, we have:

$$\begin{aligned}(\varphi_\xi \diamond_T \varphi_\eta)(x) &= \Gamma\left(\frac{1}{\sqrt{T}}\right)[\Gamma(\sqrt{T})\varphi_\xi \cdot \Gamma(\sqrt{T})\varphi_\eta](x) \\ &= \Gamma\left(\frac{1}{\sqrt{T}}\right)[\varphi_{\sqrt{T}\xi} \cdot \varphi_{\sqrt{T}\eta}](x) \\ &= \Gamma\left(\frac{1}{\sqrt{T}}\right)[e^{\langle x,\sqrt{T}\xi\rangle-(1/2)\langle\sqrt{T}\xi,\sqrt{T}\xi\rangle} \cdot e^{\langle x,\sqrt{T}\eta\rangle-(1/2)\langle\sqrt{T}\eta,\sqrt{T}\eta\rangle}] \\ &= \Gamma\left(\frac{1}{\sqrt{T}}\right)[e^{\langle x,\sqrt{T}(\xi+\eta)\rangle-(1/2)(\langle T\xi,\xi\rangle+\langle T\eta,\eta\rangle)}].\end{aligned}$$

In the exponential we subtract and add $\langle\sqrt{T}\xi, \sqrt{T}\eta\rangle$ to complete the square, and obtain:

$$\begin{aligned}(\varphi_\xi \diamond_T \varphi_\eta)(x) &= e^{\langle T\xi,\eta\rangle}\Gamma\left(\frac{1}{\sqrt{T}}\right)[e^{\langle x,\sqrt{T}(\xi+\eta)\rangle-1/2\langle\sqrt{T}(\xi+\eta),\sqrt{T}(\xi+\eta)\rangle}] \\ &= e^{\langle T\xi,\eta\rangle}\left[\Gamma\left(\frac{1}{\sqrt{T}}\right)\varphi_{\sqrt{T}(\xi+\eta)}\right] \\ &= e^{\langle T\xi,\eta\rangle}\varphi_{(1/\sqrt{T})\sqrt{T}(\xi+\eta)}(x) \\ &= e^{\langle T\xi,\eta\rangle}\varphi_{\xi+\eta}(x).\end{aligned}$$

For $T := 0$, we obtain:

$$\varphi_\xi \diamond \varphi_\eta = \varphi_{\xi+\eta}. \qquad \square$$

We also have the following functorial property.

Lemma 3.2. *Let B and T be two commuting, bounded, nonnegative, and self-adjoint operators on E_c, such that B is invertible. Then there exists a vector space V, that is dense in all the spaces $L^p(\mathcal{E}', \mu)$, with $1 \leq p < \infty$, such that for any two random variables φ and ψ in V, we have*:

$$\Gamma(B)\,(\varphi \diamond_T \psi) = \Gamma(B)\varphi \diamond_{TB^{-2}} \Gamma(B)\psi. \tag{3.17}$$

Proof. We can take V to be the vector space spanned by the exponential functions. For any φ and ψ in V, we have:

$$\begin{aligned}
\Gamma(B)\,(\varphi \diamond_T \psi) &= \Gamma(B)\Gamma\left(\frac{1}{\sqrt{T}}\right)[\Gamma(\sqrt{T})\varphi \cdot \Gamma(\sqrt{T})\psi] \\
&= \Gamma\left(\frac{B}{\sqrt{T}}\right)\left[\Gamma\left(\frac{\sqrt{T}}{B}\right)\Gamma(B)\varphi \cdot \Gamma\left(\frac{\sqrt{T}}{B}\right)\Gamma(B)\psi\right] \\
&= \Gamma(B)\varphi \diamond_{TB^{-2}} \Gamma(B)\psi. \qquad \square
\end{aligned}$$

4. An Integral Representation of the Generalized Gaussian Wick Product

In this section we prove an integral representation of the Gaussian T-Wick product for every $0 \leq T \leq 2I$, where I denotes the identity operator.

Let us denote by Exp the complex vector space generated by the exponential functions with subscripts from $\mathcal{E}_c$, that means,

$$\mathrm{Exp} = \{c_1\varphi_{\xi_1} + \cdots + c_n\varphi_{\xi_n} \,|\, n \geq 1, c_i \in \mathbb{C}, \xi_i \in \mathcal{E}_c, 1 \leq i \leq n\}. \tag{4.1}$$

We have the following lemma.

Lemma 4.1. *Let T be a self-adjoint and diagonal operator on E_c, commuting with the operator A used in the construction of the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$, such that*:

$$0 \leq T \leq 2I, \tag{4.2}$$

where I denotes the identity operator of E_c.

Let C and D be two bounded, self-adjoint and diagonal operators on E_c, of operatorial norm at most 1, commuting with A, such that:

$$(I - C^2)(I - D^2) \geq (T - I)^2 C^2 D^2. \tag{4.3}$$

That means, if $\{e_n\}_{n\geq 0}$ is the orthonormal basis of E made up of eigenfunctions of the operator A, then there are three sequences of real numbers, $\{t_n\}_{n\geq 0}$, $\{\alpha_n\}_{n\geq 0}$, and $\{\beta_n\}_{n\geq 0}$, such that:

- *For all* $f \in E$, *we have*:

$$Tf = \sum_{n=0}^{\infty} t_n \langle f, e_n \rangle e_n, \tag{4.4}$$

$$Cf = \sum_{n=0}^{\infty} \alpha_n \langle f, e_n \rangle e_n \tag{4.5}$$

and

$$Df = \sum_{n=0}^{\infty} \beta_n \langle f, e_n \rangle e_n. \tag{4.6}$$

- *For all* $n \geq 0$, $0 \leq t_n \leq 2$, $-1 \leq \alpha_n \leq 1$, *and* $-1 \leq \beta_n \leq 1$.
- *For all* $n \geq 0$, *we have*:

$$(1 - \alpha_n^2)(1 - \beta_n^2) \geq (t_n - 1)^2 \alpha_n^2 \beta_n^2. \tag{4.7}$$

Then, for all φ *and* ψ *in* Exp, *we have*:

$$\begin{aligned}
&(\Gamma(C)\varphi \diamond_T \Gamma(D)\psi)(x) \\
&\quad = \int_{\mathcal{E}'} \int_{\mathcal{E}'} \varphi\left(C^* x + P^* y + Q^* z\right) \psi\left(D^* x + R^* y + S^* z\right) d\mu(z) d\mu(y) \\
&\quad = \lim_{n \to \infty} \int_{E_n} \int_{E_n} E\left[\varphi \,|\, \mathcal{F}_n\right] (Cx + Py + Qz) \\
&\qquad \cdot E[\psi \,|\, \mathcal{F}_n] \, (Dx + Ry + Sz) \, d\mu_n(z) d\mu_n(y),
\end{aligned} \tag{4.8}$$

where P, Q, R, *and* S *are any bounded, self-adjoint, and diagonal operators commuting with* A, *such that*:

$$P^2 + Q^2 = I - C^2, \tag{4.9}$$

$$R^2 + S^2 = I - D^2 \tag{4.10}$$

and

$$PR + QS = (T - I)CD, \tag{4.11}$$

and for all $n \in \mathbb{N}$, $\mathcal{F}_n$ *denotes the smallest sigma algebra with respect to which* $\langle \cdot, e_0 \rangle$, $\langle \cdot, e_1 \rangle, \ldots, \langle \cdot, e_{n-1} \rangle$ *are measurable*,

$$E_n = \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-1}, \tag{4.12}$$

and μ_n *is the standard Gaussian probability measure on* $E_n \equiv \mathbb{R}^n$. *The limit from formula* (4.8) *is in the* L^p*-sense, for all* $1 \leq p < \infty$.

The operators C^*, D^*, P^*, Q^*, R^*, *and* S^* *map* $\mathcal{E}'$ *into* $\mathcal{E}'$ *and represent the dual operators of* C, D, P, Q, R, *and* S *viewed as operators from* $\mathcal{E}$ *to* $\mathcal{E}$.

Proof. We prove first the existence of such operators P, Q, R and S.

For every $n \geq 0$, since $1 - \alpha_n^2 \geq 0$, $1 - \beta_n^2 \geq 0$, and $(1 - \alpha_n^2)(1 - \beta_n^2) \geq (t_n - 1)^2 \alpha_n^2 \beta_n^2$, there exist two vectors (p_n, q_n) and (r_n, s_n) in $\mathbb{R}^2$, such that:

$$\|(p_n, q_n)\|_2 = \sqrt{1 - \alpha_n^2}, \tag{4.13}$$

$$\|(r_n, s_n)\|_2 = \sqrt{1 - \beta_n^2} \tag{4.14}$$

and

$$(p_n, q_n) \cdot (r_n, s_n) = (t_n - 1)\alpha_n \beta_n. \tag{4.15}$$

Geometrically, it means that the vectors (p_n, q_n) and (r_n, s_n) have length $\sqrt{1 - \alpha_n^2}$ and $\sqrt{1 - \beta_n^2}$, respectively, and the angle between them has a radian measure of $\arccos((t_n - 1)\alpha_n \beta_n / \sqrt{(1 - \alpha_n^2)(1 - \beta_n^2)})$, if $|\alpha_n| < 1$ and $|\beta_n| < 1$. If $\alpha_n = \pm 1$ or $\beta_n = \pm 1$, then (4.7) implies $(t_n - 1)\alpha_n \beta_n = 0$ and so we can take $(p_n, q_n) = (0, 0)$ or $(r_n, s_n) = (0, 0)$. Once a pair of vectors (p_n, q_n) and (r_n, s_n) is found, any rotation by an angle θ_n of these vectors will produce another pair with the same properties.

We choose for each $n \geq 0$, a pair of vectors (p_n, q_n) and (r_n, s_n) with the above properties, and define the operators P, Q, R and S in the following way. For every f in E, let:

$$Pf := \sum_{n=0}^{\infty} p_n \langle f, e_n \rangle e_n, \tag{4.16}$$

$$Qf := \sum_{n=0}^{\infty} q_n \langle f, e_n \rangle e_n, \tag{4.17}$$

$$Rf := \sum_{n=0}^{\infty} r_n \langle f, e_n \rangle e_n \tag{4.18}$$

and

$$Sf := \sum_{n=0}^{\infty} s_n \langle f, e_n \rangle e_n. \tag{4.19}$$

Of course, P, Q, R and S can be extended as linear operators from E_c to E_c. Since for all $n \geq 0$, we have:

$$p_n^2 + q_n^2 = 1 - \alpha_n^2 \leq 1 \tag{4.20}$$

and

$$r_n^2 + s_n^2 = 1 - \beta_n^2 \leq 1, \tag{4.21}$$

we conclude that P, Q, R and S are self-adjoint, diagonal, and bounded operators, of operatorial norm less than or equal to 1, on E. Moreover, being diagonalized in the same basis as A, they commute with A.

Let us observe that C, D, P, Q, R and S map $\mathcal{E}$ into $\mathcal{E}$, and are continuous from $\mathcal{E}$ to $\mathcal{E}$. Indeed, for all $\xi \in \mathcal{E}$ and $k \geq 0$, we have, for example:

$$\begin{aligned} |C\xi|_k &= |A^k C\xi|_0 \\ &= |CA^k\xi|_0 \\ &\leq |A^k\xi|_0 \\ &= |\xi|_k. \end{aligned}$$

It is clear that relations (4.9), (4.10), and (4.11) are satisfied.

Since both sides of (4.8) are bilinear in φ and ψ, to prove it for every linear combination of exponential functions, it is enough to check it for φ and ψ exponential functions.

So let $\varphi = \varphi_\xi$ and $\psi = \varphi_\eta$, for some ξ and η in $\mathcal{E}_c$. We have:

$$\begin{aligned} &\int_{\mathcal{E}'}\int_{\mathcal{E}'} \varphi_\xi\left(C^*x + P^*y + Q^*z\right)\varphi_\eta\left(D^*x + R^*y + S^*z\right) d\mu(z)d\mu(y) \\ &= \int_{\mathcal{E}'}\int_{\mathcal{E}'} \exp\left(\langle C^*x + P^*y + Q^*z, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right) \\ &\quad \cdot \exp\left(\langle D^*x + R^*y + S^*z, \eta\rangle - \frac{1}{2}\langle \eta, \eta\rangle\right) d\mu(z)d\mu(y) \\ &= \exp\left(\langle x, C\xi + D\eta\rangle - \frac{1}{2}\langle \xi, \xi\rangle - \frac{1}{2}\langle \eta, \eta\rangle\right) \\ &\quad \cdot \int_{\mathcal{E}'} \exp\left(\langle y, P\xi + R\eta\rangle\right) d\mu(y) \int_{\mathcal{E}'} \exp\left(\langle z, Q\xi + S\eta\rangle\right) d\mu(z) \\ &= \exp\left(\langle x, C\xi + D\eta\rangle - \frac{1}{2}\langle \xi, \xi\rangle - \frac{1}{2}\langle \eta, \eta\rangle\right) \\ &\quad \cdot \int_{\mathcal{E}'} \varphi_{P\xi+R\eta}(y) \exp\left(\frac{1}{2}\langle P\xi + R\eta, P\xi + R\eta\rangle\right) d\mu(y) \\ &\quad \cdot \int_{\mathcal{E}'} \varphi_{Q\xi+S\eta}(z) \exp\left(\frac{1}{2}\langle Q\xi + S\eta, Q\xi + S\eta\rangle\right) d\mu(z). \end{aligned}$$

Taking now the constant factors $\exp((1/2)\langle P\xi + R\eta, P\xi + R\eta\rangle)$ and $\exp((1/2)\langle Q\xi + S\eta, Q\xi + D\eta\rangle)$ outside from their integrals, we obtain:

$$\begin{aligned} &\int_{\mathcal{E}'}\int_{\mathcal{E}'} \varphi_\xi\left(C^*x + P^*y + Q^*z\right)\varphi_\eta\left(D^*x + R^*y + S^*z\right) d\mu(z)d\mu(y) \\ &= \exp\left(\langle x, C\xi + D\eta\rangle - \frac{1}{2}\langle (I - P^2 - Q^2)\xi, \xi\rangle\right) \\ &\quad \cdot \exp\left(\langle (PR + QS)\xi, \eta\rangle - \frac{1}{2}\langle (I - R^2 - S^2)\eta, \eta\rangle\right) \\ &\quad \cdot E\left[\varphi_{P\xi+R\eta}\right] E\left[\varphi_{Q\xi+S\eta}\right]. \end{aligned}$$

Since the expectation of every exponential function is 1, we have:

$$\begin{aligned}
&\int_{\mathcal{E}'}\int_{\mathcal{E}'} \varphi_\xi\left(C^*x + P^*y + Q^*z\right)\varphi_\eta\left(D^*x + R^*y + S^*z\right)d\mu(z)d\mu(y)\\
&\quad = \exp\left(\langle x, C\xi + D\eta\rangle - \frac{1}{2}\langle C^2\xi,\xi\rangle + \langle (T-I)CD\xi,\eta\rangle - \frac{1}{2}\langle D^2\eta,\eta\rangle\right)\\
&\quad = \exp(\langle TC\xi, D\eta\rangle)\\
&\qquad \cdot \exp\left(\langle x, C\xi + D\eta\rangle - \frac{1}{2}\langle C\xi, C\xi\rangle - \langle C\xi, D\eta\rangle - \frac{1}{2}\langle D\eta, D\eta\rangle\right)\\
&\quad = \exp(\langle TC\xi, D\eta\rangle)\varphi_{C\xi+D\eta}\\
&\quad = \varphi_{C\xi} \diamond_T \varphi_{D\eta}\\
&\quad = \Gamma(C)\varphi_\xi \diamond_T \Gamma(D)\varphi_\eta.
\end{aligned}$$

It is not hard to see that, for all $n \geq 1$ and all $u \in E$, we have:

$$E[\varphi_u \,|\, \mathcal{F}_n] = \varphi_{u_n}, \tag{4.22}$$

where

$$u_n := \sum_{k=0}^{n-1} \langle u, e_k\rangle e_k. \tag{4.23}$$

Since

$$\lim_{n\to\infty} \varphi_{u_n} = \varphi_u, \tag{4.24}$$

where this limit is computed in the L^p–norm, for all $1 \leq p < \infty$, formula (4.8) follows easily. □

Corollary 4.2. *Let T be a self-adjoint and diagonal operator on E_c commuting with the operator A, such that $0 \leq T \leq 2I$.*

Let C and D be two bounded, self-adjoint and diagonal operators on E_c, of operatorial norm at most 1, commuting with A, such that:

$$(I - C^2)(I - D^2) = (T - I)^2 C^2 D^2. \tag{4.25}$$

Then, for all φ and ψ in Exp, *we have:*

$$\begin{aligned}
&(\Gamma(C)\varphi \diamond_T \Gamma(D)\psi)(x)\\
&\qquad = \int_{\mathcal{E}'} \varphi\left(C^*x + P^*y\right)\psi\left(D^*x + R^*y\right)d\mu(y),
\end{aligned} \tag{4.26}$$

where

$$P = \operatorname{sgn}[(T - I)CD]\sqrt{I - C^2} \tag{4.27}$$

and

$$R = \sqrt{I - D^2}, \tag{4.28}$$

where sgn *is the right-continuous extension, to* $\mathbb{R}$, *of the function* $x \mapsto x/|x|$, *defined on* $\mathbb{R}\backslash\{0\}$. *Here* $\mathrm{sgn}[(T-I)CD]$ *is computed according to the Dunford functional calculus, which in this case, since* $T-I, C$ *and* D *are self-adjoint and diagonal operators, that are diagonalized in the same basis, means to simply apply the measurable function* sgn *to each eigenvalue of the operator* $(T-I)CD$.

Proof. In Lemma 4.1 take $Q = S := 0$, $P = \mathrm{sgn}[(T-I)CD]\sqrt{I-C^2}$ and $R = \sqrt{I-D^2}$. □

Observation 4.3. Lemma 4.1 remains true if the condition that each of the operators C, D and T commutes with A, is replaced by the fact that C, D, and T commute among themselves and are continuous (bounded) linear operators from $\mathcal{E}$ to $\mathcal{E}$. In particular, in the finite dimensional case (that means, if the dimension of E is a finite number d, in which case $\mathcal{E} = E = \mathcal{E}' \equiv \mathbb{R}^d$), since the condition of being bounded is automatically satisfied, the only condition required is that C, D, and T commute among themselves.

The above observation is important for people who are not interested in the infinite dimensional case, and are content with the finite dimensional one. In that case, the technicality of commuting with the operator A is removed. In fact, in that case there is no need to speak of such an operator A, since a Gaussian probability measure on a finite dimensional space can be defined without it.

5. Hölder Inequalities for Norms of Generalized Wick Products

We present now the main result of this paper. To prove our result we need the following theorem of Lieb (see Refs. 23 or 24 (p. 100)).

Theorem 5.1. *Fix* $k > 1$, *integers* $n_1, \ldots, n_k$ *and numbers* $p_1, \ldots, p_k \geq 1$. *Let* $M \geq 1$ *and let* B_i *(for* $i = 1, \ldots, k$*) be a linear mapping from* $\mathbb{R}^M$ *to* $\mathbb{R}^{n_i}$. *Let* $Z : \mathbb{R}^M \to \mathbb{R}^+$ *be some fixed Gaussian function,*

$$Z(x) = \exp[-\langle x, Jx\rangle]$$

with J *a real, positive semi-definite* $M \times M$ *matrix (possible zero).*

For functions f_i *in* $L^{p_i}(\mathbb{R}^{n_i})$ *consider the integral* I_Z *and the number* C_Z:

$$I_Z(f_1, \ldots, f_k) = \int_{\mathbb{R}^M} Z(x) \prod_{i=1}^{k} f_i(B_i x)dx, \tag{5.1}$$

$$C_Z := \sup\{I_Z(f_1, \ldots, f_k) \mid \||f_i\||_{p_i} = 1 \text{ for } i = 1, \ldots, k\}, \tag{5.2}$$

where $\||\cdot\||_{p_i}$ *denotes* $L^{p_i}(\mathbb{R}^{n_i})$ *norm with respect to the Lebesgue measure. Then* C_Z *is determined by restricting the* f's *to be Gaussian functions, i.e.*

$$\begin{aligned} C_Z = \sup\{&I_Z(f_1, \ldots, f_k) \mid \||f_i\||_{p_i} = 1 \text{ and } f_i(x) = c_i \exp[-\langle x, J_i x\rangle] \\ &\text{with } c_i > 0, \text{ and } J_i \text{ a real, symmetric, positive-definite} \\ &n_i \times n_i \text{ matrix}\}. \end{aligned}$$

The proof of the next corollary, of the above theorem, can be found in Ref. 11.

Corollary 5.2. *Let $p, q, r \geq 1$. Let B_1 and B_2 be linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, and J a real, positive-semidefinite 2×2 matrix (possible zero). For f in $L^p(\mathbb{R}^2)$ and g in $L^q(\mathbb{R}^2)$, we consider the product*:

$$(f \star_{B_1,B_2,J} g)(x) = \int_{\mathbb{R}} f(B_1(x,y))g(B_2(x,y))e^{-\langle (x,y), J(x,y)\rangle} dy. \tag{5.3}$$

We define:

$$C := \sup\{|||f \star_{B_1,B_2,J} g\,|||_r \mid |||f|||_p = |||g|||_q = 1\}. \tag{5.4}$$

Then C is determined by restricting f and g to be Gaussian functions.

We will also need the following computational lemma.

Lemma 5.3. *Let $\{a_i\}_{1\leq i\leq n}$ and $\{b_i\}_{1\leq i\leq n}$ be two finite sequences of real numbers. Then, we have*:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(a_i x + b_i)^2\right] dx \\
&\quad = \frac{1}{\sqrt{\sum_{i=1}^n a_i^2}} \exp\left[-\frac{1}{2}\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2}{\sum_{i=1}^n a_i^2}\right] \\
&\quad = \frac{1}{\sqrt{\sum_{i=1}^n a_i^2}} \exp\left[-\frac{1}{2}\frac{\sum_{1\leq i<j\leq n}(a_i b_j - a_j b_i)^2}{\sum_{i=1}^n a_i^2}\right].
\end{aligned} \tag{5.5}$$

Proof. We have:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(a_i x + b_i)^2\right] dx \\
&\quad = \exp\left[-\frac{1}{2}\sum_{i=1}^{n} b_i^2\right] \\
&\qquad \cdot \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n} a_i^2\right)x^2 + \left(\sum_{i=1}^{n} a_i b_i\right)x\right] dx.
\end{aligned}$$

Making first the substitution $x' = x\sqrt{\sum_{i=1}^n a_i^2}$ and then completing the square in the exponential, we obtain:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(a_i x + b_i)^2\right] dx \\
&\quad = \frac{\exp[-(1/2)\sum_{i=1}^n b_i^2]}{\sqrt{\sum_{i=1}^n a_i^2}} \cdot \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\frac{1}{2}x'^2 + \frac{\sum_{i=1}^n a_i b_i}{\sqrt{\sum_{i=1}^n a_i^2}}x'\right] dx'
\end{aligned}$$

$$= \frac{\exp[-(1/2)\sum_{i=1}^n b_i^2]}{\sqrt{\sum_{i=1}^n a_i^2}} \exp\left[+\frac{1}{2}\frac{(\sum_{i=1}^n a_i b_i)^2}{\sum_{i=1}^n a_i^2}\right]$$

$$\cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left[-\frac{1}{2}\left(x' - \frac{\sum_{i=1}^n a_i b_i}{\sqrt{\sum_{i=1}^n a_i^2}}\right)^2\right] dx'$$

$$= \frac{1}{\sqrt{\sum_{i=1}^n a_i^2}} \exp\left[-\frac{1}{2}\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2}{\sum_{i=1}^n a_i^2}\right].$$

Formula (5.5) follows now from Lagrange identity. □

Theorem 5.4. (Hölder–Young inequality for generalized Gaussian Wick products) *Let $\mathcal{E} \subset E \subset \mathcal{E}'$ be a Gel'fand triple given by a self-adjoint diagonal operator A on E, with increasing, greater than 1 eigenvalues, whose inverse is a Hilbert–Schmidt operator. Let μ be the Gaussian probability measure on $\mathcal{E}'$ whose existence is guaranteed by Minlos Theorem. Let T be a self-adjoint, diagonal operator on E, commuting with A, such that*:

$$0 \leq T \leq 2I, \tag{5.6}$$

where I denotes the identity operator of E. Let C and D be two self-adjoint, diagonal, and bounded operators on E, of operatorial norm less than or equal to 1, *commuting with the operator A, such that*:

$$(I - C^2)(I - D^2) \geq (T - I)^2 C^2 D^2. \tag{5.7}$$

Let $p, q, r > 1$ such that:

$$(r - 1)I \leq \frac{(p-1)(q-1)I - C^2 D^2 T^2}{(q-1)C^2 + (p-1)D^2 + 2C^2 D^2 T} \tag{5.8}$$

or equivalently:

$$\frac{1}{(r-1)I + T} \geq \frac{1}{\frac{p-1}{C^2} + T} + \frac{1}{\frac{q-1}{D^2} + T}, \tag{5.9}$$

with the convention that if $x \in Ker(C) := C^{-1}(0)$, then the first fraction operator from the right-hand side of (5.9) *evaluated at x is zero, and a similar convention for $x \in \mathrm{Ker}(D)$. Then for all φ in $L^p(\mathcal{E}', \mu)$ and ψ in $L^q(\mathcal{E}', \mu)$, $\Gamma(C)\varphi \diamond_T \Gamma(D)\psi$ belongs to $L^r(\mathcal{E}', \mu)$, and the following inequality holds*:

$$\|\Gamma(C)\varphi \diamond_T \Gamma(D)\psi\|_r \leq \|\varphi\|_p \cdot \|\psi\|_q. \tag{5.10}$$

On the other hand, if:

$$(r - 1)I \not\leq \frac{(p-1)(q-1)I - C^2 D^2 T^2}{(q-1)C^2 + (p-1)D^2 + 2C^2 D^2 T},$$

then the bilinear operator $(\varphi, \psi) \mapsto \Gamma(C)\varphi \diamond_T \Gamma(D)\psi$ is not bounded from $L^p(\mathcal{E}', \mu) \times L^q(\mathcal{E}', \mu)$ to $L^r(\mathcal{E}', \mu)$.

We present now the proof of Theorem 5.4. The proof will be subdivided into many steps. The main idea of each step will be written in Italic.

Proof. ($\Rightarrow$) Let us assume first that:

$$(r-1)I \leq \frac{(p-1)(q-1)I - C^2D^2T^2}{(q-1)C^2 + (p-1)D^2 + 2C^2D^2T}.$$

Step 0. *We may assume that* $T > 0, C^2 > 0$, *and* $D^2 > 0$.

Indeed, since $T \geq 0$ can be obtained from the case $T' > 0$, by gently passing to the limit on $\mathrm{Ker}(T) = T^{-1}(0)$ as $P_{\mathrm{Ker}(T)}T' \to 0^+$, where $P_{\mathrm{Ker}(T)}$ denotes the orthogonal projection of E on $\mathrm{Ker}(T)$, we may assume that $T > 0$. Similarly, we may assume that $C^2 > 0$ and $D^2 > 0$.

Step 1. *In the finite dimensional case, we can reduce the problem to the one-dimensional case via Minkowski integral inequality.*

Let d be a fixed finite dimension and let μ_d be the standard Gaussian probability measure on $\mathbb{R}^d$. Let T be in $(0, 2I]$, and C and D be two nonzero commuting linear self-adjoint operators from $\mathbb{R}^d$ to $\mathbb{R}^d$, and p, q, $r \geq 1$, such that relations (5.7) and (5.8) hold. Let $\{e_i\}_{1\leq i\leq d}$ be an orthogonal basis of $\mathbb{R}^d$ that diagonalizes all three operators T, C and D. If $x = x_1e_1 + x_2e_2 + \cdots + x_de_d \in \mathbb{R}^d$, then we write $x = (x_1, x_2, \ldots, x_d)$.

Let φ and ψ be two random variables in Exp (that means, φ and ψ are linear combinations of exponential functions). From Lemma 4.1, we know that:

$$\begin{aligned}&(\Gamma(C)\varphi \diamond_T \Gamma(D)\psi)(x)\\&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi\left(C^*x + P^*y + Q^*z\right)\psi(D^*x + R^*y + S^*z)d\mu_d(z)d\mu_d(y),\end{aligned}$$

where P, Q, R, and S are linear self-adjoint operators on $\mathbb{R}^d$ commuting with C and D, such that:

$$P^2 + Q^2 = I - C^2,$$

$$R^2 + S^2 = I - D^2,$$

and

$$PR + QS = (T - I)CD.$$

In fact, in this finite-dimensional case the use of the adjoint "$*$" is not necessary, because $C = C^*$, $D = D^*, \ldots, S = S^*$.

Let us assume that for a fixed triplet, $(p,\ q,\ r)$, inequality (5.10) holds for two finite dimensions $d = m$ and $d = n$, and prove that it continues to hold for $d = m + n$. Expanding the vectors x, y, and z along the orthonormal basis $\{e_i\}_{1\leq i\leq m+n}$ we can write $x = (x_m,\ x_n)$, $y = (y_m,\ y_n)$, and $z = (z_m,\ z_n)$, where $(x_m,\ 0)$ denotes the projection of x onto the vector space spanned by $\{e_i\}_{1\leq i\leq m}$, and $(0, x_n) = x-(x_m, 0)$. We also denote by C_m and C_n, the restriction of C to the spaces spanned by $\{e_i\}_{1\le i\le m}$ and $\{e_i\}_{m+1\le i\le m+n}$, respectively. Using Minkowski integral inequality we have:

$$
\begin{aligned}
&\|\Gamma(C)\varphi \diamond_T \Gamma(D)\psi\|_r \\
&\quad = \left\{\int_{\mathbb{R}^{m+n}} \left|\int_{\mathbb{R}^{m+n}} \int_{\mathbb{R}^{m+n}} \varphi(Cx+Py+Qz)\psi(Dx+Ry+Sz)\right.\right. \\
&\qquad \left.\left. d\mu_{m+n}(z)d\mu_{m+n}(y)\right|^r d\mu_{m+n}(x)\right\}^{1/r} \\
&\quad \le \left\{\int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\right.\right.\right. \\
&\qquad |\varphi\left((C_m x_m, C_n x_n) + (P_m y_m, P_n y_n) + (Q_m z_m, Q_n z_n)\right)| \\
&\qquad |\psi\left((D_m x_m, D_n x_n) + (R_m y_m, R_n y_n) + (S_m z_m, S_n z_n)\right)| \\
&\qquad \left.\left.\left. d\mu_n(z_n)d\mu_n(y_n)d\mu_m(z_m)d\mu_m(y_m)\right|^r d\mu_n(x_n)\right\} d\mu_m(x_m)\right\}^{1/r} \\
&\quad \le \left\{\int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\right.\right.\right.\right. \\
&\qquad |\varphi\left((C_m x_m, C_n x_n) + (P_m y_m, P_n y_n) + (Q_m z_m, Q_n z_n)\right)| \\
&\qquad |\psi\left((D_m x_m, D_n x_n) + (R_m y_m, R_n y_n) + (S_m z_m, S_n z_n)\right)| \\
&\qquad \left.\left.\left.\left. d\mu_n(z_n)d\mu_n(y_n)\right|^r d\mu_n(x_n)\right\}^{1/r} d\mu_m(z_m)d\mu_m(y_m)\right\}^r d\mu_m(x_m)\right\}^{1/r} \\
&\quad \le \left\{\int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \|\varphi\left((C_m x_m,\cdot) + (P_m y_m,\cdot) + (Q_m z_m,\cdot)\right)\|_p\right.\right. \\
&\qquad \left.\left. \|\psi\left((D_m x_m,\cdot) + (R_m y_m,\cdot) + (S_m z_m,\cdot)\right)\|_q\, d\mu_m(z_m)d\mu_m(y_m)\right\}^r d\mu_m(x_m)\right\}^{1/r} \\
&\quad \le \|\varphi\|_p\|\psi\|_q.
\end{aligned}
$$

Step 2. *In dimension $d = 1$, we may assume that both inequalities* (5.7) *and* (5.8) *are equalities.*

Indeed, if $d = 1$, then $T = tI$, for some $t \in (0, 2]$, $C = \alpha I$ and $D = \beta I$, for some α and β in $[-1, 1]\backslash\{0\}$. Inequality (5.7) becomes:

$$(1-\alpha^2)(1-\beta^2) \ge \alpha^2\beta^2(t-1)^2. \tag{5.11}$$

If inequality (5.11) is strict, then since

$$[1-(\pm 1)^2](1-\beta^2) \le (\pm 1)^2\beta^2(t-1)^2,$$

there exists α_0 having the same sign as α, with $|\alpha_0| \in (|\alpha|, 1]$, such that:

$$(1-\alpha_0^2)(1-\beta^2) = \alpha_0^2\beta^2(t-1)^2. \tag{5.12}$$

Let us suppose that inequality (5.10) holds for the triplet (α_0, β, t) and all functions $\varphi \in L^p(\mathcal{E}', \mu)$ and $\psi \in L^q(\mathcal{E}', \mu)$. Then since $\Gamma((\alpha/\alpha_0)I)$ is a bounded operator from $L^p(\mathcal{E}', \mu)$ to $L^p(\mathcal{E}', \mu)$ of operatorial norm equal to 1, we have:

$$\begin{aligned}\|\Gamma(\alpha I)\varphi \diamond_T \Gamma(\beta I)\psi\|_r &= \left\|\Gamma(\alpha_0 I)\Gamma\left(\frac{\alpha}{\alpha_0}I\right)\varphi \diamond_T \Gamma(\beta I)\psi\right\|_r \\ &\leq \left\|\Gamma\left(\frac{\alpha}{\alpha_0}I\right)\varphi\right\|_p \cdot \|\psi\|_q \\ &\leq \|\varphi\|_p \cdot \|\psi\|_q.\end{aligned}$$

Also inequality (5.8), in dimension $d = 1$, becomes:

$$r - 1 \leq \frac{(p-1)(q-1) - \alpha^2\beta^2t^2}{(q-1)\alpha^2 + (p-1)\beta^2 + 2\alpha^2\beta^2 t}. \tag{5.13}$$

If inequality (5.13) is strict, then since for p and q fixed, $\lim_{r\to\infty}(r-1) = \infty$, there exists $r_0 \in (r, \infty)$, such that:

$$r_0 - 1 = \frac{(p-1)(q-1) - \alpha^2\beta^2t^2}{(q-1)\alpha^2 + (p-1)\beta^2 + 2\alpha^2\beta^2 t}.$$

Suppose inequality (5.10) holds for the triplet (p, q, r_0) and all $\varphi \in L^p(\mathcal{E}', \mu)$ and $\psi \in L^q(\mathcal{E}', \mu)$. Then using Lyapunov inequality, we have:

$$\begin{aligned}\|\Gamma(\alpha I)\varphi \diamond_T \Gamma(\beta I)\psi\|_r &\leq \|\Gamma(\alpha I)\varphi \diamond_T \Gamma(\beta I)\psi\|_{r_0} \\ &\leq \|\varphi\|_p \cdot \|\psi\|_q.\end{aligned}$$

Step 3. *Change the problem of proving an inequality about the p, q, and r norms with respect to the standard Gaussian probability measure into a problem of proving an inequality about the p, q, and r norms with respect to the Lebesgue measure.*

We assume now that we are in dimension $d = 1$, $C = \alpha I$, $D = \beta I$, $T = tI$, $-1 \leq \alpha \leq 1$, $\alpha \neq 0$, $-1 \leq \beta \leq 1$, $\beta \neq 0$, $t \in (0, 2]$, and we have the following equalities:

$$(1-\alpha^2)(1-\beta^2) = \alpha^2\beta^2(t-1)^2 \tag{5.14}$$

and

$$r - 1 = \frac{(p-1)(q-1) - \alpha^2\beta^2t^2}{(q-1)\alpha^2 + (p-1)\beta^2 + 2\alpha^2\beta^2 t}. \tag{5.15}$$

Formula (5.15) is equivalent, via formula (5.14), to:

$$r = pq\frac{1 - \frac{1-\alpha^2}{p} - \frac{1-\beta^2}{q}}{(q-1)\alpha^2 + (p-1)\beta^2 + 2\alpha^2\beta^2 t}. \tag{5.16}$$

According to Corollary 4.2, for every φ and ψ that are linear combinations of exponential functions, we have:

$$[\Gamma(\alpha I)\varphi \diamond_t \Gamma(\beta I)\psi](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi\left(\alpha x + \operatorname{sgn}[(t-1)\alpha\beta]\sqrt{1-\alpha^2}y\right) \psi\left(\beta x + \sqrt{1-\beta^2}y\right) e^{-y^2/2} dy. \tag{5.17}$$

Let us define the numbers:

$$\gamma := \operatorname{sgn}[\alpha\beta(t-1)]\sqrt{1-\alpha^2} \tag{5.18}$$

and

$$\delta := \sqrt{1-\beta^2}. \tag{5.19}$$

Then we have:

$$\alpha^2 + \gamma^2 = 1, \tag{5.20}$$

$$\beta^2 + \delta^2 = 1, \tag{5.21}$$

and

$$\gamma\delta = \alpha\beta(t-1). \tag{5.22}$$

We observe that, for any $0 < u < \infty$, a measurable function $f(x)$ belongs to $L^u(\mathbb{R},\, \mu)$ if and only if $f(x)e^{-x^2/(2u)}$ belongs to $L^u(\mathbb{R},\, d_N x)$, where $d_N x$ denotes the normalized Lebesgue measure $(1/\sqrt{2\pi})dx$ on $\mathbb{R}$. With this in mind, we are preparing now to cross the bridge from $L^p(\mathbb{R},\, \mu)$ to $L^p(\mathbb{R},\, d_N x)$, from $L^q(\mathbb{R},\, \mu)$ to $L^q(\mathbb{R},\, d_N x)$, and from $L^r(\mathbb{R},\, \mu)$ to $L^r(\mathbb{R},\, d_N x)$. To do this, we multiply both sides of formula (5.17) by $e^{-x^2/(2r)}$ and rewrite the expression inside the integral in the following way:

$$\begin{aligned}(\Gamma(\alpha I)\varphi \diamond_t \Gamma(\beta I)\psi)(x)e^{-x^2/(2r)} &= \int_{\mathbb{R}} \varphi(\alpha x + \gamma y)e^{-(\alpha x+\gamma y)^2/(2p)}\psi(\beta x + \delta y)e^{-(\beta x+\delta y)^2/(2q)} \\ &\quad \cdot e^{(\alpha x+\gamma y)^2/(2p)}e^{(\beta x+\delta y)^2/(2q)}e^{-y^2/2}e^{-x^2/(2r)}d_N y.\end{aligned}$$

Let us define now the following functions:

$$f(x) := \varphi(x) \cdot e^{-x^2/(2p)} \tag{5.23}$$

and

$$g(x) := \psi(x) \cdot e^{-x^2/(2q)}. \tag{5.24}$$

In this way, our problem of proving that the bilinear operator, which is "hopefully" defined as:

$$B : L^p(\mathbb{R}, \mu) \times L^q(\mathbb{R}, \mu) \to L^r(\mathbb{R}, \mu), \tag{5.25}$$

(we said "hopefully" since we do not know whether it maps $L^p \times L^q$ into L^r),

$$B(\varphi, \psi)(x) := \int_{\mathbb{R}} \varphi(\alpha x + \gamma y)\psi(\beta x + \delta y)e^{-y^2/2} d_N y \tag{5.26}$$

is a bounded operator of operatorial norm equal to 1, becomes the equivalent problem of showing that the bilinear operator, which is "hopefully" defined as:

$$\tilde{B} \ : \ L^p(\mathbb{R}, d_N x) \times L^q(\mathbb{R}, d_N x) \to L^r(\mathbb{R}, d_N x), \tag{5.27}$$

$$\begin{aligned} \tilde{B}(f, g)(x) := \int_{\mathbb{R}} & f(\alpha x + \gamma y) g(\beta x + \delta y) \\ & \cdot e^{(\alpha x + \gamma y)^2/(2p)} e^{(\beta x + \delta y)^2/(2q)} e^{-y^2/2} e^{-x^2/(2r)} d_N y, \end{aligned} \tag{5.28}$$

is a bounded operator of operatorial norm equal to 1.

Let us define the kernel:

$$J_d(x, y) := e^{(\alpha x + \gamma y)^2/(2p)} e^{(\beta x + \delta y)^2/(2q)} e^{-y^2/2} e^{-x^2/(2r)}. \tag{5.29}$$

We can write $J_d(x, y)$ as an exponential of a quadratic form of (x, y), in the following way:

$$J_d(x, y) = e^{-(1/2)ax^2 + bxy - (1/2)cy^2}, \tag{5.30}$$

where

$$a := \frac{1}{r} - \frac{\alpha^2}{p} - \frac{\beta^2}{q}, \tag{5.31}$$

$$b := \frac{\alpha\gamma}{p} + \frac{\beta\delta}{q} \tag{5.32}$$

and

$$c := 1 - \frac{\gamma^2}{p} - \frac{\delta^2}{q}. \tag{5.33}$$

We make the following observations:

(1) $c > 0$.
Indeed, from formula (5.16), we can see that:

$$c = \frac{r}{pq}[(q - 1)\alpha^2 + (p - 1)\beta^2 + 2\alpha^2\beta^2 t] > 0.$$

(2) $b^2 = ac$.
Indeed, we have:

$$\begin{aligned} ac &= \left(\frac{1}{r} - \frac{\alpha^2}{p} - \frac{\beta^2}{q}\right)\left(1 - \frac{1 - \alpha^2}{p} - \frac{1 - \beta^2}{q}\right) \\ &= \frac{1}{r}\left(1 - \frac{1 - \alpha^2}{p} - \frac{1 - \beta^2}{q}\right) - \frac{\alpha^2}{p} - \frac{\beta^2}{q} \\ &\quad + \frac{\alpha^2(1 - \beta^2) + \beta^2(1 - \alpha^2)}{pq} + \frac{\alpha^2(1 - \alpha^2)}{p^2} + \frac{\beta^2(1 - \beta^2)}{q^2}. \end{aligned}$$

Substituting now r by the right-hand side of formula (5.16), we obtain:

$$\begin{aligned} ac = {} & \frac{\alpha^2(q-1)+\beta^2(p-1)+2\alpha^2\beta^2 t}{pq-(1-\alpha^2)q-(1-\beta^2)p}\left(1-\frac{1-\alpha^2}{p}-\frac{1-\beta^2}{q}\right) \\ & -\frac{\alpha^2}{p}-\frac{\beta^2}{q}+\frac{\alpha^2(1-\beta^2)+\beta^2(1-\alpha^2)}{pq}+\frac{\alpha^2(1-\alpha^2)}{p^2}+\frac{\beta^2(1-\beta^2)}{q^2} \\ = {} & \frac{\alpha^2(q-1)+\beta^2(p-1)+2\alpha^2\beta^2 t}{pq}+\frac{-\alpha^2 q-\beta^2 p+\alpha^2(1-\beta^2)+\beta^2(1-\alpha^2)}{pq} \\ & +\frac{\alpha^2(1-\alpha^2)}{p^2}+\frac{\beta^2(1-\beta^2)}{q^2} \\ = {} & \frac{2\alpha\beta[\alpha\beta(t-1)]}{pq}+\frac{\alpha^2(1-\alpha^2)}{p^2}+\frac{\beta^2(1-\beta^2)}{q^2}. \end{aligned}$$

Since

$$(1-\alpha^2)(1-\beta^2)=\alpha^2\beta^2(t-1)^2,$$

we have:

$$\alpha\beta(t-1)=\operatorname{sgn}(\alpha\beta(t-1))\sqrt{(1-\alpha^2)(1-\beta^2)}=\gamma\delta.$$

Thus we obtain:

$$\begin{aligned} ac &= \frac{2\alpha\beta\gamma\delta}{pq}+\frac{\alpha^2(1-\alpha^2)}{p^2}+\frac{\beta^2(1-\beta^2)}{q^2} \\ &= \frac{2\alpha\beta\gamma\delta}{pq}+\frac{\alpha^2\gamma^2}{p^2}+\frac{\beta^2\delta^2}{q^2} \\ &= \left(\frac{\alpha\gamma}{p}+\frac{\beta\delta}{q}\right)^2 \\ &= b^2. \end{aligned}$$

(3) $a \geq 0$.
Since we already know that $c > 0$ and

$$ac = b^2 \geq 0,$$

we conclude that $a \geq 0$.

The three conditions $a \geq 0$, $b^2 = ac$, and $c > 0$, imply that the quadratic form $-(1/2)ax^2 + bxy - (1/2)cy^2$ is negative semi-definite. More precisely, if we define:

$$m := \sqrt{a} \tag{5.34}$$

and

$$n := \operatorname{sgn}(b)\sqrt{c}, \tag{5.35}$$

then we have:

$$-\frac{1}{2}ax^2 + bxy - \frac{1}{2}cy^2 = -\frac{1}{2}(mx - ny)^2. \tag{5.36}$$

For all $u \geq 1$, we will denote the L^u-norm with respect to the standard Gaussian measure μ on $\mathbb{R}$ by $\|\cdot\|_u$, and the L^u-norm with respect to the normalized Lebesgue measure $d_N x$ on $\mathbb{R}$ by $|||\cdot|||_u$.

Step 4. *Apply Corollary* 5.2 *of Lieb theorem to compute the operatorial norm of the bilinear operator* $\tilde{B}$, *by computing the supremum only over the set of exponential functions.*

We are working now in dimension $d = 1$. Since the function $(x, y) \mapsto -(1/2)(mx - ny)^2$ is non-positive, we may use Lieb theorem. That means:

$$\begin{aligned}
&\|\tilde{B}\|_{L^p(\mathbb{R},d_Nx)\times L^q(\mathbb{R},d_Nx)\to L^r(\mathbb{R},d_Nx)} \\
&\quad = \sup\{|||\tilde{B}(f,g)|||_r \mid f \in L^p(\mathbb{R}, d_N x), g \in L^q(\mathbb{R}, d_N x), |||f|||_p = |||g|||_q = 1\} \\
&\quad = \sup\{|||\tilde{B}(f,g)|||_r \mid f = c_1 \exp[-(k/2)x^2], g = c_2 \exp[-(l/2)x^2], \\
&\qquad\quad c_1 > 0, c_2 > 0, k > 0, l > 0, |||f|||_p = |||g|||_q = 1\}.
\end{aligned}$$

If $h(x) = \lambda \exp[-(s/2)x^2]$, with $s > 0$, then for all $u \geq 1$, we have:

$$\begin{aligned}
|||h|||_u &= |\lambda| \left[\int_{\mathbb{R}} e^{-\frac{us}{2}x^2} d_N x\right]^{1/u} \\
(\text{let } x' := \sqrt{us}x) &= |\lambda| \left[\frac{1}{\sqrt{us}} \int_{\mathbb{R}} e^{-\frac{x'^2}{2}} d_N x'\right]^{1/u} \\
&= |\lambda| \frac{1}{(\sqrt{us})^{1/u}}.
\end{aligned}$$

Thus, in order to have $|||h|||_u = 1$, we must have

$$|\lambda| = (\sqrt{us})^{1/u}. \tag{5.37}$$

Therefore, we have:

$$\begin{aligned}
&\|\tilde{B}\|_{L^p(\mathbb{R},d_Nx)\times L^q(\mathbb{R},d_Nx)\to L^r(\mathbb{R},d_Nx)} \\
&\quad = \sup_{k>0,l>0} \left\{(pk)^{1/(2p)}(ql)^{1/(2q)}\right. \\
&\qquad \left.\cdot \left|\left|\left|\int_{\mathbb{R}} f(\alpha\cdot + \gamma y) g(\beta\cdot + \delta y) e^{-(1/2)(m\cdot - ny)^2} d_N y\right|\right|\right|_r\right\} \\
&\quad = p^{1/(2p)} q^{1/(2q)} \sup_{k>0,l>0} \left\{k^{1/(2p)} l^{1/(2q)}\right. \\
&\qquad \left.\cdot \left\{\int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-\frac{k}{2}(\alpha x + \gamma y)^2} e^{-\frac{l}{2}(\beta x + \delta y)^2} e^{-(1/2)(mx - ny)^2} d_N y\right]^r d_N x\right\}^{1/r}\right\}.
\end{aligned}$$

We apply now formula (5.5) from Lemma 5.3 and conclude that:

$$\int_{\mathbb{R}} e^{-\frac{k}{2}(\alpha x+\gamma y)^2} e^{-\frac{l}{2}(\beta x+\delta y)^2} e^{-(1/2)(mx-ny)^2} d_N y$$

$$= \frac{1}{\sqrt{\gamma^2 k+\delta^2 l+n^2}} \cdot \exp\left[-\frac{1}{2}\cdot\frac{(\alpha\delta-\beta\gamma)^2 kl+(n\alpha+m\gamma)^2 k+(n\beta+m\delta)^2 l}{\gamma^2 k+\delta^2 l+n^2} x^2\right]. \tag{5.38}$$

Thus, we have:

$$\|\tilde{B}\|_{L^p(\mathbb{R},d_N x)\times L^q(\mathbb{R},d_N x)\to L^r(\mathbb{R},d_N x)}$$

$$= p^{1/(2p)} q^{1/(2q)} \sup_{k>0,l>0}\left\{k^{1/(2p)} l^{1/(2q)} \cdot\left\{\int_{\mathbb{R}}\left[\int_{\mathbb{R}} e^{-\frac{k}{2}(\alpha x+\gamma y)^2} e^{-\frac{l}{2}(\beta x+\delta y)^2} e^{-(1/2)(mx-ny)^2} d_N y\right]^r d_N x\right\}^{1/r}\right\}$$

$$= p^{1/(2p)} q^{1/(2q)} \sup_{k>0,l>0}\left\{k^{1/(2p)} l^{1/(2q)} \frac{1}{\sqrt{\gamma^2 k+\delta^2 l+n^2}} \cdot\left\{\int_{\mathbb{R}} \exp\left[-\frac{r}{2}\cdot\frac{(\alpha\delta-\beta\gamma)^2 kl+(n\alpha+m\gamma)^2 k+(n\beta+m\delta)^2 l}{\gamma^2 k+\delta^2 l+n^2} x^2\right] d_N x\right\}^{1/r}\right\}.$$

$$= \frac{p^{1/(2p)} q^{1/(2q)}}{r^{1/(2r)}} \sup_{k>0,l>0}\left\{k^{1/(2p)} l^{1/(2q)} \frac{1}{\sqrt{\gamma^2 k+\delta^2 l+n^2}} \cdot \frac{(\gamma^2 k+\delta^2 l+n^2)^{1/(2r)}}{[(\alpha\delta-\beta\gamma)^2 kl+(n\alpha+m\gamma)^2 k+(n\beta+m\delta)^2 l]^{1/(2r)}}\right\}$$

$$= \frac{p^{1/(2p)} q^{1/(2q)}}{r^{1/(2r)}} \sup_{k>0,l>0} \frac{k^{1/(2p)} l^{1/(2q)}}{(\gamma^2 k+\delta^2 l+n^2 1)^{1/(2r')}[U^2 kl+V^2 k+W^2 l]^{1/(2r)}},$$

where r' is the Hölder conjugate of r, that means:

$$\frac{1}{r}+\frac{1}{r'}=1, \tag{5.39}$$

$$U := \alpha\delta-\beta\gamma, \tag{5.40}$$

$$V := n\alpha+m\gamma \tag{5.41}$$

and

$$W := n\beta+m\delta. \tag{5.42}$$

We can put the factors r and $U^2kl + V^2k + W^2l$ together. Thus it remains to prove that:

$$\sup_{k>0,l>0} \frac{(p\mathbf{k})^{1/(2p)}(q\mathbf{l})^{1/(2q)}}{(\gamma^2\mathbf{k} + \delta^2\mathbf{l} + n^2\mathbf{1})^{1/(2r')}[rU^2\mathbf{kl} + rV^2\mathbf{k} + rW^2\mathbf{l}]^{1/(2r)}} = 1. \tag{5.43}$$

We used some bold face letters to emphasize the idea for the next step.

Step 5. *Observe that the numerator of the left-hand side of* (5.43) *looks like a weighted geometric mean of* (k, l), *while the two factors from the denominator of the left-hand side of* (5.43) *look like weighted arithmetic means of* $(k, l, 1)$ *and* (kl, k, l), *respectively. Since each arithmetic mean dominates each geometric mean, our supremum has great chances of being finite. This observation tells us that, to continue, we have to use the reason behind the arithmetic–geometric mean inequality, which is the concavity of the logarithmic function.*

Let us make the following changes of variables:

$$K := pk \tag{5.44}$$

and

$$L := ql. \tag{5.45}$$

Thus, we have to prove that:

$$\sup_{K>0,L>0} \frac{\mathbf{K}^{1/(2p)}\mathbf{L}^{1/(2q)}}{[(\gamma^2/p)\mathbf{K} + (\delta^2/q)\mathbf{L} + n^2\mathbf{1}]^{1/(2r')}} \cdot \frac{1}{[(rU^2/(pq))\mathbf{KL} + (rV^2/p)\mathbf{K} + (rW^2/q)\mathbf{L}]^{1/(2r)}} = 1. \tag{5.46}$$

Claim 1. *We have*:

$$\frac{\gamma^2}{p} + \frac{\delta^2}{q} + n^2 = 1. \tag{5.47}$$

Indeed, we have:

$$\begin{aligned}
\frac{\gamma^2}{p} + \frac{\delta^2}{q} + n^2 &= \frac{\left[\operatorname{sgn}(\alpha\beta(t-1))\sqrt{1-\alpha^2}\right]^2}{p} + \frac{\left(\sqrt{1-\beta^2}\right)^2}{q} + c \\
&= \frac{1-\alpha^2}{p} + \frac{1-\beta^2}{q} + \left(1 - \frac{1-\alpha^2}{p} - \frac{1-\beta^2}{q}\right) \\
&= 1.
\end{aligned}$$

Claim 2. *We have*:

$$\frac{rU^2}{pq} + \frac{rV^2}{p} + \frac{rW^2}{q} = 1. \tag{5.48}$$

Indeed, we have:

$$\begin{aligned}
&\frac{rU^2}{pq} + \frac{rV^2}{p} + \frac{rW^2}{q} \\
&\quad = \frac{r(\alpha\delta - \beta\gamma)^2}{pq} + \frac{r(n\alpha + m\gamma)^2}{p} + \frac{r(n\beta + m\delta)^2}{q} \\
&\quad = r\left[\frac{(\alpha\delta - \beta\gamma)^2}{pq} + n^2\left(\frac{\alpha^2}{p} + \frac{\beta^2}{q}\right) + m^2\left(\frac{\gamma^2}{p} + \frac{\delta^2}{q}\right) + 2mn\left(\frac{\alpha\gamma}{p} + \frac{\beta\delta}{q}\right)\right] \\
&\quad = r\left[\frac{(\alpha\delta - \beta\gamma)^2}{pq} + c\left(\frac{1}{r} - a\right) + a(1-c) + 2\sqrt{ac}\,\operatorname{sgn}(b)\cdot b\right] \\
&\quad = r\left[\frac{(\alpha\delta - \beta\gamma)^2}{pq} + c\left(\frac{1}{r} - a\right) + a(1-c) + 2\sqrt{ac}\cdot\sqrt{ac}\right] \\
&\quad = r\left[\frac{(\alpha\sqrt{1-\beta^2} - \beta\operatorname{sgn}(\alpha\beta(t-1))\sqrt{1-\alpha^2})^2}{pq} + \frac{c}{r} + a\right] \\
&\quad = r\left[\frac{\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2) - 2\alpha\beta\operatorname{sgn}(\alpha\beta(t-1))\sqrt{(1-\alpha^2)(1-\beta^2)}}{pq}\right. \\
&\qquad \left. + \frac{c}{r} + a\right].
\end{aligned}$$

Using now the assumption that we have equality in condition (5.11), we get:

$$\operatorname{sgn}(\alpha\beta(t-1))\sqrt{(1-\alpha^2)(1-\beta^2)} = \alpha\beta(t-1).$$

Thus, we obtain:

$$\begin{aligned}
&\frac{rU^2}{pq} + \frac{rV^2}{p} + \frac{rW^2}{q} \\
&\quad = r\left[\frac{\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2) - 2\alpha^2\beta^2(t-1)}{pq} + \frac{c}{r} + a\right] \\
&\quad = r\left[\frac{\alpha^2 + \beta^2 - 2\alpha^2\beta^2 t}{pq} + \frac{c}{r} + \frac{1}{r} - \frac{\alpha^2}{p} - \frac{\beta^2}{q}\right] \\
&\quad = c + 1 - \frac{r}{pq}[\alpha^2(q-1) + \beta^2(p-1) + 2\alpha^2\beta^2 t].
\end{aligned}$$

Since we know that:

$$r = pq\frac{c}{\alpha^2(q-1) + \beta^2(p-1) + 2\alpha^2\beta^2 t},$$

we obtain:

$$\begin{aligned}\frac{rU^2}{pq} + \frac{rV^2}{p} + \frac{rW^2}{q} \\ &= c + 1 - c \\ &= 1.\end{aligned}$$

Applying Jensen inequality to the concave function $x \mapsto \ln(x)$, we get:

$$\ln\left(\frac{\gamma^2}{p}K + \frac{\delta^2}{q}L + n^2 1\right) \geq \frac{\gamma^2}{p}\ln K + \frac{\delta^2}{q}\ln L + n^2 \ln 1.$$

Exponentiating both sides of the last inequality, we obtain:

$$\frac{\gamma^2}{p}K + \frac{\delta^2}{q}L + n^2 1 \geq K^{\gamma^2/p}L^{\delta^2/q}. \tag{5.49}$$

Applying again Jensen inequality, we get:

$$\ln\left(\frac{rU^2}{pq}KL + \frac{rV^2}{p}K + \frac{rW^2}{q}L\right) \geq \frac{rU^2}{pq}\ln(KL) + \frac{rV^2}{p}\ln K + \frac{rW^2}{q}\ln L.$$

Exponentiating both sides of the last inequality, we obtain:

$$\frac{rU^2}{pq}KL + \frac{rV^2}{p}K + \frac{rW^2}{q}L \geq K^{r[U^2/(pq)+V^2/p]}L^{r[U^2/(pq)+W^2/q]}. \tag{5.50}$$

Thus applying inequalities (5.49) and (5.50), for all K and L positive numbers, we have:

$$\begin{aligned}&\frac{K^{1/(2p)}L^{1/(2q)}}{[(\gamma^2/p)K + (\delta^2/q)L + n^2 1]^{1/(2r')}} \\ &\quad \cdot \frac{1}{[(rU^2/(pq))KL + (r/p)V^2K + (rW^2/q)L]^{1/(2r)}} \\ &\leq \frac{K^{1/(2p)}L^{1/(2q)}}{K^{\gamma^2/(2pr')}L^{\delta^2/(2qr')}\cdot K^{U^2/(2pq)+V^2/(2p)}L^{U^2/(2pq)+W^2/(2q)}} \\ &= \left[\frac{K}{K^{\gamma^2/r'+U^2/q+V^2}}\right]^{1/(2p)}\left[\frac{L}{L^{\delta^2/r'+U^2/p+W^2}}\right]^{1/(2q)}.\end{aligned}$$

If we can prove now the following claim, then we will be done.

Claim 3. *We have*:

$$\frac{\gamma^2}{r'} + \frac{U^2}{q} + V^2 = 1 \tag{5.51}$$

and similarly,

$$\frac{\delta^2}{r'} + \frac{U^2}{p} + W^2 = 1. \tag{5.52}$$

Indeed, we have:

$$\begin{aligned}
&\frac{\gamma^2}{r'} + \frac{U^2}{q} + V^2 \\
&\quad = \frac{\gamma^2}{r'} + \frac{(\alpha\delta - \beta\gamma)^2}{q} + (n\alpha + m\gamma)^2 \\
&\quad = \frac{\gamma^2}{r'} + \frac{(\alpha\delta - \beta\gamma)^2}{q} + c\alpha^2 + 2\sqrt{ac}\,\mathrm{sgn}(b)\alpha\gamma + a\gamma^2 \\
&\quad = \frac{\gamma^2}{r'} + \frac{(\alpha\delta - \beta\gamma)^2}{q} + \left(1 - \frac{\gamma^2}{p} - \frac{\delta^2}{q}\right)\alpha^2 + 2b\alpha\gamma + \left(\frac{1}{r} - \frac{\alpha^2}{p} - \frac{\beta^2}{q}\right)\gamma^2 \\
&\quad = \left(\frac{\gamma^2}{r'} + \frac{\gamma^2}{r}\right) + \left[\frac{(\alpha\delta - \beta\gamma)^2}{q} - \frac{\alpha^2\delta^2 + \beta^2\gamma^2}{q}\right] + \alpha^2 - 2\frac{\alpha^2\gamma^2}{p} + 2b\alpha\gamma.
\end{aligned}$$

We can replace now b *by* $\alpha\gamma/p + \beta\delta/q$*, and obtain:*

$$\begin{aligned}
&\frac{\gamma^2}{r'} + \frac{U^2}{q} + V^2 \\
&\quad = \gamma^2\left(\frac{1}{r'} + \frac{1}{r}\right) - 2\frac{\alpha\beta\gamma\delta}{q} + \alpha^2 - 2\frac{\alpha^2\gamma^2}{p} + 2\left(\frac{\alpha\gamma}{p} + \frac{\beta\delta}{q}\right)\alpha\gamma \\
&\quad = \gamma^2 \cdot 1 + \alpha^2 \\
&\quad = (1 - \alpha^2) + \alpha^2 \\
&\quad = 1.
\end{aligned}$$

Thus, we have proved that for all K *and* L *positive, we have:*

$$\begin{aligned}
&\frac{K^{1/(2p)}L^{1/(2q)}}{[(\gamma^2/p)K + (\delta^2/q)L + C^2 1]^{1/(2r')}} \\
&\qquad \cdot \frac{1}{[(rU^2/(pq))KL + (r/p)V^2K + (rW^2/q)L]^{1/(2r)}} \\
&\quad \leq 1.
\end{aligned} \tag{5.53}$$

On the other hand, for $K = L = 1$, we have equality in (5.53).

Therefore, our supremum is equal to 1.

Step 6. *We extend the inequality to the infinite dimensional case.*

Let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of E made up of eigenfunctions of the operator A used in the construction of the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$. For all natural numbers n, let:

$$E_n := \mathbb{R}\langle\cdot, e_0\rangle \oplus \mathbb{R}\langle\cdot, e_1\rangle \oplus \cdots \oplus \mathbb{R}\langle\cdot, e_{n-1}\rangle,$$

where $\langle\cdot,\, e_i\rangle$ represents the L^2-normally distributed random variable generated by e_i, for all $0 \le i \le n-1$. We define $\mathcal{F}_n := \mathcal{F}(E_n)$, that means, $\mathcal{F}_n$ is the smallest sigma-algebra with respect to which $\langle\cdot,\, e_0\rangle$, $\langle\cdot,\, e_1\rangle, \ldots, \langle\cdot,\, e_{n-1}\rangle$ are measurable.

For all $\varphi \in L^p(\mathcal{E}',\, \mu)$ and $n \ge 1$, we define $\varphi_n := E[\varphi \,|\, \mathcal{F}_n]$, the conditional expectation of φ with respect to $\mathcal{F}_n$. Since $\{\mathcal{F}_n\}_{n\ge1}$ is an increasing family of sigma-algebras, and the sigma-algebra generated by them is the Borel sigma-algebra $\mathcal{F}$ of $\mathcal{E}'$, from the Martingale Convergence Theorem we have:

$$E[\varphi \,|\, \mathcal{F}_n] \to \varphi,$$

as $n \to \infty$, both almost surely and in $L^p(\mathcal{E}',\, \mu)$, for all $p \ge 1$.

Thus, the vector space:

$$L^p_f(\mathcal{E}') := \bigcup_{n=1}^{\infty} L^p(\mathcal{E}', \mathcal{F}_n, P) \tag{5.54}$$

is dense in $L^p(\mathcal{E}',\, \mathcal{F},\, P)$.

Due to our proof in the finite dimensional case, we know that the bilinear operator $B : L^p_f(\mathcal{E}') \times L^q_f(\mathcal{E}') \to L^r(\mathcal{E}',\, \mu)$, defined by:

$$B(\varphi, \psi) = \Gamma(C)\varphi \diamond_T \Gamma(D)\psi, \tag{5.55}$$

is bounded of operatorial norm 1, and since $L^p_f(\mathcal{E}') \times L^q_f(\mathcal{E}')$ is dense in $L^p(\mathcal{E}',\, \mu) \times L^q(\mathcal{E}',\, \mu)$, it has a unique bounded bilinear extension from $L^p(\mathcal{E}',\, \mu) \times L^q(\mathcal{E}',\, \mu)$ to $L^r(\mathcal{E}',\, \mu)$.

($\Leftarrow$) Let us assume now that the operator

$$B : L^p(\mathcal{E}', \mu) \times L^q(\mathcal{E}', \mu) \to L^r(\mathcal{E}', \mu)$$

defined as:

$$B(\varphi, \psi) = \Gamma(C)\varphi \diamond_T \Gamma(D)\psi, \tag{5.56}$$

is bounded. Then there is a constant $k > 0$, such that, for all $\varphi \in L^p(\mathcal{E}',\, \mu)$ and all $\psi \in L^q(\mathcal{E}',\, \mu)$, we have:

$$\|\Gamma(C)\varphi \diamond_T \Gamma(D)\psi\|_r \le k\|\varphi\|_p\|\psi\|_q. \tag{5.57}$$

Since C, D, and T are diagonalized in the same base, $\{e_i\}_{i\ge0}$ as A, for each $i \ge 0$, there exists α_i, β_i, and t_i real numbers such that:

$$Ce_i = \alpha_i e_i, \tag{5.58}$$

$$De_i = \beta_i e_i, \tag{5.59}$$

$$Te_i = t_i e_i. \tag{5.60}$$

Let $i \ge 0$ be a fixed natural number. Let u and s be arbitrary real numbers, such that $u \ne 0$. Let φ and ψ be the following exponential functions:

$$\varphi := \varphi_{sue_i} \tag{5.61}$$

and

$$\psi := \varphi_{ue_i}. \tag{5.62}$$

Then inequality (5.57) becomes:

$$\|e^{t_i\alpha_i\beta_i su^2}\varphi_{(s\alpha+\beta)ue_i}\|_r \leq k\|\varphi_{sue_i}\|_p\|\varphi_{ue_i}\|_q. \tag{5.63}$$

Since a simple computation shows that for every $l \in [1, \infty)$, and every exponential function φ_ξ, with $\xi \in E$, we have:

$$\|\varphi_\xi\|_l = e^{(l-1)|\xi|_0^2/2}, \tag{5.64}$$

inequality (5.63) becomes:

$$\exp\left(\frac{1}{2}(r-1)(s\alpha_i+\beta_i)^2u^2 + t_i\alpha_i\beta_iu^2\right)$$
$$\leq k\exp\left(\frac{1}{2}(p-1)s^2u^2 + \frac{1}{2}(q-1)u^2\right). \tag{5.65}$$

Taking first ln from both sides of the last inequality, and then dividing both sides by u^2, we obtain:

$$\frac{1}{2}(r-1)(s\alpha_i+\beta_i)^2 + t_i\alpha_i\beta_i \leq \frac{\ln k}{u^2} + \frac{1}{2}(p-1)s^2 + \frac{1}{2}(q-1), \tag{5.66}$$

for all $u \neq 0$ and $s \in \mathbb{R}$. Passing to the limit, as $u \to \infty$, in this inequality, we obtain:

$$\frac{1}{2}(r-1)(s\alpha_i+\beta_i)^2 + t_i\alpha_i\beta_i \leq \frac{1}{2}(p-1)s^2 + \frac{1}{2}(q-1), \tag{5.67}$$

for all real numbers s. Inequality (5.67) is equivalent to:

$$\frac{1}{2}\left[p-1-\alpha_i^2(r-1)\right]s^2 - (t_i+r-1)\,\alpha_i\beta_is + \frac{1}{2}\left[q-1-\beta_i^2(r-1)\right] \geq 0,$$

for all real numbers s. For this quadratic function of variable s to be nonnegative, for all real values of s, its leading coefficient $(1/2)[p-1-\alpha_i^2(r-1)]$ must be nonnegative and its discriminant must be nonpositive, that means:

$$\alpha_i^2\beta_i^2\,(t_i+r-1)^2 - \left[p-1-\alpha_i^2(r-1)\right]\left[q-1-\beta_i^2(r-1)\right] \leq 0.$$

The last inequality is equivalent to:

$$r-1 \leq \frac{(p-1)(q-1) - t_i^2\alpha_i^2\beta_i^2}{\alpha_i^2(q-1) + \beta_i^2(p-1) + 2\alpha_i^2\beta_i^2t_i}. \tag{5.68}$$

Since this inequality holds for all $i \geq 0$, we conclude that:

$$(r-1)I \leq \frac{(p-1)(q-1)I - C^2D^2T^2}{(q-1)C^2 + (p-1)D^2 + 2C^2D^2T}. \tag{5.69}$$

□

Let us define for any invertible self-adjoint operator B on E_c commuting with A, and any $p \in [1, \infty]$, the following norm:

$$\|\varphi\|_{p,B} := \|\Gamma(B)\varphi\|_p. \tag{5.70}$$

Let us also define the space:

$$L^{p,B}(\mathcal{E}', \mu) := \{\varphi \in (\mathcal{E})^* \,|\, \|\Gamma(B)\varphi\|_p \; < \; \infty\}. \tag{5.71}$$

With this notation, we have the following corollary.

Corollary 5.5. *Let $\mathcal{E} \subset E \subset \mathcal{E}'$ be a Gel'fand triple given by a self-adjoint diagonal operator A on E, with increasing, greater than 1 eigenvalues, whose inverse is a Hilbert–Schmidt operator. Let μ be the Gaussian probability measure on $\mathcal{E}'$ whose existence is guaranteed by Minlos Theorem. Let T be a self-adjoint, diagonal operator on E, commuting with A, such that*:

$$T \geq 0. \tag{5.72}$$

Let B, C and D be three invertible self-adjoint and diagonal operators on E, commuting with the operator A, such that:

$$|B| \geq \sqrt{\frac{T}{2}}, \tag{5.73}$$

$$|C| \geq |B|, \tag{5.74}$$

$$|D| \geq |B| \tag{5.75}$$

and

$$(C^2 - B^2)(D^2 - B^2) \geq (T - B^2)^2. \tag{5.76}$$

Let $p, q, r > 1$ such that:

$$\frac{1}{(r-1)B^2 + T} \geq \frac{1}{(p-1)C^2 + T} + \frac{1}{(q-1)D^2 + T}. \tag{5.77}$$

Then for all φ in $L^{p,C}(\mathcal{E}', \mu)$ and ψ in $L^{q,D}(\mathcal{E}', \mu)$, $\varphi \diamond_T \psi$ belongs to $L^{r,B}(\mathcal{E}', \mu)$, and the following inequality holds:

$$\|\varphi \diamond_T \psi\|_{r,B} \leq \|\varphi\|_{p,C} \cdot \|\psi\|_{q,D}. \tag{5.78}$$

On the other hand, if:

$$\frac{1}{(r-1)B^2 + T} \ngeq \frac{1}{(p-1)C^2 + T} + \frac{1}{(q-1)D^2 + T},$$

then the bilinear operator $(\varphi, \psi) \mapsto \varphi \diamond_T \psi$ is not bounded from $L^{p,C}(\mathcal{E}', \mu) \times L^{q,D}$ $(\mathcal{E}', \mu)$ to $L^{r,B}(\mathcal{E}', \mu)$.

Proof. Simply apply Theorem 5.4 and Lemma 3.2 to the following operators and random variables, with the convention that $X \to X'$ means that X from Theorem 5.4 is replaced by X' in the same theorem:

- $C \to BC^{-1}$
- $D \to BD^{-1}$

- $T \to TB^{-2}$
- $\varphi \to \Gamma(C)\varphi$
- $\psi \to \Gamma(D)\psi$.

Writing inequality (5.10) for these new operators and random variables, and using the fact that:

$$\|\varphi\|_{p,|B|} = \|\varphi\|_{p,B}, \tag{5.79}$$

inequality (5.78) follows. □

Corollary 5.6. *If we choose* $B := I$, $C > I, \psi := 1$, *and let either the eigenvalues of* D *go to* ∞, *or choose the eigenvalues of* D *large enough such that* (5.76) *is satisfied and let* q *go to* ∞, *then condition* (5.77) *becomes*:

$$\frac{1}{(r-1)I+T} \geq \frac{1}{(p-1)C^2+T} + \frac{1}{\infty}, \tag{5.80}$$

which is equivalent to:

$$|C| \geq \sqrt{\frac{r-1}{p-1}}I \tag{5.81}$$

and inequality (5.78) *becomes*:

$$\|\varphi\|_r \leq \|\Gamma(C)\varphi\|_p. \tag{5.82}$$

Inequalities (5.81) and (5.82) are exactly Nelson condition and hypercontractivity inequality. See Ref. 26.

Corollary 5.7. *If we choose* $B = I$, *then condition* (5.73) *becomes*:

$$0 \leq T \leq 2I, \tag{5.83}$$

and conditions (5.74) *and* (5.75) *become*:

$$|C| \geq I \tag{5.84}$$

and

$$|D| \geq I. \tag{5.85}$$

In this case the inequality:

$$\frac{1}{(r-1)I+T} \geq \frac{1}{(p-1)C^2+T} + \frac{1}{(q-1)D^2+T}, \tag{5.86}$$

and the "smoothness" conditions $\Gamma(C)\varphi \in L^p(\mathcal{E}', \mu)$ *and* $\Gamma(D)\varphi \in L^q(\mathcal{E}', \mu)$ *guarantee the fact that* $\varphi \diamond_T \psi$ *is a* true random variable (*not a merely generalized function*) *in the space* $L^r(\mathcal{E}', \mu)$.

We would like to make the following comments:

Comments.

- Inequality (5.10) in the case $T := 0$, $C := \alpha I$, $D := \beta I$, $\alpha^2 + \beta^2 = 1$, and $p = q = r := 2$, was obtained using Cauchy–Schwarz inequality in Refs. 20 and 21.

- Inequality (5.10) in the case $T := 0$, $C := \alpha I$, $D := \beta I$, $\alpha^2 + \beta^2 = 1$, and ($p = q = r := 1$ or $p = q = r := \infty$) was proven in Ref. 22.
- Inequality (5.10) in the case $T := 0$, $C := \alpha I$, $D := \beta I$, $\alpha^2 + \beta^2 = 1$, and $1/(r-1) = \alpha^2/(p-1) + \beta^2/(q-1)$ was proven in Ref. 11.
- In the case: $B = C = D = T = I$, the T-Wick product becomes the pointwise product, and condition (5.77) becomes:

$$\frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}, \tag{5.87}$$

  which is exactly the classic Hölder condition for probability measures. One should not forget, that for general measures, Hölder condition is the equality:

$$\frac{1}{r_0} = \frac{1}{p} + \frac{1}{q}, \tag{5.88}$$

  not an inequality, but if the measure is a probability measure, Lyapunov inequality: for all $0 < r \leq r_0$, we have:

$$\|f\|_r \leq \|f\|_{r_0}, \tag{5.89}$$

  relaxes the Hölder condition from a perfect equality to an inequality.
- Methods of finding the supremum over the exponential functions in Lieb theorem were provided in Refs. 6, 8 and 9. In this paper we used a method based on Jensen inequality for the concave natural logarithmic function that was also applied in Ref. 11.

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