NONLINEAR WAVES IN ADHESIVE STRINGS

G. M. COCLITE, G. FLORIO, M. LIGABÒ, AND F. MADDALENA

ABSTRACT. We study a 1D semilinear wave equation modeling the dynamic of an elastic string interacting with a rigid substrate through an adhesive layer. The constitutive law of the adhesive material is assumed elastic up to a finite critical state, beyond such a value the stress discontinuously drops to zero. Therefore the semilinear equation is characterized by a source term presenting jump discontinuity. Well-posedness of the initial boundary value problem of Neumann type is investigated.

1. INTRODUCTION

Adhesion, capillarity and wetting phenomena (see [4, 5]) constitute a challenging arena for mathematical problems due to the complexity of physical mechanisms involved. A rational understanding in the format of analytical descriptions of such problems, in addition to being in itself interesting, is relevant for both life sciences and manufacturing engineering. Indeed this is a long standing problem: a seminal paper in the field is [3] in which a free boundary approach is pursued. In some recent papers (see, e.g., [6, 7, 8, 9]) one of the authors has studied the static problem of adhesion of elastic thin structures under various constitutive assumptions on the adhesive material. The main goal of those works relies in characterizing, with the tools of the calculus of variations, the interplay of the occurrence of debonding with other constitutive properties. The study of the evolution problem related to these physical manifestations require the analysis of multidimensional hyperbolic problems involving mathematical issues not yet well understood. In this paper we address a prototypical dynamical problem by studying the adhesion of an elastic string glued to a rigid substrate, assuming a discontinuous softening behavior of the adhesive material, i.e. the adhesive stress jumps to zero when a critical value of the displacement is reached. We consider the mechanical system with the following energy density:

$$e[u] = \frac{1}{2}\rho(\partial_t u)^2 + \frac{1}{2}K_e(\partial_x u)^2 + \Phi(u), \tag{1.1}$$

where $\rho > 0$ denotes the mass density, K_e denotes the elastic stiffness of the string, and $\Phi(u)$ denotes the adhesion potential modeling the energetic contribution of the glue layer. To taking into account the possibility of debonding we assume for the potential Φ a behavior like in Fig. 1, for example

$$\Phi(u) = \begin{cases} u^2, & \text{if } |u| \le u^*, \\ (u^*)^2, & \text{if } |u| > u^*, \end{cases} \tag{1.2}$$

where u^* denotes the threshold beyond which the glue cannot sustain further stress.

We are interested in the qualitative properties of the Euler equations associated to the above energy density (1.1) given by

$$\rho\partial_{tt}^2 u - K_e\partial_{xx}^2 u = -\Phi'(u), \tag{1.3}$$

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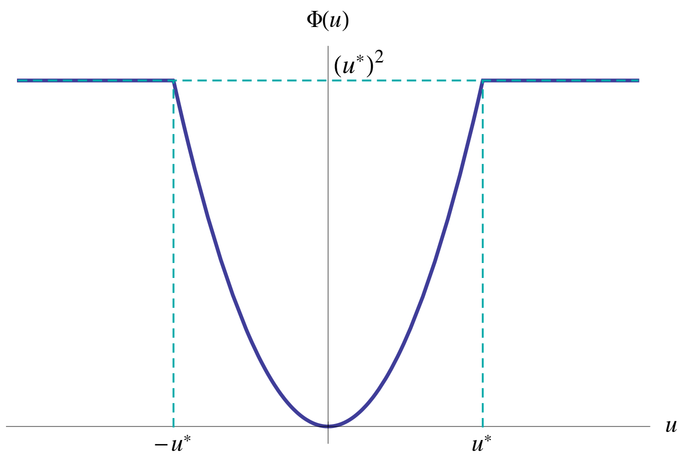

FIGURE 1. (Color online) Potential $\Phi(u)$ in Eq. 1.2.

equipped with Neumann boundary conditions.

The paper is organized as follows. In Section 2 we introduce the problem, the main assumptions and the associated energy. In Section 3 we give the definition of dissipative solution, prove existence (cf. Theorem 3.1), regularity (cf. Theorem 3.2), and non-uniqueness for the solutions of initial boundary value problem related to (1.3) (cf. Examples 3.1, 3.2, 3.3). In Section 4 we focus on the first order formulation of the problem and investigate the interplay between debonding and propagation of singularities along characteristics (cf. Theorem 4.1).

2. Statement of the problem

Let us consider a one dimensional material body, i.e. a *string*, whose rest configuration at the initial time $t = 0$ coincides with the interval $[0, L]$ and the displacement field is denoted by

$$u : [0, \infty) \times [0, L] \to \mathbb{R}.$$

The material is assumed linear elastic and, for sake of notational simplicity, the mass density ρ and the extensional stiffness K_e are assumed both equal to 1. The string interacts with an underlying rigid support through an infinitesimal layer of adhesive material characterized by an internal energy $u \mapsto \Phi(u)$ with the threshold u^* set to 1.

The balance of momentum delivers the semilinear initial boundary value problem

$$\begin{cases} \partial_{tt}^2 u = \partial_{xx}^2 u - \Phi'(u), & t > 0, 0 < x < L, \\ \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = u_0(x), & 0 < x < L, \\ \partial_t u(0, x) = u_1(x), & 0 < x < L. \end{cases} \tag{2.1}$$

We shall assume that

- (**H.1**) $\Phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{1, -1\})$, Φ is constant in $(-\infty, -1]$ and in $[1, \infty)$, convex in $[-1, 1]$, decreasing in $[-1, 0]$ and increasing in $[0, 1]$;
- (**H.2**) $u_0 \in H^1(0, L)$, $u_1 \in L^2(0, L)$.

As a consequence of (**H.1**), Φ' has a jump discontinuity in $u = \pm 1$ and

$$u \in (-\infty, -1) \cup (1, \infty) \Rightarrow \Phi'(u) = 0,$$
$$0 < u < 1 \Rightarrow 0 < \Phi'(u) \le \lim_{u \to 1^-} \Phi'(u),$$
$$-1 < u < 0 \Rightarrow 0 > \Phi'(u) \ge \lim_{u \to -1^+} \Phi'(u).$$

Assumption (**H**.**1**) characterizes the constitutive behavior of the adhesive material trough the source term $-\Phi'(u)$, i.e. when $|u| = 1$ the loss of adhesion manifests through the jump discontinuity of Φ', hence debonding of the string occurs. Differently, in [3] the authors consider two different kinds of source terms, one is independent on u and the other one mimicking adhesion along a rough plane depending on $\partial_t u$.

To fix ideas, a function satisfying such assumption is

$$\Phi(u) = \begin{cases} u^2, & \text{if } |u| \leq 1, \\ 1, & \text{if } |u| > 1. \end{cases} \tag{2.2}$$

In particular we have

$$\Phi'(u) = \begin{cases} 2u, & \text{if } |u| \leq 1, \\ 0, & \text{if } |u| > 1. \end{cases} \tag{2.3}$$

The natural energy associated to the problem (2.1) is

$$E(t) = \int_0^L \left(\frac{(\partial_t u(t,x))^2 + (\partial_x u(t,x))^2}{2} + \Phi(u(t,x)) \right) dx. \tag{2.4}$$

Due to the lack of Lipschitz continuity in the nonlinear term Φ' we cannot expect the existence of conservative solutions, i.e., solutions that preserve the energy. This is coherent with the physics behind the problem, when our material is ungluing; indeed in [7, Sec. 3.3] the authors describe the hysteresis cycles and the dissipation associated with the maximum delay strategy corresponding to the quasistatic evolution for a discrete system where the macroscopic limit (obtained by Γ-convergence [7, Appendix B]) could be viewed as the system here analyzed. Moreover, even mathematically the dissipation of energy is natural. Indeed, when we study the compactness of some approximate solutions we cannot have bounds on the second derivatives because we cannot differentiate the equation in (2.1). Therefore, we have to live with bounds on the first derivatives and then we can have only weak convergence in H^1.

3. Existence and regularity of weak solutions

This section is dedicated to the well-posedness and regularity analysis of (2.1). We show the existence of Lipshitz continuous dissipative solutions. Some examples show that those solutions are not unique and do not depend continuously on the initial conditions. Indeed, in the following section we shall focus on a qualitative analysis of the discontinuity curves of the first derivatives of the solutions. These are the loci where the dissipation of energy occurs. Therefore, it seems quite natural to introduce the concept of *dissipative solution*:

Definition 3.1. *We say that a function $u : [0,\infty) \times [0,L] \to \mathbb{R}$ is a dissipative solution of* (2.1) *if*

(*i*) $u \in C([0,\infty) \times [0,L])$;
(*ii*) $\partial_t u,\ \partial_x u \in L^\infty(0,\infty; L^2(0,L))$;
(*iii*) *for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support*

$$\begin{aligned} &\int_0^\infty \int_0^L \left(u\partial_{tt}^2\varphi + \partial_x u \partial_x \varphi + \Phi'(u)\,\varphi \right) dtdx \\ &\qquad - \int_0^L u_1(x)\varphi(0,x)dx + \int_{\mathbb{R}} u_0(x)\partial_t\varphi(0,x)dx = 0; \end{aligned} \tag{3.1}$$

(*iv*) *(energy dissipation) for almost every $t > 0$*

$$\begin{aligned} &\int_0^L \left(\frac{(\partial_t u(t,x))^2 + (\partial_x u(t,x))^2}{2} + \Phi(u(t,x)) \right) dx \\ &\qquad \leq \int_0^L \left(\frac{(u_1(x))^2 + (u_0'(x))^2}{2} + \Phi(u_0(x)) \right) dx. \end{aligned} \tag{3.2}$$

3.1. **Existence.** The main result of this subsection is the following.

Theorem 3.1 (**Existence**). *Let u_0 and u_1 be given and assume (**H.1**), (**H.2**). Then* (2.1) *admits a weak solution in the sense of Definition 3.1.*

Our argument is based on the approximation of the Neumann problem (2.1) with a sequence of Neumann problems with smooth source terms and smooth initial data.

Let $\{u_{0,n}\}_{n\in\mathbb{N}}$, $\{u_{1,n}\}_{n\in\mathbb{N}} \subset C^\infty([0,L])$, $\{\Phi_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ be sequences of smooth approximations of u_0, u_1, and Φ such that

$$\begin{aligned}
&u_{0,n} \to u_0 \quad \text{in } H^1(0,L), \quad u_{1,n} \to u_1 \quad \text{in } L^2(0,L), \quad \Phi_n \to \Phi \quad \text{uniformly in } \mathbb{R},\\
&\Phi_n' \to \Phi' \quad \text{pointwise in } \mathbb{R} \text{ and uniformly in } \mathbb{R}\setminus\{(-1-\varepsilon,-1+\varepsilon)\cup(1-\varepsilon,1+\varepsilon)\} \text{ for every } \varepsilon,\\
&|u| \geq 1+\varepsilon \Rightarrow \Phi_n'(u) = 0, \qquad \varepsilon > 0,\ n\in\mathbb{N},\\
&\|u_{0,n}\|_{H^1(0,L)} \leq C, \quad \|u_{1,n}\|_{L^2(0,L)} \leq C, \quad 0 \leq \Phi_n,\ \Phi_n' \leq C, \qquad n\in\mathbb{N},\\
&u_{0,n}'(0) = u_{0,n}'(L) = u_{1,n}(0) = u_{1,n}(L) = 0, \qquad n\in\mathbb{N},
\end{aligned} \tag{3.3}$$

where $C > 0$ denotes some constant independent on n.

Let u_n be the unique classical solution of the initial boundary value problem

$$\begin{cases}
\partial_{tt}^2 u_n = \partial_{xx}^2 u_n - \Phi_n'(u_n), & t>0, 0<x<L,\\
\partial_x u_n(t,0) = \partial_x u_n(t,L) = 0, & t>0,\\
u_n(0,x) = u_{0,n}(x), & 0<x<L,\\
\partial_t u_n(0,x) = u_{1,n}(x), & 0<x<L.
\end{cases} \tag{3.4}$$

The well-posedness of (3.4) is guaranteed for short time by the Cauchy-Kowaleskaya Theorem [10]. The solutions are indeed global in time thanks to the following a priori estimates.

Lemma 3.1 (**Energy conservation**). *The function*

$$t \mapsto E_n(t) = \int_0^L \left(\frac{(\partial_t u_n(t,x))^2 + (\partial_x u_n(t,x))^2}{2} + \Phi_n(u_n(t,x)) \right) dx$$

is constant for every n. In particular, $\{\partial_t u_n\}_{n\in\mathbb{N}}$ and $\{\partial_x u_n\}_{n\in\mathbb{N}}$ are bounded in $L^\infty(0,\infty;L^2(0,L))$.

Proof. We have that

$$\begin{aligned}
E_n'(t) =& \frac{d}{dt}\int_0^L \left(\frac{(\partial_t u_n)^2 + (\partial_x u_n)^2}{2} + \Phi_n(u_n) \right) dx\\
=& \int_0^L \left(\partial_t u_n \partial_{tt}^2 u_n + \partial_x u_n \partial_{tx}^2 u_n + \Phi_n'(u_n)\partial_t(u_n) \right) dx\\
=& \int_0^L \partial_t u_n \underbrace{\left(\partial_{tt}^2 u_n - \partial_{xx}^2 u_n + \Phi_n'(u_n) \right)}_{=0} dx = 0.
\end{aligned}$$

□

Lemma 3.2 (**L^2 estimate**). *The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(0,L))$, for every $T > 0$.*

Proof. Since

$$\begin{aligned}
\int_0^L u_n^2(t,x)dx =& \int_0^L \left(u_{0,n}(x) + \int_0^t \partial_s u_n(s,x)ds \right)^2 dx\\
\leq& 2\int_0^L u_{0,n}^2(x)dx + 2\int_0^L \left(\int_0^t |\partial_s u_n(s,x)|ds \right)^2 dx\\
\leq& 2\int_0^L u_{0,n}^2(x)dx + 2t\int_0^t \int_0^L (\partial_s u_n(s,x))^2 dsdx
\end{aligned}$$

$$\leq 2\int_0^L u_{0,n}^2(x)dx + 2t^2 \sup_{s\geq 0}\int_0^L (\partial_s u_n(s,x))^2 dx,$$

the claim follows from Lemma 3.1. □

Lemma 3.3 (L^∞ **estimate).** *The sequence* $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* $L^\infty((0,T)\times(0,L))$*, for every* $T>0$.

Proof. Fix $0<t<T$ and $0<x<L$. Lemmas 3.1 and 3.2 imply that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1(0,L))$. Since $H^1(0,L)\subset L^\infty(0,L)$ we have

$$|u_n(t,x)| \leq \|u_n(t,\cdot)\|_{L^\infty(0,L)} \leq c\,\|u_n(t,\cdot)\|_{H^1(0,L)} \leq c\,\|u_n\|_{L^\infty(0,T;H^1(0,L))},$$

for some constant $c>0$ dependeing only on L. Therefore

$$\|u_n\|_{L^\infty((0,T)\times(0,L))} \leq c\,\|u_n\|_{L^\infty(0,T;H^1(0,L))},$$

that gives the claim. □

Proof of Theorem 3.1. Thanks to Lemmas 3.1, 3.2 and [12, Theorem 5] there exists a function u satisfying (i) and (ii) of Definition 3.1 such that, passing to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H^1((0,T)\times(0,L)), \text{ for each } T\geq 0,\\ &u_n \to u \quad \text{in } L^\infty((0,T)\times(0,L)), \text{ for each } T\geq 0. \end{aligned} \tag{3.5}$$

We have to verify that u is a weak solution of (2.1). Let $\varphi\in C^\infty(\mathbb{R}^2)$ be a test function with compact support. From (3.4), for every n we have

$$\begin{aligned} \int_0^\infty\int_0^L &\left(u_n\partial_{tt}^2\varphi + \partial_x u_n\partial_x\varphi + \Phi_n'(u_n)\varphi\right)dtdx\\ &-\int_0^L u_{1,n}(x)\varphi(0,x)dx + \int_{\mathbb{R}} u_{0,n}(x)\partial_t\varphi(0,x)dx = 0. \end{aligned}$$

As $n\to\infty$, using (3.3) and (3.5), we get (3.1).

Finally, (3.2) follows from Lemma 3.1, (3.3), and (3.5). □

3.2. **Non-uniqueness.** The dissipative solutions of (2.1) are not unique. This is made clear form the following three examples. In the first example, we show that different regularizations of the discontinuous nonlinear term Φ' may lead to different dissipative solutions of (2.1). In the second example, we use only one regularization of Φ' and approximate the initial conditions in two different ways. Lastly, the third example shows that the solutions of (2.1) do not continuously depend on the initial data. Moreover, it seem quite difficult to identify a common asymptotic behavior as $t\to\infty$.

In all the following examples we assume that Φ is the one defined in (2.2).

Example 3.1. *Let* $\varepsilon>0$*. Consider the functions*

$$\widetilde{\Phi}_\varepsilon(u) = \begin{cases} u^2, & \textit{if } |u|\leq 1-\varepsilon,\\ \frac{2u-u^2}{\varepsilon} - (1-\varepsilon)\left(\varepsilon+\frac{1}{\varepsilon}\right), & \textit{if } 1-\varepsilon\leq u\leq 1,\\ -\frac{2u+u^2}{\varepsilon} - (1-\varepsilon)\left(\varepsilon+\frac{1}{\varepsilon}\right), & \textit{if } -1\leq u\leq -1+\varepsilon,\\ 1+\varepsilon^2-\varepsilon, & \textit{if } |u|\geq 1, \end{cases}$$

$$\overline{\Phi}_\varepsilon(u) = \begin{cases} u^2, & \textit{if } |u|\leq 1,\\ \frac{2(1+\varepsilon)u-u^2}{\varepsilon} - \left(1+\frac{1}{\varepsilon}\right), & \textit{if } 1\leq u\leq 1+\varepsilon,\\ -\frac{2(1+\varepsilon)u+u^2}{\varepsilon} - \left(1+\frac{1}{\varepsilon}\right), & \textit{if } -1-\varepsilon\leq u\leq -1,\\ 1+\varepsilon, & \textit{if } |u|\geq 1+\varepsilon. \end{cases}$$

We have

$$\widetilde{\Phi}'_\varepsilon(u) = \begin{cases} 2u, & \text{if } |u| \le 1-\varepsilon, \\ 2\frac{1-u}{\varepsilon}, & \text{if } 1-\varepsilon \le u \le 1, \\ -2\frac{1+u}{\varepsilon}, & \text{if } -1 \le u \le -1+\varepsilon, \\ 0, & \text{if } |u| \ge 1, \end{cases}$$

$$\overline{\Phi}'_\varepsilon(u) = \begin{cases} 2u, & \text{if } |u| \le 1, \\ 2\frac{1+\varepsilon-u}{\varepsilon}, & \text{if } 1 \le u \le 1+\varepsilon, \\ -2\frac{1+\varepsilon+u}{\varepsilon}, & \text{if } -1-\varepsilon \le u \le -1, \\ 0, & \text{if } |u| \ge 1+\varepsilon. \end{cases}$$

The functions

$$\widetilde{u}_\varepsilon(t,x) = 1, \qquad \overline{u}_\varepsilon(t,x) = \cos\left(\sqrt{2}\,t\right),$$

solve

$$\begin{cases} \partial^2_{tt}\widetilde{u}_\varepsilon = \partial^2_{xx}\widetilde{u}_\varepsilon - \widetilde{\Phi}'_\varepsilon(\widetilde{u}_\varepsilon), & t>0, 0<x<L, \\ \partial_x\widetilde{u}_\varepsilon(t,0) = \partial_x\widetilde{u}_\varepsilon(t,L) = 0, & t>0, \\ \widetilde{u}_\varepsilon(0,x) = 1, & 0<x<L, \\ \partial_t\widetilde{u}_\varepsilon(0,x) = 0, & 0<x<L, \end{cases} \tag{3.6}$$

$$\begin{cases} \partial^2_{tt}\overline{u}_\varepsilon = \partial^2_{xx}\overline{u}_\varepsilon - \overline{\Phi}'_\varepsilon(\overline{u}_\varepsilon), & t>0, 0<x<L, \\ \partial_x\overline{u}_\varepsilon(t,0) = \partial_x\overline{u}_\varepsilon(t,L) = 0, & t>0, \\ \overline{u}_\varepsilon(0,x) = 1, & 0<x<L, \\ \partial_t\overline{u}_\varepsilon(0,x) = 0, & 0<x<L. \end{cases} \tag{3.7}$$

As $\varepsilon \to 0$ we have

$$\widetilde{u}_\varepsilon(t,x) \to \widetilde{u}(t,x) = 1, \qquad \overline{u}_\varepsilon(t,x) \to \overline{u}(t,x) = \cos\left(\sqrt{2}\,t\right),$$

and $\widetilde{u}$ and $\overline{u}$ provide two different solutions of (2.1) in correspondence of the initial data

$$u_0(x) = 1, \qquad u_1(x) = 0.$$

The energies associated to (3.6) and (3.7) are

$$\widetilde{E}_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t\widetilde{u}_\varepsilon(t,x))^2 + (\partial_x\widetilde{u}_\varepsilon(t,x))^2}{2} + \widetilde{\Phi}_\varepsilon(\widetilde{u}_\varepsilon(t,x)) \right) dx = (1+\varepsilon^2-\varepsilon)L,$$
$$\overline{E}_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t\overline{u}_\varepsilon(t,x))^2 + (\partial_x\overline{u}_\varepsilon(t,x))^2}{2} + \overline{\Phi}_\varepsilon(\overline{u}_\varepsilon(t,x)) \right) dx = L,$$

respectively.

Example 3.2. *Let $\varepsilon > 0$. Consider the function*

$$\Phi_\varepsilon(u) = \begin{cases} \frac{2-\varepsilon}{2}u^2, & \text{if } |u| \le 1, \\ \frac{2-\varepsilon}{\varepsilon}\left((1+\varepsilon)\left(u-\frac{1}{2}\right) - \frac{u^2}{2}\right), & \text{if } 1 \le u \le 1+\varepsilon, \\ \frac{\varepsilon-2}{\varepsilon}\left((1+\varepsilon)\left(u+\frac{1}{2}\right) + \frac{u^2}{2}\right), & \text{if } -1-\varepsilon \le u \le -1, \\ \frac{(2-\varepsilon)(1+\varepsilon)}{2}, & \text{if } |u| \ge 1+\varepsilon. \end{cases}$$

We have

$$\Phi'_\varepsilon(u) = \begin{cases} (2-\varepsilon)u, & \text{if } |u| \le 1, \\ \frac{2-\varepsilon}{\varepsilon}(1+\varepsilon-u), & \text{if } 1 \le u \le 1+\varepsilon, \\ \frac{\varepsilon-2}{\varepsilon}(1+\varepsilon+u), & \text{if } -1-\varepsilon \le u \le -1, \\ 0, & \text{if } |u| \ge 1+\varepsilon. \end{cases}$$

The functions

$$u_\varepsilon(t,x) = (1-\varepsilon)\cos\left(\sqrt{2-\varepsilon}\, t\right), \qquad v_\varepsilon(t,x) = 1+\varepsilon$$

solve

$$\begin{cases} \partial^2_{tt}u_\varepsilon = \partial^2_{xx}u_\varepsilon - \Phi'_\varepsilon(u_\varepsilon), & t>0, 0<x<L,\\ \partial_x u_\varepsilon(t,0) = \partial_x u_\varepsilon(t,L) = 0, & t>0,\\ u_\varepsilon(0,x) = 1-\varepsilon, & 0<x<L,\\ \partial_t u_\varepsilon(0,x) = 0, & 0<x<L, \end{cases} \tag{3.8}$$

$$\begin{cases} \partial^2_{tt}v_\varepsilon = \partial^2_{xx}v_\varepsilon - \Phi'_\varepsilon(v_\varepsilon), & t>0, 0<x<L,\\ \partial_x v_\varepsilon(t,0) = \partial_x v_\varepsilon(t,L) = 0, & t>0,\\ v_\varepsilon(0,x) = 1+\varepsilon, & 0<x<L,\\ \partial_t v_\varepsilon(0,x) = 0, & 0<x<L. \end{cases} \tag{3.9}$$

As $\varepsilon \to 0$ we have

$$u_\varepsilon(t,x) \to u(t,x) = \cos\left(\sqrt{2}\, t\right), \qquad v_\varepsilon(t,x) \to v(t,x) = 1,$$

and u and v provides two different solutions of (2.1) in correspondence of the initial data

$$u_0(x) = 1, \qquad u_1(x) = 0.$$

The energies associated to (3.8) and (3.9) are

$$E_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t u_\varepsilon(t,x))^2 + (\partial_x u_\varepsilon(t,x))^2}{2} + \Phi_\varepsilon(u_\varepsilon(t,x))\right) dx = \frac{(2-\varepsilon)(1-\varepsilon)^2}{2} L,$$
$$\mathcal{E}_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t v_\varepsilon(t,x))^2 + (\partial_x v_\varepsilon(t,x))^2}{2} + \Phi_\varepsilon(v_\varepsilon(t,x))\right) dx = \frac{(2-\varepsilon)(1+\varepsilon)}{2} L,$$

respectively.

Example 3.3. *For every $\varepsilon > 0$, the solutions u_ε and v_ε of the two following problems*

$$\begin{cases} \partial^2_{tt}u_\varepsilon = \partial^2_{xx}u_\varepsilon - \Phi'(u_\varepsilon), & t>0, 0<x<L,\\ \partial_x u_\varepsilon(t,0) = \partial_x u_\varepsilon(t,L) = 0, & t>0,\\ u_\varepsilon(0,x) = 1+\varepsilon, & 0<x<L,\\ \partial_t u_\varepsilon(0,x) = \varepsilon, & 0<x<L, \end{cases} \tag{3.10}$$

$$\begin{cases} \partial^2_{tt}v_\varepsilon = \partial^2_{xx}v_\varepsilon - \Phi'(v_\varepsilon), & t>0, 0<x<L,\\ \partial_x v_\varepsilon(t,0) = \partial_x v_\varepsilon(t,L) = 0, & t>0,\\ v_\varepsilon(0,x) = 1-\varepsilon, & 0<x<L,\\ \partial_t v_\varepsilon(0,x) = 0, & 0<x<L, \end{cases} \tag{3.11}$$

are

$$u_\varepsilon(t,x) = \varepsilon t + 1 + \varepsilon, \qquad v_\varepsilon(t,x) = (1-\varepsilon)\cos(\sqrt{2}t).$$

We have

$$\|u_\varepsilon(0,\cdot) - v_\varepsilon(0,\cdot)\|_{L^2(0,L)} + \|\partial_t u_\varepsilon(0,\cdot) - \partial_t v_\varepsilon(0,\cdot)\|_{L^2(0,L)} = 3\varepsilon\sqrt{L},$$
$$\lim_{t\to\infty} u_\varepsilon(t,x) = \infty, \qquad \limsup_{t\to\infty} v_\varepsilon(t,x) = 1-\varepsilon.$$

Moreover, as $\varepsilon \to 0$,

$$u_\varepsilon(t,x) \to 1, \qquad v_\varepsilon(t,x) \to \cos(\sqrt{2}t).$$

The energies associated to (3.10) and (3.11) are

$$E_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t u_\varepsilon(t,x))^2 + (\partial_x u_\varepsilon(t,x))^2}{2} + \Phi(u_\varepsilon(t,x))\right) dx = \frac{\varepsilon^2+2}{2} L,$$

$$\mathcal{E}_\varepsilon(t) = \int_0^L \left(\frac{(\partial_t v_\varepsilon(t,x))^2 + (\partial_x v_\varepsilon(t,x))^2}{2} + \Phi(v_\varepsilon(t,x)) \right) dx = (1-\varepsilon)^2 L,$$

respectively.

3.3. **Regularity.** This subsection is devoted to the maximal regularity we can expect for the dissipative solutions of (2.1). We show that if the t and x derivative of the solutions at time $t = 0$ are bounded then we have locally Lipshitz continuous solutions. In the following section, using a first order formulation of (2.1) we will show that we cannot expect more regularity even if we consider more regular initial data.

Theorem 3.2. *Let u_0 and u_1 be given and assume (**H.1**), (**H.2**). If u is a dissipative solution of* (2.1) *and*

$$u_0 \in W^{1,\infty}(0, L), \qquad u_1 \in L^\infty(0, L), \tag{3.12}$$

then

$$u \in C([0, \infty) \times [0, L]) \cap W^{1,\infty}((0, T) \times (0, L)), \tag{3.13}$$

for every $T > 0$.

Proof. Let u be a solution of (2.1). Consider the function $\widetilde{u} : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ defined as the $2L-$periodic (in space) extension of the function $(t, x) \in [0, \infty) \times [-L, L] \mapsto u(t, |x|)$. $\widetilde{u}$ is the unique solution of the Cauchy Problem

$$\begin{cases} \partial_{tt}^2 v = \partial_{xx}^2 v - \Phi'(\widetilde{u}), & t > 0, x \in \mathbb{R}, \\ v(0, x) = \widetilde{u}_0(x), & x \in \mathbb{R}, \\ \partial_t v(0, x) = \widetilde{u}_1(x), & x \in \mathbb{R}, \end{cases} \tag{3.14}$$

where $\widetilde{u}_0$ and $\widetilde{u}_1$ are the $2L-$periodic extensions of the functions $x \in [-L, L] \mapsto u_0(|x|)$ and $x \in [-L, L] \mapsto u_1(|x|)$, respectively. Therefore, the following representation formula holds

$$\widetilde{u}(t, x) = \frac{\widetilde{u}_0(x + t) + \widetilde{u}_0(x - t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{u}_1(s)ds + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \Phi'(\widetilde{u}(s, y))dsdy. \tag{3.15}$$

We have that

$$\begin{aligned}
&|\widetilde{u}(t, x) - \widetilde{u}(t', x')| \\
&\quad= \frac{|\widetilde{u}_0(x + t) - \widetilde{u}_0(x' + t')| + |\widetilde{u}_0(x - t) - \widetilde{u}_0(x' - t')|}{2} \\
&\qquad+ \frac{1}{2}\left| \int_{x-t}^{x+t} \widetilde{u}_1(s)ds - \int_{x'-t}^{x'+t} \widetilde{u}_1(s)ds \right| + \frac{1}{2}\left| \int_{x'-t}^{x'+t} \widetilde{u}_1(s)ds - \int_{x'-t'}^{x'+t'} \widetilde{u}_1(s)ds \right| \\
&\qquad+ \frac{1}{2}\left| \int_0^t \int_{x-(t-s)}^{x+(t-s)} \Phi'(\widetilde{u}(s, y))dsdy - \int_0^{t'} \int_{x-(t-s)}^{x+(t-s)} \Phi'(\widetilde{u}(s, y))dsdy \right| \\
&\qquad+ \frac{1}{2}\left| \int_0^{t'} \int_{x-(t-s)}^{x+(t-s)} \Phi'(\widetilde{u}(s, y))dsdy - \int_0^{t'} \int_{x-(t'-s)}^{x+(t'-s)} \Phi'(\widetilde{u}(s, y))dsdy \right| \\
&\qquad+ \frac{1}{2}\left| \int_0^{t'} \int_{x-(t'-s)}^{x+(t'-s)} \Phi'(\widetilde{u}(s, y))dsdy - \int_0^{t'} \int_{x'-(t'-s)}^{x'+(t'-s)} \Phi'(\widetilde{u}(s, y))dsdy \right| \\
&\quad\le \left(\|\widetilde{u}_0'\|_{L^\infty(\mathbb{R})} + \frac{\|\widetilde{u}_1\|_{L^\infty(\mathbb{R})}}{2} + \frac{3}{2}\|\Phi'\|_{L^\infty(\mathbb{R})}(t + t') \right) (|x - x'| + |t - t'|).
\end{aligned}$$

Thanks to (3.12) we have

$$\widetilde{u} \in C([0, \infty) \times \mathbb{R}) \cap W^{1,\infty}((0, T) \times \mathbb{R}), \qquad T > 0, \tag{3.16}$$

and then (3.13). $\square$

The following simple example shows that we cannot expect C^2 regularity on the solutions. More precisely, we start with constant initial data and we explicitly construct conservative solutions exhibiting a singularity in the second derivative. The insurgence of such singularity is due to the lack of continuity of the nonlinear source Φ'.

Example 3.4. *Consider the function*

$$u(t,x)=\begin{cases}\sqrt{2}\sin(\sqrt{2}t), & \text{if } 0\le t\le \frac{\pi}{4\sqrt{2}},\\ \sqrt{2}t+1-\frac{\pi}{4}, & \text{if } t\ge \frac{\pi}{4\sqrt{2}}.\end{cases} \tag{3.17}$$

Clearly, u solves the problem

$$\begin{cases}\partial_{tt}^2 u=\partial_{xx}^2 u-\Phi'(u), & t>0, x\in\mathbb{R},\\ u(0,x)=0, & x\in\mathbb{R},\\ \partial_t u(0,x)=2, & x\in\mathbb{R},\end{cases}$$

but

$$u\in C^1\setminus C^2.$$

Indeed

$$\begin{aligned}\lim_{t\to\frac{\pi}{4\sqrt{2}}^-} u(t,x)&=1, & \lim_{t\to\frac{\pi}{4\sqrt{2}}^+} u(t,x)&=1,\\ \lim_{t\to\frac{\pi}{4\sqrt{2}}^-} \partial_t u(t,x)&=\sqrt{2}, & \lim_{t\to\frac{\pi}{4\sqrt{2}}^+} \partial_t u(t,x)&=\sqrt{2},\\ \lim_{t\to\frac{\pi}{4\sqrt{2}}^-} \partial_{tt}^2 u(t,x)&=-2, & \lim_{t\to\frac{\pi}{4\sqrt{2}}^+} \partial_{tt}^2 u(t,x)&=0.\end{aligned}$$

The energy associated to (3.17) *is*

$$E(t)=\int_0^L\left(\frac{(\partial_t u_\varepsilon(t,x))^2+(\partial_x u_\varepsilon(t,x))^2}{2}+\Phi(u_\varepsilon(t,x))\right)dx=2L.$$

4. Discontinuities, debonding and propagation of singularities

In this section we shall focus on some qualitative analysis aimed to investigate the occurence of singularities in the solutions of (2.1) and the interplay of such singularities with debonding process. Based on a first order system associated to (2.1), we give a qualitative description of the discontinuity curves of the first derivatives of the solutions. These are the loci where the dissipation of energy occurs. Moreover, we show that we cannot expect more regularity even if we consider more regular initial data.

We can rewrite the equation in (2.1) as a first order system in the following way

$$\partial_t Z+A\partial_x Z=B(Z), \tag{4.1}$$

where

$$Z=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}\partial_t u\\ \partial_x u\\ u\end{pmatrix},\qquad A=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix},\qquad B(Z)=\begin{pmatrix}-\Phi'(z_3)\\0\\z_1\end{pmatrix}.$$

Since $Z\mapsto B(Z)$ is discontinuous the solution Z of (4.1) may develop discontinuities. Let $t\mapsto (t,\gamma(t))$ be a discontinuity curve for Z. Thanks to the qualitative analysis of [2, Chapter 10] γ is locally Lipschitz continuous and the Rankine-Hugoniot condition [2, Section 4.2] holds

$$A\left(Z(t,\gamma(t)^+)-Z(t,\gamma(t)^-)\right)=\gamma'(t)\left(Z(t,\gamma(t)^+)-Z(t,\gamma(t)^-)\right),\qquad \text{a.e. } t,$$

where

$$Z(t,\gamma(t)^\pm)=\lim_{s\to\gamma(t)^\pm} Z(t,s).$$

Since the eigenvalues of the matrix A are -1, 0 and 1, we must have

$$\gamma'(t) \in \{-1, 0, 1\}, \qquad \text{a.e. } t,$$

namely $t \mapsto (t, \gamma(t))$ is a polygonal of the plane (t, x) with slopes -1, 0 and 1.

Several remarks are needed. In addition to the propagation velocities 1, -1 of the wave equation here we have one more characteristic speed. This feature is coherent with the one obtained in [1]. There the appearance of the stationary characteristics was generated by a third order hyperbolic operator and a smooth nonlinear source term $f(u)$ in one spatial dimension. In [11] the authors completed the picture showing that if the operator is of the second order and the nonlinear source term $f(u)$ is smooth we can only have two characteristic speeds. Here, we are able to obtain the third characteristic speed even with a second order wave operator because our nonlinear source term $\Phi'(u)$ is discontinuous.

System (4.1) admits the following entropy/entropy flux pair

$$\eta(Z) = \frac{|Z|^2}{2}, \qquad q(Z) = -z_1 z_2, \qquad Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{R}^3. \tag{4.2}$$

Coherently with Definition 3.1, the solutions of (4.1) satisfy the following entropy inequality

$$\partial_t \eta(Z) + \partial_x q(Z) \le \eta'(Z) B(Z), \tag{4.3}$$

in the sense of distributions. When a shock occurs the inequality in (4.3) becomes strict. Indeed, we consider dissipative solutions [13].

The interplay between the propagation of singularities and debonding is described by the following necessary condition relating the singular points in space-time with the occurrence of attachment-debonding in the characteristic cone.

Theorem 4.1. *Let u be a dissipative solution of* (2.1) *and* $(t_0, x_0) \in (0, \infty) \times (0, L)$. *We have that if u is not C^1 in (t_0, x_0) then for all $\varepsilon > 0$ there exist (t_1, x_1), $(t_2, x_2) \in \mathcal{T}_\varepsilon(t_0, x_0)$ such that $|u(t_1, x_1)| < 1 < |u(t_2, x_2)|$, where*

$$\mathcal{T}_\varepsilon(t_0, x_0) = \bigcup_{\max\{t_0 - \varepsilon, 0\} \le t \le t_0} \Big(\max\{0, x_0 - \varepsilon + (t - t_0)\}, \min\{x_0 + \varepsilon - (t - t_0), L\} \Big).$$

Proof. We argue by contradiction, namely we prove that if there exists $\varepsilon > 0$ such that for all $(t, x) \in \mathcal{T}_\varepsilon(t_0, x_0)$, $|u(t, x)| \le 1$ or for all $(t, x) \in \mathcal{T}_\varepsilon(t_0, x_0)$, $|u(t, x)| \ge 1$ then u is C^1 in (t_0, x_0).

We can always choose ε so small such that

$$t_0 - \varepsilon > 0, \qquad 0 < x_0 - 2\varepsilon < x_0 + 2\varepsilon < L, \tag{4.4}$$

in this way

$$\mathcal{T}_\varepsilon(t_0, x_0) = \bigcup_{t_0 - \varepsilon \le t \le t_0} \Big(x_0 - \varepsilon + (t - t_0), x_0 + \varepsilon - (t - t_0) \Big). \tag{4.5}$$

Assume that

$$|u(t, x)| \le 1, \qquad (t, x) \in \mathcal{T}_\varepsilon(t_0, x_0),$$

and consider a C^1 function $\overline{\Phi}$ such that

$$|u| \le 1 \Longrightarrow \overline{\Phi}(u) = \Phi(u).$$

Let $\overline{u}$ be the solution of the Cauchy problem

$$\begin{cases} \partial_{tt}^2 \overline{u} = \partial_{xx}^2 \overline{u} - \overline{\Phi}'(\overline{u}), & t > t_0 - \varepsilon, x \in \mathbb{R}, \\ \overline{u}(t_0 - \varepsilon, x) = u(t_0 - \varepsilon, x) \chi_{[x_0 - 2\varepsilon, x_0 + 2\varepsilon]}(x), & x \in \mathbb{R}, \\ \partial_t \overline{u}(t_0 - \varepsilon, x) = u(t_0 - \varepsilon, x) \chi_{[x_0 - 2\varepsilon, x_0 + 2\varepsilon]}(x), & x \in \mathbb{R}. \end{cases}$$

Due to the finite speed of propagation of the wave operator we have

$$u = \overline{u} \qquad \text{in } \mathcal{T}_\varepsilon(t_0, x_0).$$

Using the first order reformulation (4.1) of the equation for $\overline{u}$ we see that $\overline{u}$ is not developing any singularity in $\mathcal{T}_\varepsilon(t_0, x_0)$.

In the case

$$|u(t,x)| \geq 1, \qquad (t,x) \in \mathcal{T}_\varepsilon(t_0,x_0),$$

we have only to consider a C^1 function $\overline{\Phi}$ such that

$$|u| \geq 1 \Longrightarrow \overline{\Phi}(u) = \Phi(u),$$

and use the same argument. □

4.1. **Qualitative analysis.** By virtue of Theorem 4.1 we have that the debonding-attachment phenomena induces the loss of regularity of the solution u. In order to deepen the comprehension of this behavior it is important to understand which of the two physical quantities $\partial_x u$ and $\partial_t u$ is more affected from the singularity appearance. We remark that the second order derivatives of u cannot be considered due to the Definition 3.1. In order to make the discussion more transparent, we will perform the analysis for short time and compactly supported initial data. Therefore, we can consider the Cauchy problem associated to (4.1) instead of the Neumann one, indeed:

- due to the finite speed of propagation, the Cauchy and Neumann problems share the same solution for short time and compactly supported initial data;
- explicit formulas for the solutions of the Neumann problem involve Fourier series whereas the ones for the Cauchy problem only relies on the more direct D'Alembert formula.

Let us consider the system

$$\begin{cases} \partial_t Z + A\partial_x Z = 0 \\ Z(0,x) = Z_0(x) \end{cases} \tag{4.6}$$

where

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \partial_t u \\ \partial_x u \\ u \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z_0 = \begin{pmatrix} z_{1,0} \\ z_{2,0} \\ z_{3,0} \end{pmatrix} : \mathbb{R} \to \mathbb{R}^3$$

is the vector of the initial conditions. If we diagonalize the matrix A we obtain that $A = P^{-1}DP$, where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

If we define $W = PZ$ and $W_0 = PZ_0$ we gain

$$\begin{cases} \partial_t W + D\partial_x W = 0 \\ W(0,x) = W_0(x) \end{cases} \tag{4.7}$$

that can be easily solved as

$$\begin{cases} w_1(t,x) = w_{1,0}(x-t), \\ w_2(t,x) = w_{2,0}(x), \\ w_3(t,x) = w_{3,0}(x+t). \end{cases} \tag{4.8}$$

Now, since $Z = P^{-1}W$, we have that $Z = S_t Z_0$, where

$$S_t Z_0 := \frac{1}{2}\begin{pmatrix} z_{1,0}(x-t) + z_{3,0}(x-t) + z_{1,0}(x) - z_{3,0}(x) \\ 2z_{2,0}(x+t) \\ z_{1,0}(x-t) + z_{3,0}(x-t) - z_{1,0}(x) + z_{3,0}(x) \end{pmatrix}.$$

If we consider the Cauchy problem

$$\begin{cases} \partial_t Z + A\partial_x Z = B(Z), \\ Z(0,x) = Z_0(x), \end{cases} \tag{4.9}$$

we obtain

$$\begin{aligned} Z(t,x) &= S_t Z_0 + \int_0^t S_{t-s} B(Z(s,x))\, ds \\ &= S_t Z_0 + \frac{1}{2}\int_0^t \begin{pmatrix} -\Phi'(z_3(s,x-(t-s))) + z_1(s,x-(t-s)) - \Phi'(z_3(s,x)) - z_1(s,x) \\ 0 \\ -\Phi'(z_3(s,x-(t-s))) + z_1(s,x-(t-s)) + \Phi'(z_3(s,x)) + z_1(s,x) \end{pmatrix} ds, \end{aligned}$$

where

$$B(Z) = \begin{pmatrix} -\Phi'(z_3) \\ 0 \\ z_1 \end{pmatrix}.$$

The fact that the evolution of $z_2 = \partial_x u$ only involves S_t implies that the new singularities of u, due to debonding-attachment, may occur only in $\partial_t u(t,\cdot)$. From the physical point of view, this means that the stretching $\partial_x u$ is not sensitive to debonding-attachment phenomena.

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