Essays on optimal stopping in discrete and continuous time

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Stop! In the name of love.
– *Diana Ross*

Abstract

This thesis deals with optimal stopping of Markov processes in discrete and continuous time. In the first part we study discrete time random walks with infinite time horizon. Here, a main problem of interest and guiding example is the classical Chow-Robbins game, also known as the S_n/n-problem. We derive tight upper and lower bounds for the value function of the Chow-Robbins game and related problems and use these to approximate value function and continuation set of the problems. For the Chow-Robbins game we find all integer values in the continuation set for times $n \leq 489.241$.

Starting from there we analyze analytic properties of value function and stopping boundary. We show that the value function of the Chow-Robbins game is non-smooth on a dense subset of the continuation set. We also show that the continuation set is non-convex and find numeric evidence that the stopping boundary is not-smooth either. Similar results hold for discrete stopping problems in a fairly general setting.

The second part treats continuous time stopping problems with finite time horizon, namely problems with a Brownian motion as a driving process. For these we derive a new class of Fredholm-type integral equations for the stopping set. For large problem classes of interest, we show by analytical arguments that the equation uniquely characterizes the stopping boundary of the problem. We then use the integral equations to rigorously find the limit behavior of the stopping boundary close to the terminal time. We show that the leading-order coefficient is universal for wide classes of problems. We also use the representation for numerical purposes.

Zusammenfassung

Gegenstand dieser Arbeit sind Probleme des optimalen Stoppens von Markovprozessen in diskreter und stetiger Zeit. Der erste Teil befasst sich mit zeitdiskreten Random Walks mit unendlichem Zeithorizont. Zentrales Problem sowie illustratives Beispiel ist das klassische Chow-Robbins Spiel, auch bekannt als S_n/n-Problem. Wir entwickeln enge obere und untere Schranken für die Wertfunktion des S_n/n-Problems sowie verwandter Probleme. Diese nutzen wir, um Wertfunktion und Fortsetzungsgebiet des Chow-Robbins Spiels zu berechnen. Hierdurch finden wir alle ganzzahligen Punkte im Fortsetzungsgebiet für Zeiten $n \leq 489.241$.

Von hier aus untersuchen wir analytische Eigenschaften von Wertfunktion und Stoppgrenze. Wir zeigen, dass die Wertfunktion des Chow-Robbins Spiels auf einer dichten Teilmenge des Fortsetzungsgebiets nicht differenzierbar, und das Fortsetzungsgebiet nicht konvex ist. Diese Eigenschaften gelten unter relativ schwachen Voraussetzungen für allgemeine diskrete Stoppprobleme.

Der zweite Teil setzt sich mit optimalem Stoppen Brownscher Bewegungen mit endlichem Zeithorizont auseinander. Wir entwickeln eine neue Klasse von Integralgleichungen des Fredholm Typs für das Fortsetzungsgebiet. Für große Klassen von Stoppproblemen zeigen wir analytisch, dass die Stoppgrenze durch die Integralgleichungen eindeutig bestimmt wird. Des Weiteren nutzen wir die Integralgleichungen, um das Grenzverhalten der Stoppgrenze nahe dem Zeithorizont zu analysieren. Wir arbeiten heraus, dass es ein universelles Grenzverhalten für eine breite Klasse von Problemen gibt.

Contents

Chapter 1

Introduction

Repeatedly toss a fair coin and count the number of tosses and the number of *heads* and stop when the proportion of *heads* is as high as possible. What is a good strategy to maximize the expectation of this proportion? Some parts of an optimal strategy can be seen easily, e.g., if you get *heads* in the first toss, you stop since it can not get any better. If you get *tails* in the first toss, you better continue. The law of large numbers even tells us that it is never optimal to stop as long as the proportion of *heads* is lower than 1/2. Beyond that it is surprisingly difficult to find the optimal strategy, and the problem remains partly unsolved until today. This game was introduced by Yuan-Shih Chow and Herbert Robbins in 1965 as an example of an optimal stopping problem. In general, optimal stopping problems can be described as the task:

"Given a random process, find the best time to stop the process to maximize some payoff at the stopping time."

Stopping problems occur in all kinds of applications. Some examples are: clinical trials (find the right time to declare a drug effective or ineffective), epidemiology (detect when the rate of infections of a certain disease changes), mathematical finance (pricing of American-type options), as well as – idealized – everyday problems (where to park a car, when to slow down a bike). On the other hand, stopping problems are of theoretical mathematical interest. Stopping problems are used to solve other problems in probability, such as stochastic control problems, stochastic games/ Dynkin games and to derive maximal inequalities in stochastic analysis. The theory also has rich connections to other mathematical fields, such as differential equations, especially free boundary problems, potential theory, game theory and (convex) optimization.

Ideas on optimal stopping can be dated back to the 1870s where Arthur Cayley invented some *lottery problems* that have similarities to the famous *secretary problem*, see [Cay75]. First rigid formulations in the language of probability theory are from the 1940s. Abraham Wald – a German-speaking Jew from Romania, who fled Austria after the anschluss to Germany in 1938 and fellow mathematicians in the *Statistical Research Group* at Columbia University – developed some optimal stopping tools during world war II for the US Air Force, see [Wal80]. Wald then continued to develop the theory of sequential analysis. It treats questions like: We observe a random process and need to decide whether a hypothesis h_0 is true or not. When is the best time to stop the observation and decide. Here, we want to minimize the probability of a wrong decision and the time of observation simultaneously. The basis of the theory in discrete time is collected in [Wal47]. Further notable publications are [WW48] and [WW50]. The theory was later developed for continuous processes by Herman Chernoff and others. A good overview on the contributions of Chernoff to the theory of optimal stopping and some historic accounts on sequential analysis and optimal stopping can be found in [LL05]. First notions of general (discrete) optimal stopping problems date back to J. Laurie Snell in 1952 [Sne52]. Snell showed that the value of those problems can be characterized via the smallest dominating supermartingale, the *Snell envelope*. Techniques using this *supermartingale characterization* are nowadays referred to as *martingale approaches*. Eugene Dynkin showed in 1963 [Dyn63] that, if the driving process is a Markov process, then the *supermartingale characterization* translates to finding the smallest superharmonic function that dominates the payoff function. Using this fact is often referred to as the *Markovian approach*.

Further interest in the theory of optimal stopping originated from mathematical finance. Here, financial markets are modeled as stochastic processes and the pricing of options of so-called *American type* is closely related to finding optimal stopping times. Early works are from Henry P. McKean in 1965 [McK65]. A broader interest in these problems, however, emerged with the work of Black, Scholes and Merton in the 1970s, see [BS73] and [Mer73].

An overview on the earlier development in optimal stopping in discrete time with an emphasis on the martingale approach can be found in [CRS71]. The Markovian approach – in discrete and continuous time – is portrayed in [Shi78]. A more recent monograph on the topic is [PS06].

1.1 Contribution of this thesis

This thesis deals with optimal stopping and different methods of solution and analysis of optimal stopping problems. The contributions can be divided into two only loosely connected parts – optimal stopping in discrete time and optimal stopping in continuous time.

The part on problems in discrete time takes the Chow-Robbins game as a starting point. For the Chow-Robbins game we develop tight upper and lower bounds for the value function V of the problem. These are used to numerically solve the problem. The method is then generalized to random walks with subgaussian increments. We also consider some different payoff functions. These findings are going to be published in [CF22]. An early stage of some parts of this research have already been part of my master thesis "Zum S_n/n Problem" (see [Fis19]) but have since been developed further.

A related field of research is the analysis of the smoothness of discrete time value functions. Here, we let the starting point vary continuously. We show that the value function of the Chow-Robbins game – and under some conditions, the value function of discrete time stopping problems in general – is not smooth in the interior of the continuation set C. Even more, under some additional conditions we show that V is not differentiable on a dense subset of C. We give evidence that ∂C is not smooth and that C is not convex, in the Chow-Robbins game and other examples. This was published in [CF20].

The other part focuses on stopping problems in continuous time. We look at optimal stopping of a Brownian motion with finite time horizon. For these we develop a new class of Fredholm type integral equations for the stopping set. For large problem classes of interest, we show by analytical arguments that the equation uniquely characterizes the stopping boundary of the problem. Regardless of the uniqueness, we use the representation to rigorously find the limit behavior of the stopping boundary close to the terminal time and show that the leading-order coefficient is universal for wide classes of problems. We also discuss how the representation can be used for numerical purposes. These findings are submitted for publication. A preprint can be found in [CF21].

1.2 Structure of this thesis

Chapter 2 and Chapter 3 deal with discrete time stopping problems. Their content is closely related. We, however, divided the content into two chapters to display the structure of initial research and publication.

Chapter 2 treats the Chow-Robbins game and related stopping problems. In Section 2.2 the Chow-Robbins game is described and the state of research on the topic is summarized. Section 2.3 contains the main theoretical results on the problem. We develop upper and lower bounds by connecting the problem to time continuous analogue problems. We also show some direct applications, such as a new short proof for the existence of an optimal stopping time. Furthermore, we derive some generalizations and an asymptotic approximation of the errors of the bounds. In Section 2.4 these bounds are used to numerically compute the value function and the stopping boundary of the problem.

Chapter 3 analyzes the smoothness – or lack thereof – of the value function V of discrete stopping problems where we allow the starting point to vary continuously. The main result is that V is typically non-smooth on a dense subset of the continuation set. As an example we again consider the Chow-Robbins game. This is set up in Section 3.2. Section 3.3 contains the main results, and Section 3.4 discusses some further examples.

In Chapter 4 continuous stopping problems are analyzed, namely stopping of a Brownian motion with finite time horizon. In Section 4.3 we develop a Fredhom-type integral equation which is shown to hold in a fairly general setting. In Section 4.4 we use this Fredholm representation to analyze the limit behavior of the stopping boundary close to the time horizon. We show that a wide range of problems have a common limit behavior. In Section 4.5 we show that – at least for a subclass of problems – the Fredholm representation defines the continuation set uniquely. This is done using analytical arguments connected to the question of identifiability of (continuous) mixtures of random variables. Section 4.6 explores a numerical application of the Fredholm representation. Some of the proofs are stated in the Appendix 4.7. In Appendix 4.8 we review the construction of the Volterra-type integral representations for the continuation set, that our Fredholm representation is based on.

In Chapter 5 we briefly discuss open questions and possible further research as well as ideas on possible generalizations of our results.

1.3 Notation

Throughout this thesis we study stopping problems from the Markovian point of view, so we use Markovian notation accordingly. With $P_{(t,\mathbf{x})}(\cdot) = P(\cdot \mid X_t = \mathbf{x})$ we denote the probability measure for a process X started at time t in $\mathbf{x}$. With $\mathrm{E}_{(t,\mathbf{x})}$ we denote the corresponding expectation. We consider stopping problems of the form

$$V(t, \mathbf{x}) = \sup_{0 \geq \tau \geq t} \mathrm{E}_{(t,\mathbf{x})}[g(\tau, X_\tau)],$$

where V denotes the value functions and g the payoff function. We denote by $S = \{V = g\} = \{(t, \mathbf{x}) \mid V(t, \mathbf{x}) = g(t, \mathbf{x})\}$ stopping set and by $C = \{V > g\}$ the continuation set of the problem. We are often interested in the continuation set for a fixed time t, so we set $C_t := \{\mathbf{x} \in \mathbb{R}^n \mid (t, \mathbf{x}) \in C\}$ and define S_t accordingly. We denote the first entrance time to the stopping set

S by $\tau^* = \tau_S := \inf\{s \geq t \mid W_s \in S\}$. The boundary of C we denote with ∂C, for one-sided problems, where ∂C is the graph of a function, we call this function the *stopping boundary* and denote it by $b : \mathbb{R}^- \to \mathbb{R}$. If b is invertible, we set $d := b^{-1}$, for multidimensional problems we parametrize – if possible – C via $d : \mathbb{R}^n \to \mathbb{R}^-$ in the first place, i.e., $d(\mathbf{x}) := \sup\{t \leq 0 \mid (t, \mathbf{x}) \in C\}$.

We sometimes want to compare the same stopping problems for different driving processes. If the driving process is not clear from the context, we denote it by a superscript, e.g.,

$$V^W(t, \mathbf{x}) = \sup_{\tau \geq t} \mathrm{E}_{(t,\mathbf{x})}[g(\tau, W_\tau)],$$

$C^W = \{V^W = g\}$ etc. Upper and lower bounds are denoted by subscripts, e.g., V_u for an upper bound and V_l for a lower bound of V. Values in $\mathbb{R}^n$ we set in boldface if $n > 1$ (e.g. $\mathbf{x}$) and in lightface if $n = 1$ (e.g. x). The appendix contains a list of symbols used throughout this thesis.

Chapter 2

Discrete time stopping problems

Most of the research in this chapter will be published in [CF22]. Not in the publication are: Some parts of Section 2.1 and 2.2, Corollary 2 and Proposition 1. The notation has been changed slightly to fit the other chapters.

2.1 Introduction

In this section we will analyze optimal stopping problems of the form

$$V(t,x) = \sup_{\tau \leq t} E_{(t,x)}[g(\tau, X_\tau)], \tag{2.1}$$

where X is a discrete time Markov chain, usually a random walk, g is a payoff function and τ a stopping time. A main problem of interest, and a guiding example for the more general theory developed, is the Chow-Robbins game, also known as the $\frac{S_n}{n}$-problem. It can be described in the following way: We repeatedly toss a fair coin. After each toss we can either take the proportion of *heads* up to now as our reward or continue. As a stopping problem this formulates as

$$V^S(t,x) = \sup_{\tau} \mathrm{E}\left[\frac{x + S_\tau}{t + \tau}\right], \tag{2.2}$$

where S is a random walk.

For these time-dependent problems with infinite time horizon we can usually not hope to find an explicit solution, we will see some reasons for this in Chapter 3. If we, however, consider the same problems with a finite time horizon, the problems become easy – we can simply use backward induction, i.e., use the Wald-Bellman equation

$$V(t, x) = \max\{g(t, x), \mathrm{E}_{(t,x)}[g(t+1, X_{t+1})]\}$$

to successively calculate V from the time horizon backwards. A typical idea to approximate (2.1) is to choose a large number T as an *artificial* time horizon, somehow guess values of $V(T, x)$ and then use backward induction to calculate an approximation of $V(t, x)$ for $t < T$. To make this more rigorous, we can try to find upper and lower bounds for $V(T, x)$ and then successively calculate better upper and lower bounds for $V(t, x)$. So far only relatively weak bounds have been used for the $\frac{S_n}{n}$-problem in the literature, such as the *trivial* lower bound $\max\{0, \frac{x}{t}\}$. The development of tight bounds enables us to get *cheap* numerical approximations of $V(t, x)$ and to get some insight into the structure of the solution. For the Chow-Robbins game and related problems we derive tight upper and lower bounds for $V(T, x)$ in Section 2.3. Interestingly, we can do so by using the analogous continuous time problem – with infinite time horizon – that has a Brownian motion W as a driving process

$$V^W(t, x) = \sup_\tau \mathrm{E}\left[\frac{x + W_\tau}{t + \tau}\right].$$

This problem is explicitly solvable, since we can make use of its symmetries (see [She69]). Its value function V^W is an upper bound for V (see Theorem 1) and its stopping boundary is an upper bound for the stopping boundary of the $\frac{S_n}{n}$-problem. We can interpret the continuous $\frac{W_t}{t}$-problem as a modification of the game. We are allowed to stop at any time, not only discrete times, so we have more *strategies* and hence a higher value. We can also deduce a simple proof for the existence of a solution of the $\frac{S_n}{n}$-problem. We rigorously prove the upper bound for random walks with subgaussian increments in Section 2.3.1 and discuss generalizations in Section 2.3.2. In Section 2.3.3

we use a modification of the procedure for the upper bound to get a lower bound. Heuristically, it works like that: We give the process some upwards drift, making it a submartingale, and lower its starting value until it leads a lower bound for certain stopping times τ. The submartingale property then leads a lower bound for $t \leq \tau$. In Section 2.3.4 we show that these bounds have an asymptotic relative error of order $\mathcal{O}\left(\frac{1}{T}\right)$. We use these bounds to numerically solve the Chow-Robbins game for integer values $n \leq 489.241$, see Section 2.4. We also use the developed numerical methods to calculate V and b for general real values (t, x). Treating V as a function on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and b as a function on $\mathbb{R}_{\geq 0}$ leads to questions about their analytic properties. We can use the continuous problem to approximate the discrete one, and quite often discretization is used to approximate continuous time problems. From an analytical point of view, however, these problems behave quite differently. The question of limit behavior of b for $t \to 0$ is briefly treated in Section 2.4, where we see that b is approximately linear in 0, whereas the boundary of the continuous problem b^W is a square root. The question of smoothness of V and b is analyzed in more detail in Capter 3.

2.2 The $\frac{S_n}{n}$-problem

The Chow-Robbins game was first presented by Yuan-Shih Chow and Herbert Robbins in 1965 [CR65]. As a stopping problem it formulates as

$$V^S(t, x) = \sup_{\tau} \mathrm{E}\left[\frac{x + S_\tau}{t + \tau}\right], \tag{2.3}$$

where S is a random walk. In the classical version S has symmetric Bernoulli increments. It is also possible to take different random walks as driving processes. Chow and Robbins showed that an optimal stopping time exists in the Bernoulli case, later Dvoretzky [Dvo67] proved this for general centered iid. increments with finite variance. But it was (and to some extent still is) difficult to see how that solution looks like. Asymptotic results were given by Shepp in 1969 [She69], who showed that the boundary of the continuation set ∂C can be written as a function $b : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with

$$\lim_{t\to\infty} \frac{b(t)}{\alpha\sqrt{t}} = 1.$$

Here $\alpha \approx 0.8399$ is a constant and $\alpha\sqrt{t}$ is the boundary of the analogous stopping problem for a standard Brownian motion W (see Lemma 1)

$$V^W(t,x) = \sup_\tau \mathrm{E}\left[\frac{x + W_\tau}{t+\tau}\right]. \tag{2.4}$$

In 2007 Lai, Yao and AitSahlia [LLYA07] gave a second order approximation for the limit of b, that is

$$\lim_{t\to\infty} \left(\alpha\sqrt{t} - b(t)\right) = \frac{1}{2}.$$

Lai and Yao [LLY05] also calculated some approximation for values of b by using the value function (2.4), without constructing it as an upper bound. They as well gave some calculations for a random walk with standard normal increments.
A more rigorous computer analysis was given by Häggström and Wästlund in 2013 [HW13]. Using backward induction from a time horizon $T = 10^7$, they calculated lower and upper bounds for V^S. For points, reachable from $(0,0)$, they calculated if they belong to the stopping or to the continuation set and were able to do so for all but 7 points (n,x) with $n \leq 1000$. We will show that these 7 point belong to the stopping set.

Different descriptions of the $\frac{S_n}{n}$-problem There are different descriptions of the Chow-Robbins game that are used in the literature. We briefly discuss how these transform into each other. The most common version is to use S as a symmetric random walk. We use that definition, and it is used by Chow and Robbins [CR65], Shepp [She69] and Lai, Yao and AitSahlia [LLY05]. This version has the advantage that the process is a martingale, and it is more common in the literature on random walks.

Another variant is to define iid. random variables ξ_i with $P(\xi_i = 0) = P(\xi_i = 1) = \frac{1}{2}$ and set

$$S'_n = \sum_{i=1}^{n} \xi_i.$$

Again we want to maximize $\frac{S'_n}{n}$. This version has the advantage that the state space $E = \{(n, x) \in \mathbb{N} \times \mathbb{N} \mid x \leq n\}$ is irreducible and has the nice intuition as the proportion of *heads* in a series of coin tosses. This version is used by Häggström and Wästlund [HW13] and Medina and Zeilenberger [MZ09].

The different versions can be transformed into each other in the following way:

$$S_n = 2S'_n - n, \qquad S'_n = \frac{S_n + n}{2}.$$

The gain function is always the same. For the value function we have

$$\begin{aligned} V^{S'}(n,x) &= \sup_\tau \mathrm{E}\left[\frac{x + S'_\tau}{n+\tau}\right] = \sup_\tau \mathrm{E}\left[\frac{x + \frac{S_\tau + \tau}{2}}{n+\tau}\right] \\ &= \frac{1}{2}\sup_\tau \mathrm{E}\left[\frac{2x + S_\tau + \tau + n - n}{n+\tau}\right] \\ &= \frac{1}{2}V^S(n, 2x-n) + \frac{1}{2} \end{aligned}$$

and

$$V^S(n,x) = 2V^{S'}\left(n, \frac{x+n}{2}\right) - 1.$$

The boundary of the continuation set written as a function $b(n)$ and $b'(n)$ respectively can be transformed via

$$b(n) = 2b'(n) - n, \qquad b'(n) = \frac{b(n) + n}{2}.$$

Häggström and Wästlund use yet another notation. They denote points by $x - a$, where x is the number of *heads* and a is the number of *tails*. This can easily be transformed into S' notation via: $(n, x) = (x + a, x)$.

2.3 Bounds for the $\frac{S_n}{n}$-problem

2.3.1 An upper bound for the value function V^S

We construct an upper bound for the value function V^S of the $\frac{S_n}{n}$-problem, where S can be any random walk with subgaussian increments. The classical Chow-Robbins game is a special case of these stopping problems.

Definition 1 (subgaussian random variable)**.** Let $\sigma^2 > 0$. A real, centered random variable ξ is called σ^2**-subgaussian** (or subgaussian with parameter σ^2), if

$$\mathrm{E}[e^{a\xi}] \leq e^{\frac{\sigma^2 a^2}{2}} \quad \text{for all } a \in \mathbb{R}.$$

Some examples of subgaussian random variables are:

- ξ with $P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2}$ is 1-subgaussian,
- The normal distribution $\mathcal{N}(0, \sigma^2)$ is σ^2-subgaussian,
- The uniform distribution on $[-a, a]$ is a^2-subgaussian,
- Any random variable Y with values in a compact interval $[a, b]$, is $\frac{(b-a)^2}{4}$-subgaussian.

In the following we show that the value function V^W of the continuous time problem (2.4) is an upper bound for the value function V^S, whenever S is a random walk with 1-subgaussian increments. We first state the solution of (2.4).

The solution of the continuous time problem Let

$$h(t, x) := (1 - \alpha^2) \int_0^\infty e^{ax - \frac{a^2}{2}t} \,\mathrm{d}a = (1 - \alpha^2) \frac{1}{t} \frac{\Phi_t(x)}{\varphi_t(x)}, \tag{2.5}$$

where $\Phi_t(x) = \Phi(x/\sqrt{t})$ denotes the cumulative distribution function of a centered normal distribution with variance $\sigma^2 = t$, φ_t the corresponding density function and $\alpha \approx 0.839923675692373$ is the unique solution to

$$\alpha\varphi(\alpha) = (1 - \alpha^2)\Phi(\alpha).$$

Lemma 1. *The stopping problem* (2.4) *is solved by the stopping time*

$$\tau_* = \inf\{s \mid x + W_s \geq \alpha\sqrt{s+t}\}$$

and the value function

$$V^W(t,x) = \begin{cases} h(t,x) & \text{if } x \leq \alpha\sqrt{t}, \\ \frac{x}{t} & \text{else.} \end{cases}$$

The value function $V^W(t,x)$ is differentiable (smooth fit) and $h(t,x) \geq g(t,x) = \frac{x}{t}$ for all $t > 0$ and $x \in \mathbb{R}$.

This result has first been proven independently by Shepp [She69] and Walker [Wal69].

Proof that V^W is an upper bound for V^S We know from general theory that V^S is the smallest superharmonic function dominating the gain function g, see [PS06]. If we find a superharmonic function dominating g we have an upper bound for V^S.

Lemma 2. *Let ξ_i be iid. 1-subgaussian random variables and $S_n = \sum_{i=1}^n \xi_i$. The function*

$$h : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}, (t,x) \mapsto (1-\alpha^2) \int_0^\infty e^{ax - \frac{1}{2}a^2 t} \, \mathrm{d}a$$

is S-superharmonic.

Proof. We first show the claim for fixed $a \in \mathbb{R}$ and

$$f(t,x) = e^{ax - \frac{1}{2}a^2 t}.$$

We need to show that $\mathrm{E}[f(t+1, x+\xi_1)] \leq f(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$.

We calculate

$$\begin{aligned}
& \mathrm{E}\left[e^{a(x+\xi_1)-\frac{a^2}{2}(t+1)}\right] \leq e^{ax-\frac{a^2}{2}t} \\
\iff \quad & e^{ax-\frac{a^2}{2}(t+1)}\,\mathrm{E}\left[e^{a\xi_1}\right] \leq e^{ax-\frac{a^2}{2}t} \\
\iff \quad & e^{-\frac{a^2}{2}}\,\mathrm{E}\left[e^{a\xi_1}\right] \leq 1 \\
\iff \quad & \mathrm{E}\left[e^{a\xi_1}\right] \leq e^{\frac{a^2}{2}}. \qquad (2.6)
\end{aligned}$$

The last inequality (2.6) is just the defining property of a 1-subgaussian random variable.
By integration over a and multiplication with $(1-\alpha^2)$ the result follows. □

Theorem 1 (An upper bound for V^S). *Let W be a standard Brownian motion, S a random walk with 1-subgaussian increments,*

$$V^S(t,x) = \sup_\tau \mathrm{E}\left[\frac{x+S_\tau}{t+\tau}\right], \tag{2.7}$$

and

$$V^W(t,x) = \sup_\tau \mathrm{E}\left[\frac{x+W_\tau}{t+\tau}\right].$$

Then

$$V^W(t,x) \geq V^S(t,x) \text{ for all } t>0,\ x\in\mathbb{R}.$$

Proof. Let $h(t,x) = (1-\alpha^2)\int_0^\infty e^{ax-\frac{a^2}{2}t}\,\mathrm{d}a$ as in Lemma 1. We know that

- $h \geq g$ (Lemma 1),
- h is S-superharmonic (Lemma 2),
- $V^W = h\mathbb{1}_{C^W} + g\mathbb{1}_{S^W}$.

V^S is the smallest superharmonic function dominating g, therefore $V^S \leq h$. We know from Lemma 1 that

$$V^W(t,\alpha\sqrt{t}) = h(t,\alpha\sqrt{t}) = g(t,\alpha\sqrt{t}),$$

and therefore

$$g(t, \alpha\sqrt{t}) \le V^S(t, \alpha\sqrt{t}) \le h(t, \alpha\sqrt{t}) = g(t, \alpha\sqrt{t}).$$

Hence, $V^S(t, \alpha\sqrt{t}) = g(t, \alpha\sqrt{t})$ and $(t, \alpha\sqrt{t}) \in S^S$. The boundary ∂C^S is the graph of a function, therefore $(t, x) \in S^S$ for all $x \ge \alpha\sqrt{t}$. It follows that $C^S \subset C^W$ and that $V^S(t, x) \le V^W(t, x)$, for all $t > 0$, $x \in \mathbb{R}$. □

Corollary 1. *From the proof we see that*

$$C^S \subset C^W,$$

and

$$b(t) \le \alpha\sqrt{t}, \textit{ for all } t > 0.$$

From Theorem 1 we get an easy proof for the existence of an optimal stopping time.

Corollary 2. *The stopping problem* (2.7) *is solvable that is there exists an optimal stopping time that is almost surely finite.*

Proof. The first entry to the stopping set $\tau_{S^S} = \inf\{n \mid x + S_n \in S^S\}$ is an optimal stopping time, if

$$P_{(t,x)}(\tau_{S^S} < \infty) = 1 \text{ for all } (t, x) \in C^S.$$

We define $\tau' := \inf\{n > 0 \mid x + S_n \ge \alpha\sqrt{t+n}\}$. Since $C^S \subset C^W$, we have $\tau_{S^S} \le \tau'$ and it suffices to show that $P_{(t,x)}(\tau' < \infty) = 1$. If the increments ξ_i are a.s. equal 0, then $\tau \equiv 0$ is an optimal stopping time. Otherwise, ξ_i has variance $0 < \sigma^2 \le 1$ and with the law of the iterated logarithm we see that

$$\begin{aligned} & P\left(\limsup_{n\to\infty} \frac{\frac{1}{\sigma^2} S_n}{\sqrt{2n \log\log(n)}} = 1\right) = 1 \\ \Longrightarrow \quad & P\left(x + S_n \ge \alpha\sqrt{t+n} \text{ i.o.}\right) = 1 \\ \Longrightarrow \quad & P_{(t,x)}(\tau' < \infty) = 1. \end{aligned}$$

□

Application to the classical Chow-Robbins game Let $\xi_1, \xi_2 \ldots$ be iid. random variables with $P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2}$ and $S_n = \sum_{i=1}^{n} \xi_i$. The classical Chow-Robbins problem is given by

$$V^S(t,x) = \sup_\tau \mathrm{E}\left[\frac{x + S_\tau}{t + \tau}\right]. \tag{2.8}$$

The ξ_i are 1-subgaussian and have variance 1. By Corollary 2 an a.s. finite stopping time τ_* exists that solves (2.8), see also [Dvo67]. By Theorem 1 we have

$$V^S(t,x) \le V^W(t,x) = h(x,t)\mathbb{1}_{\{x \le \alpha\sqrt{t}\}} + \frac{x}{t}\mathbb{1}_{\{x > \alpha\sqrt{t}\}}.$$

We will see in Section 2.3.4 that this upper bound is very tight. We already show here in an example how it can be used to effectively compute points in the stopping set.

Example 1. For quite some time it was unclear whether it is optimal to stop in $(8,2)$ or not. It was first shown by Medina and Zeilenberger [MZ09] and later confirmed by Häggström and Wästlund [HW13] that $(8,2) \in S^S$.[1] We show how to immediately prove this with our upper bound.

We choose the time horizon $T = 9$, set $V_u^S(T,x) = V^W(T,x)$ and calculate $V_u^S(8,2)$ with one-step backward induction as an upper bound for $V^S(8,2)$. Since $3 > \alpha\sqrt{9}$ we have $(9,3) \in S$ and obtain

$$V_u^S(9,3) = \frac{3}{9} = \frac{1}{3},$$
$$V_u^S(9,1) = h(9,1) \approx 0.1642.$$

We obtain

$$V^S(8,2) \le V_u^S(8,2) = \max\left\{\frac{2}{8}, \frac{V_u^S(9,3) + V_u^S(9,1)}{2}\right\}$$
$$\frac{V_u^S(9,3) + V_u^S(9,1)}{2} = \frac{1}{6} + \frac{0.1642}{2} = 0.2488 < \frac{2}{8}.$$

Hence, we have $V^S(8,2) \le g(8,2) = \frac{2}{8}$ and it follows that $(8,2)$ is in the

[1] In [HW13] this is written as $5-3$, 5 *heads*–3 *tails*.

stopping set.[2]

2.3.2 Generalizations

In the proof of Theorem 1 we did not use the specific form of the gain function $g(t,x) = \frac{x}{t}$. Everything we needed was that:

- The value function of the stopping problem
$$V^W(t,x) = \sup_\tau \mathrm{E}\left[g(t+\tau, x+W_\tau)\right] \tag{2.9}$$
is of the form
$$V^W|_C(t,x) = \int_{\mathbb{R}} e^{ax-\frac{1}{2}a^2t}\,\mathrm{d}\mu(a),$$
for a measure μ.

- The function
$$h(t,x) = \int_{\mathbb{R}} e^{ax-\frac{1}{2}a^2t}\,\mathrm{d}\mu(a)$$
dominates g on $\mathbb{R}_{\geq 0} \times \mathbb{R}$.

- The boundary of the continuation set ∂C^S of the discrete stopping problem
$$V^S(t,x) = \sup_\tau \mathrm{E}\left[g(t+\tau, x+S_\tau)\right]$$
is the graph of a function $b : \mathbb{R}_{\geq 0} \to \mathbb{R}$.
This requirement can easily be relaxed to the symmetric case, where $\partial C^S = \mathrm{Graph}(b) \cup \mathrm{Graph}(-b)$.

- For the proof of Corollary 2 we need that $\mathrm{Var}(S_1) = 1$. We additionally required that ∂C^W is the graph of a function $b^W : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that is asymptotically dominated by the iterated logarithm, i.e., there exists $\varepsilon > 0$ such that
$$\limsup_{t\to\infty} |b^W(t)| - (1-\varepsilon)\sqrt{2t\log\log(t)} \leq 0.$$

[2]For a detailed description of the method see Section 2.4.

These requirements are not very restrictive, and there are many other gain functions and associated stopping problems for which this kind of upper bound can be constructed. A set of examples which fulfill these requirements and have an explicit solution in the continuous case (2.9) can be found in [PS06]. Some of these are:

$$g(t,x) = \frac{x^{2d-1}}{t^q}$$

with $d \in \mathbb{N}$ and $q > d - \frac{1}{2}$,

$$g(t,x) = |x| - \beta\sqrt{t}$$

for some $\beta \geq 0$, and

$$g(t,x) = \frac{|x|}{t}.$$

2.3.3 A lower bound for V^S

In this section we derive a lower bound for the value function of the Chow-Robbins game (2.8). Here S will always be a symmetric Bernoulli random walk. The basis of our construction is the following lemma.

Lemma 3 (A lower bound for V)**.** *Let X be a random walk, g be a gain function, $V(t,x) = \sup_\tau \mathrm{E}\, g(\tau + t, X_\tau + x)$ and $h : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ measurable. For a given point (t_0, x_0) let σ be a stopping time such that the stopped process $\Big(h(t \wedge \sigma + t_0, X_{t\wedge\sigma} + x_0)\Big)_{t\geq 0}$ is a submartingale and*

$$h(\sigma + t_0, X_\sigma + x_0) \leq g(\sigma + t_0, X_\sigma + x_0) \text{ a.s.}$$

Then

$$h(t_0, x_0) \leq V(t_0, x_0).$$

Proof. Since $h(\sigma + t_0, X_\sigma + x_0) \leq g(\sigma + t_0, X_\sigma + x_0)$ a.s. and

$$\int_{\{\sigma > n\}} g(n + t, X_n + x_0)\, \mathrm{d}P \overset{n\to\infty}{\to} 0,$$

we can use the optional sampling theorem and obtain

$$h(t_0, x_0) \le \mathrm{E}[h(\sigma + t_0, X_\sigma + x_0)] \le \mathrm{E}[g(\sigma + t_0, X_\sigma + x_0)] \le V(t_0, x_0).$$

□

We modify the function $h(t,x) = (1-\alpha^2)\int_0^\infty e^{ax-\frac{a^2}{2}t}\,\mathrm{d}a$ from Lemma 1 slightly to

$$h_c(t,x) := K\int_0^\infty e^{ax-\frac{c}{2}a^2t}\,\mathrm{d}a \tag{2.10}$$

for some $0 < c < 1$ and $K > 0$ to match the assumptions of Lemma 3. As a stopping time we choose

$$\tau_0 = \inf\{n \ge 0 \mid x + S_n \ge \alpha\sqrt{t+n} - 1\}. \tag{2.11}$$

Unfortunately, there is no c such that (2.10) is globally S-subharmonic, hence we have to choose c depending on the time horizon T. This makes the following result a bit technical.

Theorem 2 (A lower bound for V^S)**.** *Let*

$$h_c(t,x) := K\int_0^\infty e^{ax-\frac{c}{2}a^2t}\,\mathrm{d}a = K\frac{1}{ct}\frac{\Phi_{ct}(x)}{\varphi_{ct}(x)}$$

with

$$K = \alpha c\frac{\varphi_c(\alpha)}{\Phi_c(\alpha)}.$$

Given a time horizon $T > 0$, let c_1 be the biggest solution smaller than 1 *to*

$$\frac{1}{2}\left(h_c(T+1, \alpha\sqrt{T}-1) + h_c(T+1, \alpha\sqrt{T}+1)\right) = h_c(T, \alpha\sqrt{T}), \tag{2.12}$$

c_2 the unique positive solution of

$$h_c(T, \alpha\sqrt{T}-1) = \frac{\alpha\sqrt{T}-1}{T}, \tag{2.13}$$

and $c = \min\{c_1, c_2\}$. Let a_0 be the unique positive solution (in a) to

$$\frac{1}{2}\left(e^a + e^{-a}\right)e^{-\frac{c}{2}a^2} = 1.$$

If

$$T \geq \left(\frac{\alpha}{a_0 c}\right)^2, \tag{2.14}$$

then $h_c(t,x)$ is a lower bound for $V^S(t,x)$ for all $t \geq T$ and $x \leq \alpha\sqrt{t}$.

Remark 1. Our numerical evaluations suggest that for $T \geq 4$ equation (2.14) is always satisfied and that for $T \geq 200$ we always have $c = c_1$.

Proof. We divide the proof into tree parts:
(1.) We show that the stopped process $h_c(t \wedge \tau_0, x + S_{t\wedge\tau_0})_{t\geq T}$ is a submartingale.
(2.) We calculate K.
(3.) We show that

$$h(\tau_0, x + S_{\tau_0}) \leq g(\tau_0, x + S_{\tau_0}) \ P_{(T,x)}\text{-a.s.}$$

We then Lemma 3 to prove the statement. An illustration of the setting is given in Figure 2.1.

(1.) We have to show that

$$\frac{1}{2}\Big(h_c\big(t+1, x-1) + h_c(t+1, x+1\big)\Big) \geq h_c(t,x) \tag{2.15}$$

for every $t \geq T$ and $x \leq \alpha\sqrt{t} - 1$. We will even show (2.15) for all $x \leq \alpha\sqrt{t}$. The constant K has no influence on (2.15), so we set it equal 1 for now. We have

$$\begin{aligned} f_c(t,x) &:= \frac{1}{2}\Big(h_c(t+1, x-1) + h_c(t+1, x+1)\Big) - h_c(t,x) \\ &= \int_0^\infty \frac{1}{2} e^{a(x-1) - \frac{c}{2}a^2(t+1)}\, \mathrm{d}a \qquad (2.16) \\ &\quad + \int_0^\infty \frac{1}{2} e^{a(x+1) - \frac{c}{2}a^2(t+1)}\, \mathrm{d}a - \int_0^\infty e^{ax - \frac{c}{2}a^2 t}\, \mathrm{d}a \\ &= \int_0^\infty e^{ax - \frac{c}{2}a^2 t}\left[\frac{1}{2}(e^a + e^{-a})e^{-\frac{c}{2}a^2} - 1\right] \mathrm{d}a. \qquad (2.17) \end{aligned}$$

The function $\lambda(a) := \frac{1}{2}(e^a + e^{-a})e^{-\frac{c}{2}a^2} - 1$ has a unique positive root a_0. For

$a \in [0, a_0]$ we have $\lambda(a) \geq 0$ and for $a \geq a_0$ we have $\lambda(a) \leq 0$.
Suppose for given (t, x) we have $f_c(t, x) \geq 0$. Let $\delta \geq 0$ and $\varepsilon \in \mathbb{R}$. We have

$$
\begin{aligned}
f_c(t+\delta, x+\varepsilon) &= \int_0^{a_0} e^{ax-\frac{c}{2}a^2 t}\lambda(a)e^{\varepsilon a-\delta\frac{c}{2}a^2}\,\mathrm{d}a + \int_{a_0}^{\infty} e^{ax-\frac{c}{2}a^2 t}\lambda(a)e^{\varepsilon a-\delta\frac{c}{2}a^2}\,\mathrm{d}a \\
&\overset{(*)}{\geq} \int_0^{a_0} e^{ax-\frac{c}{2}a^2 t}\lambda(a)e^{\varepsilon a_0-\delta\frac{c}{2}a_0^2}\,\mathrm{d}a \\
&\quad + \int_{a_0}^{\infty} e^{ax-\frac{c}{2}a^2 t}\lambda(a)e^{\varepsilon a_0-\delta\frac{c}{2}a_0^2}\,\mathrm{d}a \\
&= e^{\varepsilon a_0-\delta\frac{c}{2}a_0^2} f_c(t, x) \geq 0.
\end{aligned}
$$

Here $(*)$ is true if $\varepsilon a - \delta\frac{c}{2}a^2 \geq \varepsilon a_0 - \delta\frac{c}{2}a_0^2$ for $a \leq a_0$ and $\varepsilon a - \delta\frac{c}{2}a^2 \leq \varepsilon a_0 - \delta\frac{c}{2}a_0^2$ for $a \geq a_0$ which is the case if

$$\varepsilon \leq a_0\delta\frac{c}{2}. \tag{2.18}$$

By assumption we have $f_c(T, \alpha\sqrt{T}) \geq 0$. If $c = c_1$, as in all our computational examples, this is clear. If $c = c_2 < c_1$ an inspection of f_c in (2.17) shows that $f_c > f_{c_1}$. The function $\alpha\sqrt{t}$ is concave and

$$\frac{\partial}{\partial t}\alpha\sqrt{t} = \frac{\alpha}{2\sqrt{t}},$$

so for $t \geq T$ and $x \leq \alpha\sqrt{t}$ with $(t, x) = (T+\delta, \alpha\sqrt{T}+\varepsilon)$ we have

$$\varepsilon \leq \delta\frac{\alpha}{2\sqrt{T}}.$$

Putting this into (2.18) we get the condition

$$\delta\frac{\alpha}{\sqrt{T}} \leq a_0\delta c, \text{ i.e., } T \geq \left(\frac{\alpha}{a_0 c}\right)^2$$

what is true by assumption. That concludes the first part of the proof.

(2.) We want to choose K such that $h_c(t, \alpha\sqrt{t}) = g(t, \alpha\sqrt{t}) = \frac{\alpha}{\sqrt{t}}$. We first show that this is possible and then calculate K. We have

$$h_c(t,x) = K\int_0^\infty e^{ax-\frac{c}{2}a^2t}\,\mathrm{d}a = K\frac{1}{ct}\frac{\Phi_{ct}(x)}{\varphi_{ct}(x)}$$

and

$$h_c(t,\alpha\sqrt{t}) = K\frac{1}{ct}\frac{\Phi_{ct}(\alpha\sqrt{t})}{\varphi_{ct}(\alpha\sqrt{t})} = K\frac{1}{c\sqrt{t}}\frac{\Phi_c(\alpha)}{\varphi_c(\alpha)}$$

which depends only on $\frac{1}{\sqrt{t}}$. Solving $h_c(t,\alpha\sqrt{t}) = \frac{\alpha}{\sqrt{t}}$ we get

$$K = K(c) = \alpha c\frac{\varphi_c(\alpha)}{\Phi_c(\alpha)}.$$

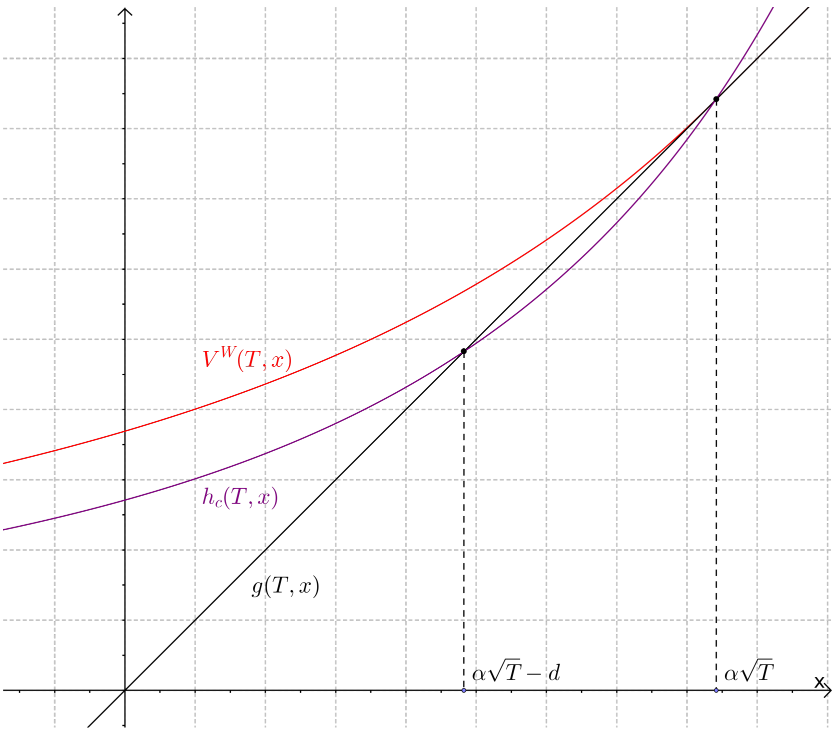

Figure 2.1: The upper bound V^W and the lower bound h_c for a fixed T. For a better illustration $c = 0.6$ is chosen very small.

(3.) We chose $\tau_0 = \inf\{n \geq 0 \mid x + S_n \geq \alpha\sqrt{t+n} - 1\}$ and need to show that $h_c(\tau_0, S_{\tau_0}) \leq g(\tau_0, S_{\tau_0})$. It is clear that $S_{\tau_0} \in [\alpha\sqrt{\tau_0} - 1, \alpha\sqrt{\tau_0}]$.

By the construction of K in (2.) we know that $h_c(t, \alpha\sqrt{t}) = g(t, \alpha\sqrt{t})$. Since h_c has strictly positive curvature, we know that $h(t, \cdot)$ has exactly one more intersection with $g(t\cdot)$ which we define as $\alpha\sqrt{t} - d(t)$. We will see that $d(t) \geq 1$ for $t \geq T$ and hence $h_c(t, x) \leq g(t, x)$ for $x \in [\alpha\sqrt{t} - d(t), \alpha\sqrt{t}]$. We have seen in (2.) that $\sqrt{t}\, h_c(t, \beta\sqrt{t})$ is constant for any $\beta > 0$. If for some x_0

$$h_c(T, x_0) \leq g(T, x_0) = \frac{x_0}{T},$$

we set $\beta := \frac{x_0}{\sqrt{T}}$ and see that for all $t > T$ we have

$$h\left(t, \frac{x_0}{\sqrt{T}}\sqrt{t}\right) \leq g\left(t, \frac{x_0}{\sqrt{T}}\sqrt{t}\right) = \frac{1}{\sqrt{t}} \frac{x_0}{\sqrt{T}}.$$

If $x_0 \leq \alpha\sqrt{T} - 1$, then $\frac{x_0}{\sqrt{T}}\sqrt{t} \leq \alpha\sqrt{t} - 1$. If $d(T) \geq 1$ then we set $x_0 := \alpha\sqrt{T} - d(T)$. We can now conclude that for $t \geq T$ we have $d(t) \geq 1$, hence it is enough to show that $d(T) \geq 1$. For $c = c_2$ this is true by assumption. In general $c \leq c_2$. The derivative

$$\frac{\partial}{\partial c} h_c(t, x) = \frac{1}{c^2} h_c(t, x) \left(K(c) - cx\frac{\varphi_{ct}(x)}{\Phi_{ct}(x)} + \frac{\alpha t^2 - x^2}{2t}\right)$$

is non-negative for $0 \leq x \leq \alpha\sqrt{t}$. Since $h_c(T, \alpha\sqrt{T}) = h_{c_2}(T, \alpha\sqrt{T})$ we have

$$h_c(T, \alpha\sqrt{T} - 1) \leq h_{c_2}(T, \alpha\sqrt{T} - 1) = \frac{\alpha\sqrt{T} - 1}{T}$$

and the statement follows. Now h_c and τ_0 fulfill the conditions of Lemma 3. This completes the proof. □

Remark 2. The only properties of S we used in the proof, are that S has limited jump sizes upwards and that

$$m_{S_1}(a) := \mathrm{E}[e^{aS_1}]$$

has only one positive intersection with $e^{\frac{c}{2}a^2}$ (i.e. $\frac{1}{2}(e^a + e^{-a})e^{-\frac{c}{2}a^2} - 1$ has only one positive root). This kind of lower bound can be constructed for any random walk with increments that fulfill these two conditions. This would of course result in different values for c.

Some values for c are given in the table below

T	**c**
10^3	0.999204
10^4	0.9999212
10^5	0.99999214
10^6	0.999999216.

2.3.4 Error of the bounds

We want to show that the relative error of the constructed bounds is of order $\mathcal{O}(1/T)$, for $x \geq 0$. First, we show that $c = c(T) \geq \frac{T-1}{T}$ for T large enough. Indeed, evaluating (2.17) for $c = \frac{T}{T+1}$ yields

$$f_{\frac{T}{T+1}}(T,x) = \int_0^\infty e^{ax-\frac{1}{2}a^2T}\left[\frac{1}{2}(e^a + e^{-a}) - e^{\frac{1}{2}\frac{1}{T+1}}\right] \mathrm{d}a, \tag{2.19}$$

which can be seen to be positive for T large enough, yielding $c_1 \geq \frac{T}{T+1} \geq \frac{T-1}{T}$. We evaluate (2.13) for $c = \frac{T-1}{T}$. With elementary estimates we obtain

$$\begin{aligned}
h_{\frac{T-1}{T}}(T, \alpha\sqrt{T}-1) &= \frac{K}{\sqrt{T-1}} \frac{\Phi\left(\frac{\alpha\sqrt{T}-1}{\sqrt{T}}\right)}{\varphi\left(\frac{\alpha\sqrt{T}-1}{\sqrt{T}}\right)} \\
&\leq \frac{K}{\sqrt{T-1}}\left(\frac{\Phi\left(\frac{\alpha\sqrt{T}}{\sqrt{T}}\right) - \frac{1}{\sqrt{T-1}}\varphi\left(\frac{\alpha\sqrt{T}}{\sqrt{T}}\right)}{\varphi\left(\frac{\alpha\sqrt{T}}{\sqrt{T}}\right)e^{\frac{\alpha\sqrt{T}-\frac{1}{2}}{T-1}}}\right) \\
&= e^{-\frac{\alpha\sqrt{T}-\frac{1}{2}}{T-1}}\left(\frac{\alpha}{\sqrt{T}} - \frac{K}{T-1}\right) \\
&\leq \frac{\alpha\sqrt{T}-1}{T}
\end{aligned}$$

for T large enough, so $c_2 \geq \frac{T-1}{T}$. We obtain $c = \min\{c_1, c_2\} \geq \frac{T-1}{T}$. We now calculate the asymptotic relative error between V_u and V_l in $x = 0$. We have

$$\frac{V_u(T,0)}{V_l(T,0)} - 1 = \frac{h(T,0)}{h_c(T,0)} - 1 = \frac{1-\alpha^2}{K}\sqrt{c} - 1 = \frac{1-\alpha^2}{\alpha}\frac{\Phi\left(\frac{\alpha}{\sqrt{c}}\right)}{\varphi\left(\frac{\alpha}{\sqrt{c}}\right)} - 1$$

where we approximate

$$\begin{aligned}\frac{\Phi\left(\frac{\alpha}{\sqrt{c}}\right)}{\varphi\left(\frac{\alpha}{\sqrt{c}}\right)} &\approx e^{-\frac{\alpha^2(c-1)}{2c}}\left(\frac{\Phi(\alpha)}{\varphi(\alpha)}+\frac{\alpha(1-\sqrt{c})}{\sqrt{c}}\right)\\ &= e^{-\frac{\alpha^2(c-1)}{2c}}\left(\frac{\alpha}{1-\alpha^2}+\frac{\alpha(1-\sqrt{c})}{\sqrt{c}}\right)\end{aligned}$$

and obtain with $c=\frac{T-1}{T}$

$$\begin{aligned}\frac{h(T,0)}{h_c(T,0)}-1 &\approx e^{-\frac{\alpha^2(c-1)}{2c}}\left(1+\frac{\alpha(1-\sqrt{c})}{\sqrt{c}}\right)-1\\ &= e^{\frac{\alpha^2}{2T-2}}\left(1+\alpha\left(\sqrt{\frac{T-1}{T}}-1\right)\right)-1=\mathcal{O}\left(\frac{1}{T}\right).\end{aligned}$$

It is now straightforward to check that for $0\le x\le\alpha\sqrt{T}$

$$\frac{h(T,0)}{h_c(T,0)}-1\ge\frac{h(T,x)}{h_c(T,x)}-1.$$

This yields

$$\frac{V(T,x)}{V_l(T,x)}-1=\mathcal{O}\left(\frac{1}{T}\right)\text{ and }\frac{V(T,x)}{V_u(T,x)}-1=\mathcal{O}\left(\frac{1}{T}\right),$$

for all $\alpha\sqrt{T}\ge x\ge 0$.

2.4 Computational results for the $\frac{S_n}{n}$-problem

In this section we show how to compute the continuation and stopping set for the Chow-Robbins game. In 2013 Häggström and Wästlund [HW13] computed stopping and continuation points starting from $(0,0)$. They choose a rather large time horizon $T=10^7$ and set

$$V_l(T,x)=\max\left\{\frac{x}{T},0\right\}$$

as a lower and

$$V_u^S(T,x) = \max\left\{\frac{x}{T}, 0\right\} + \min\left\{\sqrt{\frac{\pi}{T}}, \frac{1}{|x|}\right\}$$

as an upper bound for the value function. They use backward induction to calculate bounds for $V(n,x)$ with $n < T$. For $i \in \{u,l\}$ they calculate

$$V_i(n,x) = \max\left\{\frac{x}{n}, \mathrm{E}[V_i^S(n+1, x+\xi_i)]\right\} \tag{2.20}$$

successively. If $V_u(n,x) = \frac{x}{n}$ then $(n,x) \in S$, if $V_l(n,x) > \frac{x}{n}$ then $(n,x) \in C$. In this way they were able to decide for all but 7 points $(n,x) \in \mathbb{N} \times \mathbb{Z}$ with $n \leq 1000$, if they belong to C or S.
We use backward induction from a finite time horizon as well, but use the much sharper bounds given in Section 2.3.1 and 2.3.3. For our upper bound this has a nice intuition. We play the Chow-Robbins game up to the time horizon T, then we change the game to the favorable $\frac{W_t}{t}$-game, which slightly rises our expectation.

With a time horizon $T = 10^6$ we are able to calculate all stopping and continuation points $(n,x) \in \mathbb{N} \times \mathbb{Z}$ with $n \leq 489.241$. We show that all open points in [HW13] belong to S.

Description of the method Unlike Häggström and Wästlund we use the symmetric notation. Let ξ_i be iid. random variables with $P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2}$ and $S_n = \sum_{i=1}^n \xi_i$. We choose a time horizon T and use the function V^W given in Lemma 1 as an upper bound

$$V_u(T,x) = V^W(T,x)$$

and h_c given in Theorem 2 as a lower bound

$$V_l^S(T,x) = h_c(T,x),$$

for $x \in \mathbb{Z}$ with $x \le \alpha\sqrt{T}$. For $i \in \{u, l\}$ we now calculate recursively

$$V_i(n, x) = \max\left\{\frac{x}{n}, \frac{V_i(n+1, x+1) + V_i(n+1, x-1)}{2}\right\}.$$

If $V_l(n, x) > \frac{x}{n}$, then $(n, x) \in C$. To check if $(n, x) \in S$ we use, instead of $V_u(n, x) = \frac{x}{n}$, the slightly stronger, but numerically easier to evaluate, condition: $(n, x) \in S$ if

$$\frac{V_u(n+1, x+1) + V_u(n+1, x-1)}{2} < \frac{x}{n}.$$

We use the time horizon $T = 10^6$ to calculate $V(0, 0)$ and the integer thresholds $\hat{b}(n) := \lceil b(n) \rceil$. For 34 values $n \le 10^6$ the exact value $\hat{b}(n)$ can not be determined this way, the smallest such value is $n = 489.242$.

Theorem 3. *For the stopping problem* (2.8) *starting in* $(0, 0)$ *and for* $n \le 489.241$ *the stopping boundary* $\hat{b}$ *is given by*

$$\hat{b}(n) = \left\lceil \alpha\sqrt{n} - \frac{1}{2} + \frac{1}{7.9 + 4.54\sqrt[4]{n}} \right\rceil \tag{2.21}$$

with the following 8 exceptions:

n	$\hat{b}(n)$	n	$\hat{b}(n)$	n	$\hat{b}(n)$	n	$\hat{b}(n)$
3195	*48*	*14312*	*101*	*25257*	*134*	*51434*	*191*
12923	*96*	*24880*	*133*	*44653*	*178*	*116342*	*287*

For the value function we have

$$0.5859070128172 \le V(0, 0) \le 0.5859070128182.^3$$

Remark 3. The function $(7.9+4.54\sqrt[4]{n})^{-1}$ is constructed from our computed data. It is an interesting question whether it is indeed possible to show that

[3] In the notation of Häggström and Wästlund (denoted by V', b', ρ' etc.), our functions and values translate to $V(n, x) = 2V'(n, \frac{x+n}{2}) - 1$, $b'(n) = \left\lceil \frac{\alpha\sqrt{n}+n}{2} - \rho'(n) \right\rceil$ with $\rho'(n) = \frac{1}{4} - \frac{1}{15.8+9\sqrt[4]{n}}$, and $0.7929535064086 \le V'(0, 0) \le 0.7929535064091$. The value of $V(0, 0)$ is calculated with $T = 2 \cdot 10^6$.

$\alpha\sqrt{t} - b(t) = \frac{1}{2} - \mathcal{O}(t^{-\frac{1}{4}})$. Lai, Yao and AitSahlia introduced a method to show that

$$\lim_{t\to\infty} \alpha\sqrt{t} - b(t) = \frac{1}{2}$$

in [LLYA07]. This is reflected nicely in our calculations.

If we allow for other starting times $t \neq 0$, we can also deduce the limit for $t \to 0$. Unlike the boundary for the continuous time problem, it grows linearly, see Figure 2.2 for a plot.

Proposition 1. *We have*

$$\lim_{t\to 0} \frac{b(t)}{t} = V(0,0) \approx 0.5859.$$

Proof. We first show that V is continuous. The function $V(t,\cdot)$ is convex and hence continuous. The function $V(\cdot,x)$ is non-increasing, since $g(\cdot,x)$ is non-increasing. Since the supremum in (2.2) is obtained and the evaluation for a specific stopping time τ is continuous, $V(\cdot,x)$ is also left-continuous. To show that $V(\cdot,x)$ is right-continuous let $t > 0$ and $t > \varepsilon > 0$. We use that

$$g(t-\varepsilon,x) - g(t,x) = \frac{\varepsilon}{t^2 - t\varepsilon}$$

and obtain

$$|V(t-\varepsilon,x) - V(t,x)| \leq V(t-\varepsilon,x)\frac{\varepsilon}{t^2 - t\varepsilon} \to 0$$

as $\varepsilon \to 0$. If follows that V is continuous. We know that $V(0,0) > 0$ and $V(t,x) \to V(0,0)$ as $(t,x) \to 0$. So the stopping boundary b can be defined as the implicit function of $V(t,x) = \frac{x}{t}$. Since $b(t) \to 0$ as $t \to 0$, we get

$$\lim_{t\to 0} \frac{b(t)}{t} = \lim_{t\to 0} V(t,b(t)) = V(0,0).$$

$\square$

We calculated V and b as well for non-integer values. We did this by choosing an integer D and then calculate V^S on $\frac{1}{D}\mathbb{N} \times \frac{1}{D}\mathbb{Z}$ with the method

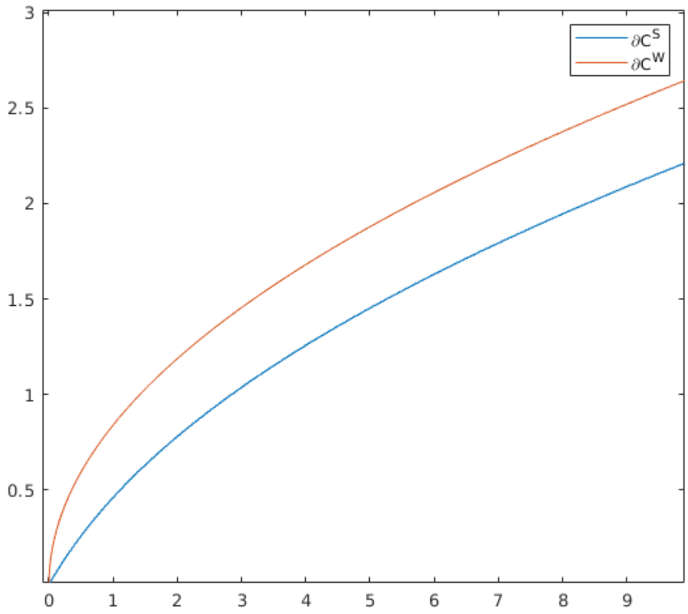

Figure 2.2: The boundaries of the continuation sets. While asymptotically similar in ∞, they behave differently close to 0. The boundary ∂C^W is a square root, whereas ∂C^S has a linear limit.

described above.[4] Some plots of b and V^S are given in Figures 2.2 - 2.4.

[4]For most plots we used $D = 300$.

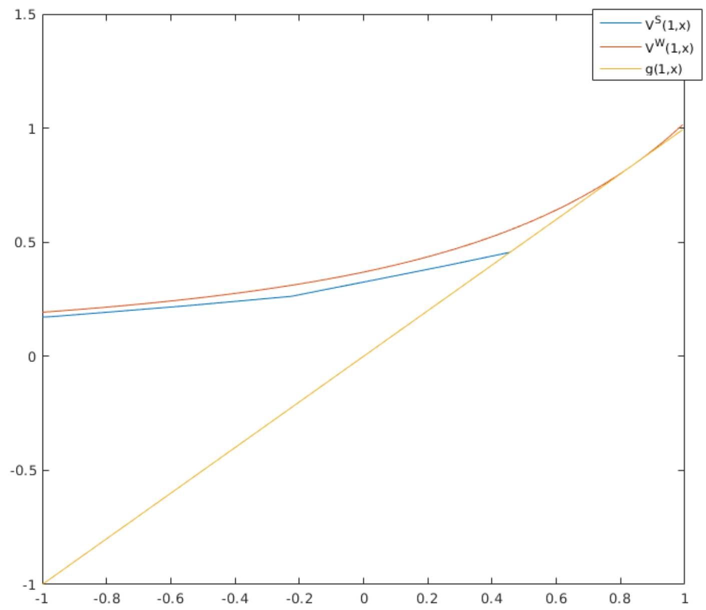

Figure 2.3: The value functions V^W and V^S in $t = 1$. V^S does not follow the smooth fit principle and is not smooth on C.

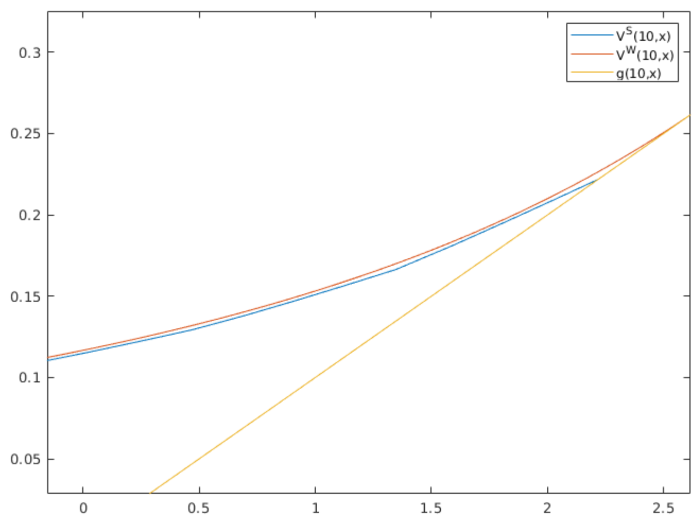

Figure 2.4: The value functions V^W and V^S in $t = 10$.

Chapter 3

On the (non-)smoothness of discrete time value functions

The research in this chapter has been published in [CF20]. The notation has been changed slightly to fit the other chapters.

Content of this chapter are again stopping problems in discrete time. We, however, allow the discrete process X to start anywhere in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ so the problem reads

$$V(t, x) = \sup_{\tau > t} E_{(t,x)} g(\tau, X_\tau). \tag{3.1}$$

We again use the Chow-Robbins game as a guiding example, but show results in more generality. We show in Section 3.3 that under some conditions on g the value function (3.1) is not smooth on C, see Theorem 4. Under additional assumptions it turns out that for every t there is a dense subset of $C \cap (\{t\} \times \mathbb{R})$ on which $V(t, \cdot)$ is not differentiable, see Theorem 5. These results lead to the conjecture that the stopping boundary ∂C is not smooth on a dense set either. We will not prove this conjecture in general, but give numerical examples in Section 3.4. We furthermore illustrate that the continuation region C of the Chow-Robbins game is not convex.
These results give an interesting qualitative characterization of V and C. They show that we can not hope to find a closed form solution for these problems.

3.1 Introduction

Let X be a Markov process on the real line and

$$V(t,x) = \sup_{\tau} \mathrm{E}_{(t,x)}\, g(\tau, X_\tau)$$

a stopping problem, where the supremum is taken over all a.s. finite stopping times $\tau \geq t$. If X is time continuous we usually want to find V on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ or a subregion. If X is time discrete, V is often defined just for discrete time points, but for many problems it seems natural that the process can be started in any point $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. In the continuous setting V is smooth under some conditions, if the smooth fit principle holds, see e.g. [SS15]. But even if smooth fit does not hold we can hope to find a solution – using the associated free boundary problem – that will be smooth on the continuation set C, see e.g. [PS06]. In the discrete setting this is however not the case. Throughout this section let $X = (X_n)_{n\in\mathbb{N}} = (\sum_{i=1}^{n} \xi_i)_{n\in\mathbb{N}}$ be a random walk, where the random variables ξ_i, $i = 1, 2, \dots,$ are i.i.d. and take discrete values with positive probability. We will at first list some properties of the Chow-Robbins game that will illustrate the general findings.

3.2 Properties of the Chow-Robbins game

As a main example we consider the classical Chow-Robbins game as described in Section 2.3. The problem was originally defined on the lattice $\mathbb{N}_0 \times \mathbb{Z}$ where stopping in $t = 0$ is ruled out. We will use $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ instead and allow for $\mathbb{N}_0$-valued τ, if $t > 0$. Only for $t = 0$ we require $\tau > 0$, i.e., $V(0,x) := \mathrm{E}\, V(1, x + \xi_1)$. We want to collect two lists of properties of (2.8) that are sufficient to show the following theorems about the non-smoothness of value functions.

Lemma 4 (Properties 1)**.** *The stopping problem* (2.8) *fulfills:*

1. *There exist* $x_1 < 0 < x_2$ *with* $P(\xi_1 = x_1) > 0$ *and* $P(\xi_1 = x_2) > 0$.

2. $g(t,\cdot)$ *is non-decreasing for every* $t > 0$.

3. *It is a one-sided problem, i.e., there exists a function* $b : \mathbb{R}_{>0} \to \mathbb{R}$*, such that* $S = \{(t,x) \mid x \geq b(t)\}$.

4. $V(t,\cdot)$ *is convex for every* t.

5. *Smooth fit does not hold, i.e., there exists* $t > 0$*, such that* $V(t,\cdot)$ *is not differentiable in* $b(t)$.

Proof. 1. and 2. are obvious, 3. can be found in [CR65].
4. $g(t,\cdot)$ is convex for every t, therefore

$$\mathrm{E}\, g(t+\tau, \cdot + X_\tau) = \mathrm{E}\left(\frac{\cdot + X_\tau}{t+\tau}\right)$$

is convex for every stopping time τ. It follows that $V(t,\cdot)$ is convex as the supremum over convex functions.
To show 5. we calculate the left and right derivative for $(t,x) \in \partial C$, where we write g' for $\frac{\partial}{\partial x} g$ and b_t for $b(t)$. The right derivative is

$$\frac{\partial_+ V}{\partial x}(t,x) = \frac{\partial_+ g}{\partial x}(t,x) = g'(t,x) = \frac{1}{t}.$$

For the calculation of the left derivative we use that $V(t, b_t - h) \in C$ for all $h > 0$ and $V(t,\cdot)$ is convex. We have

$$\begin{aligned}
\frac{\partial_- V}{\partial x}(t,x) &= \lim_{h\downarrow 0} \frac{V(t,b_t) - V(t,b_t - h)}{h} \\
&= \lim_{h\downarrow 0} \frac{1}{2h}\Big(V\big(t+1, b_t+1\big) - V\big(t+1, b_t+1-h\big) \\
&\quad + V\big(t+1, b_t - 1\big) - V\big(t+1, b_t - 1 - h\big)\Big) \\
&\leq \frac{1}{2(t+1)} + \frac{1}{2(t+1)} = \frac{1}{t+1} < \frac{1}{t}.
\end{aligned}$$

□

Lemma 5 (Properties 2)**.** *The stopping problem* (2.8) *fulfills:*

1. *Smooth fit holds nowhere, i.e. for all* $t > 0$*,* $V(t,\cdot)$ *is not differentiable in* $b(t)$.

2. *b is non-decreasing,*

3. *b is unbounded,*

4. *$(b(t+1) - b(t)) \to 0$, as $t \to \infty$,*

Proof. 1. follows from the proof of Lemma 4. 2. can be found in [CR65], 3. and 4. in [She69] and [LLY05]. □

3.3 Results on the non-smoothness

We now show that for stopping problems which have the Properties 1 the value function is not smooth. Let X be a random walk with $X_0 = 0$, $g : \mathbb{R}_{\geq 0} \times \mathbb{R}$ a measurable gain function such that the stopping problem

$$V(t,x) = \sup_{\tau} \mathrm{E}\, g(t+\tau, x+X_\tau) = \mathrm{E}\, g(t+\tau_*, x+X_{\tau_*}) \tag{3.2}$$

is solvable by an a.s. finite stopping time τ_*.

Theorem 4. *If the stopping problem* (3.2) *has Properties 1, then V is not differentiable with respect to x on C, i.e. there exists at least one $(t,x) \in C$ such that $V(t,\cdot)$ is not differentiable in x.*

We call points (t,x) where $V(t,\cdot)$ is not differentiable in x non-smoothness points.

Proof. Let $(s, b(s))$ be a point on ∂C where smooth fit does not hold. Due to convexity we therefore have

$$\frac{\partial_+ V}{\partial x}(s, b(s)) > \frac{\partial_- V}{\partial x}(s, b(s)).$$

We denote with $\tau_* = \tau_*^{t,x}$ the optimal stopping time starting in (t,x). Let be $(t,x) \in C$ with

$$P(x + X_{\tau_*} = b(s), \tau_* = s - t) > 0.$$

For $h > 0$ we have

$$\frac{V(t,x+h)-V(t,x)}{h} \geq \mathrm{E}\left(\frac{V(t+\tau_*,x+h+X_{\tau_*})-V(t+\tau_*,x+X_{\tau_*})}{h}\right) \tag{3.3}$$

and

$$\frac{V(t,x)-V(t,x-h)}{h} \leq \mathrm{E}\left(\frac{V(t+\tau_*,x+X_{\tau_*})-V(t+\tau_*,x-h+X_{\tau_*})}{h}\right). \tag{3.4}$$

The convexity of $V(t,\cdot)$ implies

$$\frac{V(t+\tau_*,x+h+X_{\tau_*})-V(t+\tau_*,x+X_{\tau_*})}{h} \geq \frac{\partial_+ V}{\partial x}(t+\tau_*,x+X_{\tau_*}), \tag{3.5}$$

$$\frac{V(t+\tau_*,x+X_{\tau_*})-V(t+\tau_*,x-h+X_{\tau_*})}{h} \leq \frac{\partial_- V}{\partial x}(t+\tau_*,x+X_{\tau_*}) \tag{3.6}$$

and

$$\frac{\partial_+ V}{\partial x}(t+\tau_*,x+X_{\tau_*}) \geq \frac{\partial_- V}{\partial x}(t+\tau_*,x+X_{\tau_*}).$$

The left-hand side in (3.5) is decreasing in h and is integrable for every h since the arguments are in S and therefore $V(t+\tau_*,x+h+X_{\tau_*}) = g(t+\tau_*,x+h+X_{\tau_*})$ and $V(t+\tau_*,x+X_{\tau_*}) = g(t+\tau_*,x+X_{\tau_*})$. The left-hand side in (3.6) is increasing and V is non-decreasing in the x component, since $g(t,\cdot)$ is non-decreasing by assumption. Therefore, the left-hand side in (3.6) is non-negative and has the integrable lower bound 0. Taking limits in (3.3) and (3.4) and using the dominated convergence theorem we get

$$\begin{aligned}\frac{\partial_+ V}{\partial x}(t,x) &\geq \mathrm{E}\left(\frac{\partial_+ V}{\partial x}(t+\tau_*,x+X_{\tau_*})\right)\\ &\geq \mathrm{E}\left(\frac{\partial_- V}{\partial x}(t+\tau_*,x+X_{\tau_*})\right) \geq \frac{\partial_- V}{\partial x}(t,x).\end{aligned}$$

Since $P(x+X_{\tau_*}=b(s),\tau_*=s-t)>0$ we have $\frac{\partial_+ V}{\partial x}(t,x) > \frac{\partial_- V}{\partial x}(t,x)$. □

Remark 4. Note that V is angled in an *upward direction* in the non-smoothness points. This is a contrast to excessive functions of diffusions. For theses – under mild assumptions – it holds that always $\frac{\partial_+ V}{\partial x}(t,x) \leq \frac{\partial_- V}{\partial x}(t,x)$, see [ST15, Corollary 3.3].

With the more restrictive additional Properties 2, the non-smoothness points lie dense in C.

Theorem 5. *If the stopping problem* (3.2) *has Properties 1 and 2, the non-smoothness points lie dense in* $C \cap (\{t\} \times \mathbb{R})$ *for every* $t > 0$.

Proof. Given $(t, x) \in C$ and $\varepsilon > 0$ such that $(t, x + \varepsilon) \in C$, we show that there is $x' \in (x - \varepsilon, x + \varepsilon)$, such that (t, x') is a non-smoothness point in the sense of Theorem 4. We show that there exists $j \in \mathbb{N}$ such that $P(x' + X_j = b(t + j), \tau_* = j) > 0$. We distinguish two cases, x_1 and x_2 from Lemma 4 either have a common multiple, or they do not.

Case 1. x_1 and x_2 from Lemma 4 have a common multiple. There exists $m^* \in \mathbb{N}$ with $P(X_{m^*} = 0) > 0$, hence we find increments $\lambda_1, \dots, \lambda_{m^*}$ such that $\sum_{i=1}^{m^*} \lambda_i = 0$ and $P(\xi_1 = \lambda_1, \dots, \xi_{m^*} = \lambda_{m^*}) > 0$. Rearranging the increments does not change the probability, i.e., we find a tuple of increments $\Lambda_{m^*} = (\lambda_{k_1}, \dots, \lambda_{k_{m^*}})$ with the same properties and $\lambda_{k_i} \le \lambda_{k_{i+1}}$ for all $i \le m^* - 1$.
We choose N such that $b(s + m^*) - b(s) < \varepsilon$ for all $s \ge N$. We choose m and y, such that $t + m \ge N$, $y \ge b(t + m)$ and $P(x + X_m = y) > 0$. Again we find a series of increments $\Lambda_y = (\lambda_1^y, \dots, \lambda_m^y)$, with $x + \sum_{i=1}^{m} \lambda_i^y = y$, $P(\xi_1 = \lambda_1^1, \dots, \xi_m = \lambda_m^y) > 0$ and $\lambda_i^y \le \lambda_{i+1}^y$, for all $i \le m - 1$. An illustration of the construction in this part is shown in Figure 3.1.
Let $k^* = \min\{k \mid b(t + m + km^*) \ge y\}$, then $b(t + m + k^*m^*) - y =: \varepsilon' \le \varepsilon$. Now $x' := x + \varepsilon'$ is our candidate starting point. We need to show that there is a path from (t, x') to $(t + m + k^*m^*, b(t + m + k^*m^*))$ that has positive probability and lies in C. The path with the increments $\underbrace{\Lambda_{m_*} \dots \Lambda_{m_*}}_{k^* \text{ times}} \Lambda_y$ has positive probability and the first k^*m^* steps clearly lie in C. The last part, however, might not lie in C, since we can not assume C to be convex. If it does not we choose an integer l and start the same procedure again with $(t+lm, l(y-x)+x)$ instead of $(t+m, y)$. Since $b(t)$ grows slower with increasing t, but the jump sizes in Λ_y does not change, the claim will hold for some l.

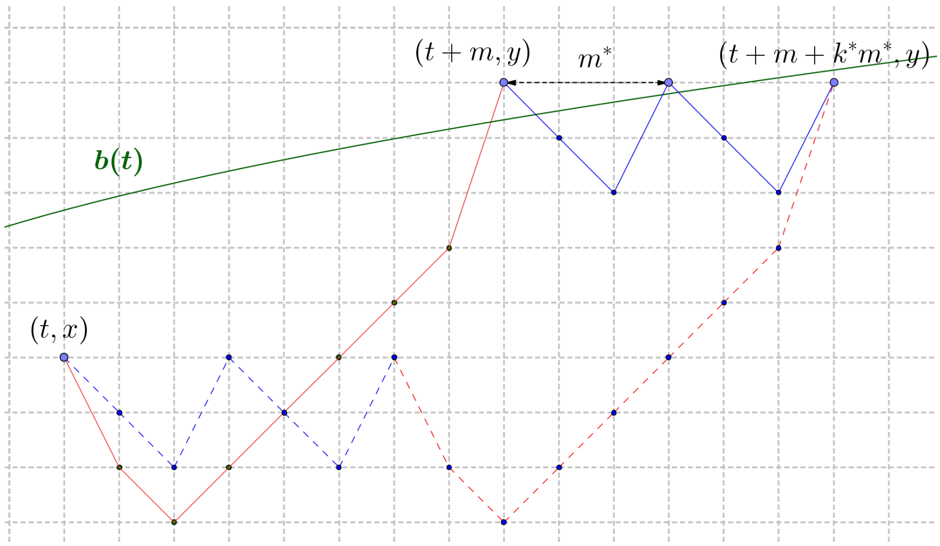

Figure 3.1: Possible paths from (t, x) to $(t+m+k^*m^*, y)$

Case 2. x_1 and x_2 have no common multiple.
We find m^*, $-\frac{\varepsilon}{2} < \tilde{x} \leq 0$ and N, such that $P(X_{m^*} = \tilde{x}) > 0$ and $b(s+m^*) - b(s) < \frac{\varepsilon}{2}$ for all $s \geq N$. The rest follows analogously to case 1. □

Remark 5. X does not need to be a random walk. The proofs work the same way if X is a discrete time Markov process such that there exist $x_1 < 0 < x_2$ with $P(X_{n+1} - X_n = x_1 \mid \mathcal{F}_n) > 0$ and $P(X_{n+1} - X_n = x_2 \mid \mathcal{F}_n) > 0$ a.s. for all $n \in \mathbb{N}_0$, where $\mathcal{F}_n$ is the natural filtration.

Remark 6. By general theory we know that $b(t) = \inf\{x \mid V(t, x) = g(t, x)\}$. If V is not smooth, there is no reason to assume that b is. In particular, if there exist t and m such that $P(b(t) + X_m = b(t+m)) > 0$ we would expect that (if existent)

$$\frac{\partial_- b}{\partial t}(t) = \lim_{h \downarrow 0} \frac{1}{h}(b(t) - b(t-h)) < \lim_{h \downarrow 0} \frac{1}{h}(b(t+h) - b(t)) = \frac{\partial_+ b}{\partial t}(t).$$

Furthermore, if the non-smooth points of V lie dense in C we expect that the non-smooth points of b lie dense in $\mathbb{R}_{>0}$. For the Chow-Robbins game this means in particular that the continuation set C is not convex. We will not prove these conjectures in general, but study numerical evidence for examples in the next section.

Remark 7. Many discrete stopping problems possess Properties 1. The condition that ∂C is the graph of a function can be relaxed quite a bit. We only need that specific points on ∂C can be reached from the interior of C with positive probability, hence similar results could be obtained for two-sided problems as well. It might be difficult to prove results in a more general setting because different effect may cancel out. Nevertheless, we believe that the observed properties are quite common for discrete time stopping problems.

If X is a random walk and the stopping problem has a one-sided solution, there is a sufficient and easy to check condition for the convexity of $V(t, \cdot)$. Let $\xi^* = \sup_\omega \xi_1^+(\omega)$. If there exists $\varepsilon > 0$ such that $g(t, \cdot)$ is convex on $[b(t) - \varepsilon, b(t) + \xi^* + \varepsilon]$ for all t, then $V(t, \cdot)$ is convex on $(-\infty, b(t) + \xi^* + \varepsilon]$ for all t. This can be proven analogously to Lemma 4. For more general processes X however, a convex gain function $g(t, \cdot)$ does not always yield a convex value function. In [Vil07] an example for a diffusion X with linear gain function is given that has a non-convex value function, see also [ES17]. Also the stronger Properties 2 are by no means restricted to the Chow-Robbins game. For example, if the increments are centered and have second moments, the slow growth condition for b, $(b(n+1) - b(n)) \to 0$, follows from the law of the iterated logarithm. The arguments in Lemma 4 can be extended as well.

3.4 Examples

The plots in this section are produced using the methods described in Section 2.4. To calculate the stopping boundary $b(t)$ we use for every t a series of M nested intervals $I_1 \supset \ldots \supset I_M$ that contain our approximation of $b(t)$ and have the length $\lambda(I_k) = C2^{-k}$. For our calculations we used $T = 5000$ and $M = 40$. The resolution in t is 500 evaluation points per plot, that is $\Delta t = 10^{-5}$ for Figure 3.6.

We have seen that the value function of the Chow-Robbins game is non-smooth on a dense subset of $C \cap (\{t\} \times \mathbb{R})$ for every $t > 0$. Figure 3.2 shows V for the fixed time $t = 1$. Some non-smoothness points can be seen in the plot:

- $x_0 = 0.46$ is the smallest value of x for which it is optimal to stop. We see that V does not follow the smooth fit principle.
- $x_1 = -0.22$ is the smallest value for which $(2, x_2 + 1)$ is in the stopping set S.
- $x_2 = -0.97$ is the smallest value for which $(3, x_3 + 2) \in S$.

In Figure 2.4 in Section 2.4 the value function V is given for $t = 10$, some non-smooth points can be seen here as well.

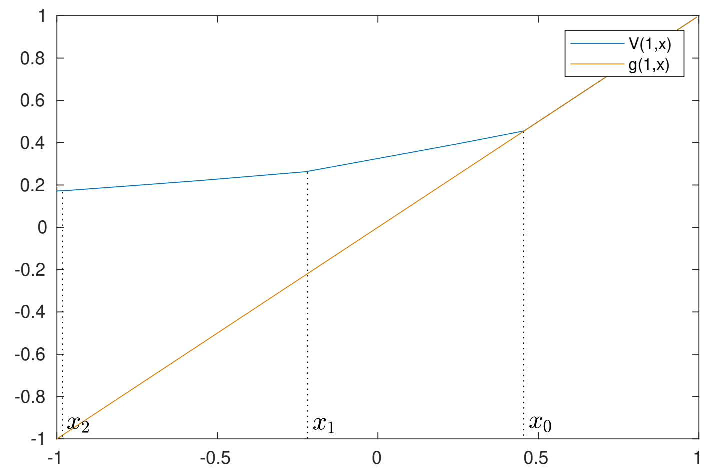

Figure 3.2: The value function of the Chow-Robbins game $V(1,\cdot)$ (blue) and the gain function $g(1,\cdot)$ (orange). Some non-smoothess points can be seen.

As mentioned in Remark 6 we expect b to be non-smooth and C to be non-convex. We found some numerical evidence for these conjectures. Figure 3.3 shows a tilted plot of b that is not smooth in $t = 0.0962$. Further examples are given in Example 2.

It is unlikely to find closed form solutions for V or b. Yet it may be helpful to study functions with the shown analytic properties in order to get a better understanding of discrete stopping problems. Examples of functions that are continuous but not differentiable on a dense subset can be found in [KK02].

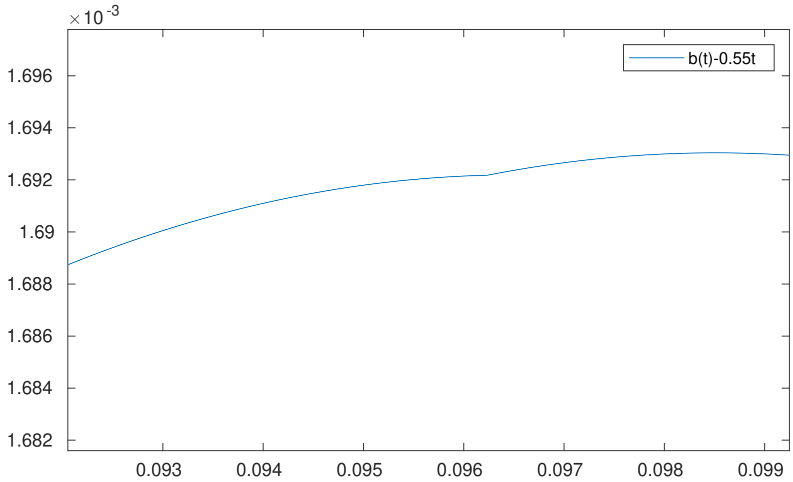

Figure 3.3: The tilted stopping boundary $b(t) - 0.55t$ of the Chow-Robbins game.

Example 2 (C is not convex)**.** We change the setting of the Chow-Robbins game slightly in order to make the effect of a non-smooth stopping boundary b stronger and more visible.
Let $\xi_1, \xi_2, \ldots,$ be i.i.d. random variables with $P(\xi_i = -1.5) = \frac{24}{85}$, $P(\xi_i = 0.2) = \frac{25}{68}$, $P(\xi_i = 1) = \frac{7}{20}$, $X_n = \sum_{i=1}^{n} \xi_i$, $g(t,x) = \frac{x}{t}$ and $V(t,x) = \sup_\tau \mathrm{E}\left(\frac{x+X_\tau}{t+\tau}\right)$. The ξ_i are centered, have unit variance and the value function V has the upper bound given in Section 2.3.1. We numerically calculate an estimate of V and the stopping boundary $b(t)$, the absolute error of our calculation is approximately 10^{-6}. In Figure 3.4 we see the stopping boundary $b(t)$. It looks smooth and concave, but if we zoom in and tilt it for better visibility, we see that this conception is misleading.
The point $(3.697, 1.089)$ lies on the boundary ∂C and $(3.697+1, 1.089+0.2)$ lies on ∂C again, hence we expect $b(t)$ to be non-smooth in $t = 3.697$. In Figure 3.5 we see a plot of $b(t) - 0.2085t$. The effect is small, but we can see clearly that C is not convex. If we zoom in further, we can see more non-smooth points shown in Figure 3.6.

Finally, we illustrate by two examples why some assumptions for Theorem 4 and Theorem 5 are necessary.

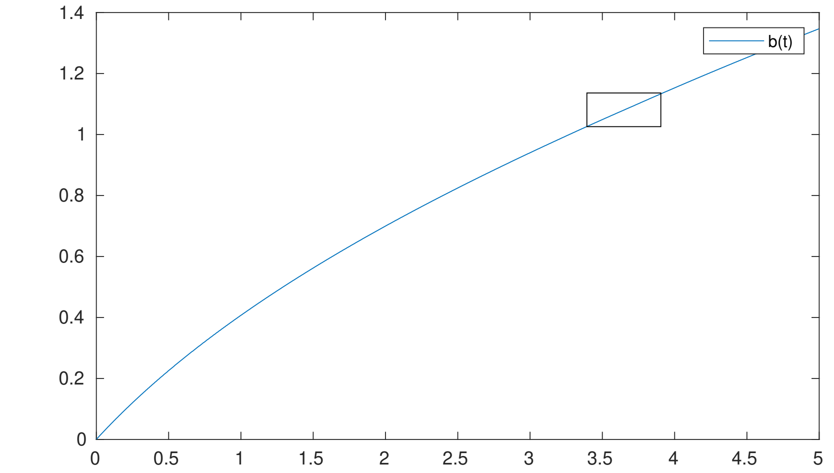

Figure 3.4: The stopping boundary $b(t)$ of the stopping problem in Example 2. The boxed part is shown in Figure 3.5.

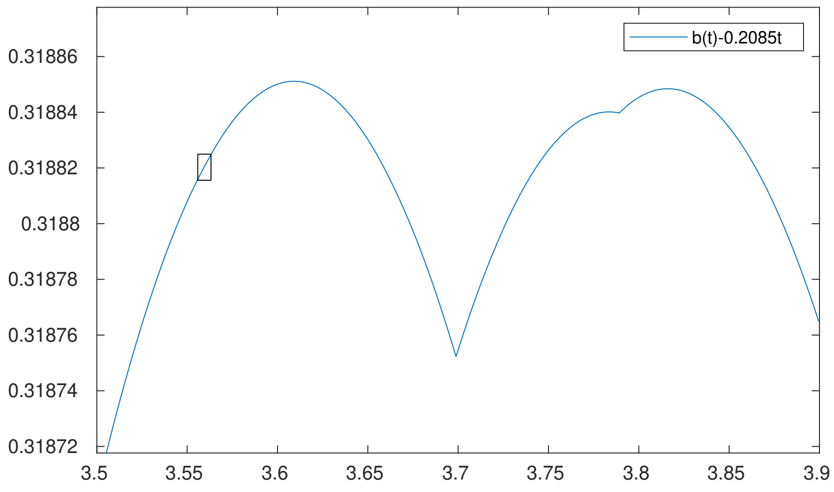

Figure 3.5: The tilted stopping boundary $b(t) - 0.2085t$ of the stopping problem in Example 2. The boxed part is shown in Figure 3.6.

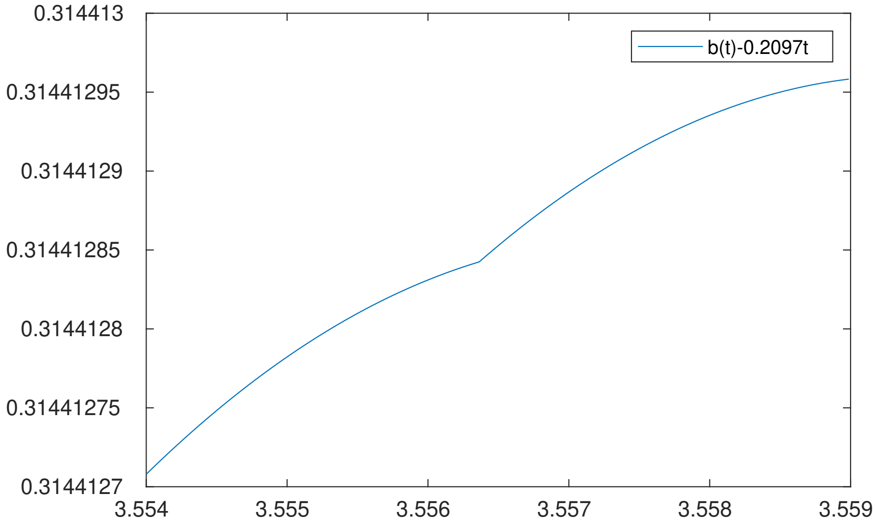

Figure 3.6: The tilted stopping boundary $b(t) - 0.2097t$ of the stopping problem in Example 2

Example 3 (Smooth fit holds)**.** Let g be smooth and non-decreasing with $g(t,x) = 0$ for $x \geq \sqrt{t}$, $g(t,x) < 0$ for $x < \sqrt{t}$ and X a Bernoulli random walk. We have $V \equiv 0$ and $b(t) = \sqrt{t}$. The value function V is smooth everywhere. This is because Property 5. of Lemma 4 (no smooth-fit) does not hold.

Example 4 (b is bounded)**.** Let X be a Bernoulli random walk with $P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2}$ and

$$g(t,x) = \begin{cases} -x^2, & x \leq 0, \\ -(\min\{\lceil x \rceil - x, x - \lfloor x \rfloor\})^2, & x > 0. \end{cases}$$

Then $b(t) = -\frac{1}{2}$ and $V(t,x) = -(\min\{\lceil x \rceil - x, x - \lfloor x \rfloor\})^2$. Clearly V has some non-smoothness points, but they do not lie dense in C, see Figure 3.7. This is because Property 3. of Lemma 5 (unboundedness of b) does not hold.

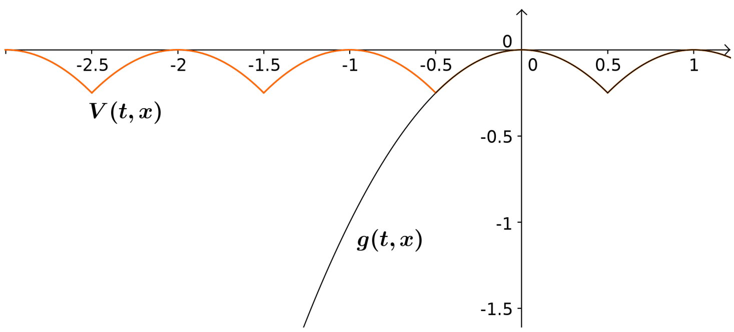

Figure 3.7: The value function V and gain function g of Example 4

Chapter 4

Continuous time stopping problems

Most parts of this chapter are submitted for publication, a preprint can be found in [CF21].

This chapter is about optimal stopping of time continuous processes, namely Brownian motions, with a finite time horizon. We consider general payoff functions dependent on time and space $g : \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}$, but put special emphasis on discounted problems, where $g(t, x) = e^{-rt}h(x)$. In the discounted case the finite time horizon introduces an additional challenge, since it makes the problem explicitly time dependent.

One ansatz to solve such problems numerically is to discretize the process and then use backward induction similar to the numeric methods in the previous chapters. This is known as *tree approximation*. A *tree approximation* can give a vague approximation with few computations, but usually has a low rate of convergence after that. It also has the problem that its complexity increases exponentially in the dimension of the problem and it is not suitable to search for analytic properties. A detailed discussion of the method used for the pricing of American options can be found in Cox, Ross and Rubinstein [CRR79]. There is a broad range of other numerical methods that we do not discuss here, see [Gla04] and [Sey09] for an overview.

Another approach that became common in the last decades is to use integral equations for the unknown stopping boundary. These are derived from stochastic analysis. These can be used for analytic and numeric evaluation of stopping problems. We will follow this approach here. We derive a new Fredholm-type integral equation for Brownian stopping problems with finite time horizon. For large problem classes of interest we show by analytical arguments that the equation uniquely characterizes the stopping boundary of the problem. Regardless of uniqueness we use the representation to rigorously find the limit behavior of the stopping boundary close to the terminal time. Interestingly, it turns out that the leading-order coefficient is universal for wide classes of problems. We also show how the representation can be used for numerical purposes.

4.1 Introduction

Let W be an n-dimensional standard Brownian motion started at time $t \leq 0$ in $\mathbf{x} \in \mathbb{R}^n$ and $g : \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}$ a payoff function with properties to be specified later. We consider the stopping problem with finite time horizon

$$V(t, \mathbf{x}) = \sup_{t \leq \tau \leq 0} \mathrm{E}_{(t,\mathbf{x})}[g(\tau, W_\tau)]. \tag{4.1}$$

Note that we use 0 as the time horizon and we start the process from $t \leq 0$. The usefulness of this convention will become clear later. Such problems arise in a wide variety of fields, including sequential statistics, change point detection and one-armed bandit problems starting in the 1960s, and option pricing and economics in the last decades. We refer to [LL05] for an overview (with a focus on the contributions by H. Chernoff). As no closed form solutions can be expected for most problems of interest, different approaches have been suggested to gain information for the solution. In the last decades, one of the most common analytical approaches is to characterize the (unknown) stopping boundary of the problem in terms of a nonlinear integral equation of Volterra-type. The main mathematical difficulty here is to prove rigorously that the nonlinear integral equation has a unique solution. In Section 4.2 we summarize how these Volterra integral equations are constructed and review some of the literature on the topic. More background is given in the appendix Section 4.8. Building upon the Volterra integral equations we will introduce and study a new integral equation for the boundary, which is – in contrast to the standard equations for finite time horizon problems – of Fredholm-type. More precisely, let $A = \frac{\partial}{\partial t} + \frac{1}{2}\Delta$ be the characteristic operator of W, C the continuation set of (4.1) and $C^- := \{(t, \mathbf{x}) \mid (-Ag)(t, \mathbf{x}) < 0\}$, then – under natural assumptions – it holds

$$-\int_{C^-} e^{\frac{\|\mathbf{c}\|^2}{2}s + \mathbf{c}\cdot\mathbf{y}}(-Ag)(s, \mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{y} = \int_{C\setminus C^-} e^{\frac{\|\mathbf{c}\|^2}{2}s + \mathbf{c}\cdot\mathbf{y}}(-Ag)(s, \mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{y}, \tag{4.2}$$

for $\mathbf{c} \in \mathbb{R}^n$ to be specified later. This is shown in Theorem 6 in Section 4.3. We derive (4.2) and show first examples illustrating how it can be used to solve optimal stopping problems in Section 4.3. Of special interest are stopping problems with discounted value functions, i.e., $g(t, \mathbf{x}) = e^{-rt}h(\mathbf{x})$. We derive versions of (4.2) for that case in Subsection 4.3.1.

Questions that arise are firstly whether the representation determines C uniquely and secondly whether it can be used to derive analytic properties of C.

The second question is tackled in Section 4.4 where we analyze the limit behavior of the continuation set C for $t \to 0$ for a large class of stopping problems. This is done independently of uniqueness, showing the properties for every set that satisfies (4.2). We derive a second order approximation for C close to $t = 0$ for discounted stopping problems. This is done rigorously in the one-dimensional case in Subsection 4.4.1. We treat the American put with high dividend as one example. In Subsection 4.4.2 we discuss how the described method can be extended to multidimensional problems.

The question of uniqueness is discussed in Section 4.5. We prove that a version of (4.2) determines C uniquely in the one-dimensional and one-sided case with discounted payoff function $g(t, x) = e^{-rt}h(x)$, see Theorem 9. For the proof we use methods known from the identifiability of certain mixtures of Gaussian laws, so our result can be interpreted as a result on identifiability for a case of a non-compact parameter space. The proof is analytical in nature, in contrast to the uniqueness results for the standard integral equations which are usually based on probabilistic arguments.

In Section 4.6 we describe numerical procedures based on (4.2) and give some examples in the one-dimensional case.

The appendix Section 4.7 contains some proofs omitted in the other sections.

4.2 Volterra integral equations

Our constructions are based on a well established integral representation of Volterra-type. In this section we briefly review how these integral representations are usually constructed.

Let $A = \frac{\partial}{\partial t} + \frac{1}{2}\Delta$ be the characteristic operator of the process (t, W_t) and p its transition kernel. In short, the construction is as follows: We apply a generalized version of Dynkin's formula to the value function V and the payoff function g and combine these into one integral equation. It is by no means easy to state in general when Dynkin's formula is applicable to a certain value function of a stopping problem. We will discuss this in the appendix Section 4.8. For now, let us assume that Dynkin's formula is applicable. Using it on (4.1) we obtain

$$\begin{aligned} V(t,\mathbf{x}) &= \mathrm{E}_{(t,\mathbf{x})}[g(0,W_0)] + \mathrm{E}_{(t,\mathbf{x})}\left[\int_t^0 \mathbb{1}_{\{(s,W_s)\in S\}}(-Ag)(s,W_s)\,\mathrm{d}s\right] \qquad (4.3)\\ &= \int_{-\infty}^{\infty} g(0,\mathbf{y})p((t,\mathbf{x}),(0,\mathbf{y}))\,\mathrm{d}\mathbf{y} \\ &\quad + \int_t^0 \int_{S_s} (-Ag)(s,\mathbf{y})p((t,\mathbf{x}),(s,\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}s. \end{aligned}$$

In the financial context, this representation is called early-exercise-premium decomposition. In the same way we obtain

$$\begin{aligned} g(t,\mathbf{x}) &= \mathrm{E}_{(t,\mathbf{x})}[g(0,W_0)] + \mathrm{E}_{(t,\mathbf{x})}\left[\int_t^0 (-Ag)(s,W_s)\,\mathrm{d}s\right] \\ &= \int_{-\infty}^{\infty} g(0,\mathbf{y})p((t,\mathbf{x}),(0,\mathbf{y}))\,\mathrm{d}\mathbf{y} \\ &\quad + \int_t^0 \int_{\mathbb{R}^n} (-Ag)(s,\mathbf{y})p((t,\mathbf{x}),(s,\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}s. \end{aligned}$$

For $(t,\mathbf{x}) \in S$ we have $V(t,\mathbf{x}) = g(t,\mathbf{x})$. We subtract the two equations above and have

$$0 = \int_t^0 \int_{C_s} (-Ag)(s,\mathbf{y})p((t,\mathbf{x}),(s,\mathbf{y}))\,\mathrm{d}\mathbf{y}\,\mathrm{d}s,\ \forall (t,\mathbf{x}) \in S. \tag{4.4}$$

This is nowadays a standard representation for stopping problems which we use as a starting point for our approach. We will discuss in Section 4.8 under what conditions it holds. We will assume silently that it holds in the following sections. Motivated by heat equations in physics, a slightly more complicated version of the integral equations was introduced by van Moerbeke in [vM76] for one-sided and one-dimensional problems. Van Moerbeke also showed local uniqueness of the solution of the integral representation. In the early 90s, Kim [Kim90] and Myneni [Myn92] derived (4.4) for the optimal exercise boundary of American options. Evaluating (4.4) for $(t,\mathbf{x})$ in the boundary of the stopping set leads to a useful equation for describing the stopping boundary. It was not until 2005, however, that it was rigorously proven by Peskir that this equation uniquely determined the stopping boundary for the case of the American put in the Black-Scholes market, see [Pes05]. The proof is based on probabilistic arguments and has later been adapted for many other classes of problems. The representation (4.4) is for example used to numerically approximate the continuation set of stopping problems. The numeric evaluation is however not always easy. One reason is that the integrand has singularities in $(t,\mathbf{x}) \in \partial C$.

4.3 Fredholm representation

In this section we derive a Fredholm-type integral representation for a large class of stopping problems with an n-dimensional Brownian motion as a driving process. We build it on the Volterra integral equation (4.4), which we silently assume to hold.

Let W be an n-dimensional standard Brownian motion with characteristic space-time operator $A = \frac{\partial}{\partial t} + \frac{1}{2}\Delta$ and transition kernel p. For the sake of simplicity, we assume $g : \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}$ to be in $C^{1,2}$. We want to analyze the stopping problem

$$V(t, \mathbf{x}) = \sup_{t \leq \tau \leq 0} \mathrm{E}_{(t,\mathbf{x})}[g(\tau, W_\tau)], \tag{4.5}$$

where the supremum is taken over all stopping times τ with $t \leq \tau \leq 0$ and $W_t = \mathbf{x}$ a.s. Note that, in contrast to most other references, our terminal time is denoted by 0 and we start the process from $t \leq 0$.

Remark 8. If a Brownian motion with drift X is the driving process of the stopping problem

$$V(t, \mathbf{x}) = \sup_{t \leq \tau \leq 0} \mathrm{E}_{(t,\mathbf{x})}[g(\tau, X_\tau)], \tag{4.6}$$

we can convert (4.6) to our setting via a measure transformation and have

$$V(t, \mathbf{x}) = \sup_{t \leq \tau \leq 0} \mathrm{E}_{(t,\mathbf{x})}[g'(\tau, W_\tau)], \tag{4.7}$$

where g' is a transformed payoff function and W is a standard Brownian motion. If the gain function is of discounted from, i.e., $g(t, x) = e^{-rt}h(x)$, then g' is of discounted from as well, e.g., $g'(t, x) = e^{-r't}h'(x)$. We refer to [CKL21] and [LU07] for details.

We denote the continuation set by

$$C := \{(t, \mathbf{x}) \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid V(t, \mathbf{x}) > g(t, \mathbf{x})\}$$

and the stopping set by $S = C^c$. For a fixed time t we set

$$C_t := \{\mathbf{x} \in \mathbb{R}^n \mid (t, \mathbf{x}) \in C\}$$

and define S_t accordingly. We denote the first entrance time to S by $\tau^* = \tau_C := \inf\{t \le s \le 0 \mid W_s \in S\}$ and note that this stopping time is optimal under minimal assumptions by general theory.

As mentioned above, the key result of [Pes05] was that it is enough to evaluate (4.4) for $(t, \mathbf{x}) \in \partial S \subseteq S$. The idea of the approach we suggest here is to use another subset of S. More precisely, we compactify the stopping region and use the *infinitely far away* boundary (the Martin boundary) instead. This leads to our Fredholm representation from (4.4).

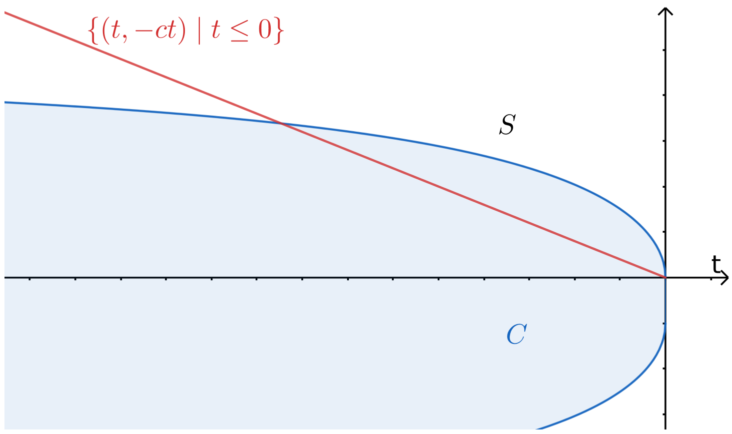

Figure 4.1: An illustration of the setting in Theorem 6

Theorem 6. *The continuation set C defined by the stopping problem* (4.5) *fulfills the equation*

$$0 = \int_{-\infty}^{0} \int_{C_s} e^{\boldsymbol{c}\cdot\boldsymbol{y} + \frac{\|c\|^2}{2} s} (-Ag)(s, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}s \tag{4.8}$$

for all $\boldsymbol{c}$ for which the integral exists and there is an $\varepsilon > 0$ such that $\{(t, -t(\boldsymbol{c} + \boldsymbol{a})) \mid t \le 0, \|a\| < \varepsilon\} \cap C$ is bounded.

Remark 9. The assumption that the cone $\{(t, -t(\mathbf{c} + \mathbf{a})) \mid t \le 0, \|a\| < \varepsilon\}$ has bounded intersection with C has technical reasons described in the appendix. For all practical purposes described here the condition that the line $\{(t, -t\mathbf{c}) \mid t \leq 0\}$ has bounded intersection with C is enough. For discounted problems for instance, we give an easy condition for this to hold later on.

Proof. We pick $\mathbf{c} \in \mathbb{R}^n$ such that $\{(t, -t\mathbf{c}) \mid t \leq 0\} \cap C$ is bounded, then $(t, -t\mathbf{c}) \in S$ for t small enough. We set $\mathbf{x} = -t\mathbf{c}$ in (4.4), divide by $p((t, -\mathbf{c}t), (0, \mathbf{0}))$ and obtain

$$0 = \int_t^0 \int_{C_s} (-Ag)(s, \mathbf{y}) \frac{p((t, -\mathbf{c}t), (s, \mathbf{y}))}{p((t, -\mathbf{c}t), (0, \mathbf{0}))} \,\mathrm{d}\mathbf{y} \,\mathrm{d}s$$

for all t small enough, if the integral exists. Taking limits we obtain

$$0 = \lim_{t \to -\infty} \int_t^0 \int_{C_s} (-Ag)(s, \mathbf{y}) \frac{p((t, -\mathbf{c}t), (s, \mathbf{y}))}{p((t, -\mathbf{c}t), (0, \mathbf{0}))} \,\mathrm{d}\mathbf{y} \,\mathrm{d}s. \tag{4.9}$$

We analyze the quotient in the integral. The kernel p is the density of an n-dimensional normal distribution, so we have

$$\begin{aligned}
\frac{p((t, -\mathbf{c}t), (s, \mathbf{y}))}{p((t, -\mathbf{c}t), (0, \mathbf{0}))} &= \frac{(2\pi)^{-\frac{n}{2}} (s-t)^{-\frac{n}{2}} \exp\left(-\frac{\|-\mathbf{c}t - \mathbf{y}\|^2}{2(s-t)}\right)}{(2\pi)^{-\frac{n}{2}} (-t)^{-\frac{n}{2}} \exp\left(-\frac{\|-\mathbf{c}t\|^2}{2(-t)}\right)} \\
&= \left(\frac{-t}{s-t}\right)^{\frac{n}{2}} \exp\left(-\frac{\|-\mathbf{c}t - \mathbf{y}\|^2}{2(s-t)} + \frac{\|-\mathbf{c}t\|^2}{2(-t)}\right) \\
&= \left(\frac{-t}{s-t}\right)^{\frac{n}{2}} \exp\left(\frac{\left(\|\mathbf{c}\|^2 t^2 + 2\mathbf{c} \cdot \mathbf{y}t + \|\mathbf{y}\|^2\right) t + \|\mathbf{c}\|^2 t^2 (s-t)}{2(t^2 - st)}\right) \\
&= \left(\frac{-t}{s-t}\right)^{\frac{n}{2}} \exp\left(\frac{2\mathbf{c} \cdot \mathbf{y}t^2 + t\|\mathbf{y}\|^2 + \|\mathbf{c}\|^2 t^2 s}{2(t^2 - st)}\right),
\end{aligned}$$

where $\cdot$ denotes the standard scalar product and $\|\cdot\|$ the corresponding norm. We calculate the limit and have

$$\lim_{t \to -\infty} \frac{p((t, -\mathbf{c}t), (s, \mathbf{y}))}{p((t, -\mathbf{c}t), (0, \mathbf{0}))} = \exp\left(\mathbf{c} \cdot \mathbf{y} + \frac{\|\mathbf{c}\|^2}{2} s\right).$$

We can pull the limit into the integral in (4.9) (for a proof of that fact see

Section 4.7) and obtain

$$0 = \int_{-\infty}^{0} \int_{C_s} e^{\mathbf{c}\cdot\mathbf{y} + \frac{\|\mathbf{c}\|^2}{2} s} (-Ag)(s, \mathbf{y}) \,\mathrm{d}\mathbf{y} \,\mathrm{d}s = \int_{C} e^{\mathbf{c}\cdot\mathbf{y} + \frac{\|\mathbf{c}\|^2}{2} s} (-Ag)(s, \mathbf{y}) \,\mathrm{d}\mathbf{y} \,\mathrm{d}s \tag{4.10}$$

for all $\mathbf{c}$ such that $\{(t, -t\mathbf{c}) \mid t \leq 0\} \cap C$ is bounded and the integral exists. □

The Fredholm-type integral equation (4.8) is highly non-linear. As far as we know, equations of this type have not been analyzed in the literature before. We illustrate how it can be used to tackle explicitly solvable problems with the following example.

Example 5. Let $n = 1$ and $g(t, x) = tx$. The corresponding stopping problem

$$V(t, x) = \sup_{t \leq \tau \leq 0} \mathrm{E}_{(t,x)}[\tau W_\tau]$$

arises from a discrete time *commodity sales* problem described by Stadje in [Sta87]. We reproduce the solution using our integral equation. Motivated by Brownian scaling, we make the ansatz that the continuation set is of the form

$$C = \{(t, x) \mid x < \alpha\sqrt{-t}\}$$

for some $\alpha > 0$. We have $-Ag(s, y) = -y$. Plugging this into (4.8) we get

$$\begin{aligned} 0 &= \int_{-\infty}^{0} \int_{-\infty}^{\alpha\sqrt{-s}} -y e^{cy + \frac{c^2}{2} s} \,\mathrm{d}y \,\mathrm{d}s \\ &= \frac{1}{c^2} \int_{-\infty}^{0} (1 - \alpha c\sqrt{-s}) e^{\frac{c^2}{2} s + \alpha c \sqrt{-s}} \,\mathrm{d}s \\ &= \frac{1}{c^2} \left(-2\alpha^3 \frac{\Phi(\alpha)}{\varphi(\alpha)} + (1 - \alpha^2) \right) \end{aligned}$$

where Φ and φ denote CDF and PDF of a standard normal distribution. The equation holds for all $c > 0$ iff

$$\begin{aligned} 0 &= -2\alpha^3 \frac{\Phi(\alpha)}{\varphi(\alpha)} + (1 - \alpha^2), \\ \text{i.e.,} \quad \alpha^3 \Phi(\alpha) &= (1 - \alpha^2)\varphi(\alpha). \end{aligned}$$

The latter equation has a unique positive solution at $\alpha_1 \approx 0.638833$. It can indeed be seen that the continuation set of the problem is

$$C = \{(t,x) \mid x < \alpha_1\sqrt{-t}\}.$$

4.3.1 The discounted case

An important special case are stopping problems with a discounted payoff function

$$g(t,\mathbf{x}) = e^{-rt}h(\mathbf{x}).$$

We now have

$$-Ag(t,\mathbf{x}) = e^{-rt}\left(rh(\mathbf{x}) - \frac{1}{2}\Delta h(\mathbf{x})\right) =: e^{-rt}\tilde{h}(\mathbf{x})$$

where Δ denotes the Laplace operator. We rewrite (4.8) to

$$0 = \int_C e^{\mathbf{c}\cdot\mathbf{y}+\frac{1}{2}\|\mathbf{c}\|^2 s - rs}\tilde{h}(\mathbf{y})\,\mathrm{d}(s,\mathbf{y}) \tag{4.11}$$

and see that the integral exists if $\|c\| > \sqrt{2r}$. We define

$$C_\infty := \bigcup_{t\leq 0} C_t.$$

Note that in cases where the infinite time horizon problem

$$V(\mathbf{x}) = \sup_{0\leq\tau} \mathrm{E}_{\mathbf{x}}[e^{-r\tau}h(W_\tau)] \tag{4.12}$$

is solvable, C_∞ is the solution to that problem (see Proposition 2). Since the dimension of (4.12) is reduced by one, C_∞ is usually easier to find than C. If C_∞ is bounded, then $\{(t,-t\mathbf{c}) \mid t \leq 0\} \cap C$ is bounded for all $\mathbf{c} \neq 0$ and (4.11) holds for all $\mathbf{c}$ with $\|\mathbf{c}\| > \sqrt{2r}$.

We now look at some properties of C_t. It is easily seen that C_t is decreasing in t, i.e., for $s \geq t$ we have $C_s \subset C_t$. We also know that $C_0 = \left\{\mathbf{x} \mid \tilde{h}(\mathbf{x}) < 0\right\}$.

We define the function

$$d : C_\infty \to \mathbb{R}_{\leq 0},\ \mathbf{x} \mapsto \sup \{t \leq 0 \mid \mathbf{x} \in C_t\}.$$

With the given properties of C_t we see that d defines C via $C_t = d^{-1}([t, 0])$. We can now simplify (4.11) using Fubini's lemma

$$\begin{aligned}
& -\int_{-\infty}^{0}\int_{C_0} e^{\frac{\|\mathbf{c}\|^2}{2}s+\mathbf{c}\cdot\mathbf{y}-rs}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s = \int_{-\infty}^{0}\int_{C_s\setminus C_0} e^{\frac{\|\mathbf{c}\|^2}{2}s+\mathbf{c}\cdot y-rs}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\
\iff \quad & -\frac{1}{\frac{\|\mathbf{c}\|^2}{2}-r}\int_{C_0} e^{\mathbf{c}\cdot\mathbf{y}}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \frac{1}{\frac{\|\mathbf{c}\|^2}{2}-r}\int_{C_\infty\setminus C_0} e^{\left(\frac{\|\mathbf{c}\|^2}{2}-r\right)d(\mathbf{y})+\mathbf{c}\cdot\mathbf{y}}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
\iff \quad & -\int_{C_0} e^{\mathbf{c}\cdot\mathbf{y}}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{C_\infty\setminus C_0} e^{\left(\frac{\|\mathbf{c}\|^2}{2}-r\right)d(\mathbf{y})+\mathbf{c}\cdot\mathbf{y}}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad (4.13)
\end{aligned}$$

for all $\mathbf{c}$ with $\|\mathbf{c}\| > \sqrt{2r}$ for which $\{(t, -t\mathbf{c}) \mid t \leq 0\} \cap C$ is bounded. The integral on the left-hand side of (4.13) is known. It is an integral transformation of $\tilde{h}$.

We can use the Fredholm representation to derive the Martin boundary representation given in [CCMS16].

Proposition 2. *If*

$$\int_{C_\infty} e^{\left(\frac{\|\boldsymbol{c}\|^2}{2}-r\right)d(\boldsymbol{y})+\boldsymbol{c}\cdot\boldsymbol{y}}\tilde{h}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0 \qquad (4.14)$$

holds for all $\boldsymbol{c} \in \mathbb{R}^n$ with $\|\boldsymbol{c}\| > \sqrt{2r}$, then

$$\int_{C_\infty} e^{\boldsymbol{c}\cdot\boldsymbol{y}}\tilde{h}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0 \qquad (4.15)$$

for all $\boldsymbol{c} \in \mathbb{R}^n$ with $\|\boldsymbol{c}\| = \sqrt{2r}$.

Proof. Assume that (4.14) holds. For $\mathbf{c}_0 \in \mathbb{R}^n$ with $\|\mathbf{c}\| = \sqrt{2r}$ let $(\mathbf{c}_i)$ be a sequence in $\mathbb{R}^n$ with $\|\mathbf{c}_i\| > \sqrt{2r}$ for all i and $\mathbf{c}_i \to \mathbf{c}_0$ as $i \to \infty$. By monotone convergence (applied to C_0 and $C_\infty \setminus C_0$ seperately) we derive

$$\begin{aligned} 0 &= \lim_{t\to\infty} \int_{C_\infty} e^{\left(\frac{\|\mathbf{c}_i\|^2}{2}-r\right)d(\mathbf{y})+\mathbf{c}_i\cdot\mathbf{y}} \tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \int_{C_\infty} \lim_{t\to\infty} e^{\left(\frac{\|\mathbf{c}_i\|^2}{2}-r\right)d(\mathbf{y})+\mathbf{c}_i\cdot\mathbf{y}} \tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \int_{C_\infty} e^{\mathbf{c}_0\cdot\mathbf{y}} \tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \end{aligned}$$

which implies (4.15). □

It was shown in [CCMS16] that under suitable assumptions (4.15) defines the continuation set $\tilde{C}$ of the infinite time horizon stopping problem (4.12) uniquely, so Proposition 2 shows that $\tilde{C} = C_\infty$.

4.3.2 The one-sided and one-dimensional case

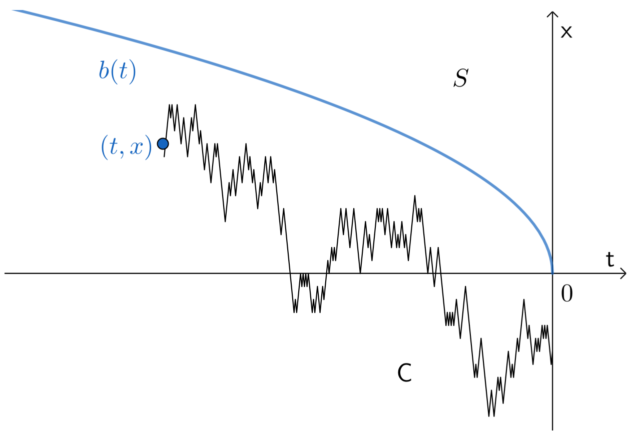

Figure 4.2: The setting in the one-dimensional and one-sided case

In one dimension an important class of stopping problems is the class of problems with one-sided solutions. These have a continuation set that can be written as

$$C = \{(t, x) \mid x < b(t)\}$$

for some function $b : \mathbb{R}_{\leq 0} \to \mathbb{R}$. In the discounted setting, b is the inverse function of d constructed above. The stopping boundary b is decreasing and we define

$$b_\infty := \lim_{t\to-\infty} b(t).$$

If the corresponding infinite time horizon stopping problem is solvable, then $b_\infty < \infty$ and $C_\infty = (-\infty, b_\infty)$. Then, (4.11) holds for all $c > \sqrt{2r}$. We can assume w.l.o.g. that $C_0 = (-\infty, 0)$. Then, the integral transformation from (4.13) is the Laplace transformation $\mathcal{L}$ and the representation can be written as

$$-\mathcal{L}\tilde{h}(c) = \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\tilde{h}(y)\,\mathrm{d}y \tag{4.16}$$

for all $c > \sqrt{2r}$.

Example 6. Let $h(x) = x^3$ and $r = 0$, then $\tilde{h}(x) = 3x$. $\tilde{h}$ is just a multiple of $-Ag$ in Example 5, so the problems have the same Fredholm representation (up to multiplication with a constant) and hence the same continuation set (see Section 4.5 for the uniqueness of the solution).

4.4 Limit behavior

In this section we show how to use the Fredholm representation to derive analytic properties of C. In particular, we study the limit behavior of C_t for $t \to 0$. We do this rigorously for the one-dimensional, one-sided and discounted case in Subsection 4.4.1 and give heuristic arguments for multi-dimensional stopping problems in Subsection 4.4.2. Our technique builds on Theorem 6 and is independent of the uniqueness discussed in Section 4.5.
For special problems the results are known, but to our knowledge they are new in the generality given below.

4.4.1 The one-dimensional case

We consider the discounted case. We assume w.l.o.g. that $C_0 = (-\infty, 0]$, so our main equation reads

$$-\mathcal{L}\tilde{h}(c) = \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\tilde{h}(y)\,\mathrm{d}y \tag{4.17}$$

for all $c > \sqrt{2r}$.

By assumption $\tilde{h}$ is continuous (this can be weakened to include cases such as American options), so we have $\tilde{h}(0) = 0$. Central for the limit behavior is the degree of h close to 0. The most common case is that $\tilde{h}$ is approximately linear, i.e., $\tilde{h}(x) = mx + o(x)$ for some $m > 0$ in a neighborhood of 0. We will stick to that case for sake of clarity but all constructions work similar as long as

$$\lim_{x \searrow 0} \frac{\tilde{h}(x)}{\tilde{h}(-x)} \in (-\infty, 0),$$

see Remark 11 for more details. For the left-hand side of (4.17) we have

$$\lim_{c \to \infty} -c^2 \mathcal{L}\tilde{h}(c) = m$$

so we obtain

$$m = \lim_{c \to \infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} \tilde{h}(y) \, \mathrm{d}y.$$

The following two lemmata show that only a small neighborhood of 0 and the limit behavior of $\tilde{h}(x)$ for $x \to 0$ are relevant for the limit behavior of C_t.

Lemma 6. *For all $\varepsilon > 0$ it holds*

$$\lim_{c \to \infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} \tilde{h}(y) \, \mathrm{d}y = \lim_{c \to \infty} c^2 \int_0^{\varepsilon} e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} \tilde{h}(y) \, \mathrm{d}y.$$

Proof. We know from the construction of the stopping problem that d is non-increasing, $d(0) = 0$ and $d(x) < 0$ for all $x > 0$. Then $d(\varepsilon) < 0$ and $d(x) \leq d(\varepsilon)$ for all $x \geq \varepsilon$. We have

$$\begin{aligned}
\left| \lim_{c \to \infty} c^2 \int_\varepsilon^{b_\infty} e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} \tilde{h}(y) \, \mathrm{d}y \right| &\leq C_1 \lim_{c \to \infty} c^2 \int_\varepsilon^{b_\infty} e^{\left(\frac{c^2}{2} - r\right) d(\varepsilon) + cy} \, \mathrm{d}y \\
&\leq C_2 \lim_{c \to \infty} c^2 e^{\frac{c^2}{2} d(\varepsilon)} \int_\varepsilon^{b_\infty} e^{cy} \, \mathrm{d}y \\
&= C_2 \lim_{c \to \infty} c^2 e^{\frac{c^2}{2} d(\varepsilon)} \frac{1}{c} (e^{cy} - 1) = 0,
\end{aligned}$$

where C_1 and C_2 are some constants. The result follows because of $[0, b_\infty] = [0, \varepsilon) \cup [\varepsilon, b_\infty]$ and the linearity of the integral. $\square$

Lemma 7. *If $\tilde{h}(x) = mx + o(x)$ in a neighborhood of 0, then*

$$\lim_{c\to\infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\tilde{h}(y)\,\mathrm{d}y = \lim_{c\to\infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy} my\,\mathrm{d}y.$$

Proof. For all $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for all $y \in [0, \varepsilon_\delta]$ we have

$$|h(y) - my| < \delta y.$$

It follows with Lemma 6 that

$$\begin{aligned}
&\left|\lim_{c\to\infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\tilde{h}(y)\,\mathrm{d}y - \lim_{c\to\infty} c^2 \int_0^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy} my\,\mathrm{d}y\right| \\
&\quad= \left|\lim_{c\to\infty} c^2 \int_0^{\varepsilon_\delta} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\left(\tilde{h}(y) - my\right)\mathrm{d}y\right| \\
&\quad\le \left|\lim_{c\to\infty} c^2 \int_0^{\varepsilon_\delta} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\delta y\,\mathrm{d}y\right| \overset{\delta\to 0}{\to} 0.
\end{aligned}$$

□

We can now state the main theorems of this section.

Theorem 7. *If $\lim_{x\searrow 0}\frac{d(x)}{x^2}$ exists in $\bar{\mathbb{R}} = [-\infty, \infty]$, then*

$$\lim_{x\searrow 0}\frac{d(x)}{x^2} = -B,$$

i.e.,

$$d(x) = -Bx^2 + o(x^2)$$

where

$$B \approx 2.4503325097411$$

is the unique solution to

$$1 = \int_0^\infty z e^{-B\frac{z^2}{2}+z}\,\mathrm{d}z.$$

Proof. Assume that $\lim_{x\searrow 0}\frac{d(x)}{x^2}$ exists in $\bar{\mathbb{R}}$. We first show that d goes to 0 in quadratic order. Let us assume that d is of lower order, i.e.,

$$\lim_{x \searrow 0} \frac{d(x)}{-x^2} = \infty.$$

For all $M < 0$ there exists $\delta > 0$ such that $d(x) < Mx^2$ for all $x \leq \delta$. By Lemma 6 we have

$$\begin{aligned} m &= \lim_{c\to\infty} c^2 \int_0^\delta e^{\left(\frac{c^2}{2}-r\right)d(y)+cy} my \, \mathrm{d}y, \\ \Longrightarrow \qquad 1 &= \lim_{c\to\infty} c^2 \int_0^\delta ye^{\left(\frac{c^2}{2}-r\right)d(y)+cy} \, \mathrm{d}y. \end{aligned}$$

We substitute $z = cy$ and obtain

$$\begin{aligned} 1 &= \lim_{c\to\infty} c \int_0^{c\delta} \frac{z}{c} e^{\left(\frac{c^2}{2}-r\right)d\left(\frac{z}{c}\right)+z} \, \mathrm{d}z \\ &= \lim_{c\to\infty} \int_0^{c\delta} ze^{\left(\frac{c^2}{2}-r\right)d\left(\frac{z}{c}\right)+z} \, \mathrm{d}z \\ &\leq \lim_{c\to\infty} \int_0^{c\delta} ze^{\left(\frac{c^2}{2}-r\right)M\frac{z^2}{c^2}+z} \, \mathrm{d}z \\ &= \lim_{c\to\infty} \int_0^{c\delta} ze^{M\frac{z^2}{2}+z} \, \mathrm{d}z =: F(M). \end{aligned} \tag{4.18}$$

We see that $F(M) \to 0$ as $M \to -\infty$. This is a contradiction, since M can be chosen arbitrarily.

Let us now assume that d is of higher order than x^2, i.e.,

$$\lim_{x \searrow 0} \frac{d(x)}{-x^2} = 0.$$

For all $M < 0$ there exists $\delta' > 0$ such that $d(x) > Mx^2$ for all $x \leq \delta'$. Note that d is non-positive. From (4.18) we see that

$$\begin{aligned}
1 &= \lim_{c\to\infty} \int_0^{c\delta'} z e^{\left(\frac{c^2}{2}-r\right) d\left(\frac{z}{c}\right)+z} \,\mathrm{d}z \\
&\geq \lim_{c\to\infty} \int_0^{c\delta'} z e^{\left(\frac{c^2}{2}-r\right) M \frac{z^2}{c^2}+z} \,\mathrm{d}z \\
&= \lim_{c\to\infty} \int_0^{c\delta'} z e^{M \frac{z^2}{2}+z} \,\mathrm{d}z =: \tilde{F}(M).
\end{aligned}$$

We see that $\tilde{F}(M) \to \infty$ as $M \to 0$. This is a contradiction, since M can be chosen arbitrarily. We have shown that $d(x) = -Bx^2 + o(x^2)$ and want to calculate B. From (4.18) we have

$$\begin{aligned}
1 &= \lim_{c\to\infty} \int_0^{c\delta'} z e^{\left(\frac{c^2}{2}-r\right) d\left(\frac{z}{c}\right)+z} \,\mathrm{d}z \\
&= \lim_{c\to\infty} \int_0^{c\delta'} z e^{\left(-\frac{c^2}{2}-r\right)\left(B\frac{z^2}{c^2}+o\left(\frac{z^2}{c^2}\right)\right)+z} \,\mathrm{d}z \\
&= \int_0^{\infty} z e^{-B\frac{z^2}{2}+z} \,\mathrm{d}z.
\end{aligned}$$

Since the integral is finite, we can solve it for B and obtain

$$B \approx 2.4503325097411.$$

□

The following theorem generalizes Theorem 7 and does not assume the existence of $\lim_{x\searrow 0} \frac{d(x)}{x^2}$.

Theorem 8. *It holds that*

$$-B \leq \limsup_{x\searrow 0} \frac{d(x)}{x^2} < 0$$

and

$$-\infty < \liminf_{x\searrow 0} \frac{d(x)}{x^2} \leq -B$$

with

$$B \approx 2.4503325097411.$$

The proof of the theorem is basically a refined version of the proof of Theorem 7. It can be found in the appendix Section 4.7.

Remark 10. We note that B does not depend on m or r, i.e., it is a universal constant for stopping problems with the same order of $\tilde{h}$ in 0. This is however not surprising. Multiplying the payoff function g by a constant does not affect C, so m should not play a role. We could achieve a change of r by Brownian scaling, i.e., scaling t by a factor a and y by $\sqrt{a}$ and multiplying g by a constant. A quadratic function such as d is invariant to that.

Remark 11. If $\tilde{h}$ is not linear in 0 but rather

$$\tilde{h}(x) = \begin{cases} mx^\beta + o(x^\beta) & \text{if } x \geq 0 \\ -m'|x|^\beta + o(|x|^\beta) & \text{if } x < 0 \end{cases}$$

for $m, m', \beta > 0$ then the same procedure as above yields

$$\frac{m'}{m}\Gamma(\beta+1) = \int_0^\infty z^\beta e^{-B_\beta \frac{z^2}{2}+z}\,\mathrm{d}z$$

which can be solved for B_β. For $m' = m$, some values are: $B_0 \approx 3.9084$, $B_{\frac{1}{2}} \approx 3.0133$, $B_2 \approx 1.7814$ and $B_3 \approx 1.3984$.
With the implicit function theorem we see that B_β is continuous as a function of β, hence other choices for $\tilde{h}$ lead to the same limit behavior, e.g., $\tilde{h}(x) = \frac{x}{\log(|x|)}$ yields the same limit behavior for C_t as $\tilde{h}(x) = x$.

Remark 12. If we look from the other direction again and state the result for the function $b = d^{-1}$ we get (if the limit exists)

$$b(t) = \alpha\sqrt{t} + o(\sqrt{t})$$

for $t \to 0$, where

$$\alpha = \frac{1}{\sqrt{B}} \approx 0.638833$$

is the unique solution to $\alpha^3\Phi(\alpha) = (1-\alpha^2)\varphi(\alpha)$. This α is the same as in Example 5, which is not surprising since the example matches our setting. The same limit behavior was earlier derived for the exercise boundary of an American put with large dividend. We treat that case in the following Example 7. For related problems, see [WDH98], [LV03] and [LL05].

Example 7 (American put with large dividend)**.** We consider an American put option in the Black-Scholes model with interest rate r and continuously paid dividend q. The problem can be reduced to the following *canonical* optimal stopping problem (see [AL99])

$$V(t,x) = \sup_{t \le \tau \le 0} \mathrm{E}_{(t,x)}[g(\tau, W_\tau)]$$

where

$$g(t,x) = e^{-\rho t}\left(1 - e^{x + (\rho - \theta\rho - \frac{1}{2})t}\right)^+$$

with $\rho = \frac{r}{\sigma^2}$ (here σ^2 is the volatility of the model that we assume to be 1 for simplicity) and $\theta = \frac{q}{r}$. We consider the case $q > r$, i.e., $\theta < 1$. For $x < 0$ we have

$$-Ag(t,x) = e^{-\rho t}\left(\rho - \theta\rho e^{x + (\rho - \theta\rho - \frac{1}{2})t}\right).$$

The function $Ag(0,x)$ has a unique negative root in $x_0 = \log(\frac{1}{\theta}) = \log(\frac{r}{q})$ and $-Ag(0,x) \le 0$ for $x \ge x_0$, so we have

$$C_0 = \left(\log\left(\frac{r}{q}\right), \infty\right).$$

By the variable transformation $z = x + \log\left(\frac{r}{q}\right) + (\rho - \theta\rho - \frac{1}{2})t$ we get the stopping problem

$$V(t,z) = \sup_{t \le \tau \le 0} \mathrm{E}_{(t,z)}[e^{-\rho\tau} h(X_\tau)]$$

where $h(z) = (\rho - \rho e^z)^+$ and X is a Brownian motion with drift. The problem now matches the assumptions of Theorem 8, so we see that the continuation set of the *canonical* problem is $C = \{(t,x) \mid t < d(x)\}$ with

$$\lim_{x \nearrow \log\left(\frac{r}{q}\right)} \frac{d(x)}{x^2} = -B.$$

This result matches the limit behavior derived for the American put (and call) in [LV03]. Note that for $q \le r$ Theorem 8 is no longer directly applicable since $g(0, x)$ has a kink in 0 right at the boundary of C_0, so $\tilde{h}$ has a point mass in 0 and is not asymptotically symmetric in 0. It seems possible to use the methods described above to analyze these cases as well, we, however, run into more technical difficulties.

4.4.2 The multi-dimensional case

We now consider the n-dimensional discounted case. We will not give rigorous proofs, but explain rather heuristically how the methods from the previous subsection can be extended.

We change the notation slightly. For notational convenience, we substitute the parameter $\mathbf{c}$ by $a\mathbf{c}$, where $\mathbf{c} \in S^{n-1}$ and $a \in \mathbb{R}^+$. Our equation now reads

$$-\int_{C_0} e^{a\mathbf{c}\cdot\mathbf{y}} \tilde{h}(\mathbf{y})\, \mathrm{d}\mathbf{y} = \int_{C_\infty \setminus C_0} e^{\left(\frac{a^2}{2} - r\right) d(\mathbf{y}) + a\mathbf{c}\cdot\mathbf{y}} \tilde{h}(\mathbf{y})\, \mathrm{d}\mathbf{y}. \tag{4.19}$$

We assume for now that C_0 is strictly convex. Then for every $\mathbf{c}$ for which $\{a\mathbf{c} \mid a \in \mathbb{R}^+\} \cap C_\infty$ is bounded, there is a unique point $\mathbf{x}_c \in \partial C_0$ realizing $\sup\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x} \in \bar{C}_0\}$. An illustration can be found in Figure 4.3. The

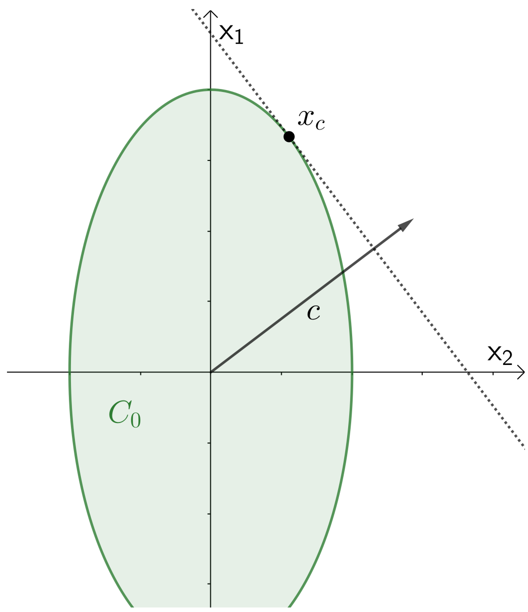

Figure 4.3: The setting in the two-dimensional case

underlying stopping problem is invariant under rotation and translation, so we can reset the coordinate system such that $\mathbf{x}_c$ lies in the origin and $\mathbf{c} = (1, 0, \ldots, 0)$, see Figure 4.4. Now (4.19) reads

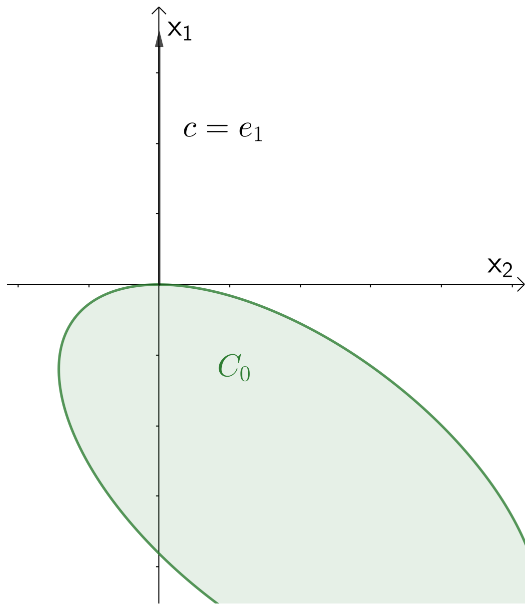

Figure 4.4: Figure 4.3 rotated and translated to $\mathbf{c} = (1, 0, \ldots, 0)$

$$-\int_{C_0} e^{ay_1}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{C_\infty \setminus C_0} e^{\left(\frac{a^2}{2}-r\right)d(\mathbf{y})+ay_1}\tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{4.20}$$

For simplicity, we assume that ∂C_0 has positive curvature, so it can be approximated in a neighborhood of 0 by $y_1 = -\sum_{i=2}^{n} \gamma_i y_i^2$ with constants $\gamma_i > 0$. The constructions are similar and lead to the same result, if we consider different approximations such as a (piecewise) constant boundary, corners or $x_1 \approx -\sum \gamma_i \operatorname{sgn}(x_i)|x_i|^p$ for $p \in \mathbb{R}_{\geq 0}$ in general.

Two-dimensional For simplicity let us look at the case $n = 2$. We set $\gamma = \gamma_2$ and calculate the limit of the left-hand side of (4.20):

$$\begin{aligned}
&\lim_{a\to\infty} -a^{\frac{5}{2}} \int_{C_0} e^{ay_1} \tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&\quad= \lim_{a\to\infty} -a^{\frac{5}{2}} \int_{-\infty}^{0} e^{ay_1} \int_{-\sqrt{\frac{y_1}{\gamma}}}^{\sqrt{\frac{y_1}{\gamma}}} m(y_1 - \gamma y_2^2)\,\mathrm{d}y_2\,\mathrm{d}y_1 = m\sqrt{\frac{\pi}{\gamma}}.
\end{aligned}$$

For the right-hand side of (4.20) we see that only neighborhoods of 0 are asymptotically relevant. It follows that

$$\begin{aligned}
&\lim_{a\to\infty} a^{\frac{5}{2}} \int_{C_\infty\setminus C_0} e^{\left(\frac{a^2}{2}-r\right)d(\mathbf{y})+ay_1} \tilde{h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&\quad= \lim_{a\to\infty} a^{\frac{5}{2}} \int_{-\infty}^{\infty}\int_{-\gamma y_2^2}^{\infty} m(y_1+\gamma y_2^2) e^{\left(\frac{a^2}{2}-r\right)d(\mathbf{y})+ay_1}\,\mathrm{d}y_1\,\mathrm{d}y_2
\end{aligned}$$

where we substitute $z_1 = ay_1$ and $z_2 = \sqrt{a}y_2$

$$= \lim_{a\to\infty} -\int_{-\infty}^{\infty}\int_{-a\gamma z_2^2}^{\infty} m(z_1+\gamma z_2^2) e^{\left(\frac{a^2}{2}-r\right)d\left(\frac{z_1}{a},\frac{z_2}{\sqrt{a}}\right)+z_1}\,\mathrm{d}z_1\,\mathrm{d}z_2$$

where we set $d(x_1, x_2) = -B(x_1 + \gamma x_2^2)^2$

$$\begin{aligned}
&= \lim_{a\to\infty} \int_{-\infty}^{\infty}\int_{-a\gamma z_2^2}^{\infty} m(z_1+\gamma z_2^2) e^{\left(\frac{a^2}{2}-r\right)\frac{-B}{a^2}(z_1+\gamma z_2^2)^2+z_1}\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \lim_{a\to\infty} \int_{-\infty}^{\infty}\int_{0}^{\infty} m z_1 e^{\left(\frac{a^2}{2}-r\right)\frac{-B}{a^2}z_1^2+z_1-\gamma z_2^2}\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} m z_1 e^{\frac{-B}{2}z_1^2+z_1-\gamma z_2^2}\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= m\sqrt{\frac{\pi}{\gamma}} \int_0^\infty z_1 e^{\frac{-B}{2}z_1^2+z_1}\,\mathrm{d}z_1.
\end{aligned}$$

Comparing the two sides we get

$$\begin{aligned} m\sqrt{\frac{\pi}{\gamma}} &= m\sqrt{\frac{\pi}{\gamma}} \int_0^\infty z_1 e^{\frac{-B}{2}z_1^2+z_1}\,\mathrm{d}z_1 \\ \iff \qquad 1 &= \int_0^\infty z_1 e^{\frac{-B}{2}z_1^2+z_1}\,\mathrm{d}z_1. \end{aligned}$$

This is the same equation as in the one-dimensional case, which is again solved by $B \approx 2.4503325$.

In higher dimensions the method works in exactly the same way.
These results have a nice interpretation that becomes clear when we write it in terms of distance to C_0. Let

$$r = r(\mathbf{y}) := \operatorname{dist}(\mathbf{y}, C_0)$$

where $\operatorname{dist}(\mathbf{y}, C_0)$ denotes the distance between $\mathbf{y}$ and C_0. Now for $\mathbf{y} \notin C_0$ close to ∂C_0 we have

$$d(\mathbf{y}) = -Br^2 + o(r).$$

So for t close to 0 we have that C_t is approximately C_0 inflated by the constant amount

$$\frac{1}{\sqrt{B}}\sqrt{t}$$

equally on each point.

4.5 Uniqueness

We show that C is defined uniquely by the Fredholm representation in the one-dimensional, discounted and one-sided case described in Section 4.3.2. We once more list all assumptions that we use for the uniqueness results.

Assumptions 1.

- The payoff function is of the form

$$e^{-rt}h(x)$$

with $r > 0$.

- The Volterra integral equation (4.4) holds.
- The integral
$$\int_{-\infty}^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\tilde{h}(y)\,\mathrm{d}y$$
exists.
- $\lambda(\tilde{h}(y)=0)=0$ where λ denotes the Lebesgue measure.
- The solution is one-sided, i.e., there exists a bounded function
$$b : \mathbb{R}_- \to \mathbb{R},$$
such that $C = \{(t,x) \mid x < b(t)\}$. Again, we set $d = b^{-1}$.

In this setting, we show that (4.11) defines b uniquely in the class of continuous and monotonic functions. The ansatz we use is not limited to this special case, in principle it can be used on multi-dimensional problems in a more general setting. This, however, brings a lot of technicalities, so we stick to the case described. Note that we do not make any assumptions on the differentiability of h. We only need that (4.4) holds. For notational convenience, we will state the proofs using $\tilde{h}$ as a function. The proofs, however, work in the same way if we understand $\tilde{h}$ as a measure and evaluate $\int_{-\infty}^{b_\infty} e^{\left(\frac{c^2}{2}-r\right)d(y)+cy}\,\mathrm{d}\tilde{h}(y)$ which includes standard American options, see [AL99].

Remark 13. Our proof is inspired by techniques introduced by Bruni and Koch in their 1985 work on the identifiability of mixtures of Gaussian densities, see [BK85]. To see the connection, one can interpret the Fredholm representation as a mixture of Gaussian densities. Indeed, we have

$$e^{cy+\frac{1}{2}c^2 s} = A_{s,y}\varphi\left(\frac{y}{-s}, \frac{1}{-s}\right)(c),$$

where $\varphi\left(\frac{y}{-s}, \frac{1}{-s}\right)$ is the density function of a normal distribution with mean $\frac{y}{-s}$ and variance $\frac{1}{-s}$ and $A_{s,y}$ is a positive function not depending on c. Bruni and Koch analyze mixtures of the form

$$f(c) = \int_D \varphi\Big(\lambda_1(x), \lambda_2(x)\Big)(c)\,\mathrm{d}\mu(x)$$

where D is compact, the mean λ_1 and the variance λ_2 are C^1 on D and bounded from above, λ_2 is additionally bounded away from 0. Under some additional assumptions they show that for given f the measure μ is defined uniquely by the equation, i.e., it is identifiable. They state that compactness of D is essential for their result. In our setting, however, the area of integration is not compact and mean and variance are not bounded, so we cannot directly use their results. Theorem 9 can be viewed as a result about identifiability in a special case of a non-compact D and unbounded λ_1 and λ_2.
We will keep in mind that we work with Gaussian mixtures with variable c, but we stick to the notation $e^{cy+\frac{1}{2}c^2 s}$.

The proof consist of the steps

1. We show that the area with the highest variances, i.e., the largest t, govern the limits $c \to \infty$, see Lemma 8.

2. We use 1. to show that if the Fredholm representation holds true for two different functions b and b', then this still holds true if we multiply the integrand by a polynomial $q(s, y)$, see Lemma 9.

3. We show that the polynomials (in fact we will use Laguerre exponential polynomials) lie dense in $L^2(\mathbb{R}^2)$, see Theorem 9.

4. We represent b and b' as part of measures μ and μ' on $\mathbb{R}^2$ and use 2. and 3. to show that $\mu = \mu'$, see Theorem 9.

We recall that we can write (4.11) as

$$-\frac{1}{\frac{c^2}{2} - r}\mathcal{L}\tilde{h}(c) = \int_{-\infty}^{0} \int_{0}^{b(s)} e^{cy+\frac{1}{2}c^2 s - rs}\tilde{h}(y)\,\mathrm{d}y\,\mathrm{d}s. \tag{4.21}$$

The integrand on the right-hand side is non-negative, the left-hand side is the known function $f(c)$. We want to show uniqueness of b in the class of positive continuous and monotone functions. We will use the following notation: Let $J \subset \mathbb{R}_{\leq 0}$ be measurable, $n, m \in \mathbb{N}_0$. We set

$$\begin{aligned}
I_J^b(n,m)(c) &:= \int_J \int_0^{b(s)} (-s+1)^n y^m e^{cy+\frac{1}{2}c^2 s - rs} \tilde{h}(y)\, \mathrm{d}y\, \mathrm{d}s, \\
I^b(n,m)(c) &:= I^b_{(-\infty,0]}(n,m)(c), \\
I_J^b(c) &:= I_J^b(0,0)(c),
\end{aligned}$$

and for two different continuous functions b and b' we define

$$\begin{aligned}
D_J(n,m)(c) &:= I_J^b(n,m)(c) - I_J^{b'}(n,m)(c) \\
D(n,m)(c) &:= D_{(-\infty,0]}(n,m)(c).
\end{aligned}$$

The following lemma states that only the parts with the largest variance, i.e., the largest t values in our setting, play a role for the limits $c \to \infty$.

Lemma 8. *Let $t \leq 0$ be fixed and $\varepsilon > 0$, then*

$$\lim_{c\to\infty} \frac{I^b_{(-\infty,t]}(n,m)(c)}{I^b_{[t-\varepsilon,t]}(n,m)(c)} = 1.$$

If additionally $b(t) \neq b'(t)$, then

$$\lim_{c\to\infty} \frac{D_{(-\infty,t]}(n,m)(c)}{D_{[t-\varepsilon,t]}(n,m)(c)} = 1.$$

Proof. Let (c_i) be a sequence with $c_i \to \infty$ for $i \to \infty$. We first show that for all $\varepsilon > 0$ and $t \leq 0$ it holds

$$\lim_{i\to\infty} \frac{I^b_{(-\infty,t]}(n,m)(c_i)}{I^b_{[t-\varepsilon,t]}(n,m)(c_i)} = 1. \tag{4.22}$$

Heuristically, this means that only the parts with the largest variances (i.e., $\frac{1}{-s}$) govern the limits. Since $I^b_{(-\infty,t]}(n,m) = I^b_{[t-\varepsilon,t]}(n,m) + I^b_{(-\infty,t-\varepsilon)}(n,m)$, equation (4.22) is equivalent to

$$\lim_{i\to\infty} \frac{I^b_{(-\infty,t-\varepsilon]}(n,m)(c_i)}{I^b_{[t-\varepsilon,t]}(n,m)(c_i)} = 0.$$

The numerator and denominator are positive. We derive an upper bound for $I^b_{(-\infty,t-\varepsilon]}(n,m)(c)$. Let $h_1 = \max\{\tilde{h}(y) \mid y \in [0, b_\infty]\}$ then

$$\begin{aligned} I^b_{(-\infty,t-\varepsilon]}(n,m)(c) &= \int_{-\infty}^{t-\varepsilon}\int_0^{b(s)} (-s+1)^n y^m e^{cy+\frac{1}{2}c^2 s - rs}\tilde{h}(y)\,\mathrm{d}y\,\mathrm{d}s \\ &\le \int_{-\infty}^{t-\varepsilon}\int_0^{b_\infty} (-s+1)^n b_\infty^m e^{cb_\infty+\frac{1}{2}c^2 s - rs} h_1\,\mathrm{d}y\,\mathrm{d}s \\ &= N_1 e^{cb_\infty} \int_{-\infty}^{t-\varepsilon} (-s+1)^n e^{\frac{1}{2}c^2 s - rs}\,\mathrm{d}s \\ &= N_1 e^{cb_\infty} \frac{1}{\frac{c^2}{2} - r} P(c,\varepsilon) e^{(\frac{c^2}{2}-r)(t-\varepsilon)} \end{aligned}$$

where N_1 is a constant and P is a polynomial in $\frac{1}{c}$ and ε.

We now derive a lower bound for $I^b_{[t-\varepsilon,t]}(n,m)(c)$. Let $0 < a_1 < a_2 \le b(t - \frac{\varepsilon}{2})$ such that $\tilde{h}(y)$ is strictly positive on $[a_1, a_2]$ and let $h_2 = \min\{\tilde{h}(y) \mid y \in [a_1, a_2]\} > 0$. Then,

$$\begin{aligned} &\int_{t-\varepsilon}^{t}\int_0^{b(s)} (-s+1)^n y^m e^{cy+\frac{1}{2}c^2 s - rs}\tilde{h}(y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad\ge \int_{t-\varepsilon}^{t-\frac{\varepsilon}{2}}\int_{a_1}^{a_2} (-t+\varepsilon+1)^n a_1^m e^{0+\frac{1}{2}c^2 s - rs} h_2\,\mathrm{d}y\,\mathrm{d}s \\ &\quad= N_2 \int_{t-\varepsilon}^{t-\frac{\varepsilon}{2}} e^{\frac{1}{2}c^2 s - rs}\,\mathrm{d}s \\ &\quad= N_2 \left(e^{(\frac{c^2}{2}-r)(t-\frac{\varepsilon}{2})} - e^{(\frac{c^2}{2}-r)(t-\varepsilon)} \right) \end{aligned}$$

where $N_2 > 0$ is a constant. Putting these results together we have

$$0 \le \lim_{i\to\infty} \frac{I^b_{(-\infty,-\varepsilon)}(n,m)(c_i)}{I^b_{[-\varepsilon,0]}(n,m)(c_i)} \le \lim_{i\to\infty} \frac{N_1 e^{c_i b_\infty} \frac{1}{\frac{c_i^2}{2}-r} P(c_i,\varepsilon) e^{(\frac{c_i^2}{2}-r)(t-\varepsilon)}}{N_2 \left(e^{(\frac{c_i^2}{2}-r)(t-\frac{\varepsilon}{2})} - e^{(\frac{c_i^2}{2}-r)(t-\varepsilon)} \right)} = 0$$

which shows the first claim.

Let now $b(t) \neq b'(t)$. We assume w.l.o.g. that $b(t) > b'(t)$. Since b and b' are continuous, there exists $\delta > 0$ such that $b(s) > b'(s)$ for all $s \in [t-\delta, t]$. Then, the term

$$D_{[t-\delta,t]}(n,m)(c_i) = \int_{t-\delta}^{t} \int_{b'(s)}^{b(s)} (-s+1)^n y^m e^{cy+\frac{1}{2}c^2 s - rs} \tilde{h}(y) \, \mathrm{d}y \, \mathrm{d}s$$

is positive and analogously to the calculations above we obtain

$$\lim_{i\to\infty} \frac{|D_{(-\infty,t-\delta]}(n,m)(c_i)|}{D_{[t-\delta,t]}(n,m)(c_i)} \leq \lim_{i\to\infty} \frac{I^{\max(b,b')}_{(-\infty,t-\delta]}(n,m)(c_i)}{D_{[t-\delta,t]}(n,m)(c_i)} = 0.$$

If $\varepsilon \leq \delta$ we can set $\delta = \varepsilon$ and we are done. If $\varepsilon > \delta$, we have

$$\lim_{i\to\infty} \frac{|D_{(-\infty,t-\varepsilon]}(n,m)(c_i)|}{D_{[t-\varepsilon,t]}(n,m)(c_i)} \leq \lim_{i\to\infty} \frac{|D_{(-\infty,t-\varepsilon]}(n,m)(c_i)|}{D_{[t-\delta,t]}(n,m)(c_i) - I^{\max(b,b')}_{(-\infty,t-\delta]}(n,m)(c_i)} = 0.$$

For c large enough the denominator is positive, hence, it follows

$$\lim_{i\to\infty} \frac{D_{(-\infty,t]}(n,m)(c_i)}{D_{[t-\varepsilon,t]}(n,m)(c_i)} = 1.$$

$\square$

The next lemma states that if the integral on the right-hand side of (4.21) is the same for two different stopping boundaries, then this property is not changed if we multiply the integrand with a polynomial $q(s, y)$.

Lemma 9. *If $I^b(c) = I^{b'}(c)$ for all $c > \sqrt{2r}$, then*

$$I^b(n,m)(c) = I^{b'}(n,m)(c)$$

for all $c > \sqrt{2r}$ and $n, m \in \mathbb{N}_0$.

Proof. We prove the claim by induction. Let $n, m = 0$ then $I^b(n,m)(c) = I^{b'}(n,m)(c)$ by assumption. Let now $n, m \neq 0$ be fixed and $I^b(l,k)(c) = I^{b'}(l,k)(c)$, for all $c > \sqrt{2r}$ and $l \leq n, k \leq m$. We multiply

$$I^b(n,m)(c) = \int_{-\infty}^{0} \int_{0}^{b(s)} (-s+1)^n y^m e^{cy+\frac{1}{2}c^2 s - rs} \tilde{h}(y) \, \mathrm{d}y \, \mathrm{d}s$$

with $e^{-\frac{c^2}{2}}$, take the derivative in c and we have

$$\begin{aligned}&\frac{\partial}{\partial c}\int_{-\infty}^{0}\int_{0}^{b(s)}(-s+1)^{-n}y^m e^{cy+\frac{1}{2}c^2(s-1)-rs}\tilde{h}(y)\,\mathrm{d}y\,\mathrm{d}s\\&=\int_{-\infty}^{0}\int_{0}^{b(s)}\Big(y+c(s-1)\Big)(-s+1)^{-n}y^m e^{cy+\frac{1}{2}c^2(s-1)-rs}\tilde{h}(y)\,\mathrm{d}y\,\mathrm{d}s\\&=e^{-\frac{c^2}{2}}\left(I^b(n,m+1)(c)-cI^b(n+1,m)(c)\right).\end{aligned}$$

For b' we obtain the same result and together we have

$$c\Big(I^b(n+1,m)(c)-I^{b'}(n+1,m)(c)\Big)=I^b(n,m+1)(c)-I^{b'}(n,m+1)(c). \quad (4.23)$$

Let us assume that $I^b(n+1,m)\neq I^{b'}(n+1,m)$. $I^b(n+1,m)$ is analytic in c, hence there exists a series (c_i) with $c_i\to\infty$ such that

$$I^b(n+1,m)(c_i)\neq I^{b'}(n+1,m)(c_i)$$

for all $i\in\mathbb{N}$. We can solve (4.23) for c_i and find

$$c_i=\frac{I^b(n,m+1)(c_i)-I^{b'}(n,m+1)(c_i)}{I^b(n+1,m)(c_i)-I^{b'}(n+1,m)(c_i)}=\frac{D(n,m+1)(c_i)}{D(n+1,m)(c_i)}. \quad (4.24)$$

We show that the right-hand side is bounded, in contradiction to the assumption $c_i\to\infty$.

Let $t^*=\sup\{t\mid b(t)\neq b'(t)\}$. If the set is empty, there is nothing to show. Let $(t_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence with $b(t_j)\neq b'(t_j)$ and $\lim_{j\to\infty}t_j=t^*$.

For every t_j we find $t_{j+1}-t_j>\varepsilon_j>0$ such that $b(s)\neq b'(s)$ for all

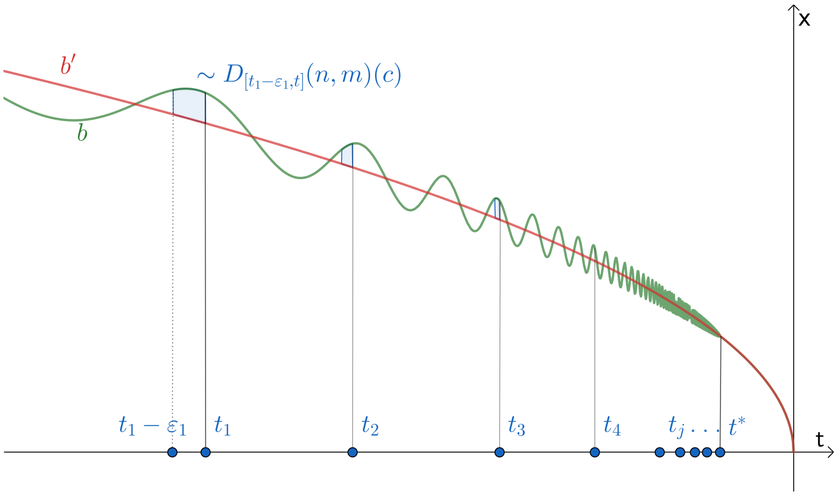

Figure 4.5: Visualization of the setting in Lemma 9

$s \in [t_j - \varepsilon_j, t_j]$. We use Lemma 8 and the fact that b is decreasing to obtain

$$\begin{aligned}
\lim_{i\to\infty} \frac{D_{(-\infty,t_j]}(n,m+1)(c_i)}{D_{(-\infty,t_j]}(n+1,m)(c_i)} &= \lim_{i\to\infty} \frac{D_{[t_j-\varepsilon_j,t_j]}(n,m+1)(c_i)}{D_{[t_j-\varepsilon_j,t_j]}(n+1,m)(c_i)} \\
&\leq \lim_{i\to\infty} \frac{b(t_j-\varepsilon_j)D_{[t_j-\varepsilon_j,t_j]}(n,m)(c_i)}{(t_j+1)D_{[t_j-\varepsilon_j,t_j]}(n,m)(c_i)} \\
&= \lim_{i\to\infty} \frac{b(t_j-\varepsilon_j)}{(t_j+1)} \\
&\leq b(t_j-\varepsilon_j).
\end{aligned}$$

The sequence $(b(t_j - \varepsilon_j))_{j\in\mathbb{N}}$ is decreasing and since b is continuous we have

$$\lim_{j\to\infty} b(t_j-\varepsilon_j) = b(t^*) < \infty.$$

By dominated convergence we have

$$\lim_{j\to\infty}\lim_{i\to\infty} \frac{D_{(-\infty,t_j]}(n,m+1)(c_i)}{D_{(-\infty,t_j]}(n+1,m)(c_i)} = \lim_{i\to\infty} \frac{D_{(-\infty,t^*]}(n,m+1)(c_i)}{D_{(-\infty,t^*]}(n+1,m)(c_i)} < \infty.$$

For all $t > t^*$ we have $b(t) = b'(t)$, hence

$$\lim_{i\to\infty} \frac{D(n,m+1)(c_i)}{D(n+1,m)(c_i)} < \infty,$$

in contradiction to (4.24). It follows $I^b(n+1,m) = I^{b'}(n+1,m)$. Plugging this into (4.23) we see that also $I^b(n,m+1) = I^{b'}(n,m+1)$.

□

We can now state our main theorem on the uniqueness of b.

Theorem 9. *Given a stopping problem that fulfills Assumptions 1. If there is $N \in \mathbb{R}$ such that the Fredholm representation* (4.11) *holds for all $c > N$, then b is determined uniquely by* (4.11) *in the class of continuous monotone functions.*

Proof. Let $I^b(c) = I^{b'}(c)$ for all $c > \sqrt{2r}$. By Lemma 9 we have $I^b(n,m)(c) = I^{b'}(n,m)(c)$ for all $c > \sqrt{2r}$ and $n, m \in \mathbb{R}$. We rewrite $I^b(n,m)(c)$ as

$$I^b(n,m)(c) = \int e^s(-s+1)^n y^m \mu(ds, dy)$$

with a measure $\mu = f \circ \lambda$ where f denotes the density function

$$f(s,y) := \mathbb{1}_{\{0 \le y \le b(t), t \le 0\}} e^{-s} e^{cy + \frac{1}{2}c^2 s - rs} \tilde{h}(y)$$

and λ denotes the Lebesgue measure on $\mathbb{R}^2$. A measure μ' with density function f' is defined analogously via b'.
Functions of the form $p(-s+1)e^s$ for a polynomial p are called *Laguerre exponential polynomials* and they lie dense in $L^2(\mathbb{R}_{\le 0})$, see [AK72, Lemma 1. (ii)]. For $B := \mathbb{R}_{\le 0} \times [0, M]$, the family $\{q(-s,y)e^s | q \text{ is a polynomial on } B\}$ lies dense in $L^2(B)$. By Lemma 9 and linearity of the integral we have

$$\int e^s q(-s,y) f(s,y)\,\mathrm{d}\lambda = \int e^s q(-s,y) f'(s,y)\,\mathrm{d}\lambda$$

for all polynomials q. Let now $g \in L^2(B)$ and q_n be a sequence of polynomials with $e^s q_n \xrightarrow{L^2} g$. For c large enough, f and f' are positive, bounded and in L^2. It follows that

$$\lim_{n\to\infty} \int e^s q_n(-s,y) f(s,y)\,\mathrm{d}\lambda = \int g(s,y) f(s,y)\,\mathrm{d}\lambda$$

and

$$\lim_{n\to\infty}\int e^s q_n(-s,y)f'(s,y)\,\mathrm{d}\lambda = \int g(s,y)f'(s,y)\,\mathrm{d}\lambda,$$

hence,

$$\int g(s,y)\mu(ds,dy) = \int g(s,y)\mu'(ds,dy)$$

for all $g \in L^2(B)$.
If follows that $\mu = \mu'$ $\tilde{h} \circ \lambda$-a.e. Since b and b' are continuous and $\lambda(\{\tilde{h}(y) = 0\}) = 0$ we have $b = b'$. $\square$

Remark 14. It is not straightforward to extend the proof immediately to the general setting (4.8), but some assumptions are easy to relax.

- We used that b is bounded in the discounted case. For the proof it is enough if b grows at most linearly for $t \to \infty$. That would then include cases like Example 6.
- We used monotonicity of b for Lemma 9. It would be enough to assume that b is bounded on compact intervals, which follows from the continuity of b.
- The assumption $\lambda(\{\tilde{h}(y) = 0\}) = 0$ is not necessary. If we do not assume this we would get uniqueness up to $-Ag \circ \lambda$ nullsets which are not necessarily λ-nullsets.
- The assumption that (4.8) holds for all $c > N$ can be relaxed to (4.8) holds for $(c_i) = (c_i)_{i\in\mathbb{N}}$ where the set of limit points of (c_i) is not bounded.

Example 8. With Theorem 9 we can see that the continuation set given in Example 6 is the correct solution without a priori argumentation via Brownian scaling.

4.6 Numerics

We show how the Fredholm representation can be used to numerically approximate the optimal stopping boundary ∂C. We stick to the one-dimensional one-sided and discounted case where we want to approximate the function d.

One ansatz is discretizing (4.11) and then minimizing the (quadratic) error via some optimization algorithm. Equation (4.11) can be written as

$$0 = \int_0^{b_\infty} e^{(\frac{c^2}{2}-r)d(y)} e^{cy} \tilde{h}(y)\,\mathrm{d}y + \int_{-\infty}^0 e^{cy}\tilde{h}(y)\,\mathrm{d}y \tag{4.25}$$

for all $c > \sqrt{2r}$. The second integral does not depend on d, so to shorten notation we write $D(c) := \int_{-\infty}^0 e^{cy}\tilde{h}(y)\,\mathrm{d}y$. We choose a number N of evaluation points $0 = y_1 < \ldots < y_N = b_\infty$ and set $d_n := d(y_n)$ for $n = 1, \ldots, N$. We want to evaluate (4.25) for M values $c_l > \sqrt{2r}$ with $l = 1, \ldots, M$. We model d as piecewise constant, i.e.,

$$d(y) \approx \sum_{n=1}^{N-1} 1_{[y_n, y_{n+1})} d_n.$$

Putting d into (4.25) we get for $l = 1, \ldots, M$

$$0 \approx D(c_l) + \sum_{n=1}^{N-1} e^{(\frac{c^2}{2}-r)d_n} \int_{y_n}^{y_{n+1}} e^{y}\tilde{h}(y)\,\mathrm{d}y.$$

It is useful to weight the error depending on the problem and on c. For every c_l we choose a convex function $F_{c_l} : \mathbb{R} \to \mathbb{R}^+$ with a unique minimum in 0 and minimize

$$f(d) = \sum_{l=1}^{M} F_{c_l}\left(D(c_l) + \sum_{n=1}^{N-1} e^{(\frac{c^2}{2}-r)d_n} \int_{y_n}^{y_{n+1}} e^{y}\tilde{h}(y)\,\mathrm{d}y\right) \tag{4.26}$$

numerically. Additionally, we can assume that $d_i \le d_{i+1}$, since d is non-increasing.

Upper and lower bounds For the numeric minimization to be stable it is very helpful to have suitable upper and lower bounds for d. Clearly, $d_1^u := 0$

is an upper bound, hence

$$\int_{-\infty}^{b_\infty} e^{(\frac{c^2}{2}-r)d_1^u(y)} e^{cy} \tilde{h}(y)\,\mathrm{d}y \geq 0$$

for all $c > \sqrt{2r}$. We define a lower bound for $x > 0$ via

$$d_1^l(x) := \inf\left\{t \mid \int_{-\infty}^{b_\infty} e^{(\frac{c^2}{2}-r)\tilde{d}_1^u(y,x)} e^{cy} \tilde{h}(y)\,\mathrm{d}y \geq 0,\ \forall c > \sqrt{2r}\right\}$$

where

$$\tilde{d}_1^u(y,x) := t1_{[x,b_\infty]}(y) \wedge d_1^u(y).$$

Since d_1^l is a lower bound we have

$$\int_{-\infty}^{b_\infty} e^{(\frac{c^2}{2}-r)d_1^l(y)} e^{cy} \tilde{h}(y)\,\mathrm{d}y \leq 0$$

for all $c > \sqrt{2r}$. We can now find a better upper bound via

$$d_2^u(x) := \sup\left\{t \mid \int_{-\infty}^{b_\infty} e^{(\frac{c^2}{2}-r)\tilde{d}_1^l(y,x)} e^{cy} \tilde{h}(y)\,\mathrm{d}y \leq 0,\ \forall c > \sqrt{2r}\right\}$$

where

$$\tilde{d}_1^l(y,x) := t1_{[0,x]}(y) \vee d_1^l(y).$$

Repeating this procedure improves the bounds slightly. They, however, do not converge to d.

Example 9. Let $h(x) = x$ and $r = 1$. We have $\tilde{h}(x) = x$, $b_0 = 0$, $b_\infty = \sqrt{\frac{1}{2}}$ and

$$D(c) = \frac{1}{c^2}.$$

We set

$$F_c(x) = \left(c^2 x + \frac{1}{1 + c^2 x}\right)^2,$$

$N = 60$, $M = 40$, $y_k = \frac{b_\infty(k-1)}{N-1}$, and $c_l = \sqrt{2r} + \frac{l}{10}$. A plot of the resulting stopping boundary is given in Figure 4.6. The plot nicely reflects the boundary behavior described in Section 4.4. The function $-Bx^2 \approx -2.4503x^2$ is plotted as a reference.

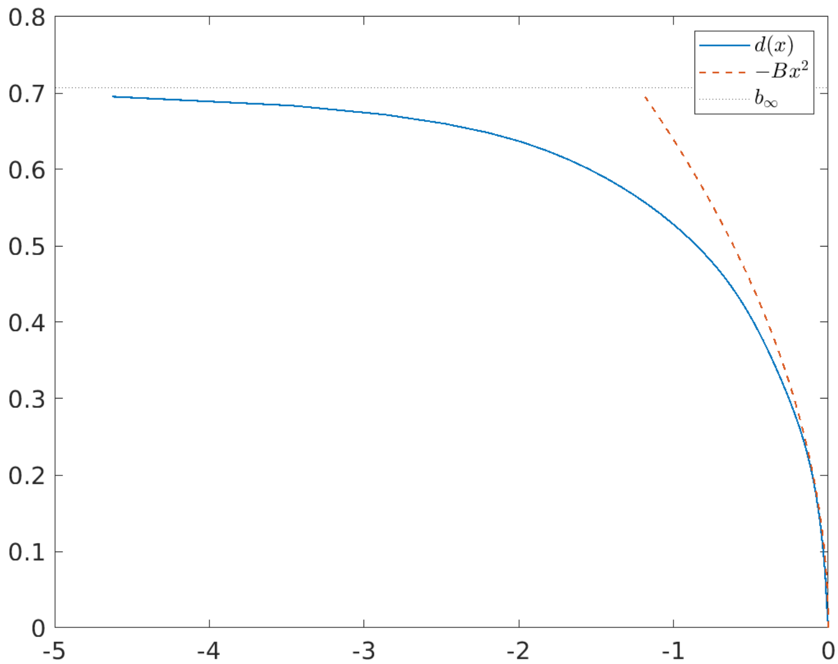

Figure 4.6: The stopping boundary from Example 9. As a reference the *limit* function $-Bx^2$ and b_∞ are plotted as well.

4.7 Appendix: Proofs

Proof of interchangeability of limit and integral in the proof of Theorem 6. We want to use the dominated convergence theorem. To do so we split the continuation set C into two parts. On one part the integrand in (4.9) is dominated by the limit function $e^{\mathbf{c}\cdot\mathbf{y}+\frac{\|\mathbf{c}\|^2}{2}s}(-Ag)(s,\mathbf{y})$. The other part we denote by A_t and we construct an integrable upper bound in the following.

For notational convenience, we set

$$G_\varepsilon := \{(t, -t(\mathbf{c}+\mathbf{a})) \mid t \leq 0, \|\mathbf{a}\| < \varepsilon\}.$$

Let $\mathbf{c} \in \mathbb{R}^n$ such that there exists $\varepsilon > 0$ such that $G_\varepsilon \cap C$ is bounded and the integral

$$0 = \int_{-\infty}^{0} \int_{C_s} e^{\mathbf{c}\cdot\mathbf{y}+\frac{\|\mathbf{c}\|^2}{2}s}(-Ag)(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s$$

exists. We define

$$A_t := \left\{ (s, \mathbf{y}) \mid \left(\frac{-t}{s-t} \right)^{\frac{n}{2}} e^{\frac{2\mathbf{c}\cdot\mathbf{y}t^2 + t\|\mathbf{y}\|^2 + \|\mathbf{c}\|^2 t^2 s}{2(t^2 - st)}} > e^{\mathbf{c}\cdot\mathbf{y} + \frac{\|\mathbf{c}\|^2}{2} s} \right\}. \tag{4.27}$$

We rearrange the defining inequality in (4.27) to

$$\begin{aligned} &\left(\frac{-t}{s-t} \right)^{\frac{n}{2}} e^{-\frac{\|\mathbf{c}s + \mathbf{y}\|^2}{2(s-t)}} > 1 \\ &\implies \quad \|\mathbf{c}s + \mathbf{y}\| < \sqrt{n(t-s) \ln \left(1 - \frac{s}{t} \right)}, \end{aligned}$$

which converges to

$$\|\mathbf{c}s + \mathbf{y}\| < \sqrt{n(-s)}$$

as $t \to -\infty$. Since the logarithm is concave, we have $A_t \subset A_u$ for $u < t$, and hence

$$A_\infty := \bigcup_{t<0} A_t = \left\{ (s, \mathbf{y}) \mid \|\mathbf{c}s + \mathbf{y}\| < \sqrt{n(-s)} \right\}.$$

We have that

$$A_\infty \setminus G_\varepsilon$$

is bounded, so $A_\infty \cap C$ is bounded, since $C \cap G_\varepsilon$ is bounded by assumption. For t small enough the functions

$$(s, \mathbf{y}) \mapsto \left(\frac{-t}{s-t} \right)^{\frac{n}{2}} e^{-\frac{\|\mathbf{c}s + \mathbf{y}\|^2}{2(s-t)}}$$

are all bounded on A_∞ by some $M < \infty$. So, on C we have that

$$M \left| e^{\mathbf{c}\cdot\mathbf{y} + \frac{\|\mathbf{c}\|^2}{2} s} (-Ag)(s, \mathbf{y}) \right|$$

is an integrable upper bound for

$$\left| \left(\frac{-t}{s-t} \right)^{\frac{n}{2}} e^{\frac{2\mathbf{c}\cdot\mathbf{y}t^2 + t\|\mathbf{y}\|^2 + \|\mathbf{c}\|^2 t^2 s}{2(t^2 - st)}} (-Ag)(s, \mathbf{y}) \right|.$$

The result follows by the dominated convergence theorem. $\square$

Proof of Theorem 8. It follows immediately from the proof of Theorem 7 that

$$\liminf_{x \searrow 0} \frac{d(x)}{x^2} \leq -B \leq \limsup_{x \searrow 0} \frac{d(x)}{x^2}.$$

Let us now assume that

$$\limsup_{x \searrow 0} \frac{d(x)}{x^2} = 0,$$

then there exists a decreasing sequence $(x_n)_{n \in \mathbb{N}} > 0$ such that $d(x_n) > -\frac{1}{n} x_n^2$. By Lemma 6 and 7 we have for $\delta > 0$ that

$$\begin{aligned} m &= \lim_{c \to \infty} c^2 \int_0^{\delta} e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} m y \, \mathrm{d}y, \\ \implies \quad 1 &= \lim_{c \to \infty} c^2 \int_0^{\delta} y e^{\left(\frac{c^2}{2} - r\right) d(y) + cy} \, \mathrm{d}y. \end{aligned}$$

We substitute $z = cy$ and obtain

$$1 = \lim_{c \to \infty} \int_0^{c\delta} z e^{\left(\frac{c^2}{2} - r\right) d\left(\frac{z}{c}\right) + z} \, \mathrm{d}z. \tag{4.28}$$

The function d is non-increasing and we can choose $\delta = x_n$. Note that a lower bound for d gives an upper bound for the integral. An illustration of the following estimate is given in Figure 4.7. We have

$$\begin{aligned} \int_0^{c\delta} z e^{\left(\frac{c^2}{2} - r\right) d\left(\frac{z}{c}\right) + z} \, \mathrm{d}z &\leq \int_0^{cx_n} z e^{-\left(\frac{c^2}{2}\right) \frac{x_n^2}{n} + z} \, \mathrm{d}z \\ &= [e^{cx_n}(cx_n - 1) + 1] e^{-\frac{c^2 x_n^2}{2n}}. \end{aligned}$$

We set $c_n := \frac{n}{x_n}$, so we have $c_n \to \infty$ as $n \to \infty$ and

$$[e^{c_n x_n}(c_n x_n - 1) + 1] e^{-\frac{c_n^2 x_n^2}{2n}} = e^{\frac{n}{2}}(n-1) + e^{-\frac{n}{2}} \overset{n \to \infty}{\to} \infty.$$

Hence, we obtain

$$\limsup_{c \to \infty} \int_0^{c\delta} z e^{\left(\frac{c^2}{2} - r\right) d\left(\frac{z}{c}\right) + z} \, \mathrm{d}z \leq \limsup_{c \to \infty} e^{\frac{n}{2}}(n-1) + e^{-\frac{n}{2}} = \infty$$

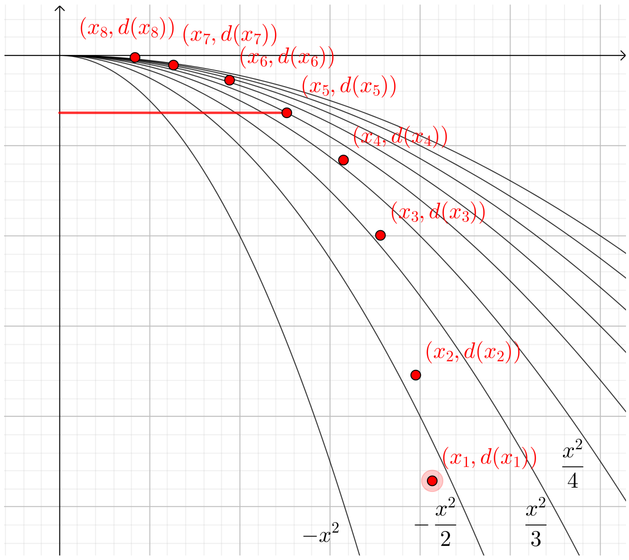

Figure 4.7: A sequence of points $(x_n, d(x_n))$ for $\limsup_{x \searrow 0} \frac{d(x)}{x^2} = 0$. The red line denotes a lower bound for d with $n = 5$.

which is a contradiction to (4.28).
So, we know that there is $M < 0$ with

$$\limsup_{x \searrow 0} \frac{d(x)}{x^2} = M.$$

We now assume that

$$\liminf_{x \searrow 0} \frac{d(x)}{x^2} = -\infty.$$

Let $(x_n)_{n \in \mathbb{N}}$ be a decreasing sequence with $x_n \to 0$ and $d(x_n) \leq Mn^2 x_n^2$. Now,

$$f_n(x) := \begin{cases} Mn^2 x_n^2, & x_n \leq x \leq n x_n, \\ Mx^2, & \text{else}, \end{cases}$$

is an upper bound for d for all $n \in \mathbb{N}$ by monotonicity of d. For an illustration see Figure 4.8. We evaluate the integral in (4.28) and obtain

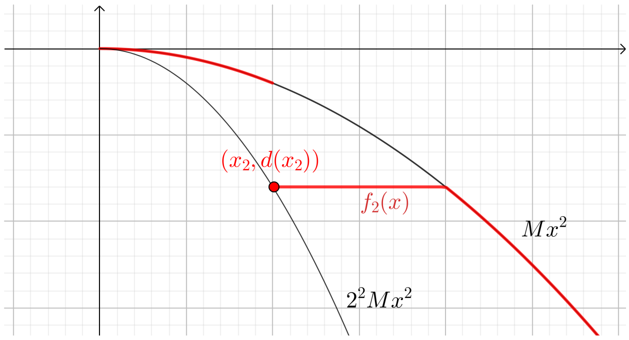

Figure 4.8: The upper bound f_n for $n = 2$. For large n the constant part takes up *most* of the relevant area.

$$\begin{aligned}\int_0^\infty ze^{\left(\frac{c^2}{2}-r\right)d\left(\frac{z}{c}\right)+z}\,\mathrm{d}z &\leq \int_0^\infty ze^{\left(\frac{c^2}{2}-r\right)f_n\left(\frac{z}{c}\right)+z}\,\mathrm{d}z \\ &= \int_0^{cx_n} ze^{M\left(\frac{1}{2}z^2-\frac{r}{c^2}\right)+z}\,\mathrm{d}z \\ &\quad + \int_{cx_n}^{ncx_n} ze^{Mn^2\left(\frac{c^2x_n^2}{2}-rx_n^2\right)+z}\,\mathrm{d}z \\ &\quad + \int_{ncx_n}^\infty ze^{M\left(\frac{1}{2}z^2-\frac{r}{c^2}\right)+z}\,\mathrm{d}z.\end{aligned}$$

We see that for $n \to \infty$ and $c \to \infty$ the discount rate r is asymptotically irrelevant, so we can approximate the right-hand side of the previous equation by

$$\begin{aligned}&\approx \int_0^{cx_n} ze^{M\frac{1}{2}z^2+z}\,\mathrm{d}z + \int_{cx_n}^{ncx_n} ze^{\frac{Mn^2c^2x_n^2}{2}+z}\,\mathrm{d}z + \int_{ncx_n}^\infty ze^{M\frac{1}{2}z^2+z}\,\mathrm{d}z \\ &= 2M^{\frac{3}{2}}\left[2\sqrt{M}e^{cx_n-\frac{Mc^2x_n^2}{2}} + \sqrt{2\pi}e^{\frac{1}{2M}}\left(\operatorname{erf}\left(\frac{Mcx_n-1}{\sqrt{2M}}\right)+\operatorname{erf}\left(\frac{1}{\sqrt{2M}}\right)\right)\right] \\ &\quad + e^{\frac{Mn^2c^2x_n^2}{2}}\left((1-cx_n)e^{cx_n}+(-1+ncx_n)e^{ncx_n}\right) \\ &\quad + 2M^{\frac{3}{2}}\left[e^{ncx_n-\frac{Mn^2c^2x_n^2}{2}} + \sqrt{2\pi}e^{\frac{1}{2M}}\left(\operatorname{erf}\left(\frac{Mncx_n-1}{\sqrt{2M}}+1\right)\right)\right] =: t_n(c).\end{aligned}$$

The function t_n has a unique minimum, whose argument is denoted by c_n and $t_n(c_n) \to 0$ as $n \to \infty$. For an illustration see Figure 4.9. Put together

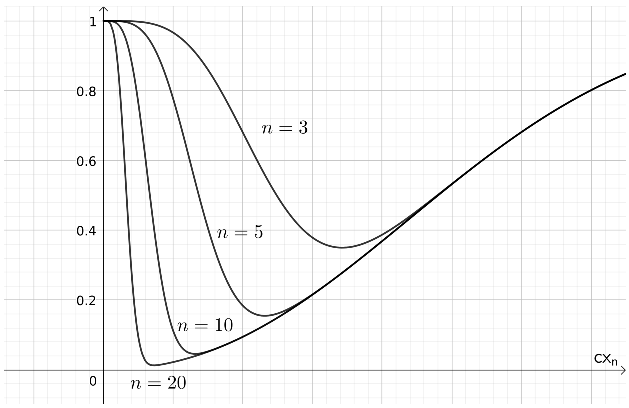

Figure 4.9: $t_n(cx_n)$ for some values of n and $M = -B$. The horizontal c-axis is stretched by x_n

we have

$$\liminf_{c\to\infty} \int_0^{c\delta} ze^{\left(\frac{c^2}{2}-r\right)d\left(\frac{z}{c}\right)+z}\,\mathrm{d}z \leq \liminf_{c\to\infty} t_n(c) \leq \lim_{n\to\infty} t_n(c_n) = 0$$

which is a contradiction to (4.28). This completes the proof. $\square$

4.8 Appendix: Applicability of Dynkin's formula

For the representation (4.4) to hold, a version of Dynkin's formula has to be applicable, we will briefly discuss here when this is the case.

4.8.1 Itô's formula

There is a rich literature on generalizations of Itô's formula. The usual way to derive generalizations of Dynkin's formula is to use theses Itô-type formulas and take their expectation. We illustrate this in an example. One of the most prominent examples of an Itô formula tailored to application in optimal stopping is due to Peskir in 2007 [Pes07]. It is stated for general semimartingales in $\mathbb{R}^n$, we state it here adapted to our setting.

Theorem 10. *Let W be an n-dimensional standard Brownian motion, $d : \mathbb{R}^n \to \mathbb{R}_{\leq 0}$ a continuous function such that $d(W)$ is a semimartingale, $C := \{(t, \boldsymbol{x}) \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid t < d(\boldsymbol{x})\}$ and $S := \{(t, \boldsymbol{x}) \in \mathbb{R}_{\leq 0} \times \mathbb{R}^n \mid t \geq d(\boldsymbol{x})\}$. Let furthermore $F : \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}$ be $C^{1,2}$ on the closed sets $\bar{C}$ and $\bar{S}$ and $A = \frac{\partial}{\partial t} + \frac{1}{2}\Delta$ be the characteristic operator of W, then the following change of variable formula for $F = F(t, \boldsymbol{x})$ holds:*

$$\begin{aligned} F(0, W_0) = F(t, W_t) &+ \int_t^0 AF(s, W_s)\,\mathrm{d}s \\ &+ \sum_{i=1}^n \int_t^0 \frac{\partial}{\partial x_i} F(s, W_s)\,\mathrm{d}W_s^i \\ &+ \frac{1}{2} \int_t^0 \left(\frac{\partial}{\partial t} F(s+, W_s) - \frac{\partial}{\partial t} F(s-, W_s) \right) \mathbb{1}_{\{W_s = b(W_s)\}}\,\mathrm{d}L_s^b \end{aligned}$$

where L^b denotes the local time of W spent on b (see [Pes07] for details).

Note that the original theorem allows a parametrization over any space component x_i as well. For us it is, however, often convenient to parametrize for the time t. For discounted problems (see Section 4.3.1) we can parametrize ∂C via a function $d : \mathbf{x} \mapsto t$.

We apply Theorem 10 to the value function V. We assume that we have a parametrization d of the stopping boundary. For many problems of interest we can assume that the smooth-fit principle holds, so the last term involving the local time disappears. The formula then reads

$$V(0, W_0) = V(t, W_t) + \int_t^0 AV(s, W_s)\,\mathrm{d}s + \sum_{i=1}^n \int_t^0 \frac{\partial}{\partial x_i} V(s, W_s)\,\mathrm{d}W_s^i. \quad (4.29)$$

On the continuations set C we have $AV = 0$, on the stopping set S we have $V = g$, so (4.29) reads

$$\begin{aligned} g(0, W_0) &= V(0, W_0) \\ &= V(t, W_t) + \int_t^0 Ag(s, W_s) \mathbb{1}_{\{(s, W_s) \in S\}}\,\mathrm{d}s \\ &\quad + \sum_{i=1}^n \int_t^0 \frac{\partial}{\partial x_i} V(s, W_s)\,\mathrm{d}W_s^i. \end{aligned} \quad (4.30)$$

If the last term is a martingale, then taking the expectation conditioned on $W_t = x$ leads to

$$\mathrm{E}_{(t,\mathbf{x})}\, g(0, W_0) = V(t, \mathbf{x}) + \mathrm{E}_{(t,\mathbf{x})} \left[\int_t^0 Ag(s, W_s) \mathbb{1}_{\{(s,W_s)\in S\}} \,\mathrm{d}s \right]$$

which can be rearranged to (4.3). For typical stopping problems the last term in (4.30) is indeed a martingale, but has to be checked for specific problems individually. A sufficient condition is

$$\sum_{i=1}^{n} \mathrm{E} \left[\int_t^0 \left(\frac{\partial}{\partial x_i} V(s, W_s) \right)^2 \mathrm{d}s \right] < \infty.$$

Often, this can be seen just by properties of the gain function g, e.g., for discounted problems a sufficient condition is that all partial derivatives $\frac{\partial}{\partial x_i} h(\mathbf{x})$ of the payoff function h are bounded. More details on these constructions can be found in [PS06].

The main problem of Theorem 10 is that we have to check whether $d(W)$ is a semimartingale. This condition holds if d is nice enough, which is, however, rather difficult to check. Sometimes properties of d can be deduced from properties of the gain function g, but this usually has to be done for specific problems. Examples of sufficient properties of d for $d(W)$ to be a semimartinagale are:

- d is C^2.
- Some generalized Itô-type formula is applicable to d itself. For a Brownian motion as the driving process – as in our setting – this is true if d is in the Sobolev space $W^{1,2}_{\mathrm{loc}}$, see [FP00]. (Weak) C^1 regularity is often easier to show than C^2 regularity.
- d is convex or concave.

There is also the question when we can parametrize ∂C in the sense of Theorem 10. In the discounted case described in Section 4.3.1 this can simply be done via the function $d(\mathbf{x}) = \sup \{t \le 0 \mid (t, \mathbf{x}) \in C\}$. In the more general case we might need to parametrize ∂C in different parts, see also the short discussion in [Pes07]. There it is also described how the C^2 regularity can be relaxed to only hold in the interior of C.

A recent version of an Itô-type formula that requires assumptions that are relatively easy to check, is due to DeAngelis and Cai [CDA21]. We again state their result broken down to our setting.

Theorem 11. *Let W be an n-dimensional standard Brownian motion, $b : \mathbb{R}^{\leq 0} \times \mathbb{R}^{n-1} \to \mathbb{R}$ a function that defines $C := \{\boldsymbol{x} \in \mathbb{R}^{\leq 0} \times \mathbb{R}^n \mid x_n < b(t, x_1, \ldots, x_{n-1})\}$ and $S := \{\boldsymbol{x} \in \mathbb{R}^{\leq 0} \times \mathbb{R}^n \mid x_{n+1} \geq b(t, x_1, \ldots, x_{n-1})\}$. Assume that*

- *$F : \mathbb{R}^{\leq 0} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1 function that is $C^{1,2,\ldots,2}$ on the open set C and on the closed set S,*
- *AF is locally bounded on C, i.e., for compact K, AF is bounded on $K \cup \partial C$,*
- *the functions $t \mapsto b(t, x_1, \ldots, x_{n-1})$ and $x_i \mapsto b(t, x_1, \ldots, x_{n-1})$ are monotonic,*

then the following change of variable formula holds:

$$F(0, W_0) = F(t, W_t) + \int_t^0 AF(s, W_s)\,\mathrm{d}s + \sum_{i=1}^{n} \int_t^0 \frac{\partial}{\partial x_i} F(s, W_s)\,\mathrm{d}W_s^i.$$

The assumption that AF is locally bounded can be seen to hold for fairly general stopping problems. The C^1 assumption is slightly stronger than the usual smooth-fit principle, but can be seen to hold in a fairly general setting as well, see [DAP20]. The assumptions on the monotonicity of b can easily be seen to hold for discounted and one-sided problems, as described in Section 4.3.2. In a more general setting we again have to check if ∂C can be parametrized in different parts.

In [CDA21] a good overview of the literature on generalized Itô formulas is given, so we mention only a few more. Interesting for our setting is the work of Protter and Föllmer [FP00], where an Itô-type formula is derived for functions in the Sobolev space $W^{1,2}_{\mathrm{loc}}$ and a multidimensional Brownian as the driving process. Their key idea is to use the quadratic covariation of W and $\partial_x F(W)$ instead of second derivatives. At least in the one-dimensional case this can also be constructed for functions with an explicit time dependence, see [FPS95].

4.8.2 Excessive functions

Another approach to deduce generalized Dynkin-type formulas is to use the Riesz decomposition of excessive functions and thereby of the value function V. We outline the idea for the discounted case where $g(t,\mathbf{x}) = e^{-rt}h(\mathbf{x})$ (see Section 4.3.1). This construction for a multidimensional geometric Brownian motion with infinite time horizon is given in [CS18] for a one-dimensional geometric Brownian motion with finite time horizon in [CS15].

The basis of the construction is the Riesz decomposition theorem.

Theorem 12. *For a standard Brownian motion, every a.s. finite r-excessive function can be decomposed uniquely into the sum of a harmonic function and a potential.*

See [CW05, Theorem 13.58] for a proof. In our context, this decomposition can be written as

$$V(t,\mathbf{x}) = \int_{(t,0]\times\mathbb{R}^d} G_r((t,\mathbf{x}),(s,\mathbf{y}))\sigma(\mathrm{d}s,\mathrm{d}\mathbf{y}), \tag{4.31}$$

where σ is a unique radon measure and the resolvent kernel G_r is given by

$$G_r((t,\mathbf{x}),(s,\mathbf{y})) = \begin{cases} e^{-r(s-t)}p((t,\mathbf{x}),(s,\mathbf{y})) & \text{if } t < s \le 0, \\ 0 & \text{if } s \le t < 0, \mathbf{x} \ne \mathbf{y}, \\ +\infty & \text{if } s = t < 0, \mathbf{x} = \mathbf{y}, \end{cases}$$

where p denotes the transition kernel, see [CS15] for the one-dimensional case and [Doo84] for a general treatment. If the measure σ in (4.31) has a Lebesgue density f on $(t,0)\times\mathbb{R}^d$, then we can deduce a Dynkin-type formula.

Proposition 3. *If σ is on $(t,0)\times\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure and has Lebesgue density f, then*

$$V(t,\boldsymbol{x}) = \mathrm{E}_{(t,x)}\left[\int_t^0 e^{-rs} f(s,W_s)\,\mathrm{d}s\right] + \mathrm{E}_{(t,x)}\, g(0,W_0). \tag{4.32}$$

Proof. We have:

$$\begin{aligned}
V(t,\mathbf{x}) &= \int_{(t,0]\times\mathbb{R}^d} G_r((t,\mathbf{x}),(s,\mathbf{y}))\sigma(\mathrm{d}s,\mathrm{d}\mathbf{y}) \\
&= \int_{(t,0)\times\mathbb{R}^d} G_r((t,\mathbf{x}),(s,\mathbf{y})) f(s,\mathbf{y})\,\mathrm{d}(s,\mathbf{y}) \\
&\quad + \int_{\mathbb{R}^d} G_r((t,\mathbf{x}),(0,\mathbf{y}))\sigma(0,\mathrm{d}\mathbf{y}) \\
&= \mathrm{E}_{(t,\mathbf{x})}\left[\int_t^\infty e^{-rs} f(s,W_s)\,\mathrm{d}s\right] + \mathrm{E}_{(t,\mathbf{x})}\, V(0,W_0) \\
&= \mathrm{E}_{(t,\mathbf{x})}\left[\int_t^0 e^{-rt} f(s,W_s)\,\mathrm{d}s\right] + \mathrm{E}_{(t,\mathbf{x})}\, g(0,W_0).
\end{aligned}$$

□

The question is under what conditions the measure σ in the Riesz decomposition of the value function $V(t,x)$ of a (Brownian) stopping problem is absolutely continuous with respect to the Lebesgue measure.

For the one-dimensional case a sufficient condition is given in [CS15].

Proposition 4. *If $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial x}$ are continuous on $(t,0)\times\mathbb{R}^d$, and $\frac{\partial V}{\partial x}$ is absolutely continuous as a function of the second argument, then σ is absolutely continuous on $(t,0)\times\mathbb{R}^d$ with respect to the Lebesgue measure.*

The authors use a generalized Itô-type formula to prove the proposition. It seems, however, that it might not be necessary to use path dependent arguments in this kind of constructions and rather deduce it directly from the smooth fit principle.

Under the conditions of Proposition 4, the measure σ is of the form

$$\sigma(\mathrm{d}s,\mathrm{d}\mathbf{y}) = (r-A)u(s,\mathbf{y})\,\mathrm{d}s\, m(\mathrm{d}\mathbf{y}) \tag{4.33}$$

on $(t,0)\times\mathbb{R}^d$ where m denotes the speed measure of the process. In our case of a standard Brownian motion m is just the Lebesgue measure. Combining Proposition 4 with (4.33) leads to equation (4.3).

Chapter 5

Conclusion

In this chapter, we briefly discuss possible generalizations and open questions that arise from the above shown.

5.1 Discrete time stopping problems

We have seen that we can use solutions to continuous time problems quite efficiently to approximate the Chow-Robbins game, and how this can be generalized. It seems that most questions about the method were answered. We numerically approximated the stopping boundary of the Chow-Robbins game with

$$\hat{b}(n) = \left\lceil \alpha\sqrt{n} - \frac{1}{2} + \frac{1}{7.9 + 4.54\sqrt[4]{n}} \right\rceil.$$

The first two terms of the equation are known to be asymptotically correct, the third term, however, arises from the numeric calculations and has so far no analytic justification. It is an open question if an asymptotic third order approximation is indeed of the form $\frac{1}{c+a\sqrt[4]{n}}$.

We have also shown that the value functions of the Chow-Robbins game is not differentiable on a dense subset of C and that this holds in a general context. We have shown non-smoothness only in the space component, but in the Chow-Robbins game and most other cases the value function will not be differentiable in the time component in the non-smoothness points neither. It furthermore seems likely that the stopping boundary $b(t)$ is not smooth on a dense set as well, but this is still to be proven rigorously. Furthermore, we restricted all the proof on the non-smoothness to specific cases, but the described phenomena seem to be typical for discrete time stopping problems. This shows that it is highly unlikely to find closed form solutions for V or b for most discrete and time dependent stopping problems with infinite time horizon. It also shows that we need to be careful with assumptions about the solutions of discrete time problems, because our intuition – often coming from problems with explicit solutions – might be misleading.

5.2 Continuous time stopping problems

We have derived a new Fredholm-type integral representation and believe that it is a useful tool for analyzing optimal stopping problems with finite time horizon. We have seen that in some cases the continuation set of a stopping problem is fully characterized by the representation. We are confident that uniqueness holds in a more general setting as well – at least in the multidimensional discounted case – but this still has to be proven. The representation also seems to be useful for numeric evaluation of stopping problems. One advantage over the standard integral representation is that the integrand has no singularities which makes numeric integration more stable.

5.2.1 Connections to Martin boundary theory

We have also seen a connection to the integral representation studied in [CCMS16] for the infinite horizon case. There, it was shown that the limit points of the parameter $\mathbf{c}$ form the Martin boundary of the (not time dependent) state space. This observation opens the door to interpreting our Fredholm integral equation as a Martin boundary representation. In fact, the parameters $\mathbf{c}$ in the Fredholm representation form the Martin boundary corresponding to the time inverse process, i.e., corresponding to the Green functions to the parabolic operator

$$\tilde{A} = \frac{1}{2}\Delta - \frac{\partial}{\partial t}.$$

The Martin boundary consist of exactly the parameters $\mathbf{c} \in \mathbb{R}^n$ with corresponding minimal harmonic functions $e^{\mathbf{c}\cdot\mathbf{y}+\frac{\|\mathbf{c}\|^2}{2}s}$ and a point ∞ corresponding to the non-minimal harmonic function $h \equiv 0$. In the construction of the Fredholm representation, this would correspond to setting $\mathbf{x} = \mathbf{c}(t)$ in (4.4) and (4.9) where $\frac{|\mathbf{c}(t)|}{-t} \to \infty$ as $t \to -\infty$ (the actual construction is $\mathbf{x} = \mathbf{c}t$ with fixed $\mathbf{c}$). These results can be found in the Sections 1.XIX.9 and 2.IX.17 of Doob's monograph on potential theory [Doo84]. An example for the use of Martin boundary theory in optimal stopping of one-dimensional diffusions can be found in [Sal85]. Martin boundary theory is not necessary to understand and use our results, it can, however, be a useful tool to gain more insight. For example, it becomes clear from Martin boundary theory why the integration kernels $e^{\mathbf{c}\cdot\mathbf{y}+\frac{\|\mathbf{c}\|^2}{2}s}$ are time reverse minimal harmonic function. To see our results in the context of Martin boundary theory can be helpful for generalizations of the representation. Whether the procedure to obtain a Fredholm representation can be fruitfully applied to other process types, is determined by whether this Martin-boundary can be found explicitly and is rich enough. This question has been discussed in the literature for different example classes. Reference is made, for example, to the case of geometric Brownian motion in [CS15]. In [Sal81] the Martin-boundary is constructed for space-time Cauchy-, d-dimensional Bessel- and Poisson processes. Where the Martin-boundary appears to be rich enough in the Poisson and Bessel case, in the case of an underlying Cauchy process the Martin-boundary consists only of the constant functions. This shows that the approach presented here is not suitable for characterizing the stopping boundary for general processes, but has to be constructed process-specific.

We have shown that the existence of the Fredholm representation implies the existence of the Martin boundary representation in the infinite time horizon case described in [CCMS16]. It, however, remains an open question, if we can derive such an implication for the uniqueness as well. Another interesting open question is, if we can obtain a similar representation for time dependent problems with infinite time horizon like the $\frac{W_t}{t}$-problem.

At last, we want to discuss possible generalizations of the results on the limit behavior in a little more detail.

5.2.2 Discussion on the limit behavior

The methods used to analyze the limit behavior is independent of the uniqueness. We showed how the representation can be used to show limit behavior of the stopping boundary in a general setting. We did not prove that the limit $\frac{d(x)}{x^2}$ always exists. Under some technical assumptions, it might be possible to extend the ideas from the proof of Theorem 8 to show the existence of the limit. It further seems possible that the results on the limit behavior are by far not the most general setting that is possible to tackle with our method, so we want to briefly discuss how further generalizations may look like.

General payoff functions g In Lemma 6 and 7 we have shown that in the discounted case for *regular* $\tilde{h}$ only the limit behavior of $\tilde{h}$ is relevant for the limit behavior of d or b. This should hold true for general stopping problems where Ag is sufficiently *regular* close to ∂C_0. If we take a look at Example 5 we see that it does not directly fit the discounted setting, but $Ag(t, x) = x$ could emerge from a discounted problem (exactly if $r = 0$ or approximately in the limit for any $r > 0$). So it is not surprising that C has the limit behavior described above.
In general, we expect a kind of similarity principle to hold true that states: if two stopping problems with the same driving process have payoff functions g and $\tilde{g}$ s.t. Ag and $A\tilde{g}$ have the same limit behavior as $t \to 0$ and $x \to \partial C$, then their continuation sets have the same limit behavior as $t \to 0$.

General diffusions Consider a Brownian motion X with volatility σ^2 and a stopping problem that matches the assumptions of Theorem 7 (except for the different driving process X) and has a stopping boundary d. We can scale the problem by $\frac{1}{\sigma}$ to obtain a problem for a standard Brownian motion W. For its stopping boundary $\tilde{d}$ we have $\tilde{d}(y) = d(\sigma y) = -By^2 + o(y^2)$ in a neighborhood of 0. So we have

$$d(x) = \frac{-B}{\sigma^2}x^2 + o(x^2)$$

in a neighborhood of 0.
Let us now assume that X is a diffusion that satisfies

$$dX = \mu(t, X)dt + \sigma(t, X)dW$$

where μ and σ are continuous, locally bounded and $\sigma > 0$. The process X locally *looks* like a Brownian motion with drift, so we could expect that for the stopping boundary d' of the corresponding stopping problem it holds

$$d'(x) = \frac{-B}{\sigma(0,0)^2}x^2 + o(x^2).$$

Possible shapes of C_0 In the n-dimensional setting, we assumed that C_0 is convex. Since limit behavior for $t \to 0$ is a local phenomenon, it would be surprising, if the global shape of C_0 was relevant. We expect the described limit behavior to hold for all parts of C_0 that are locally convex.
It would be surprising as well if the limit behavior for locally concave parts would be fundamentally different, but the Fredholm representations does not seem suitable for analyzing these areas.

Other limits In the one-dimensional and discounted case it is possible to analyze the limit behavior of C_t for $t \to -\infty$ (or limit of $d(y)$ as $y \to b_\infty$ resp.) by looking at $c \searrow \sqrt{2r}$. This leads to an exponential lower bound for b, i.e., for $t \to -\infty$ we have

$$b(t) \geq b_\infty - Me^t,$$

for some $M > 0$. We omit the details here.

In general, the Fredholm representations proved to be a valuable tool for the analysis of different kinds of limit behavior of the continuation set. It seems worthwhile to try to extend its application in different directions.

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List of Symbols

$\mathbb{R}_{\geq 0}$	Real numbers greater or equal to zero
$\mathbb{R}_{\leq 0}$	Real numbers lower or equal to zero
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$
$P_{(t,x)}$	$P(\cdot \mid X_t = x)$
$\mathrm{E}_{(t,x)}$	Expectation relative to $P_{(t,x)}$
W_t	Standard Brownian motion
S_n	Random walk
S	Stopping set
S^X	Stopping set for the driving process X
C	Continuation set
C^X	Continuation set for the driving process X
∂C	Boundary of C
V	Value function
V^X	Value function for the driving process X
τ_A	First entry time to a set A
$\tau \wedge \sigma$	Minimum of τ an σ
∂_x	$\frac{\partial}{\partial x}$
Δ	Laplace operator
$\lfloor x \rfloor$	Floor of x, the largest integer lower or equal to x
$\lceil x \rceil$	Lowest integer greater or equal to x
$\mathrm{Geom}(p)$	Geometric distribution on $\mathbb{N}$ with parameter p
$\mathcal{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$\Phi_{(\mu,\sigma^2)}$	CDF of a $\mathcal{N}(\mu, \sigma^2)$-distributed random variable
$\varphi_{(\mu,\sigma^2)}$	Density of $\Phi_{(\mu,\sigma^2)}$
$\mathcal{O}, o$	Big O notation, Bachmann–Landau notation
$\mathbb{1}_A$	Indicator function of a set A
$A \setminus B$	Set A without set B
$\mathcal{L}h$	Laplace transformation of a function h
C^n	Set of n times continuously differentiable functions
$C^{n,m}$	Set of functions that are C^n in the first and C^m in the second argument

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Erklärung

Ich habe die vorliegende Arbeit abgesehen von der Beratung durch den Betreuer meiner Promotion unter Einhaltung guter wissenschaftlicher Praxis selbstständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Des Weiteren habe ich noch keinen Promotionsversuch unternommen. Mir wurde kein akademischer Grad entzogen. Teile dieser Arbeit wurden in folgenden Artikeln vorab zur Veröffentlichung eingereicht:

- Sören Christensen and Simon Fischer. Note on the (non-) smoothness of discrete time value functions in optimal stopping. *Electron. Commun. Probab.*, 25, 2020
- Sören Christensen and Simon Fischer. A new integral equation for Brownian stopping problems with finite time horizon. *arXiv preprint arXiv:2110.02655*, 2021
- Sören Christensen and Simon Fischer. On the S_n/n-problem. *To appear in: Journal of Applied Probability*, 59.2, 2022

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