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Morse-Conley-Forman theory for generalized combinatorial multivector fields on finite topological spaces

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Streszczenie

W niniejszej pracy prezentujemy uogólnioną teorię pól multiwektorowych, która w swej pierwszej postaci została przedstawiona w [21]. Bezpośrednim poprzednikiem teorii pól multiwektorowych jest teoria kombinatorycznych pól wektorowych Robina Formana, która z kolei wywodzi się bezpośrednio z dyskretnej teorii Morse'a. Jednym z celów tej rozprawy jest stworzenie kombinatorycznego odpowiednika pól wektorowych obecnych w teorii ciągłych układów dynamicznych oraz stworzenie odpowiednich narzędzi do ich analizy.

U podstaw uogólnienia opisywanej teorii leżą trzy fundamentalne modyfikacje założeń. Po pierwsze, definiujemy pola multiwektorowe dla szerszej rodziny skończonych przestrzeni topologicznych, w przeciwieństwie do [21], gdzie konstrukcja dotyczyła kompleksów Lefschetza. Po drugie, odrzucamy wymaganie istnienia unikalnego elementu maksymalnego w multiwektorze. W rezultacie jedynym elementem definicji multiwektora jest założenie o jego lokalnej domkniętości. Uzyskujemy w ten sposób znaczną elastyczność w konstruowaniu pola multiwektorowego. Po trzecie, została uproszczona definicja odwzorowania wielowartościowego indukowanego przez pole multiwektorowe i reprezentującego kombinatoryczną dynamikę. Przekłada się to na uproszczenie dowodów i algorytmicznego aspektu wyznaczania rozkładów Morse'a oraz prowadzi do nowej interpretacji multiwektora jako dynamicznej "czarnej skrzynki".

Przy nowych założeniach teorii definiujemy kombinatoryczne odpowiedniki obiektów znanych z teorii ciągłych układów dynamicznych oraz badamy ich własności. Wśród nich mamy: zbiór izolowany niezmienniczy, parę indeksową, indeks Conley'a, zbiory graniczne, atraktor, czy rozkład Morse'a. Pokazujemy również pożądane własności jakich oczekiwalibyśmy od wymienionych wyżej obiektów, m.in. addytywność indeksu Conley'a oraz nierówności Morse'a. Nowe założenia pociągają za sobą konieczność przeprowadzenia nowych dowodów wszystkich własności.

W dalszej części pracy korzystamy z podstawowego narzędzia topologicznej analizy danych, tj. homologii persystetnych, do analizy strukturalnej trwałości zbiorów Morse'a. W tym celu konstruujemy moduł persystentny zygzak dla słabych rozkładów Morse'a oraz rozkładów Morse'a.

Następnie prezentujemy eksperymenty numeryczne bazujące na omawianej w tej rozprawie teorii. Przedstawiamy algorytm konstrukcji pola multiwektorowego z chmury wektorów. W szczególności uzyskujemy je poprzez próbkowanie wybranych ciągłych układów dynamicznych. W jednym z eksperymentów odtwarzamy graf Conley'a-Morse. Natomiast w kolejnych przykładach korzystamy z homologii persystentnych w celu zbadania ewolucji struktury zbiorów Morse'a względem wybranego parametru modyfikującego dynamikę. Eksperymenty prezentują potencjał dalszego wykorzystania wypracowanych narzędzi do analizy danych o dynamicznej naturze.

Abstract

In this work, we present a generalization of the theory of multivector fields first introduced in [21]. The direct predecessor of the multivector fields theory is the theory of combinatorial vector fields by Robin Forman. His work, in turn, is a natural consequence of a discrete Morse theory. One of the main goals of this thesis is to construct a combinatorial counterpart of vector fields induced by continuous dynamical systems and to create tools for its analysis.

The generalization involves three fundamental changes in the setting of the theory. First, we define multivector fields for a broader family of finite topological spaces, in comparison to [21] where Lefschetz complexes are used. Secondly, we lift the assumption that a multivector must have a unique maximal element. Thus, a multivector simply becomes a locally closed subset of space. This results in a greater flexibility in constructing multivector fields. Finally, we define less restrictively the multivalued map induced by a multivector field that defines a combinatorial dynamical system. Consequently, we can simplify the computational aspects of the theory, and we can introduce a new interpretation of a multivector as a dynamical "black box."

With a new setting of the multivector fields theory, we define combinatorial counterparts of multiple objects from the classical theory of dynamical systems; among others: isolated invariant set, index pair, Conley index, limit set, attractor, or Morse decomposition. We also show that the desirable properties as additivity of a Conley index and Morse inequalities hold. Even though the theory's general structure is preserved, new proves and ideas are required by the new setup.

In the further part, we use persistent homology – the topological data analysis main tool, to study the robustness of the structure of Morse sets. In particular, we construct a zigzag persistence module for weak Morse decomposition and Morse decomposition for multivector fields.

Finally, we show some numerical experiments based on the presented theory. We discuss the algorithm for constructing the multivector field from a vector cloud. As a proof of concept, we study vector clouds obtained by sampling chosen continuous vector fields. In the first experiment, we algorithmically reconstruct the Conley-Morse graph. In the further experiments, we use the persistence homology to study Morse sets' evolution with respect to a parameter modifying a dynamic. These experiments show the potential of the multivector fields theory as a new analysis tool for data with a dynamical nature.

Contents

Introduction

The combinatorial approach to dynamics is originally attributed to Robin Forman. The study of his discrete Morse theory gave rise to the concept of the combinatorial vector field on CW-complexes [11]. In the following paper [10] Forman studied a broader family of combinatorial vector fields, including non-gradient-like combinatorial flows. He also studied homological properties of such vector fields by proving Morse inequalities.

Despite being strongly inspired by dynamical systems, Forman's work does not provide a formal correspondence with the classic, continuous theory. The first efforts in this direction are presented in [15] and [4], where authors show that Forman's combinatorial vector field induces a flow-like multivalued map on a geometrical realization of a simplicial complex, i.e., an upper semi-continuous, acyclic-valued map that is homotopic to identity. Moreover, they introduce the concepts of isolated invariant set, Conley index, and Morse decomposition in the context of Forman's theory. They show that these notions correspond directly to their counterparts in classical multivalued settings.

A formal tie between continuous and combinatorial dynamics should involve connections in both directions. Thus, another important question is how to approximate a continuous dynamical system with a combinatorial model. Forman's theory is a natural starting point for that. However, one can easily produce examples of continuous vector fields which can not be modeled with Forman's combinatorial vector fields. To overcome these limitations, an extension of the idea of Forman to combinatorial multivector fields has been proposed in [21] in a more general setting of Lefschetz complexes. Both Lefschetz complexes and CW-complexes can be considered as a partial order or equivalently, by Alexandrov theorem, as a finite topological space. We can view Forman's combinatorial vector field as a partition of space into singletons and pairs of simplices where one is a face of the other of codimension 1. On the other hand, a multivector field of [21] is a partition of space into convex subsets (in the sense of order) with a unique maximal element. This extension leads to a much richer family of combinatorial vector fields. Hence, it enables us to model a greater variety of dynamical systems.

Our recent work [17], the basis for this dissertation, extends the idea of multivectors even further. There are three fundamental modifications. First, we replace Lefschetz complexes by the more general finite topological spaces. Secondly, we lift the assumption that a multivector must have a unique maximal element. Finally, we define the associated combinatorial dynamical system less restrictively, simplifying the computational aspects of the theory. The new setup preserves the general structure of the theory but requires new ideas and proofs in the study of all key dynamical concepts: Conley index, attractors, repellers, and Morse decomposition for multivector fields. By proving their properties, we also show that the Morse inequalities and Morse equations hold.

The modifications in combinatorial multivector field theory introduced in [17] simplify the theory's computational aspects. It also makes the theory better adjusted to the theory of persistence, which was presented in [7], where we study Morse decomposition's homological persistence for combinatorial dynamical systems. In particular, it sets foundations for the study of persistence of the Conley index [8].

The Morse decomposition and the associated Morse-Conley graph provide a compact global descriptor of a dynamical system. This type of qualitative summary is useful in the case of dynamical systems not available in a closed analytic form as a differential equation. This is often the case for systems known only from sampled data collected from experiments, observations or simulations.

The results of this thesis contribute to the program of building a combinatorial analogue of the classical theory of topological dynamics. In particular, the multivector field theory serves as a framework for modeling continuous vector fields, while the persistence of Morse decomposition brings additional tools for studying the Morse set's robustness.

This dissertation summarizes the process of development of the multivector field theory over the last five years. It mainly focuses on [17] and [7], in which the author of this thesis was involved.

The organization of the thesis is as follows. In Chapter 1, we set up basic definitions and concepts. We also present the basic properties of finite topological spaces (based on Section 3.4 from [17]). Chapter 2 is a short review of the algebraic topology, in particular the homology theory. We present the construction of homology groups via simplicial and singular complexes. Then we show our results on how to study the homology in the context of finite topological spaces (based on Section 3.5 from [17]). Finally, we recall the theoretical basics for zigzag persistence homology. Chapter 3 is dedicated to dynamical systems and for building the intuitions. First, we briefly review the concepts in continuous dynamics that we reconstruct in the following chapter in the combinatorial fashion. Then, we introduce the general definition of combinatorial dynamical systems and we present some examples of those. In Chapter 4, we present the main theoretical results. We define multivector fields and develop the combinatorial counterparts of the isolated invariant set, index pair, Conley index, attractor, limit sets, and Morse decomposition. We also show their properties. This chapter is extensively based on [17], particularly on Sections 4, 5, 6, and 7. Chapter 5 presents the theoretical results of [7], where we combine multivector field theory with persistent homology. It provides a tool for studying Morse sets' robustness. Finally, in Chapter 6, we consider the problem of applying the presented theory to data. We show an algorithm originally presented in [7] that translates a vector cloud into a multivector field. As a proof of a concept we present several experiments for vector fields derived from differential equations.

The author of this dissertation is the author of all proofs included in this thesis except for Proposition 2.1.2 and Theorem 2.4.3 where he is a co-author. Propositions and theorems without any reference were proved for this thesis's purposes and were not published anywhere else.

Chapter 1

Preliminaries

In this chapter we recall basic concepts and fix the notation needed in the sequel. We follow the exposition presented in [17].

1.1 Sets and maps

We denote the sets of integers, non-negative integers, non-positive integers, and positive integers, respectively, by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, and $\mathbb{N}$. Given a set A, we write $\#A$ for the number of elements in A and we denote by $\mathcal{P}(A)$ the family of all subsets of A. A *multiset* is a set where multiple instances of a single elements can occur. Formally it is a map $m : A \to \mathbb{Z}^+$ which denote the multiplicity of elements of a set A.

We say that a family $\mathcal{A}$ of nonempty subsets of X is a *partition of* X if $\bigcup \mathcal{A} = X$ and $A \cap A' = \emptyset$ for all $A, A' \in \mathcal{A}$. Given two partitions $\mathcal{A}$ and $\mathcal{B}$ of X we say that $\mathcal{A}$ is *inscribed* in $\mathcal{B}$ and write $\mathcal{A} \sqsubset \mathcal{B}$, if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$, such that $A \subseteq B$. Given a family $\mathcal{A}$ of mutually disjoint subsets of a set X, we use the notation

$$\mathcal{A}^* := \{ \bigcup \mathcal{A}' \mid \mathcal{A}' \subset \mathcal{A} \}. \tag{1.1}$$

Let $\mathcal{B}$ be another family of mutually disjoint subsets. Then we write

$$\mathcal{A} \bar{\cap} \mathcal{B} := \{ A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B} \}. \tag{1.2}$$

We write $f : X \nrightarrow Y$ for a partial map from X to Y, that is, a map defined on a subset $\operatorname{dom} f \subset X$, called the *domain* of f, and such that the set of values of f, denoted $\operatorname{im} f$, is contained in Y.

A multivalued map $F : X \multimap Y$ is a map $F : X \to \mathcal{P}(Y)$ which assigns to every point $x \in X$ a subset $F(x) \subset Y$. Given $A \subset X$, the *image* of A under F is defined by

$$F(A) := \bigcup_{x \in A} F(a).$$

For two multivalued maps $F, G : X \multimap Y$ we write $F \subset G$ if $F(A) \subset G(A)$ for all $A \in \mathcal{P}(X)$. By the *preimage* of a set $B \subset Y$ with respect to F we mean the large preimage, that is,

$$F^{-1}(B) := \{ x \in X \mid F(x) \cap B \neq \emptyset \}. \tag{1.3}$$

In particular, if $B = \{y\}$ is a singleton, we get

$$F^{-1}(\{y\}) := \{x \in X \mid y \in F(x)\}\,. \tag{1.4}$$

Thus, we have a multivalued map $F^{-1} : Y \multimap X$ given by $F^{-1}(y) := F^{-1}(\{y\})$. We call it the *inverse* of F.

1.2 Relations and digraphs

Let X and Y be finite sets. A *relation* is a subset of $X \times Y$. A map, a partial map and a multivalued map are all special cases of relations. *Inverse of a relation* $R \subset X \times Y$ is a relation $R^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in R\}$.

We say that a relation $R \subset X \times X$ is

- *reflexive* if: $\forall_{x \in X}\ (x, x) \in R$,
- *symmetric* if: $(x, y) \in R \Rightarrow (y, x) \in R$,
- *antisymmetric* if: $(x, y), (y, x) \in R \Rightarrow x = y$,
- *transitive* if: $\forall_{x,y,z \in X}\ (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$.

The *transitive closure* of relation $R \subset X \times X$ is a relation $\overline{R}$ defined as

$$\overline{R} := R \cup \{(x, x') \in X \times X \mid \exists_{x_1, x_2, \ldots, x_n \in X}\ (x, x_1), (x_1, x_2), \ldots, (x_n, x') \in R\}.$$

Relation is an *equivalence relation* if it is reflective, symmetric and transitive. An equivalence relation $R \subset X \times X$ partitions a set X into subsets of elements that are mutually related. These sets are called *equivalence classes* of a relation R. Every point $x \in X$ falls into a unique equivalence class, we denote it by $[x]_R$.

If $X = Y$ then we can identify relation $R \subset X \times X$ with a digraph $G_R = (X, R)$. The set X provides nodes of the digraph and the ordered pairs of relation R constitute directed edges (see Figure 1.1).

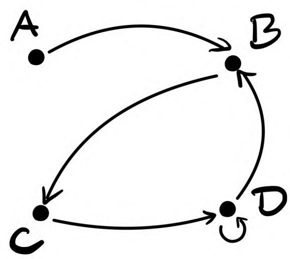


Figure 1.1: An example of a directed digraph on a set of vertices $X = \{A, B, C, D\}$. The associated relation is the family of pairs $R = \{(A, B), (C, B), (C, D), (D, C), (D, D)\}$. The digraph contains one self-loop: (D, D) and a family of closed paths of the form: $(B, C, D, D, \ldots, D, B)$, where point D can occur an arbitrary number of times. All points except A are recurrent.

A *path* in a digraph $G = (X, R)$ is a sequence of elements $\gamma = (x_0, x_1, \dots, x_n)$ such that $(x_i, x_{i+1}) \in R$ for all $i \in \{0, 1, \dots, n-1\}$. If additionally $x_0 = x_n$ then we say that γ is a *closed path*. A *(self-)loop* is a closed path consisting of only two elements ($n = 1$), that is, a path $\gamma = (x_0, x_1)$, where $x_0 = x_1$.

A vertex is *recurrent* if it belongs to a closed path. In particular, every vertex with a self-loop is recurrent. The digraph is *recurrent* if all its vertices are recurrent. Vertices x and y are in path relation if there exists a closed path containing x and y. If G is a recurrent digraph, then path relation is an equivalence relation Its equivalence classes are called *strongly connected components* of G. If a digraph consists of exactly one strongly connected component, then we say it is a *strongly connected digraph*. A subset $A \subset X$ of a digraph $G = (X, R)$ is a *strongly connected set* if a subgraph $G_A := (A, R \cap (A \times A))$ is a strongly connected digraph.

1.3 Orders and posets

Let X be a finite set. We recall that a reflexive and transitive relation $\leq$ on X is a *preorder* and the pair $(X, \leq)$ is a *preordered set*. Given a preorder $\leq$ on X we write $x < y$ meaning $x \leq y$ and $x \neq y$. If $\leq$ is also antisymmetric, then it is a *partial order* and $(X, \leq)$ is a *partially ordered set* (or *poset*). A partial order in which any two elements are comparable is a *linear* (*total*) *order*.

Given a poset $(X, \leq)$, we say that a set $A \subset X$ is *convex* if given $x, z \in A$, $y \in X$ inequalities $x \leq y \leq z$ imply $y \in A$. It is an *upper set* if $x \leq y$ with $x \in A$ and $y \in X$ implies $y \in A$. Similarly, A is a *down set* with respect to $\leq$ if given $y \in A$ and $x \in X$ inequality $x \leq y$ imply $x \in A$. A *chain* is a linearly ordered subset of a poset. We say that point $x \in X$ *covers* point $y \in X$ if $y < x$ and there is no $z \in X$ such that $y < z < x$. See Figure 1.2 for examples of convex, down and upper set.

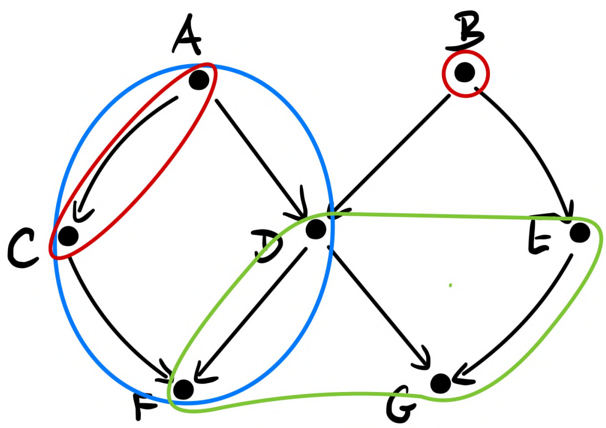


Figure 1.2: An example of a poset and a convex set (blue), a down set (green), and an upper set (red).

For $A \subset X$ we write

$$A^{\leq} := \{a \in X \mid \exists_{b \in A}\ a \leq b\},$$
$$A^{<} := A^{\leq} \setminus A.$$

Proposition 1.3.1. [17, Proposition 3.2] Let $(X, \leq)$ be a poset and let $A \subset X$ be a convex set. Then the sets $A^{\leq}$ and $A^{<}$ are down sets.

Proof. Clearly, $A^{\leq}$ is a down set directly from the definition. To see it for $A^{<}$, consider an $a \in A^{<}$ and an element $b \in X$ such that $b < a$. Then $a \notin A$. The definitions of $A^{\leq}$ and $A^{<}$ imply that there exists an element $c \in A$ such that $a < c$. Since $A^{\leq}$ is a down set we also have $b \in A^{\leq}$. We cannot have $b \in A$, because otherwise $b < a < c$ which contradicts the convexity of A. Hence, $b \in A^{<}$ which proves that $A^{<}$ is a down set. $\square$

Given preordered sets $(X, \leq)$ and $(Y, \leq)$ a map $f : X \to Y$ is called *order-preserving* if $x \leq x'$ implies $f(x) \leq f(x')$ for all $x, x' \in X$.

1.4 Topological spaces

Given a set X we say that $\mathcal{T} \subset \mathcal{P}(X)$ is a *topology* on X if the following conditions are satisfied:
(T1) $\emptyset, X \in \mathcal{T}$,
(T2) $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$,
(T3) $\mathcal{U} \subset \mathcal{T} \implies \bigcup \mathcal{U} \in \mathcal{T}$.

We say that a pair $(X, \mathcal{T})$ is a *topological space*. When the topology $\mathcal{T}$ is clear from the context we simply write X. Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ of a space X we say that $\mathcal{T}$ is a *finer* topology than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$. The elements of $\mathcal{T}$ are referred to as *open sets*. A set A is *closed* if there exists an open set $B \in \mathcal{T}$ such that $A = X \setminus B$. A set N is a *neighborhood* of a set A if there exists an open set $U \in \mathcal{T}$ such that $A \subset U \subset N$. A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is *continuous* if for every $U \in \mathcal{T}_Y$ we have $f^{-1}(U) \in \mathcal{T}_X$. A bijective map $f : X \to Y$ is a *homeomorphism* if both f and f^{-1} are continuous. The following proposition is straightforward.

Proposition 1.4.1. Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous map. If $\mathcal{T}'_X$ is a finer topology on X than $\mathcal{T}_X$ then $f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}_Y)$ is continuous.

If a topological space $(X, \mathcal{T})$ satisfy additional condition:
(T2*) $\mathcal{U} \subset \mathcal{T} \implies \bigcap \mathcal{U} \in \mathcal{T}$.
then we refer to it as an *Alexandrov (topological) space*. We say that X is a *finite topological space* if X is a finite set. It follows from (T2*) that every finite topological space is an Alexandrov space.

The *interior* of a set $A \subset X$ is the union of all open sets contained in A. The *closure* of a set $A \subset X$ is the intersection of all closed sets containing A. We denote the interior and closure of $A \subset X$ with respect to $\mathcal{T}$ by $\operatorname{int}_{\mathcal{T}} A$ and $\operatorname{cl}_{\mathcal{T}} A$, respectively. We define the *mouth* of A as the set $\operatorname{mo}_{\mathcal{T}} A := \operatorname{cl}_{\mathcal{T}} A \setminus A$. If X is finite, we also distinguish the *minimal open superset* (or *open hull*) of A as the intersection of all the open sets containing A. By (T2*) open hull is open. We denote it by $\operatorname{opn}_{\mathcal{T}} A$. We note that when X is finite then the family $\mathcal{T}^{\text{op}} := \{X \setminus U \mid U \in \mathcal{T}\}$ of closed sets is also a topology on X, called *dual* or *opposite topology*. The following proposition is straightforward.

Proposition 1.4.2. [17, Proposition 3.3] If $(X, \mathcal{T})$ is a finite topological space then for every set $A \subset X$ we have $\operatorname{opn}_{\mathcal{T}} A = \operatorname{cl}_{\mathcal{T}^{\text{op}}} A$.

If $A = \{a\}$ is a singleton, we simplify the notation $\operatorname{int}_{\mathcal{T}}\{a\}$, $\operatorname{cl}_{\mathcal{T}}\{a\}$, $\operatorname{mo}_{\mathcal{T}}\{a\}$ and $\operatorname{opn}_{\mathcal{T}}\{a\}$ to $\operatorname{int}_{\mathcal{T}} a$, $\operatorname{cl}_{\mathcal{T}} a$, $\operatorname{mo}_{\mathcal{T}} a$ and $\operatorname{opn}_{\mathcal{T}} a$. When the topology $\mathcal{T}$ is clear from the context, we drop the subscript $\mathcal{T}$ in this notation. Given a finite topological space $(X, \mathcal{T})$ we briefly write $X^{\text{op}} := (X, \mathcal{T}^{\text{op}})$ for the same space X but with the opposite topology.

We recall that a subset A of a topological space X is *locally closed* if every $x \in A$ admits a neighborhood U in X such that $A \cap U$ is closed in U. Locally closed sets are crucial in the sequel. In particular, we have the following characterization of locally closed sets.

Proposition 1.4.3. [9, Problem 2.7.1] Assume A is a subset of a topological space X. Then the following conditions are equivalent.

(i) A is locally closed,
(ii) $\operatorname{mo}_{\mathcal{T}} A := \operatorname{cl}_{\mathcal{T}} A \setminus A$ is closed in X,
(iii) A is a difference of two closed subsets of X,
(iv) A is an intersection of an open set in X and a closed set in X.

As an immediate consequence of Proposition 1.4.3(iv) we get the following propositions.

Proposition 1.4.4. [17, Proposition 3.5] The intersection of a finite family of locally closed sets is locally closed.

Proposition 1.4.5. [17, Proposition 3.6] If A is locally closed and B is closed, then $A \setminus B$ is locally closed.

Proposition 1.4.6. [17, Proposition 3.7] Let $(X, \mathcal{T})$ be a finite topological space. A subset $A \subset X$ is locally closed in the topology $\mathcal{T}$ if and only if it is locally closed in the topology $\mathcal{T}^{\text{op}}$.

We recall that the topology $\mathcal{T}$ is T_2 or *Hausdorff* if for any two different points $x, y \in X$, there exist disjoint sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. It is T_0 or *Kolmogorov* if for any two different points $x, y \in X$ there exists a $U \in \mathcal{T}$ such that $U \cap \{x, y\}$ is a singleton.

Finite topological spaces stand out from general topological spaces by the fact that the only Hausdorff topology on a finite topological space X is the discrete topology consisting of all subsets of X.

Proposition 1.4.7. [17, Proposition 3.8] Let $(X, \mathcal{T})$ be a finite topological space and let $A \subset X$. Then

$$\operatorname{cl} A = \bigcup_{a \in A} \operatorname{cl} a \quad \text{and} \quad \operatorname{opn} A = \bigcup_{a \in A} \operatorname{opn} a.$$

Proof. Let $A' := \bigcup_{a \in A} \operatorname{cl} a$. Clearly, $A \subset A' \subset \operatorname{cl} A$. Since X is finite, A' is closed as a finite union of closed sets. Therefore, also $\operatorname{cl} A \subset A'$. Since X is finite, and therefore an Alexandrov space, the second formula is dual. □

A remarkable feature of finite topological spaces is the following theorem.

Theorem 1.4.8. (Alexandrov Theorem [1]) For a preorder $\leq$ on a finite set X, there is a topology $\mathcal{T}_{\leq}$ on X whose open sets are the upper sets with respect to $\leq$. For a topology $\mathcal{T}$ on a finite set X, there is a preorder $\leq_{\mathcal{T}}$ where $x \leq_{\mathcal{T}} y$ if and only if $x \in \operatorname{cl}_{\mathcal{T}} y$. The correspondences $\mathcal{T} \mapsto \leq_{\mathcal{T}}$ and $\leq \mapsto \mathcal{T}_{\leq}$ are mutually inverse. Under these correspondences continuous maps are transformed into order-preserving maps and vice versa. Moreover, the topology $\mathcal{T}$ is T_0 (Kolmogorov) if and only if the preorder $\leq_{\mathcal{T}}$ is a partial order.

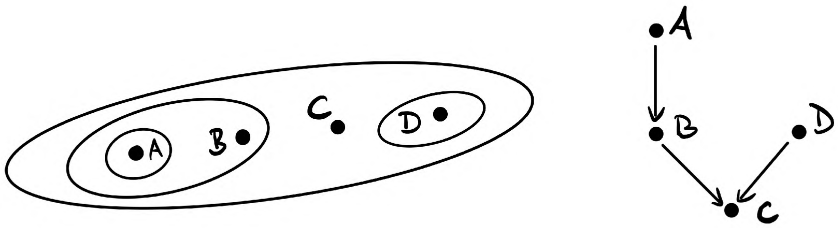


Figure 1.3: An example of the correspondence between a finite topological space and a partial order provided by Alexandrov Theorem. Left: a finite space $X = \{A, B, C, D\}$ with a topology $\mathcal{T} = \{\emptyset, \{A\}, \{D\}, \{A, B\}, \{A, B, D\}, \{A, B, C, D\}\}$. Right: the poset associated to the topological space $(X, \mathcal{T})$.

The correspondence resulting from Theorem 1.4.8 lets us translate concepts and problems between topology and order theory (see the example in Figure 1.3). It follows directly from the theorem that closed sets can be identified with down sets and open sets with upper sets. We can also easily find similar identifications for other types of sets.

Proposition 1.4.9. [17, Proposition 3.10] Let $(X, \mathcal{T})$ be a finite topological space. Then, for $A \subset X$ we have

$$\begin{aligned} \operatorname{opn}_{\mathcal{T}} A &= \{x \in X \mid \exists_{a \in A}\ x \geq_{\mathcal{T}} a\}, \\ \operatorname{cl}_{\mathcal{T}} A &= \{x \in X \mid \exists_{a \in A}\ x \leq_{\mathcal{T}} a\}, \\ \operatorname{int}_{\mathcal{T}} A &= \{a \in A \mid \forall_{x \in X}\ x \geq_{\mathcal{T}} a \ \Rightarrow x \in A\}. \end{aligned}$$

In other words, $\operatorname{cl}_{\mathcal{T}} A$ is the minimal down set with respect to $\leq_{\mathcal{T}}$ containing A, $\operatorname{opn}_{\mathcal{T}} A$ is the minimal upper set with respect to $\leq_{\mathcal{T}}$ containing A and $\operatorname{int}_{\mathcal{T}} A$ is the maximal upper set with respect to $\leq_{\mathcal{T}}$ contained in A.

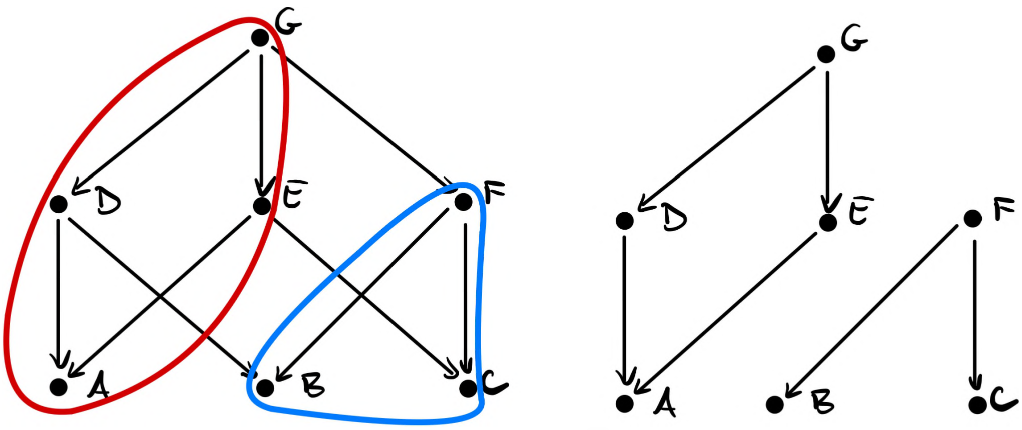


Figure 1.4: Left: a finite topological space $(X, \mathcal{T})$ with a partition into two sets $\mathcal{A} = \{\{A, D, E, G\}, \{B, C, F\}\}$. Right: the same space X with the disconnecting topology $\mathcal{T}(\mathcal{A})$. Poset associated to the new topology consists of two components.

Proposition 1.4.10. [17, Proposition 3.11] Assume X is a T_0 finite topological space and $A \subset X$. Then A is locally closed if and only if A is convex with respect to $\leq_{\mathcal{T}}$.

Proof. Assume that A is locally closed. Let $x, y \in A$. By Proposition 1.4.3 we can write $A = U \cap D$, where U is open and D is closed. By Theorem 1.4.8 we know that U is an upper set and D is a down set with respect to $\leq_{\mathcal{T}}$. Let $x, z \in A$ and let $y \in X$ be such that $x \leq_{\mathcal{T}} y \leq_{\mathcal{T}} z$. Since $x \in U$ and U is an upper set, it follows, that $y \in U$. Since $z \in D$ and D is a down set, it follows $y \in D$. Thus $y \in U \cap D = A$.

Conversely, assume that A is convex with respect to $\leq_{\mathcal{T}}$. By Proposition 1.4.3(ii) it suffices to prove that $\operatorname{mo}_{\mathcal{T}} A = \operatorname{cl}_{\mathcal{T}} A \setminus A$ is closed. Suppose the contrary. Then there exist an $x \notin \operatorname{mo}_{\mathcal{T}} A$ and a $y \in \operatorname{mo}_{\mathcal{T}} A$ such that $x \in \operatorname{cl}_{\mathcal{T}} y$, that is $x \leq_{\mathcal{T}} y$. It follows from Proposition 1.4.9 and $y \in \operatorname{mo}_{\mathcal{T}} A \subset \operatorname{cl}_{\mathcal{T}} A$ that there exists an element $z \in A$ such that $y \leq_{\mathcal{T}} z$. In consequence we get $x \leq_{\mathcal{T}} z$, and therefore $x \in \operatorname{cl}_{\mathcal{T}} A$. In view of $x \notin \operatorname{mo}_{\mathcal{T}} A$ this implies $x \in A$, and the assumed convexity of A then gives $y \in A$, which contradicts $y \in \operatorname{mo}_{\mathcal{T}} A$. $\square$

We say that sets $A, B \subset X$ are *disconnected* if there exist open and disjoint sets U and V such that $A \subset U$ and $B \subset V$, otherwise they are *connected*. A connected set $A \subset X$ is a *connected component* of X if there is no connected set $B \subset X$ such that $A \subsetneq B$. Note that the family of connected components of a topological space X forms a partition of X.

A partition $\mathcal{A}$ is $\mathcal{T}$*-disconnected* if each set $A \in \mathcal{A}$ is open in topology $\mathcal{T}$, otherwise $\mathcal{A}$ is $\mathcal{T}$*-connected*. In particular, the partition of X into connected components is $\mathcal{T}$-disconnected. Observe that $\mathcal{A}^*$ (see (1.1)) is the smallest topology in $\bigcup \mathcal{A}$ such that $\mathcal{A}$ is $\mathcal{A}^*$-disconnected.

Theorem 1.4.11. [7, Theorem 5.1] Assume $(X, \mathcal{T})$ is an arbitrary topological space and $\mathcal{A}$ is a finite family of mutually disjoint, nonempty subsets of X. Then $\mathcal{T}(\mathcal{A}) := (\mathcal{A} \bar{\cap} \mathcal{T})^*$ (see (1.2)) is a topology on $\bigcup \mathcal{A}$. Moreover,

(i) if $\mathcal{T}$ is a T_0 topology, then so is $\mathcal{T}(\mathcal{A})$;

(ii) for every $A \in \mathcal{A}$, the topology induced on A by $\mathcal{T}$ coincides with the topology induced on A by $\mathcal{T}(\mathcal{A})$;
(iii) the family $\mathcal{A}$ is $\mathcal{T}(\mathcal{A})$-disconnected;
(iv) if additionally $\bigcup \mathcal{A} = X$ and each set in $\mathcal{A}$ is $\mathcal{T}$-connected, then the connected components with respect to $\mathcal{T}(\mathcal{A})$ coincide with the sets in $\mathcal{A}$.

We say that $\mathcal{T}(\mathcal{A})$ is the *disconnecting topology* on $\bigcup \mathcal{A}$ induced by $\mathcal{T}$ and $\mathcal{A}$ (see Figure 1.4).

If $\mathcal{A}$ is a singleton, say $\mathcal{A} = \{A\}$, then we write $\mathcal{T}(A) := \mathcal{T}(\mathcal{A})$. Note that in this case the disconnecting topology and topology induced by $\mathcal{T}$ in A coincide. In particular $\mathcal{T}(A) = (\{A\} \bar{\cap} \mathcal{T})^* = \{U \cap A \mid U \in \mathcal{T}\}$. We have the following straightforward observation.

Corollary 1.4.12. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A}$ is a finite family of mutually disjoint, nonempty subsets of X. Then the disconnecting topology $\mathcal{T}(\mathcal{A})$ is finer then the topology induced by $\mathcal{T}$ in $\bigcup \mathcal{A}$, that is $\mathcal{T}(\bigcup \mathcal{A}) \subset \mathcal{T}(\mathcal{A})$.

1.5 Algebra

A pair $(G, +)$ is called a *group* if the sum operator $+ : G \times G \to G$ satisfies:
(i) $\forall_{a,b,c \in G}\ (a + b) + c = a + (b + c)$ (associativity),
(ii) $\exists_{\varepsilon \in G} \forall_{a \in G}\ \varepsilon + a = a = a + \varepsilon$ (neutral element),
(iii) $\forall_{a \in G} \exists_{b \in G}\ a + b = \varepsilon = b + a$, where ε is the neutral element (inverse elements).
If additionally
(iv) $\forall_{a,b \in G}\ a + b = b + a$ (commutativity),
then we say that $(G, +)$ is an *abelian group*. A subset $H \subset G$ is a *subgroup* if $a + b \in H$ for any $a, b \in H$. Then $(H, +_H)$ is a group, where $+_H$ denotes the restriction of the sum operator to H. A group consisting of only one element is called a *trivial group* and is denoted by 0. If $n \in \mathbb{N}$ then we write na for the sum of n copies of an element $a \in G$, that is $na = \overbrace{a + a + \ldots + a}^{n}$. A set $B \subset G$ is called a *basis* of a group G if for every $a \in G$ there exists a unique set of integers $\{n_b\}_{b \in B}$ such that

$$a = \sum_{b \in B} n_b b.$$

An abelian group G is *free* if it admits a basis. Every basis of a free abelian group G has the same number of elements [16, Theorem 7.3]. An abelian group with a finite basis is called a *finitely generated abelian group*. The *rank of a finitely generated group* G is the cardinality of its basis. We refer to elements of a basis as *generators*.

Let $G_1, \ldots, G_k$ be a finite family of groups. The *direct product* of these groups is the Cartesian product $G_1 \times G_2 \times \ldots \times G_k$ with the sum operator:

$$+ : ((h_1, h_2, \ldots, h_k), (h_1, h_2, \ldots, h_k)) \to (g_1 +_1 h_1, g_2 +_2 g_2, \ldots, g_k +_k g_k).$$

Direct product of groups gives a new group. We denote direct product by $G_1 \oplus G_2 \oplus \ldots \oplus G_k$ or $\bigoplus_i G_i$.

Let $(G, +_G)$ and $(H, +_H)$ be groups. A map $h : G \to H$ is a *homomorphism* if for every $a, b \in G$ we have $h(a +_G b) = h(a) +_H h(b)$. We call h a *monomorphism*, *epimorphism* or *isomorphism* respectively when h is injective, surjective or bijective. The *image* of a homomorphism is the set $\operatorname{im} h := \{h(a) \in H \mid a \in G\}$. The *kernel* of a homomorphism is the set $\ker h := \{a \in G \mid h(a) = \varepsilon_H\}$, that is a set of elements mapped into the neutral element of H.

A sequence of abelian groups and homomorphisms

$$X_0 \xrightarrow{h_0} X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} \dots \xrightarrow{h_n} X_{n+1}$$

is an *exact sequence* if for every $i \in \{0, 1, \dots, n-1\}$ we have $\operatorname{im} h_i = \ker h_{i+1}$. The following two propositions are straightforward.

Proposition 1.5.1. If the sequence

$$0 \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} 0$$

is exact then the map g is an isomorphism.

Proposition 1.5.2. If in the following exact sequence

$$X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} X_3 \xrightarrow{h_3} X_4 \xrightarrow{h_4} X_5$$

h_1 is surjective and h_4 is injective, then $X_3 = 0$.

Lemma 1.5.3. (The Steenrod five-lemma)[22, Lemma 24.3] Suppose one is given a commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & A_4 \arrow[r] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

in which the horizontal sequences are exact. If h_1, h_2, h_4, and h_5 are isomorphisms, so is h_3.

Let H be a subgroup of an abelian group $(G, +)$. The *quotient group* ${}^{G}/_{H}$ is the group consisting of the family of equivalence classes

$$\{[g] \mid g \in G\} := \{g + H \subset G \mid g \in G\} = \{\{g + h \mid h \in H\} \mid g \in G\}$$

with the sum operator given by

$$+ : {}^{G}/_{H} \times {}^{G}/_{H} \ni ([g], [h]) \mapsto [g + h] \in {}^{G}/_{H}.$$

A triple $(R, +, \cdot)$ with operators $+, \cdot : R \times R \to R$ is a *ring* if $(R, +)$ is an abelian group and:

(*i*) $\forall_{a,b,c \in R}\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$, (associativity)

(*ii*) $\forall_{a,b,c \in R}\ a \cdot (b + c) = a \cdot b + a \cdot c$, (distributivity)

(iii) $\forall_{a,b,c \in R}$ $(a + b) \cdot c = a \cdot c + b \cdot c$. (distributivity)

If additionally

(iv) $\exists_{\varepsilon \in R} \forall_{a \in R}$ $\varepsilon \cdot a = a = a \cdot \varepsilon$ (neutral element),

we say that R is a *ring with unity*. The neutral element of the sum operator is usually denoted by 0 and the neutral element of the product operator by 1. We say that $(R, +, \cdot)$ is a *field* if conditions (i)-(vi) are satisfied, where

(v) $\forall_{a \in R \setminus \{0\}} \exists_{b \in R}$ $a \cdot b = 1 = b \cdot a$, (inverse elements)

(vi) $\forall_{a,b \in R}$ $a \cdot b = b \cdot a$. (commutativity)

Let $(M, +)$ be an abelian group and $(R, \oplus, \odot)$ a ring with unity. We say that $(M, R, \cdot)$ is an *R-module* if $\cdot : R \times M \to M$ is an operator, called external multiplication, such that for every $x, y \in M$ and $a, b \in R$:

(i) $a \cdot (x + y) = a \cdot x + a \cdot y$,

(ii) $(a \oplus b) \cdot x = a \cdot x + b \cdot x$,

(iii) $(a \odot b) \cdot x = a \cdot (b \cdot x)$,

(iv) $1_R \cdot x = x$.

An R-module $(M, R, \cdot)$ where R is a field is called a *vector space* M over a field R. A triple $(N, R, \cdot)$, where $N \subset M$ is a *subspace of a vector space* M if for every $v, w \in N$ and $a \in R$ we have $a \cdot (v + w) \in N$. We then write $N \subseteq M$.

Chapter 2

Algebraic topology

In this chapter we recall concepts from algebraic topology needed in sequel. The Sections 2.1-2.3 are based on [22]. In Sections 2.4 and 2.5 we follow the exposition in [17] and [6], respectively.

2.1 Simplicial theory

2.1.1 Geometric simplicial complexes

A set of points $V = \{v_0, v_1, \ldots, v_n\} \subset \mathbb{R}^d$ is *affinely independent* if

$$\sum_{i=0}^{n} t_i v_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i = 0$$

implies that $t_i = 0$ for all $i \in \{0, 1, \ldots, n\}$. Let $\langle v_0, v_1, \ldots v_n \rangle \subset \mathbb{R}^d$ denote the family of *barycentric combinations* of vertices V, that is points $x \in \mathbb{R}^d$ satisfying

$$x = \sum_{i=0}^{n} t_i v_i \quad \text{such that} \quad \sum_{i=0}^{n} t_i = 1, \quad \text{where} \quad \forall_{i \in \{0,1,\ldots,n\}} t_i \geq 0. \tag{2.1}$$

We say that $\langle v_0, v_1, \ldots v_n \rangle$ is a set *spanned* by V. A set σ spanned by V, a collection of $n+1$ affinely independent points, is called an *n-simplex*. Then n is referred to as the *dimension* of σ and denoted $\dim \sigma$. Coefficients t_i uniquely define every point $x \in \sigma$ and are called the *barycentric coordinates* of x with respect to V. A simplex τ spanned by a subset $W \subset V$ is called a *face* of the simplex σ; we denote this by $\tau \prec \sigma$. In this case we also say that σ is a *coface* of τ. If $W \subsetneq V$ then τ is a *proper face* of σ and σ is a *proper coface* of τ. The union of proper faces of a simplex σ is called the *combinatorial boundary* of σ and is denoted $\operatorname{Bd} \sigma$. The *combinatorial interior* of σ is defined as $\operatorname{Int} \sigma := \sigma \setminus \operatorname{Bd} \sigma$. The *star* of a simplex σ denoted $\operatorname{st} \sigma$ is the union of the combinatorial interiors of the cofaces of σ, that is $\operatorname{st} \sigma = \bigcup \{\operatorname{Int} \tau \mid \sigma \prec \tau\}$.

A finite collection of simplices K in $\mathbb{R}^d$ is called *geometrical simplicial complex* or briefly a *simplicial complex* if the following conditions are satisfied:

1) if $\sigma \in K$ and τ is a face of σ then $\tau \in K$,

2) if $\sigma, \tau \in K$ then $\sigma \cap \tau \in K$.

The *dimension* of K, denoted $\dim K$ is the maximum of dimensions of simplices in K. A subset $L \subset K$ is said to be a *subcomplex* of K if L is a simplicial complex. In particular, the collection K_n of all simplices of K up to dimension n forms a subcomplex. We call it an *n-skeleton* of K. Note that the family of simplices of K of exactly dimension n, denoted by $K_{\{n\}}$, is not a subcomplex of K. A simplex $\sigma \in K$ is a *toplex* in K if there is no proper coface of σ in K. The union $\bigcup_{\sigma\in K} \sigma \subset \mathbb{R}^d$, denoted $|K|$, is called the *polytope* of K.

A complex L is a *subdivision* of a complex K if $|L| = |K|$ and for every simplex $\tau \in L$ there exists a $\sigma \in K$ such that $\tau \subset \sigma$. There is a standard way to construct a subdivision. To this end we define the *barycenter* of a simplex $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ as the point

$$\dot{\sigma} := \sum_{i=0}^{n} \frac{1}{n+1} v_i.$$

The collection of simplices

$$\operatorname{sd} K := \{\langle \dot{\sigma}_0, \dot{\sigma}_1, \ldots, \dot{\sigma}_p\rangle \mid \sigma_0 \prec \sigma_1 \prec \ldots \prec \sigma_p \text{ and } \sigma_i \in K \text{ for } i \in \{0, 1, \ldots, p\}\}$$

is called the *first barycentric subdivision* of complex K (see Figure 2.2, left and right panel).

Let K be a simplicial complex. Every point $x \in |K|$ is contained in the interior of a unique simplex $\langle v_0, v_1, \ldots, v_n\rangle = \sigma \in K$. This means that

$$x = \sum_{i=0}^{n} t_i v_i, \quad \text{for some} \quad t_i > 0. \tag{2.2}$$

Thus, we can generalize the concept of the barycentric coordinate of an $x \in |K|$ to a map $t_v : |K| \times V_0 \to [0, 1]$. Let $x \in \operatorname{Int} \sigma$ and let t_i be as in (2.2). Then we set

$$t(x, v) := t_v(x) := \begin{cases} t_i & \text{if } v = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

A *simplicial map* is a map $f : K_0 \to L_0$ such that

$$\langle v_0, v_1, \ldots, v_n\rangle \in K \quad \Rightarrow \quad \langle f(v_0), f(v_1), \ldots, f(v_n)\rangle \in L.$$

A simplicial map $f : K_0 \to L_0$ can be linearly extended to a continuous map $|f| : |K| \to |L|$ by the formula

$$|f|(x) := \sum_{v\in K_0} t_v(x) f(v).$$

We call $|f|$ the *linear extension of the simplicial map f*.

Lemma 2.1.1. [22, Lemma 2.8] Let $f : K_0 \to L_0$ be a bijective simplicial map such that f^{-1} is also a simplicial map. Then $|f| : |K| \to |L|$ is a homeomorphism.

2.1.2 Abstract simplicial complexes

An *abstract n-simplex* is a finite set σ of $n+1$ elements. The *dimension* of a σ is the number of its elements minus one. An *abstract simplicial complex* is a collection A of abstract simplices such that if σ is in A, then all non-empty subsets of σ are also in A. A subcollection $B \subset A$ is a *subcomplex* of A if B is an abstract simplicial complex itself.

A map $f : A_0 \to B_0$ between 0-skeletons of abstract simplicial complexes A and B is a *simplicial map* in the context of abstract simplicial complexes if

$$\{x_0, x_1, \dots, x_n\} \in A \quad \Rightarrow \quad \{f(x_0), f(x_1), \dots, f(x_n)\} \in B.$$

We say that a simplicial map $f : A_0 \to B_0$ is an isomorphism of a simplicial complexes if f is a bijection such that f^{-1} is also a simplicial map. Two abstract simplicial complexes A, B are isomorphic if there exists an isomorphism $f : A_0 \to B_0$.

We can think of an abstract simplicial complex as a regular simplicial complex with dropped geometric information. More precisely, if K is a simplicial complex, then the collection

$$\{\{v_0, v_1, \dots, v_{n_\sigma}\} \mid \langle v_0, v_1, \dots, v_{n_\sigma}\rangle = \sigma \in \mathcal{K}\}$$

is an abstract simplicial complex called the *vertex scheme* of K. We say that simplicial complex K is a *geometric realization* or *polytope* of an abstract simplicial complex A if a vertex scheme of K is isomorphic to A. We refer to an isomorphism $p : A_0 \to K_0$ as *embedding map* of an abstract simplical complex A. By Lemma 2.1.1 polytopes of any two geometric realizations of an abstract simplicial complex A are homeomorphic. Thus, the geometric realization of an abstract simplicial complex is unique up to a homeomorphism. We denote the geometric realization of an abstract simplicial complex A by $|A|$.

2.1.3 Order complex

The *nerve* of a finite topological space $(X, \mathcal{T})$ is the collection of all nonempty chains in $(X, \leq_{\mathcal{T}})$. We denote it $\mathcal{K}(X, \mathcal{T})$, or briefly $\mathcal{K}(X)$ if the topology $\mathcal{T}$ is clear from the context. It forms an abstract simplicial complex that we call the *order complex* of $(X, \mathcal{T})$ (see Figure 2.1). As already mentioned, for every point $\alpha \in |\mathcal{K}(X, \mathcal{T})|$ there exists a unique $\sigma \in \mathcal{K}(X, \mathcal{T})$ such that $\alpha \in \operatorname{Int} \sigma$. It follows that there exists a chain $x_0 < x_1 < \dots < x_n$ in $(X, \leq_{\mathcal{T}})$ such that $\sigma = \langle p(x_0), p(x_1), \dots, p(x_n)\rangle$ where $p : X \to \mathcal{K}(X, \mathcal{T})$ is an embedding map. We call this chain the *support* of α and we denote it $\operatorname{supp} \alpha := \{x_0, x_1, \dots, x_n\}$.

The construction of the order complex, similarly to the barycentric subdivision, is based on subchains. In fact, both constructions are related. Let K be a simplicial complex. It naturally induces a partial order $(K, \preceq)$ where $\preceq$ is the face relation between simplices. The vertices of the order complex of this poset are the simplices of K and the geometric realization of the order complex is the first barycentric subdivision, that is $\operatorname{sd} K = \mathcal{K}(K, \mathcal{T}_{\preceq})$ with the embedding map $p : \mathcal{K}_0(K, \mathcal{T}_{\preceq}) \to (\operatorname{sd} K)_0$, given by $p(\sigma) := \dot{\sigma}$ (see Figure 2.2).

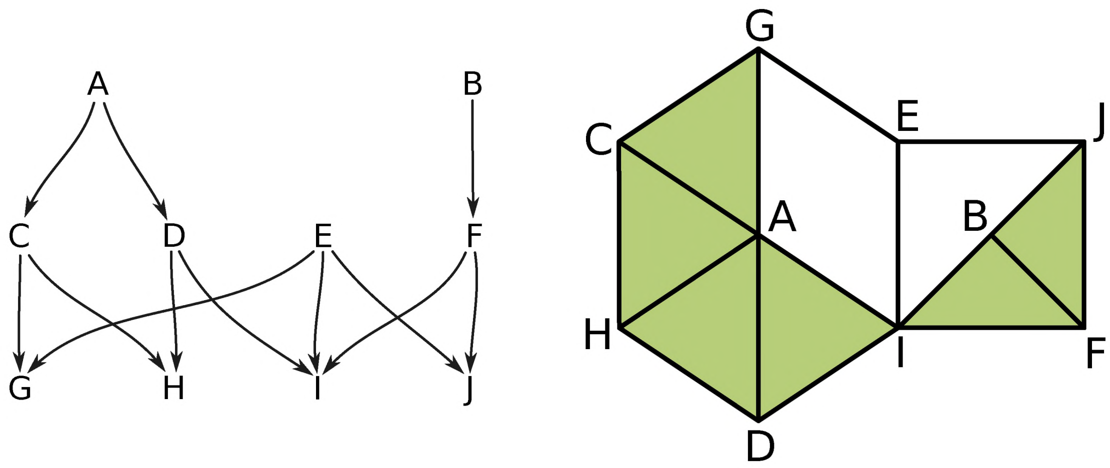


Figure 2.1: Left: an example of a poset (a finite topological space). Right: the associated order complex.

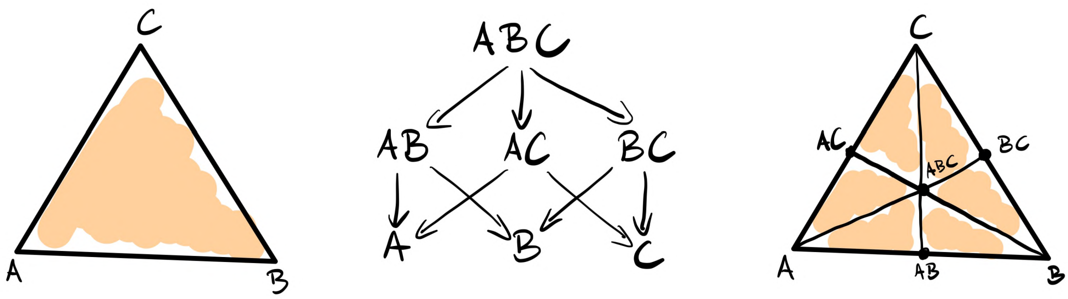


Figure 2.2: A simple simplicial complex K (left), a partial order $(K, \preceq)$ induced by the face relation of K (center) and the associated order complex $\mathcal{K}(K, \mathcal{T}_{\preceq})$ (right).

Proposition 2.1.2. [17, Proposition 3.1] Let $(X, \leq)$ be a poset and let $A, B \subset X$. Then

$$\mathcal{K}(A \cap B) = \mathcal{K}(A) \cap \mathcal{K}(B) \quad \text{and} \quad |\mathcal{K}(A \cap B)| = |\mathcal{K}(A)| \cap |\mathcal{K}(B)|. \tag{2.3}$$

Moreover, if A and B are down sets, then

$$\mathcal{K}(A \cup B) = \mathcal{K}(A) \cup \mathcal{K}(B) \quad \text{and} \quad |\mathcal{K}(A \cup B)| = |\mathcal{K}(A)| \cup |\mathcal{K}(B)|. \tag{2.4}$$

Proof. Property (2.3) and inclusions $\mathcal{K}(A) \cup \mathcal{K}(B) \subset \mathcal{K}(A \cup B)$, $|\mathcal{K}(A)| \cup |\mathcal{K}(B)| \subset |\mathcal{K}(A \cup B)|$ are straightforward. To see that $\mathcal{K}(A \cup B) \subset \mathcal{K}(A) \cup \mathcal{K}(B)$ and $|\mathcal{K}(A \cup B)| \subset |\mathcal{K}(A)| \cup |\mathcal{K}(B)|$ consider a chain $x_1 < x_2 < \cdots < x_k$ in $\mathcal{K}(A \cup B)$. Without loss of generality we may assume that $x_k \in A$. Since A is a down set, the elements of chain $x_1 < x_2 < \cdots < x_k$ are in A. Thus, $\{x_0, x_1, \ldots, x_k\} \in \mathcal{K}(A) \subset \mathcal{K}(A) \cup \mathcal{K}(B)$. Hence, $\mathcal{K}(A \cup B) \subset \mathcal{K}(A) \cup \mathcal{K}(B)$. Clearly, an embedding of $\{x_0, x_1, \ldots, x_k\}$ is in $|\mathcal{K}(A)|$, therefore the inclusion $|\mathcal{K}(A \cup B)| \subset |\mathcal{K}(A)| \cup |\mathcal{K}(B)|$ follows. $\square$

Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous map between two T_0 topological spaces. By Alexandrov Theorem (1.4.8) it preserves the partial orders $\leq_{\mathcal{T}_X}$ and $\leq_{\mathcal{T}_Y}$. Therefore, a continuous map f induces a simplicial map $\mathcal{K}(f) : \mathcal{K}(X, \mathcal{T}_X) \to \mathcal{K}(Y, \mathcal{T}_Y)$.

2.1.4 Simplicial homology

Let σ be a simplex, either abstract or regular, of dimension 1 or higher, defined or spanned by set of points $V = \{v_0, v_1, \dots, v_p\}$. Up to even permutations, we can distinguish two classes of the orderings of those points. We call these equivalence classes the *orientations* of a simplex σ. An *oriented simplex* is a simplex $\sigma = [v_0, v_1, \dots, v_p]$ with the orientation induced by the order $v_0, v_1, \dots, v_p$. We denote simplex σ with the opposite orientation by $-\sigma$. Let K be a complex, abstract or geometric. Denote by $\tilde{K}_{\{p\}}$ the set of all oriented simplices in K of dimension p. Note that $\tilde{K}_{\{0\}} = K_{\{0\}}$, because 0-simplices have only one possible orientation. Let R be a fixed commutative ring with unity. A map

$$c_p : \tilde{K}_{\{p\}} \to R$$

satisfying $c(-\sigma) = -c(\sigma)$ when $p > 0$ is called a *p-chain* on K over R. An *elementary chain* for an oriented simplex σ is defined by

$$c_\sigma(\tau) := \begin{cases} 1 & \text{if } \tau = \sigma \\ -1 & \text{if } \tau = -\sigma \\ 0 & \text{if otherwise.} \end{cases}$$

Since for $p = 0$ we have only one possible orientation, every map $c : K_0 \to R$ is a 0-chain. In particular, an elementary 0-chain associated with a 0-simplex σ is a map given by

$$c_\sigma(v) := \begin{cases} 1 & \text{if } v = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

In the sequel we identify an elementary chain c_σ with the simplex σ itself, that is we write $\sigma = c_\sigma$. For $p \in \{0, 1, \dots, \dim K\}$ the *simplicial chain group*, denoted $C_p^\Delta(K)$, is the family of all p-chains in the simplicial complex K, with pointwise addition of chains as the group operator. For $p \notin \{0, 1, \dots, \dim K\}$ we take $C_p^\Delta(K)$ as the trivial group.

Lemma 2.1.3. [22, Lemma 5.1] The simplicial chain group $C_p^\Delta(K)$ is a free abelian group with a basis consisting of all the elementary p-chains in $C_p^\Delta(K)$.

Now, we can introduce the *boundary operator*, the central homomorphism for homology theory

$$\partial_p : C_p^\Delta(K) \to C_{p-1}^\Delta(K).$$

It is defined for basis elements by

$$\partial_p \sigma = \partial_p[v_0, v_1, \dots, v_p] := \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p],$$

where $\hat{v}_i$ means the absence of the ith point. It is easy to prove that for every p we have $\partial_p \circ \partial_{p+1} = 0$. The group of *p-cycles* is defined as the kernel of ∂_p and the group of *p-boundaries* is the image of ∂_{p+1}. We denote them by

$$Z_p^\Delta(K) := \ker \partial_p \quad \text{and} \quad B_p^\Delta(K) := \operatorname{im} \partial_{p+1}.$$

The quotient group

$$H_p^{\Delta}(K) := Z_p^{\Delta}(K) \big/ B_p^{\Delta}(K)$$

is called the p-th *homology group* of K.

In particular, if $R = \mathbb{F}$ is a field, then $C_p^{\Delta}(K)$ is a vector space over $\mathbb{F}$, and so is $H_p^{\Delta}(K)$. Note that the chain group and homology groups are in fact R-modules with pointwise defined external multiplication. This is important in the persistence homology theory (see Section 2.5), which encounters technical difficulties with non-fields [24].

2.1.5 Relative simplicial homology

Let K be a simplicial complex and let K' be its subcomplex. A quotient group of p-chains $C_p^{\Delta}(K) \big/ C_p^{\Delta}(K')$ is the group of *p-chains in K relative to K'*. We denote it by $C_p^{\Delta}(K, K')$. In particular, p-chains $c, d \in C_p^{\Delta}(K)$ are considered identical in $C_p^{\Delta}(K, K')$ if $c - d \in C_p^{\Delta}(K')$.

One can easily verify that the boundary operator ∂_p induces its relative variant

$$\partial_p : C_p^{\Delta}(K, K') \to C_{p-1}^{\Delta}(K, K'),$$

which also satisfies $\partial_{p-1} \circ \partial_p = 0$.

Thus, we may define the relative group of p-cycles, p-boundaries and p-homologies respectively by,

$$\begin{aligned}
Z_p^{\Delta}(K, K') &:= \ker \partial_p, \\
B_p^{\Delta}(K, K') &:= \operatorname{im} \partial_{p+1}, \\
H_p^{\Delta}(K, K') &:= Z_p^{\Delta}(K, K') \big/ B_p^{\Delta}(K, K').
\end{aligned}$$

We present two classical results for simplicial homology theory that we will use in the sequel.

Theorem 2.1.4. (Excision theorem [22, Theorem 9.1]) Let K be a simplicial complex and let K' be its subcomplex. Assume that U is an open set contained in $|K'|$ such that $|K| \setminus U$ is a polytope of a subcomplex L of K and L' is the subcomplex of K whose polytope is $|K'| \setminus U$. Then the inclusion $(L, L') \hookrightarrow (K, K')$ induces an isomorphism

$$H^{\Delta}(L, L') \cong H^{\Delta}(K, K')$$

in simplicial homology.

Theorem 2.1.5. (Relative simplicial Mayer-Vietoris sequence [22, Chapter 25 Ex.2]) Let K be a simplicial complex. Assume that L and M are subcomplexes of K such that $K = L \cup M$. Let L' and M' be subcomplexes of L and M, respectively. Then there is an exact sequence

$$\begin{aligned}
\ldots \to H_n^{\Delta}(L \cap M, L' \cap M') \to H_n^{\Delta}(L, L') \oplus H_n^{\Delta}(M, M') \to \\
H_n^{\Delta}(L \cup M, L' \cup M') \to H_{n-1}^{\Delta}(L \cap M, L' \cap M') \ldots,
\end{aligned}$$

called the *relative Mayer-Vietoris sequence.*

2.2 Chain complexes and chain homology

The homology theory for chain complexes may be generalized to the abstract setting of chain complexes. A *chain complex* is a family $\mathcal{C} = \{C_p, \partial_p\}_{p\in\mathbb{Z}}$ of R-modules C_p and homomorphisms

$$\partial_p : C_p \to C_{p-1}$$

such that $\partial_p \circ \partial_{p+1} = 0$ for every p. Then, we define the group of *p-cycles and p-boundaries of chain complex* $\mathcal{C}$ respectively by

$$Z_p(\mathcal{C}) := \ker \partial_p \quad \text{and} \quad B_p(\mathcal{C}) := \operatorname{im} \partial_{p+1}.$$

The *homology of a chain complex* $\mathcal{C}$ with coefficients in R is defined by

$$H_p(\mathcal{C}) := Z_p(\mathcal{C}) \big/ B_p(\mathcal{C}).$$

In the case of field coefficients the rank of $H_p(\mathcal{C})$ is called the pth *Betti number* of $\mathcal{C}$ and it is denoted $\beta_p(\mathcal{C})$. Chain groups of a simplicial complex together with the boundary operators provide an example of a chain complex.

Let $\mathcal{C} = \{C_p, \partial_p^{\mathcal{C}}\}$ and $\mathcal{D} = \{D_p, \partial_p^{\mathcal{D}}\}$ be chain complexes. A collection of maps $\{\eta_p : C_p \to D_p\}$, written for short as $\eta : \mathcal{C} \to \mathcal{D}$, is a *chain map* if $\partial_p^{\mathcal{D}} \circ \eta_p = \eta_{p-1} \circ \partial_p^{\mathcal{C}}$ for all p. A chain map induces a homomorphism

$$(\eta_p)_* : H_p(\mathcal{C}) \ni [\sigma]_{\mathcal{C}} \to [\eta(\sigma)]_{\mathcal{D}} \in H_p(\mathcal{D}) \tag{2.5}$$

where $\sigma \in Z_p(\mathcal{C})$ and $[\,\cdot\,]_{\mathcal{C}}$, $[\,\cdot\,]_{\mathcal{D}}$ are equivalence classes in homology groups of $\mathcal{C}$ and $\mathcal{D}$.

In particular, a simplicial map $f : K_0 \to L_0$ induces a chain map $f_\# : C_p^\Delta(K) \to C_p^\Delta(L)$ defined on an elementary $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ by

$$f_\#(\sigma) := \begin{cases} \langle f(v_0), f(v_1), \ldots, f(v_n)\rangle & \text{if all } f(v_i) \text{ are pairwise different,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a simplicial map also induces a homomorphism in homology $f_* := (f_\#)_*$ defined as in (2.5).

2.3 Singular theory

2.3.1 Singular chain complex

Several topological spaces, in particular most finite topological spaces cannot be represented as solids of simplicial complexes. Singular homology theory, on the other hand, applies to all topological spaces; in particular to finite topological spaces. However, singular theory operates on a generally uncountable family of objects. This makes it computationally inapplicable, at least directly. Thus, a method is to encode the singular theory into the computationally feasible simplicial theory. It may be a bit counter-intuitive, but to extract some topological invariants of a finite space, we need to go through a relatively sophisticated object (singular theory) just to reduce it to a combinatorial one (simplicial theory) again.

Consider the vector space $\mathbb{R}^{\mathbb{N}}$ of real-valued infinite sequences with pointwise addition and multiplication by scalars. Let $\mathbb{R}^{\infty}$ denote the subspace consisting of sequences which are non-zero only for a finite number of arguments. Let $i > 0$, then by e_i we denote the vector in $\mathbb{R}^{\infty}$ with 0s for every coordinate except the ith one. Thus, we have

$$\begin{aligned} e_1 &:= (1, 0, 0, 0, \ldots), \\ e_2 &:= (0, 1, 0, 0, \ldots), \\ e_3 &:= (0, 0, 1, 0, \ldots), \\ &\ldots. \end{aligned}$$

Denote by Δ_p the simplex spanned by vectors $e_1, e_2, \ldots, e_{p+1}$. It is called the *standard p-simplex* (see Figure 2.3 for an example). In particular $\Delta_p = \langle e_1, e_2, \ldots, e_{p+1} \rangle$.

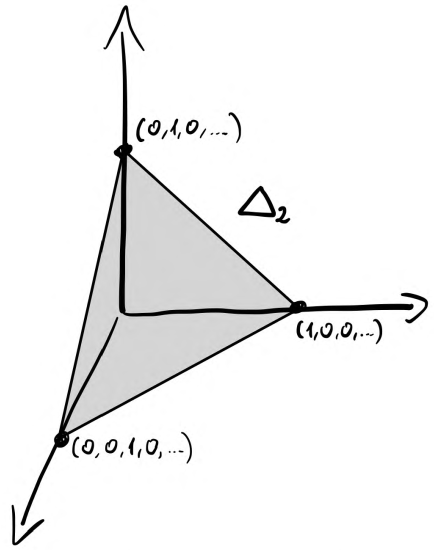


Figure 2.3: The standard 2-simplex.

Now, let X be a topological space. A continuous map $\delta_p : \Delta_p \to X$ is a *singular p-simplex*. Let $S_p(X)$ denote the free abelian group generated by the singular p-simplices. It is called the *singular chain group* of X in dimension p and its elements are referred to as *singular p-chains of X*. Hence, a singular p-chain may be written as a finite formal sum $\sum_i n_i \delta_i$ of p-simplices $\delta_i : \Delta_p \to X$ with $n_i \in \mathbb{Z}$.

To construct a chain complex we still need a boundary operator. To this end we introduce inclusion maps $l_p^i : \Delta_{p-1} \to \Delta_p$ defined for $i \in \{1, \ldots, p\}$ by

$$l_p^i((x_1, x_2, \ldots, x_p, \ldots)) := (x_1, x_2, \ldots, x_{i-1}, \underbrace{0}_{i\text{th}}, x_i, \ldots, \underbrace{x_p}_{(p+1)\text{th}}, 0, 0, \ldots).$$

The image of l_p^i is a $p-1$ dimensional face of the standard singular p simplex. We define the *singular boundary operator* $\partial_p : S_p(X) \to S_{p-1}(X)$ on the basis elements of $S_p(X)$ by

$$\partial_p \delta := \sum_{i=0}^{p} (-1)^i \delta \circ l_p^i.$$

The singular boundary operator satisfies $\partial_p \circ \partial_{p+1} = 0$. Thus, we get a well defined *singular chain complex* $\mathcal{S}(X) = \{S_p(X), \partial_p\}$. We apply it to the construction from Section 2.2 to obtain *singular homology*. We denote the pth singular homology group of a topological space X by $H_p(X) := H_p(\mathcal{S}(X))$.

In a similar way one can define singular homology modules with coefficients in a commutative ring with unity. In the case of field coefficients and finitely generated homology we denote the pth Betti number by $\beta_p(X) := \beta_p(\mathcal{S}(X))$.

Let $f : X \to Y$ be a continuous map between topological spaces. It induces a chain map $f_\# : \mathcal{S}(X) \to \mathcal{S}(Y)$ defined for $\delta_p \in \mathcal{S}(X)$ by $(f_p)_\#(\delta_p) := f \circ \delta_p$. Chain map, in turn, induces a homomorphism $H(f) : H(X) \to H(Y)$. We also write $f_* := H(f)$.

The above construction works also for relative complexes. Let A be a subset of a topological space X. A chain complex $\mathcal{S}(A)$ is a subcomplex of $\mathcal{S}(X)$. Hence, we have the quotient chain complex $\mathcal{S}(X, A) := {\mathcal{S}(X)}/{\mathcal{S}(A)}$, the induced boundary homomorphism and homology groups, denoted by $H(X, A) := H(\mathcal{S}(X, A))$.

Theorem 2.3.1. [22, Chapter 24, Exercise 1] Let $B \subset A \subset X$ be a triple of topological spaces. The inclusions induce the following exact sequence, called the *exact homology sequence of the triple*:

$$\ldots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \ldots .$$

If a topological space X has finitely generated homology, i.e. it has only a finite number of nonzero homology groups and each of them is finitely generated, we define the *Poincaré polynomial*

$$p_X(t) := \sum_{i=1}^{\infty} \beta_i(X) t^i. \tag{2.6}$$

Similarly, for a topological pair (X, A) we put

$$p_{X,A}(t) := \sum_{i=1}^{\infty} \beta_i(X, A) t^i. \tag{2.7}$$

Finally, the following theorem shows that the simplicial homology groups of a simplicial complex K are isomorphic to the singular homology groups of its polytope $|K|$.

Theorem 2.3.2. [22, Theorem 34.3 & Theorem 34.4] Let K be a simplicial complex. There exists a chain map $\eta : \mathcal{C}^\Delta(K) \to \mathcal{S}(|K|)$ that sends simplicial chains of K into singular chains of the polytope $|K|$, such that η_* is an isomorphism of simplicial and singular homology. Moreover, η_* commutes with homomorphisms induced by simplicial maps.

2.4 Homology of finite topological spaces

This section contains results from Section 3 in [17].

We have noted in Section 2.1.3 that there is a natural correspondence between finite topological spaces and simplicial complexes. It also provides a geometrical intuition for understanding finite topologies. However, if we want to consider homology of a finite topological space, we need to use singular theory, which is still well-defined for finite setting (see Figure 2.4). In order to make the notation more explicit, we use H and H^Δ for, respectively, singular and simplicial homology functor. In particular, for a finite topological space X and its order complex $\mathcal{K}(X)$, we write $H(X)$ and $H^\Delta(\mathcal{K}(X))$.

A bridge that connecting finite topological spaces and simplicial complexes was developed by McCord [18]. Let $(X, \mathcal{T})$ be a finite topological space. The map

$$\mu_{(X,\mathcal{T})} : |\mathcal{K}(X, \mathcal{T})| \ni \alpha \mapsto \max \operatorname{supp} \alpha \in (X, \mathcal{T}),$$

where the max refers to the partial order $\leq_\mathcal{T}$, is called the *McCord map*. Actually, McCord uses convection identifying down sets with open sets and puts $\mu_{(X,\mathcal{T})}(\alpha) := \min \operatorname{supp} \alpha$. However, the duality of open and closed sets in finite topological spaces (Proposition 1.4.2) justifies this modification.

The following theorem is a consequence of [3, Theorem 1.4.6 and Remark 1.4.7] and [12, Proposition 4.21].

Theorem 2.4.1. (McCord Theorem) The map $\mu_{(X,\mathcal{T})}$ is continuous. If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous map of two finite T_0 topological spaces, then the diagrams

$$\begin{tikzcd}
|\mathcal{K}(X,\mathcal{T}_X)| \arrow[r, "|\mathcal{K}(f)|"] \arrow[d, "\mu_{(X,\mathcal{T}_X)}"] & |\mathcal{K}(Y,\mathcal{T}_Y)| \arrow[d, "\mu_{(Y,\mathcal{T}_Y)}"] \\
(X,\mathcal{T}_X) \arrow[r, "f"] & (Y,\mathcal{T}_Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
H\left(|\mathcal{K}(X,\mathcal{T}_X)|\right) \arrow[r, "|\mathcal{K}(f)|_*"] \arrow[d, "\mu_{(X,\mathcal{T}_X)*}"] & H\left(|\mathcal{K}(Y,\mathcal{T}_Y)|\right) \arrow[d, "\mu_{(Y,\mathcal{T}_Y)*}"] \\
H(X,\mathcal{T}_X) \arrow[r, "f_*"] & H(Y,\mathcal{T}_Y)
\end{tikzcd}$$

commute. Moreover, the homomorphisms $\mu_{(X,\mathcal{T}_X)*}$ and $\mu_{(Y,\mathcal{T}_Y)*}$ are isomorphisms of singular homologies.

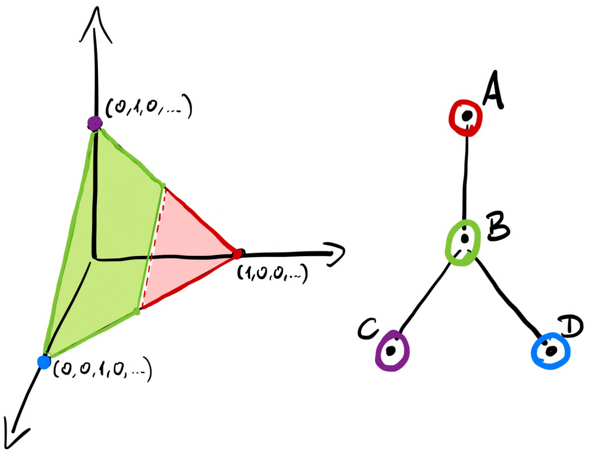
Consequently, the homology of a finite space is the same as of the polytope of its order complex. We can further combine isomorphism μ_{X*} with isomorphism η_* introduced in Theorem 2.3.2, to establish a complete bridge between the singular homology of a finite topological space and the simplicial homology of its order complex:

$$H(X) \cong H(|\mathcal{K}(X)|) \cong H^\Delta(\mathcal{K}(X)).$$

We extend this correspondence to relative homology as well (see Figure 2.5 for geometrical interpretation).

Proposition 2.4.2. [17, Proposition 3.12] Let A, B be subsets of a finite topological space X such that $B \subset A$. Then $\mathcal{K}(B)$ is a subcomplex of $\mathcal{K}(A)$ and

$$H(A, B) \cong H^\Delta(\mathcal{K}(A), \mathcal{K}(B)).$$



$$\delta : \Delta_2 \to X$$

$$\delta((x,y,z)) := \begin{cases} A & \text{if } y+z < \frac{1}{2}; \\ B & \text{if } \dfrac{1}{2} \leq y+z \leq 1 \\ & \quad \text{and } y, z \neq 1; \\ C & \text{if } y = 1; \\ D & \text{if } z = 1. \end{cases}$$

Figure 2.4: Map δ shows an example of a continuous embedding of the 2-standard simplex into a finite topological space. Thus δ is a a singular 2-simplex in a finite topological space.

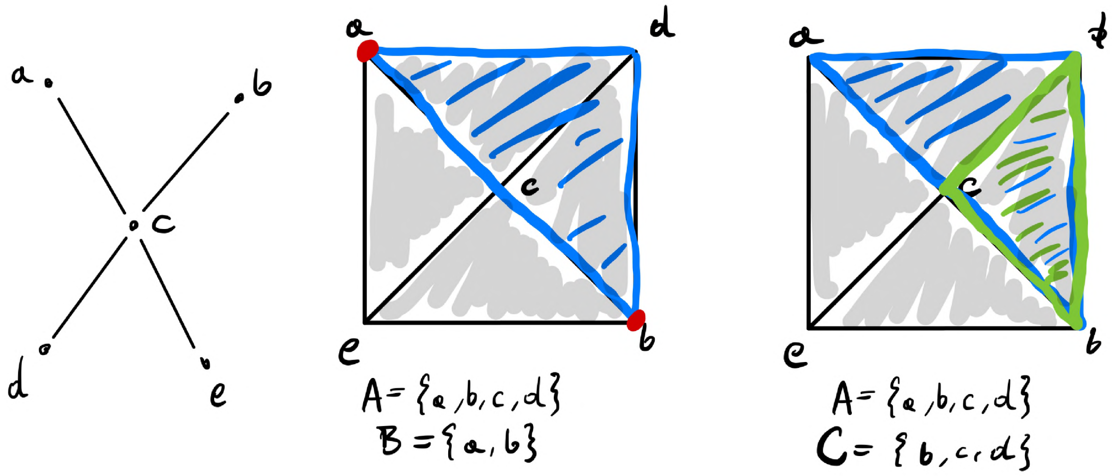


Figure 2.5: To "see" relative homology in the context of finite topological spaces it is enough to consider the simplicial homology of the corresponding order complexes. Given a finite topological space (left) and sets A, B, C we have $H(A,B) = H^{\Delta}(\mathcal{K}(A), \mathcal{K}(B))$ and $H(A,C) = H^{\Delta}(\mathcal{K}(A), \mathcal{K}(C))$. Order complexes $\mathcal{K}(A)$, $\mathcal{K}(B)$ and $\mathcal{K}(C)$ are highlighted in blue, red and green color, respectively.

Proof. The McCord map μ_X restricted to A and B naturally induces a homomorphism $\mu_{X*}(A,B)$ in relative homology. Consider the commutative diagram

$$\begin{tikzcd}
H_n(|\mathcal{K}(B)|) \arrow[r] \arrow[d, "\mu_{X*}"] & H_n(|\mathcal{K}(A)|) \arrow[r] \arrow[d, "\mu_{X*}"] & H_n(|\mathcal{K}(A)|,|\mathcal{K}(B)|) \arrow[r] \arrow[d, "\mu_{X*}(A,B)"] & H_{n-1}(|\mathcal{K}(B)|) \arrow[r] \arrow[d, "\mu_{X*}"] & H_{n-1}(|\mathcal{K}(A)|) \arrow[d, "\mu_{X*}"] \\
H_n(B) \arrow[r] & H_n(A) \arrow[r] & H_n(A,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(A)
\end{tikzcd}$$

The Five Lemma [22, Lemma 24.3] implies that $\mu_{X*}(A,B)$ is also an isomorphism. Similarly, the chain map η induces a homomorphism $\eta_*(A,B)$. Thus again, the commutative diagram

$$\begin{tikzcd}
H_n^{\Delta}(\mathcal{K}(B)) \arrow[r] \arrow[d, "\eta_*"] & H_n^{\Delta}(\mathcal{K}(A)) \arrow[r] \arrow[d, "\eta_*"] & H_n^{\Delta}(\mathcal{K}(A),\mathcal{K}(B)) \arrow[r] \arrow[d, "\eta_*(A,B)"] & H_{n-1}^{\Delta}(\mathcal{K}(B)) \arrow[r] \arrow[d, "\eta_*"] & H_{n-1}^{\Delta}(\mathcal{K}(A)) \arrow[d, "\eta_*"] \\
H_n(|\mathcal{K}(B)|) \arrow[r] & H_n(|\mathcal{K}(A)|) \arrow[r] & H_n(|\mathcal{K}(A)|,|\mathcal{K}(B)|) \arrow[r] & H_{n-1}(|\mathcal{K}(B)|) \arrow[r] & H_{n-1}(|\mathcal{K}(A)|)
\end{tikzcd}$$

together with the Five Lemma implies that $\eta_*(A,B)$ is an isomorphism. It follows that $\mu_{X*}(A,B)\circ\eta_*(A,B)$ is also an isomorphism. □

In the sequel, we also need the variants of the excision theorem and Mayer-Vietoris sequence for finite topological spaces.

Theorem 2.4.3. [17, Theorem 3.14] Let $(X,\mathcal{T})$ be a finite topological space and let A,B,C,D be closed subsets of X such that $B\subset A$, $D\subset C$ and $A\setminus B = C\setminus D$. Then $H(A,B)\cong H(C,D)$.

Proof. We first observe that $\mathcal{K}(A)\setminus\mathcal{K}(B)=\mathcal{K}(C)\setminus\mathcal{K}(D)$. Indeed, consider a chain q in A which is not a chain in B. Let q_0 be the maximal element of q. Then $q_0\notin B$, because otherwise, since B is a closed set, and therefore a down set with respect to $\leq_{\mathcal{T}}$, we get $q\subset B$. Hence, $q_0\in A\setminus B = C\setminus D$. Since C is a down set as a closed set, it follows that $q\subset C$ and clearly $q\not\subset D$. Thus, $q\in\mathcal{K}(C)\setminus\mathcal{K}(D)$ which proves that $\mathcal{K}(A)\setminus\mathcal{K}(B)\subset\mathcal{K}(C)\setminus\mathcal{K}(D)$. The proof of the opposite inclusion is analogous.

Define $\breve{B} := |\mathcal{K}(A)|\setminus|\mathcal{K}(\mathrm{cl}(A\setminus B))|$ (see Figure 2.6). Clearly, $\breve{B}$ is open in $|\mathcal{K}(A)|$. We will show that $\breve{B}\subset|\mathcal{K}(B)|$. Let $\alpha\in\breve{B}$. Set $r:=\operatorname{supp}(\alpha)$ and $r_0:=\max(r)$. Suppose that $r_0\notin B$. Then, $r_0\in A\setminus B$ and $r\subset\mathrm{cl}(A\setminus B)$ which implies $\alpha\in|r|\subset|\mathcal{K}(\mathrm{cl}(A\setminus B))|$, a contradiction. Hence, $r\subset B$ and $\alpha\in|r|\subset|\mathcal{K}(B)|$.

Moreover,

$$|\mathcal{K}(A)|\setminus\breve{B} = |\mathcal{K}(A)|\setminus\Big(|\mathcal{K}(A)|\setminus|\mathcal{K}(\mathrm{cl}(A\setminus B))|\Big) = |\mathcal{K}(\mathrm{cl}(A\setminus B))|$$

and by Proposition 2.1.2

$$\begin{aligned}|\mathcal{K}(B)|\setminus\breve{B} &= |\mathcal{K}(B)|\cap|\mathcal{K}(\mathrm{cl}(A\setminus B))| = |\mathcal{K}(B\cap\mathrm{cl}(A\setminus B))| \\ &= |\mathcal{K}(\mathrm{cl}(A\setminus B)\setminus(A\setminus B))| = \mathcal{K}(\mathrm{mo}(A\setminus B))|.\end{aligned}$$

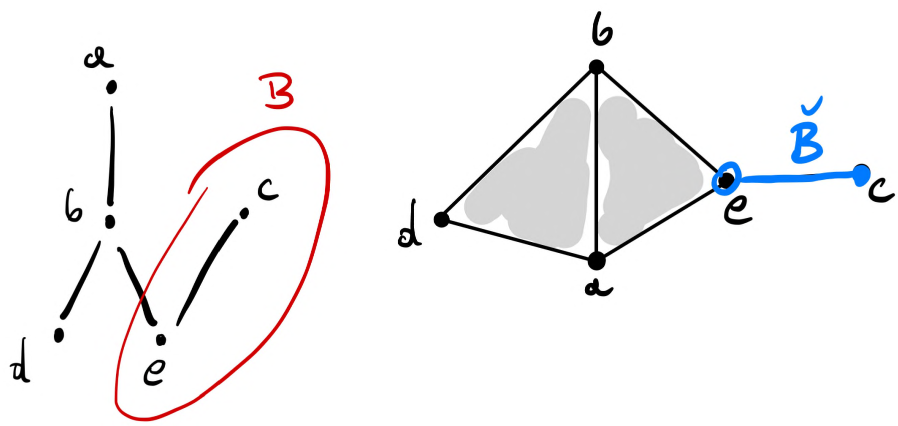


Figure 2.6: A geometrical interpretation of a set $\breve{B}$ for $A = \{a, b, c, d, e\}$ and $B = \{c, e\}$. Note that $e \notin \breve{B}$.

Analogous properties hold for $\breve{D} := |\mathcal{K}(C)| \setminus |\mathcal{K}(\mathrm{cl}(C \setminus D))|$ in $|\mathcal{K}(C)|$. Therefore, by Theorem 2.1.4 we have the following isomorphisms

$$\begin{aligned} H^{\Delta}(\mathcal{K}(A), \mathcal{K}(B)) &\cong H^{\Delta}(\mathcal{K}(\mathrm{cl}(A \setminus B), \mathcal{K}(\mathrm{mo}(A \setminus B)), \\ H^{\Delta}(\mathcal{K}(C), \mathcal{K}(D)) &\cong H^{\Delta}(\mathcal{K}(\mathrm{cl}(C \setminus D), \mathcal{K}(\mathrm{mo}(C \setminus D)). \end{aligned}$$

Note that according to $A \setminus B = C \setminus D$ we have $\mathcal{K}(\mathrm{cl}(A \setminus B)) = \mathcal{K}(\mathrm{cl}(C \setminus D))$ and $\mathcal{K}(\mathrm{mo}(A \setminus B)) = \mathcal{K}(\mathrm{mo}(C \setminus D))$. Thus, with Proposition 2.4.2 we get

$$\begin{aligned} H(A, B) &\cong H^{\Delta}(\mathcal{K}(A), \mathcal{K}(B)) \cong H^{\Delta}(\mathcal{K}(\mathrm{cl}(A \setminus B), \mathcal{K}(\mathrm{mo}(A \setminus B))) \\ &= H^{\Delta}(\mathcal{K}(\mathrm{cl}(C \setminus D), \mathcal{K}(\mathrm{mo}(C \setminus D))) \cong H^{\Delta}(\mathcal{K}(C), \mathcal{K}(D)) \cong H(C, D), \end{aligned}$$

which completes the proof of the theorem. $\square$

Theorem 2.4.4. (Relative Mayer-Vietoris sequence for finite topological spaces [17, Theorem 3.17]) Let X be a finite topological space. Assume that $Y_0 \subset X_0, Y_1 \subset X_1$ are pairs of closed sets in X such that $X = X_0 \cup X_1$. Then there is an exact sequence

$$\begin{aligned} \ldots \to H_n(X_0 \cap X_1, Y_0 \cap Y_1) \to H_n(X_0, Y_0) \oplus H_n(X_1, Y_1) \to \\ H_n(X_0 \cup X_1, Y_0 \cup Y_1) \to H_{n-1}(X_0 \cap X_1, Y_0 \cap Y_1) \ldots . \end{aligned}$$

Proof. By Proposition 2.1.2 we have $\mathcal{K}(X) = \mathcal{K}(X_0) \cup \mathcal{K}(X_1)$ and $\mathcal{K}(Y_0 \cup Y_1) = \mathcal{K}(Y_0) \cup \mathcal{K}(Y_1)$. Thus, the proof follows from the relative simplicial Mayer-Vietoris Theorem 2.1.5 and Proposition 2.4.2. $\square$

2.5 Persistent homology

In this section we summarize the zig-zag persistence theory following [6].

2.5.1 Zigzag persistence

Let $\mathbb{V}$ denote a sequence $(V_i)_{i=1}^n$ of vector spaces with coefficients in a field $\mathbb{F}$ together with connecting maps $(h_i)_{i=1}^{n-1}$. Every connecting map can be a forward map $h_i : V_i \to V_{i+1}$ or a backward map $h_i : V_{i+1} \to V_i$. We write such a sequence as

$$\mathbb{V}: \quad V_1 \xleftrightarrow{h_1} V_2 \xleftrightarrow{h_2} \dots \xleftrightarrow{h_{n-1}} V_n. \tag{2.8}$$

and we call it a *zigzag module*. Directions of maps in the sequence define its *type*. If all maps are forward (or backward) we refer to (2.8) as the *persistence module*. A module $\mathbb{W}$ is a *submodule* of $\mathbb{V}$ if both have the same type, each W_i is a subspace of V_i and for every map, depending on its direction, either $h_i(W_i) \subset W_{i+1}$ or $h_i(W_{i+1}) \subset W_i$. A submodule $\mathbb{W}$ is a *summand* of $\mathbb{V}$ if there exists another submodule $\mathbb{X}$ of $\mathbb{V}$ such that $V_i = W_i \oplus X_i$ for every i. In that case $\mathbb{V}$ is a *direct sum of zigzag modules* $\mathbb{W}$ and $\mathbb{X}$, and we write $\mathbb{V} = \mathbb{W} \oplus \mathbb{X}$. A module is said to be *decomposable* if it can be written as a direct sum of two nonzero submodules. It is *indecomposable* otherwise. An elementary example of an indecomposable module is the *interval module* $I(b, d)$

$$\mathbb{I}(b,d): \quad I_1 \longleftrightarrow I_2 \longleftrightarrow \dots \longleftrightarrow I_n, \tag{2.9}$$

where $I_i = \mathbb{F}$ (a field fixed at the beginning of this section) if $b \le i \le d$ and 0 otherwise. The integer b defining interval module is the *birth time* and d is the *death time* of the interval.

According to Gabriel Theorem [6, Theorem 2.5], we can decompose every zigzag module $\mathbb{V}$ into a direct sum of interval modules. In particular

$$\mathbb{V} \cong \mathbb{I}(b_1, d_1) \oplus \mathbb{I}(b_2, d_2) \oplus \dots \oplus \mathbb{I}(b_N, d_N). \tag{2.10}$$

We can gather births and deaths from the above decomposition into a multiset

$$\mathrm{Pers}(\mathbb{V}) = \{(b_i, d_i) \mid i \in \{1, 2, \dots, N\}\}.$$

We refer to the elements of $\mathrm{Pers}(\mathbb{V})$ as the *zigzag persistence pairs* of $\mathbb{V}$.

2.5.2 Interpretation of persistent homology

We are particularly interested in a zigzag persistence of homology vector spaces. Similarly to zigzag modules we can consider a sequence of simplicial complexes (or topological spaces) together with either forward or backward simplicial maps (or continuous maps), which we write as

$$\mathbb{X}: \quad X_1 \xleftrightarrow{f_1} X_2 \xleftrightarrow{f_2} \dots \xleftrightarrow{f_{n-1}} X_n.$$

Each simplicial map (or continuous map) f_i induces homomorphism f_{i*} in homology (see Section 2.2 and 2.3.1). Thus, for a fixed dimension p, we obtain a zigzag module

$$H_p(X_1) \xleftrightarrow{f_{1*}} H_p(X_2) \xleftrightarrow{f_{2*}} \dots \xleftrightarrow{f_{n-1*}} H_p(X_n). \tag{2.11}$$

Each interval in the decomposition (2.10) of (2.11) represents a basis element of $H_p(X_i)$ which persists trough the interval. Left end of the interval, referred to as the birth, indicates at which step the generator appears. Similarly, the right end point of the interval, referred to as the death, indicates when the generator vanishes. Consider the sequence of simplicial complexes in Figure 2.7 and the following sequence:

$$K_0 \longrightarrow K_1 \longleftarrow K_2 \longrightarrow K_3 \longleftarrow K_4 \longrightarrow K_5 \longleftarrow K_{5\cap 6} \longrightarrow K_6 \longleftarrow K_{6\cap 7} \longrightarrow K_7 \longleftarrow K_8, \tag{2.12}$$

where every map is an inclusion, $K_{5\cap 6} := K_5 \cap K_6$ and $K_{6\cap 7} := K_6 \cap K_7$. Now, by applying the homology functor, we get a zigzag module

$$\begin{aligned} \mathbb{V} : H(K_0) \to H(K_1) \leftarrow H(K_2) \to H(K_3) \leftarrow H(K_4) \to H(K_5) \leftarrow \\ \leftarrow H(K_{5\cap 6}) \to H(K_6) \leftarrow H(K_{6\cap 7}) \to H(K_7) \leftarrow H(K_8), \end{aligned}$$

and the associated multiset of persistence zigzag pairs $\mathrm{Pers}(\mathbb{V})$. We can graphically present $\mathrm{Pers}(\mathbb{V})$ using *persistence barcode* (see Figure 2.8), where every pair (b_i, d_i) is represented by a separate bar with endpoints given by birth and death. Persistence barcode sums up the evolution of homology groups as the underlying space morphs. The vertical section of a barcode gives us the exact number of homology generators for a particular parameter value (e.g., for $i = 5$, we have 2 generators in the 0th dimension and 1 in 1st dimension). Looking at the barcode in Figure 2.8, we can also deduce that one of the cycles (1st dimensional homology generator) lives for a wide range of parameters, which may be interpreted as its robustness.

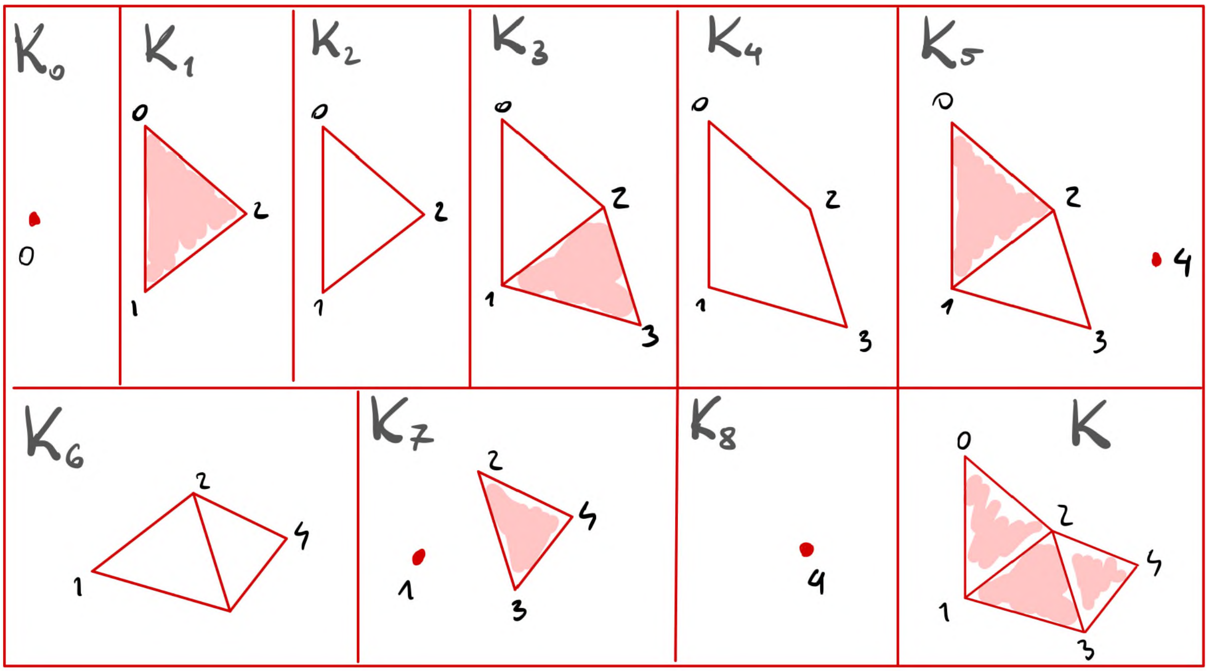


Figure 2.7: A sequence of simplicial complexes. Every complex K_i is a subcomplex of a simplicial complex K drawn in the last picture.

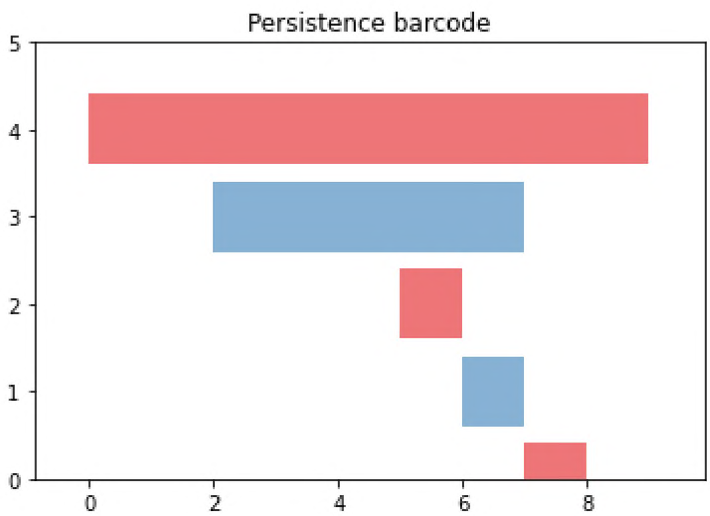


Figure 2.8: A persistence barcode for a zigzag persistence module $\mathbb{V}$ (2.12) for a filtration from Figure 2.7. Red-colored bars represent persistence pairs of dimension 0, while blue bars correspond to the 1st dimension.

Chapter 3

Dynamical systems

3.1 Continuous dynamical systems

The principal object of interest of this thesis is the theory of combinatorial dynamical systems. However, we first recall some basic concepts of the theory of continuous dynamical systems.

In this section we assume that X is a locally compact metric space.

Definition 3.1.1. A *continuous dynamical system* on X, or a *flow*, is a continuous map $\varphi : X \times \mathbb{R} \to X$ such that for every $x \in X$ and $t, s \in \mathbb{R}$ we have

$$\varphi(x, 0) = x \quad \text{and} \quad \varphi(\varphi(x, t), s) = \varphi(x, t + s).$$

A *solution* of a point $x \in X$, is the map $\varphi_x : \mathbb{R} \to X$ given by $\varphi_x(t) := \varphi(x, t)$. The *invariant part* of a set $S \subset X$ is the set $\operatorname{Inv} S := \{x \in S \mid \varphi(x, \mathbb{R}) \subset S\}$. In particular, S is an *invariant set* if $S = \operatorname{Inv} S$. A compact set $N \subset X$ is an *isolating neighborhood* if $\operatorname{Inv} N \subset \operatorname{int} N$. We say that an invariant set S is an *isolated invariant set* if there exists an isolating neighborhood N such that $S \subset \operatorname{int} N$. Figure 3.1 shows basic examples of isolated invariant sets with corresponding isolated neighborhoods marked in green.

We can algebraically describe the nature of an isolated invariant set using the Conley index. Its construction involves a technical object called the index pair. We follow here the exposition presented in [19]. A pair $P = (P_1, P_2)$ of closed sets such that $P_2 \subset P_1$ is an *index pair* for an isolated invariant set S if

(1) $S = \operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_2))$ and $\operatorname{cl}(P_1 \setminus P_2)$ is a neighborhood of S,

(2) P_2 is *positively invariant* in P_1; that is given $x \in P_2$ and $\varphi_x([0, t]) \subset N$, then $\varphi_x([0, t]) \subset P_2$,

(3) P_2 is an *exit set* for P_1; that is given an $x \in P_1$ and a $t_1 > 0$ such that $\varphi_x(t_1) \notin P_1$, there exists a $t_0 \in [0, t_1]$ for which $\varphi_x([0, t_0]) \subset P_1$ and $\varphi_x(t_0) \in P_2$.

The following theorem guarantees the existence of an index pair.

Theorem 3.1.2. [19, Theorem 3.5] Given an isolated invariant set S, there exists an index pair for S.

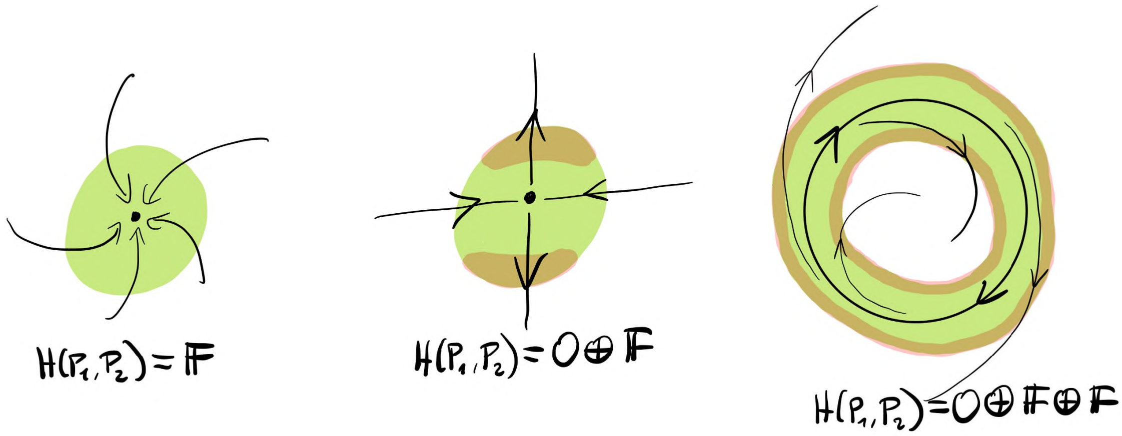


Figure 3.1: Examples of isolated invariant sets with their Conley indices: attracting point (left), saddle point (center) and repelling periodic orbit (right). Green sets represent isolating neighborhoods for given isolated invariant sets. For each example, the green set together with the beige set (it is empty in the first case) as an exit set form an index pair (P_1, P_2).

An example of an index pair is provided by Ważewski set. We say that a compact set N is a Ważewski set if $N^- := \{x \in N \mid \forall_{\varepsilon>0}\ \varphi(x, [0, \varepsilon]) \not\subset N\}$ is closed. One can show that then N is an isolating neighborhood and (N, N^-) is an index pair for $\operatorname{Inv} N$. Note that if N is a Ważewski set then $N \setminus N^-$ is locally closed.

Theorem 3.1.3. [19, Theorem 3.12] (Ważewski principle) If N is a Ważewski set and $H_*(N, N^-) \neq 0$ then $\operatorname{Inv} N \neq \emptyset$.

Even though index pairs are not unique, they carry common information. In particular, we define the *homology Conley index* of an isolated invariant set S as $H(P_1, P_2)$, where (P_1, P_2) is an index pair for S.

Theorem 3.1.4. [19, Theorem 3.8] The Conley index is well defined, that is if (P_1, P_2) and (P_1', P_2') are index pairs for an isolated invariant set S then $H(P_1, P_2) \cong H(P_1', P_2')$.

Conley index is used twofold. Firstly, by the Ważewski principle, if the Conley index for an isolating neighborhood N (that is the Conley index for $\operatorname{Inv} N$) is non-trivial, then the invariant part of N is non-empty. See Figure 3.2 that the inverse implication is not valid. Secondly, the Conley index can serve as descriptor of an isolated invariant set. Notice that every example in Figure 3.1 has a different index.

Let $x \in X$. An α- and ω-*limit sets* of x for a dynamical system φ are defined as

$$\alpha(x) := \operatorname{cl} \bigcap_{t \in \mathbb{R}^+} \varphi_x\left((-\infty, -t)\right),$$
$$\omega(x) := \operatorname{cl} \bigcap_{t \in \mathbb{R}^+} \varphi_x\left((t, +\infty)\right).$$

Limit sets capture the ultimate past and the ultimate future of a point $x \in X$ (see Figure 3.3).

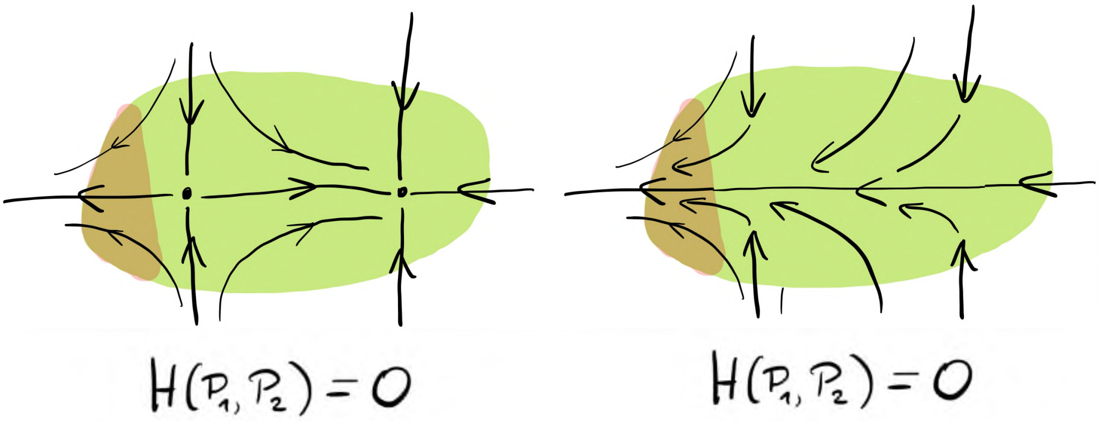


Figure 3.2: The isolated invariant set S in the left picture consists of two stationary points and the orbit connecting them. The green and the beige set forms an index pair (P_1, P_2) for S resulting in a trivial Conley index. The isolated invariant set in the right picture is empty, yet the pair (P_1, P_2) is still a valid index pair. This shows that the Conley index may fail to distinguish qualitative properties of the dynamics inside the isolating neighborhood (the green set) presented in both panels.

A compact invariant set $A \subset X$ is an *attractor* if there exists a neighborhood U of A such that $\omega(U) = A$. Similarly, a compact invariant set $R \subset X$ is an *repeller* if there exists a neighborhood U of R such that $\alpha(U) = R$.

Definition 3.1.5. Let φ be a dynamical system for a locally compact topological space X and let $(\mathbb{P}, \leq)$ be a finite partial order. A collection $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of X is a *Morse decomposition* of X if

(i) $\mathcal{M}$ is a collection of mutually disjoint isolated invariant subsets of X,

(ii) for every $x \in X \setminus \bigcup \mathcal{M}$ there exist $p, q \in \mathbb{P}$ such that $p \leq q$ and

$$\alpha(x) \subset M_p \text{ and } \omega(x) \subset M_q$$

We refer to the elements of $\mathcal{M}$ as *Morse sets*.

The second condition means that the order imposed on Morse decomposition reflects the existence of trajectories between Morse sets. The Hasse diagram of the poset labeled with Conley indices of Morse sets in its nodes is called the Morse-Conley graph of the Morse decomposition[2]. The graph captures the global behavior of the dynamical system. Figure 3.4 shows an example of a Morse decomposition and its Morse-Conley graph.

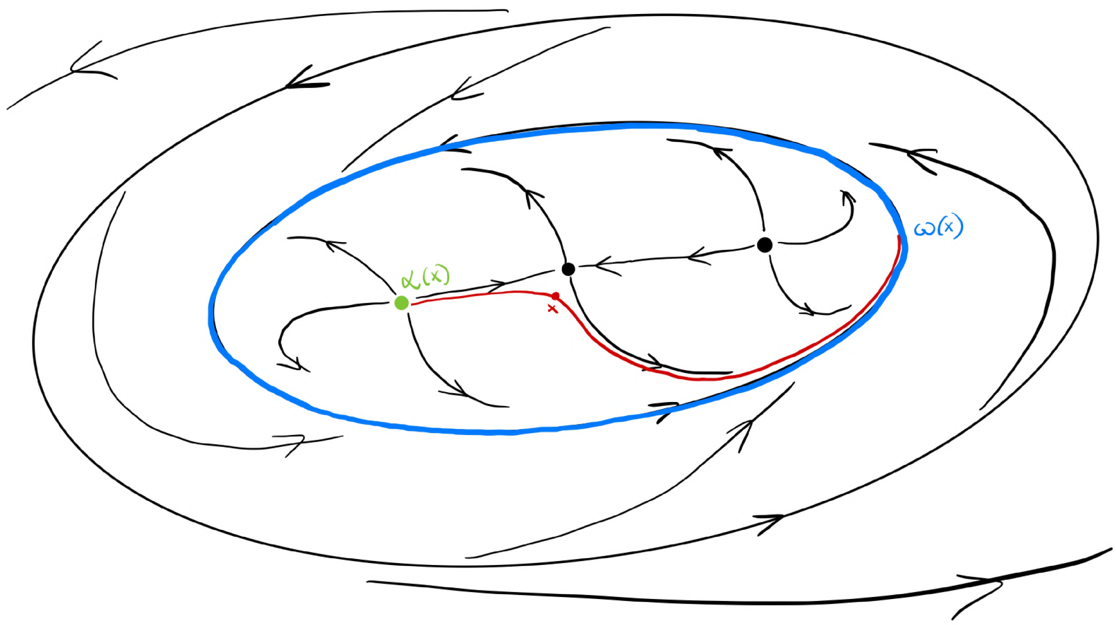


Figure 3.3: The α-limit set for a point x is the stationary point highlighted in green and its ω-limit set is the blue periodic orbit.

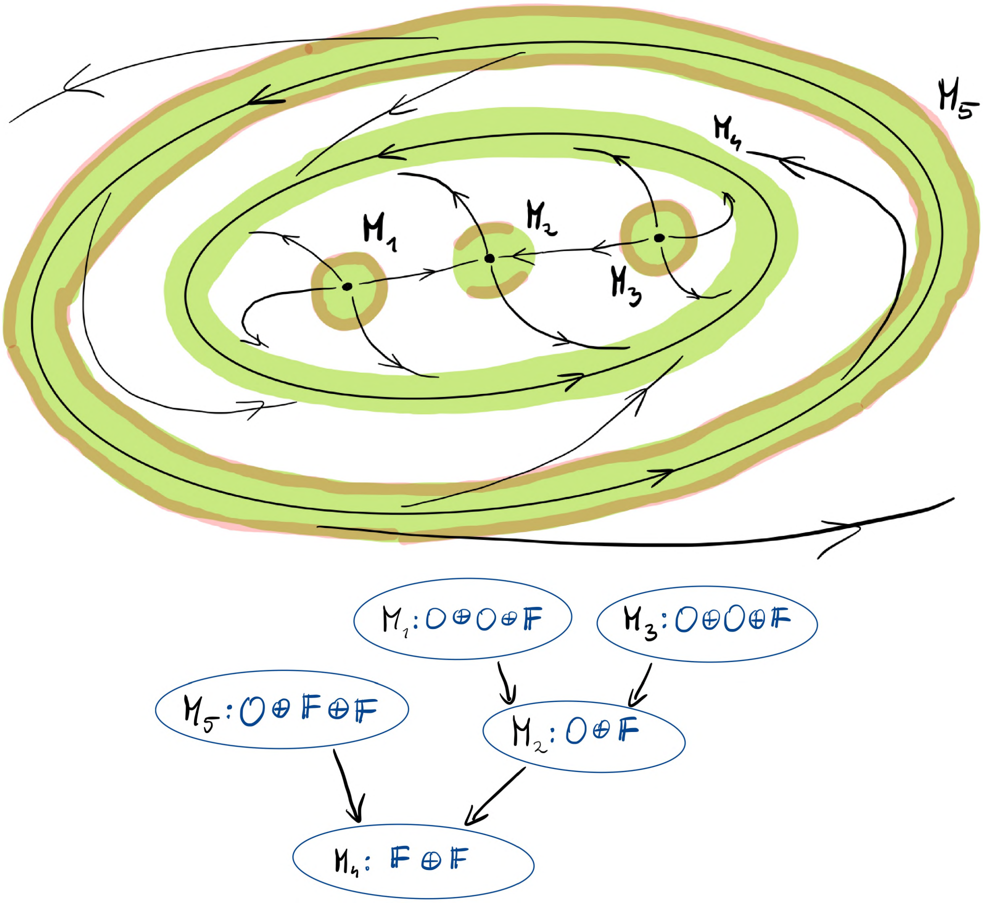


Figure 3.4: An example of a Morse decomposition and the associated Morse-Conley graph. Green and orange sets represent index pairs for each Morse set.

3.2 Combinatorial dynamical systems

Here, we present the general framework for combinatorial dynamics on finite topological spaces. It is mostly based on Section 4.1 and 4.2 in [17] and 2.3 in [7].

Assume that X is a finite topological space.

Definition 3.2.1. By a *combinatorial dynamical system* or briefly, a *dynamical system* in X we mean a multivalued map $\Pi : X \times \mathbb{Z}^+ \multimap X$ such that for every $x \in X$ and $m, n \in \mathbb{Z}^+$ we have

$$\Pi(x, 0) = x \quad \text{and} \quad \Pi\left(\Pi(x, m), n\right) = \Pi(x, m + n). \tag{3.1}$$

Let Π be a combinatorial dynamical system in X. Consider the multivalued map $\Pi^n : X \multimap X$ given by $\Pi^n(x) := \Pi(x, n)$. We call Π^1 the *generator* of the combinatorial dynamical system Π. It follows from (3.1) that the combinatorial dynamical system Π is uniquely determined by its generator. Thus, it is natural to identify a combinatorial dynamical system with its generator. In particular, we consider any multivalued map $\Pi : X \multimap X$ as a combinatorial dynamical system $\Pi : X \times \mathbb{Z}^+ \multimap X$ defined recursively by

$$\begin{aligned} \Pi(x, 1) &:= \Pi(x), \\ \Pi(x, n + 1) &:= \Pi(\Pi(x, n)), \end{aligned}$$

as well as $\Pi(x, 0) := \{x\}$. We call it the combinatorial dynamical system induced by a map Π. In particular, the inverse Π^{-1} (see Section 1.1) of Π also induces a combinatorial dynamical system. We call it the *dual dynamical system*.

3.2.1 Combinatorial solutions and paths

A set is a $\mathbb{Z}$-interval if it is of form $\mathbb{Z} \cap I$ where I is an interval in $\mathbb{R}$. If a $\mathbb{Z}$-interval has a minimum, we say it is *left bounded*; otherwise, it is *left-infinite*. It is *right bounded* if it has a maximum; otherwise it is *right-infinite*. If it is both right and left bounded, then it is *bounded*.

A *solution* of a combinatorial dynamical system $\Pi : X \multimap X$ in $A \subset X$ is a partial map $\varphi : \mathbb{Z} \nrightarrow A$ whose *domain*, denoted $\operatorname{dom} \varphi$, is a $\mathbb{Z}$-interval and for any $i, i+1 \in \operatorname{dom} \varphi$ the inclusion $\varphi(i + 1) \in \Pi(\varphi(i))$ holds. The solution φ is *stationary* if $\varphi(t) = x$ for all $t \in \operatorname{dom} \varphi$ for some $x \in X$. The solution *passes* through $x \in X$ if $x = \varphi(i)$ for some $i \in \operatorname{dom} \varphi$. The solution φ is *full* if $\operatorname{dom} \varphi = \mathbb{Z}$. It is a *backward solution* if $\operatorname{dom} \varphi$ is left-infinite. It is a *forward solution* if $\operatorname{dom} \varphi$ is right-infinite. It is a *partial solution* or simply a *path* if $\operatorname{dom} \varphi$ is bounded.

If φ is left-bounded, then we call the value of φ at the minimal value of its domain the *left endpoint* of φ. If φ is right-bounded, then we call the value of φ at the maximal value of its domain the *right endpoint* of φ. We denote the left and right endpoints of φ, respectively, by $\varphi^{\sqsubset}$ and $\varphi^{\sqsupset}$.

A full solution $\varphi : \mathbb{Z} \to X$ is *periodic* if there exists a $T \in \mathbb{N}$ such that $\varphi(t+T) = \varphi(t)$ for all $t \in \mathbb{Z}$. Note that a path φ satisfying $\varphi^{\sqsubset} = \varphi^{\sqsupset}$ maybe extended to a periodic solution.

By a *shift* of a solution φ we mean the composition $\varphi \circ \tau_n$, where the map

$$\tau_n : \{k - n \mid k \in \operatorname{dom} \varphi\} \ni m \mapsto m + n \in \operatorname{dom} \varphi$$

is an n-translation. Given a right-bounded solution φ and a left-bounded solution ψ such that $\psi^{\sqsubset} \in \Pi(\varphi^{\sqsupset})$, there is a unique shift τ_n such that

$$\mathbb{Z} \ni t \nrightarrow \begin{cases} \varphi(t) & \text{if } t \in \operatorname{dom} \varphi \\ (\psi \circ \tau_n)(t) & \text{if } t \in \operatorname{dom} \tau_n \end{cases} \in X$$

is a solution. We call this union of paths the *concatenation* of φ and ψ and we denote it by $\varphi \cdot \psi$. We also identify each $x \in X$ with the trivial solution $\varphi : \{0\} \to \{x\}$. Given a full solution φ, we denote its restrictions to $\mathbb{Z}^+$ by φ^+ and to $\mathbb{Z}^-$ by φ^-. We finish this section with the following straightforward proposition.

Proposition 3.2.2. [17, Proposition 4.1] If $\varphi : \mathbb{Z} \to X$ is a full solution of a dynamical system $\Pi : X \multimap X$, then $\mathbb{Z} \ni t \to \varphi(-t) \in X$ is a solution of the dual dynamical system induced by Π^{-1}. We call it the *dual solution* and denote it φ^{op}.

3.2.2 Examples of combinatorial dynamical systems

In this section we briefly present and compare three examples of combinatorial dynamical systems arising from a combinatorial analogue of the classical vector field. See [7] for another example of a combinatorial counterpart of a discrete dynamical system constructed from a sampled data.

The following examples show the evolution of the theory that we present in the next chapter. Note that a combinatorial dynamical system $\Pi : X \multimap X$ may be identified with the relation $\{(x, y) \in X \times X \mid y \in \Pi(x)\}$ in X. Therefore, we can visualize it as a graph (see Section 1.2).

All three presented theories have slightly different general settings: CW-complexes, Lefschetz complexes and finite topological spaces. The examples we present are constructed in a simplicial complexes, a natural common-ground for all three settings.

Forman's combinatorial vector field

Robin Forman [10] who introduced the concept of combinatorial vector field does not study combinatorial dynamical systems in our sense. However, they are present in his work implicitly via the concept of paths.

Let K be a finite simplicial complex. A *Forman's combinatorial vector field* on K is a map $V : K \to K \cup \{\emptyset\}$ such that

(i) if $V(\sigma) \neq \emptyset$, then $\dim V(\sigma) = \dim \sigma + 1$, and $\sigma \preceq V(\sigma)$;

(ii) if $V(\sigma) = \tau \neq \emptyset$, then $V(\tau) = \emptyset$;

(iii) for all $\sigma \in K$, $\# V^{-1}\sigma \leq 1$.

A combinatorial vector field induces a partition $\{\{\sigma, V(\sigma)\} \mid \sigma \in K,\ V(\sigma) \neq \emptyset\}$ for σ such that $V(\sigma) \neq \sigma$ and singletons $\{\sigma\}$ for σ such that $V(\sigma) = \emptyset$ and $\sigma \notin \operatorname{im} V$. These doubletons and singletons are called *combinatorial vectors*. We can easily visualize this idea on the simplicial complex. Consider the example in Figure 3.5 where red dots denote singletons and doubletons are represented by red arrows from σ to $V(\sigma)$. Condition (i) says that an arrow starting from σ can only go to a coface of σ of codimension 1. The second condition asserts that the head of an arrow cannot be a tail of another arrow. The third condition means that a simplex σ can be the head of at most one simplex. One can show that this is equivalent to a partitioning the finite topological space induced by the face relation in a simplicial complex into singletons and connected locally closed doubletons (Figure 3.5, middle).

Forman defines V-paths of index p to be a sequence

$$\gamma : \sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \tau_{r-1}, \sigma_r$$

such that for all $i = 0, 1, \ldots, r-1$

(i) $\dim \sigma_i = \dim \sigma_r = p$, $\dim \tau_i = p + 1$,

(ii) $\tau_i = V(\sigma_i)$,

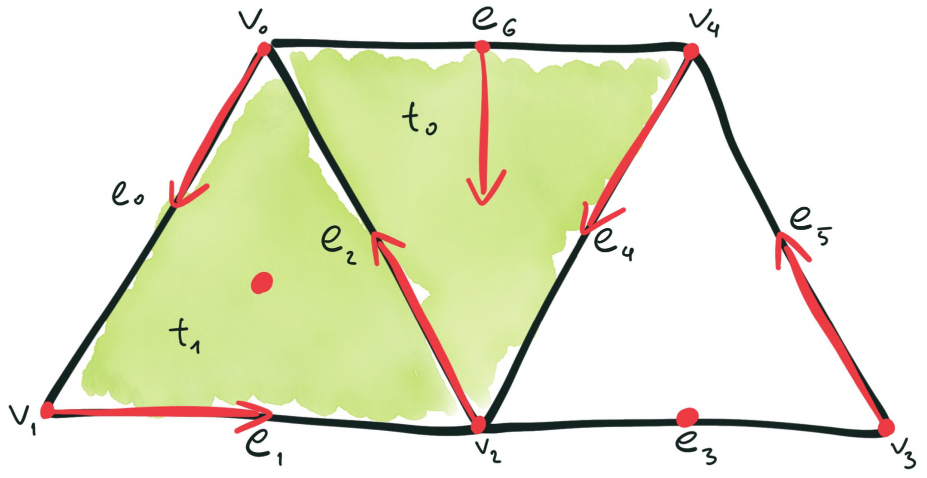
(iii) $\sigma_i \neq \sigma_{i+1} \prec \tau_i$.

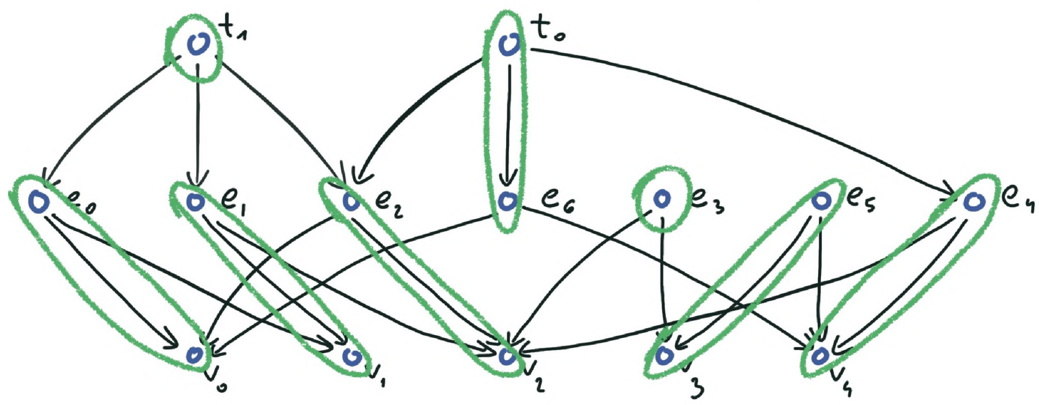
These conditions imply that a path of index p alternates between simplices of dimension p and $p+1$. The movement from σ_i to τ_i goes along the combinatorial vector, while the step from τ_i to σ_{i+1} is a descent to one of the faces of τ_i. Sequence $v_0, e_0, v_1, e_1, v_2, e_2, v_0$ is an example of a V-path of index 0 for the combinatorial vector field at Figure 3.5. We can wrap up this idea by reversing the arrows within combinatorial vectors in a poset and adding self-loops for singletons (see Figure 3.5, bottom). The resulting directed graph $G(V)$ contains all V-paths. The corresponding combinatorial dynamical system $\Pi : K \multimap K$ in the finite topological space $(K, \mathcal{T}_{\preceq})$ is given by

$$\Pi(\sigma) := V(\sigma) \cup \{\tau \in K \mid \sigma \text{ covers } \tau\} \setminus V^{-1}(\sigma).$$

Combinatorial multivector field in the sense of [21]

Theory of combinatorial multivector fields [21] is the direct predecessor of the theory presented in this dissertation. The theory is defined for Lefschetz complexes, but for the sake of this presentation, we stick to simplicial complexes. Let $(K, \mathcal{T}_{\preceq})$ be a finite topological space induced by a simplicial complex. A *multivector* V is a convex subset of K with a unique maximal element. We say that a multivector V is *critical* if $H^{\Delta}(\operatorname{cl} V, \operatorname{mo} V) \neq 0$. A *combinatorial multivector field* is a partition of K into multivectors (see Figure 3.6 middle). We denote this family of multivectors by $\mathcal{V}$. For $\sigma \in K$ we denote by $[\sigma]$ the unique multivector to which σ belongs. The combinatorial dynamical system $\Pi : K \multimap K$





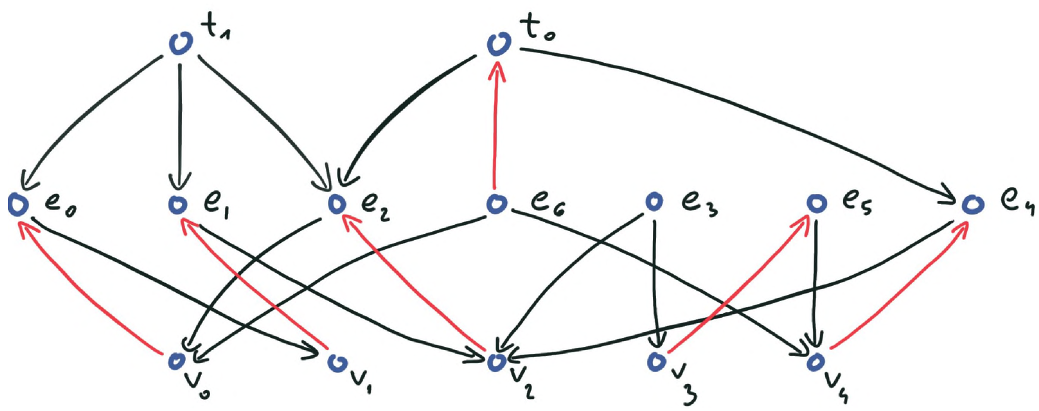


Figure 3.5: An example of a Forman's combinatorial vector field on a simplicial complex (top), its equivalent representation in terms of partitioning the poset (middle), and a directed graph capturing the combinatorial dynamics induced by the Forman's vector field.

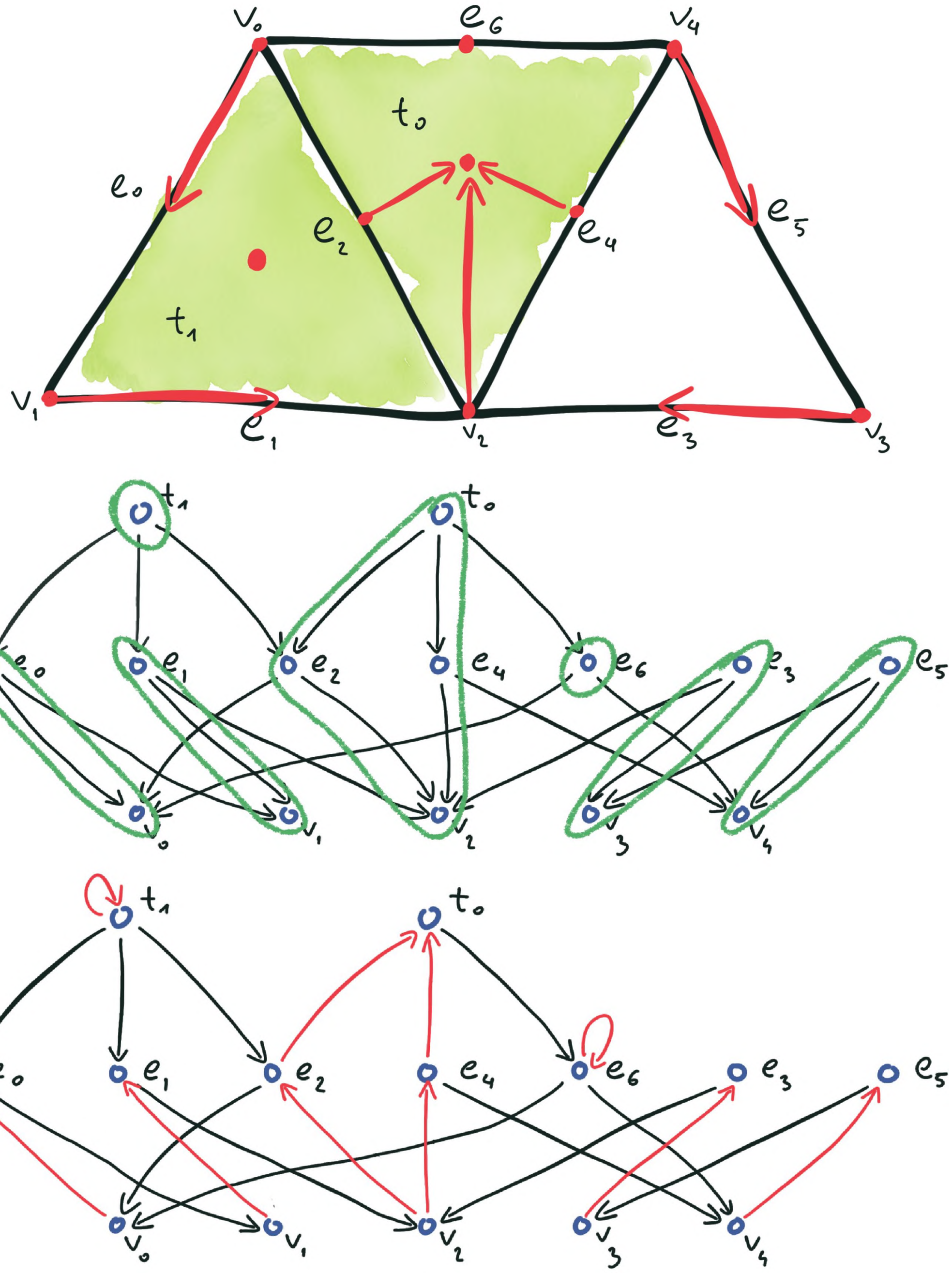


Figure 3.6: An example of combinatorial multivector vector field in the sense of [21] on a simplicial complex (top), its equivalent representation in terms of partitioning the poset (middle), and a directed graph capturing the combinatorial dynamics Π induced by $\mathcal{V}$.

associated with $\mathcal{V}$ is defined for $\sigma \in K$ by

$$\Pi(\sigma) := \begin{cases} (\operatorname{cl}\sigma \setminus [\sigma]) \cup \{\sigma\} & \text{if } \sigma \text{ is maximal in } [\sigma] \text{ and } [\sigma] \text{ is critical,} \\ \operatorname{cl}\sigma \setminus [\sigma] & \text{if } \sigma \text{ is maximal in } [\sigma] \text{ and } [\sigma] \text{ is regular,} \\ \operatorname{opn}\sigma \cap [\sigma] & \text{otherwise.} \end{cases}$$

Therefore, there are three types of arrows, the upward arrows inside multivectors, downward arrows from the maximal element in a multivector to its faces and a self-loop for a maximal element of a critical multivector. Again, we can translate this into a directed graph by starting with a poset, and then reversing arrows within multivectors, adding self-loops to the maximal elements of critical multivectors, and removing all down-arrows that do not start in a maximal element of a multivector. We also should add arrows resulting from upward or downward transitivity (e.g. from v_2 to t_0 and t_0 to v_4), but we skip them to keep the graph more readable. Figure 3.6 bottom shows an example an example of such a graph.

Combinatorial multivector field in the sense of [17]

Finally, we present an example of a combinatorial dynamical system associated to the multivector field theory which constitutes the main topic of this thesis. Let $(K, \mathcal{T}_{\preceq})$ be the finite topological space induced by a simplicial complex. The *multivector* V is simply a convex subset of K. A *combinatorial multivector field* $\mathcal{V}$ is a partition of K into multivectors (see Figure 3.7 middle). The unique multivector to which point σ belongs is denoted by $[\sigma]_{\mathcal{V}}$. Then the associated combinatorial dynamical system $\Pi_{\mathcal{V}} : K \multimap K$ is defined by

$$\Pi_{\mathcal{V}}(\sigma) := [\sigma]_{\mathcal{V}} \cup \operatorname{cl}\sigma.$$

This definition immediately translates itself into a graph build upon a poset with additional upward arrows within multivectors and self-loops at every point (see Figure 3.7 bottom).

To see one of the examples of why larger multivectors, compared to [21] can be beneficial consider the family of continuous dynamical systems on $\mathbb{R}$ given by differential equation

$$x'(t) = t - \alpha \tag{3.2}$$

with parameter $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$. For every value of the parameter we observe a repelling stationary point at coordinate α (see Figure 3.8 top). Suppose we have a fixed simplicial complex

$$K := \{\langle p \rangle \mid p \in \{-2, -1, 0, 1, 2\}\} \cup \{\langle p, p+1 \rangle \mid p \in \{-2, -1, 0, 1\}\},$$

and our goal is to construct the best combinatorial representation of (3.2) on K for different values of α. For example, if $\alpha = \frac{1}{2}$ then the repeller lies at barycenter of a 1-simplex and we can model this situation by creating multivector $\{\langle 0, 1 \rangle\}$ which behaves

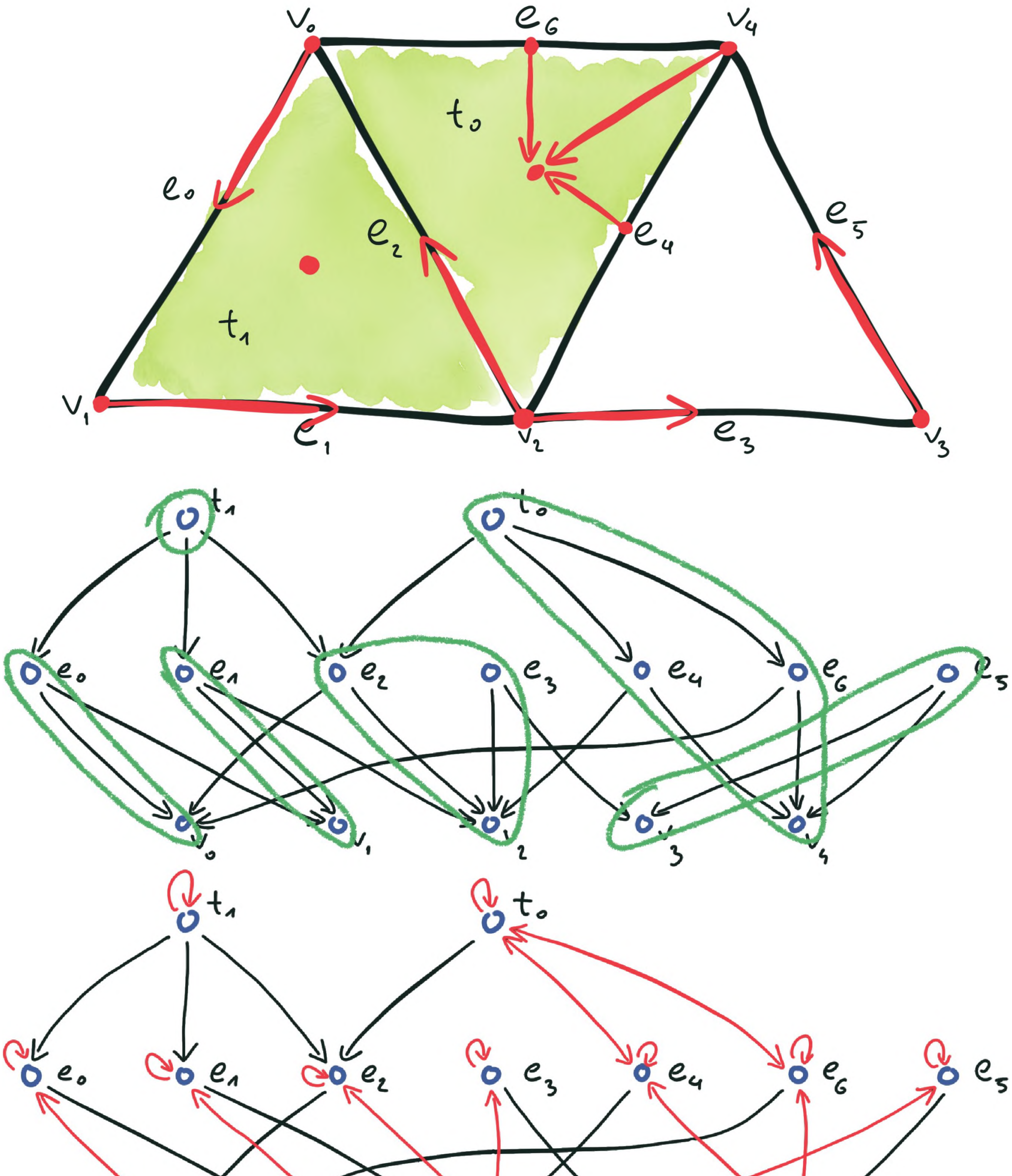


Figure 3.7: An example of a combinatorial multivector field in the sense of [17] on a simplicial complex (top), its equivalent representation in terms of partitioning the poset (middle), and a directed graph capturing the combinatorial dynamics $\Pi_{\mathcal{V}}$ induced by $\mathcal{V}$. Note that multivector $\{v_2, e_2, e_3\}$ is not a proper multivector in terms of [21] because there is no unique maximal element.

as a repeller (see $\mathcal{V}_1$ in Figure 3.8 middle). Similarly, for $\alpha = -\frac{1}{2}$ we model the repeller with $\{\langle -1, 0\rangle\}$ (see $\mathcal{V}_0$ in Figure 3.8 middle). The problem appears for $\alpha = 0$ when the repeller is located exactly at the 0-simplex $\langle 0\rangle$. We do not want to model the repeller with multivector $\{\langle 0\rangle\}$ because, by the definition of $\Pi_{\mathcal{V}}$ it would behave as an attractor. We can make an arbitrary choice and accept one of the two models obtained for $\alpha = -\frac{1}{2}$ or $\alpha = \frac{1}{2}$ as "close enough" models. However, if the subject of interest is the family of models for all values of α, then for $\alpha = 0$, we observe a sort of "discontinuity." Namely, for $\alpha \in [-\frac{1}{2}, 0)$, we have multivector field $\mathcal{V}_0$ and $\mathcal{V}_1$ for $\alpha \in (\frac{1}{2}, 1]$ (see Figure 3.8 middle). Thus, at $\alpha = 0$ we have to rearrange multivectors to get from $\mathcal{V}_0$ to $\mathcal{V}_1$. On the other hand, if we allow multivectors without a unique maximal element, then for $\alpha = 0$ we can model the repeller with multivector $\{\langle -1, 0\rangle, \langle 0\rangle, \langle 0, 1\rangle\}$ (see $\mathcal{V}_2$ in Figure 3.8). Moreover, the transition from $\mathcal{V}_0$ to $\mathcal{V}_1$ via $\mathcal{V}_2$ consists of simpler, called *atomic rearrangements*, that is, a split of a single multivector into two multivectors, or the reverse operation. This idea will be explored further in future work.

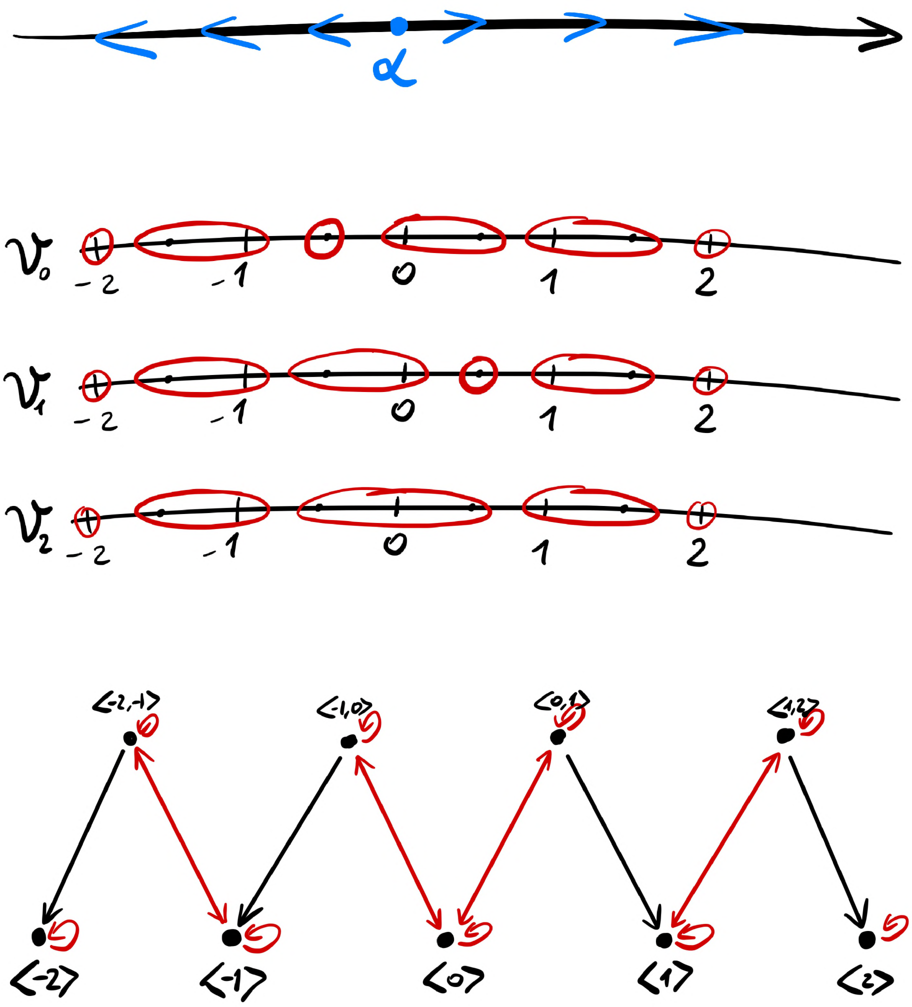


Figure 3.8: Top: a schematic representation of a vector field given by (3.2). Middle: three multivector fields, $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}_2$ corresponding to $\alpha = -\frac{1}{2}$, $\alpha = \frac{1}{2}$, $\alpha = 0$, respectively. Bottom: a direct graph representing the combinatorial dynamics induced by $\mathcal{V}_2$.

Chapter 4

Combinatorial multivector fields theory

This chapter is largely based on sections 4, 5, 6, and 7 of [17]. The author of this thesis wrote or partook substantially in writing all of the included proofs.

4.1 Multivector fields

As we already mentioned, the idea of the combinatorial multivector fields was introduced in [21]. It was inspired by Forman's combinatorial vector fields [10] informally reviewed in Section 3.2.2. Here, we present a generalization of the theory presented in [17].

The generalization, in comparison to [21] is trifold. Firstly, the Lefschetz complexes and Lefschetz homology are replaced with the more general setting of finite topological spaces and singular homology. Secondly, multivectors are no longer assumed to have a unique maximal element. This assumption was introduced in [21] to simplify proofs but turned out to be too restrictive for the further development of the theory. Thirdly, the induced combinatorial dynamical system is defined more liberally. These fundamental changes lead to a theory more convenient for applications. However, new proofs are required.

4.1.1 Combinatorial multivector fields for finite topological spaces

Let $(X, \mathcal{T})$ be a finite topological space. A *combinatorial multivector* in X is a non-empty, locally closed subset of X. We define a *combinatorial multivector field* (MVF) as a partition $\mathcal{V}$ of X into multivectors (see an example in Figure 4.1). The following proposition, an immediate consequence of Proposition 1.4.4, provides an example of a property that is not true in the setting of [21]. To see this consider $Y = X \setminus \{t_0\}$ with X and $\mathcal{V}$ as in Figure 3.6.

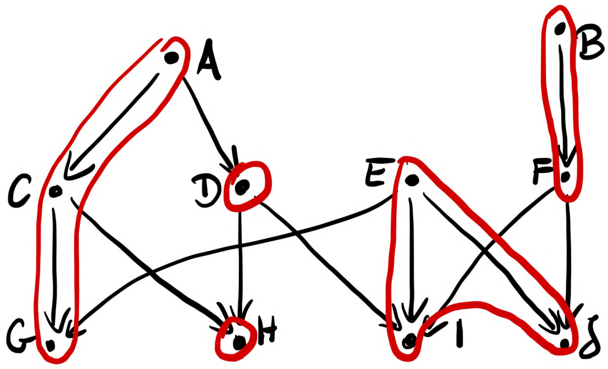


Figure 4.1: An example of a combinatorial multivector field $\mathcal{V} = \{\{A, C, G\}, \{D\}, \{H\}, \{E, I, J\}, \{B, F\}\}$ on a finite topological space consisting of ten points. There are two regular multivectors, $\{A, C, G\}$ and $\{E, I, J\}$, the others are critical.

Proposition 4.1.1. [17, Proposition 4.2] Assume $\mathcal{V}$ is a combinatorial multivector field on a finite topological space X and $A \subset X$ is a locally closed subspace. Then

$$\mathcal{V}_A := \{\, V \cap A \mid V \in \mathcal{V}, V \cap A \neq \emptyset \,\}$$

is a multivector field in A. We call it the *multivector field induced by* $\mathcal{V}$ *on* A. □

The following theorem provides more features of multivector fields not satisfied in the setting of [21].

Theorem 4.1.2. [7, Theorem 5.4 (ii),(iii)] Let X and Y be finite topological spaces. Let $\mathcal{V}$ be a combinatorial multivector field on Y. If $f : X \to Y$ is a continuous map then

$$f^{-1}(\mathcal{V}) := \{f^{-1}(V) \mid V \in \mathcal{V}\}$$

is a multivector field on X. If $\mathcal{W}$ is another multivector field on X, then $\mathcal{V} \bar{\cap} \mathcal{W}$ is also a multivector field on X such that $\mathcal{V} \bar{\cap} \mathcal{W} \sqsubset \mathcal{V}$ and $\mathcal{V} \bar{\cap} \mathcal{W} \sqsubset \mathcal{W}$.

Every point $x \in X$ has a unique multivector in $\mathcal{V}$ to which x belongs. We denote this multivector by $[x]_{\mathcal{V}}$. If the multivector field $\mathcal{V}$ is clear from the context, we write briefly $[x] := [x]_{\mathcal{V}}$. We say that a set $A \subset X$ is $\mathcal{V}$-*compatible* if for each $x \in A$ we have $[x]_{\mathcal{V}} \subset A$. The following proposition is a basic consequence of the definition of $\mathcal{V}$-compatibility.

Proposition 4.1.3. [17, Proposition 4.3] The union and the intersection of a family of $\mathcal{V}$-compatible sets is $\mathcal{V}$-compatible.

With every multivector field $\mathcal{V}$ we associate a combinatorial dynamical system on X induced by the multivalued map $\Pi_{\mathcal{V}} : X \multimap X$ given by

$$\Pi_{\mathcal{V}}(x) := [x]_{\mathcal{V}} \cup \operatorname{cl} x. \tag{4.1}$$

Note that in a single time-step a point x can jump to any other element of $V = [x]_{\mathcal{V}}$. Moreover, the only way to escape V is through $\operatorname{mo} V = \operatorname{cl} V \setminus V$. Intuitively, we can think of a multivector as a dynamical black-box. The behavior inside a multivector is unknown, but we know where flow escapes. Thus, we can interpret a multivector as a Ważewski set with $(\operatorname{cl} V, \operatorname{mo} V)$ as the associated index pair (see Section 3.1). We distinguish two types of multivectors. We say that a multivector V is *critical* if the relative singular homology $H(\operatorname{cl} V, \operatorname{mo} V)$ is non-zero. We say that a multivector is *regular* if it is not critical. Thus, a critical multivector may be interpreted as the Ważewski set with a non-empty invariant subset and a regular multivector may be interpreted as a Ważewski set with all solutions flowing through. A point $x \in X$ is *critical* (respectively *regular*) with respect to $\mathcal{V}$ if $[x]_\mathcal{V}$ is critical (respectively regular). Note that the combinatorial dynamical system associated with a multivector field in the sense of [21] depends on the criticality of points, which is not the case here.

Proposition 4.1.4. Let $\mathcal{V}$ and $\mathcal{W}$ be a multivector fields on X. If $\mathcal{V} \sqsubset \mathcal{W}$ then $\Pi_\mathcal{V} \subset \Pi_\mathcal{W}$.

Proof. Let $x \in X$. We have $\Pi_\mathcal{V}(x) = [x]_\mathcal{V} \cup \operatorname{cl} x \subset [x]_\mathcal{W} \cup \operatorname{cl} x = \Pi_\mathcal{W}(x)$. □

Recall that the preimage notation with the respect to a multivalued map means the large preimage (1.3). Note the duality of $\Pi_\mathcal{V}$ (4.1) and ${\Pi_\mathcal{V}}^{-1}$ (4.2).

Proposition 4.1.5. [17, Proposition 4.5] Let $\mathcal{V}$ be a combinatorial multivector field on $(X, \mathcal{T})$. If $A \subset X$, then

$$ {\Pi_\mathcal{V}}^{-1}(A) = \bigcup_{x \in A} ([x]_\mathcal{V} \cup \operatorname{opn} x). $$

In particular, if $A = \{x\}$ is a singleton we have

$$ \Pi^{-1}(x) := \Pi^{-1}(\{x\}) = [x]_\mathcal{V} \cup \operatorname{opn} x. \tag{4.2} $$

Proof. Assume $y \in {\Pi_\mathcal{V}}^{-1}(A)$. By (1.3) there exists an $x \in A$ such that $x \in \Pi_\mathcal{V}(y)$, that is, $x \in \operatorname{cl} y \cup [y]_\mathcal{V} = \Pi_\mathcal{V}(y)$. If $x \in \operatorname{cl} y$, then from Proposition 1.4.8 we have $x \leq_\mathcal{T} y$. It follows from Proposition 1.4.9 that $y \in \operatorname{opn} x$. If $x \in [y]_\mathcal{V}$ then $[x]_\mathcal{V} = [y]_\mathcal{V} \ni y$. Hence, $y \in \operatorname{opn} x \cup [x]_\mathcal{V}$ and consequently

$$ {\Pi_\mathcal{V}}^{-1}(A) \subset \bigcup_{x \in A} [x]_\mathcal{V} \cup \operatorname{opn} x. $$

In order to show the opposite inclusion consider an $x \in A$ and a $y \in \operatorname{opn} x \cup [x]_\mathcal{V}$. If $y \in [x]_\mathcal{V}$, then clearly $x \in [y]_\mathcal{V} \subset \Pi_\mathcal{V}(y)$ which implies $x \in \Pi_\mathcal{V}(y) \cap A \neq \emptyset$. Thus $y \in {\Pi_\mathcal{V}}^{-1}(A)$. If $y \in \operatorname{opn} x$, then by Proposition 1.4.9 we have $x \leq_\mathcal{T} y$ and therefore $x \in \operatorname{cl} y$. Thus, $x \in \Pi_\mathcal{V}(y)$ and again $\Pi_\mathcal{V}(y) \cap A \neq \emptyset$. Hence, $y \in {\Pi_\mathcal{V}}^{-1}(A)$ which completes the proof of the opposite inclusion. □

Note that by Proposition 1.4.6 a multivector in a finite topological space X is also a multivector in X^{op}, that is, in the space X with the opposite topology. Thus, a multivector field $\mathcal{V}$ in X is also a multivector field in X^{op}. We indicate this in notation by writing $\mathcal{V}^{\mathrm{op}}$ for the multivector field $\mathcal{V}$ considered with the opposite topology. It induces a combinatorial dynamical system $\Pi_\mathcal{V}^{\mathrm{op}} := \Pi_{\mathcal{V}^{\mathrm{op}}} : X^{\mathrm{op}} \multimap X^{\mathrm{op}}$ given by $\Pi_{\mathcal{V}^{\mathrm{op}}}(x) := [x]_\mathcal{V} \cup \operatorname{cl}_{\mathcal{T}^{\mathrm{op}}} x$. As an immediate consequence of Proposition 1.4.2 and Proposition 4.1.5 we get following result.

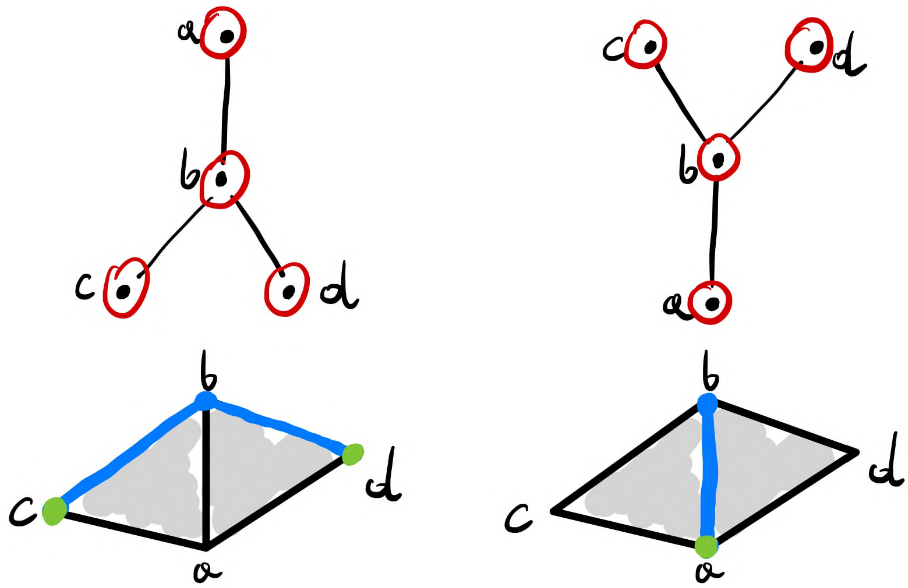


Figure 4.2: An example of a finite topological space X and X^{op} consisting of four points and with the same partition into multivectors $\mathcal{V} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. In X multivectors $\{b\}$, $\{c\}$ and $\{d\}$ are critical, while in X^{op} only $\{a\}$ is critical. Blue and green subcomplexes in the bottom row represents respectively $\mathcal{K}(\text{cl}\{b\})$ and $\mathcal{K}(\text{mo}\{b\})$ for X and X^{op}.

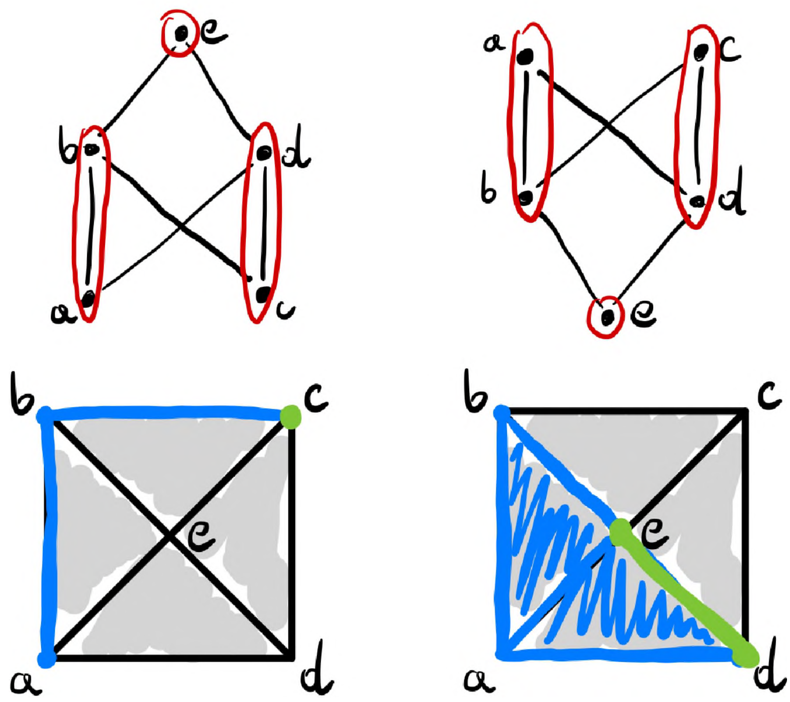


Figure 4.3: An example of a finite topological space X and X^{op} consisting of five points and with the same partition into multivectors $\mathcal{V} = \{\{a, b\}, \{c, d\}, \{e\}\}$. Either in case of $\mathcal{V}$ and $\mathcal{V}^{\text{op}}$ the only critical multivector is $\{e\}$. Multivectors $\{a, b\}$ and $\{c, d\}$ are regular in both cases. Blue and green subcomplexes in the bottom row show respectively $\mathcal{K}(\text{cl}\{a, b\})$ and $\mathcal{K}(\text{mo}\{a, b\})$ for X and X^{op}.

Proposition 4.1.6. [17, Proposition 4.6] The combinatorial dynamical system $\Pi_{\mathcal{V}}^{\mathrm{op}}$ is dual to the combinatorial dynamical system $\Pi_{\mathcal{V}}$, that is, we have $\Pi_{\mathcal{V}}^{\mathrm{op}} = \Pi_{\mathcal{V}}^{-1}$.

Note that the duality of a multivector fields $\mathcal{V}$ and $\mathcal{V}^{\mathrm{op}}$ does not mean that the regularity/criticality is preserved because some critical multivectors can become regular and vice versa due to the topology change (see examples in Figure 4.2 and 4.3).

4.1.2 Essential solutions

Let $\mathcal{V}$ be a multivector field on a finite topological space X. We say that φ is a *solution* (*full solution*, *forward solution*, *backward solution* or *path*) of $\mathcal{V}$ if φ is a solution (full solution, forward solution, backward solution or path) for $\Pi_{\mathcal{V}}$ (see Section 3.2.1). Given a solution φ of $\mathcal{V}$ we denote by $\mathcal{V}(\varphi)$ the set of multivectors $V \in \mathcal{V}$ such that $V \cap \operatorname{im}\varphi \neq \emptyset$.

Given a subset $A \subset X$, we denote the family of all paths of $\mathcal{V}$ in A by $\mathrm{Path}_{\mathcal{V}}(A)$. Similarly, we denote the family of full solutions of $\mathcal{V}$ in A (respectively backward or forward solutions in A) by $\mathrm{Sol}_{\mathcal{V}}(A)$ (respectively $\mathrm{Sol}_{\mathcal{V}}^{-}(A)$, $\mathrm{Sol}_{\mathcal{V}}^{+}(A)$). Sometimes we are interested in paths and solutions passing through a particular point or with fixed endpoints. In that case we write

$$\begin{aligned}
\mathrm{Sol}_{\mathcal{V}}(x, A) &:= \{\varphi \in \mathrm{Sol}_{\mathcal{V}}(A) \;\mid\; \varphi(0) = x\},\\
\mathrm{Path}_{\mathcal{V}}(x, A) &:= \{\varphi \in \mathrm{Path}_{\mathcal{V}}(A) \;\mid\; \varphi(0) = x\},\\
\mathrm{Path}_{\mathcal{V}}(x, y, A) &:= \{\varphi \in \mathrm{Path}_{\mathcal{V}}(A) \;\mid\; \varphi^{\sqsubset} = x \text{ and } \varphi^{\sqsupset} = y\}.
\end{aligned}$$

Note that (4.1) implies $x \in \Pi_{\mathcal{V}}(x)$. Thus, every point admits a full stationary solution and every path may be extended to a full solution. This may suggest that every point is somehow invariant. To make the theory more distinctive, we follow the "dynamical black box" intuition. We know that a critical multivector V may be interpreted as a Ważewski set with a non-empty invariant set inside. Hence, a solution entering V may stay there forever. On the other hand, a regular multivector W may be interpreted as a Ważewski set with a passing-through flow. Thus, a trajectory entering W should exit W in a finite amount of time. With this motivation in mind, we introduce the concept of the essential solution.

We say that a backward solution $\varphi : \mathbb{Z} \nrightarrow X$ is *left-essential* (respectively forward solution is *right-essential*) if for every regular point $x \in \operatorname{im}\varphi$ the set $\{\, t \in \operatorname{dom}\varphi \mid \varphi(t) \notin [x]_{\mathcal{V}} \,\}$ is *left-infinite* (respectively *right-infinite*). We say that a full solution φ is *essential* if it is both left- and right-essential. We say that a point $x \in X$ is *essentially recurrent* if an essential periodic solution passes through x. The following proposition is straightforward.

Proposition 4.1.7. A periodic solution φ is essential if and only if either $\#\mathcal{V}(\varphi) \geq 2$ or the only multivector in a singleton $\mathcal{V}(\varphi)$ is critical. □

We denote the set of all essential solutions in $A \subset X$ (respectively left- or right-essential solutions in A) by $\mathrm{eSol}_{\mathcal{V}}(A)$ (respectively $\mathrm{eSol}_{\mathcal{V}}^{-}(A)$, $\mathrm{eSol}_{\mathcal{V}}^{+}(A)$) and the set of all essential solutions in a set $A \subset X$ passing through a point x by

$$\mathrm{eSol}_{\mathcal{V}}(x, A) := \{\varphi \in \mathrm{eSol}(A) \;\mid\; \varphi(0) = x\}.$$

We have the following straightforward proposition.

Proposition 4.1.8. Let $\varphi \in \mathrm{eSol}^+_{\mathcal{V}}(A)$, $\psi \in \mathrm{eSol}^-_{\mathcal{V}}(A)$ and $\rho \in \mathrm{Path}_{\mathcal{V}}(A)$ be such that $\psi^{\sqsupset} = \rho^{\sqsubset}$ and $\rho^{\sqsupset} = \varphi^{\sqsubset}$. Then $\psi \cdot \rho \cdot \varphi \in \mathrm{eSol}(A)$. $\square$

For a given $x \in X$ we can always construct a right-essential solution starting at x. We prove this in the following proposition. However, this is not true for left-essential solutions. To see this, consider a point a in the first example in Figure 4.2 (upper-left). Since ${\Pi_{\mathcal{V}}}^{-1}(a) = a$, every left-infinite solution is eventually constant in a. But, multivector $\{a\}$ is regular which makes the solution not left-essential.

Proposition 4.1.9. Let $x \in X$, then $\mathrm{eSol}^+_{\mathcal{V}}(x, X) \neq \emptyset$.

Proof. Let $x = x_0 \in X$. If $[x_0]$ is critical then a stationary solution in x_0 is an essential solution. If $[x_0]$ is regular, we can construct right-essential solution φ in the following way. Let $\varphi(0) := x_0$. If there exists a $y_0 \in [x_0]$ such that $\mathrm{cl}\, y_0 \setminus [x_0] \neq \emptyset$ then we put $\varphi(1) := y_0$ and $\varphi(2) = x_1$ where $x_1 \in \mathrm{cl}\, y_0 \setminus [x_0]$. Again, if there exists a $y_1 \in [x_1]$ with $\mathrm{cl}\, y_1 \setminus [x_1] \neq \emptyset$, we put $\varphi(3) := y_1$ and $\varphi(4) := x_2$ for an $x_2 \in \mathrm{cl}\, y_1 \setminus [x_2]$. We proceed by induction up to infinity unless we reach an x_n such that $\mathrm{mo}[x_n] = \emptyset$ when we set $\varphi(k) := x_n$ for $k > 2n$. Then $H(\mathrm{cl}[x_n], \emptyset) \neq 0$, which means that $[x_n]$ is critical. Thus, in both cases φ is right-essential. $\square$

With the notion of essential solution, we can now introduce the concept of invariance that will not degenerate in our settings.

Definition 4.1.10. The *invariant part* of $A \subset X$ is the set

$$\mathrm{Inv}_{\mathcal{V}} A := \{x \in A \mid \mathrm{eSol}(x, A) \neq \emptyset\}. \tag{4.3}$$

Moreover, we say that A is an *invariant set* for $\mathcal{V}$ if $\mathrm{Inv}_{\mathcal{V}} A = A$. In the sequel if a multivector field $\mathcal{V}$ is clear from the context we drop the subscript $\mathcal{V}$ in $\mathrm{Sol}_{\mathcal{V}}$, $\mathrm{eSol}_{\mathcal{V}}$ and $\mathrm{Inv}_{\mathcal{V}}$.

Note that in contrast to the classical theory, the invariant part of a dynamical system to an invariant set needs not be invariant. More precisely, if A is invariant for $\mathcal{V}$ then A may not to be invariant for $\mathcal{V}_A$. Consider multivector field in Figure 4.2(left). The set $A = \{b, c\}$ is invariant because either $\{b\}$ and $\{c\}$ are critical multivectors for $\mathcal{V}$. However, the restriction $\mathcal{V}_A$ of multivector field $\mathcal{V}$ to set A changes the status of multivector $\{b\}$. In particular $\{b\}$ is regular in $\mathcal{V}_A$. Thus, we get $\{c\} = \mathrm{Inv}_{\mathcal{V}_A} A \neq \mathrm{Inv}_{\mathcal{V}} A = A$. On the other hand if $B = \{b, c, d\}$ then $\mathrm{Inv}_{\mathcal{V}} B = \mathrm{Inv}_{\mathcal{V}_B} = B$.

More generally, Proposition 4.1.13 distinguishes some situations when the restriction to a subspace does not affect the invariance. We need first the following propositions.

Proposition 4.1.11. The invariant part of a closed and $\mathcal{V}$-compatible set $A \subset X$ is closed. In particular the invariant part of the whole space X is closed.

Proof. Let $x \in \operatorname{Inv} A$, $\varphi \in \operatorname{eSol}(x, A)$ and $y \in \operatorname{cl} x \subset A$. By Proposition 4.1.9 we can find a $\psi \in \operatorname{eSol}^+(y, X)$. Since A is closed and $\mathcal{V}$-compatible we have $\Pi_\mathcal{V}(A) = A$. Thus, for every $z \in A$ we get $\Pi_\mathcal{V}(z) \in A$ and $\psi \in \operatorname{eSol}^+_\mathcal{V}(y, A)$. Hence, $\varphi^- \cdot \psi \in \operatorname{eSol}(y, A)$ and consequently $y \in \operatorname{Inv} A$. $\square$

Proposition 4.1.12. Let $A \subset X$ be a $\mathcal{V}$-compatible set. Then $\operatorname{Inv} A$ is $\mathcal{V}$-compatible.

Proof. Let $x \in \operatorname{Inv} A$ and let $y \in [x]_\mathcal{V}$. Since A is $\mathcal{V}$-compatible we have $y \in A$. Select a solution $\varphi \in \operatorname{eSol}(x, A)$. Then $\varphi^- \cdot y \cdot \varphi^+$ is a well-defined essential solution in A. Therefore, $\operatorname{eSol}(y, A) \neq \emptyset$ and $y \in \operatorname{Inv}(A)$. Hence, $\operatorname{Inv} A$ is $\mathcal{V}$-compatible. $\square$

Proposition 4.1.13. Let $A \subset X$ be closed and $\mathcal{V}$-compatible. Then $\operatorname{Inv}_\mathcal{V} A = \operatorname{Inv}_{\mathcal{V}_A} A$. In particular $\operatorname{Inv}_\mathcal{V} X = \operatorname{Inv}_{\mathcal{V}_{\operatorname{Inv} X}} X$.

Proof. Let $V \in \mathcal{V}$ such that $V \subset A$. Therefore $V \in \mathcal{V}_A$. Since A is closed we have $\operatorname{cl}_X V = \operatorname{cl}_A V$ and $\operatorname{mo}_X V = \operatorname{mo}_A V$. It follows that V is critical in $\mathcal{V}_A$ if and only if V is critical in $\mathcal{V}$. Thus, $\operatorname{eSol}_\mathcal{V}(A) = \operatorname{eSol}_{\mathcal{V}_A}(A)$ and $\operatorname{Inv}_\mathcal{V} A = \operatorname{Inv}_{\mathcal{V}_A} A$.

By Propositions 4.1.11 and 4.1.12 $\operatorname{Inv}_\mathcal{V} X$ is closed and $\mathcal{V}$-compatible. Hence, the second assertion is proved. $\square$

We have the following straightforward proposition.

Proposition 4.1.14. Let A be an invariant set. Then the family $\{V \in \mathcal{V} \mid \; V \cap A \neq \emptyset\}$ contains at least one critical multivector or two regular ones. $\square$

Proposition 4.1.15. [17, Proposition 4.7] Let $A, B \subset X$ be invariant sets. Then $A \cup B$ is also invariant.

Proof. Let $x \in A$. By the definition of an invariant set there exists an essential solution $\varphi \in \operatorname{eSol}(x, A)$. Clearly $\operatorname{eSol}(x, A) \subset \operatorname{eSol}(x, A \cup B)$. Thus, $\varphi \in \operatorname{eSol}(x, A \cup B)$ and $x \in \operatorname{Inv}(A \cup B)$. The same holds for $x \in B$. Hence, $A \cup B \subset \operatorname{Inv}(A \cup B)$. The opposite inclusion is obvious. $\square$

Proposition 4.1.16. Let $A \subset X$ be locally closed. Then $\operatorname{Inv} A$ is also locally closed.

Proof. Let $x, z \in \operatorname{Inv} A$ and $y \in A$ such that $z \leq_\mathcal{T} y \leq_\mathcal{T} x$. Let $\varphi \in \operatorname{eSol}(x, A)$ and $\psi \in \operatorname{eSol}(z, A)$. Then $\varphi^- \cdot y \cdot \psi^+ \in \operatorname{eSol}(y, A)$. Hence $y \in \operatorname{Inv} A$. $\square$

We can capture the lack of essential solutions with homology.

Theorem 4.1.17. [17, Lemma 5.8] Let A be a $\mathcal{V}$-compatible, locally closed subset of X such that there is no essential solution in A. Then $H(\operatorname{cl} A, \operatorname{mo} A) = 0$.

Proof. Let $\mathcal{A} := \{V \in \mathcal{V} \mid V \subset A\}$. Since A is $\mathcal{V}$-compatible, we have $A = \bigcup \mathcal{A}$. Let $\lesssim_\mathcal{A}$ denote the transitive closure of the relation $\preceq_\mathcal{A}$ in $\mathcal{A}$ given for $V, W \in \mathcal{A}$ by

$$V \preceq_\mathcal{A} W \;\Leftrightarrow\; V \cap \operatorname{cl} W \neq \emptyset. \tag{4.4}$$

We claim that $\lesssim_{\mathcal{A}}$ is a partial order in $\mathcal{A}$. Clearly, $\lesssim_{\mathcal{A}}$ is reflective and transitive. Hence, we only need to prove that $\lesssim_{\mathcal{A}}$ is antisymmetric. To verify this, suppose the contrary. Then there exists a cycle $V_n \preceq_{\mathcal{A}} V_{n-1} \preceq_{\mathcal{A}} \cdots \preceq_{\mathcal{A}} V_0 = V_n$ with $n > 1$ and $V_i \neq V_j$ for $i \neq j$ and $i, j \in \{1, 2, \ldots n\}$. Since $V_i \cap \operatorname{cl} V_{i-1} \neq \emptyset$ we can choose $v_i \in V_i \cap \operatorname{cl} V_{i-1}$ and $v'_{i-1} \in V_{i-1}$ such that $v_i \in \operatorname{cl} v'_{i-1}$. Then $v_i \in \Pi_{\mathcal{V}}(v'_{i-1})$ and $v'_{i-1} \in \Pi_{\mathcal{V}}(v_{i-1})$. Thus, we can construct an essential solution

$$\ldots \cdot v'_n \cdot v_1 \cdot v'_1 \cdot v_2 \cdot v'_2 \cdot \ldots \cdot v'_{n-1} \cdot v_n \cdot v'_n \cdot v_1 \cdot \ldots .$$

This contradicts our assumption and proves that $\lesssim_{\mathcal{A}}$ is a partial order.

Moreover, since a constant solution in a critical multivector is essential, all multivectors in $\mathcal{A}$ have to be regular. Thus,

$$H(\operatorname{cl} V, \operatorname{mo} V) = 0 \quad \text{for every} \quad V \in \mathcal{A}. \tag{4.5}$$

Since $\lesssim_{\mathcal{A}}$ is a partial order, we may assume that $\mathcal{A} = \{V_i\}_{i=1}^m$ where the numbering of V_i extends the partial order $\lesssim_{\mathcal{A}}$ to a linear order $\leq_{\mathcal{A}}$, that is,

$$V_1 \leq_{\mathcal{A}} V_2 \leq_{\mathcal{A}} \cdots \leq_{\mathcal{A}} V_m.$$

We claim that

$$i < j \quad \Rightarrow \quad \operatorname{cl} V_i \setminus V_j = \operatorname{cl} V_i. \tag{4.6}$$

Indeed, if this were not satisfied, then $V_j \cap \operatorname{cl} V_i \neq \emptyset$ which, by the definition (4.4) of $\preceq_{\mathcal{A}}$ gives $V_j \preceq_{\mathcal{A}} V_i$ as well as $V_j \lesssim_{\mathcal{A}} V_i$, and therefore $j \leq i$, a contradiction. For $k \in \{0, 1, \ldots m\}$ define set $W_k := \bigcup_{j=1}^k V_j$. Then $W_0 = \emptyset$ and $W_m = A$. Now fix a $k \in \{0, 1, \ldots m\}$. Observe that by (4.6) we have

$$\operatorname{cl} W_k \setminus A = \bigcup_{j=1}^k \operatorname{cl} V_j \setminus \bigcup_{j=1}^m V_j = \bigcup_{j=1}^k \operatorname{cl} V_j \setminus \bigcup_{j=1}^k V_j = \operatorname{cl} W_k \setminus W_k = \operatorname{mo} W_k.$$

Therefore,

$$\operatorname{mo} W_k = \operatorname{cl} W_k \setminus A \subset \operatorname{cl} A \setminus A = \operatorname{mo} A.$$

It follows that $W_k \cup \operatorname{mo} A = \operatorname{cl} W_k \cup \operatorname{mo} A$. Hence, the set $Z_k := W_k \cup \operatorname{mo} A$ is closed. For $k > 0$ we have

$$Z_k \setminus Z_{k-1} = W_k \setminus W_{k-1} \setminus \operatorname{mo} A = V_k \cap A = V_k = \operatorname{cl} V_k \setminus \operatorname{mo} V_k.$$

Hence, we get from Theorem 2.4.3 and (4.5)

$$H(Z_k, Z_{k-1}) = H(\operatorname{cl} V_k, \operatorname{mo} V_k) = 0.$$

Now it follows from the exact sequence of the triple $(Z_{k-1}, Z_k, \operatorname{cl} A)$ that (see Theorem 2.3.1 and Proposition 1.5.1)

$$H(\operatorname{cl} A, Z_k) \cong H(\operatorname{cl} A, Z_{k-1}).$$

Note that $Z_0 = W_0 \cup \operatorname{mo} A = \operatorname{mo} A$ and $Z_m = W_m \cup \operatorname{mo} A = A \cup \operatorname{mo} A = \operatorname{cl} A$. Therefore, we finally obtain

$$H(\operatorname{cl} A, \operatorname{mo} A) = H(\operatorname{cl} A, Z_0) \cong H(\operatorname{cl} A, Z_m) = H(\operatorname{cl} A, \operatorname{cl} A) = 0,$$

which completes the proof of the Theorem. $\square$

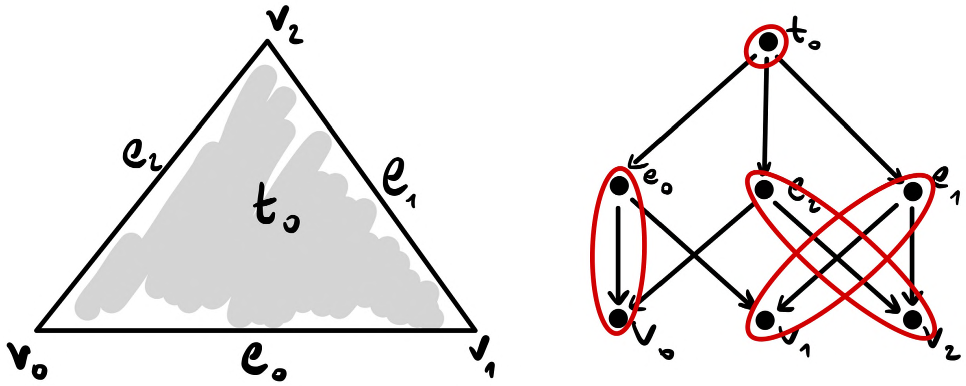


Figure 4.4: A multivector field $\mathcal{V}$ (right) for a simplicial complex (left) and two isolated invariant sets $S = \{v_0, v_1, v_2, e_0, e_1, e_2\}$ and $T = \{t_0\}$. Note that $S \subset \operatorname{cl} T$.

4.1.3 Isolated invariant sets

The next step is to define the isolation in the settings of finite topological spaces. Due to the shortage of open sets, we considerably relax, compared to the classical settings, the conditions for the isolation.

Definition 4.1.18. [17, Definition 4.8] A closed set N *isolates* an invariant set $S \subset N$, if the following two conditions hold:

(a) Every path in N with endpoints in S is a path in S,

(b) $\Pi_{\mathcal{V}}(S) \subset N$.

In this case, we also say that N is an *isolating set* for S. An invariant set S is *isolated* if there exists a closed set N meeting the above conditions.

Note that we use the notion of an "isolating set" instead of an "isolating neighborhood" to emphasize the difference with the classical definition (see Section 3.1). First, due to a finite space's sparseness, it may be the case that two isolated invariant sets do not admit disjoint neighborhoods (see Figure 4.4). Therefore Definition 4.1.18 is relative, meaning that we need to specify which isolated invariant set is isolated by a given isolating set. Secondly, the invariant part of an isolating set N does not need to be contained in the interior of N. Even more, it is possible that $\operatorname{int} N = \emptyset$ and still $\operatorname{Inv} N \neq \emptyset$ (consider $N = \operatorname{cl} S$ in Figure 4.4). Thirdly, in the finite setting, there exists a minimal isolating set. This situation is not true in the continuous case. We have the following simple yet handy observations.

Proposition 4.1.19. [17, Proposition 4.9] The whole space X isolates its invariant part $\operatorname{Inv} X$. In particular, $\operatorname{Inv} X$ is an isolated invariant set. □

Proposition 4.1.20. [17, Proposition 4.11] Let N be an isolating set for an isolated invariant set S. If M is a closed set such that $S \subset M \subset N$, then S is also isolated by M. In particular, $\operatorname{cl} S$ is the smallest isolating set for S.

Proof. The first part is straightforward. The second statement follows because $\operatorname{cl} S$ is the smallest closed set containing S and contained in N. □

There is a fundamental relation between the $\mathcal{V}$-compatibility, local closedness, and isolated invariant sets.

Proposition 4.1.21. [17, Proposition 4.10, 4.12, 4.13] Let $S \subset X$ be an invariant set. Then S is the isolated invariant set if and only if S is locally closed and $\mathcal{V}$-compatible.

Proof. Assume first that S is an isolated invariant set. Suppose S is not $\mathcal{V}$-compatible. Then there exists an $x \in S$ and a $y \in [x]_{\mathcal{V}} \setminus S$. Let N be an isolating set for S. It follows from Definition 4.1.18(b), that $y \in \Pi_{\mathcal{V}}(x) \subset N$. Since $[x]_{\mathcal{V}} = [y]_{\mathcal{V}}$ we have $x \in \Pi_{\mathcal{V}}(y)$. Thus the path $x \cdot y \cdot x$ is a path in N with endpoints in S, but it is not contained in S which in turn contradicts Definition 4.1.18(a).

Now, suppose that S is not locally closed. By Proposition 4.1.20 the set $N := \operatorname{cl} S$ is an isolating set for S. By Proposition 1.4.10 there exist $x, z \in S$ and a $y \notin S$ such that $x \leq_{\mathcal{T}} y \leq_{\mathcal{T}} z$. Hence, it follows from Theorem 1.4.8 that $x \in \operatorname{cl}_{\mathcal{T}} y$ and $y \in \operatorname{cl}_{\mathcal{T}} z$. Therefore, $x \in \Pi_{\mathcal{V}}(y)$ and $y \in \Pi_{\mathcal{V}}(z)$. In particular, $x, y, z \in \operatorname{cl} S$. Thus, $\varphi := z \cdot y \cdot x$ is a solution in $\operatorname{cl} S$ with endpoints in S. In consequence, $y \in S$, a contradiction.

To show the opposite implication assume that S is a $\mathcal{V}$-compatible and locally closed. To this end we will show that $N := \operatorname{cl} S$ isolates S. We have

$$\Pi_{\mathcal{V}}(S) = \bigcup_{x \in S} \operatorname{cl} x \cup \bigcup_{x \in S} [x]_{\mathcal{V}} = \operatorname{cl} S \cup S = \operatorname{cl} S = N.$$

This proves condition (b) of Definition 4.1.18.

We will now show that every path in N with endpoints in S is a path in S. Let $\varphi := x_0 \cdot x_1 \cdot \ldots \cdot x_n$ be a path in N with endpoints in S. Thus, $x_0, x_n \in S$. Suppose that there is an $i \in \{0, 1, \ldots, n\}$ such that $x_i \notin S$. Without loss of generality we may assume that i is maximal such that $x_i \notin S$. Then $x_{i+1} \neq x_i$ and $i < n$, because $x_n \in S$. We have $x_{i+1} \in \Pi_{\mathcal{V}}(x_i) = [x_i]_{\mathcal{V}} \cup \operatorname{cl} x_i$. Since $x_i \notin S$, $x_{i+1} \in S$ and S is $\mathcal{V}$-compatible, we cannot have $x_{i+1} \in [x_i]_{\mathcal{V}}$. Therefore, $x_{i+1} \in \operatorname{cl} x_i$. Since φ is a path in $N = \operatorname{cl} S$, we have $x_i \in \operatorname{cl} S$. Hence, $x_i \in \operatorname{cl} z$ for some $z \in S$. It follows from Proposition 1.4.10 that $x_i \in S$, because $x_{i+1}, z \in S$, $x_{i+1} \in \operatorname{cl} x_i$, $x_i \in \operatorname{cl} z$ and S is locally closed. Thus, we get a contradiction proving that also condition (a) of Definition 4.1.18 is satisfied. In consequence, N isolates S and S is an isolated invariant set. $\square$

The next corollary is a straightforward consequence of Propositions 4.1.12, 4.1.16 and 4.1.21.

Corollary 4.1.22. Let A be a locally closed and $\mathcal{V}$-compatible. Then $\operatorname{Inv} A$ is an isolated invariant set.

4.1.4 Multivector field as a digraph

Let $\mathcal{V}$ be a multivector field in X. As mentioned in Section 1.2, we can think of a multivalued map $\Pi_{\mathcal{V}}$ as a digraph. We denote it by $G_{\mathcal{V}}$. Figure 4.5 gives an example of how the partition in Figure 4.1 translates to the digraph. In particular, we can identify solutions of $\mathcal{V}$ with paths in $G_{\mathcal{V}}$. In this section we will show that some properties

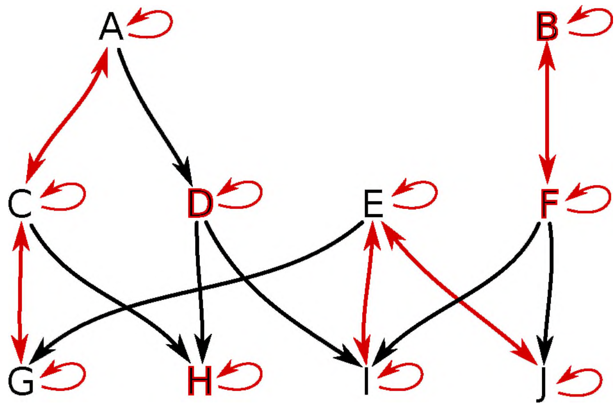


Figure 4.5: Digraph $G_{\mathcal{V}}$ for a multivector field Figure 4.1. Black edges are induced by closure relation, while the red bi-directional edges represent connections within a multivector. For clarity, we omit the edges that can be obtained by the inter-multivector transitivity (e.g., from A to G and from I to J). Nodes that are part of a critical multivector are additionally bolded in red.

of multivector field may be phrased in the language of graph theory. Nevertheless, we should keep in mind that the theory of multivector fields cannot be reduced to graph theory because we still have an additional intrinsic structure – the topology. We did not use it extensively so far, except for determining the criticality of multivectors, but this will change soon. Let $\mathcal{V}$ be a multivector field on X and let $G_{\mathcal{V}}$ be the associated digraph.

Proposition 4.1.23. [17, Proposition 4.14] Assume $A \subset X$ is strongly connected in $G_{\mathcal{V}}$. Then the following conditions are pairwise equivalent.

(i) There exists an essentially recurrent point x in A, that is, there exists an essential periodic solution in A through x,

(ii) A is non-empty and every point in A is essentially recurrent in A,

(iii) $\operatorname{Inv} A \neq \emptyset$.

Proposition 4.1.24. Let A be a strongly connected set in $G_{\mathcal{V}}$. Then there exists a full periodic solution φ in A such that $\operatorname{im} \varphi = A$.

Proof. For convenience, we index all points in A, that is $A = \{a_0, a_1, \ldots, a_n\}$. Since A is a strongly connected set we can find a path $\varphi_{i,j} \in \operatorname{Path}_{\mathcal{V}}(a_i, a_j, A)$ for $i, j \in \{0, 1, \ldots n\}$. A full periodic solution

$$\varphi = \ldots \cdot \varphi_{n,0} \cdot \varphi_{0,1} \cdot \varphi_{1,2} \cdot \ldots \cdot \varphi_{n-1,n} \cdot \varphi_{n,0} \cdot \varphi_{0,1} \cdot \ldots$$

clearly satisfies the assertion. □

Proposition 4.1.25. [17, Proposition 4.15] If $C \subset X$ is a strongly connected component of $G_{\mathcal{V}}$, then C is $\mathcal{V}$-compatible and locally closed.

Proof. Let $x \in C$ and let $y \in [x]_{\mathcal{V}}$. Then $x \cdot y \in \operatorname{Path}_{\mathcal{V}}(x, y, X)$ and $y \cdot x \in \operatorname{Path}_{\mathcal{V}}(y, x, X)$ proving that $y \in C$. Hence C is $\mathcal{V}$-compatible.

Let $x, z \in C$ and let $y \in X$ be such that $x \leq_{\mathcal{T}} y \leq_{\mathcal{T}} z$. Since C is strongly connected we can find a path ρ from x to z. By Proposition 1.4.9 and (4.1) we have $y \in \Pi_{\mathcal{V}}(z)$ and $x \in \Pi_{\mathcal{V}}(y)$. Thus $y \cdot \rho \in \mathrm{Path}_{\mathcal{V}}(y, z, X)$ and $z \cdot y \in \mathrm{Path}_{\mathcal{V}}(z, y, X)$. Since we can construct a path to/from a point of a strongly connected component, we can extend it to any other element of C. It follows that $y \in C$. Hence, C is convex and, by Proposition 1.4.10, C is locally closed. □

Theorem 4.1.26. [17, Proposition 4.16] If $C \subset X$ is a strongly connected component of $G_{\mathcal{V}}$ such that $\mathrm{eSol}(C) \neq \emptyset$, then C is an isolated invariant set.

Proof. According to Proposition 4.1.21 it suffices to prove that C is a $\mathcal{V}$-compatible, locally closed invariant set. It follows from Proposition 4.1.25 that C is $\mathcal{V}$-compatible and locally closed. Thus, we only need to show that C is invariant. Since $\operatorname{Inv} C \subset C$, we only need to prove that $C \subset \operatorname{Inv} C$. Let $y \in C$. Since $\mathrm{eSol}(C) \neq \emptyset$, we may take an $x \in C$ and a $\varphi \in \mathrm{eSol}(x, C)$. Since C is strongly connected we can find paths ρ and ρ' in C from x to y and from y to x respectively. Then by Proposition 4.1.8 the solution $\varphi^- \cdot \rho \cdot \rho' \cdot \varphi^+$ is a well-defined essential solution through y in C. Thus, $\mathrm{eSol}(y, C) \neq \emptyset$, which proves that we have $y \in \operatorname{Inv} C$. □

4.2 Index pairs and Conley index for MVF

In this section we introduce homological Conley index of an isolated invariant set of a combinatorial multivector field. To this end we need a technical concept of an index pair. We prove that for a given isolated invariant set an index pair always exists and its homology depends exclusively on the isolated invariant set.

4.2.1 Index pairs and their properties

An index pair may be considered as an isolating set P_1 with a distinguished exit set P_2. More precisely, we have the following definition.

Definition 4.2.1. Let S be an isolated invariant set. A pair $P = (P_1, P_2)$ of closed subsets of X such that $P_2 \subset P_1$, is called an *index pair for S* if
(IP1) $x \in P_2,\ y \in \Pi_{\mathcal{V}}(x) \cap P_1 \ \Rightarrow\ y \in P_2$ (positive invariance),
(IP2) $x \in P_1,\ \Pi_{\mathcal{V}}(x) \setminus P_1 \neq \emptyset \ \Rightarrow\ x \in P_2$ (exit set),
(IP3) $S = \operatorname{Inv}(P_1 \setminus P_2)$ (invariant part).

An index pair P is said to be *saturated* if $S = P_1 \setminus P_2$. The pair $(\{B, F, I, J\}, \{I, J\})$ is an example of a saturated index pair for the multivector field in Figure 4.1 and the set $\{B, F\}$.

We write $P \subset Q$ for index pairs P, Q whenever $P_i \subset Q_i$ for $i = 1, 2$. We say that index pairs P, Q of S are *semi-equal* if $P \subset Q$ and either $P_1 = Q_1$ or $P_2 = Q_2$.

Proposition 4.2.2. [17, Proposition 5.2] Let P be an index pair for an isolated invariant set S. Then P_1 isolates S.

Proof. According to our assumptions, the set P_1 is closed, and by (IP3) we have

$$S = \operatorname{Inv}(P_1 \setminus P_2) \subset P_1 \setminus P_2 \subset P_1.$$

Thus, it only remains to be shown that conditions (a) and (b) in Definition 4.1.18 are satisfied.

In order to verify (*b*) of Definition 4.1.18 consider a path $\psi := x_0 \cdot x_1 \cdot \ldots \cdot x_n$ in P_1 such that $x_0, x_n \in S$. First, we will show that $\operatorname{im} \psi \subset P_1 \setminus P_2$. To this end, suppose the contrary. Then, there exists an $i \in \{1, 2, \ldots, n-1\}$ such that $x_i \in P_2$ and $x_{i+1} \in P_1 \setminus P_2$. Since ψ is a path we have $x_{i+1} \in \Pi_{\mathcal{V}}(x_i)$. But, property (IP1) implies $x_{i+1} \in P_2$, a contradiction. Since S is invariant and $x_0, x_n \in S$, we may take a $\varphi_0 \in \operatorname{eSol}(x_0, S)$ and a $\varphi_n \in \operatorname{eSol}(x_n, S)$. The solution $\varphi_0^- \cdot \psi \cdot \varphi_n^+$ is an essential solution in $P_1 \setminus P_2$ through x_i. Thus, $x_i \in \operatorname{Inv}(P_1 \setminus P_2) = S$. This proves that every path in P_1 with endpoints in S is contained in S, and therefore Definition 4.1.18(a) is satisfied.

In order to verify (b), let $x \in S$ be arbitrary. We have already seen that then $x \in P_1 \setminus P_2 \subset P_1$. Now suppose that $\Pi_{\mathcal{V}}(x) \setminus P_1 \neq \emptyset$. Then (IP2) implies $x \in P_2$, which contradicts $x \in P_1 \setminus P_2$. Therefore, we necessarily have $\Pi_{\mathcal{V}}(x) \setminus P_1 = \emptyset$, that is, $\Pi_{\mathcal{V}}(x) \subset P_1$, which immediately implies (b). Hence, P_1 isolates S. $\square$

As we said earlier, we can exit a multivector only through its mouth. This pattern extends to invariant sets, and we capture this future in the following proposition. Moreover, the fact that there exists a minimal isolating set (Proposition 4.1.20) lets us effortlessly construct the minimal index pair.

Proposition 4.2.3. [17, Proposition 5.3] Let S be an isolated invariant set. Then the pair $(\operatorname{cl} S, \operatorname{mo} S)$ is a saturated index pair for S.

Proof. To prove (IP1) assume that $x \in \operatorname{mo} S$ and $y \in \Pi_{\mathcal{V}}(x) \cap \operatorname{cl} S$. Since by Proposition 4.1.21 S is $\mathcal{V}$-compatible we have $[x]_{\mathcal{V}} \cap S = \emptyset$. Therefore, $[x]_{\mathcal{V}} \cap \operatorname{cl} S \subset \operatorname{cl} S \setminus S = \operatorname{mo} S$. Due to Propositions 1.4.3 and 4.1.21 $\operatorname{mo} S$ is closed, therefore $\operatorname{cl} x \subset \operatorname{mo} S \subset \operatorname{cl} S$. Hence,

$$y \in \Pi_{\mathcal{V}}(x) \cap \operatorname{cl} S = ([x]_{\mathcal{V}} \cup \operatorname{cl} x) \cap \operatorname{cl} S = ([x]_{\mathcal{V}} \cap \operatorname{cl} S) \cup (\operatorname{cl} x \cap \operatorname{cl} S) \subset \operatorname{mo} S.$$

To see (IP2) note that by the $\mathcal{V}$-compatibility of S we have

$$\Pi_{\mathcal{V}}(S) = \bigcup_{x \in S} (\operatorname{cl} x \cup [x]_{\mathcal{V}}) = \bigcup_{x \in S} \operatorname{cl} x \cup S = \operatorname{cl} S.$$

Thus, if $x \in S$, then $\Pi_{\mathcal{V}}(x) \setminus \operatorname{cl} S = \emptyset$. Therefore, $\Pi_{\mathcal{V}}(x) \setminus \operatorname{cl} S \neq \emptyset$ for $x \in P_1 = \operatorname{cl} S$ implies $x \in \operatorname{cl} S \setminus S = \operatorname{mo} S$.

Finally, since S is invariant locally closed set we get $\operatorname{Inv}(\operatorname{cl} S \setminus \operatorname{mo} S) = \operatorname{Inv} S = S$. This proves (IP3), as well as the fact that $(\operatorname{cl} S, \operatorname{mo} S)$ is saturated. $\square$

Proposition 4.2.4. [17, Proposition 5.6] Assume S is an isolated invariant set. Let P be an index pair for S. Then the set $P_1 \setminus P_2$ is $\mathcal{V}$-compatible and locally closed.

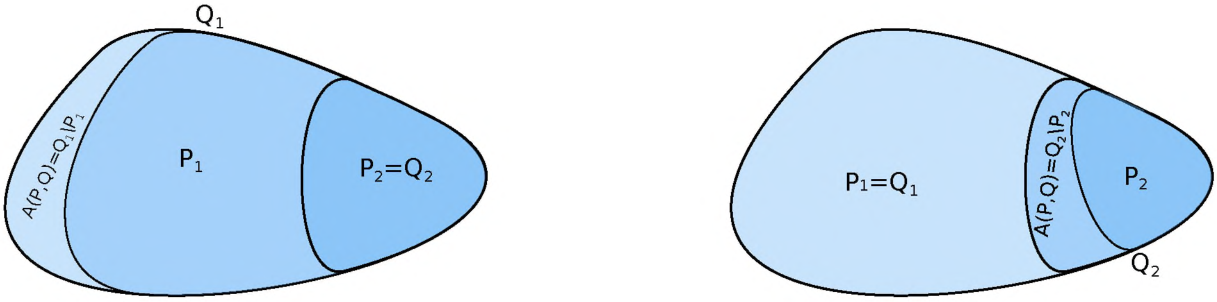


Figure 4.6: Schematic depiction of the two possible cases of set $A(P,Q)$.

Proof. Assume that $P_1 \setminus P_2$ is not $\mathcal{V}$-compatible. This means that for some $x \in P_1 \setminus P_2$ there exists a $y \in [x]_\mathcal{V} \setminus (P_1 \setminus P_2)$. Then either $y \in P_2$ or $y \notin P_1$. Consider the case $y \in P_2$. Since $[x]_\mathcal{V} = [y]_\mathcal{V}$, we have $x \in \Pi_\mathcal{V}(y)$. It follows from (IP1) that $x \in P_2$, a contradiction. Consider now the case $y \notin P_1$. Then from (IP2) one obtains $x \in P_2$, which is again a contradiction. Together, these cases imply that $P_1 \setminus P_2$ is $\mathcal{V}$-compatible.

Finally, the local closedness of $P_1 \setminus P_2$ follows immediately from Proposition 1.4.3(iii). $\square$

The remaining part of this subsection consists of a sequence of propositions and lemmas that lead us to the proof of the following theorem, which, together with Proposition 4.2.3, enables the definition of Conley index in Section 4.2.2.

Theorem 4.2.5. [17, Theorem 5.16] Let P and Q be two index pairs for an invariant set S. Then $H(P_1, P_2) \cong H(Q_1, Q_2)$.

First we will show that this property holds for saturated index pairs as well as semi-equal index pairs.

Lemma 4.2.6. [17, Lemma 5.5] Assume S is an isolated invariant set. Let P and Q be saturated index pairs for S. Then $H(P_1, P_2) \cong H(Q_1, Q_2)$.

Proof. By the definition of a saturated index pair $Q_1 \setminus Q_2 = S = P_1 \setminus P_2$. Hence, using Theorem 2.4.3 we get $H(P_1, P_2) \cong H(Q_1, Q_2)$. $\square$

To show the property semi-equal pairs consider two semiequal index pairs such that $P \subset Q$ and define

$$A(P,Q) := \begin{cases} Q_1 \setminus P_1 & \text{if } P_2 = Q_2, \\ Q_2 \setminus P_2 & \text{if } P_1 = Q_1. \end{cases}$$

Proposition 4.2.7. [17, Proposition 5.4] Let P and Q be semi-equal index pairs for S. Then there is no essential solution in the set $A(P,Q)$.

Proof. By the definition of $A(P,Q)$ we have to analyze two cases. If $P_2 = Q_2$ (see Figure 4.6, left) then

$$A(P,Q) = Q_1 \setminus P_1 \subset Q_1 \setminus P_2 = Q_1 \setminus Q_2 \tag{4.7}$$

and

$$A(P,Q) \cap (P_1 \setminus P_2) \subset A(P,Q) \cap P_1 = \emptyset. \tag{4.8}$$

Similarly, if $P_1 = Q_1$ (see Figure 4.6, right) then

$$A(P,Q) = Q_2 \setminus P_2 \subset Q_1 \setminus P_2 = P_1 \setminus P_2 \quad \text{and} \quad A(P,Q) \cap (Q_1 \setminus Q_2) = \emptyset.$$

By (IP3) and (4.7) we get in the first case that

$$\operatorname{Inv} A(P,Q) \subset \operatorname{Inv}(Q_1 \setminus Q_2) = S = \operatorname{Inv}(P_1 \setminus P_2),$$

which together with 4.8 implies $\operatorname{Inv} A(P,Q) = \emptyset$. Exactly the same arguments applies to the second case. Thus, by the definition of the invariant part (Definition 4.1.18) there is no essential solution in $A(P,Q)$. □

Proposition 4.2.8. [17, Proposition 5.7] Assume S is an isolated invariant set. Let $P \subset Q$ be semi-equal index pairs for S. Then $A(P,Q)$ is $\mathcal{V}$-compatible and locally closed.

Proof. Our assumptions give $P_2, Q_2 \subset P_1$ and $P_2, Q_2 \subset Q_1$. Therefore, if $P_2 = Q_2$, then

$$A(P,Q) = Q_1 \setminus P_1 = (Q_1 \setminus P_2) \setminus (P_1 \setminus P_2) = (Q_1 \setminus Q_2) \setminus (P_1 \setminus P_2).$$

Similarly, if $P_1 = Q_1$, then, denoting by B^c the complement of $B \subset X$ in X, we have

$$\begin{aligned} A(P,Q) &= Q_2 \setminus P_2 = Q_2 \cap P_2^c = (P_1 \cap P_2^c) \cap Q_2 = (P_1 \setminus P_2) \cap Q_2 \\ &= (P_1 \setminus P_2) \cap (Q_1 \cap Q_2^c)^c = (P_1 \setminus P_2) \cap (Q_1 \setminus Q_2)^c = (P_1 \setminus P_2) \setminus (Q_1 \setminus Q_2), \end{aligned}$$

Thus, by Proposition 4.2.4, in both cases, $A(P,Q)$ may be represented as a difference of $\mathcal{V}$-compatible sets. Therefore, it is also $\mathcal{V}$-compatible.

The local closedness of $A(P,Q)$ follows from Proposition 1.4.3. □

Lemma 4.2.9. [17, Lemma 5.9] Let $P \subset Q$ be semi-equal index pairs of an isolated invariant set S. If $P_1 = Q_1$, then $H(Q_2, P_2) = 0$, and analogously, if $P_2 = Q_2$, then $H(Q_1, P_1) = 0$.

Proof. By Proposition 4.2.8 the set $A(P,Q)$ is locally closed and $\mathcal{V}$-compatible. Hence, the conclusion follows from Proposition 4.2.7 and Lemma 4.1.17. □

Lemma 4.2.10. [17, Lemma 5.10] Let $P \subset Q$ be semi-equal index pairs of an isolated invariant set S. Then $H(P_1, P_2) \cong H(Q_1, Q_2)$.

Proof. Assume $P_2 = Q_2$. We get from Lemma 4.2.9 that $H(Q_1, P_1) = 0$. Thus, Theorem 2.3.1 applied to the triple $P_2 \subset P_1 \subset Q_1$ implies

$$H(P_1, P_2) \cong H(Q_1, P_2) = H(Q_1, Q_2).$$

Similarly, if $P_1 = Q_1$ we consider the triple $P_2 \subset Q_2 \subset Q_1$ and obtain

$$H(P_1, P_2) = H(Q_1, P_2) \cong H(Q_1, Q_2).$$

□

We define the *push-forward* and the *pull-back* of a set $A \subset B$ by

$$\pi^+_{\mathcal{V}}(A,B) := \{x \in B \mid \exists_{\varphi \in \mathrm{Path}_{\mathcal{V}}(B)}\ \varphi^{\sqsubset} \in A,\ \varphi^{\sqsupset} = x\}, \tag{4.9}$$

$$\pi^-_{\mathcal{V}}(A,B) := \{x \in B \mid \exists_{\varphi \in \mathrm{Path}_{\mathcal{V}}(B)}\ \varphi^{\sqsubset} = x,\ \varphi^{\sqsupset} \in A\} \tag{4.10}$$

Given an index pair P for S we consider the set $\hat{P} \subset P_1$ of all points $x \in P_1$ such no path starting in x enters S, that is

$$\hat{P} := \{x \in P_1 \mid \pi^+_{\mathcal{V}}(x, P_1) \cap S = \emptyset\}. \tag{4.11}$$

Consider the pairs

$$P^* := (S \cup \hat{P}, P_2),$$
$$P^{**} := (S \cup \hat{P}, \hat{P}).$$

We will show that P^* and P^{**} are index pairs. See Figure 4.7 for an example of these auxiliary index pairs.

Proposition 4.2.11. [17, Proposition 5.11] If $A \subset X$, then $\pi^+_{\mathcal{V}}(A,X)$ (respectively $\pi^-_{\mathcal{V}}(A,X)$) is closed (respectively open) and $\mathcal{V}$-compatible.

Proof. Let $x \in \pi^+_{\mathcal{V}}(A,X)$. By definition (4.9) of push-forward there exist a point $a \in A$ and a path $\varphi \in \mathrm{Path}_{\mathcal{V}}(a,x,X)$. For any $y \in [x]_{\mathcal{V}}$ the concatenation $\varphi \cdot y$ is also a path. Thus, $y \in \pi^+_{\mathcal{V}}(A,X)$ and consequently $\pi^+_{\mathcal{V}}(A,X)$ is $\mathcal{V}$-compatible.

To show closedness, consider a $z \in \mathrm{cl}\, x$. Then the path $\varphi \cdot z$ is a path from A to z, implying that $z \in \pi^+_{\mathcal{V}}(A,X)$ and $\mathrm{cl}\, x \subset \pi^+_{\mathcal{V}}(A,X)$. Since X is finite obtain

$$\mathrm{cl}\, \pi^+_{\mathcal{V}}(A,X) = \bigcup_{x \in \Pi^+_{\mathcal{V}}(A,X)} \mathrm{cl}\, x \subset \pi^+_{\mathcal{V}}(A,X),$$

and therefore $\pi^+_{\mathcal{V}}(A,X)$ is closed. The proof for $\pi^-_{\mathcal{V}}(A,X)$ is analogous. ☐

Proposition 4.2.12. [17, Proposition 5.12] If P is an index pair for an isolated invariant set S, then $S \cap \hat{P} = \emptyset$ and $P_2 \subset \hat{P}$.

Proof. We get the first assertion directly from (4.11). In order to see the other take an $x \in P_2$ and suppose that $x \notin \hat{P}$. This means that there exists a path φ in P_1 such that $\varphi^{\sqsubset} = x$ and $\varphi^{\sqsupset} \in S$. The condition (IP1) of Definition 4.2.1 implies $\mathrm{im}\, \varphi \subset P_2$. Therefore, $\varphi^{\sqsupset} \in P_2$ and $P_2 \cap S \neq \emptyset$ which contradicts $S \subset P_1 \setminus P_2$ given by (IP3). ☐

Proposition 4.2.13. [17, Proposition 5.13] If P is an index pair for an isolated invariant set S, then $\mathrm{mo}\, S \subset \hat{P}$. Moreover, $\Pi_{\mathcal{V}}(S) \subset S \cup \hat{P}$.

Proof. To prove that $\mathrm{mo}\, S \subset \hat{P}$ assume the contrary. Then there exists an $x \in \mathrm{mo}\, S$, such that $\pi^+_{\mathcal{V}}(x, P_1) \cap S \neq \emptyset$. It follows that there exists a path φ in P_1 from x to S. Since $x \in \mathrm{mo}\, S \subset \mathrm{cl}\, S$, we can take a $y \in S$ such that $x \in \mathrm{cl}\, y \subset \Pi_{\mathcal{V}}(y)$. It follows that $\psi := y \cdot \varphi$ is a path in P_1 through x with endpoints in S. Since, by Proposition 4.2.2, P_1 isolates S, which contradicts $x \in \mathrm{mo}\, S$.

Finally, by $\mathcal{V}$-compatibility of S guaranteed by Proposition 4.1.21, we have the inclusion $\Pi_{\mathcal{V}}(S) \subset \mathrm{cl}\, S \subset S \cup \mathrm{mo}\, S \cup \hat{P} = S \cup \hat{P}$ because we proved that $\mathrm{mo}\, S \subset \hat{P}$. This proves the remaining assertion. ☐

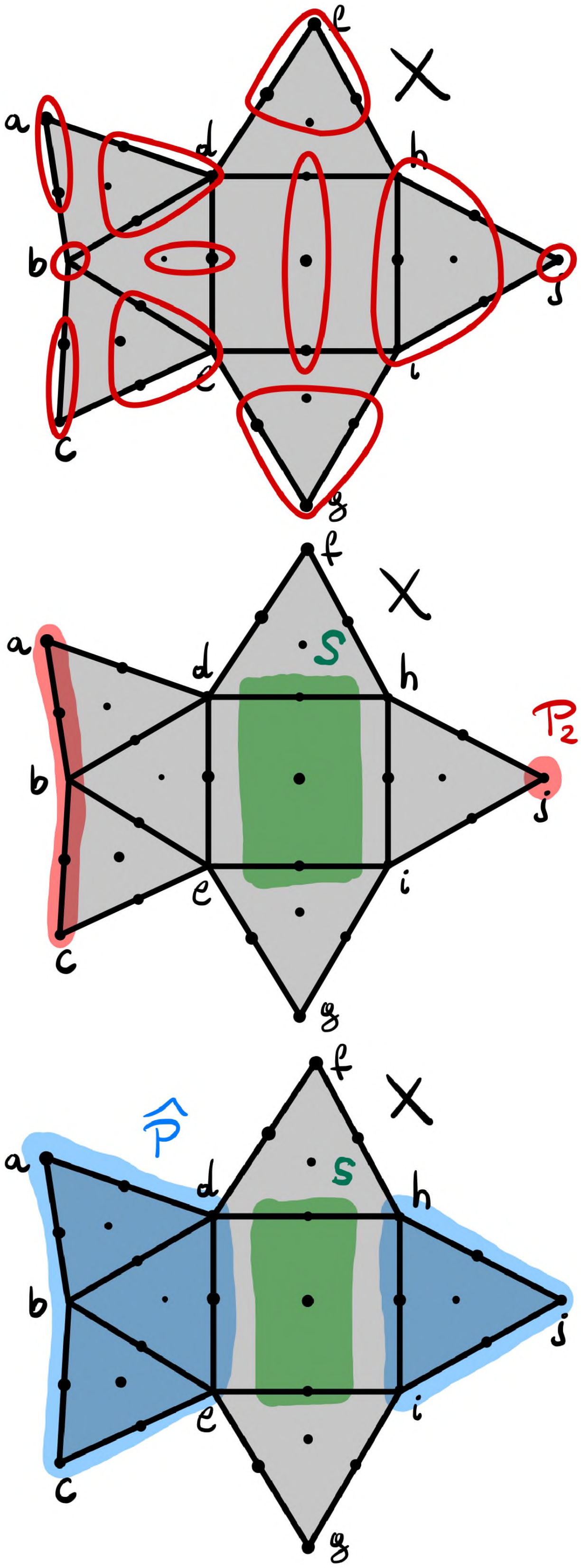


Figure 4.7: Consider a multivector field for a space X (top) and an isolated invariant set $S = \{dh, ei, deih\}$. Pair $(P_1, P_2) = (X, \{a, b, c, j, ab, bc\})$ is an index pair for S. The bottom figure shows set $\hat{P}$ which we use to construct auxiliary index pairs P^* and P^{**}.

Proposition 4.2.14. [17, Proposition 5.14] Let P be an index pair for an isolated invariant set S. Then the sets $\hat{P}$ and $\hat{P} \cup S$ are closed.

Proof. Let $x \in \hat{P}$ and let $y \in \operatorname{cl} x$. Then $y \in \Pi_{\mathcal{V}}(x)$. Moreover, $y \in P_1$, because $\hat{P} \subset P_1$ and P_1 is closed. Consider a $z \in \pi_{\mathcal{V}}^+(y, P_1)$. Let φ be a path from y to z in P_1. Then $x \cdot \varphi$ is a path from x to z in P_1. It follows that $z \in \pi_{\mathcal{V}}^+(x, P_1)$. Therefore, $\pi_{\mathcal{V}}^+(y, P_1) \subset \pi_{\mathcal{V}}^+(x, P_1)$. Since, by (4.11), the latter set is disjoint from S, so is the former one. Thus, $y \in \hat{P}$, which proves that $\hat{P}$ is closed.

Hence, using Proposition 4.2.13 we get $\operatorname{cl}(S \cup \hat{P}) = \operatorname{cl} S \cup \operatorname{cl} \hat{P} = S \cup \operatorname{mo} S \cup \hat{P} = S \cup \hat{P}$, and the closedness of $S \cup \hat{P}$ follows. □

Lemma 4.2.15. [17, Lemma 5.15] If P is an index pair for an isolated invariant set S, then $P^* := (S \cup \hat{P}, P_2)$ is an index pair for S and $P^{**} := (S \cup \hat{P}, \hat{P})$ is a saturated index pair for S.

Proof. First consider P^*. By Proposition 4.2.14 set $P_1^* = S \cup \hat{P}$ is closed. By Proposition 4.2.12 we have $P_2 \subset \hat{P} \subset S \cup \hat{P}$.

Let $x \in P_2^* = P_2$ and let $y \in \Pi_{\mathcal{V}}(x) \cap P_1^*$. Then $y \in \Pi_{\mathcal{V}}(x) \cap P_1$. It follows from (IP1) for P that $y \in P_2$. Thus, (IP1) is satisfied for P^*.

Now, let $x \in P_1^* = S \cup \hat{P}$ and suppose that there is a $y \in \Pi_{\mathcal{V}}(x) \setminus P_1^* \neq \emptyset$. We have $x \notin S$, because otherwise $\Pi_{\mathcal{V}}(x) \subset \operatorname{cl} S \subset \operatorname{cl}(S \cup \hat{P})$ and then Proposition 4.2.14 implies $\Pi_{\mathcal{V}}(x) \subset \operatorname{cl}(S \cup \hat{P}) = S \cup \hat{P} \subset P_1^*$ which contradicts $\Pi_{\mathcal{V}}(x) \setminus P_1^* \neq \emptyset$. Hence, $x \in \hat{P}$. We have $y \notin P_1$ because otherwise $y \in \pi_{\mathcal{V}}^+(x, P_1) \subset \hat{P} \subset P_1^*$, a contradiction. Thus $\Pi_{\mathcal{V}}(x) \setminus P_1 \neq \emptyset$. Since $x \in P_1^* \subset P_1$, by (IP2) for P we get $x \in P_2 = P_2^*$. This proves (IP2) for P^*.

Clearly, $P_1^* \setminus P_2^* = P_1^* \setminus P_2 \subset P_1 \setminus P_2$, and therefore we have the inclusion $\operatorname{Inv}(P_1^* \setminus P_2^*) \subset \operatorname{Inv}(P_1 \setminus P_2) = S$. To verify the opposite inclusion, let $x \in S$ be arbitrary. Since S is an invariant set, there exists an essential solution $\varphi \in \operatorname{eSol}(x, S)$. We have

$$\operatorname{im} \varphi \subset S \subset (\hat{P} \cup S) \setminus P_2 = P_1^* \setminus P_2^*,$$

because $P_2 \cap S = \emptyset$. Consequently, $x \in \operatorname{Inv}(P_1^* \setminus P_2^*)$ and $S = \operatorname{Inv}(P_1^* \setminus P_2^*)$. Hence, P^* also satisfies (IP3), which completes the proof that P^* is an index pair for S.

Now, consider the second pair P^{**}. Let $x \in P_2^{**} = \hat{P}$ be arbitrary and choose a $y \in \Pi_{\mathcal{V}}(x) \cap P_1^{**} = \Pi_{\mathcal{V}}(x) \cap (\hat{P} \cup S)$. Since $x \in \hat{P}$ we get from (4.11) that $\Pi_{\mathcal{V}}(x) \cap S = \emptyset$. Thus, $y \in \Pi_{\mathcal{V}}(x) \cap \hat{P} \subset \hat{P} = P_2^{**}$. This proves (IP1) for the pair P^{**}.

To see (IP2) take an $x \in P_1^{**} = \hat{P} \cup S$ and assume $\Pi_{\mathcal{V}}(x) \setminus P_1^{**} \neq \emptyset$. We cannot have $x \in S$, because then $\Pi_{\mathcal{V}}(x) \subset \Pi_{\mathcal{V}}(S)$ and Proposition 4.2.13 implies $\Pi_{\mathcal{V}}(x) \subset S \cup \hat{P} = P_1^{**}$, a contradiction. Hence, $x \in \hat{P} = P_2^{**}$ which proves (IP2) for P^{**}.

Since $S \cap \hat{P} = \emptyset$ by Proposition 4.2.12, we set $P_1^{**} \setminus P_2^{**} = (S \cup \hat{P}) \setminus \hat{P} = S$ and

$$\operatorname{Inv}(P_1^{**} \setminus P_2^{**}) = \operatorname{Inv} S = S.$$

This proves that P^{**} is saturated and satisfies (IP3). □

We are now ready to present the proof of Theorem 4.2.5.

Proof of Theorem 4.2.5. Let P and Q be arbitrary two index pairs for S. It follows from Lemma 4.2.15 that P^* and P as well as P^* and P^{**} are semi-equal index pairs. Hence, by Lemma 4.2.10 we have isomorphisms

$$H(P_1, P_2) \cong H(P_1^*, P_2^*) \cong H(P_1^{**}, P_2^{**}).$$

Similarly, we obtain

$$H(Q_1, Q_2) \cong H(Q_1^*, Q_2^*) \cong H(Q_1^{**}, Q_2^{**}).$$

Since both pairs P^{**} and Q^{**} are saturated, we get from Lemma 4.2.6 that $H(P_1^{**}, P_2^{**}) \cong H(Q_1^{**}, Q_2^{**})$. It follows that $H(P_1, P_2) \cong H(Q_1, Q_2)$. ☐

4.2.2 Conley index and its properties

Let S be an isolated invariant set for a combinatorial multivector field. We define the *homology Conley index* of S as $H(P_1, P_2)$, where $P = (P_1, P_2)$ is an index pair for S. We denote it by $\operatorname{Con}(S)$. Theorem 4.2.5 together with Proposition 4.2.3 guarantee that Conley index is well defined. In this subsection we show the Ważewski property and the additivity property of the Conley index.

Recall (see Section 2.4) that singular homology and Betti numbers are well defined for finite topological spaces, in particular for subsets of a finite topological space. Given a locally closed subset A of a finite topological space X we define the ith relative Betti number of A by $\bar{\beta}_i(A) := \beta_i(\operatorname{cl} A, \operatorname{mo} A) = \operatorname{rank} H_i(\operatorname{cl} A, \operatorname{mo} A)$. Furthermore, we define the relative Poincaré polynomial of set A by $\bar{p}_A(t) := p_{\operatorname{cl} A, \operatorname{mo} A}(t)$ (see (2.7)). Note, that if A is closed then $\bar{\beta}_i(A) = \beta_i(A)$ and $\bar{p}_A(t) = p_A(t)$.

Proposition 4.2.16. [17, Lemma 5.17] If (P_1, P_2) is an index pair for an isolated invariant set S, then

$$\bar{p}_S(t) + p_{P_2}(t) = p_{P_1}(t) + (1+t)q(t), \tag{4.12}$$

where $q(t)$ is a polynomial with non-negative coefficients. Moreover, if

$$H(P_1) = H(P_2) \oplus H(\operatorname{cl} S, \operatorname{mo} S)$$

then $q(t) = 0$.

Proposition 4.2.17. (Ważewski property) Let A be an locally closed and $\mathcal{V}$-compatible set. If $\operatorname{Inv} A \neq \emptyset$ then $\operatorname{Con}(\operatorname{Inv} A) = H(\operatorname{cl} A, \operatorname{mo} A)$. If $\operatorname{Inv} A = 0$ then $H(\operatorname{cl} A, \operatorname{mo} A) = 0$. In other words if $\operatorname{Con} A \neq 0$ then $\operatorname{Inv} A \neq \emptyset$.

Proof. If $\operatorname{Inv} A = \emptyset$ then $\operatorname{eSol}(A) = \emptyset$. Thus, by Theorem 4.1.17 we get $H(\operatorname{cl} A, \operatorname{mo} A) = 0$. Let $S := \operatorname{Inv} A \neq \emptyset$. By Propositions 4.1.12, 4.1.16 and 4.1.21 set S is an isolated invariant set. We will show that $(\operatorname{cl} A, \operatorname{mo} A)$ is an index pair for S. Condition (IP3) is clear because $\operatorname{Inv}(\operatorname{cl} A \setminus \operatorname{mo} A) = \operatorname{Inv} A = S$. To show (IP1) note that

$$\Pi_{\mathcal{V}}(\operatorname{mo} A) \cap \operatorname{cl} A = (\operatorname{mo} A \cup \bigcup_{y \in \operatorname{mo} A} [y]) \cap \operatorname{cl} A = \operatorname{mo} A$$

because A is $\mathcal{V}$-compatible and $[y] \cap \operatorname{cl} A \subset \operatorname{mo} A$ for every $y \in \operatorname{mo} A$. Condition (IP2) is also immediate because $\Pi_{\mathcal{V}}(A) = \operatorname{cl} A$. Therefore $\Pi_{\mathcal{V}}(x) \setminus \operatorname{cl} A = \emptyset$ for every $x \in A$. By the definition of Conley index $\operatorname{Con}(S) = H(\operatorname{cl} A, \operatorname{mo} A)$. ☐

We say that an isolated invariant set S *decomposes* into the isolated invariant sets S' and S'' if $S = S' \cup S''$ as well as $S'' \cap \operatorname{cl} S' = \emptyset$ and $S' \cap \operatorname{cl} S'' = \emptyset$.

Proposition 4.2.18. [17, Lemma 5.18] Assume an isolated invariant set S decomposes into the isolated invariant sets S' and S''. Then $\operatorname{Sol}(S) = \operatorname{Sol}(S') \cup \operatorname{Sol}(S'')$.

Theorem 4.2.19. [17, Lemma 5.19] Assume an isolated invariant set S decomposes into the isolated invariant sets S' and S''. Then we have

$$\operatorname{Con}(S) = \operatorname{Con}(S') \oplus \operatorname{Con}(S'').$$

Proof. In view of Proposition 4.2.3, the two pairs $P = (\operatorname{cl} S', \operatorname{mo} S')$ and $Q = (\operatorname{cl} S'', \operatorname{mo} S'')$ are saturated index pairs for S' and S'', respectively. Consider the following exact sequence given by Theorem 2.4.4:

$$\begin{aligned} \ldots \to & H_n(P_1 \cap Q_1, P_2 \cap Q_2) \to H_n(P_1, P_2) \oplus H_n(Q_1, Q_2) \\ \to & H_n(P_1 \cup Q_1, P_2 \cup Q_2) \to H_{n-1}(P_1 \cap Q_1, P_2 \cap Q_2) \to \ldots. \end{aligned} \tag{4.13}$$

Set S decomposes into sets S' and S'', therefore we get $S' \cap Q_2 \subset S' \cap \operatorname{cl} S'' = \emptyset$ and similarly $S'' \cap P_2 = \emptyset$. Since both P and Q are saturated and $S' \cap S'' = \emptyset$ we get

$$\begin{aligned} P_1 \cap Q_1 &= (S' \cup P_2) \cap (S'' \cup Q_2) \\ &= (S' \cap S'') \cup (S' \cap Q_2) \cup (P_2 \cap S'') \cup (P_2 \cap Q_2) = P_2 \cap Q_2. \end{aligned}$$

Thus, $H(P_1 \cap Q_1, P_2 \cap Q_2) = 0$, which together with the exact sequence (4.13) implies

$$H_*(P_1 \cup Q_1, P_2 \cup Q_2) \cong H_*(P_1, P_2) \oplus H_*(Q_1, Q_2). \tag{4.14}$$

Notice further that $S' \cap Q_2 = \emptyset$ implies $S' \setminus Q_2 = S'$. Similarly $S'' \setminus P_2 = S''$. Therefore, since P and Q are saturated, we obtain the identity

$$\begin{aligned} (P_1 \cup Q_1) \setminus (P_2 \cup Q_2) &= (P_1 \setminus P_2 \setminus Q_2) \cup (Q_1 \setminus Q_2 \setminus P_2) \\ &= (S' \setminus Q_2) \cup (S'' \setminus P_2) = S' \cup S'' = S. \end{aligned}$$

Hence, by Theorem 2.4.3,

$$H(\operatorname{cl} S, \operatorname{mo} S) \cong H(P_1 \cup Q_1, P_2 \cup Q_2). \tag{4.15}$$

Finally, from (4.14) and (4.15) we get

$$\begin{aligned} \operatorname{Con}(S) = H(\operatorname{cl} S, \operatorname{mo} S) &\cong H(P_1 \cup Q_1, P_2 \cup Q_2) \\ &\cong H(P_1, P_2) \oplus H(Q_1, Q_2) = \operatorname{Con}(S') \oplus \operatorname{Con}(S''), \end{aligned}$$

which completes the proof of the theorem. ☐

4.3 Attractors, repellers and limit sets

For the rest of Chapter 4, we fix our topological space X and a multivector field $\mathcal{V}$ on X. We also assume that X is an invariant set for $\mathcal{V}$. In general, every point admits some right-essential solution, but not necessarily a left-essential one (Proposition 4.1.9). The invariance of X allows us to build an essential solution at every point of X. It will be crucial to obtain a duality between attractors and repeller.

Note that the invariance of X is not a particularly limiting assumption because we can simply restrict a space X to its invariant part. We already know that restricting a multivector field $\mathcal{V}$ to a subspace $X' := \operatorname{Inv}_{\mathcal{V}} X$ is still a proper multivector field $\mathcal{V}_{X'}$ (see Proposition 4.1.1). Proposition 4.1.13 guarantees that X' is invariant for the restricted multivector field $\mathcal{V}_{X'}$. Moreover, an invariant part is $\mathcal{V}$-compatible (Proposition 4.1.12) and closed (Proposition 4.1.11). It follows that for a multivector $V \subset X$ we have $\operatorname{cl}_X V = \operatorname{cl}_{X'} V$ and $\operatorname{mo}_X V = \operatorname{mo}_{X'} V$. Thus no multivector will be modified and will not change its criticality.

4.3.1 Attractors, repellers and minimal sets

We define an *attractor* as an invariant set $A \subset X$ such that $\Pi_{\mathcal{V}}(A) = A$. Dually, a *repeller* is an invariant set $R \subset X$ such that $\Pi_{\mathcal{V}}^{-1}(R) = R$.

Proposition 4.3.1. The whole space X is both an attractor and a repeller.

Proof. To see that X is an attractor observe that the inclusion $\Pi_{\mathcal{V}}(X) \subset X$ is obvious and the opposite inclusion follows from the general assumption that X is invariant. The proof for the repeller is analogous. $\square$

We can state the equivalent condition for an attractor or repeller using, respectively, push-forward and pull-back defined earlier (see (4.9) and (4.10)).

Proposition 4.3.2. [17, Theorem 6.1] Let A be an invariant set. Then A is an attractor (respectively a repeller) in X if and only if $\pi_{\mathcal{V}}^{+}(A, X) = A$ (respectively $\pi_{\mathcal{V}}^{-}(A, X) = A$).

Proof. Let A be an attractor. The inclusion $A \subset \pi_{\mathcal{V}}^{+}(A, X)$ is true by the definition of push-forward. To show inclusion $\pi_{\mathcal{V}}^{+}(A, X) \subset A$, assume that there exists a $y \in \pi_{\mathcal{V}}^{+}(A, X) \setminus A$. Then by (4.9) we can find an $x \in A$ and $\varphi \in \operatorname{Path}_{\mathcal{V}}(x, y, X)$. This implies that there exists a $k \in \mathbb{Z}$ such that $\varphi(k) \in A$ and $\varphi(k+1) \notin A$. But A is an attractor and $\varphi(k+1) \in \Pi_{\mathcal{V}}(\varphi(k)) \subset \Pi_{\mathcal{V}}(A) = A$, a contradiction. Therefore, $\pi_{\mathcal{V}}^{+}(A, X) = A$. Now assume that $\pi_{\mathcal{V}}^{+}(A, X) = A$. Then $\Pi_{\mathcal{V}}(A) = \Pi_{\mathcal{V}}(\pi_{\mathcal{V}}^{+}(A, X)) = \pi_{\mathcal{V}}^{+}(A, X) = A$, proving that A is an attractor.

The proof for the repeller is analogous. $\square$

Theorem 4.3.3. [17, Theorem 6.2] The following conditions are equivalent:

(1) A is an attractor,

(2) A is closed, $\mathcal{V}$-compatible, and invariant,

(3) A is a closed isolated invariant set.

Proof. Let A be an attractor. It follows immediately from Propositions 4.3.2 and 4.2.11 that condition *(1)* implies condition *(2)*. Moreover, Proposition 4.1.21 shows that *(2)* implies *(3)*. Finally, suppose that *(3)* holds. By Proposition 4.1.21 set A is $\mathcal{V}$-compatible. Since i is also closed, we have

$$\Pi_{\mathcal{V}}(A) = \bigcup_{x \in A} \operatorname{cl} x \cup [x]_{\mathcal{V}} = \bigcup_{x \in A} \operatorname{cl} x \cup \bigcup_{x \in A} [x]_{\mathcal{V}} = \operatorname{cl} A \cup A = A,$$

which proves that A is an attractor. □

Theorem 4.3.4. [17, Theorem 6.3] The following conditions are equivalent:

(1) R is a repeller,
(2) R is open, $\mathcal{V}$-compatible, and invariant,
(3) R is an open isolated invariant set.

Proof. Assume R is a repeller. It follows from Propositions 4.3.2 and 4.2.11 that condition *(1)* implies condition *(2)*, and Proposition 4.1.21 shows that *(2)* implies *(3)*. Finally, assume that condition *(3)* holds. Then R is $\mathcal{V}$-compatible by Proposition 4.1.21. The openness of R and Proposition 4.1.5 imply

$$\Pi_{\mathcal{V}}^{-1}(R) = \bigcup_{x \in R} \operatorname{opn} x \cup [x]_{\mathcal{V}} = \bigcup_{x \in R} \operatorname{opn} x \cup \bigcup_{x \in R} [x]_{\mathcal{V}} = R,$$

which proves that R is a repeller. □

Note that Theorem 4.3.4 does not hold if X is not invariant. We can examine the example from Figure 4.2(left). Neither of the open sets is invariant because $\{a\}$ does not admit an essential solution. Actually, there is no repeller in this example. As mentioned, we can fix this by restricting X to $\operatorname{Inv} X = X \setminus \{a\}$. This modification makes singleton $\{b\}$ a repeller. In general, "forward" results hold without assuming the invariance of X (Proposition 4.1.9), while "backward" results may fail without this assumption.

Let φ be a full solution in X. We define the *ultimate backward* and *ultimate forward image* of φ respectively by

$$\operatorname{uim}^{-} \varphi := \bigcap_{t \in \mathbb{Z}^{-}} \varphi\left((-\infty, t]\right),$$
$$\operatorname{uim}^{+} \varphi := \bigcap_{t \in \mathbb{Z}^{+}} \varphi\left([t, +\infty)\right).$$

Note that in a finite space a descending sequence of sets must eventually stabilize. Therefore, we get the following result.

Proposition 4.3.5. [17, Proposition 6.4] Let φ be a full solution in X. There exists a $k \in \mathbb{N}$ such that $\operatorname{uim}^{-} \varphi = \varphi((-\infty, -k])$ and $\operatorname{uim}^{+} \varphi = \varphi([k, +\infty))$. In particular, the sets $\operatorname{uim}^{-} \varphi$ and $\operatorname{uim}^{+} \varphi$ are always non-empty. □

Proposition 4.3.6. [17, Proposition 6.5] If φ is a right-essential (respectively a left-essential) solution in X, then we can find an essential solution ψ such that $\operatorname{im}\psi \subset \operatorname{uim}^+ \varphi$ (respectively $\operatorname{im}\psi \subset \operatorname{uim}^- \varphi$). In other words $\operatorname{eSol}(\operatorname{uim}^\pm \varphi) \neq \emptyset$.

Proof. Let φ be a right-essential solution in X. By Proposition 4.3.5 there exists a $k \in \mathbb{Z}$ such that $\operatorname{uim}^+ \varphi = \varphi([k, +\infty))$. If there exists a critical vector V such that $V \cap \operatorname{uim}^+ \varphi \neq \emptyset$, then we can take as ψ the stationary essential solution through a point in $V \cap \operatorname{uim}^+ \varphi$. Otherwise, since φ is right-essential, we have at least two different regular multivectors $V, W \in \mathcal{V}$ such that $V \cap \operatorname{uim}^+ \varphi \neq \emptyset \neq W \cap \operatorname{uim}^+ \varphi$. Then there exist $t, s, u \in \mathbb{Z}$ such that $k < t < s < u$, $\varphi(t) \in V$, $\varphi(s) \in W$, $\varphi(u) \in V$ and the periodic extension $\ldots \cdot \varphi|_{[t,u]} \cdot \varphi|_{[t,u]} \cdot \ldots$ of $\varphi|_{[t,u]}$ is essential by Proposition 4.1.7.

The proof for a left-essential solution is analogous. $\square$

For a full solution φ in X. We define sets of multivectors that are visited by φ infinitely many times by

$$\mathcal{V}^-(\varphi) := \left\{ V \in \mathcal{V} \mid V \cap \operatorname{uim}^- \varphi \neq \emptyset \right\}, \tag{4.16}$$

$$\mathcal{V}^+(\varphi) := \left\{ V \in \mathcal{V} \mid V \cap \operatorname{uim}^+ \varphi \neq \emptyset \right\}. \tag{4.17}$$

We refer to a multivector $V \in \mathcal{V}^-(\varphi)$ (respectively $\mathcal{V}^+(\varphi)$) as a *backward* (respectively *forward*) *ultimate multivector* of φ. The families $\mathcal{V}^-(\varphi)$ and $\mathcal{V}^+(\varphi)$ will be used in the sequel to define combinatorial limit sets, but for now, we use them in the proof of the following theorem.

Theorem 4.3.7. [17, Theorem 6.10] Assume the whole space X is invariant. Let $A \subset X$ be an attractor. Then $A^\star := \operatorname{Inv}(X \setminus A)$ is a repeller in X. It is called the *dual repeller* of A. Conversely, if R is a repeller, then $R^\star := \operatorname{Inv}(X \setminus R)$ is an attractor in X. It is called the *dual attractor* of R. Moreover, the dual repeller (respectively the dual attractor) is nonempty, unless we have $A = X$ (respectively $R = X$).

Proof. We present the proof for an attractor. The proof for repeller is analogous.

Assume that $A^\star$ is non-empty. We will show that $A^\star$ is open. Let $x \in A^\star$. It is sufficient to prove that $\operatorname{opn} x \in A^\star$. Thus, take a $y \in \operatorname{opn} x$. Then we have $x \in \operatorname{cl} y$ by Proposition 1.4.9. Since A is closed as an attractor (Proposition 4.3.3), we get $y \notin A$. The invariance of X lets us select a $\varphi \in \operatorname{eSol}(y, X)$. Then $\operatorname{im}\varphi^- \cap A = \emptyset$, because otherwise there exists a $t \in \mathbb{Z}^-$ such that $\varphi(t) \in A$ and $\varphi(t+1) \notin A$ which leads to a contradiction with

$$\varphi(t+1) \in \Pi_{\mathcal{V}}(\varphi(t)) \subset \Pi_{\mathcal{V}}(A) = A.$$

Now, let $\psi \in \operatorname{eSol}(x, X \setminus A)$. Since $x \in \operatorname{cl} y \subset \Pi_{\mathcal{V}}(y)$, we get $\varphi^- \cdot \psi^+ \in \operatorname{eSol}(y, X \setminus A)$. It follows that $y \in \operatorname{Inv}(X \setminus A) = A^\star$ which proves that $X \subset A^\star$. Therefore, the set $A^\star$ is open.

Since A is $\mathcal{V}$-compatible, also $X \setminus A$ is $\mathcal{V}$-compatible. Therefore, by Proposition 4.1.12 the set $A^\star$ is also $\mathcal{V}$-compatible. Altogether, the set $A^\star$ is invariant, open and $\mathcal{V}$-compatible. Thus, by Theorem 4.3.4 it is a repeller.

Finally, we will show that $A^\star \neq \emptyset$ unless $A = X$. Suppose that $X \setminus A \neq \emptyset$, and let $x \in X \setminus A$. Since X is invariant, there exists a $\varphi \in \mathrm{eSol}(x, X)$. As in the first part of the proof one can show that $\operatorname{im}\varphi^- \cap A = \emptyset$, that is, we have $\operatorname{im}\varphi^- \subset X \setminus A$. According to Proposition 4.3.6 there exists an essential solution ψ such that $\operatorname{im}\psi \subset \operatorname{uim}^-\varphi \subset \operatorname{im}\varphi^- \subset X \setminus A$, and this immediately implies $\operatorname{im}\psi \subset A^\star = \operatorname{Inv}(X \setminus A) \neq \emptyset$. □

Definition 4.3.8. We say that a non-empty invariant set $A \subset X$ is *minimal* if the only non-empty attractor in A of the restriction $\mathcal{V}_A$ is the entire set A.

The following proposition is straightforward.

Proposition 4.3.9. Let $A \subset X$. Then we have $\pi^+_{\mathcal{V}}(x, A) = \pi^+_{\mathcal{V}_A}(x, A)$ and $\pi^-_{\mathcal{V}}(x, A) = \pi^-_{\mathcal{V}_A}(x, A)$.

Proposition 4.3.10. Let A be a non-empty invariant set for $\mathcal{V}$. Then A is minimal if and only if the following implication holds

$$B \subset A,\ \pi^+_{\mathcal{V}}(B, A) = B \ \Rightarrow\ B = A. \tag{4.18}$$

Proof. Let A be minimal. Assume there exists $B \subsetneq A$ such that $\pi^+_{\mathcal{V}}(B, A) = B$. By Proposition 4.3.9 we have $\pi^+_{\mathcal{V}}(B, A) = \pi^+_{\mathcal{V}_A}(B, A) = B$. Thus, B is closed in A and $\mathcal{V}_A$-compatible by Proposition 4.2.11. Set $\operatorname{Inv}_{\mathcal{V}_A} B$ is non-empty by Proposition 4.1.9 and closed by Proposition 4.1.11. Hence, by Theorem 4.3.3 set $\operatorname{Inv}_{\mathcal{V}_A} B$ is an attractor for $\mathcal{V}_A$. It follows that A is not minimal, a contradiction.

Now assume that (4.18) holds. Suppose that A is not minimal. Then there exists an attractor $B \subsetneq A$ for $\mathcal{V}_A$. By the definition of attractor and Proposition 4.3.2 we have $B = \Pi_{\mathcal{V}_A}(B) = \pi^+_{\mathcal{V}_A}(B, A)$. This contradicts (4.18). □

Proposition 4.3.11. Let $A \subset X$ be a non-empty set. Then $\operatorname{Inv}_{\mathcal{V}_A}(\pi^+_{\mathcal{V}_A}(x, A)) \neq \emptyset$ for any point $x \in A$.

Proof. Since $x \in \pi^+_{\mathcal{V}_A}(x, A)$ it is non-empty. By Proposition 4.1.9 there exists a $\varphi \in \mathrm{eSol}^+_{\mathcal{V}_A}(z, A)$. By Proposition 4.3.6 we can construct an essential solution ψ such that $\operatorname{im}\psi \subset \operatorname{uim}^+\varphi \subset \pi^+_{\mathcal{V}_A}(x, A)$. Thus, $\emptyset \neq \operatorname{im}\psi \subset \operatorname{Inv}_{\mathcal{V}_A}\pi^+_{\mathcal{V}_A}(x, A)$. □

Proposition 4.3.12. [17, Proposition 6.7] Let $A \subset X$ be a non-empty invariant set. Then A is minimal if and only if A is a strongly connected set in $G_{\mathcal{V}}$.

Proof. Let A be minimal. If A is not a strongly connected then there exist $x, y \in A$ such that $\mathrm{Path}_{\mathcal{V}}(x, y, A) = \emptyset$. It follows that $y \notin \pi^+_{\mathcal{V}}(x, A)$. Clearly $\pi^+_{\mathcal{V}}(\pi^+_{\mathcal{V}}(x, A)) = \pi^+_{\mathcal{V}}(x, A) \subsetneq A$. Thus, $\pi^+_{\mathcal{V}}(x, A)$ does not satisfies (4.18) and consequently by Proposition 4.3.10 set A is not minimal.

On the other hand, if A is a strongly connected set then for any $B \subset A$ we have $\pi^+_{\mathcal{V}}(B, A) = A$. Hence, the only set satisfying (4.18) is A. By Proposition 4.3.10 set A is minimal.

□

Corollary 4.3.13. Let $A \subset X$ be a strongly connected component of $G_{\mathcal{V}}$ with a non-empty invariant part. Then A is a minimal isolated invariant set.

Proof. The implication follows from Theorem 4.1.26 and Proposition 4.3.12. $\square$

The duality given by the invariance of X allows us to adapt the proof of Proposition 4.3.12 to get the following proposition.

Proposition 4.3.14. [17, Proposition 6.8] Let R be a non-empty invariant set. Then R is minimal if and only if the only non-empty repeller in R for the restriction $\mathcal{V}_R$ is the entire set R.

Proof. Assume that R is minimal. By Proposition 4.3.12 the set R is a strongly connected set in the graph $G_{\mathcal{V}}$. Thus, for every set $A \subset R$ we have $\pi_{\mathcal{V}_R}^{-}(A, R) = R$. Consequently, by Proposition 4.3.2 the only repeller in R for $\mathcal{V}_R$ is the entire R.

Now suppose that R is the only non-empty repeller in R for $\mathcal{V}_R$. Thus, $\operatorname{Inv}_{\mathcal{V}_R} R = R$. Let $A \subset R$ be an attractor in R for $\mathcal{V}_R$. Consider its dual repeller $A^\star$. If $A^\star \neq \emptyset$ then we get $A^\star = R$, because we assumed that R is the only possible repeller in R. Therefore, $A = \emptyset$ because $R = A^\star = \operatorname{Inv}_{\mathcal{V}_R}(R \setminus A) \subset R \setminus A$. On the other hand, if $A^\star = \emptyset$ then by Theorem 4.3.7 and Proposition 4.3.2 we have $A = R$. Hence, the only non-empty attractor in R for $\mathcal{V}_R$ is the entire set R. This proves that R is minimal. $\square$

Proposition 4.3.15. [17, Proposition 6.9] Let $S \subset X$ be a minimal invariant set and let $A \subset X$ be an attractor or a repeller. If $A \cap S \neq \emptyset$ then $S \subset A$.

Proof. Let A be an attractor. Let $x \in A \cap S$ and let $y \in S$. Since, by Proposition 4.3.12, S is strongly connected, there exists a path $\varphi \in \operatorname{Path}_{\mathcal{V}}(x, y, S)$. We will prove by induction that $\varphi(k) \in A$ for $k \in \operatorname{dom} \varphi$. This is obviously true for $k_0 := \min \operatorname{dom} \varphi$, because $\varphi(k_0) = x \in A$. Hence, assume $k, k+1 \in \operatorname{dom} \varphi$ and $\varphi(k) \in A$. Clearly,

$$\varphi(k+1) \in \Pi_{\mathcal{V}}(\varphi(k)) \subset \Pi_{\mathcal{V}}(A) = A.$$

Therefore, $y = \varphi(\max \operatorname{dom} \varphi) \in A$. This proves that $S \subset A$. The proof for a repeller is analogous. $\square$

4.3.2 Limit sets

We define the *$\mathcal{V}$-hull of a set* $A \subset X$ as the intersection of all $\mathcal{V}$-compatible, locally closed sets containing A. We denote it by $\langle A \rangle_{\mathcal{V}}$. Figure 4.8 gives an example of a set whose $\mathcal{V}$-hull is much larger than the original set. As an immediate consequence of Proposition 1.4.4 and Proposition 4.1.3 we get the following result.

Proposition 4.3.16. [17, Proposition 6.11] For every $A \subset X$ its $\mathcal{V}$-hull is $\mathcal{V}$-compatible and locally closed. $\square$

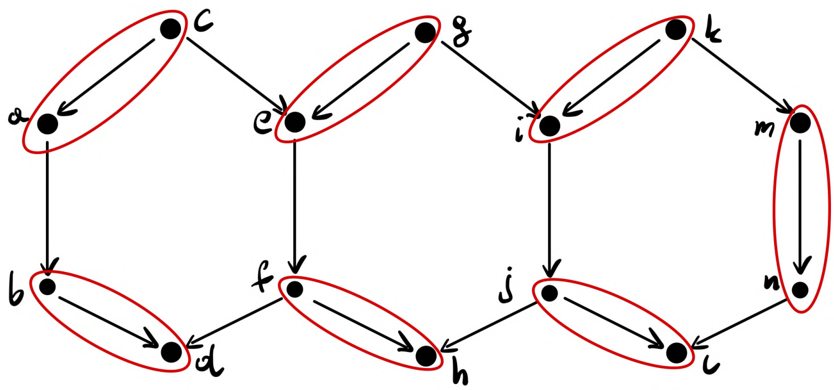


Figure 4.8: The figure shows the finite topological space X and the multivector field consisting of 7 multivectors $\mathcal{V} = \{\{a,c\},\{b,d\},\{e,g\},\{f,h\},\{i,k\},\{j,l\},\{m,n\}\}$. Let $S = \{a,b\}$. Note that the only $\mathcal{V}$-compatible, locally closed set containing S is X. Thus, $\langle S\rangle_{\mathcal{V}} = X$.

Definition 4.3.17. We define the *α- and ω-limit sets of a full solution* φ respectively by

$$\alpha(\varphi) := \left\langle \operatorname{uim}^- \varphi \right\rangle_{\mathcal{V}},$$
$$\omega(\varphi) := \left\langle \operatorname{uim}^+ \varphi \right\rangle_{\mathcal{V}}.$$

Some examples of limit sets are presented in Figure 4.9. The following proposition is a simple consequence of Proposition 1.4.6.

Proposition 4.3.18. [17, Proposition 6.12] Assume φ is a full solution of $\mathcal{V}$ and φ^{op} is the associated dual solution of $\mathcal{V}^{\mathrm{op}}$. Then

$$\alpha(\varphi) = \omega(\varphi^{\mathrm{op}}) \quad \text{and} \quad \omega(\varphi) = \alpha(\varphi^{\mathrm{op}}). \qquad \square$$

The following proposition shows that we can equivalently define limit sets in terms of ultimate multivectors.

Proposition 4.3.19. [17, Proposition 6.13] Let φ be a full solution. Then

$$\alpha(\varphi) = \left\langle \bigcup \mathcal{V}^-(\varphi) \right\rangle_{\mathcal{V}}$$

and

$$\omega(\varphi) = \left\langle \bigcup \mathcal{V}^+(\varphi) \right\rangle_{\mathcal{V}}.$$

Proof. Clearly

$$\operatorname{uim}^- \varphi \subset \bigcup \left\{ V \in \mathcal{V} \mid V \cap \operatorname{uim}^- \varphi \neq \emptyset \right\} = \bigcup \mathcal{V}^-(\varphi)$$

and therefore

$$\alpha(\varphi) = \left\langle \operatorname{uim}^- \varphi \right\rangle_{\mathcal{V}} \subset \left\langle \bigcup \mathcal{V}^-(\varphi) \right\rangle_{\mathcal{V}}.$$

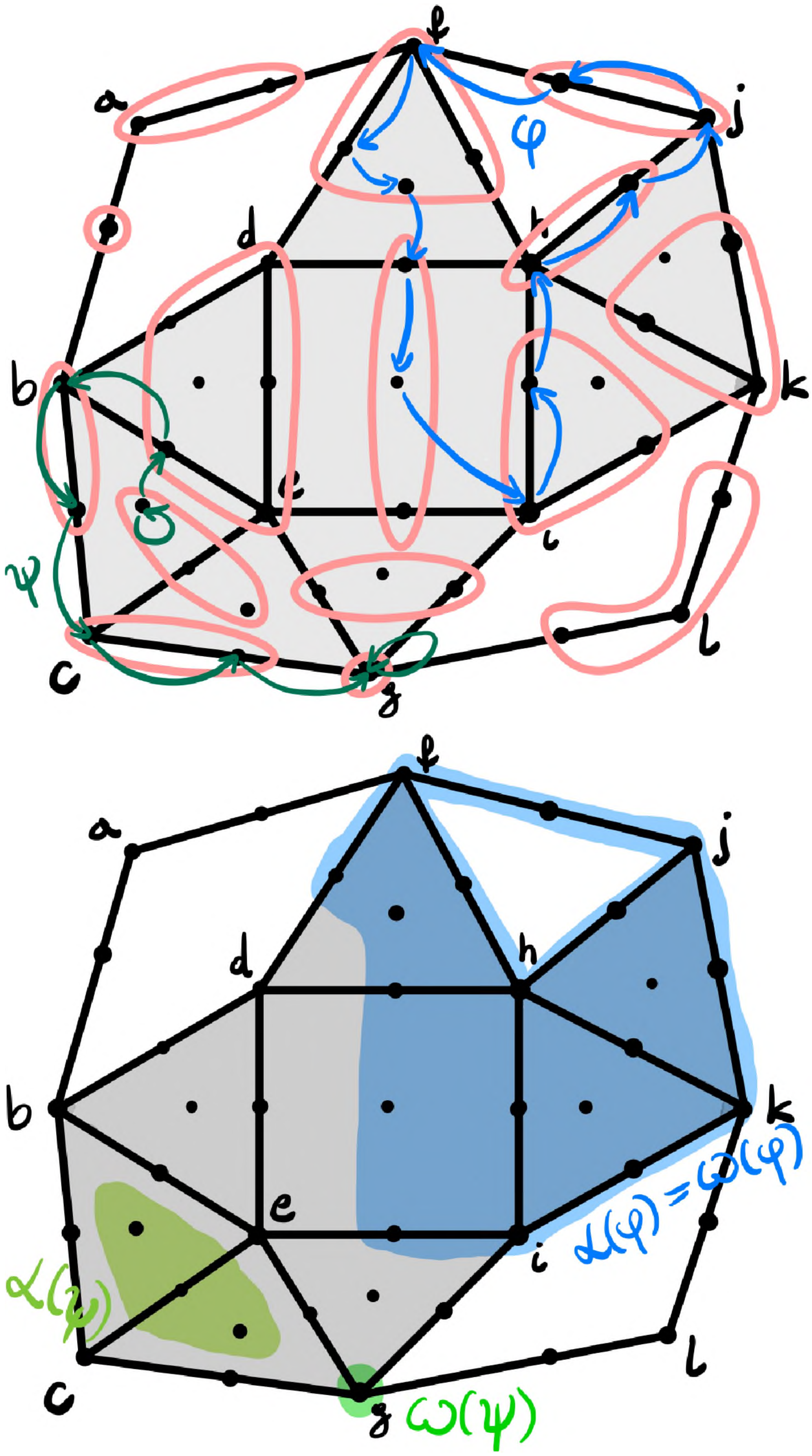


Figure 4.9: Left: a multivector field with two exemplary essential solutions φ (blue) and ψ (green). Right: limit sets for φ and ψ.

Now let $x \in \bigcup \mathcal{V}^-(\varphi)$. Then there exists a $y \in [x]_\mathcal{V}$ such that

$$y \in \operatorname{uim}^- \varphi \subset \left\langle \operatorname{uim}^- \varphi \right\rangle_\mathcal{V} = \alpha(\varphi).$$

Hence, since $\alpha(\varphi)$ is $\mathcal{V}$-compatible, we get $x \in \alpha(\varphi)$, which proves $\bigcup \mathcal{V}^-(\varphi) \subset \alpha(\varphi)$. Since $\alpha(\varphi)$ is locally closed and $\mathcal{V}$-compatible, it follows that

$$\left\langle \bigcup \mathcal{V}^-(\varphi) \right\rangle_\mathcal{V} \subset \alpha(\varphi).$$

The proof for $\omega(\varphi)$ is analogous. □

Lemma 4.3.20. [17, Lemma 6.14] Assume $\varphi : \mathbb{Z} \to X$ is a full solution of $\mathcal{V}$ and $\mathcal{V}^-(\varphi)$ (respectively $\mathcal{V}^+(\varphi)$) contains at least two different multivectors. Then for every $V \in \mathcal{V}$ such that $V \subset \alpha(\varphi)$ (respectively $V \subset \omega(\varphi)$) we have

$$(\Pi_\mathcal{V}(V) \setminus V) \cap \alpha(\varphi) \neq \emptyset \quad \text{(respectively } (\Pi_\mathcal{V}(V) \setminus V) \cap \omega(\varphi) \neq \emptyset) \tag{4.19}$$

and

$$\left(\Pi_\mathcal{V}{}^{-1}(V) \setminus V\right) \cap \alpha(\varphi) \neq \emptyset \quad \text{(respectively } \left(\Pi_\mathcal{V}{}^{-1}(V) \setminus V\right) \cap \omega(\varphi) \neq \emptyset). \tag{4.20}$$

Lemma 4.3.21. Let φ be a full solution in X. Then limit sets $\alpha(\varphi)$ and $\omega(\varphi)$ are strongly connected sets in $G_\mathcal{V}$.

Proof. Consider points $x, y \in \alpha(\varphi)$. We will show that $\operatorname{Path}_\mathcal{V}(x, y, \alpha(\varphi)) \neq \emptyset$. For the rest of the proof we use abbreviations $V_x := [x]_\mathcal{V}$ and $V_y := [y]_\mathcal{V}$. Since $\alpha(\varphi)$ is $\mathcal{V}$-compatible by the definition, both V_x and V_y are subsets of $\alpha(\varphi)$ but they need not be members of $\mathcal{V}^-(\varphi)$. Thus, we consider four cases.

Case 1. $V_x, V_y \in \mathcal{V}^-(\varphi)$. In this case we get directly from the definition of $\mathcal{V}^-$ that $\operatorname{Path}_\mathcal{V}(x, y, \alpha(\varphi)) \neq \emptyset$.

Case 2. $V_x \notin \mathcal{V}^-(\varphi)$ and $V_y \in \mathcal{V}^-(\varphi)$. By Case 1 it is enough to show that there exists at least one point $z \in \bigcup \mathcal{V}^-(\varphi)$ such that $\operatorname{Path}_\mathcal{V}(x, z, \alpha(\varphi)) \neq \emptyset$. Suppose the contrary. Then

$$\pi^+_\mathcal{V}(V_x, \alpha(\varphi)) \cap \bigcup \mathcal{V}^-(\varphi) = \emptyset. \tag{4.21}$$

By Proposition 4.2.11 set $\pi^+_\mathcal{V}(V_x, \alpha(\varphi))$ is a closed and $\mathcal{V}$-compatible subset of $\alpha(\varphi)$. By Proposition 1.4.5 set $A := \alpha(\varphi) \setminus \pi^+_\mathcal{V}(V_x, \alpha(\varphi))$ is locally closed. Clearly, A is $\mathcal{V}$-compatible and by (4.21) it contains $\bigcup \mathcal{V}^-(\varphi)$. It follows that $\alpha(\varphi) = \langle \bigcup \mathcal{V}^-(\varphi) \rangle_\mathcal{V} \subset A \subsetneq \alpha(\varphi)$, a contradiction.

Case 3. $V_x \in \mathcal{V}^-(\varphi)$ and $V_y \notin \mathcal{V}^-(\varphi)$. Similarly to the previous case it suffices to prove that there exists a point $z \in \langle \bigcup \mathcal{V}^-(\varphi) \rangle_\mathcal{V}$ such that $\operatorname{Path}_\mathcal{V}(z, y, \alpha(\varphi)) \neq \emptyset$. Assume the contrary. Then

$$\pi^-_\mathcal{V}(V_y, \alpha(\varphi)) \cap \bigcup \mathcal{V}^-(\varphi) = \emptyset. \tag{4.22}$$

By Proposition 4.2.11 the set $\pi^-_{\mathcal{V}}(V_y, \alpha(\varphi))$ is $\mathcal{V}$-compatible and open. Hence, $B := \alpha(\varphi) \setminus \pi^-_{\mathcal{V}}(V_y, \alpha(\varphi))$ is a $\mathcal{V}$-compatible, locally closed set containing $\bigcup \mathcal{V}^-(\varphi)$. It follows that $\alpha(\varphi) = \langle \bigcup \mathcal{V}^-(\varphi) \rangle_{\mathcal{V}} \subset B \subsetneq \alpha(\varphi)$, a contradiction.

Case 4. $V_x, V_y \notin \mathcal{V}^-(\varphi)$. Fix a $z \in \bigcup \mathcal{V}^-(\varphi)$. Using Case 2 we can find a path $\psi_1 \in \operatorname{Path}_{\mathcal{V}}(x, z, \alpha(\varphi))$ and by Case 3 a path $\psi_2 \in \operatorname{Path}_{\mathcal{V}}(z, y, \alpha(\varphi))$. Then, $\psi_1 \cdot \psi_2 \in \operatorname{Path}_{\mathcal{V}}(x, y, \alpha(\varphi))$. This finishes the proof that $\alpha(\varphi)$ is strongly connected.

The proof for $\omega(\varphi)$ is analogous. □

Theorem 4.3.22. [17, Theorem 6.15] Let φ be an essential solution in X. Then both limit sets $\alpha(\varphi)$ and $\omega(\varphi)$ are non-empty minimal isolated invariant sets.

Proof. The nonemptiness of $\alpha(\varphi)$ and $\omega(\varphi)$ follows from Proposition 4.3.5.

The sets $\alpha(\varphi)$ and $\omega(\varphi)$ are $\mathcal{V}$-compatible and locally closed by Proposition 4.3.16. In order to prove that they are isolated invariant sets it suffices to apply Proposition 4.1.21 as long as we prove that $\alpha(\varphi)$ and $\omega(\varphi)$ are also invariant.

To this end assume that $x \in \alpha(\varphi)$. Suppose that $\mathcal{V}^-(\varphi)$ is a singleton. Then by Proposition 4.3.19, $\alpha(\varphi) = [x]_{\mathcal{V}}$. Since φ is essential, this is possible only if $[x]_{\mathcal{V}}$ is critical. It follows that the stationary solution $\psi(t) = x$ is essential and $\alpha(\varphi)$ is an isolated invariant set.

Assume now that there are at least two different multivectors in $\mathcal{V}^-(\varphi)$. Then the assumptions of Lemma 4.3.20 are satisfied and, as a consequence of (4.19), for every $x \in \alpha(\varphi)$ there exist a point $x' \in [x]_{\mathcal{V}}$ and a point $y \in \alpha(\varphi)$ such that $y \in (\Pi_{\mathcal{V}}(x') \setminus [x]_{\mathcal{V}}) \cap \alpha(\varphi)$. Hence, we can construct a right-essential solution

$$x_0 \cdot x'_0 \cdot x_1 \cdot x'_1 \cdot x_2 \cdot x'_2 \cdot \ldots, \tag{4.23}$$

where $x_0 = x$, $x'_i \in [x_i]_{\mathcal{V}}$ such that $(\Pi_{\mathcal{V}}(x'_i) \setminus [x_i]) \cap \alpha(\varphi) \neq \emptyset$, and $x_{i+1} \in (\Pi_{\mathcal{V}}(x'_i) \setminus [x_i]_{\mathcal{V}}) \cap \alpha(\varphi)$. Property (4.20) provides a complementary left-essential solution. Concatenation of both solutions gives an essential solution in $\alpha(\varphi)$. Hence, we proved that $\alpha(\varphi)$ is invariant and consequently an isolated invariant set.

To better see why the constructed solution is of form (4.23) consider solution φ in Figure 4.9. Let $x_i = I$ then $\Pi_{\mathcal{V}}(x_i) = \{I, KI, IH, IKH\} = [x_i]$. Hence to exit $[x_i]$ we must first move, for example, to the point $x'_i = IKH$. Now $\Pi_{\mathcal{V}}(x'_i) \setminus [x_i] \neq \emptyset$ and we can choose x_{i+1} which is in a different multivector.

Finally, by Lemma 4.3.21 set $\alpha(\varphi)$ is strongly connected in $G_{\mathcal{V}}$ and the conclusion follows from Proposition 4.3.12. The proof for $\omega(\varphi)$ is analogous. □

Let $A, B \subset X$. We define the *connection set from A to B* by:

$$C(A, B) := \left\{ x \in X \mid \exists_{\varphi \in \operatorname{eSol}(x, X)}\ \alpha(\varphi) \subset A \text{ and } \omega(\varphi) \subset B \right\}. \tag{4.24}$$

Proposition 4.3.23. [17, Proposition 6.17] Assume $A, B \subset X$. Then the connection set $C(A, B)$ is an isolated invariant set.

Proof. To prove that $C(A,B)$ is invariant, take an $x \in C(A,B)$ and choose a solution $\varphi \in \operatorname{eSol}(x,X)$ from A to B as in (4.24). It is clear that $\varphi(t) \in C(A,B)$ for every $t \in \mathbb{Z}$. Thus, $\varphi \in \operatorname{eSol}(x, C(A,B))$, and this in turn implies $x \in \operatorname{Inv} C(A,B)$ and shows that $C(A,B)$ is invariant. Now consider a point $y \in [x]_{\mathcal{V}}$. Then the solution $\rho = \varphi^- \cdot y \cdot \varphi^+$ is a well-defined essential solution through y such that $\alpha(\rho) \subset A$ and $\omega(\rho) \subset B$. Thus, $C(A,B)$ is $\mathcal{V}$-compatible.

In order to prove that $C(A,B)$ is locally closed, consider $x, z \in C(A,B)$, and a $y \in X$ such that $z \leq_{\mathcal{T}} y \leq_{\mathcal{T}} x$. Select essential solutions $\varphi_x \in \operatorname{eSol}(x, C(A,B))$ and $\varphi_z \in \operatorname{eSol}(z, C(A,B))$. Then $\psi := \varphi_x^- \cdot y \cdot \varphi_z^+$ is a well-defined essential solution through y such that $\alpha(\psi) \subset A$ and $\omega(\psi) \subset B$. It follows that $y \in C(A,B)$. Thus, by Proposition 1.4.10, $C(A,B)$ is locally closed. Finally, Proposition 4.1.21 implies that $C(A,B)$ is an isolated invariant set. $\square$

Proposition 4.3.24. [17, Proposition 6.18] Assume A is an attractor. Then $C(A, A^\star) = \emptyset$. Similarly, if R is a repeller, then $C(R^\star, R) = \emptyset$.

Proof. Suppose there exists an $x \in C(A, A^\star)$. Then by (4.24) we can choose a $\varphi \in \operatorname{eSol}(x,X)$ and a $t \in \mathbb{Z}$ such that $\varphi(t) \in A$ and $\varphi(t+1) \notin A$. However, since A is an attractor, $\varphi(t) \in A$ implies $\varphi(t+1) \in A$, a contradiction. The proof for a repeller is analogous. $\square$

4.4 Morse decomposition, Morse equation, Morse inequalities

In this section we define Morse decompositions and prove the Morse inequalities for combinatorial multivector fields. We recall our general assumption that X is invariant.

4.4.1 Morse decomposition

Definition 4.4.1. Assume X is invariant and $(\mathbb{P}, \leq)$ is a finite poset. Then the collection $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ is called a *Morse decomposition of* X if the following conditions are satisfied:

(i) $\mathcal{M}$ is a family of mutually disjoint, isolated invariant subsets of X.

(ii) For every essential solution φ in X either $\operatorname{im} \varphi \subset M_r$ for an $r \in \mathbb{P}$ or there exist $p, q \in \mathbb{P}$ such that $q > p$ and

$$\alpha(\varphi) \subset M_q \quad \text{and} \quad \omega(\varphi) \subset M_p.$$

We refer to the elements of $\mathcal{M}$ as *Morse sets*.

Note that there are a few differences between the combinatorial and classical definition of Morse decomposition (see Definition 3.1.5). First, the condition (ii) is expressed in terms of solutions not points. It is caused by the multivalued nature of combinatorial dynamics. Moreover, solutions can jump from one Morse set to another in a single time-step. Hence, we need to check condition (ii) for every essential solution and every point in X, not only in between Morse sets, that is, in $X \setminus \bigcup \mathcal{M}$. Without this assumption, we could construct a degenerated example where recurrent behavior is not encapsulated in a Morse set (see Figure 4.10).

The attractor-repeller pair that is the subject of Theorem 4.3.7 is the simplest example of a non-trivial Morse decomposition of X into two Morse sets. We will show it now.

Proposition 4.4.2. [17, Proposition 7.2] Let X be an invariant set and let $A \subset X$ be an attractor. Assume $A^\star$, its dual repeller, is nonempty. Furthermore, define $M_1 := A$, $M_2 := A^\star$, and let $\mathbb{P} := \{1, 2\}$ be an indexing set with the order induced from $\mathbb{N}$. Then $\mathcal{M} = \{M_1, M_2\}$ is a Morse decomposition of X.

Proof. By Theorems 4.3.3 and 4.3.4 both A and $A^\star$ are isolated invariant sets which are disjoint by their construction. Let $x \in X$ and let $\varphi \in \mathrm{eSol}_\mathcal{V}(x, X)$. By Theorem 4.3.22 the set $\omega(\varphi)$ is a minimal, isolated invariant set. By Proposition 4.3.15 it is either a subset of A or a subset of $\mathrm{Inv}(X \setminus A) = A^\star$. The same holds for $\alpha(\varphi)$.

We therefore have four cases. The situation $\alpha(\varphi) \subset M_2$ and $\omega(\varphi) \subset M_1$ is consistent with the Morse decomposition definition. The case $\alpha(\varphi) \subset M_1$ and $\omega(\varphi) \subset M_2$ gives a contradiction, because it implies that there exists $t \in \mathbb{Z}$ such that $\varphi(t) \in M_1$ and $\varphi(t+1) \notin M_1$ while M_1 is an attractor. Now suppose that we have $\alpha(\varphi) \subset M_1$ and $\omega(\varphi) \subset M_1$. It follows that there exists a $t \in \mathbb{Z}$ such that $\varphi((-\infty, t])) \subset A$. Since A is an attractor we therefore have $\varphi(t+1) \in \Pi_\mathcal{V}(\varphi(t)) \subset A$, and an induction argument implies $\mathrm{im}\,\varphi \subset A = M_1$. The analogous argument holds if $\alpha(\varphi), \omega(\varphi) \subset M_2$. □

Corollary 4.4.3. An invariant set $S \subset X$ is minimal if and only if the only Morse decomposition $\mathcal{M}$ of S is $\mathcal{M} = \{S\}$.

We recall that $G_\mathcal{V}$ stands for the digraph associated with the multivalued map $\Pi_\mathcal{V}$ of the multivector field $\mathcal{V}$ on X. We have the following theorem which reduces the problem of constructing the Morse decomposition to graph-theoretic algorithms.

Theorem 4.4.4. [17, Theorem 7.3] Assume X is invariant. Consider the family $\mathcal{M}$ of all strongly connected components M of $G_\mathcal{V}$ with $\mathrm{eSol}(\mathcal{M}) \neq \emptyset$. Then $\mathcal{M}$ is a Morse decomposition of X.

Actually, the following proposition shows that Theorem 4.4.4 provides a recipe for the finest Morse decomposition.

Proposition 4.4.5. Let $\mathcal{M}$ be the Morse decomposition X given by Theorem 4.4.4. Then for any other Morse decomposition $\mathcal{M}'$ of X we have $\mathcal{M} \sqsubset \mathcal{M}'$. We call it the *minimal* Morse decomposition of X. We denote it by $\mathcal{M}(\mathcal{V})$.

Proof. Let $\mathcal{M}'$ be a Morse decomposition of X. Since $M \in \mathcal{M}$ is a strongly connected component of $G_\mathcal{V}$, by Proposition 4.1.24 there exists a full periodic solution φ such that $\mathrm{im}\,\varphi = M$. It follows from Proposition 4.1.14 that φ is a full essential solution. By

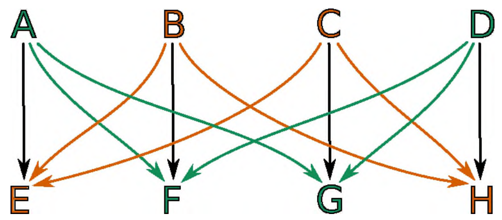


Figure 4.10: A sample combinatorial multivector field $\mathcal{V} = \{\{A, D, F, G\}, \{B, C, E, H\}\}$ on the finite topological space $X = \{A, B, C, D, E, F, G, H\}$ with Alexandrov topology induced by the partial order indicated by arrows. If we consider $\mathcal{M} = \mathcal{V}$, then one obtains a partition into isolated invariant sets with $X \setminus \mathcal{M} = \emptyset$. Note that $\ldots D \cdot H \cdot B \cdot F \cdot D \cdot \ldots$ is a periodic trajectory which passes through both "Morse sets."

Proposition 4.1.25 set M is $\mathcal{V}$-compatible and locally closed. Thus, we have $\alpha(\varphi) = M$. Hence, by Definition 4.4.1 there exists $M' \in \mathcal{M}'$ such that $M = \alpha(\varphi) \subset M'$. This proves the minimality of $\mathcal{M}$ with respect to $\sqsubset$. $\square$

Proposition 4.4.6. Let $\mathcal{M}$ be a Morse decomposition of X. Then $\mathcal{M}$ is the minimal Morse decomposition if and only if every Morse set $M \in \mathcal{M}$ is a minimal isolated invariant set.

Proof. Let $\mathcal{M}$ be the minimal Morse decomposition of X. Then the implication follows directly from the construction (Theorem 4.4.4 and Proposition 4.4.5) and Corollary 4.3.13.

Let $\mathcal{M}' := \mathcal{M}(\mathcal{V})$ be the minimal Morse decomposition and $M' \in \mathcal{M}'$ such that $M' \subset M$ for an $M \in \mathcal{M}$. By Proposition 4.4.5 Morse set M' is a strongly connected component. Morse set M is a strongly connected set (Proposition 4.3.12) such that $M \cap M' \neq \emptyset$. Thus, $M \subset M'$. It follows that $M = M'$ and $\mathcal{M} = \mathcal{M}'$. $\square$

4.4.2 Weak Morse decomposition

For reasons that will be clear in Chapter 5, we introduce a concept of the weak Morse decomposition.

Definition 4.4.7. Assume X is invariant and $(\mathbb{P}, \leq)$ is a finite poset. Then the collection $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ is called a *weak Morse decomposition of* X if the following conditions are satisfied:

(i) $\mathcal{M}$ is a family of mutually disjoint locally closed subsets of X.

(ii) For every full solution φ in X either $\operatorname{im} \varphi \subset M_r$ for an $r \in \mathbb{P}$ or there exist $p, q \in \mathbb{P}$ such that $q > p$ and

$$\alpha(\varphi) \subset M_q \quad \text{and} \quad \omega(\varphi) \subset M_p.$$

We refer to the elements of $\mathcal{M}$ as *weak Morse sets.*

In the above definition, we modified Definition 4.4.1 by replacing essential solutions with full solutions and isolated invariant sets with locally closed sets. Since every multivector admits a full stationary solution, every point belongs to a weak Morse set. Hence, we get the following corollary.

Corollary 4.4.8. Let $\mathcal{V}$ be a multivector field for X. Then a weak Morse decomposition $\mathcal{M}$ of X is a partition of X.

Proposition 4.4.9. Let $M \in \mathcal{M}$ be a weak Morse set. Then M is $\mathcal{V}$-compatible.

Proof. Let M be a weak Morse set and let $x \in M$. Assume $y \in [x]$ is such that $y \neq x$. Consider a full solution $\varphi : \ldots x \cdot y \cdot x \cdot y \cdot x \cdot \ldots$. Clearly $\alpha(\varphi) = \omega(\varphi) = [x]$. Condition (ii) of Definition 4.4.7 implies that $y \in M$. $\square$

Now we will show that the construction of the weak Morse decomposition can also be reduced to the computation of strongly connected components.

Theorem 4.4.10. Assume X is invariant. Consider the family $\mathcal{M}$ of all strongly connected components M of $G_{\mathcal{V}}$. Then $\mathcal{M}$ is the minimal weak Morse decomposition of X.

Proof. For convenience, assume that $\mathcal{M} = \{M_i \mid i \in \mathbb{P}\}$ is bijectively indexed by a finite set $\mathbb{P}$. By Proposition 4.1.25 every $M \in \mathcal{M}$ is $\mathcal{V}$-compatible and locally closed. Since sets in $\mathcal{M}$ are strongly connected components of $G_{\mathcal{V}}$ they are clearly mutually disjoint.

Define a relation $\leq$ on the indexing set $\mathbb{P}$ as

$$p \leq q \;\Leftrightarrow\; \exists_{\varphi \in \operatorname{Path}_{\mathcal{V}}(X)}\ \varphi^{\sqsubset} \in M_q \text{ and } \varphi^{\sqsupset} \in M_p.$$

It is clear that $\leq$ is reflexive. To see that it is transitive consider $M_i, M_j, M_k \in \mathcal{M}$ such that $k \leq j \leq i$. It follows that there exist paths φ and ψ such that $\varphi^{\sqsubset} \in M_i$, $\varphi^{\sqsupset}, \psi^{\sqsubset} \in M_j$ and $\psi^{\sqsupset} \in M_k$. Since M_j is strongly connected we can find $\rho \in \operatorname{Path}_{\mathcal{V}}(\varphi^{\sqsupset}, \psi^{\sqsubset}, X)$. The path $\varphi \cdot \rho \cdot \psi$ clearly connects M_i with M_k proving that $k \leq i$.

In order to show that $\leq$ is antisymmetric consider sets M_i, M_j with $i \leq j$ and $j \leq i$. It follows that there exist paths φ and ψ such that $\varphi^{\sqsubset}, \psi^{\sqsupset} \in M_i$ and $\varphi^{\sqsupset}, \psi^{\sqsubset} \in M_j$. Since the sets M_i, M_j are strongly connected we can find paths ρ and ρ' from $\varphi^{\sqsupset}$ to $\psi^{\sqsubset}$ and from $\psi^{\sqsupset}$ to $\varphi^{\sqsubset}$ respectively. Clearly, $\varphi \in \operatorname{Path}_{\mathcal{V}}(\varphi^{\sqsubset}, \varphi^{\sqsupset}, X)$ and $\rho \cdot \psi \cdot \rho' \in (\varphi^{\sqsupset}, \varphi^{\sqsubset}, X)$. This proves that M_i and M_j are the same strongly connected component.

Let φ be a full solution in X. By Lemma 4.3.21 sets $\alpha(\varphi)$ and $\omega(\varphi)$ are strongly connected sets. Hence, there exist strongly connected components $M_p, M_q \in \mathcal{M}$ such that $\alpha(\varphi) \subset M_q$ and $\omega(\varphi) \subset M_p$. We have $p \leq q$ directly from the construction of relation $\leq$. $\square$

Theorem 4.4.11. Let $\mathcal{M}$ be the weak Morse decomposition of X given by Theorem 4.4.10. Then for any other weak Morse decomposition $\mathcal{M}'$ of X we have $\mathcal{M} \sqsubset \mathcal{M}'$. We call it the *minimal weak Morse decomposition* of X. We denote it by $\mathcal{M}^{+}(\mathcal{V})$.

Proof. Let $\mathcal{M}'$ be a weak Morse decomposition of X. Since $M \in \mathcal{M}$ is a strongly connected component of $G_{\mathcal{V}}$, by Proposition 4.1.24 there exists a full periodic solution φ such that $\operatorname{im} \varphi = M$. By Proposition 4.1.25 set M is $\mathcal{V}$-compatible and locally closed. Thus, we have $\alpha(\varphi) = M = \omega(\varphi)$. Hence, by Definition 4.4.7 there exists $M' \in \mathcal{M}'$ such that $M = \alpha(\varphi) = \omega(\varphi) \subset M'$. This proves the minimality of $\mathcal{M}$ with respect to the relation $\sqsubset$. $\square$

Proposition 4.4.12. Let $\mathcal{M}$ be a weak Morse decomposition of X. Then $\mathcal{M}$ is the minimal weak Morse decomposition if and only if every weak Morse set $M \in \mathcal{M}$ is either a minimal isolated invariant set or a regular multivector.

Proof. Let $\mathcal{M}$ be the minimal weak Morse decomposition and let $M \in \mathcal{M}$. By Proposition 4.4.9 set M is $\mathcal{V}$-compatible and by Theorem 4.4.11 it is a strongly connected component of $G_{\mathcal{V}}$. Note that there are two possible cases. First, there exists a regular multivector $V \in \mathcal{V}$ such that $V = M$ which satisfies the conjecture. Otherwise, M contain a critical multivector or at least two regular ones. Thus, by Propositions 4.1.24 and 4.1.7 we can construct a full essential solution passing through all points of M. Thus, by Proposition 4.1.21 it follows that M is an isolated invariant set. Finally, by Proposition 4.3.12 it is minimal.

Now let $\mathcal{M}$ be a weak Morse decomposition of X. A weak Morse set M such that M is a regular multivector cannot be decomposed into a smaller weak Morse sets because of 4.4.9. If M is a minimal isolated invariant set then by Proposition 4.3.12 Morse set M is a strongly connected set. Suppose there is a weak Morse decomposition $\mathcal{M}'$ such that there exist $M'_p, M'_q \in \mathcal{M}'$ and $M'_p, M'_q \subset M \in \mathcal{M}$. Let $x \in M'_p$ and $y \in M'_q$. There exists a path $\varphi \in \operatorname{Path}(x, y, M)$ that can be extended to a full solution $\psi_{xy} := \ldots \cdot x \cdot x \cdot \varphi \cdot y \cdot y \cdot \ldots$. Clearly $\alpha(\psi) \subset M'_p$ and $\omega(\psi) \subset M'_q$. It follows that $q \leq p$. Similarly we can show that $q \leq p$ by constructing a full solution ψ_{yx}. This gives a contradiction with the assumption that $\mathcal{M}'$ is a weak Morse decomposition. $\square$

We show in Theorem 4.4.14 that weak Morse decomposition leads to a Conley-Morse graph that is substantially the same as the one obtained from Morse decomposition. It is mainly due to the following corollary that is an immediate consequence of Ważewski property 4.2.17.

Corollary 4.4.13. Let $\mathcal{M}$ be a weak Morse decomposition of $\mathcal{V}$ and let $M \in \mathcal{M}$. If $\operatorname{Inv} M \neq \emptyset$ then $\operatorname{Con}(\operatorname{Inv} M) = H(\operatorname{cl} M, \operatorname{mo} M)$. If $\operatorname{Inv} M = \emptyset$ then $H(\operatorname{cl} M, \operatorname{mo} M) = 0$. $\square$

Theorem 4.4.14. Let $\mathcal{M}$ be a weak Morse decomposition. Then family $\overline{\mathcal{M}} := \{\operatorname{Inv} M \mid M \in \mathcal{M},\ \operatorname{Inv} M \neq \emptyset\}$ is a Morse decomposition. If $\mathcal{M}$ is the minimal weak Morse decomposition, then $\overline{\mathcal{M}}$ is the minimal Morse decomposition. Moreover, the Morse-Conley graph of $\overline{\mathcal{M}}$ is the Morse-Conley graph of $\mathcal{M}$ restricted to the nodes corresponding to the weak Morse sets satisfying $\operatorname{Inv} M \neq \emptyset$.

Proof. Assume $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ and let $\overline{\mathbb{P}} := \{p \in \mathbb{P} \mid \operatorname{Inv} M_p \neq \emptyset\}$. Then $\overline{\mathcal{M}} = \{\operatorname{Inv} M_p \mid M_p \in \mathcal{M},\ p \in \overline{\mathbb{P}}\}$. By Corollary 4.1.22 $\operatorname{Inv} M$ is an isolated invariant set for $M \in \mathcal{M}$. Since elements of $\mathcal{M}$ are mutually disjoint, so are the elements of $\overline{\mathcal{M}}$. Thus condition (i) for Morse decomposition is satisfied.

Let $\varphi \in \operatorname{eSol}(X) \subset \operatorname{Sol}(X)$. There exist $p, q \in \mathbb{P}$ such that $\alpha(\varphi) \subset M_p$, $\omega(\varphi) \subset M_q$ and $p > q$. By Theorem 4.3.22 sets $\alpha(\varphi)$ and $\omega(\varphi)$ are non-empty isolated invariant set. Therefore, $\emptyset \neq \alpha(\varphi) \subset \operatorname{Inv} M_p$ and $\emptyset \neq \omega(\varphi) \subset \operatorname{Inv} M_q$ and $\operatorname{Inv} M_p, \operatorname{Inv} M_q \in \overline{\mathcal{M}}$. Suppose

$\operatorname{im}\varphi \subset M_p$ for some $p \in \mathbb{P}$. Then, since $\varphi \in \operatorname{eSol}(M_p)$, we get $\operatorname{im}\varphi \subset \operatorname{Inv} M_p \in \overline{\mathcal{M}}$. A partial order on $\mathbb{P}$ restricted to $\overline{\mathbb{P}}$ is clearly an order satisfying condition (ii) of a Morse decomposition definition.

The minimality follows from Propositions 4.4.12 and 4.4.6. The correspondence of Morse-Conley graphs follows from Corollary 4.4.13. $\square$

4.4.3 Properties of Morse sets

Let $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ be a Morse decomposition for $\mathcal{V}$. For a subset $I \subset \mathbb{P}$ we define the *Morse set of I* by

$$M(I) := \bigcup_{i,j\in I} C(M_i, M_j).$$

Proposition 4.4.15. Let $i \in I$ then $M_i \subset M(I)$. In particular $M_i = C(M_i, M_i)$.

Proof. Let $x \in M_i$. Since M_i is an isolated invariant set there exists $\varphi \in \operatorname{eSol}(x, M_i)$. We have $\operatorname{im}\varphi \subset M_i$. By Proposition 4.1.21 set M_i is locally closed and $\mathcal{V}$-compatible. Thus, $\alpha(\varphi), \omega(\varphi) \subset M_i$. Consequently, $x \in C(M_i, M_i)$.

Now, suppose that $x \in C(M_i, M_i)$ and $x \notin M_i$. Then there exists $\varphi \in \operatorname{eSol}(x, X)$ such that $\alpha(\varphi), \omega(\varphi) \subset M_i$. Since $\mathcal{M}$ is a Morse decomposition it follows that $i < i$. This contradicts condition (ii) of definition 4.4.1. $\square$

Theorem 4.4.16. [17, Theorem 7.4] The set $M(I)$ is an isolated invariant set.

Proof. Observe that $M(I)$ is invariant, because, by Proposition 4.3.23, every connection set is isolated invariant and $\mathcal{V}$-compatible by Proposition 4.1.21. Set $M(I)$ is $\mathcal{V}$-compatible as a union of $\mathcal{V}$-compatible sets and invariant as the union of invariant sets (Proposition 4.1.15). We will prove that $M(I)$ is locally closed. To see that, suppose the contrary. Then, by Proposition 1.4.10, we can choose $a, c \in M(I)$ and a point $b \notin M(I)$ such that $c \leq_{\mathcal{T}} b \leq_{\mathcal{T}} a$. There exist essential solutions $\varphi_a \in \operatorname{eSol}(a, X)$ and $\varphi_c \in \operatorname{eSol}(c, X)$ such that $\alpha(\varphi_a) \subset M_q$ and $\omega(\varphi_c) \subset M_p$ for some $p, q \in I$. It follows that $\psi := \varphi_a^- \cdot b \cdot \varphi_c^+$ is a well-defined essential solution such that $\alpha(\psi) \subset M_q$ and $\omega(\psi) \subset M_p$. Hence, $b \in C(M_q, M_p) \subset M(I)$ which proves that $M(I)$ is locally closed. Thus, the conclusion follows from Proposition 4.1.21. $\square$

Theorem 4.4.17. [17, Theorem 7.5] Let I be a down set in $\mathbb{P}$, then $M(I)$ is an attractor in X.

Theorem 4.4.18. [17, Theorem 7.6] If $I \subset \mathbb{P}$ is convex, then $(M(I^{\leq}), M(I^{<}))$ is an index pair for the isolated invariant set $M(I)$.

Proof. By Proposition 1.3.1 the sets $I^{\leq}$ and $I^{<}$ are down sets. Thus, by Theorem 4.4.17 both $M(I^{\leq})$ and $M(I^{<})$ are attractors. It follows that $\Pi_{\mathcal{V}}(M(I^{\leq})) \subset M(I^{\leq})$ and $\Pi_{\mathcal{V}}(M(I^{<})) \subset M(I^{<})$. Therefore, conditions (IP1) and (IP2) of an index pair are satisfied.

Let $A := M(I^{\leq}) \setminus M(I^{<})$. To prove (IP3) we first show that $M(I) \subset \operatorname{Inv}(A)$. By Theorem 4.4.16, $M(I)$ is an isolated invariant set. Therefore, it is sufficient to prove

$M(I) \subset A$. The set A is $\mathcal{V}$-compatible as a difference of $\mathcal{V}$-compatible sets. By Proposition 1.4.3 it is also locally closed, because $M(I^{\leq})$ and $M(I^{<})$ are closed as attractors (see Theorem 4.3.3). Assume that $M(I) \not\subset A$. Select an $x \in M(I) \setminus A$. By the definition of $M(I)$ we can find an essential solution φ through x such that $\omega(\varphi) \subset M_p$ for some $p \in I$. Since $M(I) \subset M(I^{\leq})$ and $x \notin A$ we get $x \in M(I^{<})$. But $M(I^{<})$ is an attractor. Therefore $\omega(\varphi) \subset M(I^{<})$, which in turn implies $p \notin I$, a contradiction.

To prove the opposite inclusion take an $x \in \operatorname{Inv}(A) = \operatorname{Inv}(M(I^{\leq}) \setminus M(I^{<}))$. Then we can find an essential solution $\varphi \in \operatorname{eSol}(x, A)$. Then $\operatorname{im} \varphi \subset M(I^{\leq}) \setminus M(I^{<})$. In particular,

$$\operatorname{uim}^{-} \varphi \cap M(I^{<}) = \emptyset \quad \text{and} \quad \operatorname{uim}^{+} \varphi \cap M(I^{<}) = \emptyset. \tag{4.25}$$

We also have $\varphi \in \operatorname{eSol}(x, M(I^{\leq}))$, which means that there exist $p, q \in I^{\leq}$ such that $p \geq q$, $\alpha(\varphi) \subset M_p$, $\omega(\varphi) \subset M_q$. We cannot have $p \in I^{<}$, because then we get $\emptyset \neq \operatorname{uim}^{-} \varphi \subset \alpha(\varphi) \subset M_p \subset M(I^{<})$ which contradicts (4.25). Therefore, $p \in I^{\leq} \setminus I^{<} = I$. By an analogous argument we get $q \in I$. It follows that $x \in C(M_p, M_q) \subset M(I)$. □

Note that if I is a down set then $I^{<} = \emptyset$. Hence, an immediate consequence of Theorem 4.4.18 we get the following Corollary.

Corollary 4.4.19. [17, Corollary 7.7] If I is a down set in $\mathbb{P}$, then $I^{\leq} = I$, $I^{<} = \emptyset$ and $(M(I), \emptyset)$ is an index pair for $M(I)$.

Theorem 4.4.20. [17, Theorem 7.8] Assume X is invariant, $A \subset X$ is an attractor and $A^{\star}$ is its dual repeller. Then we have

$$\bar{p}_A(t) + \bar{p}_{A^{\star}}(t) = \bar{p}_X(t) + (1+t)q(t) \tag{4.26}$$

for a polynomial $q(t)$ with non-negative coefficients. Moreover, if $q \neq 0$, then $C(A^{\star}, A) \neq \emptyset$.

Proof. Let $\mathbb{P} := \{1, 2\}$ with the order induced from $\mathbb{N}$. Set $M_1 := A$ and $M_2 := A^{\star}$. Then, by Proposition 4.4.2 $\mathcal{M} := \{M_1, M_2\}$ is a Morse decomposition of X. For $I := \{2\}$ one obtains $I^{\leq} = \{1, 2\}$ and $I^{<} = \{1\}$. Yet, this immediately implies that $M(I^{\leq}) = X$ and $M(I^{<}) = M(\{1\}) = A$. We have

$$\bar{p}_X(t) = \bar{p}_{M(I^{\leq})}(t) \quad \text{and} \quad \bar{p}_A(t) = \bar{p}_{M(I^{<})}(t). \tag{4.27}$$

By Theorem 4.4.18 the pair $(M(I^{\leq}), M(I^{<}))$ is an index pair for $M(I) = A^{\star}$. Thus, by substituting $P_1 := M(I^{\leq})$, $P_2 := M(I^{<})$ and $S := A^{\star}$ into (4.12) in Corollary 4.2.16 we get (4.26) from (4.27). By Proposition 4.3.24 we have the identity $C(A, A^{\star}) = \emptyset$. Therefore, if in addition $C(A^{\star}, A) = \emptyset$, then X decomposes into A and $A^{\star}$, and Theorem 4.2.19 implies

$$H(P_1) = \operatorname{Con}(X) = \operatorname{Con}(A) \oplus \operatorname{Con}(A^{\star}) = H(P_2) \oplus H(A^{\star}),$$

as well as $q = 0$ in view of Proposition 4.2.16. This finally shows that $q \neq 0$ implies $C(A^{\star}, A) \neq \emptyset$. □

4.4.4 Morse equation and Morse inequalities

The following two theorems follow from the results of the preceding section by adapting the proofs of the corresponding Theorems in [21].

Theorem 4.4.21. [17, Theorem 7.9] Assume X is invariant and $\mathbb{P} = \{1, 2, ..., n\}$ is ordered by the linear order of the natural numbers. Let $\mathcal{M} := \{M_p \mid p \in \mathbb{P}\}$ be a Morse decomposition of X and set $A_i := M(\{i\}^{\leq})$, $A_0 := \emptyset$. Then (A_{i-1}, M_i) is an attractor-repeller pair in A_i. Moreover,

$$\sum_{i=1}^{n} \bar{p}_{M_i}(t) = \bar{p}_X(t) + (1+t) \sum_{i=1}^{n} q_i(t)$$

for some polynomials $q_i(t)$ with non-negative coefficients and such that $q_i(t) \neq 0$ implies $C(M_i, A_{i-1}) \neq \emptyset$ for $i = 2, 3, ..., n$.

Theorem 4.4.22. [17, Theorem 7.10] Assume X is invariant. For a Morse decomposition $\mathcal{M}$ of X define

$$m_k(\mathcal{M}) := \sum_{r \in \mathbb{P}} \bar{\beta}_k(M_r).$$

Then for any $k \in \mathbb{Z}^+$ we have the following inequalities.

(i) The strong Morse inequalities:

$$m_k(\mathcal{M}) - m_{k-1}(\mathcal{M}) + ... \pm m_0(\mathcal{M}) \geq \beta_k(X) - \beta_{k-1}(X) + ... \pm \beta_0(X),$$

(ii) The weak Morse inequalities:

$$m_k(\mathcal{M}) \geq \beta_k(X).$$

Chapter 5

Persistence of Morse Decomposition

The first section of this chapter presents the main theoretical concepts from [7] (in particular, from Sections 5 and 6). The second section shows some simplifications leading to a better visualization.

As we mentioned in Chapter 3 isolated invariant sets are always compact in the classical setting of semiflows on locally compact Hausdorff spaces. Therefore, every Morse set forms a distinct connected component in the space obtained as the union of all Morse sets with the topology induced from the ambient space. This is because Morse sets are always disjoint and, in that case, also closed. In particular, the space between the Morse sets is "filled" with solutions connecting them.

This need not be true in the setting of finite topological spaces. In the case of multivector fields, attractors are the only closed isolated invariant sets (Theorem 4.3.3). Moreover, Morse sets are not disconnected in general (see Fig. 4.4). Thus, to study the evolution of Morse sets, we modify the topology of the space. This is where we utilize the idea of disconnecting topology introduced in Section 1.4 with Theorem 1.4.11.

5.1 Persistence modules of weak Morse decomposition

Recall that $\mathcal{T}(\mathcal{A})$ stands for the disconnecting topology on $\bigcup\mathcal{A}$ induced by $\mathcal{T}$ (see Theorem 1.4.11). Let us begin this section with the following theorem.

Theorem 5.1.1. [7, Theorem 5.2] Let X and Y be finite topological spaces. Let $\mathcal{A}$ (respectively $\mathcal{B}$) be a family of mutually disjoint subsets of X (respectively Y). Assume that a continuous map $f : X \to Y$ inscribes $\mathcal{A}$ and $\mathcal{B}$, that is $f(\mathcal{A}) \sqsubset \mathcal{B}$, where $f(\mathcal{A}) := \{\, f(A) \mid A \in \mathcal{A}\,\}$. Then, the map

$$f_{\mathcal{A},\mathcal{B}} : \left(\bigcup\mathcal{A}, \mathcal{T}(\mathcal{A})\right) \ni x \mapsto f(x) \in \left(\bigcup\mathcal{B}, \mathcal{T}(\mathcal{B})\right)$$

is well defined and continuous.

Theorem 5.1.1 is slightly more general than what we need. In particular, we are interested in the case where families $\mathcal{A}$ and $\mathcal{B}$ are weak Morse decompositions. We recall that $\mathcal{M}^+(\mathcal{V})$ stands for the minimal weak Morse decomposition of $\mathcal{V}$ and by Corollary 4.4.8 we have $\bigcup \mathcal{M}^+(\mathcal{V}) = X$.

Corollary 5.1.2. [7, Corollary 5.3] Consider two finite topological spaces X and Y with corresponding combinatorial multivector fields $\mathcal{V}$ and $\mathcal{W}$. Then the map

$$f_{\mathcal{M}^+(\mathcal{V}),\mathcal{M}^+(\mathcal{W})} : \left(\bigcup \mathcal{M}^+(\mathcal{V}), \mathcal{T}(\mathcal{M}(\mathcal{V}))\right) \ni x \mapsto f(x) \in \left(\bigcup \mathcal{M}^+(\mathcal{W}), \mathcal{T}(\mathcal{M}(\mathcal{W}))\right)$$

is continuous under the assumption that $f \circ \Pi_{\mathcal{V}} \subset \Pi_{\mathcal{W}} \circ f$, that is, $f(\Pi_{\mathcal{V}}(x)) \subset \Pi_{\mathcal{W}}(f(x))$ for every $x \in X$.

Let $(X^i, \mathcal{T}^i)_{i=1}^n$ be a family of finite topological spaces. Assume $\mathcal{V}_i$ is a combinatorial multivector field on X^i with the weak minimal Morse decomposition $\mathcal{M}_i := \mathcal{M}^+(\mathcal{V}_i)$. Let $(f_i : X^i \to X^{i+1})_{i=1}^{n-1}$ be a sequence of continuous maps such that

$$f_i \circ \Pi_{\mathcal{V}_i} \subseteq \Pi_{\mathcal{V}_{i+1}} \circ f_i \quad \text{and} \quad f_i(\mathcal{M}_i) \sqsubseteq \mathcal{M}_{i+1}.$$

Then we have continuous maps

$$\bar{f}_i := (f_i)_{\mathcal{M}_i,\mathcal{M}_{i+1}} : \left(X^i, \mathcal{T}^i(\mathcal{M}_i)\right) \to \left(X^{i+1}, \mathcal{T}^{i+1}(\mathcal{M}_{i+1})\right) \tag{5.1}$$

that induce homomorphisms in singular homology

$$H(\bar{f}_i) : H\left(X^i, \mathcal{T}^i(\mathcal{M}_i)\right) \to H\left(X^{i+1}, \mathcal{T}^{i+1}(\mathcal{M}_{i+1})\right)$$

which yield the *persistence module of the weak Morse decomposition*

$$H\left(X^1, \mathcal{T}^1(\mathcal{M}_1)\right) \xrightarrow{H(\bar{f}_1)} H\left(X^2, \mathcal{T}^2(\mathcal{M}_2)\right) \xrightarrow{H(\bar{f}_2)} \dots \xrightarrow{H(\bar{f}_{n-1})} H\left(X^n, \mathcal{T}^n(\mathcal{M}_n)\right)$$

We refer to the persistence barcode of this module as the *persistence barcode of the minimal weak Morse decomposition*. Moreover, we can replace some of the maps from the sequence $\{f_i\}$ with backwards maps, that is $f_i : X^{i+1} \to X^i$, which satisfy

$$f_i \circ \Pi_{\mathcal{V}_{i+1}} \subseteq \Pi_{\mathcal{V}_i} \circ f_i \quad \text{and} \quad f_i(\mathcal{M}_{i+1}) \sqsubseteq \mathcal{M}_i$$

and induces

$$\bar{f}_i := (f_i)_{\mathcal{M}_{i+1},\mathcal{M}_i} : \left(X^{i+1}, \mathcal{T}^{i+1}(\mathcal{M}_{i+1})\right) \to \left(X^i, \mathcal{T}^i(\mathcal{M}_i)\right). \tag{5.2}$$

Then we get the *zigzag persistence module of the minimal weak Morse decomposition*

$$H\left(X^1, \mathcal{T}^1(\mathcal{M}_1)\right) \xleftrightarrow{H(\bar{f}_1)} H\left(X^2, \mathcal{T}^2(\mathcal{M}_2)\right) \xleftrightarrow{H(\bar{f}_2)} \dots \xleftrightarrow{H(\bar{f}_{n-1})} H\left(X^n, \mathcal{T}^n(\mathcal{M}_n)\right) \tag{5.3}$$

where the direction of $H(\bar{f}_i)$ depends on the direction of an underlying map f_i.

The next step is to guarantee that we can always construct continuous maps (5.1) and (5.2) to build a persistence module for any multivector fields sequence. Note that $f^{-1}(\mathcal{W})$ and $\mathcal{V} \cap f^{-1}(\mathcal{W})$ are proper multivector fields by Proposition 4.1.2. To simplify further the notation we put $\mathcal{T}_{\mathcal{V}} := \mathcal{T}(\mathcal{M}^+(\mathcal{V}))$.

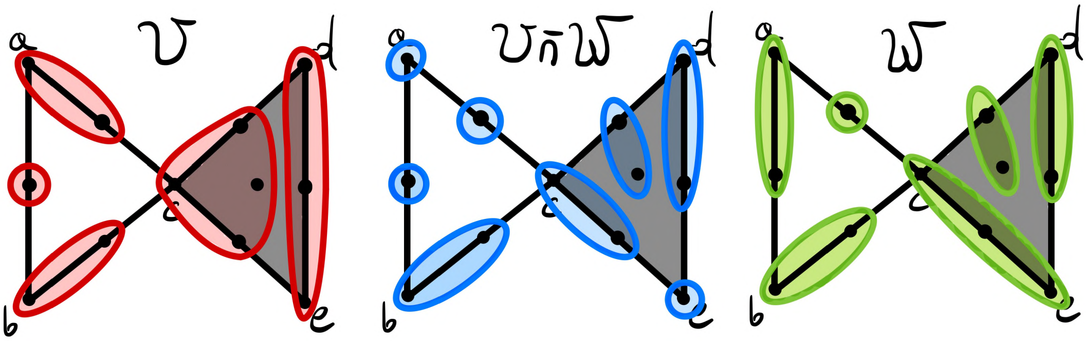


Figure 5.1: Example of two multivector fields $\mathcal{V}$ and $\mathcal{W}$, and the third multivector field $\mathcal{V}\bar{\cap}\mathcal{W}$ obtained by the intersection (middle). Minimal Morse decomposition for each multivector field are $\mathcal{M}(\mathcal{V}) = \{\{ab\}, \{d, e, de\}\}$, $\mathcal{M}(\mathcal{W}) = \{\{ac\}, \{c, e, ce\}\}$ and $\mathcal{M}(\mathcal{V}\bar{\cap}\mathcal{W}) = \{\{a\}, \{c\}, \{ab\}, \{ac\}\}$.

Proposition 5.1.3. [7, Theorem 5.4 (iv)] Let $\mathcal{V}$ be a multivector field on finite space X and $\mathcal{W}$ be a multivector field on Y. Let $f : X \to Y$ be a continuous map. Then the map induced by the identity

$$\kappa := \mathrm{id}_{\mathcal{V}\bar{\cap}f^{-1}(\mathcal{W}),\mathcal{V}} : \left(X, \mathcal{T}_{\mathcal{V}\bar{\cap}f^{-1}(\mathcal{W})}\right) \to (X, \mathcal{T}_{\mathcal{V}})$$

and the map induced by f

$$\lambda := f_{\mathcal{V}\bar{\cap}f^{-1}(\mathcal{W}),\mathcal{W}} : \left(X, \mathcal{T}_{\mathcal{V}\bar{\cap}f^{-1}(\mathcal{W})}\right) \to (Y, \mathcal{T}_{\mathcal{W}})$$

are both continuous.

We can now explain why weak Morse decomposition is useful. For simplicity assume that f is identity. If we consider two multivector fields $\mathcal{V}$ and $\mathcal{W}$ on X, we can not guarantee that any of the following inclusions hold for Morse decomposition:

$$\bigcup \mathcal{M}(\mathcal{V}) \subset \bigcup \mathcal{M}(\mathcal{V}\bar{\cap}\mathcal{W}),$$
$$\bigcup \mathcal{M}(\mathcal{V}\bar{\cap}\mathcal{W}) \subset \bigcup \mathcal{M}(\mathcal{V}).$$

Figure 5.1 shows an example when both inclusions fail simultaneously. Consequently, maps κ and λ from Proposition 5.1.3 will not be well defined. However, if we consider weak Morse decomposition, the problem disappears because then the union of all weak Morse sets will always give the entire space.

Now, using Proposition 5.1.3 we obtain the *comparison diagram* between weak Morse decompositions for arbitrary multivector fields $\mathcal{V}$ and $\mathcal{W}$:

$$(X, \mathcal{T}_{\mathcal{V}}) \xleftarrow{\kappa} \left(X, \mathcal{T}_{\mathcal{V}\cap f^{-1}(\mathcal{W})}\right) \xrightarrow{\lambda} (Y, \mathcal{T}_{\mathcal{W}}).$$

Thus, assume a sequence of finite topological spaces $(X^i, \mathcal{T}^i)_{i=1}^n$ with corresponding combinatorial multivector fields $\mathcal{V}_i$ and connecting continuous maps $f_i : X^i \to X^{i+1}$ are given. Then the diagram

$$\left(X^1,\mathcal{T}^1_{\mathcal{V}_1}\right) \xleftarrow{\kappa_1} \left(X^1,\mathcal{T}^1_{\mathcal{V}_1 \bar{\cap} f_1^{-1}(\mathcal{V}_2)}\right) \xrightarrow{\lambda_1} \left(X^2,\mathcal{T}^2_{\mathcal{V}_2}\right) \xleftarrow{\kappa_2} \left(X^2,\mathcal{T}^2_{\mathcal{V}_2 \bar{\cap} f_2^{-1}(\mathcal{V}_3)}\right) \xrightarrow{\lambda_2} \cdots$$

$$\cdots \xrightarrow{\lambda_{n-2}} \left(X^{n-1},\mathcal{T}^{n-1}_{\mathcal{V}_{n-1}}\right) \xleftarrow{\kappa_{n-1}} \left(X^{n-1},\mathcal{T}^{n-1}_{\mathcal{V}_{n-1} \bar{\cap} f_{n-1}^{-1}(\mathcal{V}_n)}\right) \xrightarrow{\lambda_{n-1}} \left(X^n,\mathcal{T}^n_{\mathcal{V}_n}\right)$$

after applying singular homology functor produces the following zigzag persistence module

$$H\left(X^1,\mathcal{T}^1_{\mathcal{V}_1}\right) \xleftarrow{H(\kappa_1)} H\left(X^1,\mathcal{T}^1_{\mathcal{V}_1 \bar{\cap} f_1^{-1}(\mathcal{V}_2)}\right) \xrightarrow{H(\lambda_1)} H\left(X^2,\mathcal{T}^2_{\mathcal{V}_2}\right) \xleftarrow{H(\kappa_2)} \cdots$$

$$\cdots \xrightarrow{H(\lambda_{n-2})} H\left(X^{n-1},\mathcal{T}^{n-1}_{\mathcal{V}_{n-1}}\right) \xleftarrow{H(\kappa_{n-1})} H\left(X^{n-1},\mathcal{T}^{n-1}_{\mathcal{V}_{n-1} \bar{\cap} f_{n-1}^{-1}(\mathcal{V}_n)}\right) \xrightarrow{H(\lambda_{n-1})} H\left(X^n,\mathcal{T}^n_{\mathcal{V}_n}\right).$$

5.2 Persistence modules of Morse decomposition

The results of the previous section show how to track all the weak Morse sets simultaneously. However, in practice, since every regular multivector may be a weak Morse set on its own, that strategy can lead to an excessive number of tracked components. In order to focus our analysis only on sets that carry the qualitative information about the flow, we relax our assumptions. In particular, we do not consider the components of the intermediate spaces to be Morse sets.

Let $\mathcal{V}$ be a multivector field on a finite topological space $(X,\mathcal{T})$. We denote the union of all Morse sets as $D_{\mathcal{V}} := \bigcup \mathcal{M}(\mathcal{V})$. We endow set $D_{\mathcal{V}}$ with the disconnecting topology $\mathcal{T}_{\mathcal{V}} := \mathcal{T}^X(\mathcal{M}(\mathcal{V}))$.

Proposition 5.2.1. Let $\mathcal{V}$ be a multivector field on a finite topological space $(X,\mathcal{T}^X)$ and let $\mathcal{W}$ be a multivector field on $(Y,\mathcal{T}^Y)$. Assume that $f:(X,\mathcal{T}^X)\to(Y,\mathcal{T}^Y)$ is a continuous map. Denote $D_{\mathcal{V},\mathcal{W}} := D_{\mathcal{V}} \cap f^{-1}(D_{\mathcal{W}})$ and $\mathcal{T}_{\mathcal{V},\mathcal{W}} := \mathcal{T}_{\mathcal{V}}\left(f^{-1}(\mathcal{M}(\mathcal{W}))\right)$ – the disconnecting topology induced by $\mathcal{T}_{\mathcal{V}}$ and the counterimages of Morse sets for $\mathcal{W}$. The map induced by the identity

$$\beta := \mathrm{id}_{\mathcal{V},\mathcal{W}} : (D_{\mathcal{V},\mathcal{W}},\mathcal{T}_{\mathcal{V},\mathcal{W}}) \to (D_{\mathcal{V}},\mathcal{T}_{\mathcal{V}})$$

and the map induced by f

$$\gamma := f_{\mathcal{V},\mathcal{W}} : (D_{\mathcal{V},\mathcal{W}},\mathcal{T}_{\mathcal{V},\mathcal{W}}) \to (D_{\mathcal{W}},\mathcal{T}_{\mathcal{W}})$$

are continuous.

Proof. Consider the diagram

$$(D_{\mathcal{V},\mathcal{W}},\mathcal{T}_{\mathcal{V},\mathcal{W}}) \xrightarrow{\ i\ } (D_{\mathcal{V},\mathcal{W}},\mathcal{T}_{\mathcal{V}}(D_{\mathcal{V},\mathcal{W}})) \xrightarrow{\ j\ } (D_{\mathcal{V}},\mathcal{T}_{\mathcal{V}})$$

where maps i and j are induced by the identity. Since $\mathcal{T}_{\mathcal{V}}(D_{\mathcal{V},\mathcal{W}})$ is the topology induced in $D_{\mathcal{V},\mathcal{W}}$ by $\mathcal{T}_{\mathcal{V}}$ and j is an inclusion of the subspace we get the continuity of j. By the construction, $\mathcal{T}_{\mathcal{V},\mathcal{W}}$ is a finer topology than $\mathcal{T}_{\mathcal{V}}(D_{\mathcal{V},\mathcal{W}})$. It follows by Proposition 1.4.1 that map i is also continuous. Hence, β is continuous as a composition of continuous maps.

Now, let $A \in \mathcal{T}_{\mathcal{W}}$. It follows that there exists a $U \in \mathcal{T}^Y$ such that $A = U \cap \bigcup \mathcal{M}(\mathcal{W})$. Since f is continuous we have $f^{-1}(U) \in \mathcal{T}^X$. Thus,

$$\gamma^{-1}(A) = f^{-1}(A) \cap D_{\mathcal{V},\mathcal{W}} = f^{-1}(U) \cap f^{-1}(D_{\mathcal{W}}) \cap D_{\mathcal{V},\mathcal{W}} = f^{-1}(U) \cap D_{\mathcal{V},\mathcal{W}} \in \mathcal{T}_{\mathcal{V},\mathcal{W}},$$

because $D_{\mathcal{V},\mathcal{W}} \subset f^{-1}(D_{\mathcal{W}})$. This proves that γ is continuous. $\square$

Using Proposition 5.2.1 we construct the *comparison diagram* between Morse decompositions for arbitrary multivector fields $\mathcal{V}$ and $\mathcal{W}$:

$$(D_{\mathcal{V}}, \mathcal{T}_{\mathcal{V}}) \xleftarrow{\ \beta\ } (D_{\mathcal{V},\mathcal{W}}, \mathcal{T}_{\mathcal{V},\mathcal{W}}) \xrightarrow{\ \gamma\ } (D_{\mathcal{W}}, \mathcal{T}_{\mathcal{W}})$$

Now, consider a sequence of finite topological spaces $(X^i, \mathcal{T}^i)_{i=1}^n$ with corresponding combinatorial multivector fields $\mathcal{V}_i$ and connecting continuous maps $f_i : X^i \to X^{i+1}$. We use the notation as in the Proposition 5.2.1. Then the diagram

$$(D_{\mathcal{V}_1}, \mathcal{T}_{\mathcal{V}_1}) \xleftarrow{\beta_1} (D_{\mathcal{V}_1,\mathcal{V}_2}, \mathcal{T}_{\mathcal{V}_1,\mathcal{V}_2}) \xrightarrow{\ \gamma_1\ } (D_{\mathcal{V}_2}, \mathcal{T}_{\mathcal{V}_2}) \xleftarrow{\quad \beta_2 \quad} (D_{\mathcal{V}_2,\mathcal{V}_3}, \mathcal{T}_{\mathcal{V}_2,\mathcal{V}_3}) \xrightarrow{\quad \gamma_2 \quad} \cdots$$

$$\cdots \xrightarrow{\ \gamma_{n-2}\ } (D_{\mathcal{V}_{n-1}}, \mathcal{T}_{\mathcal{V}_{n-1}}) \xleftarrow{\beta_{n-1}} (D_{\mathcal{V}_{n-1},\mathcal{V}_n}, \mathcal{T}_{\mathcal{V}_{n-1},\mathcal{V}_n}) \xrightarrow{\gamma_{n-1}} (D_{\mathcal{V}_n}, \mathcal{T}_{\mathcal{V}_n})$$

after applying singular homology functor produces the following *zigzag persistence module of the minimal Morse decomposition*

$$H\left(D_{\mathcal{V}_1}, \mathcal{T}_{\mathcal{V}_1}\right) \xleftarrow{H(\beta_1)} H\left(D_{\mathcal{V}_1,\mathcal{V}_2}, \mathcal{T}_{\mathcal{V}_1,\mathcal{V}_2}\right) \xrightarrow{\ H(\gamma_1)\ } H\left(D_{\mathcal{V}_2}, \mathcal{T}_{\mathcal{V}_2}\right) \xleftarrow{\quad H(\beta_2) \quad} \cdots$$

$$\cdots \xrightarrow{H(\gamma_{n-2})} H\left(D_{\mathcal{V}_{n-1}}, \mathcal{T}_{\mathcal{V}_{n-1}}\right) \xleftarrow{H(\beta_{n-1})} H\left(D_{\mathcal{V}_{n-1},\mathcal{V}_n}, \mathcal{T}_{\mathcal{V}_{n-1},\mathcal{V}_n}\right) \xrightarrow{H(\gamma_{n-1})} H\left(D_{\mathcal{V}_n}, \mathcal{T}_{\mathcal{V}_n}\right).$$

Note that theorems in this and the previous section are more general than we need for computational purposes presented in the next chapter. In the next chapter, among others, we will study the behavior of a dynamical system with respect to the change of equation or algorithm parameter. Space X will stay fixed, and only a multivector field will be changing. Thus, $X = Y$ and the map $f : X \to Y$ will always be identity. Therefore, the assumption $f(\Pi_{\mathcal{V}}(x)) \subset \Pi_{\mathcal{W}}(f(x))$ from Section 5.1 simplifies to $\Pi_{\mathcal{V}}(x) \subset \Pi_{\mathcal{W}}(x)$. Thus, in terms of multivector fields it suffices to assume that $\mathcal{V}$ is a finer multivector field than $\mathcal{W}$, that is $\mathcal{V} \sqsubset \mathcal{W}$.

Note also that in this chapter, we study the homology of a (weak) Morse set M that is $H(M)$. This homology group coincides with the Conley index $\operatorname{Con}(\operatorname{Inv} M)$ only if M is an attractor. Nevertheless, $H(M)$ carries information on whether M contains an orbit or a fixed point, but we lose the information if it is an attracting or repelling set. Let us note that the study of the persistence of the Conley index itself is undertaken in [8].

Chapter 6

Numerical experiments

In this chapter we present some possible applications of the multivector field theory. Here, as data, we take vector clouds sampled from differential equations. Previous chapters provided us with the theoretical background. However, there is still a crucial gap that has not been covered yet. Namely, there is an open question, how to construct a multivector field from a vector cloud? There is no canonical answer to that. In [7] we introduces a greedy algorithm dependent on an angle parameter. A non-parametric algorithm is mentioned in [13]; however, its details are not published. In this thesis we focus on the greedy algorithm.

6.1 From a vector cloud to a multivector field

Table 6.1 presents the greedy algorithm CVCMF that computes a multivector field from a cloud of vectors. The input for the algorithm consists of:

- a simplicial complex K with vertices in a cloud of points $P = \{p_i \mid i = 1, 2, \ldots, n\} \subset \mathbb{R}^d$,
- the associated cloud of vectors $V := \{\vec{v_i} \mid i = 1, 2, \ldots, n\} \subset \mathbb{R}^d$ such that vector $\vec{v_i}$ originates from point p_i (vectors are assumed to be normalized),
- an angular parameter α.

The algorithm builds a map $m : K \to K$ that sends simplex to one of its co-faces. The family of preimages $\{m^{-1}(\sigma) : \sigma \in K\}$ provides the requested multivector field. The idea is to map simplices consistently with the input vectors. The angular parameter controls the algorithm's tendency to assign simplices to lower-dimensional simplices — the higher the parameter, the "flatter" a multivector field is. The map m is initialized as the identity map, and the algorithm consists of three main steps of refining it:

- The first loop assigns a simplex σ to a toplex in the star of σ pointed by the vector v_σ computed as a mean of vectors in vertices spanning σ. To determine this toplex, we attach v_σ to the barycenter of σ (see Figure 6.1).
- The second loop focuses on assigning vertices. It decides if a vector v_i assigned to p_i should be aligned with a lower-dimensional co-face. Namely, if a vector v_i is almost parallel to a subspace spanned by some co-face of p_i, then it is aligned with

1: **procedure** CVCMF[7](K, V, α)
2: $m \leftarrow$ an identity map $id_K : K \to K$. ▷ Initialization of a map
3: **for all** $\sigma \in K$ **do** ▷ Initial assignment to toplexes
4: $m(\sigma) \leftarrow$ any toplex in the star of σ pointed by mean of $\{\, \vec{v_i} \in V \mid p_i \preceq \sigma \,\}$
5: **for all** $i = 1, 2, \ldots, n$ **do** ▷ Aligns vectors
6: $S \leftarrow \{\, (\dim \sigma, \measuredangle(\sigma, \vec{v_i}), \sigma) \mid \sigma \in K$ and $p_i \preceq \sigma$ and $\measuredangle(\sigma, \vec{v_i}) \leq \alpha \,\}$
7: $S' \leftarrow$ sort S using lexicographical order on first two positions ▷ $(\dim, \measuredangle, \_)$
8: $(\_, \_, \sigma) \leftarrow$ first element of S'
9: $m[p_i] \leftarrow \sigma$
10: **for all** $\sigma \in K$ in descending dimension **do** ▷ Remove convexity conflicts
11: **while** exists $\tau \preceq \sigma$ s.t. $\tau \preceq \sigma \preceq m(\tau)$ and $m(\tau) \neq m(\sigma)$ **do**
12: $m(\tau) \leftarrow \sigma$ and $m(\sigma) \leftarrow \sigma$
13: $\mathcal{V} \leftarrow \{m^{-1}(\sigma) \mid \sigma \in K\}$
14: **return** $\mathcal{V}$.

Table 6.1: An algorithm constructing a combinatorial multivector field from a sampled vector field. By $\measuredangle(\sigma, \vec{v_i})$ we denote the angle between a vector $\vec{v_i}$ and the hyperplane spanned by a simplex σ. Notation $(\_, \_, \sigma)$ means that we only retrieve the third element of the returned triple.

this subspace. The "almost" is determined by the angular parameter α, that is v_i is flattened to τ if the angle between the vector $\vec{v_i}$ and the hyperplane spanned by simplex σ is less than α. (see Figure 6.2).

- The last loop handles the convexity issue. If there is a simplex σ such that $\tau < \sigma < m(\tau)$ then both simplices are remapped to σ. In particular, this operation deals with conflicts presented in Figure 6.3.

Finally, we obtain the multivector field $\mathcal{V} = \{m^{-1}(\sigma) : \sigma \in K\}$. Note that since simplices can be assigned only to their co-faces, the algorithm produces a partition into multivectors with a unique maximal element. This means that the resulting multivector field satisfied the more restrictive definition of multivector field considered in [21].

Proposition 6.1.1. Algorithm presented in Table 6.1 terminates and produces a multivector field $\mathcal{V}$. Moreover, every multivector $V \in \mathcal{V}$ have a maximal element.

Proof. The first loop (line 3) iterates over all simplices assigning them a maximal-dimensional coface. Similarly, the second loop (line 5) iterates over all vertices. The value of $m(\sigma)$, where σ is a vertex, may be changed to some other coface of σ.

Note that the nested while-loop (line 11) terminates for any simplex $\sigma \in K$. In the extreme case it sets $m(\tau) = m(\sigma) = \sigma$ to all faces τ of σ. Thus, the third for-loop also terminates because it iterates over all simplices of K.

The goal of the third loop (line 10) is to make sure that

$$\forall_{\sigma,\tau \in K} \text{ such that } \tau \preceq \sigma \text{ either } m(\sigma) = m(\tau) \text{ or } \sigma \npreceq m(\tau). \tag{6.1}$$

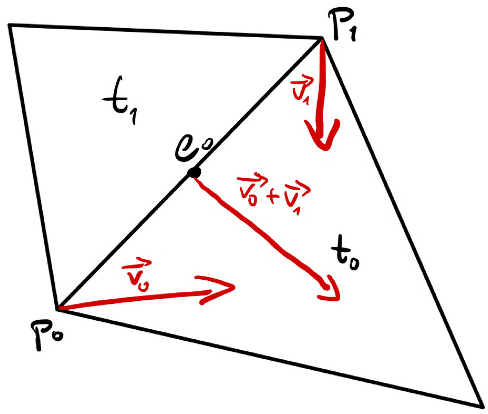


Figure 6.1: We attach vector $\vec{v}_0 + \vec{v}_1$ to the barycenter of e_0 in order to determine the value of $m(e_0)$ in the first loop of the algorithm. The pointed toplex in this example is t_0.

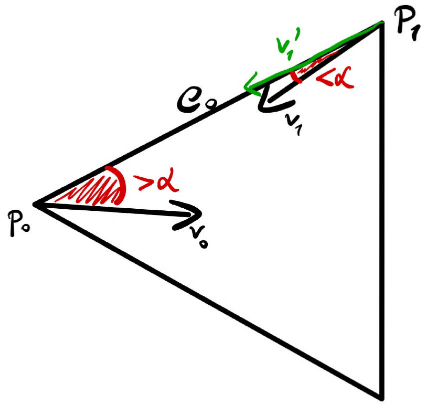


Figure 6.2: The vectors originating in vertices are flattened out if they are almost aligned with a lower dimensional simplex. In this case, vector $\vec{v}_1$ is flattened to e_0 because $\measuredangle(\vec{v}_1, e_0) \leq \alpha$, where α is an angular parameter.

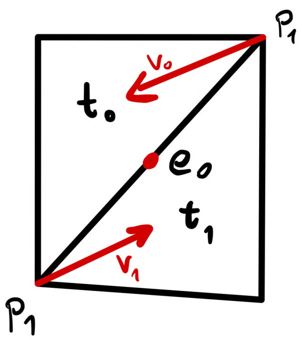


Figure 6.3: Third loop of the algorithm resolves convexity conflicts. Consider the case as in the figure with a map $m(p_0) = t_0$, $m(p_1) = t_1$, and $m(e_0) = t_1$. We will get $p_0 \prec e_0 \prec m(v_0) = t_0$ that will be resolved by setting $m(p_0) = m(e_0) = e_0$. In the next step we will also set $m(p_1) = m(e_0) = e_0$.

We will show that (6.1) is achieved after the third loop is finished. It processes simplices in descending order given by simplices' dimension. Consider $\sigma \in K$ and assume that $n := \dim \sigma = \dim K$. The condition (6.1) restricted to $K_{\{n\}}$ hold trivially. Since there is no simplex $\upsilon \in K$ such that $\sigma \prec \upsilon$ the conditions in line 11 are never satisfied. Consequently, no reassignment in line 12 of the algorithm is performed. Inductively, assume that $\dim \sigma = k < n$ and that (6.1) holds for all simplices in $K \setminus K_k$ (that is simplices of dimension higher than k). Let $\upsilon \in K \setminus K_k$ such that $\sigma \prec \upsilon$. The case when $\upsilon \preceq m(\sigma)$ would be resolved while processing υ. In particular the algorithm would set $m(\sigma) = m(\upsilon) = \upsilon$. Thus (6.1) holds for $K \setminus K_{k-1}$ before processing k-dimensional simplices. While processing k-dimensional simplex σ it may happen that the algorithm sets $m(\sigma) = \sigma$ if there exists some $\tau \prec \sigma$ such that $\sigma \prec m(\tau)$ and $m(\sigma) \neq m(\tau)$. Nevertheless, in that case, for any $\upsilon \in K \setminus K_k$ such that $\sigma \prec \upsilon$ we have $\sigma = m(\sigma) \prec \upsilon$. Hence, (6.1) is not disrupted for $K \setminus K_{k-1}$. This proves that after finishing the third loop the condition (6.1) is satisfied.

Let $\sigma \in K$. We will show that $V_\sigma := m^{-1}(\sigma)$ has maximal element if it is non-empty. At each step of the algorithm only faces of σ can be assigned to σ, that is $m(\tau) = \sigma$ where $\tau \preceq \sigma$. It follows that if $V_\sigma \neq \emptyset$ then for every $\tau \in V_\sigma$ we have $\tau \preceq \sigma$. Thus σ is maximal in V_σ.

The condition (6.1) guarantees that $\mathcal{V} := \{m^{-1}(\sigma) : \sigma \in K\}$ is a multivector field. To see that assume the contrary. Then there exists $\sigma \in K$ such that $m^{-1}(\sigma)$ is not convex. It follows that there exists $\tau, \upsilon \in K$ such that $\tau \prec \upsilon \preceq m(\tau) = \sigma$ and $m(\upsilon) \neq \sigma$ but this contradicts (6.1). □

6.2 Experimental setup

In the next section, we present examples of computations based on the developed theory. The first example presents the generation of the Morse-Conley graph from a sampled vector field. Thus, for a given simplicial complex K, sampled vector field V and angle parameter α, we use algorithm 6.1 to get multivector field $\mathcal{V}$. Then, according to results in Sections 4.4.1 and 4.4.2, it is enough to extract strongly connected components of the graph $G_\mathcal{V}$ and filter out those corresponding to a single regular multivector to obtain the minimal Morse decomposition for $\mathcal{V}$. The partial order on the collection of Morse sets is given by connections between them. Conley index for a Morse set M is computed using simplicial homology as $H^\Delta(\mathcal{K}(\operatorname{cl} M), \mathcal{K}(\operatorname{mo} M))$. We can summarize this as the following simple procedure

1: $\mathcal{V} \leftarrow$ CVCMF(K,V,α)
2: $(\mathcal{M}, G) \leftarrow$ MC-graph(K,$\mathcal{V}$)

In the first line, we use the algorithm presented in Table 6.1. The second line captures the procedure of obtaining Morse decomposition described above. It returns two elements: $\mathcal{M}$ – the obtained minimal Morse decomposition (family of subsets of K) and G – the corresponding Morse-Conley graph.

Experiments 2-4 present some of the possible applications of the persistence of Morse decomposition. To this end, we use the pipeline presented in Table 6.2. The input for this procedure is a simplicial complex K and a sequence of pairs $\{(V_i, \alpha_i)\}_{i=0}^{n-1}$ consisting of sampled vector fields and values of the angle parameter used by the greedy algorithm. In Experiments 2 and 4, we keep the angle parameter fixed across the sequence. In Experiment 3, we specifically test the influence of the angle parameter, and therefore the sampled vector field is fixed across the sequence. The first for-loop (line 3) of 6.2 computes the sequence of multivector fields $\mathcal{V}_i$, corresponding Morse decompositions $\mathcal{M}_i$ and McCord complex $\mathcal{X}_i$ corresponding to $D_{\mathcal{V}_i} = \bigcup \mathcal{M}(\mathcal{V}_i)$. In the second for-loop (line 7), we create McCord complexes corresponding to the intermediate spaces $D_{\mathcal{V},\mathcal{W}}$ (see Section 5.2). In particular, we need them for the comparison diagram between two consecutive Morse decompositions. Let us recall that $\mathcal{K}(K, \mathcal{T}_{\preceq}(\mathcal{M}))$ stands for the order complex of a finite topological space K (we consider simplicial complex as a finite topological space) with the disconnecting topology on K induced by the topology given by a face relation $\mathcal{T}_{\preceq}$ and $\mathcal{M}$. Finally, we compute the persistence barcodes for the sequence of complexes. The zigzag persistence is obtained with Dionysus software [20]. The greedy algorithm (6.1) and Morse-Conley graphs computations are part of RedHom library [14].

1: **procedure** MDPersistence(K, $\{(V_i, \alpha_i)\}_{i=0}^{n-1}$)
2: $\quad \mathcal{X}[2n+1] \leftarrow$ Initialize an empty array of simplicial complexes of size $2n+1$
3: $\quad$ **for all** $i = 0, 1, \ldots, n-1$ **do** $\qquad \triangleright$ Compute Morse decompositions
4: $\qquad \mathcal{V} \leftarrow$ CVCMF(K,V_i,α_i)
5: $\qquad (\mathcal{M}, \_) \leftarrow$ MC-graph(K,$\mathcal{V}$)
6: $\qquad \mathcal{X}[2i] \leftarrow \mathcal{K}(\bigcup \mathcal{M}, \mathcal{T}_{\preceq}(\mathcal{M}))$ $\qquad \triangleright$ Compute McCord complex
7: $\quad$ **for all** $i = 0, 1, \ldots, n-2$ **do** $\qquad \triangleright$ Compute the intermediate spaces
8: $\qquad \mathcal{X}[2i+1] \leftarrow \mathcal{X}[2i] \bar{\cap} \mathcal{X}[2i+2]$
9: $\quad$ PB $\leftarrow$ ZigZagPersistence($\{\mathcal{X}_i\}_{i=0}^{2n+1}$)
10: **return** PB.

Table 6.2: A pipeline for computing persistence of Morse decomposition for a given sequence of sampled vector fields.

We sample the following vector fields to obtain data for our experiments:

1. the double periodic orbit:

$$\begin{cases} x' = \quad cy - x(x^2+y^2-a)(x^2+y^2-b) \\ y' = -cx - y(x^2+y^2-a)(x^2+y^2-b) \end{cases}. \tag{6.2}$$

2. Lotka-Volterra prey-predator model (see [5, Chapter 2, equations (2.13) and (2.14)]):

$$\begin{cases} x' = x\left(1-\frac{x}{k}\right) - \frac{a_1 xy}{b+x} \\ y' = \frac{a_2 xy}{b+x} - gy \end{cases}, \tag{6.3}$$

where $a_1 = \left(1-\frac{1}{k}\right)(b+1)$ and $a_2 = g(b+1)$.

3. Sel'kov glycolysis model (see [23, Example 7.3.2 and 7.3.3]):

$$\begin{cases} x' = -x + ay + x^2y \\ y' = b - ay - x^2y \end{cases}. \tag{6.4}$$

We construct complexes used in the experiments by first building a regular triangular mesh, and then we perturb vertices. Next, for every vertex we compute the associated vector and we normalize it. This procedure guarantees a vaguely uniformly distributed cloud of vectors with some randomness introduced.

6.3 Experiments

6.3.1 Experiment 1: Morse-Conley graph

The first example presents a simple computation of the Morse-Conley graph for the multivector field derived from equation (6.2). The angular parameter α in the algorithm in 6.1 is set to 21 degrees, and the equations parameters are $a = 2$, $b = 1$ and $c = 1.4$.

The outcome is presented in Figure 6.4. We see that both periodic orbits and their Conley indices are correctly retrieved. The attracting point at the origin gets divided into two Morse sets: M_3 (green) and M_4 (purple) with the Conley index of an attracting periodic orbit and a repelling stationary point, respectively. At first glance, this may look incorrect. However, by merging M_3 and M_4 into a single Morse set $M_3 \cup M_4$ we get an isolated invariant set with the index of an attracting fixed point. If we zoom in and analyze the vector field close to the origin, we will notice a strong rotation (see Figure 6.5). With the given mesh resolution and the assumption that we do not know the real equation, the outcome with two Morse sets is acceptable.

6.3.2 Experiment 2: influence of noise

In the second experiment, we test the influence of noise on the Morse decomposition reconstruction. We analyze the system (6.2), and we set again the angle parameter to $\alpha = 21$. We fix a simplicial complex and we normalize vectors as before. Then, we perturb every vector by taking $v_i' = v_i + e_i * r$, where e_i is a randomly chosen vector such that $||e_i|| \leq 1$ and $r \in [0, 1]$ is a noise level. We get a sequence of vector fields by changing the value of r. In the next step we use the algorithm to build a sequence of multivector fields and then a sequence of Morse decompositions. We summarize the analysis with the persistence of this sequence.

Results are presented in Figure 6.6. As one can expect, the higher the noise level, the more artificial Morse sets emerge, and the less accurate reconstruction of the dynamic is. Below the noise level $r = 0.37$, we can see that results are consistent with the Morse decomposition obtained in the Experiment 1. There are four 0-dimensional generators responsible for four Morse sets. Since three of those sets are periodic orbits, we get three 1-dimensional generators. At $r = 0.4$ both periodic orbits merge together and split

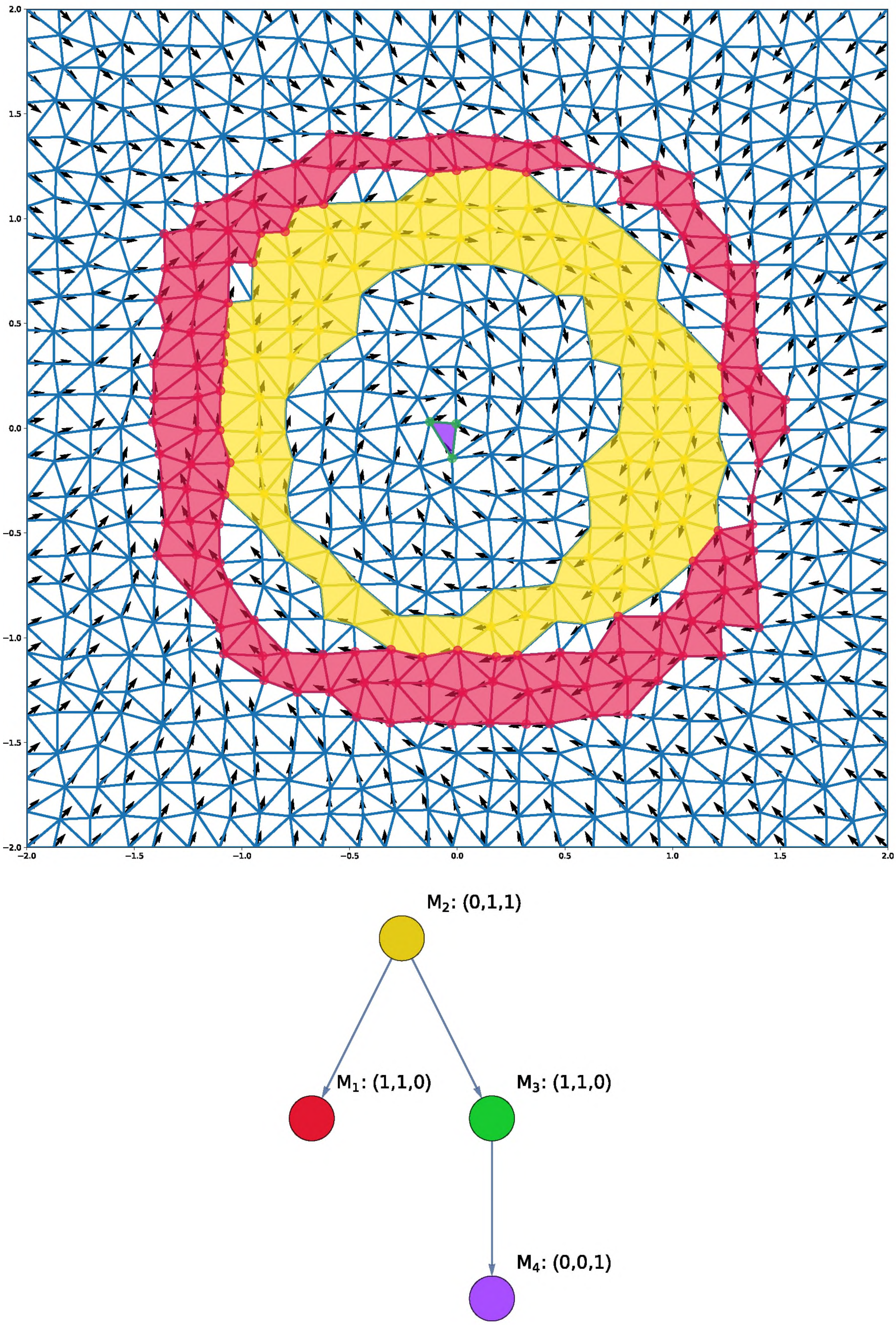


Figure 6.4: Experiment 1: Morse-Conley graph. Top: Four Morse sets detected by the algorithm. Bottom: computed Morse-Conley graph.

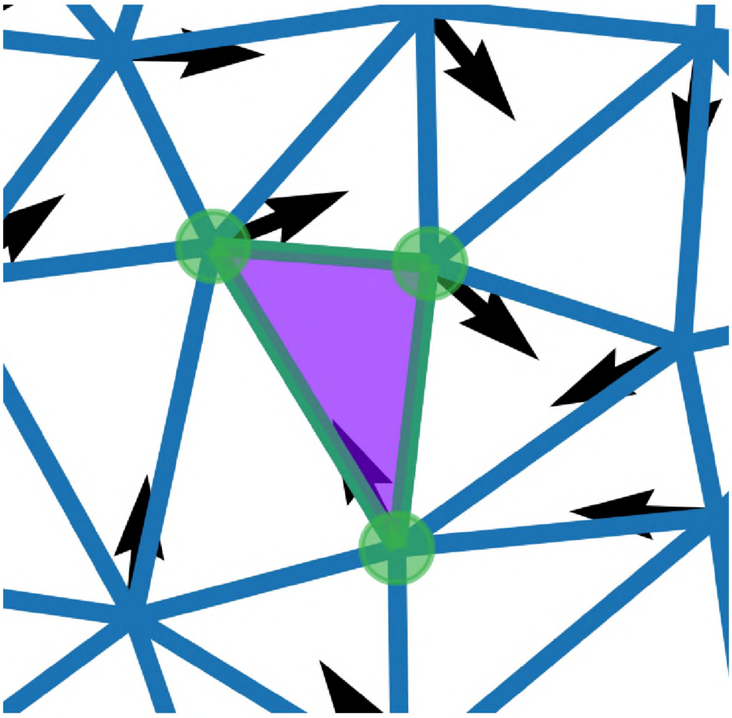


Figure 6.5: Zoom to the stationary attracting point for the example in Figure 6.4. Green simplices corresponds to Morse set M_3 (attracting periodic orbit) and the purple simplex corresponds to set M_4 (repelling stationary point). The given resolution and strong rotation around the attracting point do not facilitate the proper distinction of the attracting point.

again at $r = 0.45$. Above $r = 0.71$, the noise dominates, and one of the periodic orbits disappears completely. The additional "hole" (1st-dimensional generator) emerges above $r = 0.75$ because a new critical cell is created within a periodic Morse set.

The outcome of this experiment would vary significantly depending on the random seed. We use it just to demonstrate the noise impact.

6.3.3 Experiment 3: angle parameter

In the third experiment, we study the Lotka-Volterra model (6.3) with parameters $a = 3.9$, $b = 1.2$, and $c = 0.5$. We vary the angle parameter from 0 to 40 in steps of 2 to obtain another Morse decompositions sequence. Using persistence, we try to analyze the robustness of Morse sets. Moreover, we can observe the algorithm's behavior depending on the angular parameter α.

The barcode (Figure 6.7, bottom-right) shows that three generators, two 0-dimensional and one 1-dimensional, survive the entire filtration. Indeed, they correspond to the repelling stationary point in the center and the attracting periodic orbit. We can see other short-living orbits in the intermediate steps (Figure 6.7, top-right and middle-right). However, they have a trivial Conley index, which confirms them as the algorithm's artifacts.

Note that the small values of the angle parameter produce a thin and more precise orbit. However, it also increases the chances of producing small, critical multivectors -

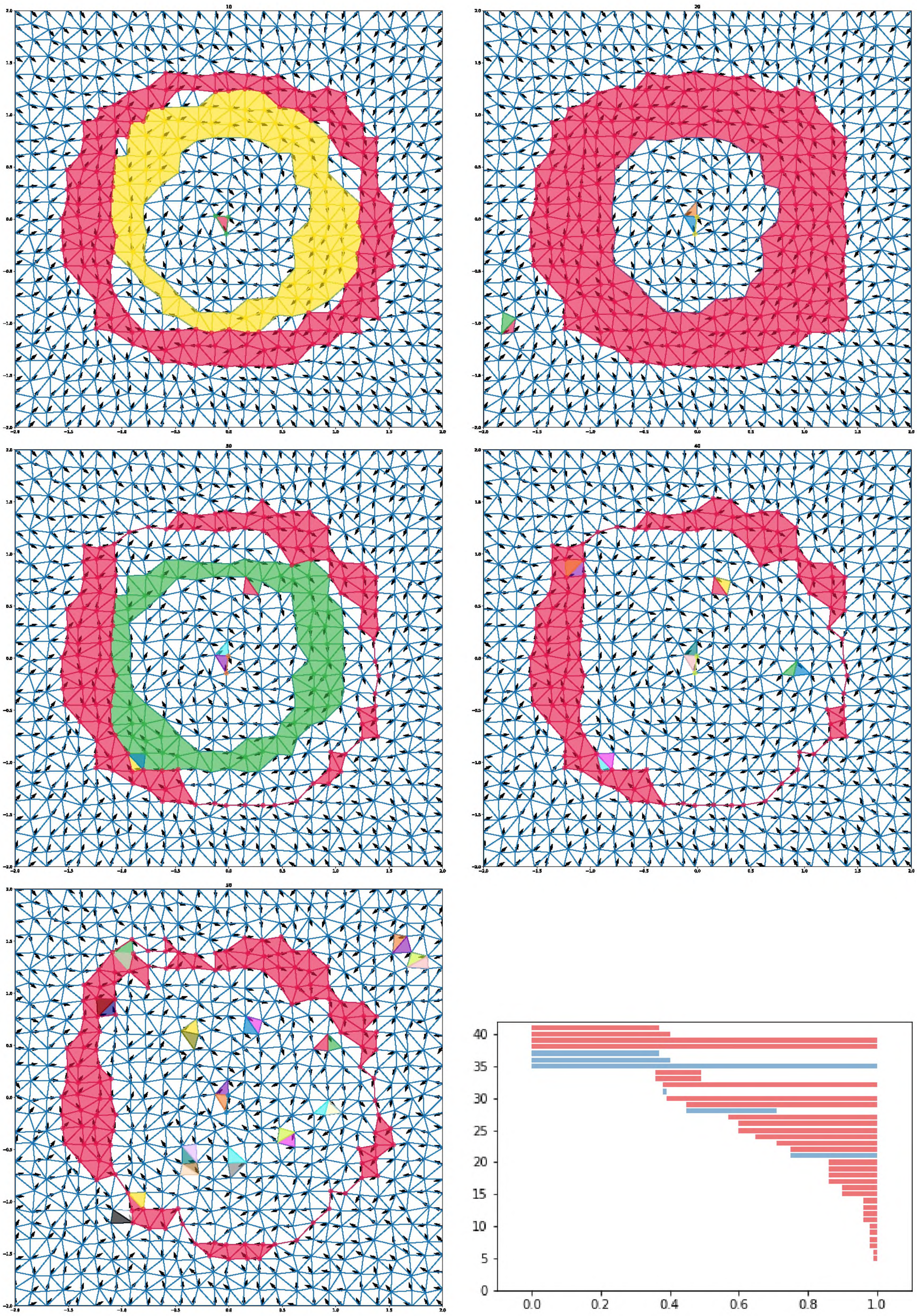


Figure 6.6: Experiment 2: noise influence. Morse sets detected by the algorithm for the double periodic orbit equation (6.2). Noise level by rows starting from top-left: $r = 0.2$, $r = 0.4$, $r = 0.6$, $r = 0.8$, $r = 1.0$. Bottom-right: persistence barcode for Experiment 2 with coordinates on X-axis given by the noise level; red bars correspond to persistence pair of dimension 0, while blue bars correspond to the 1st dimension.

artifacts, as in the center and the bottom left part of the top-right panel in Figure 6.7. On the other hand, increasing the parameter leads to more expansive multivector fields, where the orbit is highly overestimated.

6.3.4 Experiment 4: Hopf bifurcation

In the last experiment, we study the evolution of a system going through the Hopf bifurcation. The Hopf bifurcation refers to the situation where a periodic orbit emerges from a stationary point [23, Chapter 8.2]. For a fixed parameter $a = 0.08$, Syl'kov system (equation 6.4) admits an attracting limit cycle for approximately $b \in (0.346, 0.848)$ with a repelling stationary point in the middle [23, Example 7.3.3]. At the boundary values of this interval, the orbit collapses into an attracting fixed point. Thus, sliding the parameter b from 0 to 1.3, we should observe the Hopf bifurcation twice.

A few steps of the filtration for selected values of a are presented in Figure 6.8. In the barcode, we can see a long single 0-dimensional generator that survives most of the filtration and a single 1-dimensional interval $[0.425, 0.95]$. Both of them are responsible for a detected periodic orbit. The 1-dimensional generator persists for a shorter time because as the continuous vector field evolves, the multivector field generated by the algorithm changes in a discrete fashion. For some transitions, orbits may differ significantly. As a result, the intermediate multivector fields created for the comparison diagram (see Chapter 5) may fail to detect continuation of an orbit. Nevertheless, the interval in which the orbit is the most stable, approximately coincides with the expectations from the analytical computations. Moreover, for the higher values of the parameter a, where we know that the periodic orbit does not exists, the rotation still dominates the attraction. Consequently, we can still observe a Morse set with the Conley index of an attracting orbit (Figure 6.8, bottom-left). This is a similar case to the one observed in Experiment 1.

Note that this framework could be used to analyze the evolution of a dynamical system in time. The X-axis of the barcode may be interpreted as a timeline, and persistent barcodes represent the long-lasting isolated invariant sets that form during the evolution.

6.4 Further research

Experiment 3 shows that the angle parameter used by algorithm 6.1 can significantly affect the results. The low values of the parameter lead to large multivectors containing maximal-dimensional simplices. Such a multivector field tends to create an expanding combinatorial flow, which may highly overestimate the real flow as presented in Figure 6.9. On the other hand, too aggressive aligning (i.e., small values of the angular parameter) may fail to detect periodic orbits or create artificial critical multivectors. Figure 6.10 shows an example where aligning vectors to lower-dimensional simplices misinterprets a simple passing-through flow.

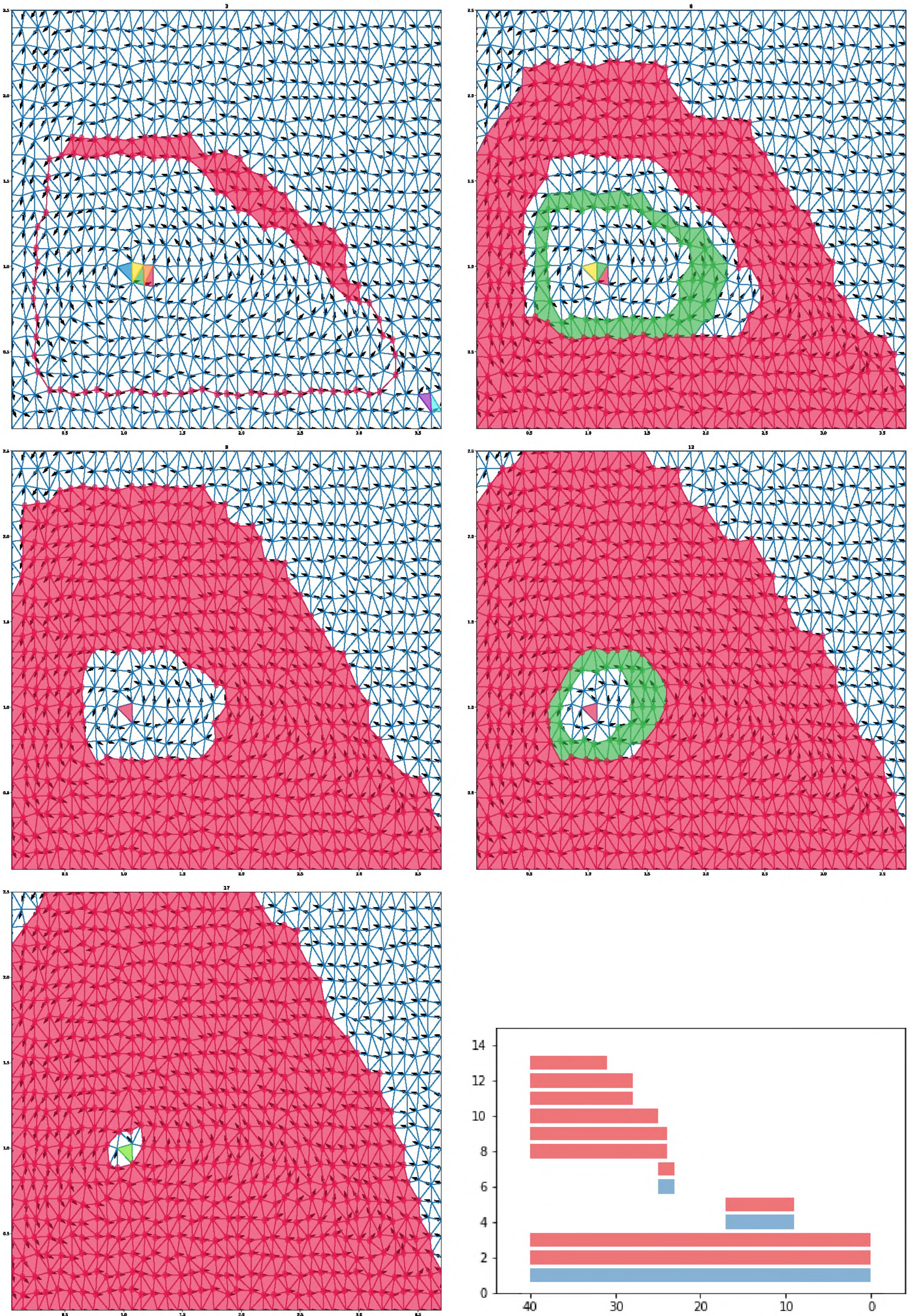


Figure 6.7: Experiment 3: angular parameter. Morse sets detected by the algorithm for the Lotka-Volterra model (6.3) for different values of angular parameter. Starting from top-left: $\alpha = 34$, $\alpha = 24$, $\alpha = 18$, $\alpha = 16$, $\alpha = 6$. Bottom-right: persistence barcode for Experiment 3 with coordinates on X-axis given by values of the angular parameter; red bars correspond to persistence pair of dimension 0, while blue bars correspond to the 1st dimension.

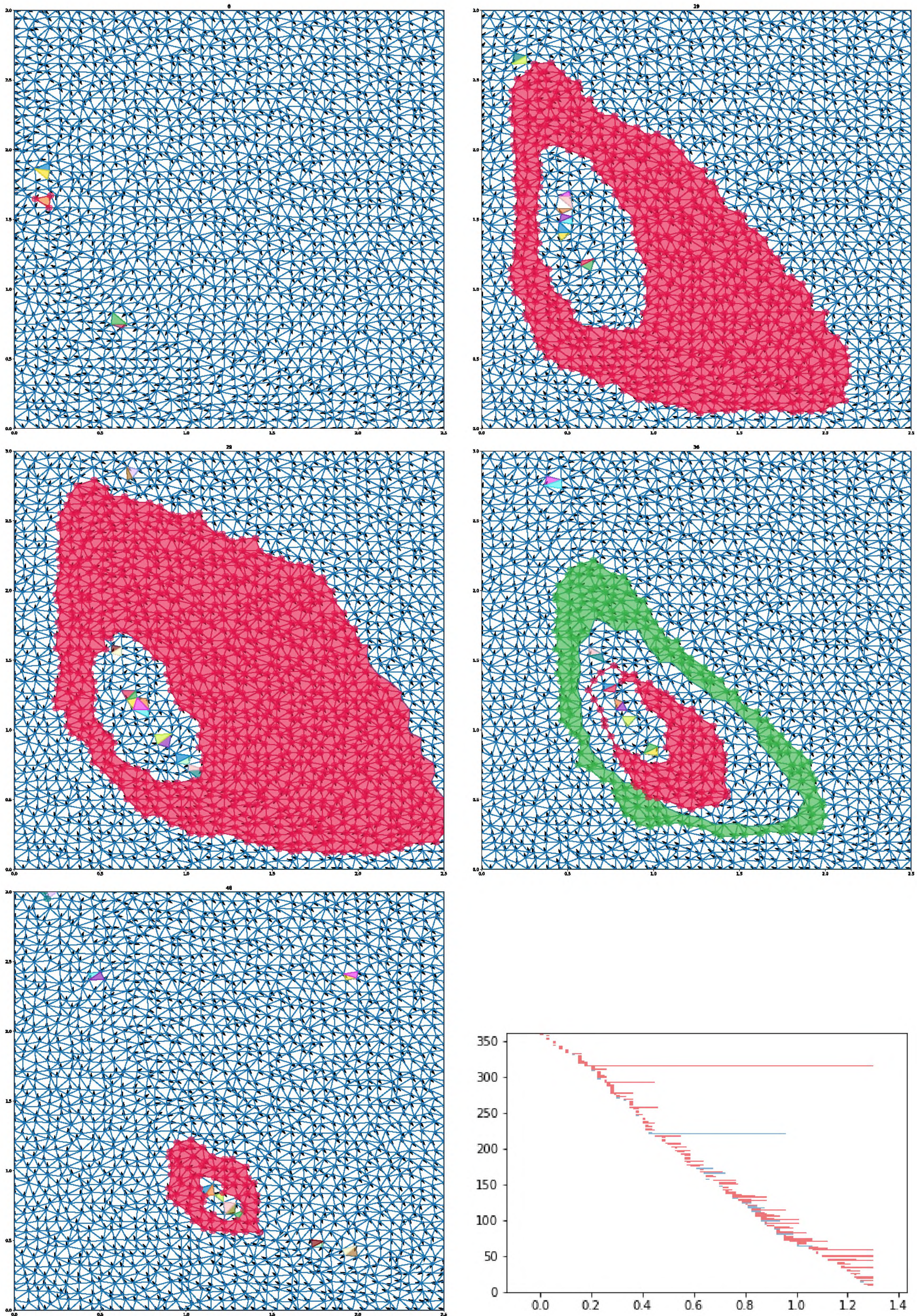


Figure 6.8: Experiment 4: Hopf bifurcation. Morse sets detected by the algorithm for the Sel'kov model (6.4) for different values of parameter a. Starting from top-left: $a = 0.225$, $a = 0.5$, $a = 0.75$, $a = 0.925$, $a = 1.225$. Bottom-right: persistence barcode for Experiment 4 with coordinates on X-axis given by values of parameter a; red bars correspond to persistence pair of dimension 0, while blue bars correspond to the 1st dimension.

The above observations raise the question if a more efficient, adaptive and possibly parameterless algorithm for constructing a multivector field from data is possible. This would be especially useful in the framework of Experiment 4 where we can use the idea of persistence of Morse decomposition to study the evolution of a dynamical system in time.

Algorithm 6.1 creates a multivector field in the sense of [21]. Another natural step in developing the algorithm is a modification that will allow for constructing the generalized multivector fields. As explained in this thesis, it should provide more flexibility in modeling dynamical systems.

With the generalized theory of multivector fields and the tools it provides, it should be easier to investigate the topic of continuation in the combinatorial setting raised in [21] and mentioned briefly at the end of Section 3.2.2.

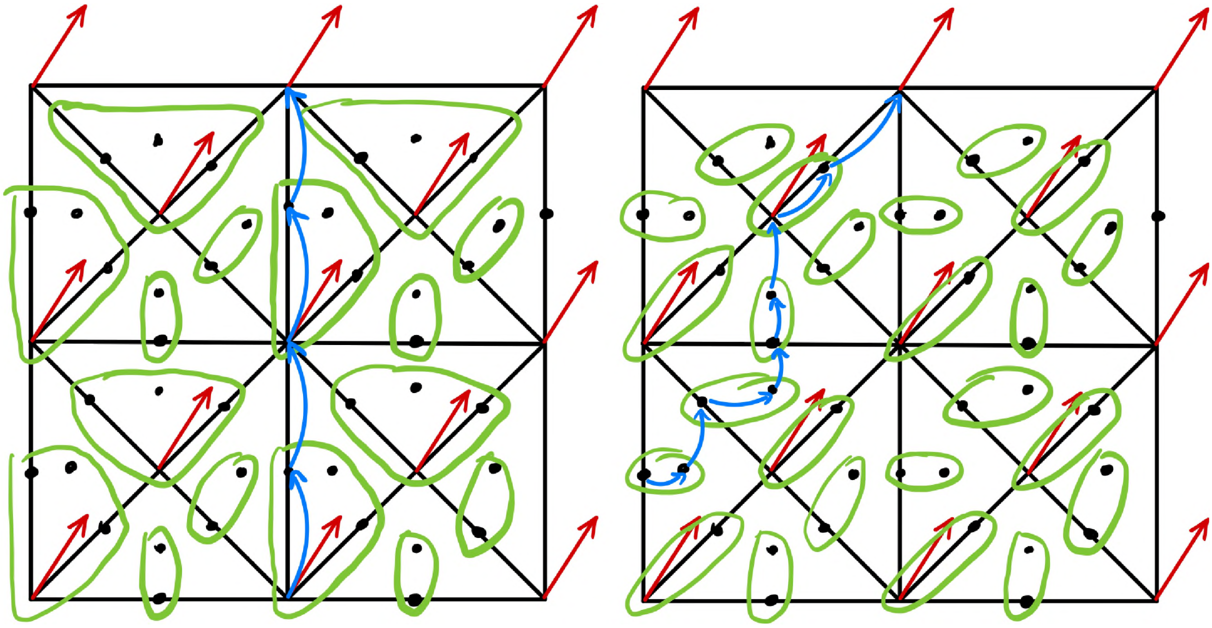


Figure 6.9: An effect of the expansion caused by large multivectors. Both pictures present a multivector field for a parallel flow. For low values of the angular parameter we get large multivectors that allow solution less consistent with the actual flow (blue, vertical trajectory). Flattened vectors (right) produce a more adequate representation.

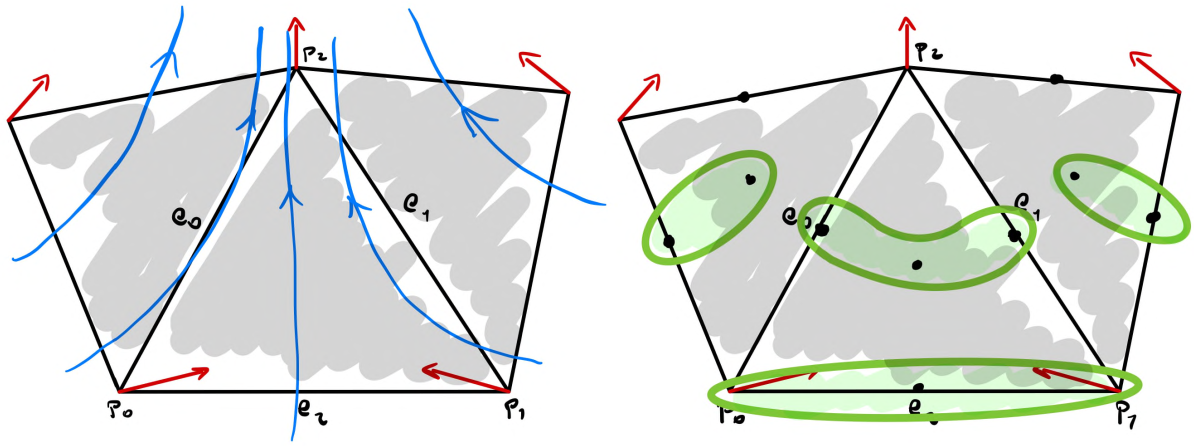


Figure 6.10: Artificial critical cells may be produced with too aggressive flattening of vectors. In the picture we can see a simple flow with no invariant sets, yet aligning vectors leads to creating an attractor and a saddle.

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