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EVALUATION OF SOME EXPLICIT SUMMATION FORMULAE FOR TRUNCATED GAUSS FUNCTION AND APPLICATIONS

M. I. QURESHI[1], S. JABEE[1], D. AHAMAD[1], §

Abstract. In this paper, some explicit expressions for truncated Gauss' hypergeometric series ${}_2F_1$ with argument 2, when one numerator and one denominator parameters are negative integers such as ${}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m$; $m > j$ and ${}_2F_1\left[\begin{array}{c} -m, a; \\ -2m-j; \end{array} 2\right]_m$; $m, j \in \mathbb{N}$ are obtained with the help of Whipple and Vidunas identities respectively. Further, we derive some results as applications of these summation theorems.

Keywords: Gauss' hypergeometric function, generalized hypergeometric function, truncated series.

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1. Introduction

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \dots\}$; $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$; $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0, -1, -2, -3, \dots\}$.
The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.
The Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ ([9, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [11, p.23, eq(22) and eq(23)]), is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1 & ; (p = 0; \alpha \in \mathbb{C} \setminus \{0\}) \\ \alpha(\alpha+1)\dots(\alpha+n-1) & ; (p = n \in \mathbb{N}; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ \frac{(-1)^n k!}{(k-n)!} & ; (\alpha = -k; p = n; n, k \in \mathbb{N}_0; 0 \le n \le k) \\ 0 & ; (\alpha = -k; p = n; n, k \in \mathbb{N}_0; n > k) \\ \frac{(-1)^n}{(1-\alpha)_n} & ; (p = -n; n \in \mathbb{N}; \alpha \in \mathbb{C} \setminus \mathbb{Z}), \end{cases} \tag{1}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

[1] Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia, New Delhi-110025, India.
e-mail: miqureshi_delhi@yahoo.co.in; ORCID: https://orcid.org/0000-0002-5093-1925.
e-mail: saimajabee007@gmail.com; ORCID: https://orcid.org/0000-0001-7564-8694.
e-mail: dlshdhmd4@gmail.com; ORCID: https://orcid.org/0000-0002-4583-483X.

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The generalized hypergeometric function ${}_pF_q$ ([9, Art.44, pp.73-74], see also [1]), is defined by

$$ {}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\dots,\alpha_p;\\ \\ \beta_1,\beta_2,\dots,\beta_q;\end{array} z\right] = {}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{2}$$

By convention, a product over the empty set is unity.
$\big(p,q\in\mathbb{N}_0;\ p\le q+1\ ;\ p\le q$ and $|z|<\infty;\ \ p=q+1$ and $|z|<1;\ p=q+1,|z|=1$ and $\Re(\omega)>0;\ p=q+1,|z|=1,z\ne 1$ and $-1<\Re(\omega)\le 0\big)$,
where

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

$$\big(\alpha_j\in\mathbb{C}\ (j=1,2,\dots,p);\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-(j=1,2,\dots,q)\big).$$

The truncated hypergeometric series is given by

$$\text{The sum of first } (m+1)-\text{terms of infinite series } {}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array} z\right]$$

$$\begin{aligned} &= {}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array} z\right]_m = \sum_{n=0}^{m}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}\\ &= \frac{[(\alpha_p)]_m z^m}{[(\beta_q)]_m m!}\,{}_{q+2}F_p\left[\begin{array}{c}-m,1-(\beta_q)-m,1;\\ \\ 1-(\alpha_p)-m;\end{array} \frac{(-1)^{p+q+1}}{z}\right], \end{aligned} \tag{3}$$

where $(\alpha_p),(\beta_q),1-(\alpha_p)-m,1-(\beta_q)-m\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ m\in\mathbb{N}_0$, and

$$[(\alpha_p)]_m = (\alpha_1)_m(\alpha_2)_m\dots(\alpha_p)_m = \prod_{i=1}^{p}(\alpha_i)_m = \prod_{i=1}^{p}\frac{\Gamma(\alpha_i+m)}{\Gamma(\alpha_i)} \tag{4}$$

with similar interpretation for others.
The terminating hypergeometric series (the hypergeometric polynomial) is given by

$$ {}_{p+1}F_q\left[\begin{array}{c}-m,(\alpha_p);\\ \\ (\beta_q);\end{array} z\right] = \frac{[(\alpha_p)]_m(-z)^m}{[(\beta_q)]_m}\,{}_{q+1}F_p\left[\begin{array}{c}-m,1-(\beta_q)-m;\\ \\ 1-(\alpha_p)-m;\end{array} \frac{(-1)^{p+q}}{z}\right], \tag{5}$$

where $(\alpha_p), (\beta_q), 1-(\alpha_p)-m, 1-(\beta_q)-m \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $m \in \mathbb{N}_0$.
In 1928, Whipple [14, p.89, eq.(8.41)] gave the following summation theorems

$$
\begin{aligned}
&{}_2F_1\left[\begin{array}{r} A, B; \\ C; \end{array} -1\right] \\
&\qquad = \frac{\Gamma(C)}{2\Gamma(A)} \sum_{k=0}^{\infty} \left\{ (-1)^k \frac{(C-A+B-1)_k}{k!} \frac{\Gamma\left(\frac{A+k}{2}\right)}{\Gamma\left(C-\frac{A}{2}+\frac{k}{2}\right)} \right\}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&{}_2F_1\left[\begin{array}{r} A, B; \\ C; \end{array} -1\right] \\
&\qquad = \frac{\Gamma(C)\Gamma(1-B)}{2\Gamma(A)\Gamma(C-A)} \sum_{k=0}^{\infty} \left\{ \frac{(A-B-C+1)_k}{k!} \frac{\Gamma\left(\frac{A+k}{2}\right)}{\Gamma\left(\frac{A+k+2-2B}{2}\right)} \right\},
\end{aligned} \tag{7}
$$

where $A, C, 1-B, C-A \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $\Re(C-A-B) > -1$.
In 2007, Choi-Rathie-Malani gave the following result (see for example [2, pp.1523-1524, eq.(2.2)])

$$
\begin{aligned}
&{}_2F_1\left[\begin{array}{r} A, B; \\ 1+A-B-P; \end{array} -1\right] \\
&\qquad = \frac{\Gamma(1+A-B-P)}{2\Gamma(A)} \sum_{r=0}^{P} \left\{ \binom{P}{r} \frac{\Gamma\left(\frac{r+A}{2}\right)}{\Gamma\left(\frac{r+A}{2}+1-B-P\right)} \right\},
\end{aligned} \tag{8}
$$

where $\Re(B) < \left(\frac{2-P}{2}\right); A, 1+A-B-P \in \mathbb{C}\setminus\mathbb{Z}_0^-; P \in \mathbb{N}_0$.
In 2011, Rakha and Rathie established the following result (see for example [10, p.828, Th.(4)])

$$
\begin{aligned}
&{}_2F_1\left[\begin{array}{r} A, B; \\ 1+A-B-P; \end{array} -1\right] \\
&\quad = \frac{2^{-A}\Gamma\left(\frac{1}{2}\right)\Gamma(1+A-B-P)}{\Gamma\left(\frac{1+A-P}{2}-B\right)\Gamma\left(\frac{A-P}{2}+1-B\right)} \sum_{r=0}^{P} \left\{ \binom{P}{r} \frac{\Gamma\left(\frac{A-P+r+1}{2}-B\right)}{\Gamma\left(\frac{1+A+r-P}{2}\right)} \right\},
\end{aligned} \tag{9}
$$

where $\Re(B) < \left(\frac{2-P}{2}\right); 1+A-B-P, 1+A-P-2B \in \mathbb{C}\setminus\mathbb{Z}_0^-; P \in \mathbb{N}_0$.
Pfaff-Kummer's Linear transformation [11, p.33, eq.(19)] is given by

$$
{}_2F_1\left[\begin{array}{r} \alpha, \beta; \\ \gamma; \end{array} z\right] = (1-z)^{-\alpha} {}_2F_1\left[\begin{array}{r} \alpha, \gamma-\beta; \\ \gamma; \end{array} \frac{-z}{1-z}\right], \tag{10}
$$

where $\alpha, \beta \in \mathbb{C}; \gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $|\arg(1-z)| < \pi$.

2. Ratio of Pochhammer symbols

Definition 2.1. For $m, k, \ell \in \mathbb{N}$ and $r \in \mathbb{N}_0$, we have the following results ([11, p.23, eq.(23)], see also [9, p.32,Q.No.8]) When $0 \le r \le m$, then

$$
\frac{(-m)_r}{(-m-k)_r} = \frac{m!(m+k-r)!}{(m-r)!(m+k)!}. \tag{11}
$$

When $m+1 \le r \le m+k$, then

$$
\frac{(-m)_r}{(-m-k)_r} = 0. \tag{12}
$$

When $r = m + k + \ell$, then

$$\frac{(-m)_r}{(-m-k)_r} = \frac{(-1)^k(\ell)_k}{(1+m)_k}. \tag{13}$$

The indeterminate form $\left(\frac{0}{0}\right)$ in (13) is evaluated by cancelling the common factors in numerator and denominator, when we apply the definition of Pochhammer symbol.
Note. When $\ell = 1$ in equation (13), we get

$$\frac{(-m)_{m+k+1}}{(-m-k)_{m+k+1}} = \frac{(-1)^k k!}{(1+m)_k}, \tag{14}$$

where $m, k \in \mathbb{N}$.

3. General series identity

Theorem 3.1. Suppose $\{\Phi(r)\}_{r=0}^{\infty}$ is a bounded sequence of arbitrary real and complex numbers. Then following general series identity holds true

$$\begin{aligned}\sum_{r=0}^{\infty} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} &= \sum_{r=0}^{m} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} + \\ &+ \frac{(-1)^k k! z^{m+k+1}}{(1+m)_k (2)_{m+k}} \sum_{r=0}^{\infty} \frac{(k+1)_r \Phi(r+m+k+1)}{(2+m+k)_r} \frac{z^r}{r!},\end{aligned} \tag{15}$$

provided that involved infinite series of right-hand side of equation (15) is absolutely convergent and $m, k \in \mathbb{N}$.
Proof. Consider the left-hand side of equation (15)

$$\begin{aligned}&\sum_{r=0}^{\infty} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} \\ &= \sum_{r=0}^{m} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} + \sum_{r=m+1}^{m+k} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} + \sum_{r=m+k+1}^{\infty} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!}.\end{aligned} \tag{16}$$

By the ratios of two Pochhammer's symbols given by (13), the second finite series of right hand side of equation (16) will be zero, and replace r by $r+m+k+1$ in the third infinite series of right hand side of equation (16), and after simplification, we get

$$\begin{aligned}\sum_{r=0}^{\infty} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} &= \sum_{r=0}^{m} \frac{(-m)_r \Phi(r)}{(-m-k)_r} \frac{z^r}{r!} + \\ &+ \frac{(-m)_{m+k+1} z^{m+k+1}}{(-m-k)_{m+k+1} (2)_{m+k}} \sum_{r=0}^{\infty} \frac{(k+1)_r \Phi(r+m+k+1)}{(2+m+k)_r} \frac{z^r}{r!}.\end{aligned} \tag{17}$$

Now using the result (14) in equation (17), we get the desired result (15).

Corollary 3.1. If we set $\Phi(r) = \frac{[(\alpha_p)]_r}{[(\beta_q)]_r} = \left(\prod_{i=1}^{p} (\alpha_i)_r\right) \left(\prod_{j=1}^{q} (\beta_j)_r\right)^{-1}$ in equation (15), we get

$$
{}_{p+1}F_{q+1}\left[\begin{array}{c}-m,(\alpha_p);\\-m-k,(\beta_q);\end{array}z\right]={}_{p+1}F_{q+1}\left[\begin{array}{c}-m,(\alpha_p);\\-m-k,(\beta_q);\end{array}z\right]_m+
$$
$$
+\frac{(-1)^k k! z^{m+k+1}[(\alpha_p)]_{m+k+1}}{(1+m)_k(2)_{m+k}[(\beta_q)]_{m+k+1}}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}k+1,(\alpha_p)+m+k+1;\\2+m+k,(\beta_q)+m+k+1;\end{array}z\right], \tag{18}
$$

where $\alpha_1,\alpha_2,\ldots,\alpha_p,\beta_1,\beta_2,\ldots,\beta_q\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $m,k\in\mathbb{N}$ and $p\le q+1$.
The hypergeometric series ${}_{p+1}F_{q+1}(.)$ of left-hand side of equation (18) is convergent in following cases:
(i) When $p\le q$, then ${}_{p+1}F_{q+1}(.)$ of right-hand side of equation (18) is convergent for all finite values of z.
(ii) When $p=q+1$, then ${}_{q+2}F_{q+1}(.)$ of right-hand side of equation (18) is convergent when $|z|<1$.
(iii) When $p=q+1$, $|z|=1$, then ${}_{q+2}F_{q+1}(.)$ of right-hand side of equation (18) is absolutely convergent, if

$$
\Re\left\{\sum_{i=1}^{q}\beta_i-\sum_{i=1}^{q+1}\alpha_i\right\}>k.
$$

(iv) When $p=q+1$, $|z|=1$, and $z\neq1$, then ${}_{q+2}F_{q+1}(.)$ of right-hand side of equation (18) is conditionally convergent, if

$$
-1<\Re\left\{\sum_{i=1}^{q}\beta_i-\sum_{i=1}^{q+1}\alpha_i\right\}-k\le0.
$$

Motivated by the work [2, 5, 6, 7, 10], we derive some summation theorems for truncated Gauss' hypergeometric series.

Remark 3.1. *From equation* (18)*, it is clear that , the following non-terminating hypergeometric series*

$$
{}_2F_1\left[\begin{array}{c}-m,a;\\-2m+j;\end{array}2\right]\quad(m\ge j)
$$

and

$$
{}_2F_1\left[\begin{array}{c}-m,a;\\-2m-j;\end{array}2\right]\quad(m,j\in\mathbb{N}_0)
$$

are divergent in nature due to the argument greater then 1*. Therefore, we are interested to find the sum of first* $(m+1)$ *terms of infinite series* ${}_2F_1\left[\begin{array}{c}-m,a;\\-2m+j;\end{array}2\right]$ $(m\ge j)$ *and* ${}_2F_1\left[\begin{array}{c}-m,a;\\-2m-j;\end{array}2\right]$ $(m,j\in\mathbb{N}_0)$.

We have the following formula given by Kim et al. (see for example [4, p.2570, eq.(41)])

$$
{}_2F_1\left[\begin{array}{c}-m,a;\\-2m-1;\end{array}2\right]_m\mathrel{\mathring{=}}\frac{2^{2m+1}m!}{(2m+1)!}\left(\frac{a+1}{2}\right)_{m+1}
$$
$$
\mathrel{\mathring{=}}\frac{2^{2m}m!}{(2m+1)!}(a+1)\left(\frac{a+3}{2}\right)_m. \tag{19}
$$

The symbol $\mathring{=}$ exhibits the fact that above equation (19) does not hold true as stated.

4. Sum of truncated Gauss hypergeometric series of argument 2

Theorem 4.1. *Suppose m and j are positive integers such that $m > j$ or $j \geq 2m+1$ and a is a complex parameter. Then we have*

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m = \frac{a(m-j)!(1+a)_{2m-j}}{(2m-j)!} \sum_{r=0}^{j} \binom{j}{r} \frac{1}{(r+a)\left(\frac{2+r+a}{2}\right)_{m-j}}. \qquad (20) $$

Proof. The left-hand side of equation (20) can be written as

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m = \sum_{r=0}^{m} \frac{(-m)_r (a)_r 2^r}{(-2m+j)_r r!}. \qquad (21) $$

Now, replacing r by $m-r$ in the above equation (21), we get

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m = \frac{(a)_m 2^m (-1)^m}{(-2m+j)_m} {}_2F_1\left[\begin{array}{c} -m, 1+m-j; \\ 1-a-m; \end{array} \frac{1}{2}\right]. \qquad (22) $$

On using Pfaff-Kummer's linear transformation formula (10), we have

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m = \frac{(a)_m (-1)^m}{(-2m+j)_m} {}_2F_1\left[\begin{array}{c} -m, -a-2m+j; \\ 1-a-m; \end{array} -1\right]. \qquad (23) $$

The hypergeometric function on the right-hand side of the above equation (23) can be summed by the Whipple's identity (6) by taking $A = -a-2m+j, B = -m$ and $C = 1-a-m$ or Choi-Rathie-Malani (8) with $P = j$, we get

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m+j; \end{array} 2\right]_m = \frac{\Gamma(1-a)}{2(j-2m)_m \Gamma(j-a-2m)} \sum_{r=0}^{j} \frac{(-1)^r (-j)_r}{r!} \frac{\Gamma\left(\frac{j+r-a-2m}{2}\right)}{\Gamma\left(\frac{2+r-a-j}{2}\right)}. \qquad (24) $$

Again replacing r by $j-r$, and after simplification, we get the desired result (20). □

Theorem 4.2. *Suppose m and j are positive integers and a is a complex parameter. Then we have*

$$ \begin{aligned} &{}_2F_1\left[\begin{array}{c} -m, a; \\ -2m-j; \end{array} 2\right]_m \\ &= \frac{m! 2^{2m+j} \Gamma\left(\frac{1+a+j}{2}+m\right) \Gamma\left(\frac{2+a+j}{2}+m\right)}{(2m+j)! \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{1+a}{2}\right)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{1}{\left(\frac{a+r}{2}\right)_{m+1}}. \end{aligned} \qquad (25) $$

Proof. The left-hand side of equation (25) can be written as

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m-j; \end{array} 2\right]_m = \sum_{r=0}^{m} \frac{(-m)_r (a)_r 2^r}{(-2m-j)_r r!}. \qquad (26) $$

Now, replacing r by $m-r$ in the above equation (26), we get

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m-j; \end{array} 2\right]_m = \frac{(a)_m 2^m (-1)^m}{(-2m-j)_m} {}_2F_1\left[\begin{array}{c} -m, 1+m+j; \\ 1-a-m; \end{array} \frac{1}{2}\right]. \qquad (27) $$

On using Pfaff-Kummer's linear transformation formula (10), we have

$$ {}_2F_1\left[\begin{array}{c} -m, a; \\ -2m-j; \end{array} 2\right]_m = \frac{(a)_m (-1)^m}{(-2m-j)_m} {}_2F_1\left[\begin{array}{c} -m, -a-2m-j; \\ 1-a-m; \end{array} -1\right]. \qquad (28) $$

The hypergeometric function on the right-hand side of the above equation (28) can be summed by the Whipple's identity (7) by taking $A = -a - 2m - j, B = -m$ and $C = 1 - a - m$ or Choi-Rathie-Malani (8), we get

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m - j; \end{matrix} 2\right]_m = \frac{(-1)^m (a)_m \Gamma(1-a-m)\, m!}{2(-2m-j)_m \Gamma(-j-a-2m)\,(m+j)!} \sum_{r=0}^{j} \frac{(-j)_r}{r!} \frac{\Gamma\left(\frac{r-a-j-2m}{2}\right)}{\Gamma\left(\frac{2+r-a-j}{2}\right)}. \quad (29) $$

Now, replacing r by $j - r$ and after simplification we get the desired result (25). □

Remark 4.1. *We have verified summation formulae* (20) *and* (25) *for suitable numerical values of* a, m *and* j.

5. Special cases.

If we put $j = 0, 1, 2$ in equations (20) and (25) respectively, we get the following summations formulae for truncated Gauss' hypergeometric series

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m; \end{matrix} 2\right]_m = \frac{\left(\frac{1+a}{2}\right)_m}{\left(\frac{1}{2}\right)_m}, \quad (30) $$

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m+1; \end{matrix} 2\right]_m = \frac{1}{\left(\frac{1}{2}\right)_m}\left[\left(\frac{1+a}{2}\right)_m + \left(\frac{a}{2}\right)_m\right], \quad (31) $$

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m+2; \end{matrix} 2\right]_m = \frac{1}{(1-m)\left(-\frac{1}{2}\right)_m}\left[\frac{(1-a-m)}{(1-a)}\left(\frac{a-1}{2}\right)_m + \left(\frac{a}{2}\right)_m\right] \quad (m \neq 1), \quad (32) $$

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m-1; \end{matrix} 2\right]_m = \frac{1}{\left(\frac{3}{2}\right)_m}\left[(a+1)\left(\frac{3+a}{2}\right)_m - a\left(\frac{2+a}{2}\right)_m\right], \quad (33) $$

$$ {}_2F_1\left[\begin{matrix} -m, a; \\ -2m-2; \end{matrix} 2\right]_m = \frac{1}{(m+1)\left(\frac{3}{2}\right)_m}\left[(a+1)(a+m+1)\left(\frac{a+3}{2}\right)_m - a(a+2)\left(\frac{a+4}{2}\right)_m\right], \quad (34) $$

We have verified the special cases (30), (31), (32), (33) and (34) for suitable numerical values of m and a. The summation formula (33) is the correct form of equation (19) (see for example [4, p.2570, eq.(41)]).

6. Conclusions

In previous sections, we derived some summation theorems for truncated Gauss' hypergeometric series ${}_2F_1$ with argument 2, when one numerator and one denominator parameters are negative integers. Also, we have discussed their applications. It is expected that these summation formulae will be of wide interest and will help to advance research in the field of special functions.

We conclude our present investigation by observing that several hypergeometric summation theorems can be derived from a known summation theorem in an analogous manner.

References

[1] Bailey, W. N., (1964), Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935, Reprinted by Stechert Hafner, New York.

[2] Choi, J., Rathie, A. K. and Malani, S., (2007), Kummer's Theorem and its Contiguous Identities, Taiwan. J. Math., 11(5), pp. 1521-1527.

[3] Choi, J., Rathie, A. K., Srivastava, H. M., (2011), A generalization of a formula due to Kummer, Integral Transforms Spec. Funct., 22, pp. 851-859.

[4] Kim, Y. S., Rathie, A. K. and Paris, R. B., (2018), Evaluations of some terminating hypergeometric ${}_2F_1(2)$ series with applications, Turk. J. Math., 42, pp. 2563-2575.

[5] Luke Y.L., (1969), The special functions and their approximations, Vol. I, Academic Press, New York.

[6] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I., (1990), Integrals and Series, Vol. III, Gordon and Breach Science Publishers, New York.

[7] Qureshi, M. I. and Baboo, M. S., (2016), Some Unified and Generalized Kummer's First Summation Theorems with Applications in Laplace Transform Technique, Asia Pac. J. Math., 3(1), pp. 10-23.

[8] Qureshi, M. I. and Jabee, S., General series identity, summation theorems for Clausen's function with negative integral numerator, denominator parameters and applications (communicated).

[9] Rainville, E. D., (1971), Special Functions, Reprint of 1960 first edition. Chelsea Publishig Co., Bronx, New York.

[10] Rakha, M. A. and Rathie, A. K., (2011), Generalizations of Classical Summation Theorems for the Series ${}_2F_1$ and ${}_3F_2$ with Applications, Integral Transforms Spec. Funct., 22(11), pp. 823-840.

[11] Srivastava, H. M. and Manocha, H. L., (1984), A treatise on generating functions, Halsted Press (Ellis Horwood limited), New York.

[12] Vidunas, R., (2002), A generalization of Kummer's identity, Rocky Mt. J. Math., 32(2), pp. 919-936.

[13] Whipple, F. J. W., (1924), A group of generalized hypergeometric series: Relations between 120 allied series of type ${}_2F_1\left[\begin{array}{c} a,b,c; \\ e,f; \end{array} 2\right]$, Proc. London Math. Soc., 23, pp. 104-114.

[14] Whipple, F. J. W., (1928), On series allied to the hypergeometric series with argument -1, Proc. London Math. Soc., 30, pp. 81-94.

M. I. Qureshi for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.4.

Saima Jabee is an assistant professor in the Department of Applied Sciences and Humanities, Jamia Millia Islamia, New Delhi, India. She received her Ph.D. from the same university and department in 2019. Her research interests include special functions, integral transforms, operational techniques, multiple hypergeometric functions, differential equations and summation theorems.

D. Ahamad for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.4.