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Brief Paper

A set-theoretic generalization of dissipativity with applications in Tube MPC ☆

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ABSTRACT

This paper introduces a framework for analyzing a general class of uncertain nonlinear discrete-time systems with given state-, control-, and disturbance constraints. In particular, we propose a set-theoretic generalization of the concept of dissipativity for systems that are affected by external disturbances. The corresponding theoretical developments build upon set based analysis methods and lay a general theoretical foundation for a rigorous stability analysis of economic tube model predictive controllers.

1. Introduction

Dissipativity theory can be regarded as one of the most fundamental tools for analyzing the stability of control systems (Byrnes, Isidori, & Willems, 1991). Its origins can be traced to the work of Willems (1971, 1972), who analyzed the theoretical properties of dissipative systems and formalized the concepts of energy supply and energy storage for general control systems.

Recent work on dissipativity theory has focused on its application to optimally operated control systems. For example, Angeli, Amrit, and Rawlings (2012) established a link between dissipativity of a control system and the existence of optimal steady-states. In Faulwasser, Grüne, Müller, et al. (2018), a thorough review of economic model predictive control (MPC) schemes is presented. Unlike standard tracking problems, economic MPC controllers are based on objective functions which are, in general, not positive definite. For such controllers a number of stability conditions are available (Angeli et al., 2012; Müller, Angeli, & Allgöwer, 2015; Müller & Grüne, 2016; Zanon, Grüne, & Diehl, 2017), which all rely on dissipativity theory.

In order to understand why one may wish to develop a generalization of dissipativity for set-valued systems, one must be aware of set-valued analysis (Aubin & Frankowska, 2009) and its importance in the development and analysis of robust control methods (Bertsekas & Rhodes, 1971; Blanchini, 1999). Among the various set-theoretic control methodologies, Tube model predictive control strategies have been analyzed exhaustively during the past two decades (Langson, Chryssochoos, Raković, & Mayne, 2004; Mayne, Seron, & Raković, 2005). Here, the main idea is to replace trajectories by robust forward invariant tubes (RFITs), i.e., set-valued functions in the state space enclosing all future system states, independently of the uncertainty realization. A great variety of methods for Tube MPC synthesis can be found in the overview article (Raković, 2012).

Notice that there is a large body of work regarding the stability of nominal (certainty-equivalent) MPC schemes (Chen & Allgöwer, 1998; Grüne, 2009; Rawlings & Mayne, 2009). Of course, if a parameterized version of a Tube MPC problem can be written as a standard MPC problem, such stability results can be applied. For example, in the so-called Rigid Tube MPC (Raković, Munoz-Carpintero, Cannon & Kouvaritakis, 2012; Zeilinger, Raimondo, Domahidi, Morari, & Jones, 2014) one computes offline both the tube cross-section as well as an ancillary feedback law in order to add robustness margins to all constraints. Thus, in this case, the robust reformulation is equivalent to a nominal MPC scheme with tightened constraints and standard stability results for MPC can be applied (Raković, 2012; Zeilinger et al., 2014). For Rigid Tube MPC schemes with economic objectives, stability results can be obtained using tools from the field of dissipativity theory (Bayer, Müller, & Allgöwer, 2014, 2018; Broomhead, Manzie, Shekhar, & Hield, 2015).

As rigid tubes may be rather conservative, multiple strategies have been proposed to increase the accuracy of RFITs. These include the use of homothetic (Raković, Kouvaritakis, & Cannon, 2013; Raković, Kouvaritakis, Findeisen & Cannon, 2012) and elastic tube parameterizations (Raković, Levine, & Açıkmeşe, 2016),

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which are based on polytopic sets with a constant, pre-specified number of facets. The use of ellipsoidal parameterizations, Villanueva, Quirynen, Diehl, Chachuat, and Houska (2017), has also been proposed for tube MPC. In general, the question of which set parameterization is the best has no unique answer, as Tube MPC formulations face an inherent tradeoff between computational tractability and conservatism (Raković, 2012). Roughly, whenever one attempts to increase the accuracy of the set representation, the computational procedures become more demanding in terms of their memory and run-time requirements (Houska & Villanueva, 2019).

In this context, one of the main contributions of this paper is the development of a rigorous mathematical framework for the stability analysis of a rather general class of set-valued control systems. Towards this aim, Section 2 introduces a set-based generalization of cost-to-travel functions, which have originally been developed for certainty-equivalent control systems (Houska & Müller, 2017). Section 3 builds on this construction to propose a set-theoretic generalization of dissipativity for a particular class of storage functions. The practical applicability of these rather abstract concepts is discussed in Section 4, which establishes set-theoretic stability conditions for a large class of Tube MPC controllers with possibly economic objectives and no assumptions on the feedback structure. These controllers can be based, in the most general case, on parameterizations where the set-valued cross-sections of the tube itself are free optimization variables. The theoretical developments of this paper are illustrated throughout the paper using a series of academic examples. Section 5 concludes the paper.

1.1. Notation and preliminaries

We use the symbols $\mathbb{K}^n$ and $\mathbb{K}^n_{\mathrm{C}}$ to denote the sets of compact and compact convex subsets of $\mathbb{R}^n$, respectively. The Hausdorff distance between two sets $A, B \in \mathbb{K}^n$ is denoted by

$$d_{\mathrm{H}}(A,B) = \max\left\{ \max_{x\in A}\min_{y\in B}\|x-y\|,\ \max_{y\in B}\min_{x\in A}\|x-y\| \right\}.$$

Notice that $(\mathbb{K}^n, d_{\mathrm{H}})$ is a metric space (Srivastava, 2008).

As this paper uses functions whose arguments are sets in $\mathbb{K}^n$, we introduce the following definitions.

Definition 1. Let the domain $\mathcal{D} \subseteq \mathbb{K}^n$ be given. A function $L : \mathcal{D} \to \mathbb{R}$ is called

(1) continuous on $\mathcal{D}$ if there exists for every $A \in \mathcal{D}$ and every $\epsilon > 0$ a $\delta > 0$ such that $|L(A) - L(B)| < \epsilon$, for all $B \in \mathcal{D}$ with $d_{\mathrm{H}}(A,B) \leq \delta$,
(2) lower semi-continuous on $\mathcal{D}$ if there exists for every $A \in \mathcal{D}$ and every $\epsilon > 0$ a constant $\delta > 0$ such that $L(B) > L(A) - \epsilon$, for all $B \in \mathcal{D}$ with $d_{\mathrm{H}}(A,B) \leq \delta$, and
(3) monotonous if $A \subseteq B$ implies $L(A) \leq L(B)$.

We also introduce the generalized Hausdorff distance,

$$H(\mathcal{D},\mathcal{E}) = \max\left\{ \max_{A\in\mathcal{D}}\min_{B\in\mathcal{E}} d_{\mathrm{H}}(A,B),\ \max_{B\in\mathcal{E}}\min_{A\in\mathcal{D}} d_{\mathrm{H}}(A,B) \right\},$$

which is defined for any $\mathcal{D}, \mathcal{E} \subseteq \mathbb{K}^n$. The symbol $\mathcal{K}^n$ is used to denote the topological space of all nonempty subsets of $\mathbb{K}^n$ that are compact in $2^{\mathbb{K}^n}$—the power set of $\mathbb{K}^n$. Recalling that d_{H} induces a metric in $\mathbb{K}^n$, one can show that the generalized Hausdorff distance H induces a metric in $\mathcal{K}^n$ (Rockafellar & Wets, 2005).

The following definition is useful for the analysis of difference inclusions, as needed in the context of Tube MPC.

Definition 2. Consider the function $F : \mathbb{K}^n \to \mathcal{K}^n$. It is called continuous if there exists for every $\epsilon > 0$ a $\delta > 0$ such that

$$H(F(A), F(B)) < \epsilon,$$

for all $A, B \in \mathbb{K}^n$ with $d_{\mathrm{H}}(A,B) \leq \delta$.

2. Set-based cost-to-travel functions

The main goal of this paper is to analyze uncertain discrete-time control systems of the form

$$x_{k+1} = f(x_k, u_k, w_k). \tag{1}$$

Here, $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $w_k \in \mathbb{R}^{n_w}$ denote the state, control, and disturbance vectors at time k. The disturbance sequence w is unknown, but assumed to take values in the given set $\mathbb{W} \in \mathbb{K}^{n_w}$. The associated state- and control constraint sets, $\mathbb{X} \in \mathbb{K}^{n_x}$ and $\mathbb{U} \in \mathbb{K}^{n_u}$, are also assumed to be given.

Since (1) depends on an uncertain disturbance sequence, its reachable set is, in general, not a singleton. Hence, Section 2.1 briefly reviews some concepts from robust forward invariance (Blanchini, 1999), used for the analysis. Section 2.2 introduces a novel set-theoretic generalization of cost-to-travel functions (Houska & Müller, 2017), whose properties are analyzed in Section 2.3.

2.1. Difference inclusions and robust invariance

Recalling that the focus of this paper is on set-based methods for analyzing (1), we introduce the map $F : \mathbb{K}^{n_x} \to \mathcal{K}^{n_x}$, given by

$$F(A) = \left\{ B \in \mathbb{K}^{n_x} \;\middle|\; \begin{array}{l} \forall x \in A,\ \exists u \in \mathbb{U}:\ \forall w \in \mathbb{W}, \\ f(x,u,w) \in B \end{array} \right\} \tag{2}$$

for all $A \in \mathbb{K}^{n_x}$. This transition map F is the basis for the construction of control invariant sets and tubes for (1).

Definition 3. A sequence $X = (X_0, X_1, \ldots)$ of compact sets is called a robust forward invariant tube (RFIT) for (1) if it satisfies the difference inclusion

$$\forall k \in \mathbb{N}, \quad X_{k+1} \in F(X_k)\ .$$

If $X = (X^\star, X^\star, \ldots)$ is a time-invariant RFIT, $X^\star$ is called a robust control invariant (RCI) set.

Notice that F maps a set to a set of sets. This notation may appear rather abstract on the first view, but it has the advantage that we do not have to introduce notation for the underlying possibly set-valued feedback law and the associated closed-loop reachability sequences, which are parametric on the feedback law.

2.2. Set-based cost-to-travel functions

Let $\mathcal{D} \subseteq \mathbb{K}^n$ be a given domain and $L : \mathcal{D} \to \mathbb{R}$ a given lower semi-continuous function on $\mathcal{D}$. The cost-to-travel function $V_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \times \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ of (1) on $\mathcal{D}$ is given by

$$\begin{aligned} V_{\mathcal{D}}(A,B,N) = \quad & \min_{X \in \mathcal{D}^{N+1}} \quad \sum_{k=0}^{N-1} L(X_k) \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \forall k \in \{0,1,\ldots,N-1\}, \\ X_{k+1} \in F(X_k) \\ X_k \subseteq \mathbb{X}, \\ X_0 = A, \quad X_N = B\ , \end{array} \right. \end{aligned} \tag{3}$$

which is defined for all sets $A, B \in \mathcal{D}$ and all $N \in \mathbb{N}$. In order to ensure that $V_{\mathcal{D}}$ is well-defined, the following assumption is needed.

Assumption 1. The domain $\mathcal{D}$ and the functions f and L have the following properties:

(1) the right-hand side function f is continuous in all of its arguments,
(2) the set $\mathcal{D} \subseteq \mathbb{K}^{n_x}$ is closed in the metric space $(\mathbb{K}^{n_x}, d_H)$, and
(3) the function $L : \mathcal{D} \to \mathbb{R}$ is lower semi-continuous and monotonous on $\mathcal{D}$, in the sense of Definition 1.

Proposition 1. *Let Assumption 1 be satisfied. Then, the right-hand side of (3) either admits a minimizer or has an empty feasible set.*

Proof. First, notice that F is continuous in the sense of Definition 2. This is a direct consequence of the definition of F in (2), the continuity of f as well as the compactness of $\mathbb{U}$ and $\mathbb{W}$—see Aubin and Frankowska (2009) for details. Since $\mathbb{X}$ is compact and $\mathcal{D}$ closed, the feasible set of (7) is compact in $(\mathbb{K}^{n_x}, d_{\mathrm{H}})$. Since L is lower semi-continuous, the right-hand side of (3) either admits a minimizer or has an empty feasible set. □

If the sets A and B are such that the right-hand side of (3) is infeasible, we set $V_{\mathcal{D}}(A,B,N) = \infty$. This guarantees that the function $V_{\mathcal{D}}$ is well-defined for all $A, B \in \mathbb{K}^{n_x}$.

Example 1. Let us consider a dynamic system given by

$$f(x,u,w) = \begin{pmatrix} u \\ \frac{1}{2}x_2 + u + w \end{pmatrix} ,$$

with $\mathbb{X} = [-5,5] \times [-5,5]$, $\mathbb{U} = [-5,5]$, and $\mathbb{W} = [-1,1]$. Moreover we consider the 2-dimensional interval domain

$$\mathcal{D} = \left\{ [a_1,a_2] \times [a_3,a_4] \subseteq \mathbb{R}^2 \;\middle|\; \begin{array}{l} a_1,a_2,a_3,a_4 \in \mathbb{R} \\ (a_1 \le a_2) \wedge (a_3 \le a_4) \end{array} \right\}$$

as well as the stage cost

$$L([a_1,a_2] \times [a_3,a_4]) = 2a_2 + \frac{1}{20}\left(3a_1^2 + a_2^2 + 2a_3^2 + a_4^2\right) .$$

In this setting, the cost-to-travel function $V_{\mathcal{D}}(\cdot,\cdot,1)$ can be constructed explicitly. In fact, it is given by

$$V_{\mathcal{D}}(A,B,1) = \begin{cases} L([a_1,a_2] \times [a_3,a_4]) & \text{if } \;(a,b) \in G \\ \infty & \text{otherwise.} \end{cases}$$

for all intervals $A = [a_1,a_2] \times [a_3,a_4] \in \mathcal{D}$ and all intervals $B = [b_1,b_2] \times [b_3,b_4] \in \mathcal{D}$. Here, we have used the shorthand notation

$$G = \left\{ (a,b) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \exists v_1, v_2 \in [-5,5] : \\ \;b_3 \le \frac{1}{2}a_3 + v_1 - 1 \\ \;b_4 \ge \frac{1}{2}a_4 + v_2 + 1 \\ \;a_4 \ge 2(v_1 - v_2) + a_3 \\ \;b_1 \le v_1 \le b_2 \\ \;b_1 \le v_2 \le b_2 \\ -5 \le a_1 \le a_2 \le 5 \\ -5 \le a_3 \le a_4 \le 5 \end{array} \right\} .$$

2.3. Properties of cost-to-travel functions

The following propositions summarize basic properties of the cost-to-travel function $V_{\mathcal{D}}$.

Proposition 2 (*Monotonicity*)**.** *Let Assumption 1 be satisfied. Then,*

$$V_{\mathcal{D}}(A,C,N) \le V_{\mathcal{D}}(A',C,N) \quad \textit{and} \quad V_{\mathcal{D}}(A,C,N) \ge V_{\mathcal{D}}(A,C',N)$$

for all sets $A, A', C, C' \in \mathcal{D}$ with $A \subseteq A'$ and $C \subseteq C'$ and all $N \in \mathbb{N}$.

Proof. As discussed above, Assumption 1 ensures that $V_{\mathcal{D}}$ is well-defined. The definition of F implies that

$$C \in F(A') \quad \Longrightarrow \quad C \in F(A)$$
$$C \in F(A) \quad \Longrightarrow \quad C' \in F(A)$$

hold for all sets $A, A', C, C' \in \mathcal{D}$ with $A \subseteq A'$ and $C \subseteq C'$. Moreover, Assumption 1 requires L to be monotonous; that is,

$$A \subseteq A' \quad \Longrightarrow \quad L(A) \subseteq L(A'). \tag{4}$$

The statement of the proposition is a direct consequence of these three implications recalling the definition of $V_{\mathcal{D}}$ in (3). □

Proposition 3 (*Continuity*)**.** *Let Assumption 1 be satisfied. Then, the function $V_{\mathcal{D}}(\cdot,\cdot,N)$ is lower semi-continuous on its domain*

$$\{(A,B) \in \mathcal{D} \times \mathcal{D} \mid V_{\mathcal{D}}(A,B,N) < \infty\} .$$

Proof. Assumption 1 ensures that F is continuous and L lower semi-continuous. Since $\mathbb{X}$ is compact, it follows, from standard arguments from set-valued analysis (Aubin & Frankowska, 2009), that $V_{\mathcal{D}}$ is lower semi-continuous. For example, one can use an indirect argument, as follows.

If $V_{\mathcal{D}}$ was not lower-semi-continuous, we could find a sequence of sets (A_i, B_i) with

$$V_{\mathcal{D}}(A_i,B_i,N) < V_{\mathcal{D}}(A,B,N) - \epsilon ,$$

for some $\epsilon > 0$ as well as a feasible pair (A,B), such that (A_i,B_i) converges to (A,B) for $i \to \infty$. But this means that there exists a sequence of associated feasible points X^i of (3) with A and B replaced by A_i and B_i; and

$$\sum_{k=0}^{N-1} L(X_k^i) < V_{\mathcal{D}}(A,B,N) - \epsilon .$$

Since $\mathbb{X}$ is compact, this sequence must have a convergent subsequence, whose limit sequence X^∞ is feasible too, and satisfies

$$\sum_{k=0}^{N-1} L(X_k^\infty) \le V_{\mathcal{D}}(A,B,N) - \epsilon .$$

This is a contradiction, as we have $X_0^\infty = A$ as well as $X_N^\infty = B$ by construction. Thus, $V_{\mathcal{D}}(\cdot,\cdot,N)$ is lower semi-continuous. □

The set-based cost-to-travel function $V_{\mathcal{D}}$, satisfies a functional equation, as stated in the following proposition.

Proposition 4 (*Functional Equation*)**.** *Let Assumption 1 be satisfied. Then, $V_{\mathcal{D}}$ satisfies the functional equation*

$$V_{\mathcal{D}}(A,C,M+N) = \min_{B \in \mathcal{D}} V_{\mathcal{D}}(A,B,M) + V_{\mathcal{D}}(B,C,N)$$

for all $A, C \in \mathcal{D}$ and all $M, N \in \mathbb{N}$.

Proof. This statement follows from the definition of $V_{\mathcal{D}}$ and Proposition 3. This ensures that, either a minimizer exists for the optimization problem over B or that the expressions on both sides of the functional equation are equal to ∞. □

3. A set-theoretic generalization of dissipativity

This section introduces a generalization of dissipativity in the context of discrete-time set-valued inclusions.

Definition 4. System (1) is called set-dissipative on its domain $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ with respect to a given supply rate $S : \mathcal{D} \to \mathbb{R}$ on $\mathcal{D}$, if there exists a nonnegative storage function $\Lambda : \mathcal{D} \to \mathbb{R}_+$ such that the inequality

$$\Lambda(B) - \Lambda(A) \le S(A) ,$$

holds for all $A, B \in \mathcal{D}$ with $A, B \subseteq \mathbb{X}$ and $B \in F(A)$.

Notice that for the special case that $\mathbb{W}$ is a singleton and $\mathcal{D}$ is the set of singletons in $\mathbb{K}^{n_x}$, set-dissipativity is equivalent to dissipativity for deterministic systems with control-invariant supply rates, as introduced by Willems (1971, 1972). To explain how set-dissipativity relates to the ongoing developments in this paper, we introduce the following definition.

Definition 5. A set $X^\star \in \mathcal{D}$ is called an optimal robust control invariant set, if

$$V_{\mathcal{D}}^\star = V_{\mathcal{D}}(X^\star, X^\star, 1) = \min_{A\in\mathcal{D}} V_{\mathcal{D}}(A, A, 1)\ .$$

The following assumption is introduced to guarantee that $V_{\mathcal{D}}^\star$ is well-defined.

Assumption 2. The set $\{A \in \mathcal{D} \mid A \in F(A),\ A \subseteq \mathbb{X}\}$ has a non-empty interior in $\mathcal{D}$.

Proposition 5. *Let Assumptions 1 and 2 hold. Then, there exists at least one optimal robust control invariant set $X^\star \in \mathcal{D}$.*

Proof. Assumption 2 implies that there exists at least one set $A \in \mathcal{D}$ with $A \subseteq \mathbb{X}$ and $A \in F(A)$, which ensures that the domain

$$\{(A, A) \in \mathcal{D} \times \mathcal{D} \mid V_{\mathcal{D}}(A, A, 1) < \infty\}$$

is non-empty. Now, the statement of this proposition is a direct consequence of Proposition 3 and Weierstrass' theorem, which can be applied here as $\mathbb{X}$ is compact. □

Example 2. Consider the setting from Example 1. Here, the optimal robust control invariant set can be found by solving

$$\min_{(a,b)\in\mathbb{R}^4} L([a_1, a_2] \times [a_3, a_4]) \quad \text{s.t.} \quad \begin{cases} (a, b) \in G \\ a = b. \end{cases} \tag{5}$$

Notice that (5) is a strictly convex quadratic program with an unique minimizer $a^\star = b^\star = (-1, -1, -4, 0)^\intercal$. Thus, the optimal robust control invariant set is given by the line segment $X^\star = \{-1\} \times [-4, 0]$ with $V_{\mathcal{D}}^\star = -\frac{1}{5}$.

Definition 6. The function $V_{\mathcal{D}}(\cdot, \cdot, N)$ is called separable on $\mathcal{D}$ if it admits a non-negative separable lower bound $W : \mathcal{D} \to \mathbb{R}_+$ satisfying

$$\forall A, B \in \mathcal{D}, \quad V_{\mathcal{D}}(A, B, N) - NV_{\mathcal{D}}^\star \geq W(B) - W(A)\ .$$

The following theorem establishes the link between set-dissipativity and cost-to-travel functions.

Theorem 1. *Let Assumptions 1 and 2 be satisfied. System (1) is set-dissipative on its domain $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ with respect to the supply rate $S(A) = L(A) - L(X^\star)$ on $\mathcal{D}$ if and only if $V_{\mathcal{D}}(\cdot, \cdot, 1)$ is separable on $\mathcal{D}$.*

Proof. Proposition 5 implies that the constant offset $L(X^\star) = V_{\mathcal{D}}^\star < \infty$ is well-defined. If the system (1) is set dissipative and A and B are such that $V(A, B, 1) < \infty$, we have

$$\begin{aligned} V_{\mathcal{D}}(A, B, 1) - V_{\mathcal{D}}^\star &= L(A) - L(X^\star) \\ &\geq \Lambda(A^+) - \Lambda(A) \end{aligned}$$

for all sets $A^+ \in \mathcal{D}$ with $A^+ \in F(A)$ and $A^+ \in \mathbb{X}$. In particular, this inequality must hold for $A^+ = B$, which implies

$$V_{\mathcal{D}}(A, B, 1) - V_{\mathcal{D}}^\star \geq \Lambda(B) - \Lambda(A)\ .$$

This inequality also holds whenever $V(A, B, 1) = \infty$. Therefore, $W = \Lambda$ is a non-negative separable lower bound of $V_{\mathcal{D}}(\cdot, \cdot, 1)$ on $\mathcal{D}$. Therefore, if (1) is set-dissipative on $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ with respect to the supply rate $L(\cdot) - L(X^\star)$ on $\mathcal{D}$, then $V_{\mathcal{D}}(\cdot, \cdot, 1)$ is separable on the domain $\mathcal{D}$.

In order to establish the converse implication, we use the fact that $L(A) = V(A, B, 1)$ for all $A, B \in \mathcal{X}$ with $A, B \subseteq \mathbb{X}$ and $B \in F(A)$. Hence, for all such A, B we obtain

$$W(B) - W(A) \leq V_{\mathcal{D}}(A, B, 1) - V_{\mathcal{D}}^\star = L(A) - L(X^\star),$$

which implies that (1) is set-dissipative with storage function $\Lambda = W$, as long as $V_{\mathcal{D}}(\cdot, \cdot, 1)$ is separable on $\mathcal{D}$ with separable lower bound W. □

Example 3. Here, we continue discussing Examples 1 and 2. In this setting, the function

$$W([a_1, a_2] \times [a_3, a_4]) = \begin{cases} 16 + \frac{8}{5}(a_3 - a_2) & \text{if } A \subseteq \mathbb{X} \\ 0 & \text{otherwise} \end{cases}$$

happens to be a non-negative separable lower bound on $V_{\mathcal{D}}(\cdot, \cdot, 1)$. Here, the offset $16 \geq \frac{8}{5}a_2 - a_3$ is chosen such that W is non-negative on $\mathbb{X} = [-5, 5] \times [-2, 2]$. To verify that W is indeed a separable lower bound, we can compute the minimum of the right-hand side of the inequality

$$V_{\mathcal{D}}(A, B, 1) - V_{\mathcal{D}}^\star - W(B) + W(A) \geq 0 \tag{6}$$

over the domain of $V_{\mathcal{D}}(A, B, 1)$. Here, we notice that the minimum of the convex quadratic program

$$\begin{aligned} \min_{a,b,v_1}\ & L([a_1, a_2] \times [a_3, a_4]) - \frac{8}{5}(b_3 - b_2) + \frac{8}{5}(a_3 - a_2) \\ \text{s.t.}\ & \begin{cases} b_3 \leq \dfrac{1}{2}a_3 + v_1 - 1 \\ v_1 \leq b_2\ ,\ \ a_1 \leq a_2\ , \end{cases} \end{aligned}$$

is $-\frac{1}{5}$, with unique minimizer given by $(a^\star)^\intercal = (-1, -1, -4, 0)^\intercal$, $(b^\star)^\intercal = (0, 3, 0, 0)^\intercal$ and $v_1^\star = 3$. Since we have $V_{\mathcal{D}}^\star = -\frac{1}{5}$, the inequality (6) must be satisfied on the domain of $V_{\mathcal{D}}(\cdot, \cdot, 1)$.

Definition 7. The function $V_{\mathcal{D}}(\cdot, \cdot, N)$ is called strictly separable on $\mathcal{D}$ if it is separable and the point $(X^\star, X^\star)$ is the unique minimizer of

$$\min_{A,B\in\mathcal{D}} \left(V_{\mathcal{D}}(A, B, N) - NV_{\mathcal{D}}^\star - W(B) + W(A)\right)\ .$$

Notice that $V_{\mathcal{D}}(\cdot, \cdot, 1)$ is strictly separable if and only if (1) is set-dissipative with respect to the supply rate $S(A) = L(A) - L(X^\star)$ and the storage function Λ is such that

$$\Lambda(B) - \Lambda(A) < S(A)$$

for all $A, B \in \mathcal{D}$ with $A, B \subseteq \mathbb{X}$, $B \in F(A)$, and $(A, B) \neq (X^\star, X^\star)$. In this sense, one may state that strict separability of $V_{\mathcal{D}}(\cdot, \cdot, 1)$ is equivalent to "strict dissipativity" of (1).

4. Set-dissipativity and stability of tube MPC

4.1. Tube model predictive control

Tube MPC methods proceed by solving, in a receding-horizon manner, optimal control problems of the form

$$\begin{aligned} \min_{X\in\mathcal{D}^{N+1}}\ & E(X_0) + \sum_{k=0}^{N-1} L(X_k) + M(X_N) \\ \text{s.t.}\ & \begin{cases} \forall k \in \{0, 1, \ldots, N-1\}, \\ X_{k+1} \in F(X_k)\ , \qquad z \in X_0, \\ X_k \subseteq \mathbb{X}\ , \qquad X_N \subseteq T \end{cases} \end{aligned} \tag{7}$$

with $z \in \mathbb{R}^{n_x}$ being the current state-measurement and $T \in \mathcal{D}$ a terminal set. Here, $E : \mathcal{D} \to \mathbb{R}$, $L : \mathcal{D} \to \mathbb{R}$, and $M : \mathcal{D} \to \mathbb{R}$

denote lower semi-continuous initial, stage, and terminal costs, respectively. It is well-known (Rawlings & Mayne, 2009) that this tube MPC controller (7) is recursively feasible if $T \in F(T)$ and $T \subseteq \mathbb{X}$.

Remark 1. If one is interested in adding a decoupled control penalty to the objective of the MPC controller, one can always introduce discrete-time states that satisfy

$$\tilde{x}_{k+1} = u_k \ ,$$

and append them to the state vector, such that the next state is equal to the current control input. In this sense, it is not restrictive to assume that the objective in (7) does not explicitly depend on the control input.

Remark 2. There is a close relation between the tube MPC problem (7) and set-based cost-to-travel functions. In particular, as a direct consequence of Proposition 4, (7) can be equivalently written as

$$\begin{aligned} \min_{X \in \mathcal{D}^{N+1}} \ & E(X_0) + \sum_{k=0}^{N-1} V_{\mathcal{D}}(X_k, X_{k+1}) + M(X_N) \\ \text{s.t.} \quad & y \in X_0 \ , \quad X_N \subseteq T \ . \end{aligned}$$

4.2. Tube MPC feedback law

Notice that, any feasible point X of (7) is an RFIT. Thus, we can construct a control law, $\mu[X] : \mathbb{N} \times \mathbb{R}^{n_x} \to \mathbb{U}$, associated to this RFIT such that the state of any closed-loop system

$$\forall k \in \mathbb{Z}, \qquad x_{k+1} = f(x_k, \mu[X](k, x_k), w_k)$$

satisfies the implication

$$x_k \in X_k \quad \Longrightarrow \quad x_{k'} \in X_{k'}$$

for all $k' \geq k$ with $k, k' \in \{0, 1, \ldots, N\}$. This is implication is a direct consequence of the definition of the transition map F.

Remark 3. Consider an RFIT $X = (X_0, X_1, \ldots)$ as well as point $z \in X_k$. One can evaluate the feedback law $\mu[X](k, z)$ by solving the robust feasibility problem

$$\min_{u_k} \ 0 \quad \text{s.t.} \quad f(z, u_k, w) \in X_{k+1}, \quad \forall w \in \mathbb{W}$$

In particular, the signal $\mu[X](k, z) = u_k^\star$—with $u_k^\star$ being a solution of the above feasibility problem, will drive z to X_{k+1} regardless of the uncertainty realization.

Now, in contrast to this control law $\mu[X]$ associated to the RFIT, the Tube MPC feedback law $\nu : \mathbb{X} \to \mathbb{U}$ is time-invariant and given by

$$\nu(z) = \mu[\Xi](0, z). \tag{8}$$

Here, $\Xi(z)$ denotes a minimizing sequence of (7) as a function of the current measurement z. In the following, we use $y = (y_0, y_1, \ldots)$ to denote the closed-loop state recursion of the Tube MPC controller (7), given by

$$y_{k+1} = f(y_k, \nu(y_k), w_k) \tag{9}$$

with $k \in \mathbb{N}$. That is, we first set $z = y_k$, solve (7), update the system using the feedback (8), and repeat. In the next section we present an analysis of the stability properties of this closed-loop sequence using set-dissipativity.

4.3. Stability analysis

The goal of this section is to analyze stability of Tube MPC in the enclosure sense. Our definition of stability is motivated by the fact that the closed-loop trajectory y, given by (9), depends on the uncertainty sequence w.

Definition 8. The closed-loop state sequence y is said to admit a stable enclosure, if there exists a sequence $Y = (Y_0, Y_1, \ldots)$ of compact sets, $Y_k \in \mathbb{K}^{n_x}$, such that

(1) $y_k \in Y_k$ for all $k \in \mathbb{N}$, and
(2) the sequence $d_{\mathrm{H}}(Y_k, X^\star)$ is stable (in the sense of Lyapunov).

If, additionally,

$$\lim_{k \to \infty} d_{\mathrm{H}}(Y_k, X^\star) = 0 \ ,$$

then y admits an asymptotically stable enclosure Y.

Remark 4. Notice that Y is not necessarily an RFIT, since the set sequence Y is only required—under the above definition—to contain the actual closed-loop sequence y.

The following theorem establishes a stability result for the Tube MPC controller (7) under the assumption that the initial cost function E is a strictly separable lower bound of $V_{\mathcal{D}}(\cdot, \cdot, 1)$. Equivalently, E must be a storage function that establishes strict dissipativity of (1) on $\mathcal{D}$ with respect to the supply rate $S(A) = L(A) - L(X^\star)$. The statement is based on the additional assumption that the strictly separable lower bounding function E is also lower semi-continuous. At this point it has to be mentioned that a precise characterization of dissipative systems for which such a lower semi-continuous storage function exists, is still an open problem. However, there exist sufficient conditions under which one can assert the existence of continuous storage functions (Polushin & Marquez, 2002)—at least for nominal (not set-valued) systems.

Theorem 2. *Let Assumptions 1 and 2 be satisfied. Let the terminal region be an optimal robust control invariant set, $T = X^\star$, and let y_0 be such that (7) is feasible for $y = y_0$. If (1) is strictly set-dissipative on $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ with respect to the supply rate $S(\cdot) = L(\cdot) - L(X^\star)$ on $\mathcal{D}$ with E being an associated lower semi-continuous storage function and $M = 0$, then the closed-loop sequence y of the tube MPC controller (7) admits an asymptotically stable enclosure.*

Proof. We start the proof by constructing a sequence of compact sets, $Y = (Y_0, Y_1, Y_2, \ldots)$, as follows.
For all $j \in \mathbb{N}$:

(a) Measure the state, y_j
(b) Set $X^j = \Xi(y_j)$, where $\Xi(y_j)$ is the optimal solution sequence of the jth tube MPC problem

$$\begin{aligned} \min_{X \in \mathcal{D}^{N+1}} \ & E(X_0^j) + \sum_{k=0}^{N-1} V_{\mathcal{D}}(X_k^j, X_{k+1}^j, 1) \\ \text{s.t.} \quad & y_j \in X_0^j \ , \quad X_N^j = X^\star \ . \end{aligned}$$

(c) Set $Y_j = X_0^j$
(d) Evaluate $\nu(y_j)$, cf. Remark 3, send the feedback signal to the system, and go to (a).

For the construction in Step (b), we recall the relation between the tube MPC problem (7) and cost-to-travel functions in Remark 2.

Since $y_j \in X_0^j$ holds, the relation $y_j \in Y_j$ also holds by construction. In order to show that the sets Y_j are well defined, we introduce the shifted sequence

$$\widetilde{X}_j = \left(X_1^j, X_2^j, \ldots, X_{N-1}^j, X^\star, X^\star\right) \in \mathcal{D}^{N+1} .$$

Since the inclusion $y_{j+1} \in X_1^j$ holds independently of the uncertainty realization, the sequence $\widetilde{X}_j$ is a feasible point of the $(j+1)$th Tube MPC problem. Thus, recursive feasibility holds and Y_j is well defined.

Let $\mathcal{R}_\mathcal{D} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ denote the rotated cost-to-travel function, defined by

$$\mathcal{R}_\mathcal{D}(A, B) = E(A) - E(B) + V_\mathcal{D}(A, B, 1) - V_\mathcal{D}^\star$$

for all $A, B \in \mathcal{D}$. Thus, the tube MPC problem in Step (b) can be written in the equivalent form

$$\min_{X_j \in \mathcal{D}^{N+1}} \sum_{k=0}^{N-1} \mathcal{R}_\mathcal{D}(X_k^j, X_{k+1}^j) \quad \text{s.t.} \quad \begin{cases} y_j \in X_0^j \\ X_N^j = X^\star. \end{cases}$$

The key idea of this proof is to establish the claim that the function $\mathcal{L}_\mathcal{D} : \mathcal{D}^{N+1} \to \mathbb{R}$, given by

$$\forall Z \in \mathcal{D}^N, \qquad \mathcal{L}_\mathcal{D}(Z) = \sum_{k=0}^{N-1} \mathcal{R}_\mathcal{D}(Z_k, Z_{k+1}) ,$$

can be used as a Lyapunov function for the iterates X^j of the tube MPC controller.

Our first goal is to show that the sequence X^j is stable and converges to the limit point

$$\widehat{X}^\star = (X^\star, X^\star, \ldots, X^\star) \in \mathcal{D}^{N+1} .$$

Let us establish the following properties of the candidate Lyapunov function $\mathcal{L}_\mathcal{D}$.

P1 The function $\mathcal{L}_\mathcal{D}$ is lower semi-continuous (in the sense of Definition 1).

P2 The function $\mathcal{L}_\mathcal{D}$ is positive definite, i.e., it satisfies $\mathcal{L}_\mathcal{D}(Z) = 0$ if and only if $Z = \widehat{X}^\star$ and $\mathcal{L}_\mathcal{D}(Z) > 0$ otherwise.

P3 The sequence X^j satisfies

$$\mathcal{L}_\mathcal{D}\left(X^{j+1}\right) < \mathcal{L}_\mathcal{D}\left(X^j\right)$$

for all j, whenever $X_0^j \neq X^\star$.

Notice that P1 follows from Proposition 3. Moreover, P2 follows from the definition of $\mathcal{L}_\mathcal{D}$ and the assumption that $V_\mathcal{D}(\cdot, \cdot, 1)$ is strictly separable with non-negative and strictly separable lower bound E. Thus, it remains to establish P3. As discussed above, the proposed tube MPC controller is recursively feasible. This implies that

$$\begin{aligned}\mathcal{L}_\mathcal{D}\left(X^{j+1}\right) &\leq \mathcal{L}_\mathcal{D}\left(\widehat{X}^{j+1}\right) \\ &= \mathcal{L}_\mathcal{D}\left(X^j\right) - \mathcal{R}_\mathcal{D}\left(X_0^j, X_1^j\right) \\ &< \mathcal{L}_\mathcal{D}\left(X^j\right)\end{aligned}$$

whenever $X_0^j \neq X^\star$. Here, we have used our assumption that $\mathcal{V}_\mathcal{D}(\cdot, \cdot, 1)$ is strictly dissipative, which implies that

$$\mathcal{R}_\mathcal{D}\left(X_0^j, X_1^j\right) > 0 \quad \text{whenever} \quad X_0^j \neq X^\star.$$

These properties are sufficient to conclude that $\mathcal{L}_\mathcal{D}$ is a Lyapunov function proving asymptotic stability of X^j to $\hat{X}^\star$ with respect to the Hausdorff metric. This implies that the sequence Y is an asymptotically stable enclosure of y, converging to X^*. □

Similar to existing results for economic MPC (see Faulwasser et al., 2018 and references therein), Theorem 2 establishes asymptotic stability for the proposed Tube MPC controller under a dissipativity condition. But—in contrast to nominal, certainty-equivalent, economic MPC schemes—the storage function E is not only needed for analysis purposes. In fact, the proposed Tube MPC controller makes explicit use of the initial cost E, as the initial tube is not fixed but an optimization variable.

Remark 5. Theorem 2 specializes—for simplicity of presentation—on the case $T = X^\star$ and $M(A) = 0$. However, a generalization of this stability result for any terminal region $T \in \mathcal{D}$ with $T \subseteq \mathbb{X}$ is possible under the additional assumption that the function M is lower semi-continuous and satisfies the condition

$$\forall A \subseteq T, \exists B \in F(A), \quad \begin{cases} A, B \in \mathcal{D} \\ B \subseteq T \\ M(B) - E(B) \leq M(A) + L(A) - E(A) , \end{cases}$$

see also Angeli et al. (2012) for details. An in-depth discussion on how to construct such set-based terminal costs is, however, beyond the scope of this paper.

Example 4. Let us return to the setting from Examples 1 and 2—recalling that the optimal RCI set is given by $X^\star = \{-1\} \times [-4, 0]$. Let us attempt to set up a robust MPC controller without initial cost and $N = 2$, i.e.

$$\min_{X \in \mathcal{D}^3} \; L(X_0) + L(X_1) \quad \text{s.t.} \begin{cases} \forall k \in \{0, 1\} \\ X_{k+1} \in F(X_k) \\ X_k \subseteq \mathbb{X} \\ z \in X_0 \\ X_2 = X^\star . \end{cases} \tag{10}$$

Using the notation from Examples 1 and 2, the optimization problem (10) can be formulated as the strictly convex parametric quadratic program

$$\begin{aligned} \min_{a,b,c \in \mathbb{R}^4} \quad & L([a_1, a_2] \times [a_3, a_4]) + L([b_1, b_2] \times [b_3, b_4]) \\ \text{s.t.} \quad & \begin{cases} (a, b) \in G , \; (b, c) \in G \\ c^\intercal = x^\star , \qquad z \in [a_1, a_2] \times [a_3, a_4] \end{cases} \end{aligned} \tag{11}$$

with $(x^\star)^\intercal = (-1, -1, -4, 0)^\intercal$. Having Remark 3 in mind, we can introduce a decision variable $u_0 \in [-5, 5]$ and augment (10) with the constraints

$$\forall w \in [-1, 1], \quad f(z, u_0, w) \in [b_1, b_2] \times [b_3, b_4] ,$$

which hold, whenever

$$\begin{aligned} & b_1 \leq u_0 \leq b_2 , \\ & b_3 \leq \frac{1}{2} z_2 + u_0 - 1 , \quad \text{and} \quad b_4 \geq \frac{1}{2} z_2 + u_0 + 1 \end{aligned} \tag{12}$$

hold.

Now, the parametric optimizer of (10) (augmented with (12)) is a piecewise linear function defined on 22 critical regions (non-overlapping interval boxes).

Let us consider the region $[-5, 0] \times [-4, 0]$, containing $X^\star$. An associated parametric optimal set sequence is given by

$$\Xi_0(z) = \{z_1\} \times [z_2, 0] , \quad \Xi_1(z) = \left\{-\frac{1}{2} z_2 - 3\right\} \times [-4, 0] ,$$

and $\Xi_2(z) = X^\star$, for all $z \in [-5, 0] \times [-4, 0]$. An optimal feedback law in this region is given by

$$\forall z \in [-5, 0] \times [-4, 0], \quad \nu(z) = u_0^\star(z) = -\frac{1}{2} z_2 - 3 .$$

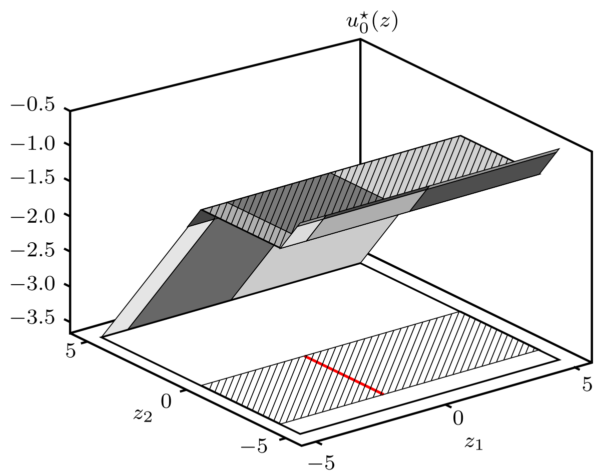

Fig. 1. Component u_0 of the parametric optimizer of (14). The region $[-5,5]\times[-4,0]$ is shown hatched while the set $X^\star$ is shown as a red solid line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

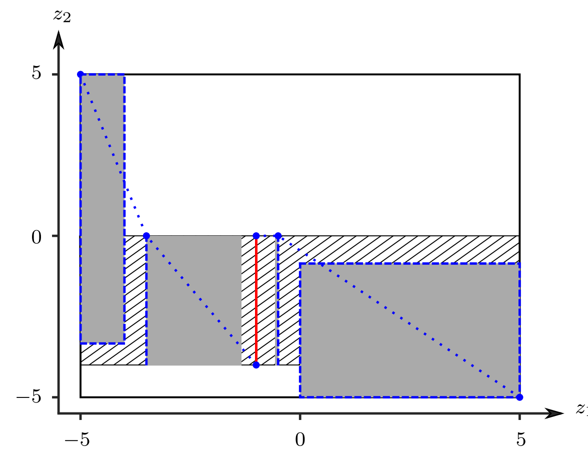

Fig. 2. Closed-loop behavior under the feedback law (15). The figure shows the closed-loop sequences $y=(y_0,y_1,y_2)$ (blue dots joined by blue dotted lines), and the optimal sequences $\Xi(y_0)$ (gray) starting from two initial conditions. The boundaries of their asymptotically stable enclosures are shown as blue dashed lines. The terminal set is shown as a red solid line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This feedback law is recursively feasible, but unstable in the enclosure sense. Consider a closed-loop sequence with initial condition $y_0=(-1,-2)^\intercal$. The initial condition is in the optimal RCI set and $Y_0=\Xi(y_0)=\{-1\}\times[-3,0]\subset X^\star$. Now, at the next time instance we have, by construction of the RFIT, $y_1\in\Xi_1(y_0)=\{-2\}\times[-4,0]$—regardless of the uncertainty realization. Notice that $\Xi_1(y_0)\cap X^\star=\varnothing$. Since $y_1\in Y_1$ must hold by construction, no matter how the uncertainty is realized, the closed-loop system must be unstable in the enclosure sense.

This instability issue can be fixed by adding the initial cost term $E=W$ from Example 3. Now, the robust MPC formulation is given by

$$\min_{X\in\mathcal{D}^3}\; E(X_0)+\sum_{k=0}^{1}L(X_k)\quad \text{s.t.}\;\begin{cases}\forall k\in\{0,1\}\\ X_{k+1}\in F(X_k)\\ X_k\subseteq\mathbb{X}\\ y\in X_0\\ X_2=X^\star\,.\end{cases}\tag{13}$$

Again, we can formulate this as the quadratic program

$$\begin{aligned}\min_{a,b,c\in\mathbb{R}^4}\;& W([a_1,a_2]\times[a_3,a_4])+L([a_1,a_2]\times[a_3,a_4])\\ &+L([b_1,b_2]\times[b_3,b_4])\\ \text{s.t.}\;&\begin{cases}(a,b)\in G\,,\ (b,c)\in G\\ c^\intercal=x^\star\,,\qquad y\in[a_1,a_2]\times[a_3,a_4]\end{cases}\end{aligned}\tag{14}$$

augmented with the decision variable $u_0\in[-5,5]$ and the constraints (12). The optimizer is, again, a piecewise affine function defined over 24 critical regions. Fig. 1 shows the component u_0 of the parametric optimizer.

The tube MPC feedback law ν, leading to the minimal stage cost, is given by

$$\nu(z)=u_0^\star(z)=\begin{cases}-\frac{1}{2}z_2-3 & \text{if } z_2\in[-5,-4]\\ -1 & \text{if } z_2\in[-4,0]\\ -\frac{1}{2}z_2-1 & \text{otherwise}\,,\end{cases}\tag{15}$$

for all $y\in[-5,5]\times[-5,5]$. This feedback law is not only recursively feasible, but also asymptotically stable in the enclosure sense.

Notice that the region $[-5,5]\times[-4,0]$ – depicted with a hatched pattern in Fig. 1 – is forward reachable in at most one step, for any initial feasible initial condition and any $w_0\in\mathbb{W}$. Moreover, any closed-loop sequence satisfies $y_{k+1}\in X^\star$, whenever $y_k\in[-5,5]\times[-4,0]$—irrespective of w_k. Since we have $Y_k\subseteq X^\star$ for all $k\geq 0$, any closed loop sequence admits a stable enclosure. In addition, the associated optimal set sequence is given by

$$\Xi_0(z)=\begin{cases}[z_1,-4]\times[-4,0] & \text{if } z_1\in[-5,-4]\\ \{z_1\}\times[-4,0] & \text{if } z_1\in[-4,0]\\ [0,z_1]\times[-4,0] & \text{otherwise}\,,\end{cases}$$

and $\Xi_1(z)=\Xi_2(z)=X^\star$ for all $z\in[-5,5]\times[0,4]$. Based on the previous reachability argument, it is clear that any closed loop sequence under the feedback law (15) admits an asymptotically stable enclosure.

Fig. 2 depicts closed-loop sequences (blue dots with blue dotted lines) starting from the lower right and upper left corners – $(5,-5)^\intercal$ and $(-5,5)^\intercal$ respectively – of the constraint set $\mathbb{X}$. The disturbance sequence has been constructed so as to maximize the cost. The gray sets denote the optimal sequences $\Xi(y_0)$, while the blue dashed lines denote the boundary of the enclosure sequences Y. Notice the closed-loop system reaches the region $[-5,5]\times[-4,0]$ (hatched), in at most 1 step, and the terminal set (red continuous line) in at most 2 steps—remaining there, as predicted.

5. Conclusions

This paper has introduced a set theoretic generalization of dissipativity in order to establish stability conditions for a general class of Tube MPC controllers (cf. Theorem 2). Here, the focus has been on robust MPC controllers, whose compact set-valued states are either entirely free optimization variables, or belong to a finite dimensional, parametric subset $\mathcal{D}$ of all compact sets in the state space. The analysis has shown why the usual requirements for asymptotic stability of certainty-equivalent MPC controllers—namely invariance of the terminal region, a strict dissipativity condition and feasibility of the initial point—are not sufficient to guarantee asymptotic stability (see the first part in Example 4). In fact, Example 4 shows that a tube MPC controller requires an initial cost term, which corresponds to the storage function in the set-dissipativity condition.

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