TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1448-1458

A NEW APPROACH TO FIND AN APPROXIMATE SOLUTION OF LINEAR INITIAL VALUE PROBLEMS WITH HIGH DEGREE OF ACCURACY

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Abstract. This work investigates a new approach to find closed form solution to linear initial value problems (IVP). Classical Bernoulli polynomials have been used to derive a finite set of orthonormal polynomials and a finite operational matrix to simplify derivatives in IVP. These orthonormal polynomials together with the operational matrix of relevant order provides a robust approximation to the solution of a linear initial value problem by converting the IVP into a set of algebraic equations. Depending upon the nature of a problem, a polynomial of degree n or numerical approximation can be obtained. The technique has been demonstrated through four examples. In each example, obtained solution has been compared with available exact or numerical solution. High degree of accuracy has been observed in numerical values of solutions for considered problems.

Keywords:approximate solution, Bernoulli polynomials, initial value problems, orthonormal polynomials.

AMS Subject Classification: 34A45, B4B05, 11B68.

1. Introduction

Initial value problems (IVPs) for ordinary differential equations arise in a natural way in real life problems and modelling of science and engineering problems. Some examples of such modelling problems are heat conduction, wave propagation, diffusion problems, gas dynamics, nuclear physics, atomic structures, fluid flow and chemical reactions, continuum mechanics, electricity and magnetism, geophysics, antenna, synthesis problem, population genetics communication theory, mathematical modelling of economics, radiation problems and astrophysics. Many times, the exact analytic solution of such problems are not available which give rise a need to find numerical solutions. Bulk of literature is available to explore exact and numerical solutions of initial and boundary value problems [1, 2, 3, 4]. Many researchers have focused their attention to find approximate solutions differential and integral equations. Xu [5] adopted method of variational iteration, Pandey, et. al. [6] applied homotopic perturbation and method of collocation. Cheon [7] discussed possible applications of Bernoulli polynomials and functions in numerical analysis. Some other

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§ Manuscript received: July 26, 2020; accepted: April 25, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.4

latest investigations include uses of Chebyshev polynomials [8], Legendre polynomials [9], Laguerre polynomials and Wavelet Galerkin method [10], Legendre wavelets [11], the operational matrix [12].

Numerical solution to boundary value problems has been also presented by Shiralashetti and Kumbinarasaiah [13, 14], Abd-Elhameed et al. [15], Iqbal et al. [16] and, Kumar and Singh [17]. Bernoulli polynomials and its properties have been also discussed by many authors [18, 19]. Singh et.al. [20] used operational matrix to solve some integro-differential equations arising in theory of anomalous relaxation processes in vicinity of singular point. Tohidi et. al. [21] obtained numerical approximation for generalized pantograph equation using Bernoulli matrix method, Tohidi and Khorsand [22] to solve second-order linear system of partial differential equations, Mohsenyzadeh [23] used Bernoulli polynomials to solve Volterra type integral equations. Recently, Singh et al. [?] used Bernoulli polynomials to develop a trigonal operational matrix to solve Abel-Volterra type integral equations. Devi et. al. [24] obtained solution for Lane-Emden, Riccati's and Bessel's equations with Lagrange operational matrix and, a similar approach was recently adopted by Maurya et. al. [25] to solve some Abel integral and integro-differential equations.

In this work, it is proposed to solve linear initial value problems (IVPs) by application of a new operational matrix developed from a set of modified orthonormal Bernoulli polynomials.

2. Bernoulli Polynomials

The word *BernoulliPolynomials* was first coined by J. L. Raabe in 1851 while discussing the formula $\sum_{n=0}^{m-1} B_n\left(x+\frac{k}{m}\right) = m^{-(n+1)}B_n(mx)$, however, the polynomials $B_n(x)$ were already introduced by Jakob Bernoulli in 1690 in his book "*ArsConjectandi*" [26]. A thorough study of these polynomials was first done by Leonhard Euler in 1755, who showed in his book "Foundations of differential calculus" that these polynomials satisfy the finite difference relation:

$$B_n(\zeta+1) - B_n(\zeta) = n\zeta^{n-1}, \quad n \geq 1 \tag{1}$$

and proposed the method of generating function to calculate $B_n(x)$. Following Leonhard Euler, recently Costabile and Dell'Accio [26] showed that Bernoulli Polynomials are monic which can be extracted from its generating function

$$\frac{\gamma\, e^{\zeta\gamma}}{e^{\gamma}-1} = \sum_{n=0}^{\infty} B_n(\zeta)\,\frac{\gamma^n}{n!} \quad (|\zeta| < 2\pi) \tag{2}$$

and represented in the simple form:

$$B_n(\zeta) = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} B_j(0)\,\zeta^{n-j}, \quad n = 0, 1, 2, ... \quad 0 \leq \zeta \leq 1 \tag{3}$$

where, $B_n(0)$ are the Bernoulli numbers, which can also be calculated with Kronecker's formula $B_n(0) = -\sum_{j=1}^{n+1} \frac{(-1)^j}{j} \begin{pmatrix} n+1 \\ j \end{pmatrix} \sum_{k=1}^{j} k^n$; $n \geq 0$ [27]. Thus, first few Bernoulli polynomials can be written as $B_0(\zeta) = 1, B_1(\zeta) = \zeta - \frac{1}{2}, B_2(\zeta) = \zeta^2 - \zeta + \frac{1}{6}, B_3(\zeta) = \zeta^3 - \frac{3}{2}\zeta^2 + \frac{1}{2}\zeta, B_4(\zeta) = \zeta^4 - 2\zeta^3 + \zeta^2 - \frac{1}{30}$.

These Bernoulli Polynomials form a complete basis over $[0,1]$ [28] and show some interesting properties [29, 30]:

$$\left.\begin{array}{l} B_n'(\zeta) = nB_{n-1}(\zeta), \;\; n \geq 1 \\ \int_0^1 B_n(z)dz = 0, \qquad n \geq 1 \\ B_n(\zeta+1) - B_n(\zeta) = n\zeta^{n-1}, \; n \geq 1 \end{array}\right\}. \tag{4}$$

3. The Orthonormal Polynomials

The second relation of property (4) shows that the polynomials $B_n(x)\,(n \geq 1)$ (3) are orthogonal to $B_o(x)$ with respect to standard inner product on $L^2 \in [0,1]$ defined as:

$$< f_1, f_2 >= \int_0^1 f_1(x)\bar{f_2}(x)dx\,; \;\; f_1, f_2 \in L^2[0,1] \tag{5}$$

With inner product (5), an orthonormal set of $n+1$ polynomials is derived for any B_n with Gram-Schmidt orthogonalization. For illustration, a set of ten such orthonormal polynomials are obtained for $n = 9$ in Appendix A:

4. Approximation of Functions

Theorem 4.1. *Let $H = L^2[0,1]$ be a Hilbert space and $Y = span\{y_0, y_1, y_2, ..., y_n\}$ be a subspace of H such that $dim(Y) < \infty$, every $f \in H$ has a unique best approximation out of Y* [28]*, that is, $\forall y(t) \in Y$, $\exists \hat{f}(t) \in Y$ s.t. $\| f(t) - \hat{f}(t) \|_2 \leq \| f(t) - y(t) \|_2$. This implies that, $\forall\, y(t) \in Y, < f(t) - f(t), y(t) >= 0$, where $<,>$ is standard inner product on $L^2 \in [0,1]$ (c.f. Theorems 6.1-1 and 6.2-5, Chapter 6* [28]*).*

Remark 4.1. *Let $Y = span\{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$, where $\phi_k \in L^2[0,1]$ are orthonormal Bernoulli polynomials. Then, from Theorem 4.1, for any function $f \in L^2[0,1]$,*

$$f \approx \hat{f} = \sum_{k=0}^{n} c_k \phi_k, \tag{6}$$

where $c_k = \langle f, \phi_k \rangle$, and $<,>$ is the standard inner product on $L^2 \in [0,1]$.

For numerical approximation, series (5) can be written as:

$$f(\zeta) \simeq \sum_{k=0}^{n} c_k \phi_k = C^T \phi(\zeta) \tag{7}$$

where $C = (c_0, c_1, c_2, ..., c_n)\,, \phi(\zeta) = (\phi_0, \phi_1, \phi_2, ..., \phi_n)$ are column vectors, and number of polynomials n can be chosen to meet required accuracy.

5. Construction of operational matrix

The orthonormal polynomials, as derived in Appendix A, can be expressed as:

$$\int_0^\zeta \phi_o(\eta)d\eta = \frac{1}{2}\phi_o(\zeta) + \frac{1}{2\sqrt{3}}\phi_1(\zeta) \tag{8}$$

$$\begin{aligned} \int_0^\zeta \phi_i(x)dx = \; & \frac{1}{2\sqrt{(2i-1)(2i+1)}}\phi_{i-1}(\zeta) \\ & + \frac{1}{2\sqrt{(2i+1)(2i+3)}}\phi_{i+1}(\zeta), \quad (\,for\, i = 1\,, 2, \ldots, n) \end{aligned} \tag{9}$$

Relations (8-9) can be represented in closed form as:

$$\int_0^{\zeta} \phi(\eta)d\eta = \Theta\,\phi(\zeta) \tag{10}$$

where $\zeta \in [0,1]$ and Θ is operational matrix of order $(n+1)$ given as :

$$\Theta = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{1.3}} & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{1.3}} & 0 & \frac{1}{\sqrt{3.5}} & \cdots & 0 \\ 0 & -\frac{1}{\sqrt{3.5}} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{\sqrt{(2n-1)(2n+1)}} \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{(2n-1)(2n+1)}} & 0 \end{bmatrix} \tag{11}$$

6. Solution of Initial Value Problems

First, we will show the process of finding solution for general form of second order linear IVP for simpler understanding, and a generalization of the same to linear IVP of any order will be done later. Let us consider,

$$\frac{d^2y}{d\zeta^2} + P(\zeta)\frac{dy}{d\zeta} + Q(\zeta)y = r(\zeta),\;\; y(0) = \alpha, \left(\frac{dy}{d\zeta}\right)_{\zeta=0} = \beta \tag{12}$$

where $y = y(\zeta); p(\zeta)$ and $q(\zeta)$ and $r(\zeta)$ are continuous functions defined $[0,1]$. It is further assumed that equation (12) admits a unique solution on $[0,1]$, otherwise, a suitable transformation $z = z(x)$ may be applied to change the domain of y, p, q and r.

Case 6.1. *Coefficients of y and its derivatives are constant*

In this case, taking p for $P(x)$ and q for $Q(x)$, equation (12), is written as:

$$\frac{d^2y}{d\zeta^2} + p\frac{dy}{d\zeta} + q(\zeta)y = r(\zeta),\;\; y(0) = \alpha, \left(\frac{dy}{d\zeta}\right)_{\zeta=0} = \beta \tag{13}$$

Let $R = (r_o, r_1, ..., r_n)$ be a real column vector such that the function $r(\zeta)$ can be approximated in terms of first $n+1$ orthonormal Bernoulli polynomials as:

$$r(\zeta) = R\,\phi(\zeta) \tag{14}$$

Let $C = (c_o, c_1, c_2, ..., c_n)-$ be a column vector of $n+1$ unknown quantities. Taking

$$\frac{d^2y}{d\zeta^2} = C^T\,\phi(\zeta) \tag{15}$$

equation (12) can be re-written as:

$$C^T\phi(\zeta) + \; p\; C^T\,\Theta\,\phi(\zeta) + \; q\; C^T\,\Theta^2\,\phi(\zeta) = R^T\,\phi(\zeta) \tag{16}$$

which gives,

$$C^T \; = R^T\left[I + p\;\Theta + q\;\Theta^2\right]^{-1} \tag{17}$$

Substituting equation (17) back into equation (15), an approximation for $y(\zeta)$ can be obtained as:

$$y(\zeta) = C^T\,\Theta^2\,\phi(\zeta). \tag{18}$$

Case 6.2. *Generalization of Case−6.1*

With the method discussed in Case−6.1, solution to linear IVP of order $n-$

$$a_n \frac{d^n y}{d\zeta^n} + a_{n-1} \frac{d^{n-1} y}{d\zeta^{n-1}} + \cdots + a_1 \frac{dy}{d\zeta} + a_o y = r(\zeta), \quad y(0) = y_o, y'(0) = y_1, \ldots, y_n(0) = y_n \tag{19}$$

can be obtained by taking $\frac{d^n y}{d\zeta^n} = C^T \phi(\zeta)$ as:

$$y(\zeta) = C^T \Theta^n \phi(\zeta). \tag{20}$$

where,

$$C^T \ = R^T \left[a_n I + a_{n-1} \Theta + a_{n-2} \Theta^2 + \ldots a_o \Theta^n \right]^{-1} \tag{21}$$

Case 6.3. *Coefficients of y and its derivatives are functions of independent variable*

Taking $P(\zeta) = a^T \phi(\zeta)$ and $Q(\zeta) = b^T \phi(\zeta)$ together with equations. (14-15), equation (12) can be written as:

$$C^T \phi(\zeta) + \left(a^T \phi(\zeta)\right) \left(C^T \Theta \phi(\zeta)\right) + \left(b^T \phi(\zeta)\right) \left(C^T \Theta^2 \phi(\zeta)\right) = R^T \phi(\zeta) \tag{22}$$

Because $a^T \phi(\zeta)$ and $b^T \phi(\zeta)$ in second and third terms respectively on left side of equation (22) are just the polynomials or degree $2n$, equation (22) can be re-written as,

$$C^T \phi(\zeta) + C^T \Theta \left[\phi(\zeta) \left(a^T \phi(\zeta)\right)\right] + C^T \Theta^2 \left[\phi(\zeta) \left(b^T \phi(\zeta)\right)\right] = R^T \phi(\zeta) \tag{23}$$

Here, $\phi(\zeta) \left(a^T \phi(\zeta)\right)$ is a vector of type

$$\left(\phi_o(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta),\ \phi_1(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta),\ \ldots,\ \phi_n(\zeta) \sum_{k=0}^{k=n} a_k \phi_k(\zeta) \right)$$

$$\equiv (\psi_o(\zeta), \psi_1(\zeta), \ldots, \psi_n(\zeta)) = \psi(\zeta), \quad (say!) \tag{24}$$

In equation (24), each $\psi_k(\zeta)$ can be approximated as a linear combination of orthonormal polynomials in the form $\psi_k(\zeta) = A_k^T \phi(\zeta)$, where A_k^T are vectors of form $1 \times (n+1)$ for $k = 1, 2, \ldots, n$, and, therefore, $\phi(\zeta) \left(a^T \phi(\zeta)\right) = \psi(\zeta) = A\phi(\zeta)$, where $A = \left(A_o^T, A_1^T, \ldots, A_n^T\right)_{(\underline{n+1} \times 1)}$. Similarly, $\phi(\zeta) \left(b^T \phi(\zeta)\right)$ can be approximated as $B^T \phi(\zeta)$ for some vector $B^T = (B_o, B_1, \ldots, B_n)_{(\underline{n+1} \times 1)}$ such that B_k^T are real vectors of form $1 \times (n+1)$. With these intermediate approximations, equation (23) can be written as :

$$C^T \phi(\zeta) + C^T \Theta A \phi(\zeta) + C^T \Theta^2 B \phi(\zeta) = R^T \phi(\zeta) \tag{25}$$

From equation (25), the required coefficient vector C is obtained as:

$$C^T = R^T \left(I + \theta A + \theta^2 B\right)^{-1}, \tag{26}$$

where, I is identity matrix of order n. The expression for $y(\zeta)$ is obtained as:

$$y(\zeta) = C^T \Theta^2 \phi(\zeta) \tag{27}$$

7. Examples

In order to discuss and establish the accuracy and efficacy of the present method, following examples have been taken.

Example 7.1. *Let us consider the IVP*

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = e^{-x}; \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \tag{28}$$

which has exact solution $y(x) = e^{-\frac{5}{2}x}\left(\cosh(\frac{\sqrt{13}}{2}x) + \frac{3}{\sqrt{13}}\sinh(\frac{\sqrt{13}}{2}x)\right) - e^{-x}$.

Comparing equation (28) with equation (12) and taking $n = 6$, equations (14 - 17) yield

$$C^T = (0.08086, -0.02459, -0.00156, -0.00240, -0.001046, -0.000425, -0.00017, 0) \tag{29}$$

Using value of C^T in equation (18), an approximate solution is obtained as:

$$y(x) \approx -0.00009x + 0.25057x^2 - 0.08655x^3 + 0.07415x^4 \\ -0.10716x^5 + 0.0930x^6 - 0.03382x^7 \tag{30}$$

A comparison of approximation (30) with exact solution of IVP (28) has been shown in Table 1. Maximum magnitude of the error between the two solutions is of order 10^{-6} for $n = 6$.

TABLE 1. Comparison of exact solution $y_e(x)$ and present approximation $y_a(x)$ for $n = 6$ (example 7.1).

x	$y_e(x)$	$y_a(x)$	Absolute Error
0.1	0	5.50233×10^{-18}	5.50233×10^{-18}
0.1	0.004107110	0.004107109	0.000000001
0.2	0.013580700	0.013580702	0.000000002
0.3	0.025412500	0.025412513	0.000000013
0.4	0.037786300	0.037786295	0.000000005
0.5	0.049646000	0.049645986	0.000000014
0.6	0.060415500	0.060415530	0.000000030
0.7	0.069818500	0.069818515	0.000000015
0.8	0.077762700	0.077762500	0.000000200
0.9	0.084265300	0.084265100	0.000000200
1	0.089405800	0.089407400	0.000001600

Example 7.2. *Consider the IVP*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \tan(x); \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \tag{31}$$

which is linear in nature but its not easy to solve manually in terms of simply known mathematical functions. We will compare the present solution of this IVP with the one generated by Mathematica.

Comparing equation (31) with equation. (12) and taking $n = 9$, equations (14 - 17) yield

$$C^T = (5.1220,\ 5.5181,\ 2.9304,\ 1.0668,\ 0.2958,\ 0.0663,\ 0.0125,\ 0.0020,\ 0.0003) \tag{32}$$

Using value of C^T (32) in equation (18), an approximate solution is obtained as:

$$y(x) \approx 0.0001x - 0.0025x^2 + 0.1942x^3 + 0.04799x^4 + 0.7521x^5 \\ -0.9599x^6 + 1.5043x^7 - 0.9351x^8 + 0.3669x^9 \tag{33}$$

TABLE 2. Comparison of exact solution $y_e(x)$ and present approximation $y_a(x)$ for $n = 9$, (Example 7.2)

x	$y_e(x)$	$y_a(x)$	Absolute Error
0	6.02836×10^{-17}	-0.000000913	0.000000913
0.1	0.000189754	0.000189495	0.000000259
0.2	0.001745100	0.001745300	0.000000200
0.3	0.006838420	0.006838370	0.000000050
0.4	0.019013800	0.019013700	0.000000100
0.5	0.044018900	0.044019100	0.000000200
0.6	0.091126400	0.091126300	0.000000100
0.7	0.175237000	0.175236000	0.000001000
0.8	0.320222000	0.320217000	0.000005000
0.9	0.564237000	0.564210001	0.000026999
1	0.968188000	0.968176383	0.000011617

This solution (33) is compared with exact solution of IVP (31) and observed absolute error of order 10^{-5} for $n = 9$ (*cf*. Table 2).

Example 7.3. *In order to demonstrate the case 6.3, let us take the IVP*

$$\frac{d^2y}{dx^2} + \tan(x)\frac{dy}{dx} + 2\cos^2(x)y = 2\cos^4(x); \quad y(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \tag{34}$$

The exact solution of this IVP is $y(x) = 2 - 2\cos(\sqrt{2}\sin^2 x) - \sin x$.

Using the method discussed in *section* 6.3, coefficient vector C^T and approximate solution $y(x)$ of example (34) is obtained for $n = 6$ as:

$$C^T = (0.78730, 0.62821, 0.10352, -0.02660, 0.00101, 0.00136, 0.00011) \tag{35}$$

$$\begin{aligned} y(x) \approx & -0.00025 + 0.00718x + 2.94625x^2 + 0.16696x^3 \\ & -1.57951x^4 + 0.17065x^5 + 0.33315x^6 \end{aligned} \tag{36}$$

Polynomial solution (36) and exact solution of IVP (34) are compared in Table 3. It can be observed that the maximum absolute error is less than 1.5% for $n = 6$.

TABLE 3. Comparison of exact solution $y_e(x)$ and present approximation $y_a(x)$ for $n = 6$ (Example 7.3)

x	$y_e(x)$	$y_a(x)$	Absolute Error
0	-0.000250225	-0.000250000	0.000000225
0.1	0.029959465	0.029941500	0.000017965
0.2	0.117931792	0.117920000	0.000011792
0.3	0.259440594	0.259438000	0.000002594
0.4	0.447388474	0.447384000	0.000004474
0.5	0.672604452	0.672591000	0.000013452
0.6	0.924898498	0.924880000	0.000018498
0.7	1.194399717	1.194340000	0.000059717
0.8	1.473007286	1.472860000	0.000147286
0.9	1.769937120	1.755890000	0.014047120
1	2.083274170	2.044430000	0.038844170

Example 7.4. *For the shake of demonstration of Case−6.2, let us consider an IVP of order 4.*

$$\frac{d^4y}{dx^4}+6\frac{d^3y}{dx^3}-64\frac{d^2y}{dx^2}-54\frac{dy}{dx}+495y=\left(1+x-3x^2\right)\tan(x);$$
$$y(0)=\left(\frac{dy}{dx}\right)_{x=0}=\left(\frac{d^2y}{dx^2}\right)_{x=0}=\left(\frac{d^3y}{dx^3}\right)_{x=0}. \tag{37}$$

Comparing equation (37) with equation (19) and proceeding as in examples (7.1-7.3) for $n = 11$, equations (20-21) gives−

$$C^T=(2.09613,\ 2.21139,\ 1.16833,\ 0.43614,\ 0.11821,\ 0.02785,$$
$$0.00462,\ 0.00091\,,0.00007,\ 0.00002,\ 0.00000,\ 0.00000) \tag{38}$$
$$y(x)\approx -0.000095x^2+0.001101x^3-0.006754x^4+0.032691x^5-0.059672x^6$$
$$+0.08871x^7-0.07522x^8+0.04063x^9-0.01085x^{10}+0.00144x^{11} \tag{39}$$

The polynomial approximation (39) is compared with the *Mathematica* generated numerical solution of (37) in Table 4 and found that the present approximation is close enough to high degree numerical solution by *Mathematica*.

Furthermore, the errors of approximation between present and *Mathematica* generated numerical solution to example 7.4 is compared for for different values of n ($n = 5, 8,\ 11$) through graphical representation, figure (1). It can be observed that errors decrease very fast with increase of n−the degree of Bernoulli polynomials. Absolute error for $n = 5, 8$ and 11 are of order $10^{-4}, 10^{-6}$ and 10^{-9} respectively. This also shows the convergence of approximation for this example.

Table 4. Comparison of exact solution $y_e(x)$ and present approximation $y_a(x)$ for $n = 11$ (Example 7.4).

x	$y_e(x)$	$y_a(x)$	Absolute Error
0	0.000000000000000	-0.000000031941000	0.000000031941000
0.1	0.000000079123100	0.000000006947800	0.000000072175300
0.2	0.000002463150000	0.000002462140000	0.000000001010000
0.3	0.000018617100000	0.000018617099966	0.000000000000034
0.4	0.000079649800000	0.000079649799952	0.000000000000048
0.5	0.000251261000000	0.000251261000060	0.000000000000060
0.6	0.000657129000000	0.000657128999930	0.000000000000070
0.7	0.001516360000000	0.001516359999000	0.000000000001000
0.8	0.003203730000000	0.003203730000000	0.000000000000000
0.9	0.006346460000000	0.006346460000000	0.000000000000000
1	0.011979500000000	0.011979717086757	0.000000217086757

8. Conclusion

In this work, a new method has been presented and demonstrated to find fast and approximate solution of linear initial value problems with help of orthogonal Bernoulli polynomials. A set of n orthonormal polynomials based on Bernoulli polynomials up to degree n and an operational matrix developed with these new family of polynomials were used as a tool to solve an ordinary differential equations with initial value conditions. The present method converts a given initial value problem into a system of algebraic equations with unknown coefficients, which are easily obtained with the help of operational matrix,

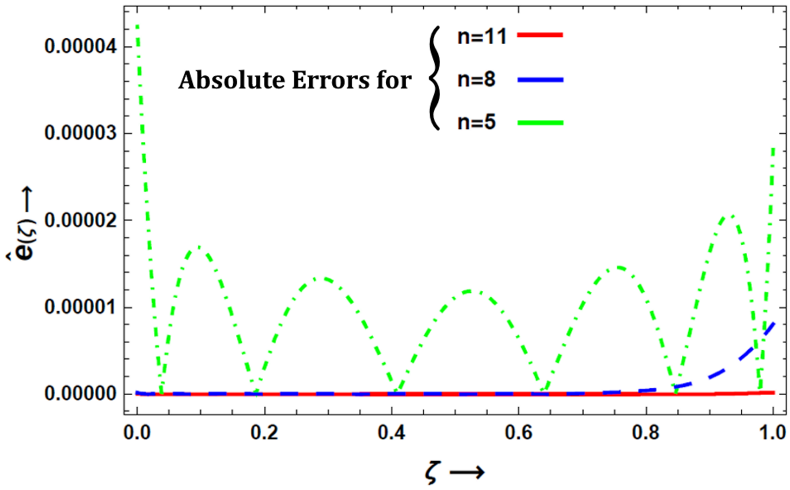

FIGURE 1. Comparison errors of approximation to solution of IVP 7.4 for $n = 5, 8$ and 11 .

and finally an approximate solution is obtained in form of a polynomial of degree n. The method has been demonstrated with four examples including both type of linear differential equations with constant and variable coefficients. With the last example (example 7.4), the absolute errors for different values of n has been compared to show the convergence of solution and decrease of errors with the degree of Bernoulli polynomials $B_n(x)$. The main features of this method can be summarized as:

- the method is programmable, fast and easily adaptable.
- solutiion is obtained in form of a polynomial of degree n which can be easily used for various applications.
- error can be minimized up to required accuracy because error decreases quickly with increase of n-the degree of Bernoulli polynomials.
- error is negligible for simple IVPs with constant coefficients.

APPENDIX A

First ten orthonormal polynomials derived with Bernoulli polynomials are:

$$\phi_0(\zeta) = 1 \tag{40}$$

$$\phi_1(\zeta) = \sqrt{3}(-1 + 2\zeta) \tag{41}$$

$$\phi_2(x) = \sqrt{5}\left(1 - 6x + 6x^2\right) \tag{42}$$

$$\phi_3(\zeta) = \sqrt{7}(-1 + 12\zeta - 30\zeta^2 + 20\zeta^3) \tag{43}$$

$$\phi_4(\zeta) = 3(1 - 20\zeta + 90\zeta^2 - 140\zeta^3 + 70\zeta^4) \tag{44}$$

$$\phi_5(\zeta) = \sqrt{11}(-1 + 30\zeta - 210\zeta^2 + 560\zeta^3 - 630\zeta^4 + 252\zeta 5^) \tag{45}$$

$$\phi_6(\zeta) = \sqrt{13}\begin{pmatrix} 1 - 42\zeta + 420\zeta^2 - 1680\zeta^3 \\ +3150\zeta^4 - 2772\zeta^5 + 924\zeta^6 \end{pmatrix} \tag{46}$$

$$\phi_7(x) = \sqrt{15}\begin{pmatrix} -1 + 56x - 756x^2 + 4200x^3 \\ -11550x^4 + 16632x^5 - 12012x^6 + 3432x^7 \end{pmatrix} \tag{47}$$

$$\phi_8(x) = \sqrt{17}\begin{pmatrix} -1 + 72x - 1260x^2 + 9240x^3 - 34650x^4 \\ +72072x^5 - 84084x^6 + 51480x^7 - 12870x^8 \end{pmatrix} \tag{48}$$

$$\phi_9(x) = \sqrt{19}\begin{pmatrix} -1 + 90x - 1980x^2 + 18480x^3 - 90090x^4 + 252252x^5 \\ -420420x^6 + 411840x^7 - 218790x^8 + 48620x^9 \end{pmatrix} \quad (49)$$

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