Gravitational Closure of Matter Field Equations

General Theory and Perturbative Solutions

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation von

M.Sc. (hon.) Florian Wolz

2022

Mitglieder der Promotionskommission:

Prof. Dr. Elmar Schrohe (Vorsitzender)

Prof. Dr. Domenico Giulini (Betreuer)

Prof. Dr. Frederic P. Schuller

Gutachter:

Prof. Dr. Domenico Giulini

Prof. Dr. Frederic P. Schuller

Prof. Dr. Björn-Malte Schäfer

Tag der Promotion: 20. Mai 2022

KURZZUSAMMENFASSUNG

Wenn wir die Geometrie der Raumzeit und Materie gemeinsam in einem kanonischen Formalismus beschreiben wollen, bleiben uns nur wenige Möglichkeiten den gravitativen Freiheitsgraden Diffeomorphismen-invariante Dynamik zu verleihen. Konkret ist es möglich, wie wir im Rahmen der Arbeit zeigen werden, die gravitative Dynamik anhand jeder Materietheorie *herzuleiten*, welche drei physikalisch unabdingbare Bedingungen erfüllt und auf einer tensoriellen Geometrie formuliert ist. Hierfür löst man ein abzählbares System aus linearen partiellen Differentialgleichungen, den gravitativen Abschlussgleichungen, und erhält das gravitative Wirkungsfunktional auf konstruktive Art und Weise. Wie in dieser Arbeit demonstriert, ist es möglich die allgemeine Relativitätstheorie, ausgehend von den Materiefeldern des Standardmodels der Teilchenphysik, durch Lösen dieses Systems von partiellen Differentialgleichungen herzuleiten. In Anwendungsfällen in denen das Verhalten von schwachen Feldern von Interesse ist vereinfacht sich dieses System weiter zu linearen algebraischen Gleichungen für die Expansionskoeffizienten des Wirkungsfunktionals. Dies wurde in dieser Arbeit genutzt um die linearen Bewegungsgleichungen einer Gravitationstheorie herzuleiten welches die Doppelbrechung vom Licht im Vakuum erlaubt.

SCHLAGWÖRTER

Modifizierte Gravitationstheorie, Klassische Materiefelder, Perturbationstheorie

ABSTRACT

If we require that both the geometry of spacetime and matter canonically evolve together, we are left with only few options to provide diffeomorphism-invariant dynamics to the gravitational degrees of freedom. Concretely, as we will show in this thesis, it is possible to *derive* the gravitational dynamics from any matter theory, which fulfils three physically essential conditions and is formulated on a tensorial geometry. For this, one solves a countable set of linear partial differential equations, the gravitational closure equations, and obtains the gravitational action functional in a constructive fashion. As demonstrated in this thesis, one can obtain general relativity starting from the matter fields of the standard model of particle physics by solving this system of partial differential equations. In applications where the behaviour of weak fields is of interest, this system further simplifies to linear algebraic equations for the expansion coefficients of the action functional. This was used in this thesis to derive the linear equations of motion of a gravitational theory that allows for birefringence of light in vacuo.

KEYWORDS

Modified Gravitational Theory, Classical Matter Fields, Perturbation Theory

CONTENTS

LIST OF FIGURES

LIST OF TABLES

"Gravity is working against me,
Gravity is trying to bring me down
Oh, twice as much ain't twice as good
And can't sustain like one half could
It's wanting more that's gonna send me to my knees"
— John Mayer

CHAPTER 1
INTRODUCTION

After centuries of progress in understanding the nature of the universe, the benevolent estimate is that we can successfully describe around 5% of its content. The standard model of particle physics, which allows for extremely high precision predictions such as the gyromagnetic factor of the electron, seems to describe all but the gravitational interaction. The latter is then presumably described by Einstein's theory of general relativity, with similar success as certified by indirect measurements of binary pulsars and direct measurements of gravitational waves of black hole mergers. Combined to the cosmological ΛCDM model, one can then describe the evolution of the entire universe since its inception.

Assuming all of these theories, both the standard model and general relativity, are the correct descriptions, one surprisingly finds that the ordinary matter can only account for 5 percent of all the matter content; 25 percent must be attributed to what is called dark matter, that is matter that does not interact via electromagnetic interaction. The remaining enormous amount of 70 percent is due to what is called dark energy (compare figure 1.1 for an illustration). As of today, we have no proper candidates for both dark matter and energy, such that for decades we have tried to gain more insight into their exact nature.

While the standard model and ΛCDM are quite successful in their classical scopes of application, there is a steadily growing stack of effects that it fails to explain. For instance, there is the *satellite plane problem*: for satellite galaxies of the Milky Way and the Andromeda, galaxy one finds that they align in flattened planes (Pawlowski, 2018). This is unexpected since from simulations based on the ΛCDM paradigm, one finds that the experimentally found distributions are extremely unlikely to occur.

Historically, one of the main reasons dark matter was introduced was to explain the rotation curves in galaxies. Recent studies such as Piña et al. (2019) and the analysis in Piña et al. (2021) of the galaxy AGC114905 indicate that the galaxies identified by the authors contain close to no dark matter. Also, in this case, simulations show that the probability for such galaxies is rather unlikely.

Another observation, while typically stated as evidence for dark matter, provides a scenario that is difficult to explain: in the so-called *bullet cluster* (1E 0657-56) that consists of two colliding clusters of galaxies (compare figure 1.2), one finds from separate experiments that the observed gravitational pull from spacetime curvature is stronger than expected from the visible matter. However, from X-ray observations, one infers that the necessary collision speed in the Bullet cluster must have been around $3000\mathrm{km\,s^{-1}}$ to be compatible with the ΛCDM model. Such an enormous velocity is again extremely unlikely: one recent

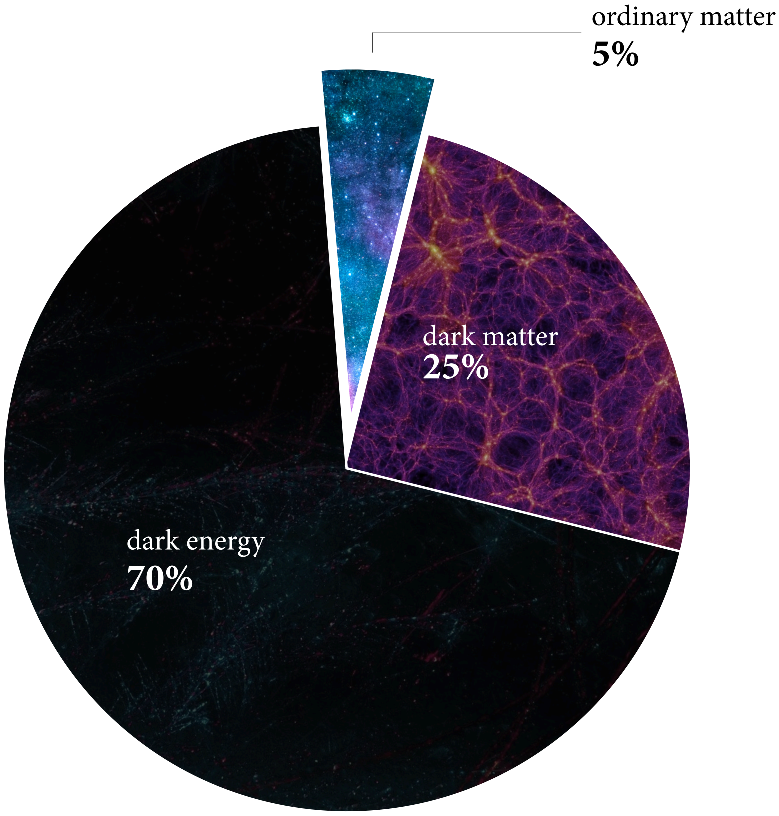


Figure 1.1 Energy content of the universe according to the standard model of particle physics and general relativity

*S*ources: Max-Planck-Institute for Astrophysics Garching and Pixabay

simulation estimated the probability for this to be around 6.4×10^{-6} (Bouillot et al., 2015).

One alternative to the introduction of additional particles in the standard model in the form of dark matter is provided by *modified Newtonian dynamics* (MOND) (Milgrom, 1994) that, as the name indicates, supplies a modification to Newton's law of gravity at low accelerations. One of its predictions is the *external field effect* that, due to the non-linearity in the acceleration in a modified Newton's law, states that an external acceleration influences the internal dynamics of a system. This effect is not compatible with the strong equivalence principle. Recent studies such as Chae et al. (2020) and Chae et al. (2021) provide observational evidence in favour of the external field effect in a MOND theory.

While some of the results are dependent on the particular MOND model, there seems to be an increasing amount of evidence *against* dark matter and the ΛCDM model. This points in the direction that we may need another explanation on how to fill in the mysterious gaps in figure 1.1. See for example the references Perivolaropoulos and Skara (2021), Kroupa (2015), Bullock and Boylan-Kolchin (2017) or McGaugh et al. (2016) for further details on problems of the standard model of cosmology.

But if dark matter may, in fact, not be the correct approach to fix the problems such as the rotational curves, what alternative ways exist? One possibility that has attracted a lot of research interest in recent years is modified gravity that modifies the theory for the interactions themself instead of introducing additional particles to accommodate for the observational mismatches for the gravitational interaction.

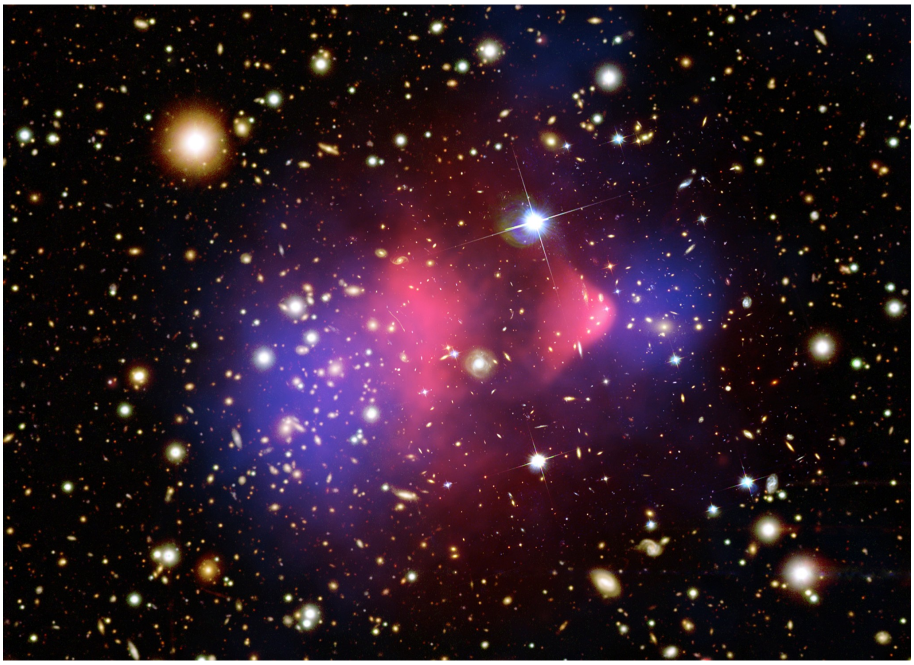


Figure 1.2 The Matter of the Bullet Cluster. The red colored regions show the distribution of hot gas (as inferred via X-ray measurements). The blue-colored regions indicate the space-time curvature (measured by gravitational lensing)

Source: NASA APOD August 24 2006

GRAVITATIONAL BIREFRINGENCE

One physically interesting effect we can introduce to modify the nature of gravitational interaction is the so-called *gravitational birefringence*, which occurs once we allow for inhomogeneous geometries. This effect was, for example, analysed in the effective setup of standard model extension (Kostelecký and Mewes, 2002, 2006) that allows obtaining a global bound on birefringence to a sensitivity of 10^{-38}. Note, however, that this reduces to a search for *global deviations* from Lorentz symmetry, which is not a necessity since these deviations very well may be *local*. In order to correctly derive constraints on these effects – and if it may be for trying to rule out its existence – it is necessary first to set up a consistent description that *also* takes gravitational interaction into account, since the effect itself may only occur in the presence of non-negligible gravitational fields.

But how do we get the dynamics of a theory allowing for gravitational birefringence? The first step is to analyse the light propagation with gravitational birefringence. It turns out, as shown in Hehl et al. (1999); Hehl and Obukhov (2003), that one theory that still obeys a superposition principle of solutions is given in the form of a Yang-Mills type theory called *general linear electrodynamics*. Its action functional reads

$$\mathcal{S}_{\mathrm{GLED}}[G, A] = -\frac{1}{4}\int \mathrm{d}x\, \psi(G) G^{abcd} F_{ab}[A] F_{cd}[A]\,, \tag{1.1}$$

where F denotes the typical curvature 2-form, i.e. containing both the electric fields $E_\alpha = F_{0\alpha}$ as well as the magnetic field density $B^\alpha = \epsilon^{\alpha\beta\gamma}F_{\beta\gamma}$. What is new in this theory is that, instead of the usual metric term $\sqrt{-g}g^{a[c}g^{d]b}$ of Maxwellian electrodynamics, there appears a more general term G^{abcd} and a de-densitization $\psi(G)$ that is built with the help of G. This object G, that we will refer to as *area metric* in the following, provides the most general linear constitution relation between excitations $H = (\mathcal{D}, \mathcal{H})$, that are measurable via *charges*, and the fields strengths F. The gravitational interaction we must then obtain, by some means, is an additional action functional for the area metric.

One might expect that birefringence in vacuo, if possible, would appear on cosmological scales. However, it turns out (Düll et al., 2020) that, due to the homogeneity, a cosmological area metric almost entirely reduces to the expressions one finds in general relativity: one obtains the scale factor $a(t)$ known from a Friedman-Robertson-Walker spacetime, but also an additional scalar $c(t)$ that measures the deviation from metric geometry. As a result, vacuum birefringence on the cosmological scales is suppressed and only arises for local anisotropies: These, however, can only be adequately judged once the dynamical equations for an area metric are known.

While seemingly controversial, such a geometry is not entirely new: as was shown in Drummond and Hathrell (1980) one encounters an area metric in the effective action of photons once the one-loop vacuum polarizations are taken into account on a general curved background. The effective action takes the form

$$\mathcal{S}_{\text{photon}}[A] \propto \int \mathrm{d}^4x \, \sqrt{-g} \left(g^{a[c}g^{d]b} + \lambda C^{abcd} \right) F_{ab}F_{cd} + \mathcal{O}(\lambda^2) \,. \tag{1.2}$$

As stated before, the dynamical equations of such an area metric are *not* known. The pedestrian approach would be to write down the most general linear combination of all possible terms one could come up with and then rule out the separate terms by experiments. This will be cumbersome since, depending on the number of constants of nature that we end up with, we need to perform *at least* as many experiments until we end up with a predictive theory.

Pre-Metric 2-form gravity

We will now try to follow this pedestrian approach for the area metric. One "natural" framework to write down all possible terms one could end up with is the language of differential forms. For a four-dimensional spacetime, a Lagrangian is a 4-form that we need to construct with the help of the area metric. In order to obtain differential equations rather than algebraic equations for the gravitational field, we need to come up with a way to differentiate. For differential forms, this is performed with the help of the exterior derivative that takes a n-forms into $n+1$ forms. Furthermore, we can combine forms with the help of the wedge product. The general Lagrangian is then a linear combination of all the 4-forms we can construct using those operations, i.e. differentiating the configuration variables and combining the differential forms into 4-forms via the wedge product. Since there is only so much we can write down using these operations, this seems like the perfect setup for approaching the search for gravitational field equations for the area metric.

This, of course, requires our configuration variables to be some n-form on spacetime, which is a priori not the case for the area metric. Due to its symmetries, it induces a metric on the six-dimensional space of 2-forms that can be used to classify the different area metrics. As was shown in Schuller et al. (2010), the subspace of area metrics that includes the ones induced by Lorentzian metrics g of signature

$(1, -1, -1, -1)$, is the one with signatures $(-1, -1, -1, 1, 1, 1)$.

We can then bring any of the area metrics of this class into a normal form $N = \text{diag}(-1, -1, -1, 1, 1, 1)$ at each spacetime point with the help of frames and their duals. This procedure is similar to the vielbeins one introduces in the Palatini formulation of general relativity. We write

$$G^{abcd} = N^{AB} e_A^{ab} e_B^{cd} \qquad \text{and} \qquad G_{abcd} = N_{AB} \epsilon^A_{ab} \epsilon^B_{cd} , \tag{1.3}$$

with the duality relation $\epsilon^A_{ab} e_B^{ab} = \delta^A_B$ and the indices $a, b, c, d = 1, \ldots, 4$ and $A, B = 1, \ldots, 6$.

These frames are determined up to $\mathrm{SO}(3,3)$ transformations, which we can treat as gauge freedom of our configuration variables. In particular this means that the Lagrangian 4-form we construct must be equivariant under $\mathrm{SO}(3,3)$. Note that this is a quite similar construction to the one made in the tetrad formulation of general relativity. In fact, given vielbeins $e^a_{(i)}$ and their inverses $\epsilon^{(i)}_a$ the 2-forms can be induced in the following fashion by

$$\begin{aligned} \epsilon^1 &= \epsilon^{(1)} \wedge \epsilon^{(2)} & \epsilon^2 &= \epsilon^{(1)} \wedge \epsilon^{(3)} & \epsilon^3 &= \epsilon^{(1)} \wedge \epsilon^{(4)} \\ \epsilon^4 &= \epsilon^{(2)} \wedge \epsilon^{(3)} & \epsilon^5 &= \epsilon^{(2)} \wedge \epsilon^{(4)} & \epsilon^6 &= \epsilon^{(3)} \wedge \epsilon^{(4)} \end{aligned} \tag{1.4}$$

and their respective duals constructed analogously by antisymmetrizations of the $e^a_{(i)}$. Note that this, however, only exists if we are equipped with the vielbeins in the first place.

Using the frames, as well as the de-densitization $\psi(G)$, we can also define a family of metrics ε_ψ and its inverses on our space of frames via

$$\varepsilon_\psi^{AB} = \psi(G) \cdot \epsilon^A_{ab} \epsilon^B_{cd} \varepsilon^{abcd} , \tag{1.5a}$$

$$\varepsilon^\psi_{AB} = \psi(G)^{-1} \cdot e_A^{ab} e_B^{cd} \varepsilon_{abcd} , \tag{1.5b}$$

where we used the totally antisymmetric density ε_{abcd} and its inverse. If we could only employ these objects, the only viable candidates for a 4-form are given by

$$N_{AB}\, \epsilon^A \wedge \epsilon^B \qquad , \qquad \varepsilon^\psi_{AB}\, \epsilon^A \wedge \epsilon^B . \tag{1.6}$$

The first one, in the metric limit where the frames ϵ^A are induced by existing vielbeins as in (1.4), corresponds to a cosmological constant term. The second one essentially provides us with a term proportional to the de-densitization in the Lagrangian. Clearly, from the lack of any derivatives in the expressions, none of those terms by themselves describe a particularly exciting universe.

Differentiation

We will now rectify this by introducing derivatives of the frames. Since each ϵ^A is a 2-form, the natural derivatives are the 3-forms obtained by exterior differentiation $(\mathrm{d}\epsilon^A)$. It suffices to consider first derivatives since, due to Schwarz's theorem, we have that $\mathrm{d}^2 \equiv 0$. This provides us with finitely many terms we could write down.

However, it is easy to see that under a $\mathrm{SO}(3,3)$ gauge transformation $U^A{}_B$ the 3-form we obtained via exterior differentiation transforms non-trivially, i.e.

$$\mathrm{d}\left(U^A{}_B \epsilon^B\right) = U^A{}_B \left(\mathrm{d}\epsilon^B\right) + \left(\mathrm{d}U^A{}_B\right) \epsilon^B . \tag{1.7}$$

The solution to this is well-known: we need to introduce the covariant derivative D for a connection 1-form $\omega^A{}_B$, where the connection transforms under a gauge transformation such that the covariant derivative transforms like a $\mathrm{SO}(3,3)$ tensor, i.e. for the covariant derivative defined by

$$(\mathrm{D}\epsilon)^A := \mathrm{d}\epsilon^A + \omega^A{}_B \wedge \epsilon^B\,, \tag{1.8}$$

we have under a transformation

$$\omega^A{}_B \longrightarrow U^A{}_M\,\omega^M{}_N\,\left(U^{-1}\right)^N{}_B - \left(U^{-1}\right)^M{}_B\,\mathrm{d}U^A{}_M\,, \tag{1.9}$$

$$(\mathrm{D}\epsilon)^A \longrightarrow U^A{}_B\,(\mathrm{D}\epsilon)^B\,. \tag{1.10}$$

This, however, comes with the price that we need to introduce another object – the connection $\omega^A{}_B$, in the action that we, as a consequence, need to consider in the Euler-Lagrange equations as well. Furthermore, for the covariant derivative, we have $\mathrm{D}^2 \neq 0$ in general. This means that we can obtain further "building blocks" for our Lagrangian by repeated application of the covariant derivative. Since it produces a $n+1$-form from a n-form, we luckily know that there will be finitely many candidates.

For the covariant derivative we find the new object

$$(\mathrm{D}\omega)^A{}_B = \mathrm{d}\omega^A{}_B + \omega^A{}_M \wedge \omega^M{}_B =: \Omega^A{}_B\,, \tag{1.11}$$

which one easily recognizes as the standard *curvature 2-form* to the connection $\omega^A{}_B$. For the second derivative of the frame 2-form we find that it is given in terms of the curvature 2-form and the frame, i.e.

$$(\mathrm{D}^2\epsilon)^A = \Omega^A{}_B \wedge \epsilon^B\,. \tag{1.12}$$

As a result, this does not give any new information, and we do not need to add it to our list of possible building blocks. Moreover, one also finds that the second derivative of the connection vanishes:

$$(\mathrm{D}^2\omega)^A{}_B = (\mathrm{D}\,\Omega)^A{}_B \equiv 0\,. \tag{1.13}$$

As a result, the second derivatives do not give any new information, and it suffices to build our Lagrangian form from the 2-frame, the connection 1-form and the curvature 2-form. This gives us three terms in total

$$\varepsilon^{\psi}_{AB}\epsilon^A \wedge \epsilon^B \qquad , \qquad N_{AB}\,\epsilon^A \wedge \epsilon^B \qquad , \qquad \Omega^A{}_B \wedge \Omega^B{}_A\,. \tag{1.14}$$

All three do not lead to particularly interesting dynamics: the first and second terms do not contain any derivatives, i.e. leads to an algebraic equation of motion that eliminates the frames. The curvature terms do lead to a differential equation; however, the equations are independent of the frame ϵ^A and are, as a result, also not of any use. As long as we cannot find another operation to produce additional building blocks for the Lagrangian, we cannot produce any meaningful Lagrangian for the area metric degrees of freedom.

Hodge operator

In the vielbein formulation of general relativity, given in terms of the Palatini-Holst action (Palatini, 1919; Holst, 1996), one employs the Hodge duality for the Lagrangian form. For the area metric, we can try to define a similar operation to obtain further differential forms on spacetime build by our configuration variables. Indeed, it is possible with the help of the two objects $N^{\cdot\cdot}$ and $\varepsilon^{\cdot\cdot}$ to define the following operations

$$\star^A{}_B := \frac{1}{2}\varepsilon^{\psi}_{NM} N^{MA}\,, \tag{1.15a}$$
$$\widetilde{\star}^A{}_B := \frac{1}{2}\varepsilon_{\psi}^{AM} N_{MB}\,. \tag{1.15b}$$

It is easy to verify that $\widetilde{\star}$ is the inverse of $\star$ and that $\star$ reduces to the usual definition of the Hodge operation for a metric induced area metric. However, while in metric geometry, one finds that $\star^2 = -\mathrm{id}$ this is not true for the operators defined above in our area metric spacetime. As a result, it does not suffice to apply our operations once on the building blocks we constructed so far via differentiation, but in fact, we also need to consider higher powers.

Luckily, due to the Cayley-Hamilton theorem, it is possible to write $\star^6$ in terms of $\star^i$ for $i < 6$, and by extension, all higher powers. This (apparently) guarantees that we only need to consider finitely many terms. Furthermore, it suffices to concentrate on $\star$ since, also by application of Cayley-Hamilton, it is clear that $\widetilde{\star}$ can be expressed as a polynomial in $\star$. This leaves us, in total, with finitely many building blocks that can be obtained by applying our Hodge operation.

Collecting the results we find that we can write the Lagrangian in the form

$$\mathcal{L} = \sum_{k=0}^{5}\sum_{l=0}^{k}\left[a_{kl}\left(\star^l\Omega^A{}_B\right)\wedge\left(\star^k\Omega^B{}_A\right) + b_{kl}N_{AB}\left(\star^k\epsilon^A\right)\wedge\left(\star^l\epsilon^B\right) + c_{kl}\varepsilon_{AB}\left(\star^k\epsilon^A\right)\wedge\left(\star^l\epsilon^B\right)\right]\,, \tag{1.16}$$

with the coefficients a_{kl}, b_{kl} and c_{kl}. In total this gives 63 coefficients that we need to derive by experiments, as stated above.

Scalar functions

There is, however, an unfortunate complication in case of an area metric: with the definition of $\star^A{}_B$ we can calculate the trace to obtain a scalar function from the area metric, i.e.

$$\star^A{}_A = \varepsilon^{AB}N_{AB} = \psi(G)\epsilon_{abcd}G^{abcd}\,. \tag{1.17}$$

As a result, we can write *any* function $f(\star^A{}_A)$ in front of each of the building blocks and obtain a valid Lagrangian form, which a priori turns the coefficients above into arbitrary *functions* of the possible scalar terms. Effectively, this leaves us with infinitely many possibilities or undetermined contributions in the equations of motion.

Even if we choose $\psi(G) := \left(\epsilon_{abcd}G^{abcd}\right)^{-1}$ as our de-densitization we can construct scalar functions since the de-densitization is not unique: for an area metric we can construct

$$\widetilde{\psi}(G) = \left|\det\left(\mathrm{Petrov}(G)\right)\right|^{\frac{1}{6}}\,, \tag{1.18}$$

which is a scalar density of weight 1 (Witte, 2009; Schuller et al., 2010). The tensor $\mathrm{Petrov}(G)$, being 6×6 dimensional matrix of the independent components of the area metric, reads

$$\mathrm{Petrov}(G) = \begin{pmatrix} G_{0101} & G_{0102} & G_{0103} & G_{0112} & G_{0131} & G_{0123} \\ & G_{0202} & G_{0203} & G_{0212} & G_{0231} & G_{0223} \\ \ddots & & G_{0303} & G_{0312} & G_{0331} & G_{0323} \\ & \ddots & & G_{1212} & G_{1231} & G_{1223} \\ & & \ddots & & G_{3131} & G_{3123} \\ & & & \ddots & & G_{2323} \end{pmatrix}. \tag{1.19}$$

As a result, the expression

$$\left|\det\left(\mathrm{Petrov}(G)\right)\right|^{\frac{1}{6}} \epsilon_{abcd} G^{abcd} =: \lambda(G) \tag{1.20}$$

is a scalar function of the geometry and so an arbitrary function $f(\lambda)$ may appear in front of the Lagrangian, or in other words, the objects a_{kl}, b_{kl} and c_{kl} in the Lagrangian are dependent on λ. A priori no choice of de-densitization is favoured over the other and thus we must draw the conclusion that we cannot determine a finite expression for the area metric Lagrangian since we are left with the residual arbitrariness of scalars that can be built from the geometry. This type of argument is not limited to an area metric: If one considers a bi-metric theory, i.e. two Lorentzian metrics g and h, we can construct the scalar $f(g,h) := \sqrt{gh^{-1}}$ which can occur with unlimited complexity in an action functional.

In summary, we can conclude that spelling out independent contributions to the Lagrangian and fixing the constants from experiments will not pay off for the area metric. Ultimately, infinitely many experiments would be necessary to rule out one action over the other. Already from a practical point of view, it should be clear that this will not be an option.

Gravitational Closure

There is, however, an alternative way of finding an appropriate action functional for the gravitational sector that we will derive throughout this thesis. This approach can be used to derive the gravitational Lagrangian for *any* geometry: requiring that a prescribed matter field dynamics – given by an action functional – is predictive and canonically quantisable, as well as demanding that the gravitational dynamics is diffeomorphism invariant in a canonical description, a system of partial differential equations called the *gravitational closure equations* for the coefficients appearing in the gravitational Lagrangian can be derived. The remarkable result is that this boils down a physical problem, the search for gravitational dynamics for the desired geometry, to a purely mathematical problem where one solves a system of partial differential equations. For an illustration of the idea, compare figure 1.3.

In practice, finding a solution to this system may become arbitrarily complex. However, even in these cases, the gravitational closure equations can still be employed to derive the Lagrangian in specific situations: for example, using some symmetry assumption such as cosmological symmetries, it is possible to derive the symmetry reduced Lagrangian (Düll et al., 2020). Similarly, as we will see in this thesis, one can consider the gravitational closure equations in a perturbative regime and derive the field equations

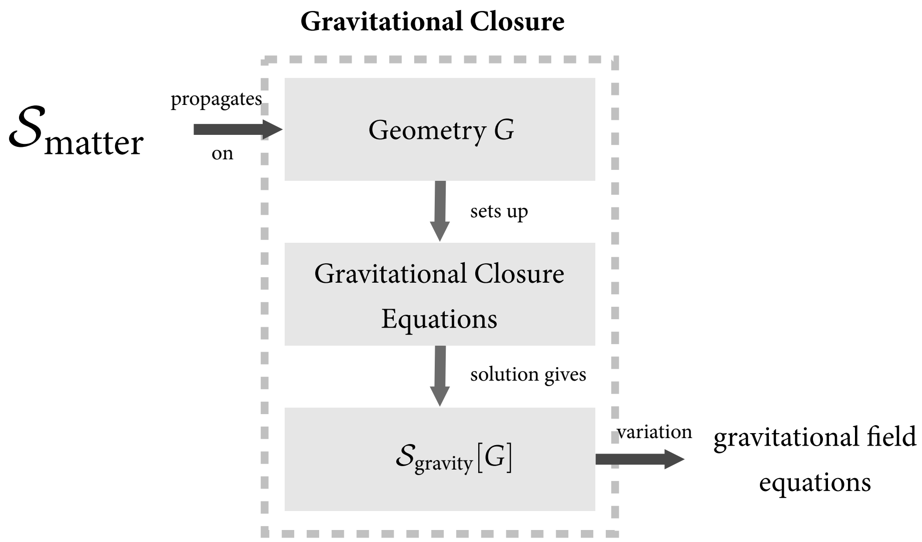


Figure 1.3 Gravitational closure of prescribed matter field equations. The geometry is extracted from the matter field equations. Once the geometry is known it is possible to setup the gravitational closure equations, a system of partial differential equations whose solution gives the gravitational Lagrangian. Once solved, the "closed" system can be used to analyze the common field equations.

for weak perturbations around a Minkowskian background. This equips us with a powerful tool that can be used to study modified gravity theories beyond general relativity.

STRUCTURE OF THIS THESIS

This thesis is structured in the following fashion: we start with a purely kinematical analysis of matter theories. One can impose three conditions on classical field theories that are, from a physical perspective, non-negotiable and ensure that the theory under investigation is both predictive and canonically quantisable. Once imposed, these conditions allow us to obtain a notion of observers and massless as well as massive dispersion relations.

In chapter 3 we will lay out the details of the gravitational closure mechanism. We first review a 3+1 decomposition of spacetime in section 3.1 and present how the action of hypersurface deformations on functionals the foliations reflect the time evolution of the geometry. Afterwards, we can mimic these deformations on a canonical phase space in section 3.2 and show that their behaviour can be cast into a system of linear partial differential equations, whose solutions give the gravitational Lagrangian. Once derived, we continue with discussing some properties of their general solution space and possible simplifications that can be made when solving the system.

In chapter 4 we will then employ the system of partial differential equations to show that the gravitational closure of the geometry of Maxwellian electrodynamics – in fact of the whole standard model of particle physics – yields general relativity with only two constants of integration that need to be fixed by experiments: the gravitational constant G and the cosmological constant Λ. Afterwards, we also set up all the required input coefficients appearing in the gravitational closure equations for the area metric and work towards the solution of the system of differential equations.

We then dedicate chapter 5 to the presentation of an alternative road towards a solution for settings with weak gravitational interactions. In these cases, we can use a perturbative treatment that turns the system of linear partial differential equations into a system of linear *algebraic* equations for the Taylor series expansion coefficients of the gravitational Lagrangian. Additional simplifications can be made by employing the available background geometry to construct the independent tensorial ansätze we can construct. This leaves us with linear equations for scalar coefficients that we can solve to derive the perturbative Lagrangian. We can then apply this to derive the linear field equations for general linear electrodynamics, which physically corresponds to a setting of weakly birefringent electrodynamics.

The thesis then concludes with a brief summary and discussion of the topics of this thesis and open questions and directions worth investigating in the future.

Notation and conventions

Some conventions and notations may need some explanation, and we will quickly take the time to explain the most important of them. When we encounter a hyperbolic metric tensor, we will always assume it to be of $(+,-,-,-)$ signature, i.e. West Coast signature. We will always work in four-dimensional spacetime: there will be multiple ways that we try to modify gravity; however, moving to extra dimensions will be none of them.

Moreover, there will be different types of indices appearing in equations:

$\alpha, \beta, \ldots$	spatial indices
$a, b, \ldots$	spacetime indices
$\mathcal{A}, \mathcal{B}, \ldots$	spatial multi-indices
$A, B, \ldots$	indices for the (gravitational) degrees of freedom

On top, some symbols serve multiple purposes. In particular in the discussion of the formal analysis of involutive system partial differential equation, we adopt the notation used in Seiler (2009), that uses several greek indices to denote different notions:

α	indices labeling the dependent variables
μ	multi-indices in the independent variables
τ	indices labeling the different equations appearing in the system

Wherever there is cause for confusion due to ambiguous notation, we try to explain them in context further. Ultimately, the intended meaning should be clear from the equations.

CHAPTER 2
SPACETIME KINEMATICS

We begin our discussion by considering the kinematics of any suitable matter field theory. It turns out that the equations of motion of a matter field already contain all information about the background geometry the matter field propagates on. Even further, if we require that the dynamics is predictive – in the sense that the system of differential equations is *hyperbolic* – we obtain conditions the geometry must satisfy. As an example, it is well-known that the hyperbolic spacetime geometry, given by a Lorentzian metric with signature $(3, 1, 0)$ – or $(1, 3, 0)$ depending on favoured sign convention – is essential to rendering Maxwell's equations of electrodynamics predictive.

In the very same fashion, one can identify two additional physically quite reasonable – and for a canonical quantisation even necessary – conditions: the first condition essentially describes a duality between the velocities and momenta of massless modes, and the second condition the existence of a split into positive and negative energy modes. The advantage is that, once these three conditions on the geometry are imposed, we have a well-defined notion of observers, light-, space- and time-like vectors and covectors, and even a map between light-like vectors and covectors. While common wisdom tells us that this requires a metric, one can construct such a duality for any geometry that satisfies the three stated conditions. However, in contrast to the metric, this is generally *not* a linear operation.

Our discussion of the kinematical aspects is structured in the following way: we will first expand upon the notion of matter field equations that we will use in this thesis and then move on to the derivation of an object called the *principal polynomial*. Although these constructions are already in use essentially since Hörmander's analysis of linear partial differential equations (Hörmander, 1955), we will see that certain subtleties exist that must be carefully dealt with. Afterwards, we make the mentioned three matter conditions precise before we finally take a closer look at how to construct the object of interest for the gravitation closure mechanism – the Legendre maps between velocities and momenta of massive particle modes.

The results presented in this chapter, in particular the derivation of the principal polynomial in the presence gauge symmetries in section 2.2.1, have already been published as

M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational Closure of Matter Field Equations
Phys. Rev. D97 (2018), 084036

and the description of the Cartan-Kuranishi algorithm and its relation to the principal polynomial, as well as the example presented in 2.2.2 is set to appear in

F. Wolz
Causal Structure of Matter Field Equations
Proceedings of the 15$^{\text{th}}$ Marcel Grossmann Meeting on General Relativity

but have been expanded on several occasions for a self-complete discussion.

2.1 MATTER FIELD EQUATIONS

Before we start, it is important to note that we will restrict ourselves to a specific class of matter theories throughout this thesis. Although the assumptions we impose for these matter theories may certainly be considered weak, it is still possible to construct theories where they are not valid anymore. The five assumptions we make are the following:

Field theories All degrees of freedom are encoded in tensor fields (or densities of a certain weight) on a spacetime manifold $\mathcal{M}$, with the dynamics given by a coupled system of differential equations for those field degrees of freedom.

Local action functionals All the equations of motion, as well as the coupling to other fields, can be obtained from a local action functional.

Local dependence The Lagrangian of the field equations, whose existence is guaranteed by the fact that we have a local action functional, depends only on finitely many derivatives of the fields, all evaluated at the same spacetime point.

Quasi-linear The highest derivative order of the matter field appears *linearly* in the equations of motion.

Ultra-local coupling to geometry The degrees of freedom couple to a background geometry in an ultra-local fashion, i.e. without dependencies on the derivatives of the geometry.

The first condition is the most conservative of them since most modern physical theories are formulated in terms of field theories. The second condition is more restrictive and excludes some theories since there are matter theories that can not be derived from an action functional, at least without introducing Lagrange multipliers as additional degrees of freedom. Luckily, given some equations of motion, it can be decided by the *Helmholtz conditions* if a Lagrangian, and by this a local action functional, can be constructed (Douglas, 1939).

The third condition is required to decide on the highest derivative order coefficient in the equation. The fourth is a quite weak assumption in the setup of gravitational closure: we want to examine the dynamics of a geometry, and for this, it suffices to consider the dynamics of test matter that allows us to probe the background geometry while keeping the backreaction on the geometry arbitrarily small. Such test matter can always be obtained by taking a suitable linearisation of the dynamics.

Last of all, as stated in the fifth condition, it is clearly necessary for the degrees of freedom to couple to the geometry if we want to probe the geometry with the help of test matter. While this may seem trivial, examples of theories such as the Chern-Simons theory exist that can be formulated without any use of a background geometry. For this apparent reason, we exclude such theories. Furthermore, we require that the action functional of the matter field does not depend on the derivatives but only the components of the geometry at each spacetime point.

Now, starting with our action functional for a whole theory describing the dynamics of *all* available degrees of freedom, we split the action into two distinct sectors of interest

$$\mathcal{S}_{\text{universe}}[A_1, \ldots, A_N, G] = \mathcal{S}_{\text{geometry}}[G] + \mathcal{S}_{\text{matter}}[A_1, \ldots, A_N; G)\,, \tag{2.1}$$

where the split encodes our requirement that all matter is coupled to the geometric degrees of freedom, while the geometry itself includes an action for self-coupling, describing the non-linear nature of gravity. Moreover, by the third and fifth assumption, this coupling occurs in an ultra-local fashion, i.e. for a particular matter field A (restricting to $N = 1$ for now), the functional dependencies of the Lagrangian can be spelled out as

$$\mathcal{S}_{\text{matter}}[A; G) = \int_{\mathcal{M}} \mathrm{d}^4x\, \mathcal{L}_{\text{matter}} \left(A(x), \partial A(x), \ldots, \partial^F A(x); G(x) \right)\,, \tag{2.2}$$

for some *finite* F. For the geometry, we only assume that the Lagrangian depends locally on the field, i.e. considers the same spacetime point x in $\mathcal{M}$.

The equations of motion can be obtained from the action by variation and yield two sets of partial differential equations that need to be solved simultaneously. Abstractly, they read

$$0 = \frac{\delta \mathcal{S}_{\text{matter}}}{\delta A^{\mathcal{A}}(x)}\,, \tag{2.3}$$

$$0 = \frac{\delta \mathcal{S}_{\text{geometry}}}{\delta G^{\mathfrak{B}}(x)} + \frac{\delta \mathcal{S}_{\text{matter}}}{\delta G^{\mathfrak{B}}(x)}\,, \tag{2.4}$$

where the multi-indices $\mathcal{A}$ and $\mathfrak{B}$ depends on the specific matter field and geometric field under consideration, respectively. More generally, we will assume $\mathcal{A} = 1, \ldots, R$ and $\mathfrak{B} = 1, \ldots, \overline{R}$ with $\mathcal{A}$ and $\mathcal{B}$ labeling a basis of a R-dimensional / $\overline{R}$-dimensional representation of $\mathrm{GL}(4)$ under which the components of the matter field or the geometry, respectively, transform.

Due to the quasi-linearity – see assumption 4 – we can always bring the equations of motion into the following form

$$Q^{i_1 \ldots i_F}_{\mathcal{AB}} \left(G(x), A(x), \partial A(x), \ldots, \partial^{F-1} A(x) \right) \left(\partial_{i_1} \cdots \partial_{i_F} A^{\mathcal{B}} \right)(x) \\ + \text{terms of lower derivative order in A} = 0\,, \tag{2.5}$$

where the symbol $Q^{i_1 \ldots i_F}_{\mathcal{AB}} \left(G(x), A(x), \partial A(x), \ldots, \partial^{F-1} A(x) \right)$ is *guaranteed* to be a tensor density of weight 1 since the equations of motion are derived from an action functional. Note that, even though these equations are not formulated in terms of a covariant but by partial derivatives the equation as whole is diffeomorphism invariant if it stems from a diffeomorphism invariant action. Any terms that arise from ∂ under a coordinate transformation are compensated by the lower order coefficients $Q^{i_1 \ldots i_p}_{\mathcal{AB}}$ for $p < F$

that do *not* transform as tensors (or densities). However, we will ultimately only need the highest order coefficient and, thus, the construction transforms properly.

In the case of quasi-linear equations, as shown above, the underlying causal structure of the equations is not purely given by the geometric background but also the matter field itself. As a result, the field impacts its own causal structure, or in other words, the traditional separation of matter and geometry breaks down. As a result, we will restrict ourselves to linear test matter for which the field A does *not* appear in the highest order coefficient. In this case, we have that for a solution $A(x)$ of the field equations, we can obtain another valid solution by scaling down A by some $0 < \varepsilon \ll 1$. Physically this encodes that we can make the backreaction to the spacetime geometry G arbitrarily small and use the test matter to probe the properties of the geometry. Note, however, that the construction can also be applied for the quasi-linear case, as we will see in an explicit example in section 2.2.2. Before we continue with the general discussion, we will spell out some of the examples we consider throughout this thesis.

Example 1: Maxwellian electrodynamics

The first well-known example is Maxwell's theory of electrodynamics in its modern covariant formulation that describes the linear propagation of light in a Lorentzian spacetime, also known as Yang-Mills theory with $\mathrm{U}(1)$ gauge symmetry for a covector field A.

DEFINITION MAXWELLIAN ELECTRODYNAMICS

The action functionals of Maxwellian electrodynamics and the Einstein-Hilbert action for the geometry g are given by

$$\begin{aligned} \mathcal{S}_{\mathrm{Maxwell}}[A,g) &= -\frac{1}{4}\int \mathrm{d}^4x\,\sqrt{-g}\,g^{ab}g^{cd}F_{ac}[A]F_{bd}[A]\,, \\ \mathcal{S}_{\mathrm{EH}}[g] &= \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,(R[g]-2\Lambda)\,, \end{aligned}$$

with the curvature 2-form $F = \mathrm{d}A$, i.e. $F_{ab} = \partial_a A_b - \partial_b A_a$, and g and R being a metric and Ricci curvature scalar, respectively. Λ is the cosmological constant.

Calculating the equations of motion by variation of the action yields the two familiar sets of partial differential equations

$$\partial_b\left(\sqrt{-g}F^{ba}\right) = 0\,, \tag{2.6a}$$

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa\left(g^{mn}F_{am}F_{bn} - \frac{1}{4}g_{ab}F^{mn}F_{mn}\right)\,, \tag{2.6b}$$

where the first equation encodes the two inhomogeneous Maxwell equations and the second one being Einstein's field equation with a cosmological constant coupled to the stress-energy-momentum tensor of the electromagnetic field. The homogeneous Maxwell equations are implemented trivially since $\mathrm{d}F = \mathrm{d}^2A \equiv 0$.

Reading off the highest order coefficient of the matter field equation, one obtains

$$Q^{abij}(g) = 2\sqrt{-g}g^{a[b}g^{i]j}\,. \tag{2.7}$$

In this example, we, of course, already know that the action functional of the geometrical degrees of freedom is given by the Einstein-Hilbert action. However, it was shown in Kuchar (1974) and Hojman et al. (1976), as well as Schuller and Witte (2014), that we can also arrive at this action by construction from the Maxwell action. We will recover this in chapter 4.1 by performing gravitational closure of the matter theory $\mathcal{S}_{\text{Maxwell}}$. In some sense, this is precisely what Einstein did: giving the background coefficients of Maxwell's equations consistent dynamics that allow common canonical evolution.

Example 2: General linear electrodynamics

Our second example again considers the more refined theory[1] of light propagation on an area metric spacetime. It was derived based on the following five axioms and describes the most general linear electrodynamics (Hehl et al., 1999; Hehl and Obukhov, 2003):

- **Conservation of electric charge**
- **Existence of the Lorentz force**
- **Conservation of magnetic flux**
- **Local energy-momentum distribution**
- **Existence of an electromagnetic spacetime relation**

Similar to Maxwellian electrodynamics, as presented in the previous section, the degrees of freedom of the electromagnetic field are given by the covector A. However, the notable difference is that one obtains a more general constitutive relation between the field strength two-form $F = \mathrm{d}A$ and the excitation two-form H, i.e.

$$H_{ab} = \frac{1}{4}\epsilon_{abmn}\,\chi^{mnpq}\,F_{pq}\,, \tag{2.8}$$

with the constitutive tensor density χ. By choosing a suitable de-densitization $\omega(G)$, we can turn this tensor density into the rank four tensor field G called the *area metric* we discussed in the introduction. It is equipped with the algebraic symmetries of a Riemann tensor, i.e.

$$\begin{aligned} G^{abcd} &= -G^{bacd}\,, \\ G^{abcd} &= G^{cdab}\,. \end{aligned}$$

This leaves 21 independent degrees of freedom in an area metric. Phenomenologically, an area metric allows for several interesting effects that are not possible in a "traditional" Lorentzian spacetime, most importantly the effect of birefringence in vacuo. If one excludes vacuum birefringence, it turns out that the area metric is induced by a metric (Lämmerzahl and Hehl, 2004). The corresponding action functional of general linear electrodynamics is described in the following box:

[1] As a matter of fact, we deal with a class of theories since an expression for the de-densitization $\omega(G)$ has to be provided. In the end, however, the results should be independent of the chosen de-densitization.

DEFINITION GENERAL LINEAR ELECTRODYNAMICS

The action functional for general linear electrodynamics is given by

$$\begin{aligned}\mathcal{S}_{\text{GLED}}[A,G] &= -\frac{1}{4}\int \mathrm{d}^4x\, \omega(G) G^{abcd} F_{ab}[A] F_{cd}[A] \\ \mathcal{S}_{\text{area metric}}[G] &= \ ?\end{aligned}$$

with the curvature 2-form $F = \mathrm{d}A$, i.e. $F_{ab} = \partial_a A_b - \partial_b A_a$. $\omega(G)$ denotes a choice of de-densitization. An action for the area-metric, however, is unknown.

Since, as of now, to our best knowledge, no exact theory for the gravitational dynamics of an area metric is known, and the initial attempt in the previous chapter left us with a large undetermined part of the Lagrangian. This makes it the perfect testing ground to perform gravitational closure, as laid out in the following chapters, to derive the action of the area metric from the action of general linear electrodynamics.

Keep in mind that, unlike in general relativity, there is certain freedom in the choice of a de-densitization. One popular choice for the de-densitization that we will use in the following is given by

$$\omega(G) := \left(\frac{1}{24}\epsilon_{abcd} G^{abcd}\right)^{-1}, \tag{2.9}$$

with $\epsilon_{....}$ being the Levi-Civita tensor density.

Spelling out the equations of motion for general linear electrodynamics gives a similar expression compared to Maxwell's field equations in their covariant formulation

$$0 \ = \partial_n \left(\omega(G) G^{abmn} F_{ab}\right) . \tag{2.10}$$

The highest order derivative coefficient reads

$$Q^{abij}(G) \ = 2\,\omega(G) G^{aijb} . \tag{2.11}$$

Example 3: Klein-Gordon fields coupled to separate metrics

Another interesting example to consider is the dynamics of two Klein-Gordon fields ϕ and ψ that are coupled to two separate metrics g and h, respectively, without having any interaction terms. The naïve assumption would be that the two sectors are entirely independent of each other. It turns out that this assumption is wrong.

If observers shall be able to describe a common evolution of the fields ϕ and ψ, the background geometry is composed of *both* metrics g and h. The dynamics of this "super"-geometry must occur in such a fashion that the initial data surfaces the observer sees are compatible for all of the fields involved. As a result, it is quite clear from the beginning that the gravitational dynamics *cannot* be given by simply adding together the Einstein-Hilbert actions of both respective metrics, since in this case, there will be solutions of the equations of motion that do not share common initial data surfaces. How its precise Lagrangian has to look like can be derived by performing gravitational closure.

DEFINITION BI-KLEIN-GORDON THEORY

$$\begin{aligned}\mathcal{S}_{\text{bi-KG}}[g,h,\phi,\psi] &= \frac{1}{2}\int \mathrm{d}^4x\,\left[\sqrt{-g}\left(g^{ab}\,\partial_a\,\phi\partial_b\phi + m_\phi^2\,\phi^2\right) + \sqrt{-h}\left(h^{ab}\,\partial_a\psi\,\partial_b\psi + m_\psi^2\,\psi^2\right)\right]\\ \mathcal{S}_{\text{bi-metric}}[g,h] &= ?\end{aligned}$$

For future reference, we also write down the equations of motion, which read

$$\sqrt{-g}\left(\Box_g\,\phi - m_\phi^2\phi\right) = 0\,, \tag{2.12a}$$

$$\sqrt{-h}\left(\Box_h\,\psi - m_\psi^2\psi\right) = 0\,, \tag{2.12b}$$

with the D'Alembert operator of the metric g defined via $\Box_g := \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\,g^{ab}\,\partial_b\right)$ and in the same fashion for the metric h[2]. The highest order coefficients reads

$$Q^{ab} = \begin{pmatrix}\sqrt{-g}\,g^{ab} & 0\\ 0 & \sqrt{-h}\,h^{ab}\end{pmatrix}\,. \tag{2.13}$$

2.2 THE PRINCIPAL POLYNOMIAL

We will now show how to extract the causal structure underlying given matter field equations of motion from their highest-order coefficient Q. For this, we make a Wentzel-Kramers-Brillouin (WKB) expansion to see that we can obtain a dispersion relation for modes with practically infinite frequency, which are physically indistinguishable from massless modes. This approximation, sometimes also called the *geometrical optical limit*, is obtained by inserting the following expansion

$$A^{\mathcal{A}}(x) = \mathrm{Re}\left(\exp\left(\frac{iS(x)}{\lambda}\right)\left(a^{\mathcal{A}} + \mathcal{O}(\lambda)\right)\right) \tag{2.14}$$

into the equations of motion for the matter field A. We then obtain, to lowest order λ^{-F} that

$$Q_{\mathcal{AB}}^{i_1\ldots i_F}\left(G(x)\right)k_{i_1}\cdots k_{i_F}a^{\mathcal{B}} = 0\,, \tag{2.15}$$

with the covector $k_a = -(\partial_a S)(x)$ being the gradient of the eikonal function S. For finite frequency information we would need to consider higher-order terms in λ.

This linear system needs to be solved for the amplitudes $a^{\mathcal{B}}$. Naturally, all the amplitudes lying in the kernel of the operator $Q\cdot k\cdots k$ give rise to the rather uninteresting solutions $a^{\mathcal{B}} \equiv 0$. As a result, we look for all non-trivial modes for the amplitudes $a^{\mathcal{B}}$. Basic linear algebra tells us that this is the case, given that the $R\times R$ dimensional matrix

$$T_{\mathcal{AB}}(G,k) := Q_{\mathcal{AB}}^{i_1\ldots i_F}(G)k_{i_1}\cdots k_{i_F} \tag{2.16}$$

[2]Typically one would divide by the density term in each equation to turn the equation into a scalar equation instead of a scalar density. In order to keep in line with the general procedure presented in the following we will refrain from doing so and simply note that this step is possible and does not impact the results.

is *not* invertible. In other words, this is the case if the determinant of the quadratic matrix T is vanishing, which gives conditions on the possible spacetime covector directions k in which physical massless modes can propagate. This polynomial *density*, defined by

$$P_G(k) := \det_{\mathcal{AB}} \left(T_{\mathcal{AB}}(G,k) \right) , \tag{2.17}$$

gives a *dispersion relation* for massless particles, i.e. modes in the infinite-frequency limit of our expansion, propagating on the geometry G, since only particles with covectors k being a solution of $P_G(k) = 0$ can propagate.

Example: Klein-Gordon field in Lorentzian spacetime

A quite simple example that illustrates this procedure, is a massless Klein-Gordon field in a Lorentzian spacetime. The action is given by

$$\mathcal{S}_{\mathrm{KG}}[\phi;g] = \frac{1}{2}\int \mathrm{d}^4x \left(\sqrt{-g}g^{ab}(\partial_a\phi)(\partial_b\phi)\right)(x)\,, \tag{2.18}$$

and the equations of motion are the well-known massless Klein-Gordon equation

$$\sqrt{-g}\,\Box_g\phi(x) = 0\,. \tag{2.19}$$

with the D'Alembert operator defined as in the previous section. It is easy to see that the highest order coefficient is given by

$$Q^{ij} = \sqrt{-g}g^{ij}\,, \tag{2.20}$$

and the polynomial density in this case becomes the well-known

$$P_g(k) = \sqrt{-g}g^{ab}k_ak_b\,, \tag{2.21}$$

which gives the dispersion relation of massless particles. This tells us that they are propagating on the light-cone of the metric g^{ab}, i.e. propagating with a light-like covector k, as encouraged by terminology.

Historically, this polynomial density was taken to define the principal polynomial we will be chasing after in this section to analyse the causal structure of the matter field equations. Later it was used because this object provided the capability to generalise the geometrodynamics approach to any geometry. Although this definition suffices for many examples, three complications can arise in calculating the principal polynomial that we need to deal with.

The first one is that the matter field equations may contain gauge symmetries. In this case, it does not suffice to identify covectors k that seemingly have non-vanishing amplitudes $a^{\mathcal{B}}$ as they may lie in the gauge orbit of a zero solution. This would imply that we can find a gauge transformation to eliminate this solution. Instead, we need to refine the definition to only find covectors k that are true physical solutions. Mathematically, we need to carefully separate the kernel of the matrix $T[k]$ into the gauge contribution and the physical covectors.

The second complication is related to the notion of *involutivity*. Given that the equations of motion are in non-involutive form, whose precise definition will be given in section 2.2.2, one can find hidden integrability conditions in the system of differential equations that can impact the causal structure – and in some cases, even the differential order of the differential equation. Only once all hidden information is explicitly added to the system we can correctly derive the dispersion relations and all the constructions the gravitational closure mechanism will depend upon. However, given that the system is involutive, all the information is contained in the highest order coefficient. We will make this more precise in section 2.2.2.

The third and final complication arises if the highest order coefficient of the system of differential equations – our equations of motion – is not square. While this cannot occur if we obtain the equations of motion by variation of an action functional, making hidden information explicit requires us to add additional equations to the system, making the symbol $T[k]$ non-square. We will deal with this in section 2.2.3.

2.2.1 Gauge symmetries

As a matter of fact, in almost all field theories we currently use in the standard model of particle physics, some gauge symmetry is present. In the case of the standard model, this is the local $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry, while general relativity has the more complex group $\mathrm{Diff}\mathcal{M}$ of spacetime diffeomorphisms.

In all cases, employing a gauge transformation for a field theory possessing a s-dimensional gauge symmetry in the WKB ansatz used above we find that the amplitude transforms as

$$a^{\mathcal{A}} \longrightarrow a^{\mathcal{A}} + \sum_{\sigma=1}^{s} k_a \chi^{a\mathcal{A}}_{(\sigma)}\,, \tag{2.22}$$

with $\chi^{a\mathcal{A}}_{(\sigma)}$ being s linearly independent coefficient fields that span the s-dimensional linear subspace of the solutions of equation (2.15) that is pure gauge. As a result the solution of the matrix equation $T[k] \cdot a = 0$ will always be at least s dimensional. In order to find at least one non-vanishing solution that is not purely gauge, we must require that the kernel of $T[k]$ is at least $s+1$-dimensional – or equivalently that $\mathrm{rank} T[k]$ is $R-s$-dimensional. A necessary and sufficient condition is that the adjunct, which collects all minors of order $R-s$ vanishes, i.e.

$$T_{\mathrm{adj}}^{[\mathcal{A}_1\cdots\mathcal{A}_s][\mathcal{B}_1\cdots\mathcal{B}_s]}(x,k) := \frac{\partial^s\left(\det T\right)}{\partial T_{\mathcal{A}_1\mathcal{B}_1}\cdots\partial T_{\mathcal{A}_s\mathcal{B}_s}}(x,k) = 0\,, \tag{2.23}$$

for all $\binom{R}{s}$ independent coefficients of the bilinear map defined by T_{adj} on the space of s-forms over the R dimensional representation space in which the gauge field takes its value. By generalization of the argument applied by Itin (2009), one can show that the adjunct matrix T_{adj} can be expressed in terms of the independent coefficient fields from the gauge symmetry and a common homogeneous polynomial density $\widetilde{P}(k)$ of order $F \cdot R - (F+2) \cdot s$:

$$T_{\mathrm{adj}}^{[\mathcal{A}_1\cdots\mathcal{A}_s][\mathcal{B}_1\cdots\mathcal{B}_s]}(x,k) = \epsilon^{\sigma_1\cdots\sigma_s}\epsilon^{\tau_1\cdots\tau_s}\chi^{a_1\mathcal{A}_1}_{(\sigma_1)}\chi^{b_1\mathcal{B}_1}_{(\tau_1)}\cdots\chi^{a_s\mathcal{A}_s}_{(\sigma_s)}\chi^{b_s\mathcal{B}_s}_{(\tau_s)}k_{a_1}\cdots k_{a_s}\cdot k_{b_1}\cdots k_{b_s} \times \widetilde{P}(k)\,. \tag{2.24}$$

The condition for finding a non-vanishing solution that is not gauge reduces to covectors that lie in the null space of the polynomial density

$$\widetilde{P}(x,k) = 0\,. \tag{2.25}$$

Being a homogeneous polynomial in k, we can express the polynomial density it in terms of totally symmetric components

$$\widetilde{P}(x,k) = \widetilde{P}^{a_1 \ldots a_{\deg \widetilde{P}}}(x) k_{a_1}(x) \cdots k_{a_{\deg \widetilde{P}}}(x)\,, \tag{2.26}$$

where $\deg \widetilde{P}$ is the degree of the polynomial. Since the polynomial is, moreover, only defined up to a spacetime function in the first place, we can choose a suitable non-vanishing scalar density $\rho(x)$ of opposite weight to turn the polynomial into a scalar function of same degree, i.e.

$$P(x,k) := \rho(x) \cdot \widetilde{P}(x,k)\,. \tag{2.27}$$

Example 1: Maxwellian electrodynamics

In the Maxwellian electrodynamics case, we have the $\mathrm{U}(1)$ gauge symmetry for transformations of the form

$$A_a \longrightarrow A_a + (\partial_a \Lambda)\,, \tag{2.28}$$

for some arbitrary spacetime function Λ. Moving to the WKB ansatz, this becomes

$$A_a \longrightarrow A_a + k_b \delta^b_a \cdot \lambda\,, \tag{2.29}$$

for some real number λ. As expected for the one-dimensional gauge symmetry, we find one linear independent coefficient

$$\chi^b_{a\,(1)} = \delta^b_a\,, \tag{2.30}$$

which spans the pure gauge subspace of the kernel of $T[k]$. Calculating the adjoint matrix of $T[k]$ that can be read of equation (2.7) we explicitely find that the covector k factors out twice

$$\begin{aligned} T^{\mathrm{adj}}_{ab}[k] &= \sqrt{-g}\, k_a k_b \left(g^{mn} k_m k_n\right)^2 \\ &= \sqrt{-g}\, k_i k_j\, \delta^i_a \delta^j_b \cdot \left(g^{mn} k_m k_n\right)^2\,. \end{aligned} \tag{2.31}$$

Indeed the coefficient for the gauge symmetry appears in the expression as expected from equation (2.23). This tells us that the covectors of the physical massless modes are given by the solution of the polynomial equation

$$\widetilde{P}(k) = \sqrt{-g}\, \left(g^{ab} k_a k_b\right)^2 = 0\,, \tag{2.32}$$

and by multiplying by the scalar density $\rho(x) = \sqrt{-g}^{-1}$

$$P(k) = \left(g^{ab} k_a k_b\right)^2 = 0\,, \tag{2.33}$$

which is a homogeneous polynomial in k of degree 4 as expected and, of course, recovers the well-known result. We will, once we arrived at our final definition of the principal polynomial, always make the convention that the polynomial is de-densitized and repeated factors are removed from the polynomial. In this example, the principal polynomial that we would obtain after de-densitization and removal of repeated factors is the polynomial of degree 2, namely $P(k) = g^{ab} k_a k_b$.

Example 2: General linear electrodynamics

In the case of general linear electrodynamics, the calculation becomes a bit more tedious. The gauge symmetry remains unchanged since we still deal with the $\mathrm{U}(1)$ gauge group. As shown in Itin (2009) one again finds that the adjunct matrix splits into

$$T^{\mathrm{adj}}_{ab}[k] \quad \propto \quad k_a k_b P(k)\,, \tag{2.34}$$

where the polynomial $P(k)$ (already de-densitized here) reads

$$P(k) = -\frac{1}{24}\omega(G)^2\, \epsilon_{mnpq}\epsilon_{rstu} G^{mnri} G^{jpsk} G^{lqtu}\, k_i k_j k_k k_l\,, \tag{2.35}$$

which was first obtained by Rubilar (Rubilar, 2002; Rubilar et al., 2002). Due to the de-densitization $\omega(G)$ we deal with a non-polynomial dependence on the geometry, which provides a rather complicated kinematical structure, compared to the well-known Lorentzian one.

To summarise, equation (2.24) equips us with a way to correctly calculate the dispersion relation of massless particles in the presence of gauge symmetries. However, we still need to ensure that all required information is present in the highest order symbol, which requires that the differential equation is in involutive form. We will dedicate the following section to a more precise explanation of this.

2.2.2 Involutivity

The statement often made that all the causal structure is contained in the highest order coefficient of the equations of motion technically only holds for a specific class of differential equations. In all other cases, there may be information in the lower order coefficients that may also contribute to the causal structure of the equations in a subtle fashion. Luckily, an algorithm exists that allows us to bring *any* system of differential equations into this particular form so that we can indeed derive our principal polynomial again from the highest order coefficient with the techniques described in the previous sections.

This is because there may be hidden integrability conditions in a differential equation of order q, that is, further equations of order q that can be obtained by linear superposition of derivatives of the original. This information is already implicitly present, and any solution of our original differential equation already solves these integrability conditions. However, such an equation does contribute to the causal structure and must be considered in calculating the principal polynomial, for otherwise, wrong conclusions are drawn.

Example for integrability conditions

We start by considering a simple two-dimensional example where the equations of motion are given by two second-order partial differential equations

$$0 \quad = \quad \ddot{f} + \partial_x f\,, \tag{2.36a}$$
$$0 \quad = \quad \partial_x \dot{f} + \dot{f}\,. \tag{2.36b}$$

By differentiating the first equation in x-direction and the second one in t-direction and subtracting both equations we, however, find another differential equation of second-order the solution f must solve, namely the two-dimensional wave equation

$$0 = \ddot{f} - \partial_{xx} f \,. \tag{2.37}$$

No new information was revealed, that is, any solution $f(x,t)$ of the equations (2.36a) and (2.36b) is already a solution of the wave equation. However, if we were to calculate the principal polynomial to derive the equation's causal structure, we need to add this equation to our system to make this information explicit.

Example 2: Modified electrodynamics by Velo-Zwanziger

Another example was given – although for other reasons – in Velo and Zwanzinger (1969). They constructed a Lagrangian in a fully Lorentz-invariant fashion. Still, they showed that, contrary to the common view that Lorentz-invariance in the Lagrangian already guarantees to give the Lorentzian causal structure of special relativity, the causal structure is not Lorentzian anymore. The analysed Lagrangian is a modification of covariant Maxwellian electrodynamics by a Proca mass term and a quartic term in the covector field A, i.e.

$$\mathcal{L}_{\mathrm{VZ}}[A] = -\frac{1}{4}\eta^{ac}\eta^{bd}F_{ab}[A]F_{cd}[A] + \frac{1}{2}m^2\,\eta^{ab}A_aA_b + \frac{1}{2}\lambda\left(\eta^{ab}A_aA_b\right)^2 \,. \tag{2.38}$$

Since the additional terms do not contain any derivatives of A, the highest order coefficient coincides with the one from standard Maxwellian electrodynamics and the expectation is that the causal structure, as a result, also coincides. However, one can derive the lower order constraint equation from the equations of motion,

$$0 = \partial_a\left(\left(1 + \lambda m^{-2}(\eta^{ij}A_iA_j)\right)\eta^{ab}A_b\right) , \tag{2.39}$$

that leads to another second-order equation by differentiation. They found that this term contributes to the principal polynomial in the following fashion

$$P(k) = (\eta^{ab}k_ak_b)^3 \cdot \left[\eta^{ij}k_ik_j + \lambda\, m^{-2}\left((\eta^{ij}k_ik_j)(\eta^{mn}A_mA_n) + 2(\eta^{ij}k_iA_j)^2\right)\right] . \tag{2.40}$$

This is a product of the standard term $\eta^{ab}k_ak_b$ and a modification term, resulting in massless modes propagating with another speed of light that depends on the particular solution A_a to the equations of motion.

Differential equations and their geometric and principal symbol

In order to systematically deal with this issue of revealing hidden integrability conditions, we take a closer look at the theory of *involutive differential equations* and the *Cartan-Kuranishi algorithm*. This requires some initial definitions to be able to formulate the algorithm correctly. The following discussion closely follows the terminology presented in Seiler (2009). We start with the definition of a differential equation (or rather a system of differential equations):

DEFINITION DIFFERENTIAL EQUATION

A (system of) differential equations $\mathcal{R}_q$ is a collection of implicit functions

$$\Phi^\tau(x^i, u^\alpha, p^\alpha_\mu) = 0 \qquad \tau = 1, \ldots, l\,,$$

that depend on *n independent variables* $x^1, \ldots, x^n$ and *m dependent variables* $u^1, \ldots, u^m$, i.e. functions of the independent variables. $p^\alpha_\mu = \frac{\partial^{|\mu|} u^\alpha}{\partial x^\mu}$ are the corresponding derivatives in the direction of the independent variables, and μ denotes a multi-index. The order of the differential equations, given by the largest number of elements in the multi-indices μ, will be denoted by q. τ labels the $1, \ldots, l$ different equations in the system of differential equations.

Geometrically, we consider $\mathcal{R}_q$ to be a fibered submanifold of the jet bundle $J_q\mathcal{E}$ where the dependent variables take their values in.

For point particle equations of motions, the only independent variable is the time coordinate t and the generalised coordinates q^i are the dependent variables. For field theories, like the ones we treated in the previous sections, the independent variables are the spacetime point coordinates (in some chosen coordinate system), and the field values constitute the dependent variables. Another example that we will see in chapter 3 are the gravitational closure equations: Here, the independent variables correspond to the local degrees of freedom of the gravitational field and their derivatives, and the dependent variables are the expansion coefficients appearing in the gravitational Lagrangian.

One object of great importance is the *geometric symbol* of a differential equation, since it contains much information about the solution space of the differential equation[3]:

DEFINITION GEOMETRIC SYMBOL

The *geometric symbol* of a differential equation is defined by the linear equation

$$\mathcal{M}_q : \quad \sum_{|\mu|=q} \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu} v^\alpha_\mu = 0\,,$$

which is the kernel of the coefficient $Q^{\tau\mu}{}_\alpha := \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu}$, with $|\mu| = q$.

We can bring this matrix Q into row echelon form by Gaussian elimination to read off for which coefficients p^α_μ we can solve for in the differential equation. The remaining derivatives have to be provided by initial values.

The second important object for the differential equation is a familiar object called the *principal symbol*. While it generally contains the same information as the geometric symbol, it presents it in a different fashion and is favourable in some situations.

[3]Revealing this information, however, may become tedious in practice

DEFINITION PRINCIPAL SYMBOL

The principal symbol of an equation is obtained by contraction with covectors k

$$T^{\tau}_{\alpha}[k] := \sum_{|\mu|=q} \frac{\partial \Phi^{\tau}}{\partial p^{\alpha}_{\mu_1 \ldots \mu_q}} k_{\mu_1} \cdots k_{\mu_q} .$$

As the name already indicates, this is of course intimately connected to the principal polynomial we defined before. In fact, both the geometric symbol as the highest order coefficient of the differential equation, as well as the principal symbol itself, already appeared in the previous sections, although they now appear in a more general setting.

Beta-coefficients

From the geometric symbol of the equation, we can read off some coefficients that are essential for the notion of involutivity and the Cartan-Kuranishi algorithm. In simple terms, we count which Taylor coefficients we can solve for in the differential equation at order q. For involutive equations, it will turn out that we can derive the number for higher orders already in terms of the coefficients of the lower orders. This means we can completely predict the behaviour at higher orders. As a result, one could, in principle, even count the number of functions appearing in the solution (Seiler, 1995).

To be able to define those coefficients, we need to introduce some fixed order of the columns of the geometric symbol (labelling the different Taylor coefficients at the highest order q). For this, we associate to each multi-index a class via

DEFINITION CLASS OF A MULTI-INDEX

For a multi-index $\mu = [i_1, \ldots, i_n]$ we define its *class* as the index k of the first non-vanishing i_k.

Using this, we can sort the columns of Q in a class respecting order – from highest to lowest – and after bringing the matrix into row echelon form by Gaussian elimination search for the first non-vanishing entry in each row. The corresponding p^{α}_{μ} of this entry is called the *leader* of the row of class k. We can now count the leaders of each row of class k and denote this $\beta^{(k)}_q$. With these coefficients, we can define the notion of an *involutive symbol*:

DEFINITION INVOLUTIVE SYMBOL

The geometric symbol of a differential equation at order q is called *involutive* given that the rank of the geometric symbol $\mathcal{M}_{q+1}$, obtained by differentiating the differential equation in the direction of each independent coordinate, satisfies the relation

$$\operatorname{rank} \mathcal{M}_{q+1} = \sum_{k=1}^{n} k \cdot \beta_q^{(k)} .$$

Given that the symbol at order q is involutive one can derive a recurrence relation to calculate the coefficients $\beta_{q+1}^{(k)}$ directly from the coefficients $\beta_q^{(k)}$ (Seiler, 1994). This means that we can completely predict what coefficients we can solve for in the prolongations of the differential equation, i.e. the derivatives in the direction of the independent coordinates. One finds (Seiler, 1995) that, given that the symbol of order q is involutive, that at order $q + r$ the beta coefficients read

$$\beta_{q+r}^{(k)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \beta_q^{(i)} . \tag{2.41}$$

Even further, once the symbol $\mathcal{M}_q$ is involutive, it turns out that the same will be true for all further prolongations $\mathcal{M}_{q+r}$, for any $r \geq 0$. The rank of the geometric symbol at order $q + r$ can be obtained inductively and one finds that

$$\operatorname{rank} \mathcal{M}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \beta_q^{(k)} . \tag{2.42}$$

Moreover, it can also be shown that even if at order q the symbol was not involutive *yet*, there always exist an integer r such that the symbol $\mathcal{M}_{q+r}$ will be involutive, and thus so will be all further prolongations. Note that this, however, does increase the differential order of the equation since we will add all the prolongations explicitly to the system until the symbol is involutive. In practice, this can lead to the scenario where a seemingly second-order field equation is in fact of higher order and is then possibly plagued with Ostrogradsky ghosts (Ostrogradsky, 1850; Woodard, 2015).

For a thorough treatment of involutive differential equations and the formal study of differential equations, we refer the interested reader to Seiler (2009) and the references therein.

REMARK

With the help of the $\beta_q^{(k)}$-coefficients one can define the so-called *Cartan characters*

$$\alpha_q^{(k)} := m \cdot \binom{n+q-k-1}{q-1} - \beta_q^{(k)} \qquad k = 1, \dots, n\,,$$

where m is the number of dependent variables in the system. Those symbols encode much of the information about the solution space. Given that $\alpha_q^{(n)}$ is non-vanishing we immediately know that there has to be *at least* one undetermined function in the solution space. In order to have only finitely many constants, we need to find that $\alpha_q^{(k)} = 0$ for all k.

Example Let us quickly apply this on the example for equations (2.36a) – ((2.36b)). In this case, the geometric symbol of the equations is easily found to be

$$\mathcal{M}_2 : \quad \begin{array}{c} \begin{array}{ccc} \ddot{f} & \partial_x \dot{f} & \partial_{xx} f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \tag{2.43}$$

where we chose our order for the independent variables such that $\ddot{f}$ (which corresponds to the multi-index $[2,0]$) is of class 2, while $\partial_x \dot{f}$ (corresponding to $[1,1]$) is of class 1. As a result, we have that

$$\beta_2^{(1)} = 1 \qquad , \qquad \beta_2^{(2)} = 1\,. \tag{2.44}$$

In order for the geometric symbol $\mathcal{M}_2$ to be involutive, we must confirm that the rank of the geometric symbol of the prolongations – that is taking all derivatives in the direction of all independent variables – is equal to $1 \cdot \beta_2^{(1)} + 2 \cdot \beta_2^{(2)} = 3$. If we calculate this, one indeed finds that

$$\mathrm{rank} \begin{array}{c} \begin{array}{cccc} \dddot{f} & \partial_x \ddot{f} & \partial_{xx} \dot{f} & \partial_{xxx} f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} = 3\,. \tag{2.45}$$

The symbol is, as a result, involutive. However, we have already seen before that there exists a hidden integrability condition, namely equation (2.37), that we need to consider.

Hidden integrability conditions

As found in the example, "hidden information" may still be contained in the differential equations. If we can algebraically manipulate the prolongation of the differential equations – equations of order $q+1$, such that we obtain an equation of order q, then this is considered to be an *integrability condition* and needs to be added to the system (Seiler, 2009). Of course, no new information is added; it is simply made explicit and can then properly be considered in all further calculations – as in the analysis of the causal structure encoded in the principal polynomial.

In the geometric formulation we can tackle this by obtaining the differential equation $\mathcal{R}_q^{(1)} \subseteq J_q\mathcal{E}$ that is obtained by first taking the derivative in the direction of each independent variable and then, after bringing the equations into row echelon form, projecting back to order q by keeping only equations that contain coefficients up to order q. Using the geometric symbol $\mathcal{M}_{q+1}$ one can show that in general

$$\dim \mathcal{R}_q^{(1)} = \dim\ \mathcal{R}_{q+1} - \operatorname{rank} \mathcal{M}_{q+1}\,, \tag{2.46}$$

where it holds that the dimension of the projection fulfills

$$\dim \mathcal{R}_q^{(1)} \leq \dim \mathcal{R}_q\,. \tag{2.47}$$

This allows one, even for a rather complicated system of differential equations, to systematically check for the presence of integrability conditions in practice.

Given that integrability conditions were identified and added to the system, it turns out that an involutive symbol can suddenly fail to be involutive. In this case, it is necessary to increase the differential order of the equation until the symbol becomes involutive again, after which it is required to again check for any further integrability conditions. Performing all those steps is the content of the Cartan-Kuranishi algorithm, which we will describe in the subsequent section.

The Cartan-Kuranishi algorithm

As stated above, identifying hidden integrability conditions may cause the geometric symbol of the equation to becoming non-involutive. As a result, we need to differentiate to make the symbol involutive again at some higher differential order before looking for further integrability conditions again. This raises the question of whether the seemingly Sisyphean task of making the symbol involutive again, as well as projecting back to reveal hidden integrability, may actually stop at some point.

Luckily, a reassuring answer was given by the Cartan-Kuranishi theorem that states that, in any case, we end up with a differential equation where the symbol is involutive, as well as all the integrability conditions were added to the system, in *finitely many steps*.[4] Once the sequence stops, we say that the system is *involutive* (Seiler, 2009). Once made involutive, all hidden information is made explicit.

The algorithm to bring the differential equation into involutive form is the so-called *Cartan-Kuranishi* algorithm and can be performed for any differential equations (Kuranishi, 1957). Once this step has been performed, all information that can contribute to the characteristic surfaces, and by this to the causal structure of the differential equation, is now explicitly present. We summarise the steps of the algorithm in the following box.

[4] Arriving there in practice may still feel rather Sisyphean than finite.

CARTAN-KURANISHI ALGORITHM

Input: Differential equation $\mathcal{R}_q \subseteq J_q\mathcal{E}$
Output: Equivalent involutive differential equation $\mathcal{R}^{(s)}_{q+r} \subseteq J_{q+r}\mathcal{E}$
$r \leftarrow 0$, $s \leftarrow 0$;
repeat
  while $\mathcal{M}^{(s)}_{q+r}$ *not involutive* **do**
    $r \longleftarrow r+1$
  end
  integabilityConditions $\longleftarrow \mathcal{R}^{(s+1)}_{q+r} \subset \mathcal{R}^{(s)}_{q+r}$;
  if *integrabilityConditions* **then**
    $s \longleftarrow s+1$
  end
until $\neg$ *integrabilityConditions*;
return $\mathcal{R}^{(s)}_{q+r}$

As a result, we can then correctly calculate the principal polynomial of the differential equation and obtain its correct principal polynomial and its underlying causal structure. Suppose one applies all the steps of the Cartan-Kuranishi algorithm for the Euler-Lagrange equations of the Lagrangian considered by Velo-Zwanziger. In that case, it turns out that there are indeed hidden integrability conditions. Calculation of the principal polynomial then recovers their result. Note, however, that the result was obtained in a completely algorithmic fashion.

One notable fact is that physicists are quite familiar with the algorithm by itself, as it is identical to the Dirac-Bergmann algorithm for constrained systems (Dirac, 1950, 1958, 1964) in the case of point particles. For field theories, it presents a more general algorithm that considers not only the derivatives in temporal direction and looks for additional constraints but also in the spatial direction (Seiler and Tucker, 1999). What the Dirac-Bergmann procedure offers on top is the classification scheme of integrability conditions into first and second classes. From the geometrical perspective, the second class constraints restrict to a symplectic submanifold of the phase space, while first class constraints will act as the generators of gauge transformations.

Example: Generalizations of Maxwellian electrodynamics

We now want to put the algorithm presented above to good use by analyzing a specific example that is still interesting enough from a physical perspective (Wolz, 2021). The setup we consider is the following Lagrangian:

$$\mathcal{L}[A;\eta] = -\frac{1}{4}\eta^{ac}\eta^{bd}F_{ab}[A]F_{cd}[A] + V(A)\,, \tag{2.48}$$

i.e. we take the Lagrangian of Maxwellian electrodynamics on a flat Minkowskian background η and add a not further specified function $V(a)$ of the covector field A_m.

The equations of motions are calculated by variation of the action with respect to A, which yields a differential equation of 2nd order

$$0 = 2\,\eta^{m[n}\eta^{p]a}\partial_{mn}A_p + \frac{\partial V}{\partial A_a}(A)\,. \tag{2.49}$$

In this case, because our function V does not depend on derivatives of the convector field, we see that the geometric symbol coincides with the term obtained for standard electrodynamics.

To obtain the four $\beta_2^{(i)}$ coefficients, we first pick an order for our independent variables. In general it proves useful to solve for as many $\ddot{A}_m$ terms as possible as it allows us to distinguish between evolutionary- and constraint equations. Thus, we make the choice (x^α, t) for our classes, i.e. time comes *after* the spatial components and has the highest class. As the next step, we need to write down the geometric symbol and sort the columns in our chosen order. As the expression is rather tedious we will not state it here explicitly and directly present the beta coefficients

$$\beta_2^{(1)} = 0 \quad , \quad \beta_2^{(2)} = 0 \quad , \quad \beta_2^{(3)} = 1 \quad , \quad \beta_2^{(4)} = 3\,, \tag{2.50}$$

which reflects the well-known fact that in Maxwell's equations we have three evolutionary equations and one constraint. In order to check if the geometric symbol is involutive, we need to calculate the rank at the next order and see if it is given by

$$\operatorname{rank}\mathcal{M}_3 = \sum_{k=1}^{4} k\cdot\beta_2^{(k)} = 15\,. \tag{2.51}$$

Although tedious, the calculation can easily be performed by computer algebra software, with the result that the symbol is indeed involutive.

The next step is to check for hidden integrability conditions. By considering the divergence of the equations of motion (2.49) we find, due to the anti-symmetrization in the first term in (2.49) the integrability condition

$$0 = \frac{\partial^2 V}{\partial A_m \partial A_n}(A)(\partial_n A_m) =: H^{mn}(A)(\partial_n A_m)\,, \tag{2.52}$$

which is of 1st order. This means we have to add it to the system to make the information explicit.

For our new system, we repeat the steps from above, i.e. check if the symbol is involutive. Since the new equation (2.52) is of 1st order, the symbol remains unchanged and so it is still involutive. But clearly, if we prolong the integrability condition (2.52) we find another 2nd order equation that also has to be added to the system. The geometric symbol of the system has then changed, so we need to read off the $\beta_2^{(k)}$ coefficients again. We find that

$$\beta_2^{(1)} = 1 \quad , \quad \beta_2^{(2)} = 1 \quad , \quad \beta_2^{(3)} = 2 \quad , \quad \beta_2^{(4)} = 4\,, \tag{2.53}$$

and that the geometric symbol is still involutive. If we now check for integrability conditions we find that there are no further conditions. As a result our final system

$$\begin{aligned} 0 &= 2\eta^{m[n}\eta^{p]a}(\partial_{mn}A_p) + \frac{\partial V}{\partial A_a}(A)\,,\\ 0 &= H^{mn}(A)(\partial_n A_m)\,,\\ 0 &= H^{mn}(A)(\partial_{np}A_m) + (\partial_p H^{mn}(A))(\partial_n A_m) \end{aligned} \tag{2.54}$$

is involutive and thus, all hidden information is made explicit.

We can now analyse the causal structure of this system. By making a Wentzel-Kramers-Brillouin approximation to lowest order again (compare equation (2.14)), and calculating the principal polynomial[5] we find the admittedly non-Lorentzian polynomial

$$P(k) = \left(\eta^{mn} k_m k_n\right)^3 \cdot \left(H^{pq}(A) k_p k_q\right) . \tag{2.55}$$

If we eliminate repeated factors, this gives a principal polynomial of degree 4. In general, we see that this is not the standard Lorentzian causal structure but resembles the causality of a bi-metric theory, with the second metric being constructed by the four-potential A itself. This particularly indicates that in this case, the dichotomy of matter fields and a geometric field η breaks down since A acts in both sectors.

Only for two special cases do we recover the principal polynomial of the standard model: Since only the 2nd derivative of V appears in the principal polynomial, any term linear in the gauge field leaves the causal structure unchanged. This is not surprising since this just contributes a coupling of the covector field A_m to a current j, i.e.

$$V(A) = j^m A_m . \tag{2.56}$$

The other case occurs in case the second derivative of V with respect to the covector field A is proportional to the flat metric η, i.e.

$$\frac{\partial^2 V}{\partial A_p \partial A_q} \propto \eta^{pq} . \tag{2.57}$$

In this case V only contributes as a repeated factor to the principal polynomial. This corresponds to a Proca mass term

$$V(A) = m^2 \, \eta^{ab} A_a A_b . \tag{2.58}$$

While this does break the $U(1)$ gauge invariance – which could be seen in the according to involutive system from the $\beta_2^{(4)}$ coefficient since we now have four evolution equations and no constraint – the causal structure is not altered.

2.2.3 Non-square systems

A third complication may arise once the equations of motion were put into involutive form by performing the Cartan-Kuranishi algorithm: by explicitly adding new equations, the hidden integrability conditions, to the system, the principal symbol $T[k]$ is generally not a square matrix anymore. As a result, the covectors k of propagating massless modes cannot be obtained by taking the principal symbol's determinant.

This can, luckily, be rectified rather easily: remember that we were identifying those covectors, whose zero-set give non-vanishing amplitudes in a WKB approximation, and thus the geometrical optical limit of the matter theory. Even in involutive form, the equations of motion still read to the lowest order in the expansion

$$T_{\tau\mathcal{B}}[k] a^{\mathcal{B}} = 0 \qquad \text{for} \qquad \tau = 1, \ldots, \text{number of equations} , \tag{2.59}$$

[5] We have to be careful since we deal with a so-called non-square system, as will be elaborated in more detail the following subsection.

with the subtle, but most important, difference that the rows of the principal symbol are no longer labelled by the index $\mathcal{A}$ arising from varying the action with respect to the matter field. But also, in this case, common linear algebra knowledge tells us that we can again derive a polynomial equation in order to identify non-trivial amplitudes: this is the case given that the determinant of the matrix $T^t[k] \cdot T[k]$ vanishes. This finally motivates the following definition of the principal polynomial that can be applied for non-square systems:

DEFINITION PRINCIPAL POLYNOMIAL (WITHOUT ANY GAUGE FREEDOM)

The principal polynomial of a non-square system of differential equations is given by

$$P(x,k) = \rho(x) \det \left(T^t[k] \cdot T[k] \right) ,$$

where $T[k]$ denotes the principal symbol of the equations of motion.

Note that for a square system this definition reduces to the square of the original definition of the polynomial, that we hinted at in equation (2.17), since

$$P(x,k) \propto \det \left(T^t[k] \cdot T[k] \right) = \det \left(T^t[k] \right) \cdot \det \left(T[k] \right) = \left(\det \left(T[k] \right) \right)^2 , \tag{2.60}$$

which describes precisely the same zero-set and thus, massless covectors. Since we will settle on the convention to remove repeated factors from the polynomials in the next section, it is clear that both definitions will in this case yield the same result for square systems.

Furthermore, it should be noted again that this definition is only valid, given that no gauge symmetries are present: this can be either achieved by a suitable gauge fixing or by performing the steps described in section 2.2.1 with the adjunct matrix of $T^t[k]T[k]$ instead of $T[k]$. This will, again, yield a homogeneous polynomial density that gives the dispersion relation of physical massless modes.

Examples Although not really motivated by a proper physical theory, we go back to the example from equations (2.36a) – (2.36b) after addition of the integrability condition. One can easily verify that, once the condition is added, this system is involutive. The principal symbol $T[k]$ is then given by the vector

$$T[k] = \begin{pmatrix} k_t^2 \\ k_x \, k_t \\ k_t^2 - k_x^2 \end{pmatrix} . \tag{2.61}$$

Inserting this into definition, we find that the principal polynomial P is then given by

$$P(k) = 2\, k_t^4 - k_x^2 k_t^2 + k_x^4 . \tag{2.62}$$

A more interesting example is the generalization of Maxwellian electrodynamics that we completed to involution in the previous section. The principal symbol, removing trivial rows, becomes

$$T[k] = \begin{pmatrix} k^2 \cdot \eta - \eta(k,\cdot) \otimes \eta(k,\cdot) \\ H(k,\cdot) \otimes k \end{pmatrix} , \tag{2.63}$$

from which we can calculate the determinant of $T^t[k] \cdot T[k]$ with the help of computer algebra and verify that, indeed, we obtain the expression (2.55) stated in the previous section.

Lagrange multipliers

Another way to derive the principal polynomial for systems that are non-square after Cartan-Kuranishi completion is to add the integrability conditions that were discovered by the algorithm to our initial Lagrangian with Lagrange multipliers (Wierzba, 2018).

Let us write our original system of differential equation, obtained from a Lagrangian functional $\mathcal{L}[A]$ by variation, as

$$D[\partial]A = 0\,, \tag{2.64}$$

with the operator valued symbol $D[\partial]$. After bringing it into involutive form we add the conditions in the form $C[\partial]A = 0$ to the initial system. The principal symbol of the involutive differential equation then takes the form

$$T[k] = \begin{pmatrix} T_D[k] \\ T_C[k] \end{pmatrix}\,, \tag{2.65}$$

with $T_D[k]$ being the principal symbol of the original equation and $T_C[k]$ the principal symbol of the integrability conditions. Then we can define a modified Lagrangian of the form $\mathcal{L}[A] + \lambda C[\partial]A$ that, by variation of both A and λ, leads to the equation of motion

$$D[\partial]A + C[-\partial]^t\lambda = 0\,, \tag{2.66}$$

$$C[\partial]A = 0\,. \tag{2.67}$$

From this we get, again, the quadratic geometric symbol of the modified Lagrangian

$$\widetilde{T}[k] = \begin{pmatrix} T_D[k] & T_C[-k]^t \\ T_C[k] & 0 \end{pmatrix}\,. \tag{2.68}$$

For this principal symbol, we can calculate the principal polynomial by "simply" taking the determinant or decomposing the adjunct matrix in the presence of a gauge symmetry by the prescription given in section 2.2.1. In the end, the result is independent of the chosen method.

Let us quickly summarise the results of the previous sections: given some matter field equations of motion, it is first essential to bring the system of differential equations into involutive form to guarantee that all the information that contributes to the causal structure is present in the highest order coefficient of the differential equation. Afterwards, in the absence of gauge symmetries, the principal polynomial can be calculated in the WKB approximation from the highest order coefficient. In case gauge symmetries are present, it can be calculated with the steps laid out in 2.2.1. In any case, we end up with a polynomial homogeneous in the covectors k, the principal polynomial, that contains the information about the characteristic surfaces of the differential equations. By this, the causal structure is dictated by the background geometry.

2.3 THE MATTER CONDITIONS

As will be presented in detail in the next chapter, the gravitational closure framework allows deriving the dynamics of the gravitational fields. The required input is a generalised way of turning the conormal to a hypersurface into a "time-like" vector with any geometry. As it turns out, this information is encoded in the principal polynomial already – given that some specific conditions are imposed on the principal polynomial. However, as mentioned before, these conditions are quite reasonable for a physical theory, in fact even necessary for any canonical quantisation of the classical dynamics of a given matter theory.

The first condition ensures that the equations of motion describe an initial value problem; in other words, the matter field dynamics is predictive. This translates into a specific condition on the principal polynomial of the equations of motion. The second condition is required in order to be able to move between a Lagrangian and Hamiltonian description of massless modes propagating with the dispersion relation defined by the principal polynomial of the given matter field. The third and last condition on the polynomial requires the spacetime to be energy-distinguishing, meaning that we can distinguish positive energy modes from negative energy modes – and there are no other modes.

All those rather physical conditions – in the following referred to as *matter conditions* – translate into requirements on the principal polynomial, and by this, into requirements on the background geometry that is contained in the matter field equations themselves. We will now lay out these conditions in more detail.

2.3.1 Condition 1: Predictivity

As stated above, the first matter condition requires that the matter dynamics be formulated as a proper initial value formation. The mathematical version of this statement is that the equations of motion need to be *hyperbolic* differential equations for only then they are solved by prescription of initial values that will then evolve along with some evolution parameter t. It can then be shown that this is precisely the case, given that the principal polynomial is hyperbolic:

DEFINITION HYPERBOLIC POLYNOMIAL

The polynomial $P(x,k)$ is called *hyperbolic* if there exists a covector $h \in \mathrm{T}^*_x\mathcal{M}$ such that

- $P(x,h) \neq 0$
- $P(x,q+\lambda \cdot h) = 0$ has $\deg P$-many real solutions λ, for all covectors $q \in \mathrm{T}^*_x\mathcal{M}$

In case such a hyperbolic covector h exists, there always exists an open and convex cone $C_x(P,h)$ called the *hyperbolicity cone* that contains all hyperbolic covectors that lie together with h in one connected set.

For an illustration of this idea, see figure 2.1 where we present the null cone of a 2$^{\text{nd}}$ degree principal polynomial together with a hyperbolic covector h.

Given a hyperbolic covector h, one easily sees that $-h$ is also a hyperbolic covector. However, one finds that it does not belong to the same connected component and $C_x(P,h) \cap C_x(P,-h) = \emptyset$. As a result, there is always an even number of hyperbolicity cones at each spacetime point.

It turns out (Raetzel et al., 2011) that in case the principal polynomial is reducible, i.e. it can be written in terms of factors

$$P(x) = P_1(x) \cdots P_f(x)\,. \tag{2.69}$$

Now, P is hyperbolic if and only if all of its lower-degree factors P_i are hyperbolic on their own. The hyperbolicity cones are then given by the intersection of the hyperbolicity cones of each factor, i.e.

$$C(P,h) = C(P_1,h) \cap \cdots \cap C(P_f,h)\,. \tag{2.70}$$

Clearly, $C(P,h) = \emptyset$ unless h is a hyperbolic covector for each factor.

We can furthermore make the choice of a *time orientation* of spacetime by prescription of a smooth and everywhere hyperbolic vector field n. This singles out a hyperbolicity cone C_x at each space time point via

$$C_x = C_x\left(P_x, n_x\right)\,. \tag{2.71}$$

Since the boundary of the hyperbolicity cones is merely defined by the roots of P, while its interior has a constant sign, we are free to pick a sign convention. In this thesis we will always choose a *positive sign* for the hyperbolicity cone selected by a time orientation, i.e. $P(C_x) > 0$.

Given a polynomial where repeated factors occur – as they, for example, did in the case of Maxwellian electrodynamics – we see that those factors provide no new information: the hyperbolicity cones identically match each other. As a result, we can remove those repeated factors from the polynomial without any loss of generality. This will be important for the formulation of the second matter condition.

Example: Maxwellian electrodynamics

We saw in the previous sections that the principal polynomial of Maxwellian electrodynamics – once de-densitized, dealt with all gauge ambiguities and eliminated all repeated factors – is given by

$$P(x,k) = g^{ab}(x)\,k_a(x)\,k_b(x)\,. \tag{2.72}$$

For the polynomial to be hyperbolic, we search for the roots of the equation

$$\begin{aligned} P(k+\lambda\cdot n) &= g^{ab}\left(k_a+\lambda n_a\right)\left(k_b+\lambda n_b\right) \\ &= g^{ab}n_a n_b\,\lambda^2 + 2\,g^{ab}k_a n_b\,\lambda + g^{ab}k_a k_b = 0\,. \end{aligned} \tag{2.73}$$

This has $\deg P = 2$ real solutions given that the discriminant is larger than zero,

$$\left(g^{ab}k_a n_b\right)^2 - \left(g^{ab}k_a k_b\right)\,\left(g^{cd}n_c n_d\right) > 0\,.$$

Now choosing, without loss of generality, a coordinate system such that $n = (1,0,0,0)$ and $g^{0\alpha} = 0$ we can simplify this to

$$g^{00}\cdot g^{\alpha\beta}k_\alpha k_\beta < 0\,.$$

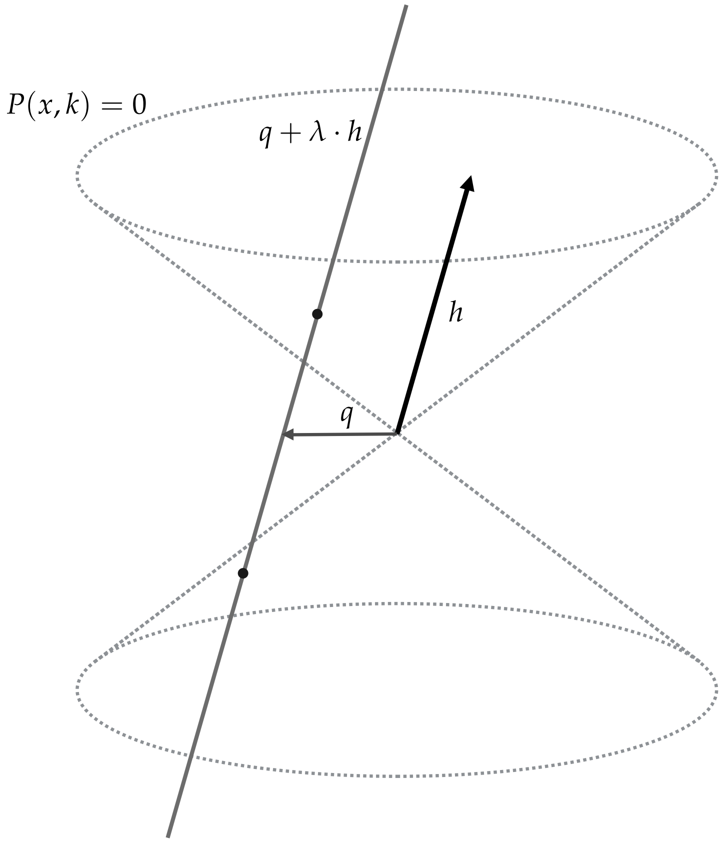


Figure 2.1 Hyperbolic covector $h \in T^*_x\mathcal{M}$ with respect to a hyperbolic polynomial $P(x,k)$ of degree 2. The line $q + \lambda \cdot h$ intersects the null cone exactly twice.

In case $g^{00} > 0$ we find that $g^{\alpha\beta}k_\alpha k_\beta < 0$ and for $g^{00} < 0$ we find that $g^{\alpha\beta}k_\alpha k_\beta > 0$, which has to hold for all covectors k. But this then implies that g has to be of Lorentzian signature, which is the well-known result that only a Lorentzian geometry renders Maxwell's equation predictive. With our convention that $P(n) > 0$ we find that $g^{00} > 0$, i.e. the metric has $(+,-,-,-)$ signature.

2.3.2 Condition 2: Momentum-velocity duality of massless modes

The second matter condition establishes a duality between the momentum and velocity of massless particle modes that satisfies the dispersion relation defined by the principal polynomial. For this we first write down the totally constrained Helmholtz action for the massless modes as

$$\mathcal{S}_{\text{massless}}[x,k,\rho] := \int \mathrm{d}\lambda \, \Big(k_a(\lambda)\dot{x}^a(\lambda) - \rho(\lambda)\, P\left(x(\lambda), k(\lambda)\right) \Big) \,, \tag{2.74}$$

with ρ being a Lagrange multiplier that enforces the dispersion relation P for the massless particle. We can then calculate the equations of motion of the massless mode by variation of the action by x, k and ρ

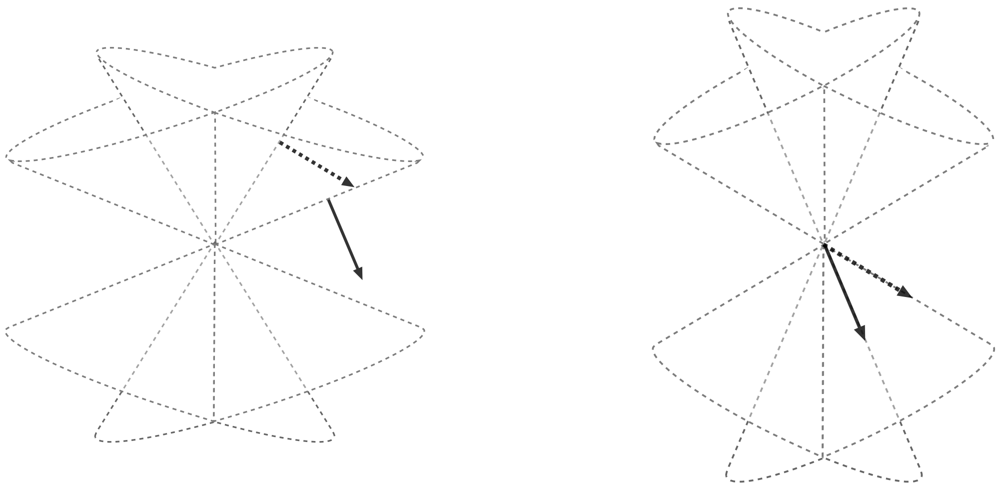


Figure 2.2 Gauss map sending P-null covectors to $P^\sharp$-null vectors. (a) Null-surface of a hyperbolic reducible principle polynomial P in cotangent space; with typical gradient (co-co-)vectors, and (b) null-surface of the dual polynomial $P^\sharp$ in tangent space; containing, by definition, the gradient vectors to the P-null surface.

and obtain the following system

$$\dot{x}^a(\lambda) = \rho(\lambda) \cdot \frac{\partial P}{\partial k_a}\left(x(\lambda), k(\lambda)\right), \tag{2.75a}$$

$$\dot{k}_a(\lambda) = -\rho(\lambda)\frac{\partial P}{\partial x}\left(x(\lambda), k(\lambda)\right), \tag{2.75b}$$

$$0 = P\left(x(\lambda), k(\lambda)\right). \tag{2.75c}$$

In order to express the momenta k in terms of the particle velocity $\dot{x}$, we need the inverse of the derivative of the principal polynomial with respect to its fibre argument. It can then be shown (Raetzel et al., 2011; Rivera, 2012) that such an inverse (up to a projective factor) exists in case P is hyperbolic and is given by the gradient of a *dual polynomial* in tangent space for which

$$P^\sharp\left(x, \frac{\partial P}{\partial k}(x,k)\right) = 0 \quad \text{for all} \quad k \in N^{\text{smooth}}(x), \tag{2.76}$$

with the cone N^{smooth} being defined as

$$N^{\text{smooth}}(x) := \left\{k \in \mathrm{T}_x^* \,|\, P(x,k) = 0 \quad \text{and} \quad \frac{\partial P}{\partial k}(x,k) \neq 0\right\}. \tag{2.77}$$

While the dual polynomial $P^\sharp$ is just defined up to a real factor function, its roots are unaffected by this ambiguity and well-defined.

The dual polynomial of a factor polynomial $P(x,h) = P_1(x,h) \cdots P_f(x,h)$ is then also defined as

$$P^\sharp(x,v) := P_1^\sharp(x,v) \cdots P_f^\sharp(x,v), \tag{2.78}$$

where each factor of the dual polynomial $P_i^\sharp$ is defined according equation (2.76) for each factor P_i.

The mathematical content of our second matter condition is that we also require the dual polynomial to be hyperbolic. The physical consequence of this is then that the following projective relation holds

$$\left[\mathrm{D}P^\sharp\left(\mathrm{D}P(k)\right)\right] = [k]\,, \tag{2.79}$$

where $\mathrm{D}P$ and $\mathrm{D}P^\sharp$ denote the derivative of the polynomial with respect to their fibre argument and $[\cdot]$ denotes projective equivalence. This establishes a duality between the $P^\sharp$-null ray directions, which we obtain by taking the geometrical-optical (infinite frequency) limit of a theory, and the P-null covectors (compare figure 2.2).

Using this, one can finally show that the corresponding Lagrange action functional for the massless particle is given by

$$\mathcal{S}_{\text{massless}}[x,\mu] := \int \mathrm{d}\lambda\, \mu(\lambda)\, P^\sharp\left(x(\lambda), \dot{x}(\lambda)\right)\,, \tag{2.80}$$

with the Lagrange multiplier $\mu(\lambda)$. Furthermore, the condition that $P^\sharp$ also be hyperbolic can also be understood as a sufficient condition that $P(x)$ may be recoverable from $P^\sharp$ as the double dual, such that for each spacetime point $x \in \mathcal{M}$ we have that

$$P(x) \sim P^{\sharp\sharp}(x)\,.$$

Example: Maxwellian electrodynamics

For the principal polynomial of Maxwellian electrodynamics, $P(x,k) = g^{ab}(x)k_a(x)k_b(x)$, with the Lorentzian signature of g being enforced by the hyperbolicity condition from the first matter condition, one finds that the dual polynomial is given by

$$P^\sharp(x,v) = g_{ab}(x)\, v^a(x)\, v^b(x)\,. \tag{2.81}$$

As $g_{\cdot\cdot}$ is already of Lorentzian signature – since we required it for its inverse $g^{\cdot\cdot}$ – the second matter condition, the hyperbolicity of $P^\sharp$ is already implemented.

2.3.3 Condition 3: Energy distinction

Now, having established the physical meaning of the first and second matter condition as the predictivity of the matter field equation and the duality between momenta and velocities of massless modes, we can ask for an observer-independent way to split the energies of those modes into either positive or negative energies. Mathematically, we look for an open set $\mathcal{O}_x$ in each tangent space $\mathrm{T}_x\mathcal{M}$ of the spacetime manifold that contains all tangent vectors U to *observer worldlines* such that for all massless modes k

$$\text{either} \qquad k \in \mathcal{O}_x^+ \qquad \text{or} \qquad k \in -\mathcal{O}_x^+\,. \tag{2.82}$$

The closed dual cone $\mathcal{O}_x^+$ is obtained via

$$\mathcal{O}_x^+ := \left\{k \in \mathrm{T}_x^*\mathcal{M} \mid P(k) = 0 \text{ and } U(k) > 0 \quad \text{for all } U \in \mathcal{O}_x\right\}\,. \tag{2.83}$$

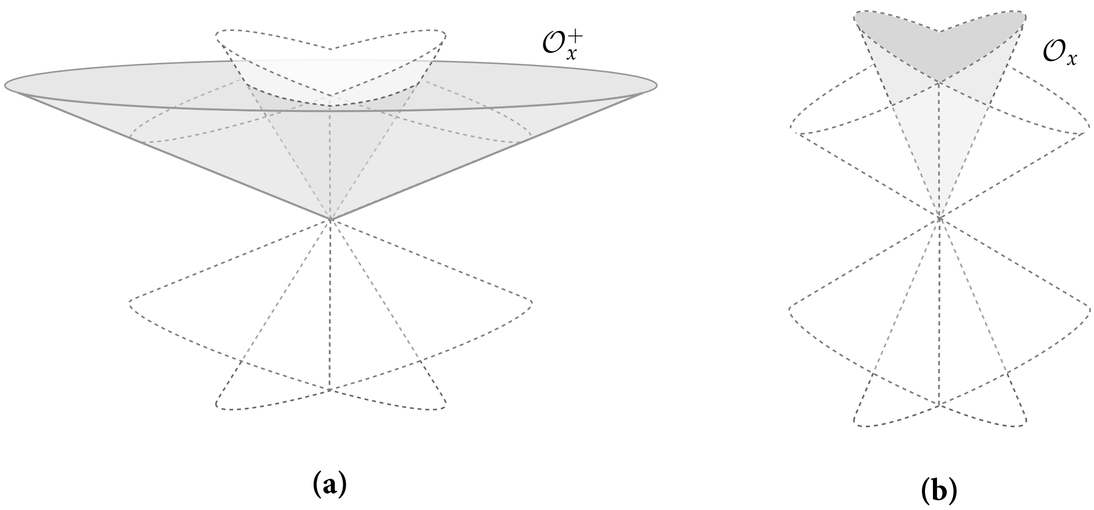


Figure 2.3 Positive energy cone $\mathcal{O}_x^+$ as the dual of the observer cone $\mathcal{O}_x$. (a) Cone covering all momenta of positive energy as unanimously judged by all observers, and (b) cone containing all tangent vectors to observer worldlines through one point.

This implements the idea that all observers – represented by their worldline tangent U – agree on the sign of the energy (see figure 2.3a for an illustration of this).

Our 3rd matter condition now is that the cone of massless momenta N_x decomposes into disjoint pieces

$$N_x \setminus \{0\} = N_x^+ \mathbin{\dot\cup} N_x^- \tag{2.84}$$

at each spacetime point x, where $N_x^\pm = N_x \cap \left(\pm\mathcal{O}_x^+\right)$.

The only remaining question is what the largest set $\mathcal{O}_x$ would be that one can choose – if it exists – that fulfills the energy condition (2.84). It turns out that this is given by any of the hyperbolicity cones of $P^\sharp$ (see figure 2.3b), so by choosing a smooth vector field T (a time orientation) that is everywhere hyperbolic we can define the observer cones as

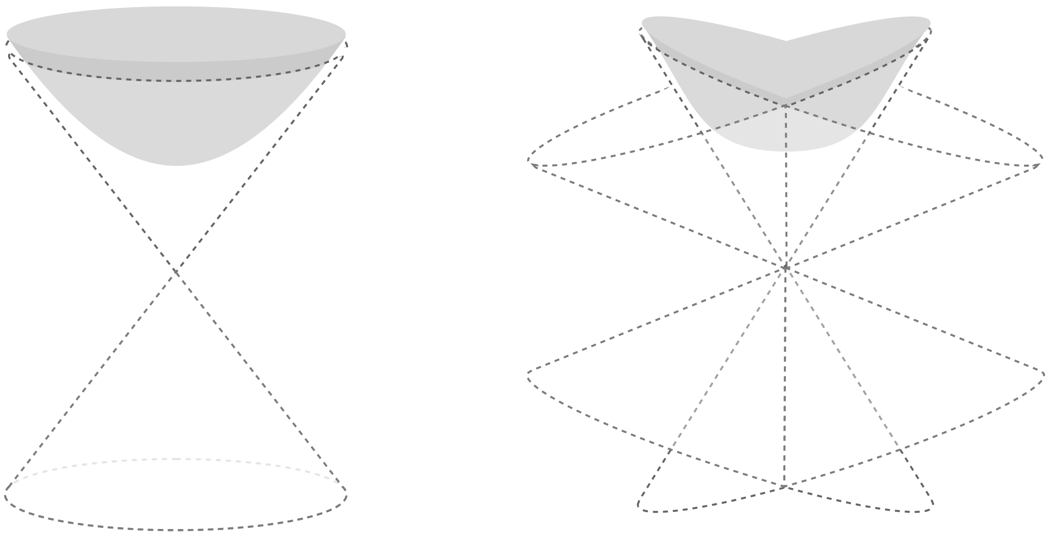
$$\text{future-directed observer cones } O_x = C_x(P^\sharp, T)\,. \tag{2.85}$$

However, we then obtain a subtle condition on the principal polynomial that has to be implemented in practice for it to exist. Physically, this condition is more than reasonable: ultimately, observers need to be able to decide whether a decay involving a massless particle is kinematically possibly or not.

Note that, so far, all three matter conditions only use the roots of the principal polynomial P and the dual $P^\sharp$. While it is not directly evident from the definition of the observer cones, even here, its structure is completely given in terms of the roots of $P^\sharp$, even though the tangent vectors in O_x are non-roots.

2.4 LEGENDRE MAPS

Having implemented all three matter conditions, we can finally define a map between the cotangent and tangent spaces in this section. As it turns out, this map, which we will in the following refer to as Legen-



(a) **(b)**

Figure 2.4 Examples for positive mass shells. (a) Quadric mass shell of 2nd degree principal polynomial P_x satisfying the three matter conditions, and (b) quartic mass shell of 4th degree principal polynomial P_x satisfying the three matter conditions.

dre map, can be defined in two fashions: for the first map, we introduce a dispersion relation for massive particles with the help of the principal polynomial. Once defined we can always derive an invertible duality between massive momenta and velocities. For the second, alternative, map we implement that the spatial directions seen by an observer at each point of the wordline are compatible with the initial data hypersurfaces of the matter field dynamics.

2.4.1 Legendre maps from massive point particle modes

Due to the positive sign convention we chose, we know that the principal polynomial has a constant, that is positive, sign for the interior of the hyperbolicity $C_x \subseteq O_x^{\perp}$. As a result, we can employ that P is a homogeneous polynomial of degree $\deg P$ to assign a real number m to each covector inside the observer cone. This represents the *mass* of a particle and allows us to make the following definition that gives us a general prescription to obtain the mass shells (compare figure 2.4)

DEFINITION MASSIVE DISPERSION RELATION

The dispersion relation of any massive modes $k \in C_x$ is given as

$$P(x,k) = m^{\deg P}\,,$$

for a real number m that we will refer to as *mass* in the following.

At this point, it is crucial that we de-densitized the principal polynomial. Before, we only looked at the roots, which are insensitive to the particular choice of scalar density $\rho(x)$. However, here the choice does impact the definition of the mass but does so in a global fashion such that, once a choice was made, masses are expressed in a coordinate independent manner.

The totally constrained Hamilton action of a particle with such a dispersion relation can easily be formulated as

$$\mathcal{S}_{\text{massive}}[x,k,\mu] := \int \mathrm{d}\lambda \left(k_a(\lambda)\dot{x}^a(\lambda) - \mu(\lambda) \ln\left(P\left(x(\lambda), \frac{k(\lambda)}{m} \right)\right)\right) . \tag{2.86}$$

We then find the following equations of motion

$$\dot{x}^a(\lambda) = \mu(\lambda) \frac{\partial \ln P}{\partial k}\left(x(\lambda), \frac{k(\lambda)}{m} \right) , \tag{2.87a}$$

$$\dot{k}_a(\lambda) = -\mu(\lambda) \frac{\partial \ln P}{\partial x}\left(x(\lambda), \frac{k(\lambda)}{m} \right) , \tag{2.87b}$$

$$P\left(x(\lambda), k(\lambda)\right) = m^{\deg P} . \tag{2.87c}$$

As in the case of the massless modes, we want to solve for the momentum k in terms of the velocity to guarantee duality between a Lagrangian and Hamiltonian formulation. Luckily, having implemented all three matter conditions, one can show that a Legendre map ℓ_x between k and a corresponding tangent vector exists, is invertible and given by the following definition:

DEFINITION LEGENDRE MAP FROM MASSIVE POINT PARTICLE MODES

Given that all three matter conditions are implemented, the following injective Legendre map

$$\begin{aligned} \ell_x : C_x &\longrightarrow \mathrm{T}_x\mathcal{M} \\ k &\longmapsto \frac{1}{\deg P} \frac{\partial \ln P}{\partial k}(x,k) \end{aligned}$$

and its inverse $\ell_x^{-1} : \ell_x(C_x) \longrightarrow C_x$ exist.

Using this, it turns out that the corresponding Lagrange action is given by

$$\mathcal{S}_{\text{massive}}[x] := \int \mathrm{d}\lambda\, m \sqrt[\deg P]{P^\star\left(x(\lambda), \dot{x}(\lambda)\right)} , \tag{2.88}$$

where we introduced the decidedly non-polynomial object $P^\star$ as

$$\begin{aligned} P^\star_x : \ell_x(C_x) &\longrightarrow \mathbb{R} \\ v &\longmapsto \left(P\left(x, \ell_x^{-1}(v)\right)\right)^{-1} . \end{aligned} \tag{2.89}$$

Note that this action is invariant under strictly monotonously increasing reparametrizations. Using this, we can always reparametrize such that the polynomial $P^\star$ gives

$$P^\star\left(x(\lambda), \dot{x}(\lambda)\right) = 1\,, \tag{2.90}$$

in which case the momenta k take the particularly simple form

$$k(\lambda) = m\,\ell^{-1}_{x(\lambda)}\left(\dot{x}(\lambda)\right)\,. \tag{2.91}$$

Such paramatrizations for which these two conditions, i.e.

$$\dot{x} \in O_{x(\lambda)} \qquad \text{and} \qquad P^\star\left(x(\lambda), \dot{x}(\lambda)\right) = 1 \tag{2.92}$$

are true for the observer wordline $x(\lambda)$ physically correspond to the trajectories of freely falling, non-rotating observers. We will refer to a frame where both conditions are implemented as *local observer frame* in the following.

By variation of the action with respect to the world-line, and using the chosen parametrization we find that the massive particles obey the geodesics equation

$$0 = m\left[\ddot{x}^m g_{am}(x,\dot{x}) + \dot{x}^m\dot{x}^n\partial_n g_{am}(x,\dot{x}) - \frac{1}{2}\dot{x}^m\dot{x}^n\partial_a g_{mn}(x,\dot{x})\right]\,, \tag{2.93}$$

for the Finsler metric g_{ab} obtained from the polynomial $P^\star$ via

$$g_{mn}(x,v)\,u^m w^n := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left(P^\star(x, v + s\cdot u + t\cdot w)\right)^{2/\deg P}\Big|_{s=0,t=0}\,. \tag{2.94}$$

Remarkably, it can be shown (Rivera, 2012) that this metric is of Lorentzian signature, given that the matter conditions are implemented. Furthermore, since $P^\star$ is a homogenous function of degree $\deg P$, we find that the metric is homogenous of degree zero in its directional argument v and that

$$\begin{aligned} g_{mn}(x,v)v^m v^n &= \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left(P^\star(x,(1+s+t)\cdot v)\right)^{2/\deg P}\Big|_{s=0,t=0} \\ &= \frac{1}{2}\left(P^\star(x,v)\right)^{2/\deg P}\cdot\frac{\partial^2}{\partial s\partial t}\left(1+s+t\right)^2\Big|_{s=0,t=0} \\ &= P^\star(x,v)^{2/\deg P}\,. \end{aligned} \tag{2.95}$$

This allows us to write the Lagrange action of the massive point particle in terms of the Finsler metric as

$$\mathcal{S}_{\text{massive}}[x] = \int \mathrm{d}\lambda\, m\sqrt{g_{mn}\left(x(\lambda),\dot{x}(\lambda)\right)\,\dot{x}^m(\lambda)\dot{x}^n(\lambda)}\,, \tag{2.96}$$

which looks remarkably similar to the well-known expression from general relativity with the notable difference that the metric itself is dependent on the direction $\dot{x}(\lambda)$. Note that in general the Finsler metric is non-polynomial in its directional argument and is built from the geometry G.

The above construction of an observer frame moreover defines the purely spatial directions $S(\lambda) \subset T_{x(\lambda)}\mathcal{M}$ seen by the observer at $x(\lambda)$ via

$$\ell^{-1}_x(\dot{x}\,(\lambda))\,\left(S(\lambda)\right) = 0\,. \tag{2.97}$$

Note that the three matter conditions are sufficient for this formulation of observer frames that are compatible with the causality of the original matter field equations. Moreover, they are necessary for the matter field dynamics to be canonically quantisable. See for example Rivera and Schuller (2011) for the canonical quantisation of general linear electrodynamics using all of those ideas.

2.4.2 Legendre map from initial data surface compatibility

However, there is another definition of a Legendre map using the dual polynomial $P^\sharp$ instead of the principal polynomial. In this case, it is not necessary to assume that massive modes have the dispersion relation described above. Instead, we require that the spatial directions that an observer sees at $x(\lambda)$, defined via analogous constructions employing an alternative Legendre map, are compatible with the initial data surfaces of the matter field dynamics.

To make this precise, we first observe that we can make an analogous definition of a Legendre map $\widetilde{\ell}_x$ for a hyperbolicity cone $C_x^\sharp$ of the dual polynomial by choice of a time orientation. This map exists and is invertible for elements in the hyperbolicity cone:

DEFINITION LEGENDRE MAP FROM INITIAL DATA SURFACE COMPATIBILITY
Given that all three matter conditions are implemented, the following injective Legendre map

$$\begin{aligned} \widetilde{\ell}_x(v) : C_x^\sharp &\longrightarrow \mathrm{T}_x^*\mathcal{M} \\ v &\longmapsto \frac{1}{\deg P^\sharp}\frac{\partial \ln P^\sharp}{\partial v}(x,v) \end{aligned}$$

and its inverse $\widetilde{\ell}_x^{-1} : \ell_x(C_x^\sharp) \to C_x^\sharp$ exist.

Afterwards we can make another choice for the largest set for the observer cone $\mathcal{O}_x$ using this Legendre map. We first observe that the hyperbolicity cone C_x of P, i.e. the set of conormals n to the possible hypersurfaces is always contained in the image of our new Legendre map, i.e.

$$C_x \subseteq \widetilde{\ell}_x\left(C_x^\sharp\right) . \tag{2.98}$$

We now require that the Legendre map, applied to a tangent to an observer worldline, also gives a conormal that lies in the hyperbolicity cone C_x to be suitable as an initial value surface of the matter field dynamics. This then leads to the result that the largest possible observer cone compatible with this requirement is given by

$$\widetilde{O}_x = \widetilde{\ell}_x^{-1}\left(C_x\right) . \tag{2.99}$$

Both approaches to the Legendre map have proper physical motivation and are equally suitable for use in the derivation of the gravitational closure program. We will, as a result, use both of them. Luckily, it turns out that the calculations are mostly independent of our particular choice of Legendre map.

This concludes our treatment of the kinematical aspect, and we are finally equipped to start our approach to derive the dynamics of the geometry as seen by a particular matter field theory.

CHAPTER 3
GRAVITATIONAL CLOSURE

Having analysed the kinematical aspects of spacetime as seen by matter field theory, it is time to move to the dynamical aspects. In this chapter, we will see how the dynamics of the geometrical degrees of freedom can be *derived* once a particular matter theory was chosen and the principal polynomial and its dual implement the three matter conditions we laid out in the previous chapter. This extends the well-known result from Hojman et al. (1976), where it was shown that general relativity could, in fact, be derived by representing the algebra of hypersurface deformations on a phase space[1].

The discussion of this generalisation will consist of three parts: We will start in section 3.1 with a description of how to perform a 3+1 split of spacetime into hypersurfaces, even when no metric is at hand, and how to pull back *spacetime* tensor fields to tensor fields on the three-dimensional slices of our spacetime split. Here we can observe how the fields change under infinitesimal deformation of these hypersurfaces. Dynamics, in the sense of how the values of these fields change infinitesimally between hypersurfaces, can be understood precisely in terms of hypersurface deformations.

Afterwards, we will show in section 3.2 how this allows us to mimic these deformations in a canonical Hamiltonian formulation which will give us compatibility conditions that need to be fulfilled in order for both formulations, the hypersurface deformation picture, as well as the canonical phase space formulation, to agree on their description of dynamics of geometry.

The last step, presented in section 3.3.1 and the succeeding sections, consists of finally distilling these into a general system of linear homogeneous partial differential equations, called the gravitational closure equations, that allows to calculate the gravitational action. We conclude the chapter with a discussion of the general properties of the solution space and possible simplifications that can be made.

kappa

The results presented in this chapter have already been published as

M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational Closure of Matter Field Equations
Phys. Rev. D97 (2018), 084036

but are expanded on several occasions to allow for a self-complete description.

[1] Ignoring about minor topological subtleties on superspace (Giulini, 2009).

3.1 HYPERSURFACE DEFORMATIONS

We first approach dynamics from a spacetime perspective, where we split our four-dimensional spacetime manifold into a collection of three-dimensional hypersurfaces that are suitable as initial data hypersurfaces. In this picture, dynamics appears by observing how the fields change along the hypersurfaces. Since we, however, pull the spacetime geometry back to the hypersurfaces – where the precise technical notion of pulling back the degrees of freedom will be defined in the following – no new information is revealed: Instead, the existing information is simply presented in a different fashion. Yet this reformulation is incredibly helpful, as it reveals that the notion of diffeomorphism invariance is – for any geometry – encoded in an algebra.

3.1.1 3+1 decomposition and observer frames

Mathematically, the idea of a 3+1 decomposition of spacetime is to find a one-parameter family of smooth embeddings, i.e.

$$X_t : \Sigma \longrightarrow X_t(\Sigma) \subset \mathcal{M}\,, \tag{3.1}$$

where Σ is a three-dimensional manifold. Over the course of this thesis, we will assume Σ to be orientable, i.e. it is equipped with a volume form.

The space of embeddings from the three-dimensional manifold Σ to $\mathcal{M}$ is an infinite dimensional manifold we will denote as $\mathrm{Emb}(\Sigma, \mathcal{M})$. For each embedding to be suitable as initial data hypersurfaces compatible with the surfaces from given matter dynamics, its co-normal needs to be hyperbolic, i.e. it must lie in the hyperbolicity cone of the principal polynomial P at each point in $X_t(\Sigma)$.

Given such an embedding, as well as coordinates x^a on $\mathcal{M}$ and coordinates y^α on Σ, we can try to obtain an orthonormal basis of the tangent and cotangent space at each spacetime point. For this, we first observe that for a specific $t \in \mathbb{R}$ the push-forward of the coordinate-induced vector field ∂_α on Σ along the embedding map X_t naturally provides linearly independent tangential vectors to the embedded hypersurface $X_t(\Sigma)$, i.e.

$$e_\alpha|_{X_t(x)} := X_{t*}\left(\partial_\alpha|_x\right) \qquad \text{for } \alpha = 1\ldots 3,\ x \in \Sigma\,. \tag{3.2}$$

Then, we also naturally obtain the co-normal vector field ϵ^0 to the hypersurface as the annihilator of the tangential vectors, i.e.

$$\epsilon^0(e_\alpha) = 0 \qquad \text{for } \alpha = 1\ldots 3\,. \tag{3.3}$$

This is, however, not unique since we can still scale ϵ^0 by a scalar function f on the hypersurface $X_t(\Sigma)$. To make this unique, we need to provide a normalisation condition to select a specific co-normal field. Since we are equipped with the principal polynomial P, we can use it to enforce the condition

$$P(\epsilon^0) = 1\,, \tag{3.4}$$

which is physically equivalent to restricting to the local observer frames in the sense laid out in equation (2.90) and (2.91) in section 2.4.

In order to obtain a complete frame, we also need the normal vector field dual to ϵ^0, or informally speaking, we need to raise the index. In the absence of a metric, no *a priori* method exists to obtain this object. However, with the requirement that the hypersurface be hyperbolic, we can do so by the use of the Legendre map $\ell_{X_t(x)}$. This shows the power of the kinematic constructions made in the previous chapter since otherwise, no natural map between the conormal and the normal exists in the absence of a metric tensor. The only requirements that must be imposed are the three matter conditions, which are extremely mild. Moreover, it is also important to emphasise that the Legendre map can only be applied to covectors in the hyperbolicity cone; that is what we usually refer to as time-like covectors. Only in the metric case, due to linearity, this can also be extended to space-like covectors.

Finally raising the index of the co-normal ϵ^0 using the Legendre map we get the following expression for the normal vector e_0

$$e_0 := \ell_x(\epsilon^0_x)\,. \tag{3.5}$$

In our chosen coordinate system on $\mathcal{M}$ this reads

$$(e_0)^a\,(x) = P^{am_2\ldots m_{\deg P}}(x)\,\epsilon^0_{m_2}(x)\;\cdots\;\epsilon^0_{m_{\deg P}}(x)\,, \tag{3.6}$$

where we used the normalisation condition $P(\epsilon^0) = 1$.

This can then finally be completed into a complete orthonormal basis by introduction of dual tangential frames ϵ^α by the conditions

$$\epsilon^\alpha(e_0) = 0 \qquad \text{for } \alpha = 1\ldots 3\,, \tag{3.7}$$

$$\epsilon^\alpha(e_\beta) = \delta^\alpha_\beta \qquad \text{for } \alpha,\beta = 1\ldots 3\,. \tag{3.8}$$

The annihilation condition (3.7) translates into a condition on three components of the principal polynomial

$$P\left(\epsilon^\alpha, \epsilon^0, \ldots, \epsilon^0\right) = 0\,, \tag{3.9}$$

where we contract each slot of the coefficient of the principal polynomial by the corresponding covector, respectively.

Once this is implemented, one can check that the completeness relation

$$\delta^a_b = e^a_0\,\epsilon^0_b \,+\, e^a_\alpha\,\epsilon^\alpha_b \tag{3.10}$$

holds by contracting both the left and right hand side with all the individual frame fields. Before we proceed we quickly summarize the construction in the following definition:

DEFINITION OBSERVER FRAME

The orthonormal observer frame is obtained via

$$e_\alpha := X_{t*}(\partial_\alpha) \qquad \text{and} \qquad e_0 = \ell(\epsilon^0)\,, \tag{3.11}$$

and the dual basis defined via the conditions

$$\epsilon^0(e_\alpha) = 0 \qquad \text{and} \qquad \epsilon^\alpha(e_\beta) = \delta^\alpha_\beta\,, \tag{3.12}$$

which is unique once the normalisation and annihilation conditions on the principal polynomial are implemented as

$$P(\epsilon^0, \ldots, \epsilon^0) = 1\,, \tag{3.13}$$

$$P(\epsilon^\alpha, \epsilon^0, \ldots, \epsilon^0) = 0\,. \tag{3.14}$$

With the help of the orthonormal frame, we can decompose the spacetime tangent vector field $\dot{X}_t$ along the foliation uniquely into its purely normal and tangential components, i.e.

$$\dot{X}_t = N\,e_0 + N^\alpha\,e_\alpha\,, \tag{3.15}$$

with the *lapse* N being the normal projection of the vector and *shift* $\vec{N}$ being the tangential projections defined in terms of our basis covectors via

$$N := \epsilon^0(\dot{X}_t) \qquad \text{and} \qquad N^\alpha := \epsilon^\alpha(\dot{X}_t)\,. \tag{3.16}$$

The idea of the embeddings and the decomposition into lapse and shift is visualised in figure 3.1. Up to this point, we chose a foliation X_t of spacetime and can, using the orthonormal frame constructed from it with the help of our Legendre transform, derive lapse and shift from the foliation tangential vectors. Since the foliation is arbitrary – which encodes the diffeomorphism invariance of our theory since given a specific foliation X_t and a spacetime diffeomorphism φ we get another foliation via $\varphi \circ X_t$ – we will, however, later adapt the viewpoint that for a given initial data hypersurface $X_{t_0}(\Sigma)$ we can choose arbitrary lapse functions and shift vectors to construct a foliation for $t \neq t_0$.

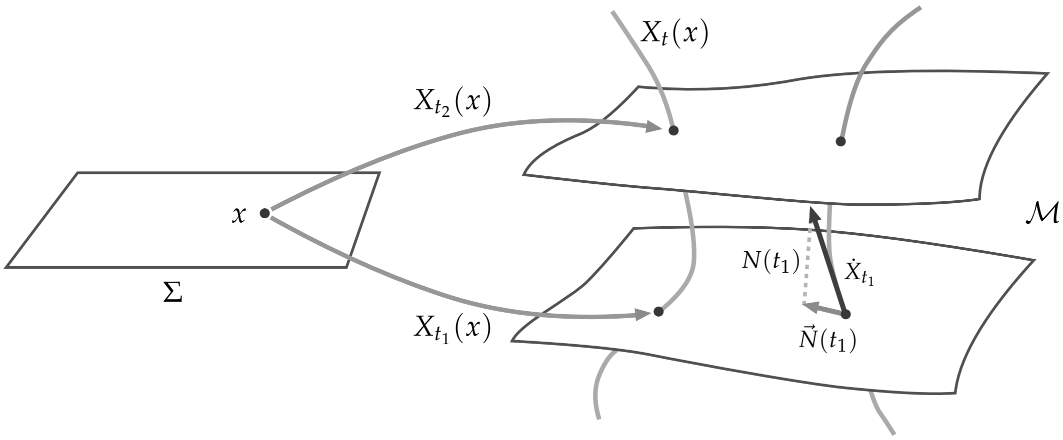


Figure 3.1 A foliation of spacetime into three-dimensional hypersurfaces.

REMARK

One may think that the normality and orthogonality conditions on the principal polynomial give restrictions on the choices of co-normal ϵ^0 and co-tangentials ϵ^α. However, it turns out that in a canonical description, one must restrict the geometric degrees of freedom such that the conditions are fulfilled in the coordinates adapted to the constructed observer frame. This seemingly eliminates four degrees of freedom from the geometry. However, since we add lapse and shift to the system and treat them as arbitrary objects, we actually reinstate those four degrees of freedom – just in a different form.

Observer Frame using the alternative Legendre map

The construction of the observer frame requires the Legendre map constructed from the principal polynomial. As we saw in the previous chapter, it is possible to also define another Legendre map $\widetilde{\ell}$ with the help of the dual polynomial $P^\sharp$ that makes sure that observer worldlines are mapped to hyperbolic covectors. Using this Legendre map, we can, again, define an orthonormal frame by similar constructions as above. As the required steps to obtain the basis are almost identical, we directly summarise the definition.

DEFINITION ALTERNATIVE OBSERVER FRAME

The orthonormal observer frame is obtained via

$$e_\alpha := X_{t*}(\partial_\alpha) \qquad \text{and} \qquad e_0 = \widetilde{\ell}^{-1}(\epsilon^0)\,, \tag{3.17}$$

and the dual basis defined via the conditions

$$\epsilon^0(e_\alpha) = 0 \qquad \text{and} \qquad \epsilon^\alpha(e_\beta) = \delta^\alpha_\beta\,, \tag{3.18}$$

which is unique once the normalisation and annihilation conditions on the principal polynomial are implemented as

$$P^\sharp(e_0, \ldots, e_0) = 1\,, \tag{3.19}$$

$$P^\sharp(e_\alpha, e_0, \ldots, e_0) = 0\,. \tag{3.20}$$

Using any of these two definitions of the observer frames allows us to project any tensor field to the three-dimensional manifold Σ. Depending on the parameter $t \in \mathbb{R}$ of our foliation, we obtain different tensor fields on Σ. In a sense, Σ becomes a cinema screen on which the evolution of the four-dimensional spacetime geometry is shown as a movie in the foliation parameter t. In this spirit, we will, from now on, refer to Σ as the *screen manifold*.

Conceptually, these constructions are quite standard in general relativity, with the notable difference that the Legendre map – whichever definition is chosen – becomes highly non-trivial for geometries beyond a Lorentzian metric.

Frame adapted to the foliation

We can also construct another frame that will be useful in some situations: Since we are equipped, given the foliation X_t, with the vector $\dot{X}_t = N \cdot e_0 + N^\alpha e_\alpha$, as well as the tangential vectors e_α we can determine the dual vectors $(\mathrm{d}t), \epsilon^\alpha$ by the usual conditions

$$(\mathrm{d}t)_a\, \dot{X}^a_t = 1\,, \tag{3.21a}$$

$$(\mathrm{d}t)_a\, e^a_\alpha = 0\,, \tag{3.21b}$$

$$e^a_\alpha\, \widetilde{\epsilon}^\beta_a = \delta^\alpha_\beta\,, \tag{3.21c}$$

$$\dot{X}^a_t\, \widetilde{\epsilon}^\alpha_a = 0\,. \tag{3.21d}$$

Solving this, it is easy to see that the duals are given by

$$(\mathrm{d}t)_a = \frac{1}{N}\epsilon^0_a\,, \tag{3.22a}$$

$$\widetilde{\epsilon}^\alpha_a = \epsilon^\alpha_a - \frac{1}{N}N^\alpha \epsilon^0_a\,. \tag{3.22b}$$

This frame can be used to express the components of the geometry in a coordinate system where time evolution is described along the foliation parameter t. The components then turn out to be given in terms of the Lagrange multipliers N and $\vec{N}$ and multiple projected geometric fields on the hypersurface – which we will take a closer look at in the next section.

Let us demonstrate this for the example of Maxwellian electrodynamics, i.e. a principal polynomial of degree 2 given by a Lorentzian metric. Expressing the components of the metric in the frame adapted to the foliation by contraction with the covectors, we obtain

$$g^{ab} = \left(\begin{array}{c|c} \frac{1}{N^2} & -\frac{1}{N}N^\beta \\ \hline -\frac{1}{N}N^\alpha & \gamma^{\alpha\beta} + \frac{1}{N^2}N^\alpha N^\beta \end{array}\right)^{ab} , \tag{3.23}$$

where we made use of the normalisation condition $P(\epsilon^0) = g^{ab}\epsilon^0_a\epsilon^0_b = 1$ and the annihilation condition $P(\epsilon^\alpha, \epsilon^0) = g^{ab}\epsilon^\alpha_a\epsilon^0_b = 0$ and introduced the Riemannian metric on the 3-manifold defined by the pull-back, i.e. $\gamma^{\alpha\beta} = g^{ab}\epsilon^\alpha_a\epsilon^\beta_b$. Unsurprisingly, this coincides with the ADM decomposition of the inverse metric (Arnowitt et al., 1959).

With this frame, we can further express the following two components of the principal polynomial and find that in general

$$P(\mathrm{d}t, \dots, \mathrm{d}t) = \frac{1}{N^{\deg P}} , \tag{3.24a}$$

$$P(\epsilon^\alpha, \mathrm{d}t, \dots, \mathrm{d}t) = -\frac{1}{N^{\deg P}}N^\alpha . \tag{3.24b}$$

This allows us in general to derive the relation between the tensor components of the spacetime geometry and lapse and shift. For the principal polynomial of a metric spacetime one finds that $N = 1/\sqrt{g^{00}}$ and $N^\alpha = -g^{0\alpha}/g^{00}$. For a bi-metric theory as in our example of two coupled Klein-Gordon fields from the previous chapter we find the relations

$$N = \frac{1}{\sqrt[4]{g^{00}h^{00}}} , \tag{3.25a}$$

$$N^\alpha = -\frac{1}{2}\left(\frac{g^{0\alpha}}{g^{00}} + \frac{h^{0\alpha}}{h^{00}}\right) , \tag{3.25b}$$

with the tensor components in an arbitrarily chosen frame. While it is easy to recognize the structural similarity to the definition of lapse and shift in the metric case it is, however, important to realize that lapse and shift are given in terms of *both* fields, rather than there being two separate sets of lapses and shifts (as proposed for instance in Hassan and Rosen (2012) and Klusoň (2014)).

3.1.2 Hypersurface projections

Equipped with an orthonormal frame constructed from the principal polynomial (or dual polynomial), we can now use them to project any spacetime tensor (density) to tensor fields (or densities) on the screen manifold. We will quickly present the procedure for some common fields.

Scalar field

In the case of a scalar field ϕ the projection is performed in a straight-forward fashion by directly employing the foliation to define the screen manifold scalar ψ, namely

$$\psi(x) := \phi\left(X_t(x)\right) . \tag{3.26}$$

Rank one tensors

For a vector field V or a covector field A, projection gives one scalar and vector or covector field each, i.e.

$$V_\perp := \epsilon^0(V)\,, \qquad V_\parallel^\alpha := \epsilon^\alpha(V)\,, \tag{3.27a}$$

$$A^\perp := A(e_0)\,, \qquad A_\alpha^\parallel := A(e_\alpha)\,. \tag{3.27b}$$

The spacetime fields can be reconstructed as

$$V = V_\perp\, e_0 + V_\parallel^\alpha\, e_\alpha\,, \tag{3.28a}$$

$$A = A^\perp\, \epsilon^0 + A_\alpha^\parallel\, \epsilon^\alpha\,. \tag{3.28b}$$

An example for this are Yang-Mills theories where the covector is given by a Lie-algebra valued covector field A, that is projected to a Lie-algebra valued scalar and a Lie-algebra valued vector on the hypersurface.

Rank two tensors

For a tensor of rank two $T^{\cdot\cdot}$, we have in total sixteen degrees of freedom that can be projected to the hypersurface. Here we can make use of the fact that we have a volume form on the hypersurface and, as a result, can project any two-form into a vector on the screen manifold. The resulting screen manifold fields are

$$\left(T_{(\text{scalar})}\right) := T^{ab}\, \epsilon_a^0 \epsilon_b^0\,, \tag{3.29a}$$

$$\left(T_{(\text{vector},1)}\right)^\alpha := T^{(ab)}\, \epsilon_a^0 \epsilon_b^\alpha\,, \tag{3.29b}$$

$$\left(T_{(\text{vector},2)}\right)^\alpha := T^{[ab]}\, \epsilon_a^0 \epsilon_b^\alpha\,, \tag{3.29c}$$

$$\left(T_{(\text{covector})}\right)_\alpha := \frac{1}{2}(\omega_G)_{\alpha\beta\gamma} T^{[ab]}\, \epsilon_a^\beta \epsilon_b^\gamma\,, \tag{3.29d}$$

$$\left(T_{(\text{tensor})}\right)^{\alpha\beta} := T^{(ab)}\, \epsilon_a^\alpha \epsilon_b^\beta\,, \tag{3.29e}$$

with the volume form ω_G obtained from the density $\epsilon_{\ldots}$ by de-densitization with the geometry G. This factor is in a sense arbitrary and simply has to be fixed once and for all. Note that the same construction can be repeated for a tensor T_{ab}.

Here the reconstruction reads

$$\begin{aligned} T^{ab} &= \left(T_{(\text{scalar})}\right) e_0^a e_0^b + \left(T_{(\text{vector},1)} + T_{(\text{vector},2)}\right)^\alpha e_0^a e_\alpha^b + \left(T_{(\text{vector},1)} - T_{(\text{vector},2)}\right)^\alpha e_0^b e_\alpha^a \\ &\quad + (\omega_G)^{\alpha\beta\gamma} \left(T_{(\text{covector})}\right)_\alpha e_\beta^a e_\gamma^b + \left(T_{(\text{tensor})}\right)^{\alpha\beta} e_\alpha^a e_\beta^b\,. \end{aligned} \tag{3.30}$$

In case we are given a metric $g^{\cdot\cdot}$, the projection simplifies to one scalar, one vector and one tensor. If the principal polynomial is the one dictated by Maxwellian electrodynamics we find that the normalisation and annihilation conditions determine the values of the scalar and vector projections during the construction of the observer frame, such that only the tensor projection on the manifold Σ remains.

However, it should be noted that it is not true in general that the annihilation and normalisation conditions fix certain components. This is only the case for linear annihilation and normalisation conditions. This is, of course, the case for Maxwellian electrodynamics (and more generally for any degree 2 principal polynomial). However, already for general linear electrodynamics, as we will see in the following section, this fails to be true and leads to many complications that need to be dealt with. One of these complications is that it is impossible to implement the annihilation and normalisation conditions by omitting certain hypersurface projections.

For an endomorphism $T^{\cdot}{}_{\cdot}$ and a covariant tensor $T_{\cdot\cdot}$ one proceeds similarly by employing the frames to either directly obtain a scalar, vector or a metric. Furthermore, we could use the spatial metric obtained from the principal polynomial $p^{\alpha\beta}$ to raise and lower indices. This metric is invertible for a bi-hyperbolic polynomial, and two-forms can be dualized with the help of the volume form.

An example for the covariant tensor is the curvature tensor of electrodynamics F_{ab}. Since it is antisymmetric only two vector modes $F_{(\text{vector},2)}$ and $F_{(\text{vector},3)}$ survive. Those vectors are the electric field E_α and (de-densitized) magnetic field B^α.

General linear electrodynamics

The last example we will present is for an area metric, that is, the geometry of general linear electrodynamics (see section 2.1). Remember that, due to the symmetry of the Faraday tensor $F = \mathrm{d}A$ in the action, we see that G possesses the algebraic symmetries of a Riemann tensor, i.e.

$$G^{abcd} = G^{[cd][ab]} \,. \tag{3.31}$$

With the help of the three matter conditions, one finds that the area metric must lie in one of seven (out of total 23) algebraic classes (Raetzel et al., 2011). In principle we could simply write down all nonvanishing components that we obtain by plugging the different covectors of our basis into the slots of the area metric. However, in practice it has proven itself to be useful to employ the volume form on Σ to dualize all tensor fields with rank larger than two. The three resulting fields are

$$\bar{g}^{\alpha\beta} := -G(\epsilon^0, \epsilon^\alpha, \epsilon^0, \epsilon^\beta)\,, \tag{3.32a}$$

$$\bar{\bar{g}}_{\alpha\beta} := \frac{1}{4}\frac{1}{\det \bar{g}^{\cdot\cdot}} \varepsilon_{\alpha\mu\nu}\varepsilon_{\beta\lambda\kappa} G(\epsilon^\mu, \epsilon^\nu, \epsilon^\lambda, \epsilon^\kappa)\,, \tag{3.32b}$$

$$\bar{\bar{\bar{g}}}^{\alpha}{}_{\beta} := \frac{1}{2}\frac{1}{\sqrt{\det \bar{g}^{\cdot\cdot}}} \varepsilon_{\beta\mu\nu} G(\epsilon^0, \epsilon^\alpha, \epsilon^\mu, \epsilon^\nu) - \delta^\alpha_\beta\,, \tag{3.32c}$$

where we used $\omega(G)_{\alpha\beta\gamma} = \left(\det \bar{g}^{\cdot\cdot}\right)^{-\frac{1}{2}} \epsilon_{\alpha\beta\gamma}$. Note that the normalisation and annihilation conditions lead to

$$\bar{\bar{\bar{g}}}^{\alpha}{}_{\alpha} = 0 \qquad \text{and} \qquad \bar{g}^{\mu[\alpha}\bar{\bar{\bar{g}}}^{\beta]}{}_{\mu} = 0\,. \tag{3.33}$$

This removes 4 out of the 21 independent components of the area metric in four dimensions. However, due to the non-linearity of the second condition, it becomes inherently complicated to implement this condition such that it remains valid under time evolution. In particular, these conditions cannot be implemented by a simple omission of degrees of freedom since the annihilation condition is quadratic in the fields.

De-densitizations

For any theory, it is, in fact, possible to de-densitize the projections to the hypersurface. Since we are equipped with the principal polynomial the following de-densitization factor can always be defined:

$$\chi_P = \frac{1}{\sqrt{-\det P(\epsilon^\alpha, \epsilon^\beta, \epsilon^0, \dots, \epsilon^0)}} \tag{3.34}$$

With the principal polynomial being hyperbolic, this ensures that the projection of the principal polynomial is an inverse metric with $\det P(\epsilon^\alpha, \epsilon^\beta, \epsilon^0, \dots, \epsilon^0) < 0$. This allows us to de-densitize any projection of the geometric fields to the hypersurface.

3.1.3 Hypersurface deformations

Up to now, we considered a single hypersurface of the embedding for some fixed $t \in \mathbb{R}$. In order to obtain a notion of dynamics in this setup, we can compare the projected values after an infinitesimal increase $t + \mathrm{d}t$ and observe how the fields changed. Mathematically, this corresponds to the derivative in the foliation parameter t of the projected fields, being *functionals* of the embedding X_t via the observer frames that are used in the projections, as well as the considered point on the hypersurface $X_t(x)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} F[X_t] &= \int_\Sigma \mathrm{d}^3x\, \dot{X}_t(x) \frac{\delta F}{\delta X_t^a(x)} \\ &= \int_\Sigma \mathrm{d}^3x\, \left(N e_0^a + N^\alpha e_\alpha^a\right)(x) \frac{\delta F}{\delta X_t^a(x)}\,,\end{aligned}$$

where we first used the functional chain rule and then plugged in the decomposition of the foliation vector field into the orthonormal basis. Geometrically, this is a vector field on $\mathrm{Emb}(\Sigma, \mathcal{M})$ that acts on the functionals of a chosen embedding that is lifted from the vector field $\dot{X}_t^a$. In fact, any vector field V on $\mathcal{M}$ defines such a vector field $X(V)$ on the space of embeddings in the following fashion:

$$X(V) := \int_\Sigma \mathrm{d}^3x\, V^a\left(X_t(x)\right) \frac{\delta}{\delta X_t^a(x)} \tag{3.35}$$

With the help of the observer frame we can decompose this into the tangential and normal direction and define the functional differential operators

$$\mathcal{H}(N) := \int_\Sigma \mathrm{d}^3x\, N(x) e_0^a(x) \frac{\delta}{\delta X_t^a(x)}\,, \tag{3.36a}$$

$$\mathcal{D}(\vec{N}) := \int_\Sigma \mathrm{d}^3x\, N^\alpha(x) e_\alpha^a(x) \frac{\delta}{\delta X_t^a(x)}\,. \tag{3.36b}$$

Note that here we assume N and $\vec{N}$ to be some functions that parametrize the arbitrary spacetime vector field V and are unrelated to a chosen foliation.

Using these definitions, we can elegantly rewrite the time evolution as the action of the tangential and normal deformation functional differential operators on our functional F, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}F[X_t] = \mathcal{H}\left(N(X_t)\right) F[X_t] + \mathcal{D}\left(\vec{N}(X_t)\right) F[X_t] \,.$$

Note that for the projected fields, the functional dependency on the embedding is solely due to the observer frame and spacetime point on the hypersurface, i.e. the fields are functions of the form

$$F\left(X_t(x), e_0[X_t(x)], e_\alpha[X_t(x)], \epsilon^0[X_t(x)], \epsilon^\alpha[X_t(x)]\right) \,. \tag{3.37}$$

This allows us to further analyze the action of the functional differential operators defined above by first analyzing the functional derivative of the frame and afterwards taking a closer look for the tangential and normal directions separately.

Functional derivatives of the observer frame

In order to calculate how the frames change with the foliation we start with the functional derivative of the tangential frames. Here it follows by straight-forward calculation from the definition that

$$\frac{\delta e^a_\alpha(x)}{\delta X^b_t(y)} = \delta^a_b(\partial_\alpha \delta_y)(x) \,. \tag{3.38}$$

For the remaining vectors and covectors, the situation is more complicated, as the objects are defined implicitly in relation to the tangential vectors and by the annihilation and normalisation conditions. Still, we can use the equations defining the frames (3.9) and the completeness relation (3.10) to derive the tangential and normal components of the functional derivatives separately.

Starting with the co-normal ϵ^0, we use the fact that $\epsilon^0(e_\alpha) = 0$ to derive that

$$0 = \frac{\delta \epsilon^0_a(x)}{\delta X^b_t(y)} e^a_\alpha(x) + \epsilon^0_b(x)(\partial_\alpha \delta_y)(x) \,. \tag{3.39}$$

This allows us to express the tangential projection of the functional derivative. Similarly, we can obtain the normal projection with the help of the normalisation condition, which yields

$$e^a_0(x)\frac{\delta \epsilon^0_a(x)}{\delta X^b_t(y)} = -\frac{1}{\deg P}(\partial_b P^{a_1\ldots a_{\deg P}})(X_t(x))\, \epsilon^0_{a_1}(x)\ldots\epsilon^0_{a_{\deg P}}(x)\ \delta_y(x) \,. \tag{3.40}$$

Combined, this gives for the functional derivative of the co-normal

$$\begin{aligned}\frac{\delta \epsilon^0_a(x)}{\delta X^b_t(y)} = &-\frac{1}{\deg P}(\partial_b P^{a_1\ldots a_{\deg P}})(X_t(x))\, \epsilon^0_a(x)\epsilon^0_{a_1}(x)\ldots\epsilon^0_{a_{\deg P}}(x)\ \delta_y(x) \\ &- \epsilon^0_b(x)\epsilon^\alpha_a(x)\ (\partial_\alpha\delta_y)(x) \,.\end{aligned} \tag{3.41}$$

In a similar fashion one can derive, with the definitions of the normal vector and the tangential covectors, their functional derivative with respect to the embedding:

$$\begin{aligned}\frac{\delta e_0^a(x)}{\delta X_t^b(y)} &= -(\deg P - 1) p^{\alpha\beta}(x) e_\alpha^a(x) \epsilon_b^0(x)\ (\partial_\beta \delta_y)(x) \\ &\quad + (\partial_b P^{a m_2 \dots m_{\deg P}})(X_t(x)) \epsilon_{m_2}^0(x) \dots \epsilon_{m_{\deg P}}^0(x)\ \delta_y(x) \\ &\quad - \frac{\deg P - 1}{\deg P} (\partial_b P^{m_1 \dots m_{\deg P}})(X_t(x)) e_0^a(x) \epsilon_{m_1}^0(x) \dots \epsilon_{m_{\deg P}}^0(x)\ \delta_y(x)\,, \qquad (3.42a) \\ \frac{\delta \epsilon_a^\alpha(x)}{\delta X_t^b(y)} &= -\epsilon_b^\alpha(x) \epsilon_a^\beta(x)\ (\partial_\beta \delta_y)(x) \\ &\quad + (\deg P - 1) \epsilon_a^0(x) \epsilon_b^0(x) p^{\alpha\beta}(x)\ (\partial_\beta \delta_y)(x) \\ &\quad - (\partial_b P^{m_1 \dots m_{\deg P}})(X_t(x)) \epsilon_a^0(x) \epsilon_{m_1}^\alpha(x) \epsilon_{m_2}^0(x) \dots \epsilon_{m_{\deg P}}^0(x)\ \delta_y(x)\,. \qquad (3.42b)\end{aligned}$$

Using these functional derivatives we can move on to inspect the action of the functional differential operators $\mathcal{D}(\vec{N})$ and $\mathcal{H}(N)$ in further detail.

Action of tangential deformations

For an arbitrary functional of the embedding, we can expand the functional derivative in terms of the derivatives of the lapse and some characteristic coefficients for the field, i.e.

$$\begin{aligned}\mathcal{D}(\vec{N}) F[X_t](y) &:= \int_\Sigma \mathrm{d}^3x\, N^\mu(x) e_\mu^a(x) \sum_{k=0}^\infty (-1)^k \frac{\partial F}{\partial \partial_{\gamma_1 \dots \gamma_k} X_t^a}(y) (\partial_{\gamma_1 \dots \gamma_k} \delta_y)(x) \\ &= \sum_{k=0}^\infty \sum_{l=0}^k \binom{k}{l} (\partial_{\gamma_1 \dots \gamma_l} N^\mu)(y) (\partial_{\gamma_{l+1} \dots \gamma_k} e_\mu^a)(y) \frac{\partial F}{\partial \partial_{\gamma_1 \dots \gamma_k} X_t^a}(y) \\ &=: \sum_{k=0}^\infty (-1)^k (\partial_{\gamma_1 \dots \gamma_k} N^\mu)(y) \boldsymbol{F}_\mu^{\gamma_1 \dots \gamma_k}(y)\,, \qquad (3.43)\end{aligned}$$

with the coefficients reading

$$\boldsymbol{F}_\mu^{\gamma_1 \dots \gamma_k}(x) := \sum_{l=0}^\infty (-1)^k \binom{k+l}{l} \left(\partial_{\alpha_1 \dots \alpha_l} e_\mu^a\right)(x) \frac{\partial F}{\partial \partial_{\alpha_1 \dots \alpha_l \gamma_1 \dots \gamma_k} X_t^a}(x)\,. \qquad (3.44)$$

The archaic $(-1)^k$ terms both in all summands and the coefficients are introduced for historical reasons.

The first coefficient turns out to be the chain rule in disguise

$$\begin{aligned}\boldsymbol{F}_\mu &= \sum_{l=0}^\infty \left(\partial_{\alpha_1 \dots \alpha_l} e_\mu^a\right) \frac{\partial F}{\partial \partial_{\alpha_1 \dots \alpha_l} X_t^a} \\ &= \sum_{l=0}^\infty \left(\partial_{\mu \alpha_1 \dots \alpha_l} X_t^a\right) \frac{\partial F}{\partial \partial_{\alpha_1 \dots \alpha_l} X_t^a} \\ &= \partial_\mu F\,. \qquad (3.45)\end{aligned}$$

The remaining coefficient have to be calculated, case by case, for any of our functionals of interest. For the projected fields we deal with functionals of the form (3.37), such that only the derivatives in X_t^a and ∂X_t^a contribute and the action of the tangential deformation operator simplifies to

$$\mathcal{D}(\vec{N}) F[X_t] = N^\mu (\partial_\mu F) - (\partial_\gamma N^\mu) \boldsymbol{F}_\mu^\gamma\,. \qquad (3.46)$$

Due to the functional derivatives of the frames, as presented in the previous section, we can furthermore derive that the second coefficient takes a rather simple form since only the derivatives in the slots belonging to the tangential directions contribute:

$$\mathbf{F}_\mu^\gamma = \epsilon_m^\gamma \frac{\partial F}{\partial \epsilon_m^\mu} - e_\mu^m \frac{\partial F}{\partial e_\gamma^m} \tag{3.47}$$

From this we can read off that the action of the tangential deformation operator in fact gives the Lie derivative along the lapse $\vec{N}$, i.e.

$$\mathcal{D}(\vec{N})F = N^\mu \partial_\mu F + (\partial_\gamma N^\mu) \left(e_\mu^m \frac{\partial F}{\partial e_\gamma^m} - \epsilon_m^\gamma \frac{\partial F}{\partial \epsilon_m^\mu} \right) \equiv (\mathcal{L}_{\vec{N}} F) \,. \tag{3.48}$$

In other words we see that $\mathcal{D}$ acts as the generator of spatial diffeomorphisms.

We can also obtain this result with an abstract argument for a specific class of functionals, that is, all functionals where a diffeomorphism φ on Σ acts via the pull-back, i.e.

$$F[X_t \circ \varphi] = \varphi^* F[X_t] \,. \tag{3.49}$$

We pick an one-parameter diffeomorphism φ_ϵ and calculate the derivative of the functional F evaluated at the embedding $X_t \circ \varphi_\epsilon$. Then on the one hand, straight forward evaluation yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F[X_t \circ \varphi_\epsilon] = \int_\Sigma \mathrm{d}^3 x \, \frac{\delta F}{\delta X_t^a(x)} e_\alpha^a(x) M^\alpha(x) = \mathcal{D}(\vec{M})\, F[X_t] \,, \tag{3.50}$$

where $\vec{M}$ is the vector field generating the diffeomorphism. On the other hand, if we spell out the derivative on the left hand side we have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F[X_t \circ \varphi_\epsilon] &= \lim_{\epsilon \to 0} \left(\frac{F[X_t \circ \varphi_\epsilon] - F[X_t]}{\epsilon} \right) \\ &= \lim_{\epsilon \to 0} \left(\frac{\varphi_\epsilon^* F[X_t] - F[X_t]}{\epsilon} \right) \\ &= \left(\mathcal{L}_{\vec{M}} F \right) \,. \end{aligned} \tag{3.51}$$

This is typically the case. However, when we later reintroduce the tangential deformation coefficient in the derivation and discussion of gravitational closure, note that none of the arguments is crucially dependent on the deformation operator acting as Lie derivative. This, for example, would allow us also to treat spinorial fields by straight-forward calculation of the coefficient.

DEFINITION ACTION OF TANGENTIAL DEFORMATIONS

The tangential deformation operator acts on functionals of the embedding by a Lie derivative along $\vec{N}$

$$\mathcal{D}(\vec{N})F[X_t] = N^\mu \partial_\mu F - \left(\partial_\gamma N^\mu\right) \mathbf{F}^\gamma_\mu = \left(\mathcal{L}_{\vec{N}} F\right) ,$$

with the tangential deformation coefficient defined as

$$\begin{aligned} \mathbf{F}^\gamma_\mu =& e^a_\mu \frac{\partial F}{\partial \partial_\gamma X^a} \\ =& \epsilon^\gamma_a \frac{\partial F}{\partial \epsilon^\mu_a} - e^a_\mu \frac{\partial F}{\partial e^a_\gamma} . \end{aligned}$$

Action of normal deformations

In a similar fashion, we can analyse the action of the normal deformation operator $\mathcal{H}(N)$. We start by, again, expanding the functional into partial derivatives:

$$\begin{aligned} \mathcal{H}(N)F[X_t](y) :=& \int_\Sigma \mathrm{d}^3x\, N(x) e^a_0(x) \sum_{k=0}^{\infty} (-1)^k \frac{\partial F}{\partial \partial_{\alpha_1 \ldots \alpha_k} X^a_t}(y) (\partial_{\alpha_1\ldots\alpha_k} \delta_y)(x) \\ =& \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} (\partial_{\alpha_1\ldots\alpha_l} N)(y) (\partial_{\alpha_{l+1}\ldots\alpha_k} e^a_0)(y) \frac{\partial F}{\partial \partial_{\alpha_1\ldots\alpha_k} X^a_t}(y) \\ =:& N(y)k(y) + \sum_{k=1}^{\infty} (\partial_{\gamma_1\ldots\gamma_k} N)(y) \boldsymbol{M}^{\gamma_1\cdots\gamma_k}(y) , \end{aligned} \tag{3.52}$$

where k captures the local behavior of the deformation and will be identified with the notion of the fields' *velocity*. The coefficients $\boldsymbol{M}^{\gamma_1\cdots\gamma_k}$ capture the non-local response of the functional to the normal deformation, and are defined in the following fashion

$$\boldsymbol{M}^{\gamma_1\cdots\gamma_k}(x) := \sum_{l=0}^{\infty} \binom{k+l}{l} (\partial_{\alpha_1\ldots\alpha_l} e^a_0)(x) \frac{\partial F}{\partial \partial_{\alpha_1\ldots\alpha_l \gamma_1\ldots\gamma_k} X^a_t}(x) . \tag{3.53}$$

Given that the functional is obtained by the projection (3.37), the situation again simplifies significantly and the only non-vanishing non-local coefficient is $\boldsymbol{M}^\gamma$, for which the expression then reads

$$\boldsymbol{M}^\gamma(x) := e^a_0(x) \frac{\partial F}{\partial \partial_\gamma X^a_t}(x) . \tag{3.54}$$

Note that this coefficient is kinematical due to the fact that no derivative of the normal vector appears in its definition. Explicitely spelling it out will then always yield a projection of the geometric fields to the hypersurface and requires no dynamical data.

For our projected fields we can further expand this expression by calculating the derivative in each slot of the frame vectors and covectors and plugging in their respective functional derivatives

$$\mathbf{M}^\gamma = \left(\deg P - 1\right) p^{\alpha\gamma} \left(\epsilon^0_a \frac{\partial F}{\partial \epsilon^\alpha_a} - e^a_\alpha \frac{\partial F}{\partial e^a_0} \right) + e^a_0 \frac{\partial F}{\partial e^a_\gamma} - \epsilon^\gamma_a \frac{\partial F}{\epsilon^0_a} . \tag{3.55}$$

Furthermore, in case the projected field F is dualized into another hypersurface field $\widetilde{F}$, then due to the chain rule we see that the non-local deformation coefficient transforms as follows:

$$\widetilde{\mathbf{M}}^{\gamma} = \frac{\partial \widetilde{F}}{\partial F} \mathbf{M}^{\gamma} \tag{3.56}$$

This proves to be quite useful in practice when moving back and forth between different screen manifold projections of the gravitational field in calculations. The action of the normal hypersurface deformations is again summarised in the following definition:

DEFINITION ACTION OF NORMAL HYPERSURFACE DEFORMATIONS

For fields where the components are projected with the help of the orthogonal frames , the action of the normal deformation operator reads

$$\mathcal{H}(N)F[X_t](y) := N(y)k(y) + (\partial_\gamma N)(y)\boldsymbol{M}^{\gamma}(y)\,,$$

with the non-local normal deformation coefficient defined as

$$\begin{aligned}\boldsymbol{M}^{\gamma} &= e_0^a \frac{\partial F}{\partial \partial_\gamma X_t^a} \\ &= \big(\deg P - 1\big)\, p^{\alpha\gamma} \left(\epsilon_a^0 \frac{\partial F}{\partial \epsilon_a^\alpha} - e_\alpha^a \frac{\partial F}{\partial e_0^a} \right) + e_0^a \frac{\partial F}{\partial e_\gamma^a} - \epsilon_a^\gamma \frac{\partial F}{\epsilon_a^0}\,.\end{aligned}$$

Commutator algebra

The functional differential operators turn out to also possess a group structure that encodes the action of space diffeomorphisms. This can be seen by analysing the commutator algebra of the operators.

As seen in equation (3.35), any spacetime vector field induces a functional differential operator (Giulini, 2009), that is a vector field on the space of embeddings $\mathrm{Emb}(\Sigma, \mathcal{M})$. As such, it is straight-forward to obtain that the commutator is given by

$$\big[X(V), X(W)\big] = X([V, W])\,.$$

This tells us that the operator is a Lie homomorphism and $X(V)$ indeed corresponds to a left action of $\mathrm{Diff}(\mathcal{M})$ on $\mathrm{Emb}(\Sigma, \mathcal{M})$. Similarly, the tangential and normal deformation operators can be found to fulfill a commutator algebra:

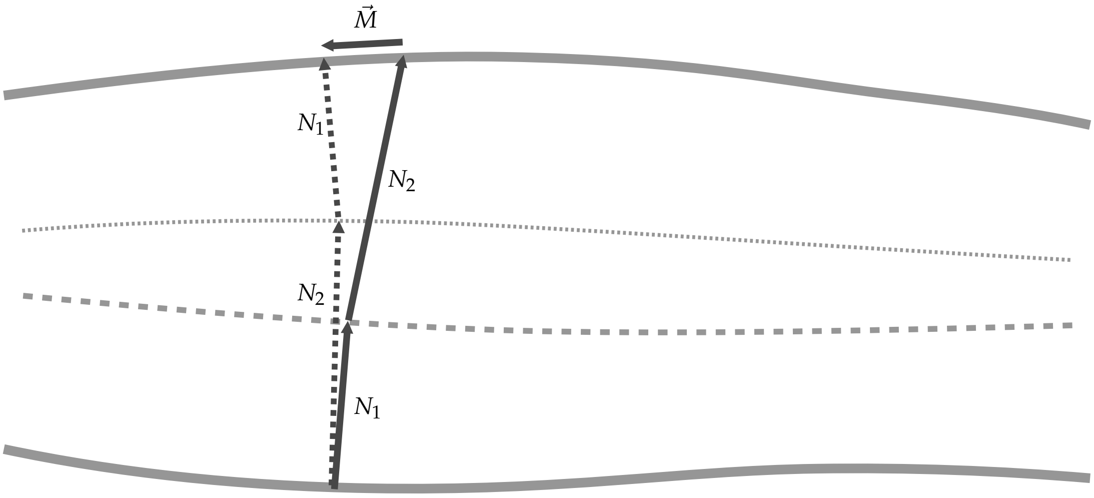


Figure 3.2 Illustration of the hypersurface deformation algebra. Two different hypersurfaces are obtained from the initial hypersurface by deforming with lapses $N_1(x)$ and $N_2(x)$ along the normal of the hypersurface. These hypersurfaces are then deformed with the other lapse, that is $N_2(x)$ and $N_1(x)$ along the new hypersurface normals. The resulting hypersurface points differ by a shift $\vec{M}$ in the final hypersurface

DEFINITION **COMMUTATOR ALGEBRA OF HYPERSURFACE DEFORMATIONS**

$$\begin{aligned}
\left[\mathcal{H}(N),\mathcal{H}(M)\right] &= -\mathcal{D}\left((\deg P-1)p^{\alpha\beta}(M\partial_\alpha N - N\partial_\alpha M)\partial_\beta\right)\\
\left[\mathcal{D}(\vec{N}),\mathcal{H}(M)\right] &= -\mathcal{H}\left(\mathcal{L}_{\vec{N}}M\right)\\
\left[\mathcal{D}(\vec{N}),\mathcal{D}(\vec{M})\right] &= -\mathcal{D}\left(\mathcal{L}_{\vec{N}}\vec{M}\right)
\end{aligned}$$

This has a simple geometric interpretation that is depicted in figure 3.2. Using the two functional operators, we first deform an initial hypersurface $X_{t_i}(\Sigma)$ along a lapse N and shift $\vec{N}$ into the hypersurface $X_{t_i+\delta t}(\Sigma)$. Afterwards we deform this hypersurface along yet another lapse M and shift $\vec{M}$ and end up with another hypersurface $X_{t_e}(\Sigma)$. If we, on the other hand, would have applied the operators in the opposite order, i.e. first evolve using M and $\vec{M}$ and then by N and $\vec{N}$, we would have ended up with a hypersurface that differs from $X_{t_e}(\Sigma)$. By how much the hypersurfaces differ is encoded in the commutator algebra, i.e. although being separate hypersurfaces they lack the application of a lapse $(\mathcal{L}_{\vec{M}}N - \mathcal{L}_{\vec{N}}M)$, as well as an additional shift vector $\left([\vec{N},\vec{M}] + (\deg P-1)\, p^{\alpha\beta}(N\partial_\alpha M - M\partial_\alpha N)\partial_\beta\right)$.

The idea behind gravitational closure is that we can obtain the gravitational Lagrangian by constructing a representation of this algebra on a canonical phase space that mimics time evolution as described by hypersurface deformation. Any such representation is a solution of a system of linear partial differential equations and can be constructively obtained by solving these equations. This is a generalisation of results in the geometrodynamics community for a Lorentzian metric $g_{\cdot\cdot}$ (Hojman et al., 1976).

Note that in the previous sections, we carried out all calculations for the observer frame constructed with the help of the principal polynomial. In the very same fashion we could redo the calculations for our alternative frame that is based on the dual polynomial $P^\sharp$. Interestingly, one finds almost entirely the same algebra, with the exception that we need to substitute

$$\deg P \longrightarrow \frac{1}{\deg P^\sharp}, \tag{3.57a}$$

$$p^{\alpha\beta} \longrightarrow \left(p^{\sharp -1}_{\cdot\cdot}\right)^{\alpha\beta}, \tag{3.57b}$$

with the inverse of $p^\sharp_{\alpha\beta} := P^\sharp(e_\alpha, e_\beta, e_0, \ldots, e_0)$. Due to hyperbolicity of the dual polynomial – enforced by the second matter condition – it can be shown that $p^\sharp_{\alpha\beta}$ is always invertible. The same transformations appear in the expression for the non-local normal deformation coefficient $\mathbf{M}^\gamma$. Ultimately, we expect that the result will be independent of the choice of observer frame we started with.

3.2 CANONICAL FORMULATION

The following section is dedicated to finally explaining the idea – and the technical realisation – of gravitational closure in more detail. Intuitively, we will see in the following section how one can construct a consistent Hamiltonian formulation that mimics the hypersurface deformation algebra presented in the previous section.

This is necessary for the following reason: The description presented above is dependent on the values of the field at *any* spacetime point; the fields are merely described in a different way with the help of the foliation. This means that the formulation only reveals information about the dynamics of the field in all of spacetime. We dub this view the *divine view* [2]. In contrast, the more practical and realistic view, let us call it the *human view*, is much more limited: the information available to us is limited in both temporal and spatial directions (see figure 3.3).

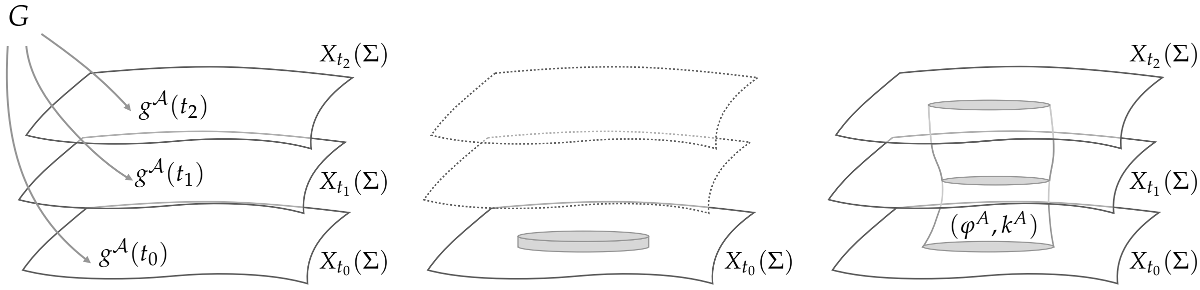
In order to move to a formulation that is tailored to our practically available tools and data, we will carry over the information that we have gathered in the analysis of the hypersurface deformation functional operators into a canonical phase space formulation. This means that we construct a representation of the hypersurface deformation algebra such that the equations of motion are *compatible* with the dynamics as seen in the divine view. This will amount to solving a system of countably many partial differential equations to obtain the gravitational Lagrangian.

3.2.1 Canonical phase space

We start the definition of our canonical phase space for the gravitational degrees of freedom by looking at the configurational variables. Naïvely, we would use the projected geometry fields, following our description from the previous sections, directly. However, this fails in general for multiple reasons.

Given that the tensor fields underly any symmetries, one immediately recognises that not all tensor components can be independent of each other. For example, a metric that is projected to the screen manifold Σ has nine components. However, only six of them are independent since the metric is symmetric

[2] "Only God has the luxury of knowing the values of the fields at all spacetime points".



(a) "Divine view" **(b)** "Human view" **(c)** "Physicist's view"

Figure 3.3 Illustration of the three different views. a) The "divine" view on the geometry G. The induced geometry on each hypersurface can be directly calulcated from the spacetime object everywhere; b) The "human" view on the spacetime geometry that includes our restricted view, both spatially and temporally, to a confined region of spacetime; c) The "physicist's" view, where we translated our restricted knowledge from the human view into a set of initial data (φ^A, k^A) on an initial hypersurface and are able to recover all further information on other hypersurfaces $X_t(\Sigma)$ with the help of equations of motion of the geometry.

in its indices. One may work around this complication by a careful setup of the phase space with the tensor components as configuration variables, as these symmetry conditions are linear conditions on the components which means that we can deal with them by omission of some components.

The second complication arises from the normalisation and annihilation conditions on the principal polynomial that must be required in order for the observer frame to be unique. Recall that they generally lead to non-linear conditions, as became evident for the conditions on the principal polynomial of degree 4 of an area-metric, i.e. birefringent electrodynamics. Any endeavour to carefully work around this complication with tensor components as configuration variables is doomed to fail from the beginning.

One possible route to solve this is to add the annihilation and normalisation to the equations of motions as constraints, such that on-shell, the phase space reduces to the subspace for which the frame conditions are implemented. In the Hamiltonian formulation, this means that we add the constraints with Lagrange multipliers to the Hamiltonian and perform the Dirac-Bergmann algorithm (or equivalently the Cartan-Kuranishi algorithm) to extract all the secondary constraints that may appear. The constraint surface is also a symplectic manifold, which would allow us to formulate the algebra with the help of the Dirac bracket. However, the corresponding partial differential equations would be rather complicated to solve.

Instead, we try to parametrize the constraints on our degrees of freedom imposed by symmetry conditions and the two frame conditions directly, in the same fashion as one would use one-dimensional generalised coordinates for a particle in Euclidian space that is confined to motions on a circle of fixed radius in classical mechanics.

Parametrizations

We start by giving a definition for a parametrization. The basic idea is that it expresses the independent tensor components of our geometric field – or multiple fields – such that the annihilation and normalisation conditions are fulfilled trivially. The degrees of freedom can then also be understood as generalised tensor components.

DEFINITION PARAMETRIZATION

A parametrization of the F geometric degrees of freedom on a F-dimensional manifold Φ is given by

- a map $\widehat{g}^{\mathcal{A}} : \Phi \longrightarrow \mathbb{R}$ such that $\widehat{g}^{\mathcal{A}}(\varphi^1, \ldots, \varphi^F)$ has all the symmetries of the projected tensor field and fulfills the two frame conditions for any $\varphi \in \Phi$.
- an inverse map $\widehat{\varphi}^A : \mathrm{T}^p_q\Sigma \longrightarrow \mathbb{R}$, that allows to extract the F degrees of freedom from the components of a (p, q) tensor field (even though the tensor field may fail to fufill the symmetry and frame conditions), with

$$\left(\widehat{\varphi}^A \circ \widehat{g}^{\mathcal{A}}\right)(\varphi^1, \ldots, \varphi^F) = \varphi^A \qquad , \text{for } A = 1, \ldots, F\,. \tag{3.58}$$

Calculating the derivative of equation (3.58) with respect to φ^B we find

$$\frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{A}}}\left(\widehat{g}^{\mathcal{A}}(\varphi)\right)\frac{\partial \widehat{g}^{\mathcal{A}}}{\partial \varphi^B}(\varphi) = \delta^A_B\,, \tag{3.59}$$

for the two maps that will frequently show up in all of the following calculations

$$\frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{A}}}\left(\widehat{g}^{\mathcal{A}}(\varphi)\right) \qquad \text{and} \qquad \frac{\partial \widehat{g}^{\mathcal{A}}}{\partial \varphi^A}(\varphi)\ . \tag{3.60}$$

We will refer to them as *intertwiners* in the following. Moreover, it can easily be shown that the object

$$\mathcal{T}^{\mathcal{A}}{}_{\mathcal{B}}(\varphi) := \frac{\partial \widehat{g}^{\mathcal{A}}}{\partial \varphi^A}(\varphi)\,\frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{B}}}\left(\widehat{g}^{\mathcal{M}}(\varphi)\right) \tag{3.61}$$

is a projector due to equation (3.59).

Example: Lorentzian metric For our typical example of Maxwellian electrodynamics, we can parametrize the Lorentzian metric in the following fashion:

$$\widehat{g}^{00}(\varphi) = 1 \qquad \widehat{g}^{0\alpha}(\varphi) = 0 \qquad \widehat{g}^{\alpha\beta}(\varphi) = \begin{pmatrix} \varphi^1 & \frac{\varphi^2}{\sqrt{2}} & \frac{\varphi^3}{\sqrt{2}} \\ \frac{\varphi^2}{\sqrt{2}} & \varphi^4 & \frac{\varphi^5}{\sqrt{2}} \\ \frac{\varphi^3}{\sqrt{2}} & \frac{\varphi^5}{\sqrt{2}} & \varphi^6 \end{pmatrix}^{\alpha\beta} \tag{3.62a}$$

$$\widehat{\varphi}^A(g^{00}, g^{0\alpha}, g^{\alpha\beta}) = \left(g^{11}, \frac{1}{\sqrt{2}}(g^{12}+g^{21}), \frac{1}{\sqrt{2}}(g^{13}+g^{31}), g^{22}, \frac{1}{\sqrt{2}}(g^{23}+g^{32}), g^{33}\right)^A \tag{3.62b}$$

and one can easily convince oneself that this fulfils all the properties of a parametrization. In particular, we implemented the annihilation and normalisation condition by setting the corresponding components of the metric to a constant.

This parametrization may seem rather unnatural, since we very well could also have chosen the simple parametrization

$$\widehat{g}^{00}(\varphi) = 1 \qquad \widehat{g}^{0\alpha}(\varphi) = 0 \qquad \widehat{g}^{\alpha\beta}(\varphi) = \begin{pmatrix} \varphi^1 & \varphi^2 & \varphi^3 \\ \varphi^2 & \varphi^4 & \varphi^5 \\ \varphi^3 & \varphi^5 & \varphi^6 \end{pmatrix}^{\alpha\beta} \tag{3.63a}$$

$$\widehat{\varphi}^A(g^{00}, g^{0\alpha}, g^{\alpha\beta}) = \left(g^{11}, g^{12}, g^{13}, g^{22}, g^{23}, g^{33}\right)^A \tag{3.63b}$$

which also does fulfill all required properties of a parametrization. The reason we will chose the first parametrization over the second in this thesis is that, besides historic reasons, in this case the intertwiners and its inverse, respectively, are matrix transposes of each other. This idea traces back to the construction of intertwiners from projection operators, see Reiß (2014) for further details.

Example 2: Bi-metric theory If we deal with a bi-metric theory, the principal polynomial reads $P^{abcd} = g^{(ab}h^{cd)}$. The annihilation and normalisation conditions can easily be read off from this as

$$g^{00}h^{00} = 1\,, \tag{3.64a}$$

$$g^{00}h^{0\alpha} + h^{00}g^{0\alpha} = 0\,. \tag{3.64b}$$

As a result, a parametrization can be constructed by

$$\widehat{g}^{00}(\varphi) = \varphi^1\,, \qquad \widehat{g}^{0\alpha}(\varphi) = \begin{pmatrix} \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{pmatrix}^{\alpha}, \qquad \widehat{g}^{\alpha\beta}(\varphi) = \begin{pmatrix} \varphi^5 & \varphi^6 & \varphi^7 \\ \varphi^6 & \varphi^8 & \varphi^9 \\ \varphi^7 & \varphi^9 & \varphi^{10} \end{pmatrix}^{\alpha\beta} \tag{3.65a}$$

$$\widehat{h}^{00}(\varphi) = \frac{1}{\varphi^1}\,, \qquad \widehat{h}^{0\alpha}(\varphi) = -\frac{1}{(\varphi^1)^2}\begin{pmatrix} \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{pmatrix}^{\alpha}, \qquad \widehat{h}^{\alpha\beta}(\varphi) = \begin{pmatrix} \varphi^{11} & \varphi^{12} & \varphi^{13} \\ \varphi^{12} & \varphi^{14} & \varphi^{15} \\ \varphi^{13} & \varphi^{15} & \varphi^{16} \end{pmatrix}^{\alpha\beta}, \tag{3.65b}$$

as well as the inverse

$$\begin{aligned}\widehat{\varphi}^A(g^{00}, g^{0\alpha}, g^{\alpha\beta}, h^{00}, h^{0\alpha}, h^{\alpha\beta}) = \Big(&g^{00}, g^{01}, g^{02}, g^{03}, g^{11}, g^{12}, g^{13}, g^{22}, g^{23}, g^{33}, \\ &h^{11}, h^{12}, h^{13}, h^{22}, h^{23}, h^{33}\Big)^A\,.\end{aligned} \tag{3.66}$$

Symplectic structure and canonically conjugate momenta

Having defined the configuration variables that we are going to use for our canonical description of gravitational dynamics, we can associate canonically conjugate momenta $\pi_A(x)$ to these degrees of freedom. The phase space is, as usual, equipped with a field-theoretic Poisson bracket defined by

$$\{F, G\} := \int_\Sigma \mathrm{d}^3x \left(\frac{\delta F}{\delta\varphi^A(x)}\frac{\delta G}{\delta\pi_A(x)} - \frac{\delta F}{\delta\pi_A(x)}\frac{\delta G}{\delta\varphi^A(x)}\right), \tag{3.67}$$

for arbitrary functionals $F[\varphi, \pi]$ and $G[\varphi, \pi]$ on phase space. This has an ambiguity in the definition of the momentum: given that π_A is a valid choice, then so is $\pi_A + \Lambda_A[\varphi]$ for any closed covector field $\Lambda_A \delta\varphi^A$, i.e. for Λ_A being a solution of

$$\frac{\delta\Lambda_A(x)}{\delta\varphi^B(y)} - \frac{\delta\Lambda_B(y)}{\delta\varphi^A(x)} = 0\,. \tag{3.68}$$

In order to make this definition well-defined we need to make sure that the constructions are well-behaved under coordinate changes on the screen manifold Σ. For this, we first observe that a coordinate change $\partial\tilde{z}/\partial z$ induces the following action on the degrees of freedom

$$\rho^A\left(\frac{\partial\tilde{z}}{\partial z}, \varphi\right) := \widehat{\varphi}^A\left(\mathcal{R}^{\mathcal{A}}{}_{\mathcal{B}}\left(\frac{\partial\tilde{z}}{\partial z}\right)\,\widehat{g}^{\mathcal{B}}(\varphi)\right), \tag{3.69}$$

with $\mathcal{R}^{\mathcal{A}}{}_{\mathcal{B}}(\partial\tilde{z}/\partial z)$ being the standard tensorial action of the $\mathrm{GL}(3)$-transformation on the various tensors we projected to Σ, which we collectively labeled by the multi-index $\mathcal{B}$. This allows us to read off the $\mathrm{GL}(3)$ group action on the canonical conjugate momenta π_A, being sections over the associated Π-fiber bundle over the manifold Σ. In order for the Poisson bracket above to be well-defined, we need to impose the group action $\rho^* : \mathrm{GL}(3) \times \Pi \longrightarrow \Pi$ in the following fashion

$$\rho^*_A\left(\frac{\partial\tilde{z}}{\partial z}, \pi\right) := \det\left(\frac{\partial\tilde{z}}{\partial z}\right)\cdot\frac{\partial\widehat{\varphi}^B}{\partial g^{\mathcal{A}}}\left(\widehat{g}^{\alpha}(\varphi)\right)\cdot\left(\mathcal{R}^{-1}\right)^{\mathcal{A}}{}_{\mathcal{B}}\left(\frac{\partial\tilde{z}}{\partial z}\right)\cdot\frac{\partial\widehat{g}^{\mathcal{B}}}{\partial\varphi^A}(\varphi)\cdot\pi_B\,. \tag{3.70}$$

It is simple to check that this makes the Poisson bracket, as stated in equation (3.67), well-defined: the functional derivative $\delta F/\delta\varphi^A(z)$ has density weight one – since $\varphi^A(z)$ has density weight zero – while the fact that π_A already has density weight one cancels the density weight from the functional differentiation in $\delta G/\delta\pi(z)$. This renders the latter of weight zero. Thus the integrand of the Poisson bracket can be shown to be a scalar density of weight one and as a result, the integral to be indeed well-defined.

As we have already seen in the example for the parametrization of the Lorentzian metric, there is ambiguity in the definition of the configuration variables. In the end, the results must not depend on a particular parametrization. To make this more precise, we see that for a given parametrization $(\widehat{g}, \widehat{\varphi})$ and a diffeomorphism f on Φ we can lift this into a new parametrization $(\widetilde{g}, \widetilde{\psi})$ by setting

$$\widetilde{g}^{\mathcal{A}}(\psi) := \left(\widehat{g}^{\mathcal{A}} \circ f^{-1}\right)(\psi)\,, \tag{3.71a}$$

$$\widetilde{\psi}^A(g^{\mathcal{A}}) := \left(f \circ \widehat{\varphi}^A\right)(g^{\mathcal{A}})\,. \tag{3.71b}$$

By calculating the derivative, we find that the intertwiners transform as

$$\frac{\partial\widetilde{g}^{\mathcal{A}}}{\partial\psi^A} = \frac{\partial\widehat{g}^{\mathcal{A}}}{\partial\varphi^B}\frac{\partial\varphi^B}{\partial\psi^A} \qquad \text{and} \qquad \frac{\partial\widetilde{\psi}^A}{\partial g^{\mathcal{A}}} = \frac{\partial\psi^A}{\partial\varphi^B}\frac{\partial\widehat{\varphi}^B}{\partial g^{\mathcal{A}}}\,. \tag{3.72}$$

We furthermore see that such a reparametrization constitutes a canonical transformation on our phase space given that the canonical conjugate momenta transform as

$$\pi_A \longrightarrow \frac{\partial\varphi^B}{\partial\psi^A}\pi_B\,. \tag{3.73}$$

This makes the symplectic structure of our phase space invariant under reparametrizations, and, as a result, we can adequately formulate the algebra relations and compatibility conditions in the following. This guarantees equivalent results for all parametrizations related by a diffeomorphism $\psi(\varphi)$.

Algebra relations

Having established a phase space for our configuration variables, we introduce the two phase space functionals

$$\widehat{\mathcal{H}}(N) = \int_\Sigma \mathrm{d}^3x\, \widehat{\mathcal{H}}(x) N(x)\,,$$
$$\widehat{\mathcal{D}}(\vec{N}) = \int_\Sigma \mathrm{d}^3x\, \widehat{\mathcal{D}}_\alpha(x) N^\alpha(x)\,,$$

that shall mimic the hypersurface deformation operators we employed in the hypersurface picture. We will refer to $\widehat{\mathcal{H}}$ as *superhamiltonian* and $\widehat{\mathcal{D}}$ as *supermomentum*. In order for the functionals to be a proper representation of the deformation operators, they must fulfill the hypersurface deformation algebra as a Poisson algebra, i.e. we must have that

$$\left\{\widehat{\mathcal{H}}(N), \widehat{\mathcal{H}}(M)\right\} = \widehat{\mathcal{D}}\left((\deg P - 1)\mathrm{p}^{\alpha\beta}(M\partial_\alpha N - N\partial_\alpha M)\partial_\beta\right)\,, \tag{3.74a}$$
$$\left\{\widehat{\mathcal{D}}(\vec{N}), \widehat{\mathcal{H}}(M)\right\} = \widehat{\mathcal{H}}\left(\mathcal{L}_{\vec{N}} M\right)\,, \tag{3.74b}$$
$$\left\{\widehat{\mathcal{D}}(\vec{N}), \widehat{\mathcal{D}}(\vec{M})\right\} = \widehat{\mathcal{D}}\left(\mathcal{L}_{\vec{N}} \vec{M}\right)\,. \tag{3.74c}$$

However, to accurately and consistently represent the algebra on our phase space, we need to consider a subtlety that arises due to the fact that this algebra has structure *functions* instead of constants since $\mathrm{p}^{\alpha\beta}$ appears in the argument of the supermomentum in equation (3.74a). This makes the algebra structure if implemented naïvely, dependent on the considered phase space point. We will deal with this subtlety in the following section.

Totally constrained action for the gravitational degrees of freedom

In order to accurately represent the hypersurface deformation algebra on our phase space, at least *on-shell*, it is necessary for the supermomentum and superhamiltonian to be constraints that vanish on solutions of the equations of motion. We can see this by the following argument (Teitelboim, 1973):

We can write the action of the supermomentum and superhamiltonian functionals as covectors on phase space in the following fashion

$$\alpha_{\widehat{\mathcal{H}}}(N) := \int_\Sigma \mathrm{d}^3x\, \left\{-, \widehat{\mathcal{H}}(x)\right\} N(x)\,, \tag{3.75a}$$
$$\alpha_{\widehat{\mathcal{D}}}(\vec{N}) := \int_\Sigma \mathrm{d}^3x\, \left\{-, \widehat{\mathcal{D}_\alpha}(x)\right\} N^\alpha(x)\,, \tag{3.75b}$$

with the yet-to-be-determined localized functional objects $\widehat{\mathcal{H}}(x) = \widehat{\mathcal{H}}(\delta_x)$ and $\widehat{\mathcal{D}}_\mu = \widehat{\mathcal{D}}(\delta_x\partial_\mu)$ and $\widehat{\mathcal{H}}(N)$ and $\widehat{\mathcal{D}}(\vec{N})$ are our phase space functional versions of the deformation operators. Lapse and shift appear as external objects that need to be provided and parametrize the dynamics of the gravitational degrees of freedom.

Using the Jacobi identity, one finds for a Hamiltonian vector field X_F associated to the phase space functional F

$$\left\{F, \left\{\widehat{\mathcal{H}}(N), \widehat{\mathcal{H}}(M)\right\}\right\} = \left[\alpha_{\widehat{\mathcal{H}}}(N)\,,\, \alpha_{\widehat{\mathcal{H}}}(M)\right](X_F)\,. \tag{3.76}$$

Since we required the functionals to fulfill the algebra relations – up to a minus sign due to our choice in $\alpha_{\widehat{\mathcal{H}}}$ and $\alpha_{\widehat{\mathcal{D}}}$ to let the superhamiltonian and supermomentum functionals act from the right in the Poisson bracket – this means that the equation reads

$$\begin{aligned}\left[\alpha_{\widehat{\mathcal{H}}}(N)\,,\,\alpha_{\widehat{\mathcal{H}}}(M)\right](X_F) &= \left\{F, \widehat{\mathcal{D}}\left((\deg P-1)\ \mathrm{p}^{\alpha\beta}(\varphi)\,(M\partial_\alpha N - N\partial_\alpha M)\right)\right\}\\ &= \alpha_{\widehat{\mathcal{D}}}\left((\deg P-1)\ \mathrm{p}^{\alpha\beta}(\varphi)\,(M\partial_\alpha N - N\partial_\alpha M)\right)(X_F)\\ &\quad + \widehat{\mathcal{D}}\left(\left\{F, (\deg P-1)\ \mathrm{p}^{\alpha\beta}(\varphi)\,(M\partial_\alpha N - N\partial_\alpha M)\right\}\right)\,, \end{aligned} \tag{3.77}$$

where the last term appears since the coefficient $\mathrm{p}^{\alpha\beta}$, as a function of the gravitational degrees of freedom, is dependent on the considered phase space point. This prevents our functionals to be homomorphisms from the hypersurface deformation operators to the derivations of phase space functionals *unless* the functional $\widehat{\mathcal{D}}$ vanishes on solutions of the equations of motion. In other words, we need to satisfy the constraint

$$\widehat{\mathcal{D}}_\mu\left[\varphi(x), \pi(x)\right] = 0\,. \tag{3.78}$$

But for this to be stable under time evolution, we furthermore find from the hypersurface deformation algebra that we have an additional constraint for $\widehat{\mathcal{H}}$

$$\widehat{\mathcal{H}}\left[\varphi(x), \pi(x)\right] = 0\,. \tag{3.79}$$

If the constraints were not implemented on the solutions of the equations of motion, the last term in (3.77) would prevent us from interpreting the supermomentum and superhamiltonian functionals as generators of *spacetime* evolution via the chosen lapse and shift functions. Note that the constraint nature of the functionals also makes sure that the *principle of path independence* is realised (compare Hojman et al. (1976)): the evolution of observables with the help of the equations of motion from an initial time t_i, as they would appear in the hypersurface picture on the hypersurface $X_{t_i}(\Sigma)$, to the values at final time t_f (that is on $X_{t_f}(\Sigma)$) will be independent of the intermediate leaves of the foliation.

As a result, just as is the case for general relativity, the Hamilton-Jacobi equations in our canonical description need to be supplied by four constraint equations, i.e.

$$\dot{\varphi}^A(t,x) = \int_\Sigma \mathrm{d}^3y\,\left(\left\{\varphi^A(t,x), \widehat{\mathcal{H}}(t,y)\right\}\,N(t,y) + \left\{\varphi^A(x), \widehat{\mathcal{D}_\alpha}(t,y)\right\}\,N^\alpha(t,y)\right)\,, \tag{3.80a}$$

$$\dot{\pi}_A(t,x) = \int_\Sigma \mathrm{d}^3y\,\left(\left\{\pi_A(t,x), \widehat{\mathcal{H}}(t,y)\right\}\,N(t,y) + \left\{\pi_A(t,x), \widehat{\mathcal{D}_\alpha}(t,y)\right\}\,N^\alpha(t,y)\right)\,, \tag{3.80b}$$

$$0 = \widehat{\mathcal{D}}_\mu\left[\varphi(t,x), \pi(t,x)\right]\,, \tag{3.80c}$$

$$0 = \widehat{\mathcal{H}}\left[\varphi(t,x), \pi(t,x)\right]\,. \tag{3.80d}$$

This set of equations of motion can always be obtained by variation of the totally constrained action functional

$$\begin{aligned}\mathcal{S}_{\mathrm{grav}}\left[\varphi, \pi, N, \vec{N}\right] = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x\,\Big(&\dot{\varphi}^A(t,x)\pi_A(t,x)\\ &- N\,\widehat{\mathcal{H}}\left[\varphi(x), \pi(x)\right]\\ &- N^\mu\,\widehat{\mathcal{D}}_\mu\left[\varphi(x), \pi(x)\right]\Big)\,,\end{aligned} \tag{3.81}$$

where lapse and shift appear as Lagrange multipliers.

At first sight, it is rather difficult to see how these complicated constructions of our phase space and the functionals $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{D}}$ represent spacetime diffeomorphisms. Historically, already for general relativity, this has been up to some heated debate. It turns out, however, that one can establish a correspondence between path independence and diffeomorphism invariance by the constructions presented in Isham and Kuchar (1985a,b) for parametrized theories, where one extends the phase space to include the coordinate fields X^a of the foliation map and associated canonically conjugate momenta and adds these new variables to the Hamiltonian such that lapse and shift are correctly related on-shell to the frame expressions we presented in the construction of our orthonormal frame. In this setup, it is indeed possible to find that we obtain a true representation of infinitesimal space diffeomorphisms.

Having established how the action of the geometric degrees of freedom arises from the phase space functionals, we can finally start by implementing conditions that fix the information we obtained about dynamics in the hypersurface deformation picture. Note that, although we already required that the algebra relations have to be fulfilled, we can extract further information – in fact, it will turn out to be one of the most crucial points of the whole construction – by turning them into a system of partial differential equations.

3.2.2 Towards the hypersurface deformation functionals

In order to obtain the complete set of Hamilton-Jacobi equations, we need the functional form of the supermomentum and superhamiltonian. A large portion of these can be obtained by enforcing compatibility of the first equation of motion (as presented in (3.80a)) with the action of the hypersurface deformation operators on the projected fields to the screen manifold we described in sections 3.1.3 and 3.1.3.

This compatibility requirement is formulated in the sense that

$$\mathcal{H}(N)g^{\mathcal{A}}[X_t] \simeq \left\{\widehat{g}^{\mathcal{A}}(\varphi), \widehat{\mathcal{H}}(N)\right\}, \tag{3.82a}$$

$$\mathcal{D}(\vec{N})g^{\mathcal{A}}[X_t] \simeq \left\{\widehat{g}^{\mathcal{A}}(\varphi), \widehat{\mathcal{D}}(\vec{N})\right\}. \tag{3.82b}$$

At this point, the two equations have to be understood in a figurative sense[3], as the objects on the left-hand side are objects living on entirely different spaces than the objects on the right-hand side of the equations. We can make this more precise by decomposing the velocity of the degrees of freedom into the different parts that mimic the various contributions we found for the actions of the hypersurface deformation operators:

$$\begin{aligned}\dot{\varphi}^A &= N\,k^A(\varphi,\pi) + (\partial_\gamma N)\mathrm{M}^{A\gamma}(\varphi) + \frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}(\widehat{g}(\varphi))\left(\mathcal{L}_{\vec{N}}\widehat{g}\right)^{\mathcal{A}}(\varphi) \\ &= N\,k^A(\varphi,\pi) + (\partial_\gamma N)\mathrm{M}^{A\gamma}(\varphi) + N^\mu\varphi^A{}_{,\mu} - (\partial_\gamma N^\mu)\mathrm{F}^A{}_\mu{}^\gamma(\varphi)\,.\end{aligned} \tag{3.83}$$

The only *dynamic* contribution on the right hand side is given by the – so far undetermined – velocities k^A. The kinematical coefficients $\mathrm{M}^{A\gamma}$ and $\mathrm{F}^A{}_\mu{}^\gamma$ are the equivalents of the coefficients we obtained by

[3]This compatibility can be seen as the point of contact between the *human view* and the *divine view*, just as depicted in the famous fresco *Creation of Adam* (Italian: *Creazione di Adamo*) by Michelangelo in the Sistine chapel (compare figure 3.4). For that reason, the two compatibility equations are also sometimes referred to as the *Michelangelo equations*.

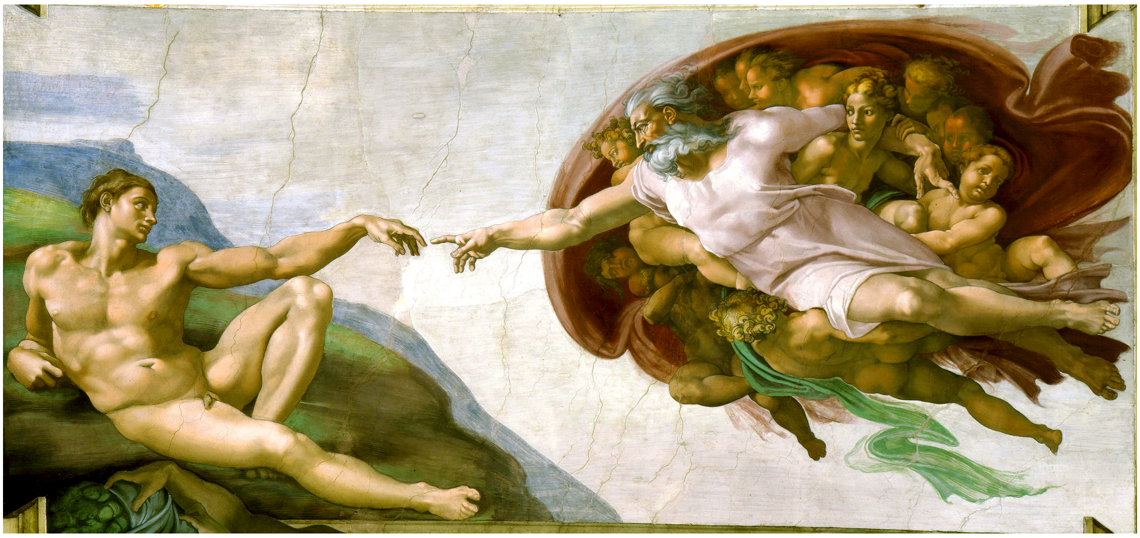


Figure 3.4 The Creation of Adam (Italian: Creazione di Adamo) by Michelangelo.

application of the hypersurface deformation operators on the screen manifold fields, i.e.

$$\mathrm{M}^{A\gamma}(\varphi) = \frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}(\widehat{g}(\varphi))\mathbf{M}^{\mathcal{A}\gamma}(\widehat{g}(\varphi))\,, \tag{3.84a}$$

$$\mathrm{F}^{A}{}_{\mu}{}^{\gamma}(\varphi) = \frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}(\widehat{g}(\varphi))\mathbf{F}^{\mathcal{A}}{}_{\mu}{}^{\gamma}(\widehat{g}(\varphi))\,, \tag{3.84b}$$

where we calculate the coefficients first in the deformation picture, replace the projected fields by the objects constructed with our parametrization and use the inverse intertwiner to pull the expression to $\mathrm{T}\Phi$. These coefficients and the screen manifold projection $p^{\alpha\beta}$ contain all the input of the theory under consideration and will be, in the following, referred to as *input coefficients*.

Note that the Lie derivative term in (3.83) can also be understood as the lifted action of the Lie derivative on the screen manifold to $\mathrm{T}\Phi$. By using that $\mathrm{id}_\Phi = \widehat{\varphi}\circ\widehat{g}$ we get that

$$\left(\mathcal{L}_{\vec{N}}\varphi\right)^A = \frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}\left(\widehat{g}(\varphi)\right)\left(\mathcal{L}_{\vec{N}}\widehat{g}(\varphi)\right)^{\mathcal{A}}. \tag{3.85}$$

We can now use the two compatibility conditions (3.82a) and (3.82b) to obtain (at least parts of) the functional form of the supermomentum $\widehat{\mathcal{D}}$ and the superhamiltonian $\widehat{\mathcal{H}}$. It turns out that this, in fact, already completely fixes the former and the non-local part of the latter. The remaining, local, part can then be further restricted with the help of the algebra relations (3.74a) and (3.74b).

Supermomentum regained

Making the compatibility condition (3.82b) precise, we require that the supermomentum acts on any functionals of the geometric degrees of freedom via Lie derivative on the screen manifold, i.e.

$$\left\{F(\varphi),\widehat{\mathcal{D}}(\vec{N})\right\} = \left(\mathcal{L}_{\vec{N}}F\right)(\varphi)\,, \tag{3.86}$$

with the Lie derivative understood as in equation (3.85), where we use the parametrization to lift the Lie derivative to screen manifold tensors and then projecting back with the help of the inverse intertwiner. All information involved is kinematical.

By specializing to $F(\varphi) = \varphi$, we obtain a functional differential equation for the supermomentum $\mathcal{D}(\vec{N})$

$$\frac{\delta \widehat{\mathcal{D}}(\vec{N})}{\delta \pi_A(x)} = \frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{A}}} \left(\widehat{g}(\varphi)\right) \left(\mathcal{L}_{\vec{N}} \widehat{g}(\varphi)\right)^{\mathcal{A}} . \tag{3.87}$$

Since the right hand side is independent of the canonically conjugate momenta π_A we can directly integrate the equation. This fixes the supermomentum up to a functional $F[\varphi]$ obtained as constant of integration. However, this functional can be eliminated with the help of the algebra relation (3.74c). The resulting expression for the supermomentum reads

$$\widehat{\mathcal{D}}(\vec{N}) = \int_\Sigma \mathrm{d}^3x \, \pi_A(x) \left(\frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{A}}} \left(\widehat{g}(\varphi)\right) \left(\mathcal{L}_{\vec{N}} \widehat{g}(\varphi)\right)^{\mathcal{A}} \right)(x) . \tag{3.88}$$

This term then indeed fulfils the algebra relation (3.74c), as can be easily checked by direct calculation.

Non-local superhamiltonian regained

We can follow the same approach to evaluate the compatibility equation (3.82a). Spelling out the separate contributions explicitly we find that the action of the superhamiltonian on the degrees of freedom φ^A reads

$$\left\{\varphi^A(x), \widehat{\mathcal{H}}(N)\right\} = \underbrace{N(x)\, k^A(\varphi(x), \pi(x))}_{\text{local}} + \underbrace{\left(\partial_\gamma N\right)(x)\, \mathrm{M}^{A\gamma}\left(\varphi(x)\right)}_{\text{non-local}} . \tag{3.89}$$

This equation can again be turned into a functional differential equation that fixes (parts of) the superhamiltonian. The first step is to split the superhamiltonian into two parts that are responsible for the local and non-local part, respectively:

$$\widehat{\mathcal{H}}(N) = \int_\Sigma \mathrm{d}^3x \, N(x) \left(\widehat{\mathcal{H}}_{\text{local}}\left[\varphi(x); \pi(x)\right) + \widehat{\mathcal{H}}_{\text{non-local}}\left[\varphi(x), \pi(x)\right] \right) . \tag{3.90}$$

This split is well-defined since the local part is defined to be ultralocal with respect to π, while all remaining functional dependencies on spatial derivatives of the momenta are then contained in the non-local part. Using this split, we see that the last element in the compatibility equation (3.89) is generated entirely by the non-local superhamiltonian, i.e.

$$\frac{\delta \widehat{\mathcal{H}}_{\text{non-local}}(N)}{\delta \pi_A(x)} = \left(\partial_\gamma N\right)(x)\, \mathrm{M}^{A\gamma}(\varphi(x)) . \tag{3.91}$$

Since the right hand side is again independent of the canonical conjugate momenta, we can integrate the functional differential equation and obtain an expression for the non-local superhamiltonian[4]

$$\widehat{\mathcal{H}}_{\text{non-local}}\left[\varphi(x), \pi(x)\right] = -\partial_\gamma \left(\pi_A \, \mathrm{M}^{A\gamma}\right)(x) . \tag{3.92}$$

[4]The constant of integration can be absorbed into the local superhamiltonian and will be further restricted in the upcoming sections.

The remaining part of the the compatibility condition (3.89) gives as an expression for the velocity functionals k^A in terms of the local part of the superhamiltonian

$$k^A[\varphi;\pi) = \frac{\partial \widehat{\mathcal{H}}_{\text{local}}[\varphi;\pi)}{\partial \pi_A}\,. \tag{3.93}$$

At this point, we have used all the information we can extract from the two compatibility conditions between the hypersurface deformation picture and our phase space formulation. The remaining information that we can use to further determine the possible expressions for the superhamiltonian is the two algebra relations (3.74a) and (3.74b). We will see in the following that this, indeed, gives further restrictions on the functional $\widehat{\mathcal{H}}_{\text{local}}$.

One may be tempted from (3.93) to expand the local superhamiltonian in terms of the momenta and enter the two algebra relations. However, this proves to be not particularly promising as the superhamiltonian enters the functional differential equations quadratically. Instead, it proves useful to go the same road as presented in Kuchar (1974) and convert the problem into a linear problem by taking the Legendre transform of the local superhamiltonian via

$$\mathcal{L}[\varphi;k) := \pi_A[\varphi;k)\,k^A \;-\; \widehat{\mathcal{H}}_{\text{local}}\Big[\varphi;\pi[\varphi;k)\Big)\,. \tag{3.94}$$

The functional dependency on φ is inherited from the local superhamiltonian, i.e.

$$\frac{\delta \widehat{\mathcal{H}}_{\text{local}}\Big[\varphi(x);\pi[\varphi(x);k(x))\Big)}{\delta\varphi^A(y)} = -\frac{\delta\mathcal{L}\Big[\varphi(x);k(x)\Big)}{\delta\varphi^A(y)}\,. \tag{3.95}$$

Since k and π are Legendre duals one also finds that the momenta, in terms of the velocities, are given by the derivative of the Lagrangian function as

$$\frac{\partial\mathcal{L}}{\partial k^A} = \pi_A[\varphi;k)\,. \tag{3.96}$$

Using this we can enter the two algebra relations (3.74a) and (3.74b) to further determine the functional $\mathcal{L}$. The resulting (functional) differential equations are the *gravitational closure equations*, with the former equation being referred to as the *selective part* and the latter as the *covariance part* of the closure equations[5]. Before we (finally) move to the algebra relations, we will show that the Lagrangian $\mathcal{L}$ allows us to write down a spacetime action functional for the geometric degrees of freedom.

3.2.3 Lagrangian spacetime action

Remember that the action of the degrees of freedom has to be totally constrained to implement the principle of path independence. We can now use the functional expressions we obtained for the supermomentum and the non-local part of the superhamiltonian to explicitely write down the contributions in the

[5]This is mainly for historical reasons. The $\left\{\widehat{\mathcal{D}},\widehat{\mathcal{H}}\right\}$ bracket encodes that the Lagrangian function $\mathcal{L}$ properly transforms as a scalar density of weight 1, which guarantees the covariance of the Hamiltonian. The $\left\{\widehat{\mathcal{H}},\widehat{\mathcal{H}}\right\}$ bracket, on the other hand, can be seen as the biggest physical input of the gravitational closure program that determines the Lagrangian since it relates the local geometry that is seen to the spatial metric on the screen manifold obtained from the principal polynomial P / dual polynomial $P^\sharp$.

action:

$$\begin{aligned}\mathcal{S}_{\text{grav}}[\varphi,\pi,N,\vec{N}] &= \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left(\dot{\varphi}^A(t,x)\pi_A(t,x) - N(t,x)\widehat{\mathcal{H}}_{\text{local}}\left[\varphi(t,x);\pi(t,x)\right)\right. \\ &\quad + N(t,x)\,\partial_\gamma\left(\pi_A(t,\cdot)\mathrm{M}^{A\gamma}(\varphi(t,\cdot))\right)(x) \\ &\quad \left. -\pi_A(t,x)\left(\mathcal{L}_{\vec{N}}\widehat{\varphi}\right)^A(\varphi(t,x))\right) \qquad (3.97)\end{aligned}$$

If we then use the expression for $\dot{\varphi}$ from equation (3.83) we see that almost all but the terms containing k and the local part of the superhamiltonian drop out, i.e. we obtain

$$\mathcal{S}_{\text{grav}}[\varphi,\pi,N,\vec{N}] = \int \mathrm{d}t \int \mathrm{d}^3x\, N(t,x)\left(\pi_A(t,x)\,k^A(t,x) - \widehat{\mathcal{H}}_{\text{local}}\left[\varphi(t,x);k[\varphi(t,x);\pi(t,x)\right)\right)\,.$$

Moreover, we see that the term in brackets is the same as the local Lagrangian we defined in the previous section. This means, that the Lagrangian spacetime action that describes the evolution of the geometric degrees of freedom is given by

$$\begin{aligned}\mathcal{S}_{\text{grav}}[\varphi,N,\vec{N}] &= \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x\, N(t,x)\,\mathcal{L}\left[\varphi^A(t,x);\frac{1}{N(t,x)}\left(\dot{\varphi}^A(t,x)\right.\right. \\ &\quad -(\partial_\gamma N)(t,x)\,\mathrm{M}^{A\gamma}\left(\varphi(t,x)\right) \\ &\quad \left.\left. -\left(\mathcal{L}_{\vec{N}}\widehat{\varphi}\right)^A\left(\varphi(t,x)\right)\right)\right.\,. \qquad (3.98)\end{aligned}$$

One can then show by variation of the action that we obtain the same set of equations of motion for both the Lagrangian action (3.98) and the totally constrained spacetime action functional (3.81). Note that this, still, is a canonical formulation with t being an external parameter that labels the evolution of the degrees of freedom on the screen manifold Σ. Diffeomorphism invariance is implemented once the Lagrangian $\mathcal{L}$ is determined such that all hypersurface deformation algebra relations are fulfilled.

3.2.4 Time-reversibility

Before we continue with the derivation of the gravitational closure equations, we will take a close look at how the gravitational theories in our framework transform under time reversal and show the following:

THEOREM TIME-REVERSABLE GRAVITATIONAL THEORY

The gravitationally closed theory is *time reversable* if, and only if

- the local-superhamiltonian is even in the momenta π_A.
- the $\mathrm{M}^{A\gamma}$ is vanishing.

To make this precise, we introduce time reversal as the operation defined by

$$T : t \longrightarrow -t\,. \qquad (3.99)$$

In the hypersurface picture, this operation acts on our foliation, i.e. $\overline{X}_{\bar{t}} := X_{T(t)} = X_{-t}$. It is easy to see that the tangents to the curves along t flip the directions, i.e.

$$\begin{aligned}\dot{\overline{X}}_{\bar{t}}{}^{a} &= -Ne_0^a - N^\alpha e_\alpha^a \\ &=: \overline{N}\bar{e}_0^a + \overline{N}^\alpha \bar{e}_\alpha^a\,. \end{aligned} \tag{3.100}$$

It should be clear that such a split is ill-defined without further specifying what we mean by a time reversal. We will make this more precise now. First, we observe that at $t = 0$ the hypersurfaces described by our two foliations X_t and $\overline{X}_{\bar{t}}$ match. By the definition of the tangential basis vectors, we thus find that $\bar{e}_\alpha = e_\alpha$. As a result, the shift vector flips its sign under time reversal.

In order to make the definition well-defined for the normal component, we remember that the principal polynomial is defined in such a fashion that $P(\epsilon^0) > 0$ means that ϵ^0 lies in the future-directed cone. We now make the natural choice that the future-directed and past-directed cones are swapped under the time-reversal operation, i.e. the principal polynomial obtains a minus sign under a time reversal. But then, we find that $\bar{e}_0^a = -e_0^a$ and as a result, the lapse needs to remain invariant under the time-reversal. In summary, under time-reversal lapse and shift transform via

$$\overline{N} = N \qquad \text{and} \qquad \overline{N}^\alpha = -N^\alpha\,. \tag{3.101}$$

Using this, we can move back to our canonical description and analyze how time-reversal acts on the phase space variables. Since we have for the velocities of the degrees of freedom that

$$\begin{aligned}\dot{\overline{\varphi}}^A &= -\dot{\varphi}^A\,, \\ \overline{N}\overline{k}^A + (\partial_\gamma \overline{N})\overline{\mathrm{M}}^{A\gamma} + (\mathcal{L}_{\vec{\overline{N}}}\overline{\varphi})^A &= -Nk^A - (\partial_\gamma N)\mathrm{M}^{A\gamma} - (\mathcal{L}_{\vec{N}}\varphi)^A\,,\end{aligned} \tag{3.102}$$

we can read off that our phase space variables must transform in the following fashion

$$\overline{\varphi}^A = \varphi^A \qquad , \qquad \overline{\pi}_A = -\pi_A\,. \tag{3.103}$$

If we want a time reversible system, this means that our transformed phase space variables are solutions of the equations of motion with time evolution in $\bar{t}$, i.e. for a the solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^A(x) = \left\{\varphi^A(x), \mathcal{H}(N) + \mathcal{D}(\vec{N})\right\}_{\varphi,\pi}\,, \tag{3.104a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_A(x) = \left\{\pi_A(x), \mathcal{H}(N) + \mathcal{D}(\vec{N})\right\}_{\varphi,\pi}\,, \tag{3.104b}$$

the time-reversed initial data are a solution of the time reversed Hamilton equations

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\overline{\varphi}^A(x) = \left\{\overline{\varphi}^A(x), \overline{\mathcal{H}}(N) + \overline{\mathcal{D}}(-\vec{N})\right\}_{\overline{\varphi},\overline{\pi}}\,, \tag{3.105a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\overline{\pi}_A(x) = \left\{\overline{\pi}_A(x), \overline{\mathcal{H}}(N) + \overline{\mathcal{D}}(-\vec{N})\right\}_{\overline{\varphi},\overline{\pi}}\,, \tag{3.105b}$$

with the time-reversal of the superhamiltonian and supermomentum defined appropriately. We call our gravitational theory *reversible* if the superhamiltonian and supermomentum match functionally, i.e. if

$$\overline{\mathcal{H}}(N)[\overline{\varphi},\overline{\pi}] = \mathcal{H}(N)[\varphi,-\pi]\,, \tag{3.106a}$$

$$\overline{\mathcal{D}}(-\vec{N})[\overline{\varphi},\overline{\pi}] = \mathcal{D}(\vec{N})[\varphi,-\pi]\,. \tag{3.106b}$$

Spelling out the Poisson-brackets above, and inserting the transformation of the phase space variables, it is easy to verify that

$$\frac{\delta\overline{\mathcal{H}}(N)}{\delta\varphi^A(x)} = \frac{\delta\mathcal{H}(N)}{\delta\varphi^A(x)} \quad , \quad \frac{\delta\overline{\mathcal{H}}(N)}{\delta\pi_A(x)} = \frac{\delta\mathcal{H}(N)}{\delta\pi_A(x)}, \tag{3.107a}$$

$$\frac{\delta\overline{\mathcal{D}}(\vec{N})}{\delta\varphi^A(x)} = -\frac{\delta\mathcal{D}(\vec{N})}{\delta\varphi^A(x)} \quad , \quad \frac{\delta\overline{\mathcal{D}}(\vec{N})}{\delta\pi_A(x)} = -\frac{\delta\mathcal{D}(\vec{N})}{\delta\pi_A(x)}. \tag{3.107b}$$

Integrating the functional derivatives by π again, we immediately find the non-local part of the reversed superhamiltonian and the reversed supermomentum, i.e.

$$\overline{\mathcal{H}}_{\text{non-local}}(N)[\overline{\varphi},\overline{\pi}] = -\int_\Sigma \mathrm{d}^3x\, N(x)\partial_\gamma \left(\overline{\pi}_A \mathrm{M}^{A\gamma}(\overline{\varphi})\right)(x)\,, \tag{3.108a}$$

$$\overline{\mathcal{D}}(\vec{N})[\overline{\varphi},\overline{\pi}] = \int_\Sigma \mathrm{d}^3x\, \overline{\pi}_A(x) \left(\frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}\left(\widehat{g}(\overline{\varphi})\right)\left(\mathcal{L}_{\vec{N}}\widehat{g}(\overline{\varphi})\right)^{\mathcal{A}}\right)(x)\,. \tag{3.108b}$$

While the reversed supermomentum fulfills the condition (3.106b), for the non-local superhamiltonian we obtain

$$\overline{\mathcal{H}}_{\text{non-local}}(N)[\overline{\varphi},\overline{\pi}] = -\mathcal{H}_{\text{non-local}}(N)[\varphi,-\pi]\,. \tag{3.109}$$

The reason for this is obvious: Both the supermomentum and the non-local superhamiltonian are linear in the momentum. As a result, they obtain a minus sign under time reversal. In the case of the supermomentum, this is "compensated" because the shift vector also flips its sign under time reversal. As the lapse is invariant under the transformation, the only way the gravitational theory has a chance of being time-reversible is if the non-local superhamiltonian vanishes. This is the case only if the $\mathrm{M}^{A\gamma}$ vanishes.

Similarly, we see that if we expand the local part of the superhamiltonian in the momenta, all odd coefficients must vanish in order to fulfil the condition (3.106a). Note that this does not tell us that time-reversal is not a symmetry of the equations; it tells us that the functional form of the Hamiltonian (and equivalently the Lagrangian) is dependent on the chosen time direction. As long as the functionals change such that (3.107) are fulfilled, solutions get mapped into solutions of the time-reversed theory. □

By extension, this also tells us that for any theory with non-vanishing $\mathrm{M}^{A\gamma}$ coefficient, even without having an expression for the local superhamiltonian, we already know that the gravitationally closed theory cannot be time-reversible. A typical example for reversible theory is, of course, Einstein's general relativity, for which both conditions are indeed met.

However, for our example of the gravitational closure of general linear electrodynamics, we see that the $\mathrm{M}^{A\gamma}$ of the area metric degrees of freedom is non-vanishing. This particularly means that it is impossible to solve the local superhamiltonian only for the even coefficients and restrict ourselves on time-reversible solutions.

3.3 THE GRAVITATIONAL CLOSURE EQUATIONS

In this section, we will finally obtain two functional differential equations for the gravitational Lagrangian from the hypersurface deformation algebra relations. This is a remarkable result, as it tells us that the requirement of diffeomorphism invariance, combined with the matter conditions from the kinematical considerations in chapter 2, translates the search for gravitational Lagragians into a purely mathematical question: solving a differential equation produces the gravitational Lagrangian, or at least a family of Lagrangians.

Practically, solving functional differential equations is, however, a rather complicated endeavour. Already in the case of general relativity, this is highly non-trivial due to the non-linearity introduced by the density factor $\sqrt{-\det g_{\cdot\cdot}}$, as well as the fact that both the metric and its inverse appear in the solution. As a result, it is rather the expectation than the exception that the solution of the functional differential equations will be highly non-linear and cannot be easily derived on the level of functional differential equations.

However, the problem can be simplified by performing two additional steps: the first one employs the fact that the gravitational Lagrangian is an ultra-local function in the velocities k. Therefore, we assume that the Lagrangian can be expanded into a polynomial of the velocities and write the Lagrangian in terms of several coefficients. In the second step, we turn the functional derivatives with respect to the degrees of freedom into partial derivatives of the $\varphi^A{}_{,\mu_1 \ldots \mu_N}$. The price we pay is that the two functional differential equations translate into a system of countably infinite partial differential equations that must be solved. While this initially does not sound like a real improvement over the two compact, functional differential equations, much stronger statements about the systems of partial differential equations can be made. For instance, in section 3.5.1 we can apply the Cartan-Kuranishi algorithm, as presented in 2.2.2, to the subsystem posed by the $\left\{\widehat{\mathcal{D}}, \widehat{\mathcal{H}}\right\}$ relation to analyze the solution space of the gravitational closure equations and derive the number of functionally independent curvature invariants from combinatorial considerations.

3.3.1 Functional differential closure equations

We start the derivation of the gravitational closure equations by spelling out the two algebra relations (3.74b) and (3.74a) in terms of the gravitational Lagrangian. In the two Poisson bracket relations, various functional derivatives of the supermomentum and superhamiltonian will appear. We will thus briefly present them here for future reference. For this, we first split the Lie derivative term in the supermomentum into its two separate contributions

$$\frac{\partial \widehat{\varphi}^A}{\partial g^{\mathcal{A}}} \left(\widehat{g}(\varphi)\right) \left(\mathcal{L}_{\vec{N}} \widehat{g}\right)^{\mathcal{A}} \quad = \quad N^\mu \, \varphi^A{}_{,\mu} - (\partial_\gamma N^\mu) \, \mathrm{F}^A{}_\mu{}^\gamma \,, \tag{3.110}$$

with the kinematical coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ parametrizing the behavior of the geometric degrees of freedom under spatial diffeomorphisms. Note that the coefficient is a local function of the degrees of freedom φ for any gravitational theory under consideration. Using this, one can easily verify that the functional derivatives of the supermomentum are given by

$$\frac{\delta\widehat{\mathcal{D}}(\vec{N})}{\delta\varphi^A(x)} = -\left(\partial_\gamma N^\mu\right)(x)\,\pi_B(x)\,\mathrm{F}^B{}_\mu{}^\gamma{}_{:A}(x) - \partial_\mu\left(N^\mu\,\pi_A\right)(x)\,, \tag{3.111}$$

$$\frac{\delta\widehat{\mathcal{D}}(\vec{N})}{\delta\pi_A(x)} = N^\mu(x)\,\varphi^A{}_{,\mu}(x) - \left(\partial_\gamma N^\mu\right)(x)\,\mathrm{F}^A{}_\mu{}^\gamma(x)\,, \tag{3.112}$$

where we introduced the short-hand symbol $\mathrm{F}^A{}_\mu{}^\gamma{}_{:B} := \partial\mathrm{F}^A{}_\mu{}^\gamma/\partial\varphi^B$ for the partial derivative of the input coefficient with respect to the degrees of freedom.

Similarly, we can derive the functional derivatives of the non-local part of the superhamiltonian

$$\frac{\delta\widehat{\mathcal{H}}_{\text{non-local}}(N)}{\delta\varphi^A(x)} = (\partial_\gamma N)(x)\,\pi_B(x)\,\mathrm{M}^{B\gamma}{}_{:A}(x)\,, \tag{3.113}$$

$$\frac{\delta\widehat{\mathcal{H}}_{\text{non-local}}(N)}{\delta\pi_A(x)} = \left(\partial_\gamma N\right)(x)\,\mathrm{M}^{A\gamma}(x)\,. \tag{3.114}$$

Using equations (3.93) and (3.95) we see that the functional derivatives of the local part of the superhamiltonian turn into

$$\frac{\delta\widehat{\mathcal{H}}_{\text{local}}(N)}{\delta\varphi^A(x)} = -\int_\Sigma \mathrm{d}^3y\,N(y)\,\frac{\delta\mathcal{L}(y)}{\delta\varphi^A(x)}\,, \tag{3.115}$$

$$\frac{\delta\widehat{\mathcal{H}}_{\text{local}}(N)}{\delta\pi_A(x)} = N(x)\,k^A(x)\,. \tag{3.116}$$

The algebra relations have to be valid for any lapse N and shift $\vec{N}$. In other words, they act as test functions that can be used to extract partial differential equations from the functional differential equations that correspond to the different derivative orders of lapse and shift. As a result, we can eliminate them from the equations by replacing the lapses with $N \to \delta_x$ and the shifts with $\vec{M} \to \delta_y\,\partial_\mu$. This turns the relation into a distributional equation. From these, we can then recover several partial differential equations. x We first observe that we expect the bracket (3.74b) to be trivially solved for the non-local part of the superhamiltonian $\widehat{\mathcal{H}}_{\text{non-local}}$ since we required in the compatibility condition (3.82b) that the supermomentum acts on all functionals of φ by a spatial diffeomorphism. With (3.86) imposed, we can check that the contributions of the non-local superhamiltonian cancel from the algebra relation. As a result, the only remaining condition that we need to impose is that the local part of the superhamiltonian has to fulfill the functional differential equation

$$\left\{\widehat{\mathcal{D}}(\delta_x\partial_\mu), \widehat{\mathcal{H}}_{\text{local}}(\delta_y)\right\} = \widehat{\mathcal{H}}_{\text{local}}\left(\delta_x(\partial_\mu\delta_y)\right)\,. \tag{3.117}$$

Inserting all the functional derivatives we spelt out in the previous section, we find the first functional differential equation for the Lagrangian $\mathcal{L}$. Since the physical content of the bracket (3.74b) is to enforce that $\widehat{\mathcal{H}}$ must transform as scalar density of weight 1, we will refer to them as *covariance part of the closure equation* in the following.

DEFINITION COVARIANCE PART OF THE FUNCTIONAL CLOSURE EQUATIONS

The gravitational Lagrangian $\mathcal{L}$ is a solution of the functional differential equation

$$\begin{aligned} 0 \;=\; & \left(\frac{\partial \mathcal{L}}{\partial k^B}\right)(y)\, k^A(y) \left(\delta^B_A \delta^\gamma_\mu + \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right)(y)\, \left(\partial_\gamma \delta_y\right)(x) + \partial_\mu \left(k^A \frac{\partial \mathcal{L}}{\partial k^A} - \mathcal{L}\right)(y)\, \delta_y(x) \\ & - k^A(y)\, \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial k^B}\right)(y)\, \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}(y)\, \delta_y(x) - \left(k^A \frac{\partial \mathcal{L}}{\partial k^A} - \mathcal{L}\right)(y)\, \left(\partial_\mu \delta_y\right)(x) \\ & + \left(\varphi^A{}_{,\mu} + \mathrm{F}^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x)\, \frac{\delta \mathcal{L}(y)}{\delta \varphi^A(x)} + \mathrm{F}^A{}_\mu{}^\gamma(x)\, \partial_\gamma \left(\frac{\delta \mathcal{L}(y)}{\delta \varphi^A(\cdot)}\right)(x)\,. \end{aligned}$$

This guarantees that the gravitational Lagrangian transforms as scalar density of weight 1.

Similarly, we can localize the algebra relation (3.74a) by setting $N = \delta_x$ and $M = \delta_y$ and extract another functional differential equation. Inserting these functional derivatives of the superhamiltonian and the supermomentum into the algebra relation (3.74a), we find the second functional differential equation for the gravitational Lagrangian. As expected, this equation is linear in the Lagrangian. In contrast to the first closure equation that merely ensures that the constructed Lagrangian transforms properly as scalar density of weight 1, the second functional differential equation will select the physically viable gravitational Lagrangians. We will refer to them as *selective part of the closure equations* in the following.

DEFINITION SELECTIVE PART OF THE FUNCTIONAL CLOSURE EQUATIONS

The gravitational Lagrangian $\mathcal{L}$ is a solution of the functional differential equation

$$\begin{aligned} 0 \;=\; & -k^B(y) \frac{\delta \mathcal{L}(x)}{\delta \varphi^B(y)} + \left(\partial_\gamma \delta_x\right)(y)\, k^B(y)\, \mathrm{M}^{A\gamma}{}_{:B}(x)\, \frac{\partial \mathcal{L}}{\partial k^A}(x) + \partial_\mu \left(\frac{\delta \mathcal{L}(x)}{\delta \varphi^B(\cdot)}\, \mathrm{M}^{B\mu}\right)(y) \\ & + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial k^A}\right)(x) \left[(\deg P - 1)\, \mathrm{p}^{\rho\mu}\, \mathrm{F}^A{}_\rho{}^\nu - \mathrm{M}^{B[\mu|}\, \mathrm{M}^{A|\nu]}{}_{:B}\right](x)\; \left(\partial_\nu \delta_x\right)(y) \\ & - \frac{\partial \mathcal{L}}{\partial k^A}(x) \left[(\deg P - 1)\, \mathrm{p}^{\rho\nu} \left(\varphi^A{}_{,\rho} + \mathrm{F}^A{}_\rho{}^\gamma{}_{,\gamma}\right) + \partial_\mu \left(\mathrm{M}^{B[\mu|}\, \mathrm{M}^{A|\nu]}{}_{:B}\right)\right](x)\, (\partial_\nu \delta_x)(y) \\ & - (x \leftrightarrow y) \end{aligned}$$

Observe that in both cases we deal with a linear problem. In principle, we could obtain the gravitational dynamics now by solving the two functional differential equations. However, it turns out to be almost impossible practically. Instead, we turn the equations into an equivalent system of linear partial differential equations.

3.3.2 Input coefficient identities

Note that the contributions of the non-local superhamiltonian in (3.74b) still give useful information in an abstract treatment of gravitational theories since we can derive an *identity* for the kinematical coefficients.

That is, in any theory under consideration, the coefficients must fulfil this identity trivially. However, when dealing with the general solution space of the gravitational closure equations, such identities can prove helpful. Even further, another identity can be obtained from the algebra relation (3.74c) that needs to hold in practice and encodes the Lie algebra structure of the group of spatial diffeomorphisms.

We will now derive the first of these two input coefficient identities[6] from the $\left\{\widehat{\mathcal{D}},\widehat{\mathcal{D}}\right\}$ bracket: by simply spelling out the bracket and integration by parts we can bring all terms in the form that no derivative acts on the momenta π_A. From this we can then read off that

$$\mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\mathrm{F}^A{}_\nu{}^\epsilon - \mathrm{F}^B{}_\nu{}^\epsilon{}_{:A}\mathrm{F}^A{}_\mu{}^\gamma = \mathrm{F}^B{}_\mu{}^\epsilon\delta^\gamma_\nu - \mathrm{F}^B{}_\nu{}^\gamma\delta^\epsilon_\mu\,. \tag{3.118}$$

One can indeed verify by brute-force computation for the setup and input coefficients of Maxwellian electrodynamics; see chapter 4.1 for further details.

In the same fashion we can spell out the $\left\{\widehat{\mathcal{H}}_{\text{non-local}},\widehat{\mathcal{D}}\right\}$ bracket to obtain the second identity for the two input coefficients. Careful calculation yields

$$\mathrm{F}^A{}_\mu{}^\gamma\,\mathrm{M}^{B\nu}{}_{:A} - \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\mathrm{M}^{A\nu} = \mathrm{M}^{B\gamma}\delta^\nu_\mu\,. \tag{3.119}$$

DEFINITION INPUT COEFFICIENT IDENTITIES

The input coefficients fulfill the following two identities

$$\mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\mathrm{F}^A{}_\nu{}^\epsilon - \mathrm{F}^B{}_\nu{}^\epsilon{}_{:A}\mathrm{F}^A{}_\mu{}^\gamma = \mathrm{F}^B{}_\mu{}^\epsilon\delta^\gamma_\nu - \mathrm{F}^B{}_\nu{}^\gamma\delta^\epsilon_\mu\,, \tag{3.120a}$$

$$\mathrm{F}^A{}_\mu{}^\gamma\,\mathrm{M}^{B\nu}{}_{:A} - \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\mathrm{M}^{A\nu} = \mathrm{M}^{B\gamma}\delta^\nu_\mu\,. \tag{3.120b}$$

3.3.3 Covariance part of the closure equations

Let us now further work on the covariance part of the functional closure equations. For this we use that the Lagrangian is ultra-local in the velocities and series expand it in k:

$$\mathcal{L}\left[\varphi;k\right) = \sum_{N=0}^{\infty} \mathrm{C}_{A_1\dots A_N}\left[\varphi\right]\; k^{A_1}\cdots k^{A_N}\,, \tag{3.121}$$

where we introduced the countably infinitely many *output coefficient* functionals $\mathrm{C}_{A_1\dots A_N}[\varphi]$. By plugging them into the covariance part of the functional closure equations, we obtain a polynomial expression in

[6] These two identities are sometimes also referred to as the *mad flow identities*, after collaborator Maximilian Düll and the author, who derived them.

the velocities

$$
\begin{aligned}
0 = & \sum_{N=0}^{\infty} (N+1)\, \mathrm{C}_{BA_1\ldots A_N}\left[\varphi(y)\right] k^A(y) k^{A_1}(y)\cdots k^{A_N}(y) \left(\delta^B_A \delta^\gamma_\mu + \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right)(y) \left(\partial_\gamma \delta_y\right)(x) \\
& + \sum_{N=0}^{\infty} (N-1)\, \partial_\mu \left(\mathrm{C}_{A_1\ldots A_N}\left[\varphi(\cdot)\right] k^{A_1}\cdots k^{A_N}\right)(y)\, \delta_y(x) \\
& - \sum_{N=0}^{\infty} (N+1) \partial_\gamma \left(\mathrm{C}_{BA_1\ldots A_N}\left[\varphi(\cdot)\right] k^{A_1}\cdots k^{A_N}\right)(y) k^A(y)\, \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}(y)\, \delta_y(x) \\
& - \sum_{N=0}^{\infty} (N-1)\, \mathrm{C}_{A_1\ldots A_N}\left[\varphi(y)\right] k^{A_1}(y)\cdots k^{A_N}(y) \left(\partial_\mu \delta_y\right)(x) \\
& + \sum_{N=0}^{\infty} \frac{\delta \mathrm{C}_{A_1\ldots A_N}\left[\varphi(y)\right]}{\delta \varphi^A(x)} k^{A_1}(y)\cdots k^{A_N}(y) \left(\varphi^A{}_{,\mu} + \mathrm{F}^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x) \\
& + \sum_{N=0}^{\infty} \partial_\gamma \left(\frac{\delta \mathrm{C}_{A_1\ldots A_N}\left[\varphi(x)\right]}{\delta \varphi^A(\cdot)}\right)(x)\, \mathrm{F}^A{}_\mu{}^\gamma(x)\, k^{A_1}(y)\cdots k^{A_N}(y)\,.
\end{aligned}
\tag{3.122}
$$

Since this equation has to be valid for any velocity k all orders must vanish independently. We can extract the separate orders of the equations by application of the following functional derivative operators and evaluating the result at $k = 0$

$$
\left.\frac{\delta^M}{\delta k^{B_1}(x_1)\cdots \delta k^{B_M}(x_M)}\right|_{k=0} \quad \text{, for } M \geq 0\,. \tag{3.123}
$$

Extracting the terms for $k = 0$ first, we obtain the functional differential equation

$$
\begin{aligned}
0 = & -\partial_\mu \left(\mathrm{C}\left[\varphi(\cdot)\right]\right)(y)\, \delta_y(x) + \mathrm{C}\left[\varphi(y)\right] \left(\partial_\mu \delta_y\right)(x) \\
& + \frac{\delta \mathrm{C}\left[\varphi(y)\right]}{\delta \varphi^A(x)} \left(\varphi^A{}_{,\mu} + \mathrm{F}^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x) + \partial_\gamma \left(\frac{\delta \mathrm{C}\left[\varphi(y)\right]}{\delta \varphi^A(\cdot)}\right)(x)\, \mathrm{F}^A{}_\mu{}^\gamma(x)\,.
\end{aligned}
\tag{3.124}
$$

Similarly, we extract the equations for the higher-order terms in the velocities. The functional differential equations for the velocities of power N read

$$
\begin{aligned}
0 = & N \cdot N!\, \mathrm{C}_{A(B_1\ldots B_{N-1}|}\left[\varphi(y)\right] \left(\delta^A_{|B_N)} \delta^\gamma_\mu + \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right)(y) \left(\partial_\gamma \delta_y\right)(x)\, \delta_y(x_1)\cdots \delta_y(x_N) \\
& + (N-1) N!\, \partial_\mu \left(\mathrm{C}_{B_1\ldots B_N}\left[\varphi(\cdot)\right]\right)(y) \delta_y(x) \delta_y(x_1)\cdots \delta_y(x_N) \\
& + (N-1) N!\, \mathrm{C}_{B_1\ldots B_N}\left[\varphi(y)\right] \delta_y(x) \sum_{J=1}^{N} \delta_y(x_1)\cdots \widetilde{\delta_y(x_J)}\cdots \delta_y(x_N) \left(\partial_\mu \delta_y\right)(x_J) \\
& - (N+1)!\, \partial_\gamma \left(\mathrm{C}_{A(B_1\ldots B_{N-1}|}\left[\varphi(\cdot)\right]\right)(y)\, \mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)}(y)\, \delta_y(x) \delta_y(x_1)\cdots \delta_y(x_N) \\
& - (N-1) N! \sum_{J=1}^{N} \mathrm{C}_{AB_J(B_1\ldots \widetilde{B_J}\ldots B_{N-1}|}\, \mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)}(y)\, \delta_y(x) \delta_y(x_1)\cdots \widetilde{\delta_y(x_J)}\cdots \delta_y(x_N) \left(\partial_\mu \delta_y\right)(x_J) \\
& - (N-1)\, N!\, \mathrm{C}_{B_1\ldots B_N}\left[\varphi(y)\right] \left(\partial_\mu \delta_y\right)(x) \delta_y(x_1)\cdots \delta_y(x_N) \\
& + N!\, \frac{\delta \mathrm{C}_{A_1\ldots A_N}\left[\varphi(y)\right]}{\delta \varphi^A(x)} \left(\varphi^A{}_{,\mu} + \mathrm{F}^A{}_\mu{}^\gamma{}_{,\gamma}\right)(x)\, \delta_y(x_1)\cdots \delta_y(x_N) \\
& + N!\, \partial_\gamma \left(\frac{\delta \mathrm{C}_{A_1\ldots A_N}\left[\varphi(x)\right]}{\delta \varphi^A(\cdot)}\right)(x)\, \mathrm{F}^A{}_\mu{}^\gamma(x)\, \delta_y(x_1)\cdots \delta_y(x_N)\,.
\end{aligned}
\tag{3.125}
$$

where the $\sim$ symbol over terms instructs us to omit the corresponding term.

Furthermore, making the weak assumption that the coefficients $C_{A_1\ldots A_N}$ are determined uniquely at every hypersurface point by the geometric degrees of freedom and their partial derivatives, we can express the functional derivatives in terms of partial derivatives as

$$\frac{\delta C_{A_1\ldots A_N}\left[\varphi(x)\right]}{\delta\varphi^B(y)} = \sum_{j=0}^{\infty} C_{A_1\ldots A_N:B}{}^{\alpha_1\ldots\alpha_j}\left[\varphi(x)\right]\left(\partial_{\alpha_1\ldots\alpha_j}\delta_y\right)(x)\,, \tag{3.126}$$

where we introduced the following notation for the partial derivatives to simplify the expressions

$$C_{A_1\ldots A_N:B}{}^{\mu_1\ldots\mu_K} := \frac{\partial C_{A_1\ldots A_N}}{\partial\varphi^B{}_{,\mu_1\ldots\mu_K}}\,. \tag{3.127}$$

In order to extract information from the distribution expressions above, we integrate against test functions $f(x,y,x_1,\ldots,x_N)$ of $N+2$ variables. This allows us to strip the equations from their distributional character since we can integrate all δ-distributions. Afterwards, we read of the independent partial differential equations since the test functions – and almost all their partial derivatives – are independent and can be chosen arbitrarily. The only subtle problem that occurs, but can be dealt with simply, is that after the integration, all derivatives are evaluated at the same screen manifold point x. As a result, not all derivatives are independent.

This can be seen from the total derivative on the function $f(x,x)$

$$\partial_\mu f(x,x) = \left(\partial_{1\mu}f\right)(x,x) + \left(\partial_{2\mu}f\right)(x,x)\,, \tag{3.128}$$

where $\left(\partial_{1\mu}\right)$ indicates that the derivative acts on the first slot of the function f, and $\left(\partial_{2\mu}\right)$ on the second. But by solving this for $\left(\partial_{2\mu}f\right)$, inserting into the smeared equation again and integrating the total derivative by parts, we can separate the equation into its independent parts. Afterwards wex can read off all the separate contributions. For higher-order derivatives we find that we can express all derivatives that act on the second slot of the test function as

$$\left(\partial^N_{2\alpha_1\ldots\alpha_N}f\right)(x,x) = \sum_{K=0}^{N}(-1)^K\binom{N}{K}\left(\partial^{N-K}_{(\alpha_1\ldots\alpha_{N-K}|}\partial^K_{1|\alpha_{N-K+1}\ldots\alpha_K)}f\right)(x,x)\,. \tag{3.129}$$

Generalisations for test functions with more than two slots exist (compare Witte (2014)).

We will now present the results that can be obtained by carefully reading of the terms for the equations at each power of k directly. Starting with the functional differential equation (3.124), we integrate against a test function $f(x,y)$ and apply equation (3.129) to extract the different independent contributions from the functional differential equation. While no term appears for $f(x,x)$, we find the following partial differential equation from terms that belong to $\left(\partial_{1\gamma}f\right)(x,x)$:

$$0 = -\,C\,\delta^\gamma_\mu + \sum_{K=0}^{\infty}\left[(K+1)\,C_{:A}{}^{\gamma\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - C_{:A}{}^{\alpha_1\ldots\alpha_K}\,F^A{}_\mu{}^\gamma{}_{,\alpha_1\ldots\alpha_K}\right]\,. \tag{3.130}$$

For the higher-order derivatives $\left(\partial_{1\beta_1\ldots\beta_N}f\right)(x,x)$, with $N\geq 2$, we find

$$0 = \sum_{K=0}^{\infty}\left[\binom{K+N}{K}C_{:A}{}^{\beta_1\ldots\beta_N\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \binom{K+N-1}{K}C_{:A}{}^{\alpha_1\ldots\alpha_K(\beta_1\ldots\beta_{N-1}|}\,F^A{}_\mu{}^{|\beta_N)}{}_{,\alpha_1\ldots\alpha_K}\right]\,. \tag{3.131}$$

We repeat the same procedure for the functional differential equation (3.125). After integrating against the test function and eliminating the dependent slot derivatives we can again read off the different relations, depending on the derivative orders of f. One finds that $f(x, x, x, \ldots, x)$ does not appear in the result. For the coefficients of f with one free spatial index γ, we find the following equation

$$\begin{aligned} 0 = &- \mathrm{C}_{A(B_1 \ldots B_{N-1}|} \left(\delta^A_{|B_N)} \delta^\gamma_\mu + N \cdot \mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)} \right) \\ &+ \sum_{K=0}^{\infty} \left[(K+1)\, \mathrm{C}_{B_1 \ldots B_N : A}{}^{\gamma \alpha_1 \ldots \alpha_K}\, \varphi^A{}_{,\mu \alpha_1 \ldots \alpha_K} - \mathrm{C}_{B_1 \ldots B_N : A}{}^{\alpha_1 \ldots \alpha_K}\, \mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha_1 \ldots \alpha_K} \right] . \end{aligned} \tag{3.132}$$

For the remaining relations that we all obtain from $\left(\partial_{1 \beta_1 \ldots \beta_N} f \right)$ for $N \geq 2$ we find the analogous result as for the coefficients C in equation (3.131), that is

$$\begin{aligned} 0 = \sum_{K=0}^{\infty} \Bigg[&\binom{K+N}{K} \mathrm{C}_{B_1 \ldots B_N : A}{}^{\beta_1 \ldots \beta_N \alpha_1 \ldots \alpha_K}\, \varphi^A{}_{,\mu \alpha_1 \ldots \alpha_K} \\ &- \binom{K+N-1}{K} \mathrm{C}_{B_1 \ldots B_N : A}{}^{\alpha_1 \ldots \alpha_K (\beta_1 \ldots \beta_{N-1}|}\, \mathrm{F}^A{}_\mu{}^{|\beta_N)}{}_{,\alpha_1 \ldots \alpha_K} \Bigg] . \end{aligned} \tag{3.133}$$

This is not surprising: remember that the physical content of the covariance part of the closure equations was to make sure that the local superhamiltonian transforms as scalar density of weight 1. As such, all output coefficients must also properly transform under spatial diffeomorphism. If we take a closer look, we in fact see that the equations encode precisely that the output coefficients transform as

$$\begin{aligned} \left\{ \mathrm{C}_{A_1 \ldots A_K}, \widehat{\mathcal{D}}(\vec{N}) \right\} &= (\mathcal{L}_{\vec{N}} C)_{A_1 \ldots A_N} \\ &= N^\mu \partial_\mu \mathrm{C}_{A_1 \ldots A_K} + K \cdot (\partial_\gamma N^\mu) \mathrm{F}^B{}_\mu{}^\gamma{}_{:(A_1|} \mathrm{C}_{|A_2 \ldots A_K)B} + (\partial_\mu N^\mu) \mathrm{C}_{A_1 \ldots A_K} , \end{aligned}$$

where we can expand the left hand side and compare the different derivative orders of the lapse.

This is the farthest we get with simplifications of the covariance part of the closure equations. We will repeat the same derivation for the second functional differential equation and derive another set of partial differential equations before collecting all equations in the common system. These will be finally portrayed on page 84.

3.3.4 Selective part of the closure equations

In the very same fashion, we can now extract the second set of partial differential equations from the selective part of the closure equations in their functional form. The required steps are almost identical to the derivation of the selective part of the closure equations in the previous section, so we keep the treatment brief and only present the most important intermediate steps.

After insertion of the series expansion of the gravitational Lagrangian the selective part of the closure equations read

$$
\begin{aligned}
0 =& - \sum_{N=0}^{\infty} \frac{\delta C_{A_1\dots A_N}\left[\varphi(x)\right]}{\delta \varphi^B(y)} k^B(y) k^{A_1}(x) \cdots k^{A_N}(x) \\
&+ \sum_{N=0}^{\infty} (N+1)\, C_{AA_1\dots A_N}\left[\varphi(x)\right]\; M^{A\gamma}{}_{:B}(x)\; \left(\partial_\gamma \delta_x\right)(y)\, k^B(y) k^{A_1}(x) \cdots k^{A_N}(x) \\
&+ \sum_{N=0}^{\infty} \partial_\mu \left(\frac{\delta C_{A_1\dots A_N}\left[\varphi(x)\right]}{\delta \varphi^B(\cdot)} M^{B\mu} \right)(y)\, k^{A_1}(x) \cdots k^{A_N}(x) \\
&+ \sum_{N=0}^{\infty} (N+1)\, \partial_\mu \left(C_{AA_1\dots A_N}\left[\varphi(\cdot)\right]\; k^{A_1} \cdots k^{A_N} \right)(x) \Big[(\deg P - 1)\, p^{\rho\mu}\, F^A{}_\rho{}^\nu \\
&\qquad - M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \Big](x)\; \left(\partial_\nu \delta_x\right)(y) \\
&- \sum_{N=0}^{\infty} (N+1)\, C_{AA_1\dots A_N}\left[\varphi(x)\right]\; k^{A_1}(x) \cdots k^{A_N}(x) \Big[(\deg P - 1)\, p^{\rho\nu} \left(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma} \right) \\
&\qquad + \partial_\mu \left(M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \right) \Big](x)\, (\partial_\nu \delta_x)(y) \\
&- \left(x \leftrightarrow y \right) .
\end{aligned}
\tag{3.134}
$$

Extracting all terms that contain no velocities, we find

$$
\begin{aligned}
0 =& \partial_\mu \left(\frac{\delta C\left[\varphi(x)\right]}{\delta \varphi^B(\cdot)} M^{B\mu} \right)(y) + \partial_\mu \left(C_A\left[\varphi(\cdot)\right] \right)(x) \left[(\deg P - 1)\, p^{\rho\mu}\, F^A{}_\rho{}^\nu - M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \right](x) \\
&\times \left(\partial_\nu \delta_x\right)(y) - C_A\left[\varphi(x)\right] \left[(\deg P - 1)\, p^{\rho\nu} \left(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma} \right) + \partial_\mu \left(M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \right) \right](x)\, (\partial_\nu \delta_x)(y) \\
&- \left(x \leftrightarrow y \right) .
\end{aligned}
\tag{3.135}
$$

In a similiar fashion, we can extract the N^{th} order contribution for $N \geq 1$ by application of the functional derivative (3.123) and find

$$
\begin{aligned}
0 =& - \sum_{K=1}^{N} \frac{\delta C_{B_1\dots\widetilde{B_K}\dots B_N}\left[\varphi(x)\right]}{\delta \varphi^{B_K}(y)} \delta_y(x_k) \delta_x(x_1) \cdots \widetilde{\delta_x(x_k)} \cdots \delta_x(x_N) \\
&+ \sum_{K=1}^{N} N \cdot N!\, C_{A(B_1\dots B_{N-1}|}\left[\varphi(x)\right]\; M^{A\gamma}{}_{:|B_N))}(x)\; \delta_y(x_K)\, \delta_x(x_1) \cdots \widetilde{\delta_x(x_K)} \cdots \delta_x(x_N)\; \left(\partial_\gamma \delta_x\right)(y) \\
&+ N!\, \partial_\mu \left(\frac{\delta C_{B_1\dots B_N}\left[\varphi(x)\right]}{\delta \varphi^B(\cdot)} M^{B\mu} \right)(y)\, \delta_x(x_1) \cdots \delta_x(x_N) \\
&+ (N+1)! \Bigg[\partial_\mu \left(C_{AB_1\dots B_N}\left[\varphi(\cdot)\right] \right)(x)\, \delta_x(x_1) \cdot \delta_x(x_N) \\
&\qquad - \sum_{K=1}^{N} C_{AB_1\dots B_N}\left[\varphi(x)\right] \delta_x(x_1) \cdots \widetilde{\delta_x(x_K)} \cdots \delta_x(x_N) \left(\partial_\mu \delta_x \right)(x_K) \Bigg] \\
&\quad \times \left[(\deg P - 1)\, p^{\rho\nu} \left(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma} \right) + \partial_\mu \left(M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \right) \right](x)\, (\partial_\nu \delta_x)(y) \\
&- N!\, C_{AB_1\dots B_N}\left[\varphi(x)\right] \left[(\deg P - 1)\, p^{\rho\nu} \left(\varphi^A{}_{,\rho} + F^A{}_\rho{}^\gamma{}_{,\gamma} \right) + \partial_\mu \left(M^{B[\mu|}\, M^{A|\nu]}{}_{:B} \right) \right](x)\, (\partial_\nu \delta_x)(y) \\
&\qquad \times \delta_x(x_1) \cdots \delta_x(x_N) - \left(x \leftrightarrow y \right) .
\end{aligned}
\tag{3.136}
$$

Compared to the covariance part of the closure equations, the derivation will be more involved for the selective part. We will thus present the results that can be obtained by carefully reading of the terms for the equations at each power of k in further detail.

Equations obtained from N=0

For the contributions in the selective part of the closure equations without a local velocity k, we need to distinguish between four cases. From all terms that appear in front of the underived test function $f(x,x)$ we get the following partial differential equation

$$\begin{aligned}0 = &-\partial_\nu\Bigg[\left(\partial_\mu \mathrm{C}_A\right)\left((\deg P-1)\,\mathrm{p}^{\rho\mu}\mathrm{F}^A{}_\rho{}^\nu - \mathrm{M}^{B[\mu|}\mathrm{M}^{A|\nu]}{}_{:B}\right)\\ &\qquad - \mathrm{C}_A\left((\deg P-1)\mathrm{p}^{\rho\nu}\left(\varphi^A{}_{,\rho}+\mathrm{F}^A{}_\rho{}^\gamma{}_{,\gamma}\right)+\partial_\mu\left(\mathrm{M}^{B[\mu|}\mathrm{M}^{A|\nu]}{}_{:B}\right)\right)\Bigg]\\ &+\sum_{K=0}^{\infty}\sum_{J=0}^{K}(-1)^J\binom{K}{J}\partial_{\gamma\alpha_1\dots\alpha_J}\left(\mathrm{C}_{:A}{}^{\beta_1\dots\beta_{K-J}(\alpha_1\dots\alpha_J|}\,\mathrm{M}^{A|\gamma)}{}_{,\beta_1\dots\beta_{K-J}}\right)\,,\end{aligned} \tag{3.137}$$

whereas all terms in front of the $\left(\partial_{1\gamma}f\right)(x,x)$ give the equation

$$\begin{aligned}0 = {}&\partial_\mu\left(\mathrm{C}_A\,\mathrm{M}^{A[\mu|}{}_{:B}\,\mathrm{M}^{B|\gamma]}\right) - 2(\deg P-1)\,\mathrm{p}^{\rho\gamma}\left[\mathrm{C}_A\,\varphi^A{}_{,\rho}+\partial_\mu\left(\mathrm{C}_A\,\mathrm{F}^A{}_\rho{}^\mu\right)\right]\\ &+\sum_{K=0}^{\infty}\sum_{J=0}^{K}(-1)^J\binom{K}{J}(J+1)\,\partial_{\alpha_1\dots\alpha_J}\left(\mathrm{C}_{:A}{}^{\beta_1\dots\beta_{K-J}(\alpha_1\dots\alpha_J|}\,\mathrm{M}^{A|\gamma)}{}_{,\beta_1\dots\beta_{K-J}}\right)\\ &+\sum_{K=0}^{\infty}\mathrm{C}_{:A}{}^{\alpha_1\dots\alpha_K}\,\mathrm{M}^{A\gamma}{}_{,\alpha_1\dots\alpha_K}\,.\end{aligned} \tag{3.138}$$

For the higher-order derivatives of the test function, i.e. $\left(\partial_{1\mu_1\dots\mu_w}f\right)(x,x)$ we get two different partial differential equation, depending on whether w is even

$$0 = \sum_{K=w}^{\infty}\sum_{J=w+1}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{w}\partial_{\alpha_1\dots\alpha_{J-w}}\left(\mathrm{C}_{:A}{}^{\beta_J\dots\beta_K(\alpha_1\dots\alpha_{J-w}\mu_1\dots\mu_{w-1}}\,\mathrm{M}^{A|\mu_w)}{}_{,\beta_J\dots\beta_K}\right)\,, \tag{3.139}$$

or w is odd

$$\begin{aligned}0 = {}&2\sum_{K=w-1}^{\infty}\binom{K}{w-1}\mathrm{C}_{:A}{}^{\beta_w\dots\beta_K(\mu_1\dots\mu_{w-1}|}\,\mathrm{M}^{A|\mu_w)}{}_{,\beta_N\dots\beta_K}\\ &-\sum_{K=w}^{\infty}\sum_{J=w+1}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{w}\partial_{\alpha_1\dots\alpha_{J-w}}\left(\mathrm{C}_{:A}{}^{\beta_J\dots\beta_K(\alpha_1\dots\alpha_{J-w}\mu_1\dots\mu_{w-1}}\,\mathrm{M}^{A|\mu_w)}{}_{,\beta_J\dots\beta_K}\right)\,.\end{aligned} \tag{3.140}$$

Combining this, for $w=3$, with equations (3.137) and taking the gradient of (3.138) we find that we can further simplify equation (3.137) into

$$0 = \sum_{K=2}^{\infty}\sum_{J=2}^{K}(-1)^J\binom{K}{J}(J-1)\,\partial_{\gamma\alpha_1\dots\alpha_J}\left(\mathrm{C}_{:A}{}^{\beta_1\dots\beta_{K-J}(\alpha_1\dots\alpha_J|}\,\mathrm{M}^{A|\gamma)}{}_{,\beta_1\dots\beta_{K-J}}\right)\,. \tag{3.141}$$

This is the farthest we get in simplifying the partial differential equations obtained from the selective part of the closure equations for $N=0$.

Remaining equations for N > 0

With the help of the usual procedure we can again extract the partial differential equations from equation (3.136). As before, $f(x, x, x, \dots, x)$ does not yield any information. For the first derivative acting on the first slot $\left(\partial_{1\mu} f\right)(x, x, x, \dots, x)$ we find the following equation

$$\begin{aligned} 0 = &(N+1)!\ (\deg P - 1)\ \mathrm{C}_{AB_1\dots B_N}\left(\mathrm{p}^{\mu\nu}\varphi^A{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\mathrm{F}^A{}_\nu{}^\gamma\right) \\ &- N\cdot N!\,\mathrm{C}_{A(B_1\dots B_{N-1}|}\,\mathrm{M}^{A\mu}{}_{:|B_N)} - N!\sum_{K=0}^{\infty}\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_K}\,\mathrm{M}^{A\mu}{}_{,\alpha_1\dots\alpha_K} \\ &- (N-1)!\sum_{J=1}^{N-1}\mathrm{C}_{B_1\dots\widetilde{B_J}\dots B_N:B_J}{}^{\mu} - (N-1)!\sum_{K=0}^{\infty}(-1)^K(K+1)\partial_{\alpha_1\dots\alpha_K}\left(\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu}\right)\,, \end{aligned} \tag{3.142}$$

where $N \geq 1$. Similarly, for two spatial derivatives μ and ν we obtain the expression

$$\begin{aligned} 0 = &(N+1)!\,(\deg P - 1)\mathrm{C}_{AB_1\dots B_N}\,\mathrm{p}^{\rho(\mu|}\mathrm{F}^A{}_\rho{}^{|\nu)} + N!\sum_{K=0}^{\infty}(K+1)\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_K(\mu|}\mathrm{M}^{A|\nu)}{}_{,\alpha_1\dots\alpha_K} \\ &+ (N-1)!\sum_{J=1}^{N-1}\mathrm{C}_{B_1\dots\widetilde{B_J}\dots B_N:B_J}{}^{\mu\nu} - (N-1)!\sum_{K=0}^{\infty}(-1)^K\binom{K+2}{2}\partial_{\alpha_1\dots\alpha_K}\left(\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu\nu}\right)\,. \end{aligned} \tag{3.143}$$

Another equation is obtained by collecting the remaining terms where s derivatives appear on the second slot and no derivatives on any of the other slots. In this case we find for $N \geq 1, s \geq 3$ that

$$\begin{aligned} 0 = &N\cdot\sum_{K=0}^{\infty}\binom{K+s-1}{s-1}\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_K(\mu_1\dots\mu_{s-1}|}\mathrm{M}^{A|\mu_s)}{}_{,\alpha_1\dots\alpha_K} + \sum_{J=1}^{N-1}\mathrm{C}_{B_1\dots\widetilde{B_J}\dots B_N:B_J}{}^{\mu_1\dots\mu_s} \\ &- \sum_{K=0}^{\infty}(-1)^{K+s}\binom{K+s}{s}\partial_{\alpha_1\dots\alpha_K}\left(\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu_1\dots\mu_s}\right)\,. \end{aligned} \tag{3.144}$$

Extracting the coefficient for $\left(\partial_{1\mu}\partial_{2\nu} f\right)$ we get

$$0 = \mathrm{C}_{AB_1\dots B_N}\left(\mathrm{M}^{B[\mu|}\,\mathrm{M}^{A|\nu]}{}_{:B} + (\deg P - 1)\,\mathrm{p}^{\rho[\mu|}\,\mathrm{F}^A{}_\rho{}^{|\nu]}\right)\,. \tag{3.145}$$

Fortunately, only two types of equations are left. The first one is obtained by collecting all equations where at least one derivative appears on the second slot of f and all the other derivatives on one other slot $J = 1\dots N-1$. In this case we find for $N \geq 2$ and $T \geq 2$ that

$$\mathrm{C}_{B_1\dots\widetilde{B_J}\dots B_N:B_J}{}^{\mu_1\dots\mu_T} = \sum_{K=0}^{\infty}(-1)^{K+T}\binom{K+T}{T}\partial_{\alpha_1\dots\alpha_K}\left(\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu_1\dots\mu_T}\right)\,. \tag{3.146}$$

All remaining derivatives must be of the following form, with $N \geq 3, T \geq 3$

$$0 = \sum_{K=0}^{\infty}(-1)^{K+T}\binom{K+T}{T}\partial_{\alpha_1\dots\alpha_K}\left(\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu_1\dots\mu_T}\right)\,. \tag{3.147}$$

Obviously, we can combine the last two equations for $N \geq 3$, which then yields the remarkable result that all coefficients $\mathrm{C}_{A_1\dots A_N}$ do only depend up to the second derivatives of the degrees of freedom, i.e.

$$\mathrm{C}_{A_1\dots A_N:B}{}^{\mu_1\dots\mu_M} = 0 \qquad \text{for } N \geq 2, M \geq 3\,. \tag{3.148}$$

For the coefficients C and C_A, however, no such statement can be made a priori. The equivalent relation, stemming from the same terms in the functional differential equation, is equation (3.144) for $N = 1$. In this case we see that C appears together with the coefficient C_A. Only in case the input coefficient $\mathrm{M}^{A\gamma}$ is vanishing, as we will discuss in more detail in section 3.5.3, we would indeed obtain (3.147) for $N = 1$ and, by assuming that there is *some* maximal derivative order, one ultimately recovers that $\mathrm{C}(\varphi, \partial\varphi, \partial\partial\varphi)$. However, in general this fails to be the case. Instead, the number of derivatives appearing in C is dependent on the number of derivatives appearing in C_A: if we, for example, would *assume* that $\mathrm{C}_A(\varphi, \partial\varphi, \partial\partial\varphi)$, equation (3.144) gives that $\mathrm{C}(\varphi, \partial\varphi, \partial\partial\varphi, \partial\partial\partial\varphi)$. Generally, by assuming that C_A depends up to order q on derivatives of the degrees of freedom, one recovers that C depends up to order $q + 1$.

Such an assumption, however, is not justified a priori for the following reason: Already in the case of general relativity, it turns out that C_A can depend to arbitrary order on the spatial derivatives of the degrees of freedom. Evaluation of the closure equations then gives that

$$\mathrm{C}_A(x) = \frac{\delta\Lambda}{\delta\varphi^A(x)}, \tag{3.149}$$

and that C_A is divergence-free. Such a coefficient decouples in the equations of motion, rendering them *effectively* second-order. However, no a priori reason exists why the scalar density Λ can only be constructed from $\varphi, \partial\varphi$ and $\partial\partial\varphi$.

Having finally extracted all partial differential equations from the functional form of the gravitational closure equations, we present the summary of the whole system of linear partial differential equations on the following two pages, where we in total obtained seven individual linear equations and fourteen sequences of equations that hold for $N \geq 2$, respectively. The gravitational Lagrangian, being determined solely by the output coefficients $\mathrm{C}_{A_1 \ldots A_N}$, is the solution of this system.

One particularly remarkable fact is that this is an entirely local system in the sense that it has to be fulfilled at each screen manifold x separately. On the other hand, the functional form of the equations considers two separate points x and y, reflecting that the hypersurface deformation algebra implements the action of two separate sets of lapses and shifts, i.e. considers two separate infinitesimal paths along whom we can evolve the initial data.

THE GRAVITATIONAL CLOSURE EQUATIONS

The seven individual equations

$$(\mathbf{C1}) \qquad 0 = -\mathrm{C}\,\delta^\gamma_\mu + \sum_{K=0}^{\max}\Big[(K+1)\,\mathrm{C}_{:A}{}^{\gamma\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \mathrm{C}_{:A}{}^{\alpha_1\ldots\alpha_K}\,\mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha_1\ldots\alpha_K}\Big]$$

$$(\mathbf{C2}) \qquad 0 = -\mathrm{C}_A\left(\delta^A_B\delta^\gamma_\mu + \mathrm{F}^A{}_\mu{}^\gamma{}_{:B}\right) + \sum_{K=0}^{\max}\Big[(K+1)\,\mathrm{C}_{B:A}{}^{\gamma\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K}\,\mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha_1\ldots\alpha_K}\Big]$$

$$(\mathbf{C3}) \qquad 0 = 2\left(\deg P - 1\right)\mathrm{p}^{(\mu|\rho}\,\mathrm{C}_{AB}\,\mathrm{F}^A{}_\rho{}^{|\nu)} + \sum_{K=0}^{\max}(K+1)\,\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K(\mu|}\,\mathrm{M}^{A|\nu)}{}_{,\alpha_1\ldots\alpha_K}$$
$$- \sum_{K=0}^{\max}(-1)^K\binom{K+2}{K}\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{:B}{}^{\alpha_1\ldots\alpha_K\mu\nu}\right)$$

$$(\mathbf{C4}) \qquad 0 = 2\left(\deg P - 1\right)\mathrm{C}_{AB}\left(\mathrm{p}^{\mu\nu}\varphi^A{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\,\mathrm{F}^A{}_\nu{}^\gamma\right) - \mathrm{C}_A\,\mathrm{M}^{A\mu}{}_{:B} - \sum_{K=0}^{\max}\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K}\,\mathrm{M}^{A\mu}{}_{,\alpha_1\ldots\alpha_K}$$
$$- \sum_{K=0}^{\max}(-1)^K(K+1)\,\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{:B}{}^{\alpha_1\ldots\alpha_K\mu}\right)$$

$$(\mathbf{C5}) \qquad 0 = 2\,\partial_\mu\left(\mathrm{C}_A\,\mathrm{M}^{A[\mu|}{}_{:B}\,\mathrm{M}^{B|\gamma]}\right) - 2(\deg P - 1)\,\mathrm{p}^{\rho\gamma}\left[\mathrm{C}_A\,\varphi^A{}_{,\rho} + \partial_\mu\left(\mathrm{C}_A\,\mathrm{F}^A{}_\rho{}^\mu\right)\right]$$
$$+ \sum_{K=0}^{\max}\sum_{J=0}^{K}(-1)^J\binom{K}{J}(J+1)\,\partial_{\alpha_1\ldots\alpha_J}\left(\mathrm{C}_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|}\,\mathrm{M}^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}}\right)$$
$$+ \sum_{K=0}^{\max}\mathrm{C}_{:A}{}^{\alpha_1\ldots\alpha_K}\,\mathrm{M}^{A\gamma}{}_{,\alpha_1\ldots\alpha_K}$$

$$(\mathbf{C6}) \qquad 0 = 6\left(\deg P - 1\right)\mathrm{C}_{AB_1B_2}\left(\mathrm{p}^{\mu\nu}\,\varphi^A{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\,\mathrm{F}^A{}_\nu{}^\gamma\right) - 4\,\mathrm{C}_{A(B_1}\,\mathrm{M}^{A\mu}{}_{:B_2)} - 2\,\mathrm{C}_{B_1B_2:A}\,\mathrm{M}^{A\mu}$$
$$- 2\,\mathrm{C}_{B_1B_2:A}{}^\alpha\,\mathrm{M}^{A\mu}{}_{,\alpha} - 2\,\mathrm{C}_{B_1B_2:A}{}^{\alpha\beta}\,\mathrm{M}^{A\mu}{}_{,\alpha\beta} - \mathrm{C}_{B_2:B_1}{}^\mu$$
$$- \sum_{K=0}^{\max}(-1)^K(K+1)\,\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{B_1:B_2}{}^{\mu\alpha_1\ldots\alpha_K}\right)$$

$$(\mathbf{C7}) \qquad 0 = \sum_{K=2}^{\max}\sum_{J=2}^{K}(-1)^J\binom{K}{J}(J-1)\,\partial_{\gamma\alpha_1\ldots\alpha_J}\left(\mathrm{C}_{:A}{}^{\beta_1\ldots\beta_{K-J}(\alpha_1\ldots\alpha_J|}\,\mathrm{M}^{A|\gamma)}{}_{,\beta_1\ldots\beta_{K-J}}\right)$$

The fourteen sequences of equations for $\mathrm{N} \geq 2$

$$(\mathbf{C8_N}) \qquad 0 = \sum_{K=0}^{\max}\left[\binom{K+N}{K}\mathrm{C}_{:A}{}^{\beta_1\ldots\beta_N\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \binom{K+N-1}{K}\mathrm{C}_{:A}{}^{\alpha_1\ldots\alpha_K(\beta_1\ldots\beta_{N-1}|}\,\mathrm{F}^A{}_\mu{}^{|\beta_N)}{}_{,\alpha_1\ldots\alpha_K}\right]$$

$$(\mathbf{C9_N}) \qquad 0 = \sum_{K=0}^{\max}\left[\binom{K+N}{K}\mathrm{C}_{B:A}{}^{\beta_1\ldots\beta_N\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \binom{K+N-1}{K}\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K(\beta_1\ldots\beta_{N-1}|}\,\mathrm{F}^A{}_\mu{}^{|\beta_N)}{}_{,\alpha_1\ldots\alpha_K}\right]$$

The fourteen sequences of equations for $\mathrm{N} \geq 2$ (continued)

$$(\mathbf{C10_N}) \qquad 0 = -\mathrm{C}_{B_1\ldots B_N}\,\delta^\gamma_\mu - N\,\mathrm{C}_{A(B_1\ldots B_{N-1}|}\,\mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)} - \mathrm{C}_{B_1\ldots B_N:A}\,\mathrm{F}^A{}_\mu{}^\gamma + \mathrm{C}_{B_1\ldots B_N:A}{}^\gamma\,\varphi^A{}_{,\mu}$$
$$- \mathrm{C}_{B_1\ldots B_N:A}{}^\alpha\,\mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha} + 2\,\mathrm{C}_{B_1\ldots B_N:A}{}^{\gamma\alpha}\,\varphi^A{}_{,\mu\alpha} - \mathrm{C}_{B_1\ldots B_N:A}{}^{\alpha\beta}\,\mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha\beta}$$

$$(\mathbf{C11_N}) \qquad 0 = \mathrm{C}_{B_1\ldots B_N:A}{}^{\beta_1\beta_2}\,\varphi^A{}_{,\mu} - \mathrm{C}_{B_1\ldots B_N:A}{}^{(\beta_1|}\,\mathrm{F}^A{}_\mu{}^{|\beta_2)} - 2\,\mathrm{C}_{B_1\ldots B_N:A}{}^{\alpha(\beta_1|}\,\mathrm{F}^A{}_\mu{}^{|\beta_2)}{}_{,\alpha}$$

$$(\mathbf{C12_N}) \qquad 0 = \mathrm{C}_{B_1\ldots B_N:A}{}^{(\alpha\beta|}\,\mathrm{F}^A{}_\mu{}^{|\gamma)}$$

$$(\mathbf{C13_N}) \qquad 0 = \mathrm{C}_{B_1\ldots B_N:A}{}^{(\alpha\beta|}\,\mathrm{M}^{A|\gamma)}$$

$$(\mathbf{C14_N}) \qquad 0 = \mathrm{C}_{AB_1\ldots B_{N-1}}\left(\mathrm{M}^{B[\mu|}\,\mathrm{M}^{A|\nu]}{}_{:B} + (\deg P - 1)\,\mathrm{p}^{\rho[\mu|}\,\mathrm{F}^A{}_\rho{}^{|\nu]}\right)$$

$$(\mathbf{C15_N}) \qquad 0 = \mathrm{C}_{B_1\ldots\widehat{B_J}\ldots B_{N+1}:B_J}{}^{\mu\nu} - \mathrm{C}_{B_1\ldots B_N:B_{N+1}}{}^{\mu\nu} \qquad \text{for} \qquad J = 1\ldots N+1$$

$$(\mathbf{C16_N}) \qquad 0 = N\cdot(N+1)(\deg P - 1)\,\mathrm{C}_{AB_1\ldots B_N}\,\mathrm{p}^{\rho(\mu|}\,\mathrm{F}^A{}_\rho{}^{|\nu)} + N\,\mathrm{C}_{B_1\ldots B_N:A}{}^{(\mu|}\,\mathrm{M}^{A|\nu)}$$
$$+ 2N\,\mathrm{C}_{B_1\ldots B_N:A}{}^{\alpha(\mu|}\,\mathrm{M}^{A|\nu)}{}_{,\alpha} + (N-2)\,\mathrm{C}_{B_1\ldots B_{N-1}:B_N}{}^{\mu\nu}$$

$$(\mathbf{C17_N}) \qquad 0 = (N+2)(N+1)(\deg P - 1)\,\mathrm{C}_{AB_1\ldots B_{N+1}}\left(\mathrm{p}^{\mu\nu}\varphi^A{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\,\mathrm{F}^A{}_\nu{}^\gamma\right)$$
$$- (N+1)^2\,\mathrm{C}_{A(B_1\ldots B_N|}\,\mathrm{M}^{A\mu}{}_{:|B_{N+1})} - (N+1)\,\mathrm{C}_{B_1\ldots B_{N+1}:A}\,\mathrm{M}^{A\mu}$$
$$- (N+1)\,\mathrm{C}_{B_1\ldots B_{N+1}:A}{}^\alpha\,\mathrm{M}^{A\mu}{}_{,\alpha} - (N+1)\,\mathrm{C}_{B_1\ldots B_{N+1}:A}{}^{\alpha\beta}\,\mathrm{M}^{A\mu}{}_{,\alpha\beta}$$
$$- \sum_{K=1}^{N+1}\mathrm{C}_{B_1\ldots\widehat{B_K}\ldots B_{N+1}:B_K}{}^\mu + 2\left(\partial_\gamma \mathrm{C}_{B_1\ldots B_N:B_{N+1}}{}^{\gamma\mu}\right)$$

$$(\mathbf{C18_N}) \qquad 0 = \mathrm{C}_{A:B}{}^{\mu_1\ldots\mu_N} - \sum_{K=0}^{\max}(-1)^{K+N}\binom{K+N}{K}\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_N}\right)$$

$$(\mathbf{C19_N}) \qquad 0 = \sum_{K=0}^{\max}\binom{K+N}{K}\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K(\mu_1\ldots\mu_N|}\,\mathrm{M}^{A|\mu_{N+1})}{}_{,\alpha_1\ldots\alpha_K}$$
$$+ \sum_{K=0}^{\max}(-1)^{K+N}\binom{K+N+1}{K}\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{:B}{}^{\alpha_1\ldots\alpha_K\mu_1\ldots\mu_{N+1}}\right)$$

$$(\mathbf{C20_{N\,even}}) \qquad 0 = \sum_{K=N}^{\max}\sum_{J=N+1}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{N}\partial_{\alpha_1\ldots\alpha_{J-N}}\left(\mathrm{C}_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}}\,\mathrm{M}^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K}\right)$$

$$(\mathbf{C21_{N\,odd}}) \qquad 0 = 2\sum_{K=N-1}^{\max}\binom{K}{N-1}\mathrm{C}_{:A}{}^{\beta_N\ldots\beta_K(\mu_1\ldots\mu_{N-1}|}\,\mathrm{M}^{A|\mu_N)}{}_{,\beta_N\ldots\beta_K}$$
$$- \sum_{K=N}^{\max}\sum_{J=N+1}^{K+1}(-1)^J\binom{K}{J-1}\binom{J}{N}\partial_{\alpha_1\ldots\alpha_{J-N}}\left(\mathrm{C}_{:A}{}^{\beta_J\ldots\beta_K(\alpha_1\ldots\alpha_{J-N}\mu_1\ldots\mu_{N-1}}\,\mathrm{M}^{A|\mu_N)}{}_{,\beta_J\ldots\beta_K}\right)$$

with the coefficients defined by

$$\mathcal{L}_{\text{geometry}} = \sum_{N=0}^{\infty}\mathrm{C}_{A_1\ldots A_N}[\varphi]\,k^{A_1}\cdots k^{A_N}$$

$$(\mathcal{L}_\xi G)^{\mathcal{A}} = \frac{\partial\widehat{g}^{\mathcal{A}}}{\partial\varphi^A}(\varphi)\left(\varphi^A{}_{,\mu}\xi^\mu - \mathrm{F}^A{}_\mu{}^\gamma\xi^\mu{}_{,\gamma}\right) \qquad \mathrm{M}^{A\gamma} = \frac{\partial\widehat{\varphi}^A}{\partial g^{\mathcal{A}}}\left(g^{\mathcal{B}}(\varphi)\right)e^a_0\frac{\partial g^{\mathcal{A}}}{\partial\partial_\gamma X^a_t}$$

$$\mathrm{p}^{\mu\nu} = P^{a_1\ldots a_{\deg P}}\epsilon^\mu_{a_1}\epsilon^\nu_{a_2}\epsilon^0_{a_3}\cdots\epsilon^0_{a_{\deg P}}$$

3.4 GRAVITATIONAL FIELD EQUATIONS

Suppose we have solved the gravitational closure equations presented on the previous pages and obtained all the output coefficients appearing in the gravitational Lagrangian for a gravitational theory of interest. Then, the obvious next question is what form the equations of motion take and what their underlying principal polynomial is.

3.4.1 Equations of motion

For the former question, we can employ the action functional of both the matter and gravitational sectors

$$\begin{aligned}\mathcal{S}_{\text{total}}[\psi, \varphi, N, \vec{N}] &= \mathcal{S}_{\text{matter}}[\psi; \varphi, N, \vec{N}) \\ &\quad + \int \mathrm{d}t \int \mathrm{d}^3x\, N(t,x)\mathcal{L}\left[\varphi; \frac{1}{N}\left(\dot{\varphi}^A - (\partial_\gamma N)\mathrm{M}^{A\gamma} - N^\mu \varphi^A{}_{,\mu} + (\partial_\gamma N^\mu)\mathrm{F}^A{}_\mu{}^\gamma\right)\right), \end{aligned} \tag{3.150}$$

and calculate the variation by lapse, shift and the F-many degrees of freedom φ. One then finds the four constraint equations

$$\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N(x,t)} = \frac{\partial\mathcal{L}}{\partial k^A}k^A - \mathcal{L} - \partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial k^A}\mathrm{M}^{A\gamma}\right), \tag{3.151a}$$

$$\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N^\alpha(x,t)} = \frac{\partial\mathcal{L}}{\partial k^A}\varphi^A{}_{,\mu} + \partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial k^A}\mathrm{F}^A{}_\mu{}^\gamma\right), \tag{3.151b}$$

and F evolutionary equations, namely

$$\begin{aligned}\frac{\delta\mathcal{S}_{\text{matter}}}{\delta\varphi^A(x,t)} &= \left(\partial_t - N^\mu\partial_\mu\right)\frac{\partial\mathcal{L}}{\partial k^A} \\ &\quad + \frac{\partial\mathcal{L}}{\partial k^B}\left((\partial_\gamma N)\mathrm{M}^{B\gamma}{}_{:A} - \left(\partial_\gamma N^\mu\right)\left(\delta^B_A\delta^\gamma_\mu + \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right)\right) \\ &\quad - \sum_{n=0}^{\infty}\int \mathrm{d}^3y\, N(t,y)\frac{\delta\mathrm{C}_{B_1\ldots B_n}[\varphi(t,y)]}{\delta\varphi^A(t,x)}k^{B_1}(t,y)\ldots k^{B_n}(t,y)\,. \end{aligned} \tag{3.151c}$$

We can further flesh this out by inserting the series expansion of the Lagrangian into the equations. For example, in case of the scalar constraint one finds

$$\begin{aligned}\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N(x)} &= \sum_{n=0}^{\infty}(n-1)\,\mathrm{C}_{B_1\ldots B_n}k^{B_1}\cdots k^{B_n} - \sum_{n=0}^{\infty}(n+1)\mathrm{C}_{AB_1\ldots B_n}\mathrm{M}^{A\gamma}{}_{,\gamma}\,k^{B_1}\cdots k^{B_n} \\ &\quad - \sum_{k=0}^{\infty}\mathrm{C}_{A:B}{}^{(\alpha_1\ldots\alpha_k|}\,\mathrm{M}^{A|\gamma)}\;\varphi^B{}_{,\gamma\alpha_1\ldots\alpha_k} \\ &\quad - \sum_{n=0}^{\infty}\sum_{k=0}^{2}(n+2)\,\mathrm{C}_{AMB_1\ldots B_n:N}{}^{(\alpha_1\ldots\alpha_k|}\,\mathrm{M}^{A|\gamma)}\;\varphi^N{}_{,\gamma\alpha_1\ldots\alpha_k}k^M k^{B_1}\cdots k^{B_n} + \\ &\quad - \sum_{n=0}^{\infty}(n+2)(n+1)\,\mathrm{C}_{AMB_1\ldots B_n}\,\mathrm{M}^{A\gamma}\left(\partial_\gamma k^M\right)k^{B_1}\cdots k^{B_n}\,. \end{aligned} \tag{3.152}$$

Observe that no acceleration term $\ddot{\varphi}$ appears in the equation, as expected for a constraint equation.

Even without having solved the closure equations explicitly, we can make an interesting observation immediately: the fourth term in the equation is of third derivative order for $k = 2$, meaning that the differential order would be higher than two (ignoring C and C_A for now). However, due to $(\mathbf{C15_N})$ and $(\mathbf{C13_N})$, we have

$$0 = C_{B_1 \dots B_N : A}{}^{(\alpha\beta} M^{B_1|\gamma)} \tag{3.153}$$

and thus see that the summand at $k = 2$ disappears for any gravitational Lagrangian that is a solution of the gravitational closure relations. This eliminates the third spatial derivative contributions that stem from $C_{B_1 \dots B_N}$ for $N \geq 2$.

The same observation can be made for the vector constraint as well. Here, the differential equation reads

$$\begin{aligned}
\frac{\delta \mathcal{S}_{\text{matter}}}{\delta N^\mu(x)} = & \sum_{n=0}^{\infty} (n+1)\, C_{AB_1 \dots B_n} \left(\delta^A_B \delta^\gamma_\mu + F^A{}_\mu{}^\gamma{}_{:B} \right) \varphi^B{}_{,\gamma}\, k^{B_1} \cdots k^{B_n} \\
& + \sum_{k=0}^{\infty} C_{A:B}{}^{(\alpha_1 \dots \alpha_k|} F^A{}_\mu{}^{|\gamma)} \varphi^B{}_{,\gamma\alpha_1 \dots \alpha_k} \\
& + \sum_{n=0}^{\infty} \sum_{k=0}^{2} (n+2) C_{AMB_1 \dots B_n : N}{}^{(\alpha_1 \dots \alpha_k|} F^A{}_\mu{}^{|\gamma)} \varphi^N{}_{,\gamma\alpha_1 \dots \alpha_k} k^M k^{B_1} \cdots k^{B_n} \\
& + \sum_{n=0}^{\infty} (n+2)(n+1) C_{AMB_1 \dots B_n} F^A{}_\mu{}^\gamma \left(\partial_\gamma k^M \right) k^{B_1} \cdots k^{B_n} \,.
\end{aligned} \tag{3.154}$$

If we now apply the closure equation $(\mathbf{C12_N})$ to the fourth term in (3.154), we find that all higher-order spatial derivatives to the partial differential equation enter only due to the output coefficients C and C_A. Last but not least, we can insert the expansion in the evolutionary equations

$$\begin{aligned}
\frac{\delta \mathcal{S}_{\text{matter}}}{\delta \varphi^A(x,t)} = & \sum_{n=0}^{\infty} (n+1) \Bigg[(n+2)\, C_{ABM_1 \dots M_n} \left(\dot{k}^B - N^\mu k^B{}_{,\mu} \right) \\
& + \sum_{m=0}^{\infty} C_{AM_1 \dots M_n : B}{}^{\alpha_1 \dots \alpha_m} \left(\dot{\varphi}^B_{,\alpha_1 \dots \alpha_m} - N^\mu \varphi^B_{,\mu\alpha_1 \dots \alpha_m} \right) \\
& + C_{BM_1 \dots M_n} \left((\partial_\gamma N) M^{B\gamma}{}_{:A} - (\partial_\gamma N^\mu) \left(\delta^B_A \delta^\gamma_\mu + F^B{}_\mu{}^\gamma{}_{:A} \right) \right) \Bigg] k^{M_1} \cdots k^{M_n} \\
& - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \partial_{\alpha_1 \dots \alpha_m} \left(N\, C_{M_1 \dots M_n : A}{}^{\alpha_1 \dots \alpha_m} k^{M_1} \cdots k^{M_n} \right) \,.
\end{aligned} \tag{3.155}$$

Due to the $\dot{k}^B$ term, we finally obtain 2nd order time derivatives of the gravitational degrees of freedom – as is expected for evolutionary equations.

In principle, it is possible to use the gravitational closure equations to further simplify the evolutionary equations – even at this abstract level. However, there is no clear benefit as the coefficient functions appearing in the differential equations will still be largely undetermined and need to be derived then, case by case, with the help of the closure equations. We will thus leave the result (3.155) as is and yet again summarise the equations of motion again in the following box on the next page:

DEFINITION GRAVITATIONAL EQUATIONS OF MOTION

The gravitational equations of motion, coupled to a matter action $\mathcal{S}_{\text{matter}}$ are given by the system of partial differential equations posed by four contraint equations

$$\begin{aligned}
\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N(x,t)} &= \frac{\partial\mathcal{L}}{\partial k^A}k^A - \mathcal{L} - \partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial k^A}\mathrm{M}^{A\gamma}\right), \\
\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N^\alpha(x,t)} &= \frac{\partial\mathcal{L}}{\partial k^A}\varphi^A{}_{,\mu} + \partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial k^A}\mathrm{F}^A{}_\mu{}^\gamma\right),
\end{aligned}$$

and F evolutionary equations

$$\begin{aligned}
\frac{\delta\mathcal{S}_{\text{matter}}}{\delta\varphi^A(x,t)} = &\left(\partial_t - N^\mu\partial_\mu\right)\frac{\partial\mathcal{L}}{\partial k^A} \\
&+ \frac{\partial\mathcal{L}}{\partial k^B}\left((\partial_\gamma N)\mathrm{M}^{B\gamma}{}_{:A} - (\partial_\gamma N^\mu)\left(\delta^B_A\delta^\gamma_\mu + \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right)\right) \\
&- \sum_{n=0}^{\infty}\int \mathrm{d}^3y\, N(t,y)\frac{\delta\mathrm{C}_{B_1\ldots B_n}[\varphi(t,y)]}{\delta\varphi^A(t,x)}k^{B_1}(t,y)\ldots k^{B_n}(t,y)\,.
\end{aligned}$$

3.4.2 Causal compatibility requirement

Once the equations of motion are derived, the obvious next step is to derive the principal polynomial of the gravitational sector. While the idea that gravity shall possess the same principal polynomial as the matter sector has very well flown into the derivation of the gravitational closure equations, it is a priori not clear whether this is generally ensured by them.

Clearly, for there to be a chance that the principal polynomial of the gravitational field equation is fixed by the gravitational closure equations, all components of the matter polynomial need to be contained in some form in the differential equations. We will thus analyze in detail what information of the principal polynomial enters the gravitational closure equations. Obviously, the three projections three projections $P(\epsilon^0,\cdot,\epsilon^0) = 1$ and $P(\epsilon^\alpha,\epsilon^0,\ldots,\epsilon^0) = 0$ (implicitly) and $\mathrm{p}^{\alpha\beta} = P(\epsilon^\alpha,\epsilon^\beta)$ (explicitly) appeared as we used them in our derivation in the previous sections. For the higher-order components it turns out that we can employ the normal deformation coefficient to relate them to the lower-order components. For the component with N spatial indices, i.e. defined via

$$\mathrm{p}^{\alpha_1\ldots\alpha_N}(x) := P^{m_1\ldots m_{\deg P}}\left(X_t(x)\right)\epsilon^{\alpha_1}_{m_1}(x)\cdots\epsilon^{\alpha_N}_{m_N}(x)\,\epsilon^0_{\alpha_{N+1}}(x)\cdots\epsilon^0_{\alpha_{\deg P}}(x)\,, \tag{3.156}$$

straight-forward calculation using equation (3.55) yields that the normal deformation coefficient reads

$$\mathbf{M}^{\alpha_1\ldots\alpha_N\gamma} = N\left(\deg P - 1\right)\mathrm{p}^{\gamma(\alpha_1}\mathrm{p}^{\alpha_2\ldots\alpha_N)} - \left(\deg P - N\right)\mathrm{p}^{\gamma\alpha_1\ldots\alpha_N} \tag{3.157}$$

On the other hand, we know that once we spell out the components of the principal polynomial, we find that these are functions of our geometric fields on the screen manifold, i.e.

$$\mathrm{p}^{\alpha_1\ldots\alpha_N}\left(g^A\right)\,. \tag{3.158}$$

As a result, we know from the chain rule of the normal deformation coefficient $\mathbf{M}^{\gamma}$, compare equation (3.56), that this can be written as

$$\mathbf{M}^{\alpha_1 \ldots \alpha_N \gamma} = \frac{\partial \mathrm{p}^{\alpha_1 \ldots \alpha_N}}{\partial g^{\mathcal{A}}} \mathbf{M}^{\mathcal{A}\gamma} \,. \tag{3.159}$$

This can be carried over to our canonical phase space formulation such that we obtain the quite interesting differential equation between the components of the principal polynomial:

$$\frac{\partial \mathrm{p}^{\alpha_1 \ldots \alpha_N}}{\partial \varphi^A}(\varphi) \mathrm{M}^{A\gamma}(\varphi) = N \left(\deg P - 1\right) \mathrm{p}^{\gamma(\alpha_1|}(\varphi) \mathrm{p}^{|\alpha_2 \ldots \alpha_N)}(\varphi) - \left(\deg P - N\right) \mathrm{p}^{\gamma\alpha_1 \ldots \alpha_N}(\varphi) \,. \tag{3.160}$$

For $N = 0$ and $N = 1$ all terms vanish due to the normalization and annihilation condition, such that no information appears. For $N > 2$, equation (3.160) can then be employed to express the higher-order components in terms of the lower order ones and by the normal deformation coefficient $\mathrm{M}^{A\gamma}$. For instance, we find that

$$\mathrm{p}^{\alpha\beta\gamma}(\varphi) = -\frac{1}{\deg P - 2} \frac{\partial \mathrm{p}^{(\alpha\beta|}}{\partial \varphi^A}(\varphi) \mathrm{M}^{A|\gamma)}(\varphi) \,. \tag{3.161}$$

Similarly, this procedure can be repeated for the remaining components such that we can recontruct the entire principal polynomial. In particular, this calculation indicates that indeed *all components* of the principal polynomial are present in the gravitational closure equations, at least implicitely, due to our two input coefficients $\mathrm{p}^{\mu\nu}$ and $\mathrm{M}^{A\gamma}$.

Let us still suppose now for a moment that the gravitational closure equations do not fix the principal polynomial P_G of the gravitational field equations to the expression P_M obtained from the matter fields: This would imply, from a causal perspective, that if we prescribe our matter fields and our geometry on a common initial data surface, they could or would evolve to separate surfaces – making a common description of dynamics impossible, unless the cone $\mathcal{C}_{\mathrm{grav},x}$ of our gravitational theory contains at each spacetime point *the whole cone* of the matter sector $\mathcal{C}_{\mathrm{matter},x}$. We thus suspect that if we further impose that the two principal polynomials agree, a condition that can be at least symbolically summarised as

$$P_G \stackrel{!}{=} P_M \,, \tag{3.162}$$

we end up with further restrictions on the gravitational Lagrangian. The favourable situation would be to turn this condition, again, into a set of partial differential equations for the output coefficients.

However, this is a rather non-trivial condition for three reasons: the first one is that the gravitational equations of motion are non-linear and generally not even quasi-linear, i.e. linear in the highest derivative order of the degrees of freedom. This complicates the derivation of the principal polynomial, already in the case of general relativity. The second reason is that we deal with a gauge symmetry and, as a result, need to repeat the steps from 2.2.1 for spacetime diffeomorphisms. The third and last reason is that, in order to accurately judge the principal polynomial, we must apply the Cartan-Kuranishi to bring the gravitational equations of motion to involutive form. We leave this up for future research.

3.5 GENERAL PROPERTIES OF SOLUTIONS

We will now dedicate the last part of this chapter to a general discussion of the gravitational closure equations and their solution space before moving on to exact solutions in chapter 4 and a perturbative treatment in chapter 5. While this system of countably infinite linear partial differential equations is a quite challenging system to solve in practice, still, some statements can be made already at an abstract level.

We will proceed in the following fashion: first, we will discuss the covariance part of the closure equations in more detail and see that they constrain the solution space to finitely many curvature invariants, that is, differential invariants that can be constructed from the geometry and its derivatives.

Second, we can enter a general discussion of the selective part of the closure equations. One then finds that the system posed by the $\{\mathcal{H},\mathcal{H}\}$ bracket has a more involved structure than the covariance part of the closure equations that stem from the $\{\mathcal{D},\mathcal{H}\}$ bracket. Since they establish a link between the different output coefficient orders, they prove to be *the* essential component to further restrict the gravitational Lagrangian. Equipped with everything learned from the covariance part, as well as the selective part of the closure equations, we can then set up a general solution strategy for how to approach the system of partial differential equations.

Lastly, we will make use of the arbitrariness of the parametrization. We will discuss cases where a reparametrization of the gravitational fields on the screen manifold to a normal form can be performed, formulated in terms of a metric, a vector and multiple scalar fields. Although it is not possible to provide the general solution to the gravitational closure equations for this normal form parametrization, we will present some helpful insights that can be put to good use when solving the closure equations.

3.5.1 Analysis of the covariance part of the closure equations

The covariance part of the closure equations – that is the subsystem of the closure equations ($\mathbf{C1}$), ($\mathbf{C2}$), ($\mathbf{C8_N}$), ($\mathbf{C9_N}$), ($\mathbf{C10_N}$), ($\mathbf{C11_N}$) and ($\mathbf{C12_N}$) – pose an interesting subsystem that is worth analyzing on its own[7]. These are linear, first-order differential equations, for which it is a much simpler task to analyze the solution space. In particular, it will allow us to show that this subsystem is, in the formulation presented on page 84, already an involutive system. The involutivity will enable us then to derive the number of functionally independent curvature invariants from combinatorial considerations.

For this analysis, we will first see that the equations can be simplified by introducing a de-densitization and rewriting the output coefficient. We will, in particular, analyze the space of all scalar densities of weight w in more detail. Afterwards, we can tackle the problem of showing that the covariance part of the closure equations are involutive – which ultimately stems from the $\{\mathcal{D}(\vec{N}),\mathcal{D}(\vec{M})\}$ relation.

De-densitization of all output coefficients

In general relativity, the fact that the density factor $1/\sqrt{-\det g^{\cdot\cdot}}$ is non-polynomial in the degrees of freedom is responsible for much of the complexity in the solution of the gravitational closure equations, and we expect very much the same for any gravitational theory. We can, of course, take much out of this

[7] Its four-dimensional analogue for the Lagrangian of first-order field theories was for instance already discussed in Gotay and Marsden (2001).

complexity by constructing a de-densitization for our geometry under investigation and de-densitize our output coefficients first and solve the closure equations than for the obtained objects.

In order to construct a de-densitization for any geometry, we will look at the transformational behavior of a density χ of weight w. From the definition of the Lie derivative, as well as equation (3.86), it is clear that we get

$$\begin{aligned} \left\{\chi(\varphi), \widehat{\mathcal{D}}(\vec{N})\right\} &= \left(\mathcal{L}_{\vec{N}}\chi\right) \\ N^\mu \chi_{:A}\varphi^A{}_{,\mu} + \chi_{:A}\mathrm{F}^A{}_\mu{}^\gamma &= N^\mu\, \chi_{:A}\varphi^A{}_{,\mu} - w \cdot \left(\partial_\mu N^\mu\right)\chi\,. \end{aligned} \tag{3.163}$$

The respective first terms on both sides of the equation obviously cancel each other. What remains is a linear partial differential equation for the function χ that all scalar densities of weight w need to solve[8]. The solution will not be unique since we, for instance, can multiply the solution by any scalar function and obtain another solution. This is precisely the same issue as in our 2-form treatment of the area metric in chapter 1 in a different disguise.

This linear first-order partial differential equation can be solved with the typical solution methods for a given input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$. Of course, this analysis of densities as solutions of a differential equation is rather of theoretical interest for the general study of the gravitational closure equations. Typically, it is usually rather well-understood what is meant by scalar densities.

We will now investigate this system, and its solution space, in more detail before we generalise the steps to the covariance part of the closure equations. For this, we again rest our discussion on the formal analysis of involutive systems and the Cartan-Kuranishi algorithm as described in section 2.2.2.

DEFINITION SCALAR DENSITIES

We say a function χ of the gravitational degrees of freedom is a *scalar density of weight w* if it is a solution of the partial differential equations

$$\chi_{:A}\mathrm{F}^A{}_\mu{}^\gamma = -w \cdot \chi\delta^\gamma_\mu\,.$$

A *scalar* is a density of weight 0.

Suppose we want to understand the solution space from a formal perspective and without having a concrete expression for the input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$. In that case, it is essential for the analysis that the differential equation is involutive. We will thus perform the Cartan-Kuranishi algorithm explicitly. Remember that this corresponds to successive steps of prolonging the differential equation to higher orders and checking if any integrability conditions can be formed.

Step 1: The rank condition of the geometric symbol For the first step in the algorithm, we have to read off the number $\beta_1^{(k)}$ of principal coefficients of class k, i.e. the number of formal expansion coefficients $\chi_{:A}$ of a solution that one can solve for in the equation.

[8] Clearly, these are simply the covariance part of the closure equations for a functional that only depends on the degrees of freedoms and none of their spatial derivatives.

In our case, as we deal with a first-order system, there are F classes in total and one class per degree of freedom. As a result, one finds that the coefficient $\beta_1^{(k)}$ for a principal coefficient of class k can either be zero or one. Which coefficients we can solve for in the differential equation is governed by the input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$. From linear algebra it is clear that the rank $r := \mathrm{rank}(\mathrm{F}^A{}_\mu{}^\gamma)$ of the coefficient can be *at most* nine, such that we can solve for at most r of the Taylor coefficients at order 1. Since we are free to label the classes, we will now assume that, without loss of generality, we have ordered them such that we can solve for the r coefficients of highest classes. The sum of the coefficients that appears in the rank condition for integrability thus reads

$$\begin{aligned}\sum_{k=1}^{F} k \cdot \beta_1^{(k)} &= \sum_{k=F-r}^{F} k \\ &= \frac{F(F+1)}{2} - \frac{(F-r)(F-r+1)}{2}\,. \end{aligned} \tag{3.164}$$

If we want to check whether the geometric symbol, i.e. our input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$, is involutive, we need to calculate the rank of the geometric symbol of the prolongation. The prolonged differential equation reads

$$\chi_{:AB}\mathrm{F}^A{}_\mu{}^\gamma + \left(\mathrm{F}^A{}_\mu{}^\gamma{}_{:B} + w \cdot \delta^A_B \delta^\gamma_\mu\right) \chi_{:A} = 0\,. \tag{3.165}$$

from which one finds the highest order coefficient

$$\delta_B^{(M}\mathrm{F}^{N)}{}_\mu{}^\gamma\,. \tag{3.166}$$

The condition for the symbol to be involutive is that the rank of the geometric symbol of the prolongation equals (compare section 2.2.2)

$$\mathrm{rank}\left(\delta_B^{(M}\mathrm{F}^{N)}{}_\mu{}^\gamma\right) \stackrel{!}{=} \frac{F(F+1)}{2} - \frac{(F-r)(F-r+1)}{2}\,. \tag{3.167}$$

In principle, we can then check this condition for each case by explicit straight-forward calculation. At this abstract level, without knowing the components of the input coefficients, this is, however, non-trivial. Luckily, we do not need to calculate the rank explicitly: it turns out that one can come up with an abstract argument for any first-order system of differential equations for a single dependent variable (compare remark 7.1.29 in Seiler (2009)). Since both conditions are fulfilled for our scalar density, we can conclude that the symbol is involutive.

Step 2: Integrability conditions The second step in the Cartan-Kuranishi algorithm is to show that no integrability condition exists, that is, a first-order equation that can be obtained by a linear combination of the prolongation. This is clearly a more complex task since we must rule out that we cannot manipulate the second-order prolongations into an additional first-order equation.

For first-order equations, this can be elegantly checked geometrically: the left-hand side of (3.163) corresponds to the vector field $\mathrm{F}^A{}_\mu{}^\gamma\,\partial_A$ that is applied to the function χ. An integrability condition appears if the commutator cannot be written as a linear combination of the vector fields of our system. Given that such a condition appears, we can add it to our system and repeat the procedure. Since there are at most F many linearly independent vector fields, it is clear that the procedure will terminate after some iterations.

If we now spell out the commutator explicitly, we obtain the following expression

$$\left[\mathrm{F}^{A}{}_{\mu}{}^{\gamma}\,\partial_{A},\mathrm{F}^{B}{}_{\nu}{}^{\epsilon}\,\partial_{B}\right] = \left(\mathrm{F}^{A}{}_{\mu}{}^{\gamma}\,\mathrm{F}^{B}{}_{\nu}{}^{\epsilon}{}_{:A} - \mathrm{F}^{A}{}_{\nu}{}^{\epsilon}\,\mathrm{F}^{B}{}_{\mu}{}^{\gamma}{}_{:A}\right)\partial_{B}\,. \tag{3.168}$$

Comparing the right hand side with the input coefficient identity (3.120a), we see that the commutator reads

$$\left[\mathrm{F}^{A}{}_{\mu}{}^{\gamma}\,\partial_{A},\mathrm{F}^{B}{}_{\nu}{}^{\epsilon}\,\partial_{B}\right] = \left(\mathrm{F}^{A}{}_{\nu}{}^{\gamma}\delta^{\epsilon}_{\mu} - \mathrm{F}^{A}{}_{\mu}{}^{\epsilon}\delta^{\gamma}_{\nu}\right)\partial_{A}\,, \tag{3.169}$$

which is indeed a linear combination of our original equations. If we take into account the inhomogeneous part of the equation $-w\cdot\chi\,\delta^{\gamma}_{\mu}$ both terms cancel each other. As a result, no integrability conditions are obtained, the Cartan-Kuranishi algorithm terminates after a single iteration and the differential equation is already involutive.

Next, we can analyse the size of the formal solution space, i.e. the space of formal power series solutions of the differential equations for scalar densities. For this, one basically counts the number of parametric coefficients – coefficients we cannot solve for at the considered differential orders – and combinatorially relates this to the Taylor expansion of smooth functions of one or more variables.

In our case, following chapter 8 of Seiler (2009) (and in particular corollary 8.2.12), one calculates that the number f_k of smooth functions of k variables is given by

$$f_F = 1-\beta_1^{(F)}\,, \tag{3.170a}$$

$$f_k = \beta_1^{(k+1)} - \beta_1^{(k)} \qquad \text{for} \qquad 1\leq k < F\,. \tag{3.170b}$$

Since we have defined our classes such that we can solve for the r coefficients of highest class, that is in the terminology of Seiler (2009) a δ-regular coordinate system, we know that

$$\beta_1^{(k)} = \begin{cases} 1 & \text{for} \quad F-r\leq k\leq F\,,\\ 0 & \text{else}\,,\end{cases} \tag{3.171}$$

and we can directly calculate the number of arbitrary functions. Then plugging the $\beta_1^{(k)}$ coefficients into equation (3.170) gives that the general solution is a single undetermined function of $F-r$ slots. This can also be understood from a geometric perspective: the differential equation (3.163) restricts r of the F first-order components. The remaining $F-r$ ones must be provided as boundary data. As a result, we find that we can construct $F-r$ functionally independent solutions to the differential equation and the general solution will be an undetermined function f of these $F-r$ densities χ^I $(I=1,\ldots,F-r)$ with

$$\frac{\partial f}{\partial \chi^I}\chi^I = f\,. \tag{3.172}$$

An interesting situation occurs for theories where the kernel of $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ is trivial, i.e. we have at most 9 degrees of freedom. In this case, the formal solution space is finite in the sense that the solution contains no undetermined functions. For general relativity, this is exactly the case: it can be checked by straightforward calculation that the rank of the input coefficient is equal to 6, i.e. the kernel is indeed trivial. As a result, one finds that *all* scalar densities will be proportional to $\left(-\det g^{\cdot\cdot}\right)^{-\frac{w}{2}}$. In general, this is not true and – just as for the pre-metric 2-form treatment in the introduction – different density terms can be formed.

Note that we can trace this arbitrariness, again, to the occurrence of scalars built from the geometry. Clearly, if we scale a density χ of weight w by a scalar function σ we obtain another density of weight w, since

$$\partial_A(\sigma \cdot \chi)\mathrm{F}^A{}_\mu{}^\gamma = \chi \cdot \sigma_{:A}\mathrm{F}^A{}_\mu{}^\gamma + \sigma\chi_{:A}\mathrm{F}^A{}_\mu{}^\gamma = -w\,(\sigma \cdot \chi)\,\delta^\gamma_\mu\,. \tag{3.173}$$

Conversely, given two scalar densities χ, ξ of weight w they yield a scalar via $\sigma := \chi/\xi$. We will look at these scalar functions now in some more detail.

From the definition, we see that a function σ is a scalar given that the gradient $\sigma_{:A}$ is an element from the kernel of $\mathrm{F}^A{}_\mu{}^\gamma$. As stated before, in the case of general relativity, we find that the kernel is trivial, which is the well-known fact that we cannot construct any scalar from the metric degrees of freedom[9]. For an area metric, the input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ is a 17×9 matrix of rank 9, such that we deal with an eight-dimensional kernel. Indeed, we can construct eight distinct scalars from the three projections of the area metric to the screen manifold by their traces and determinants (this will be described in further detail in chapter 4.2).

Conversely, one may ask whether it is possible to construct a scalar from a given element $n_A(\varphi)$ from the kernel of $\mathrm{F}^A{}_\mu{}^\gamma$. A necessary compatibility condition for the existence of such a solution is that

$$n_{A:B} - n_{B:A} = 0\,. \tag{3.174}$$

Since it is rather involved to find a basis for the kernel of the input coefficient for a gravitational setup and typically more straightforward to derive an expression for the different scalars we can construct from the geometric fields, we will leave this open for further research. It is nonetheless remarkable how the input coefficient allows us to approach the definition of scalar densities constructively.

We can now move back to our output coefficients and use everything we learnt above. Clearly, with the help of the input coefficient $\mathrm{p}^{\cdot\cdot}(\varphi)$ we can construct the following scalar density of weight 1 for *any* physically viable gravitational theory:

$$\chi(\varphi) = \frac{1}{\sqrt{-\det \mathrm{p}^{\cdot\cdot}(\varphi)}}\,. \tag{3.175}$$

Any other density ξ of weight 1 then differs by an arbitrary $R := \max(0, F - r)$ dimensional scalar function Ψ of the functionally independent screen manifold scalars $\sigma^{(i)}$, with $i = 1, \ldots, R$.

All this means that we can ultimately write the output coefficients as

$$\mathrm{C}_{A_1 \ldots A_N}[\varphi] = \frac{\Psi_{(N)}\left(\sigma^{(1)}(\varphi), \ldots, \sigma^{(R)}(\varphi)\right)}{\sqrt{-\det \mathrm{p}^{\cdot\cdot}(\varphi)}}\,\widetilde{\mathrm{C}}_{A_1 \ldots A_N}[\varphi]\,, \tag{3.176}$$

with each containing an, so far, undetermined function of the $R := \max(0, F - r)$ scalars.

[9] One may, of course, take the trace of the metric. However, this gives the dimension of the screen manifold and is, thus, independent of the degrees of freedom.

We can then transform the covariance part of the closure equations into differential equations for $\widetilde{\mathrm{C}}_{A_1\ldots A_N}$. The covariance part of the closure equations $(\mathbf{C8_N})$, $(\mathbf{C9_N})$, $(\mathbf{C11_N})$, $(\mathbf{C12_N})$ will be unaffected by this transformation. However, we can eliminate the $\mathrm{C}_{B_1\ldots B_N}\delta^\gamma_\mu$ terms from the equations $(\mathbf{C1})$, $(\mathbf{C2})$ and $(\mathbf{C10_N})$ by using equation (3.163). This will, in particular, turn the system for the scalar output coefficient into a homogeneous one.

Before we move on to the involutivity analysis of the covariance part of the closure equations, let us quickly consider a term of the form

$$\Lambda(\varphi) := \frac{\Psi_{(0)}\left(\sigma^{(1)}(\varphi),\ldots,\sigma^{(R)}(\varphi)\right)}{\sqrt{-\det \mathrm{p}^{\cdot\cdot}(\varphi)}}\,, \tag{3.177}$$

that represents a generalization of the cosmological constant term of general relativity to a cosmological *function* and depends on the geometric degrees of freedom. One immediately finds that this does trivially solve all gravitational closure equations but $(\mathbf{C5})$. This equation seems to relate the term to the output coefficient C_A and the input coefficient $\mathrm{M}^{A\gamma}$ via

$$0 = 2\,\partial_\mu\left(\mathrm{C}_A\mathrm{M}^{A[\mu|}{}_{:B}\mathrm{M}^{B|\gamma]}\right) - 2(\deg P - 1)\mathrm{p}^{\rho\gamma}\left[\mathrm{C}_A\varphi^A{}_{,\rho} + \partial_\mu\left(\mathrm{C}_A\mathrm{F}^A{}_\rho{}^\gamma\right)\right] + \frac{\partial\Lambda}{\partial\varphi^A}(\varphi)\mathrm{M}^{A\gamma}(\varphi)\,. \tag{3.178}$$

However, by inspecting the equation closely, one finds that the terms with C_A all contain at least one $\varphi^A{}_{,\sigma}$ term, such that we, in fact, can conclude that the cosmological function term must vanish separately, i.e.

$$0 = \frac{\partial\Lambda}{\partial\varphi^A}\mathrm{M}^{A\gamma}\,. \tag{3.179}$$

This further constrains the gradient of the function Λ to the kernel of the input coefficient $\mathrm{M}^{A\gamma}$. This again shows the power of the gravitational closure equations compared to a naïve construction of terms that merely transform properly as tensorial expressions.

Involutivity of the covariance part of the closure equations

We will now generalise the arguments made in the previous section, which allowed us to show that the differential equation (3.163) for scalar densities is involutive and apply it to the covariance part of the closure equations. It is not so surprising that this is possible since the covariance part of the closure equations are, in fact, simply the generalisation to tensorial objects constructed from the degrees of freedom *and their spatial derivatives*.

Before we perform the Cartan-Kuranishi algorithm, we will first rewrite the covariance part of the closure equations for the de-densitized output coefficients introduced in the previous section. By insertion of the definition (3.176), we can eliminate the density term in the covariance part of the closure equations and drop the contributions by the scalar term $\Psi_{(N)}$. Collecting all terms, we find the following system of partial differential equations for all of the output coefficients:

$$0 = -\,N\,\widetilde{\mathrm{C}}_{A(B_1\ldots B_{N-1}}\mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)} + \sum_{K=0}^{\infty}\left[(K+1)\,\widetilde{\mathrm{C}}_{B_1\ldots B_N:A}{}^{\gamma\alpha_1\ldots\alpha_K}\,\varphi^A{}_{,\mu\alpha_1\ldots\alpha_K} - \widetilde{\mathrm{C}}_{B_1\ldots B_N:A}{}^{\alpha_1\ldots\alpha_K}\,\mathrm{F}^A{}_\mu{}^\gamma{}_{,\alpha_1\ldots\alpha_K}\right]\,, \tag{3.180a}$$

and for $L \geq 2$

$$0 = \sum_{K=0}^{\infty} \left[\binom{K+L}{K} \widetilde{\mathrm{C}}_{B_1 \dots B_N : A}{}^{\beta_1 \dots \beta_L \alpha_1 \dots \alpha_K} \, \varphi^A{}_{,\mu \alpha_1 \dots \alpha_K} - \binom{K+L-1}{K} \widetilde{\mathrm{C}}_{B_1 \dots B_N : A}{}^{\alpha_1 \dots \alpha_K (\beta_1 \dots \beta_{L-1}|} \, \mathrm{F}^A{}_\mu{}^{|\beta_L)}{}_{,\alpha_1 \dots \alpha_K} \right] , \tag{3.180b}$$

for $N \geq 0$ – that is for any of the output coefficients. Here we ignored for a moment that for the output coefficients $\widetilde{\mathrm{C}}_{AB}, \widetilde{\mathrm{C}}_{ABC}, \dots$ we already know that they depend up to 2nd derivatives of φ^A in order to treat all coefficients in the same fashion.

Suppose we introduce the differential operators, for $L \geq 1$, as

$$\mathbf{X}_\mu^{\gamma_1 \dots \gamma_L} = \sum_{K=0}^{\infty} \left[\binom{K+L}{K} \varphi^A{}_{,\mu \alpha_1 \dots \alpha_K} \partial_A{}^{\gamma_1 \dots \gamma_L \alpha_1 \dots \alpha_K} - \binom{K+L-1}{K} \mathrm{F}^A{}_\mu{}^{(\gamma_1|}{}_{,\alpha_1 \dots \alpha_K} \, \partial_A{}^{|\gamma_2 \dots \gamma_L) \alpha_1 \dots \alpha_K} \right] , \tag{3.181}$$

with the derivative in directions of the jet coordinates being defined as $\partial_A{}^{\alpha_1 \dots \alpha_K} := \partial / \partial \varphi^A{}_{,\alpha_1 \dots \alpha_K}$. Then we can compactly write the covariance part of the closure equations for the de-densitized output coefficients as

$$0 = \mathbf{X}_\mu^{\gamma} \, \widetilde{\mathrm{C}}_{B_1 \dots B_N} - N \, \widetilde{\mathrm{C}}_{A(B_1 \dots B_{N-1}} \mathrm{F}^A{}_\mu{}^\gamma{}_{:|B_N)} \, , \tag{3.182a}$$

$$0 = \mathbf{X}_\mu^{\gamma_1 \dots \gamma_L} \, \widetilde{\mathrm{C}}_{B_1 \dots B_N} \qquad , \text{for} \qquad L \geq 2 \, . \tag{3.182b}$$

The inhomogeneous part in (3.182a), for $N \geq 1$, is needed to guarantee that each slot of the output coefficient transforms according to the lifted action of spatial diffeomorphisms to $\mathrm{T}^*\Phi$. We can now perform Cartan-Kuranishi completion as presented in the previous section. This is particularly simple for the scalar output coefficient since we deal with a linear first-order differential equation in a single dependent variable. Additionally, the inhomogeneous part vanishes, which further simplifies the derivations.

Step 1: Involutivity of the geometric symbol The first step is, again, to analyze the geometric symbol of the first-order equation and check the rank condition of the prolongation. The geometric symbol, in the following denoted by $\eta_\mu^{\gamma_1 \dots \gamma_L A}{}_{\lambda_1 \dots \lambda_K}$, is the matrix with $\frac{3L(L+1)}{2}$ rows from the indices μ and the totally symmetric multi-index $(\gamma_1 \dots \gamma_L)$ and columns of the first-order jet variables $\varphi^A{}_{,\lambda_1 \dots \lambda_M}$ with $M \geq 0$. Its components are given by

$$\eta_\mu^{\gamma_1 \dots \gamma_L A}{}_{\lambda_1 \dots \lambda_M} = \begin{cases} -\mathrm{F}^A{}_\mu{}^{(\gamma_1} \delta^{\gamma_2}_{(\lambda_1} \cdots \delta^{\gamma_L)}_{\lambda_{L-1})} & \text{for} \quad M = L-1 \, , \\ \binom{M}{M-L} \varphi^A{}_{,\mu(\lambda_1 \dots \lambda_{M-L}} \delta^{(\gamma_1}_{\lambda_{M-L+1}} \cdots \delta^{\gamma_L)}_{\lambda_M)} & \text{for} \quad M \geq L \, , \\ 0 & \text{else} \, . \end{cases} \tag{3.183}$$

We now need to calculate the number of principal coefficients of the different classes. Since we also need to account for the spatial derivatives of the degrees of freedom (in principle in the limit to infinitely many derivatives) here, we deal with more classes than in the previous section. If we count the number of independent variables, restricting to q many spatial derivatives at most, we find

$$\begin{aligned}
\#(\text{independent variables})_q &= \#(\varphi^A) + \#(\varphi^A{}_{,\mu}) + \#(\varphi^A{}_{,\mu\nu}) + \dots \\
&= F + 3\cdot F + 6 \cdot F + \dots \\
&= F \cdot \sum_{k=0}^{q} \binom{k+2}{k} \\
&= \frac{F}{6}(q+1)(q+2)(q+3)\,,
\end{aligned} \tag{3.184}$$

which equals the number of classes we deal with. What simplifies the discussion is the realisation that the analysis is independent of the output coefficient that is considered: since the geometric symbol is the same for all of the coefficients, we can restrict ourselves, without loss of generality, to the scalar output coefficient $\widetilde{\mathrm{C}}$ since all the linear algebraic steps necessary in the Gauss Jordon algorithm to bring the geometric symbol to reduced row echelon form are independent of the considered output coefficient.

Now the arguments are exactly the same as in the previous section: we have one formal series expansion coefficient of the solution for each class and, thus, have that $\beta_1^{(k)}$ is either zero or one. Even further, since we have a first-order differential equation with a single dependent variable – our scalar functional $\widetilde{\mathrm{C}}[\varphi]$ – the geometric symbol is automatically involutive (again by the argument by Seiler (2009)).

From the perspective of the first differential order terms in each covariance part of the closure equation, the components of the output coefficients are treated independently from each other. As a result, this means that the rank condition is fulfilled for all equations in the covariance part.

Step 2: Integrability conditions Showing that no integrability conditions occur in the Cartan-Kuranishi completion of the covariance part of the closure equations requires two steps: we again show that the first-order term in the differential equation, given by the operator $\mathbf{X}_\mu^{\gamma_1\dots\gamma_L}$ possesses a group structure that allows us to express the commutator by a linear combination of the first-order equations. The second step then requires us to show that the application of the differential operator on the inhomogeneous part in the $L=1$ equation does not yield additional first-order equations. We will see in the following that both statements are, indeed, fulfilled for the covariance part of the closure equations.

For the first part, we can derive the commutator with the help of the $\left\{\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})\right\}$ algebra relation.

THEOREM

The commutator of the first-order differential operator $\mathbf{X}_\mu^{\gamma_1\dots\gamma_N}$, for $N \geq 1$, is given by

$$\begin{aligned}
[\mathbf{X}_\mu^{\alpha_1\dots\alpha_N}, \mathbf{X}_\nu^{\beta_1\dots\beta_M}] =& \binom{N+M-1}{M} \delta_\nu^{(\alpha_1}\mathbf{X}_\mu^{\alpha_2\dots\alpha_N)\beta_1\dots\beta_M} \\
&- \binom{N+M-1}{N} \delta_\mu^{(\beta_1}\mathbf{X}_\nu^{\beta_2\dots\beta_M)\alpha_1\dots\alpha_N}\,.
\end{aligned} \tag{3.185}$$

In order to prove that the expression is correct, we move back to the action of the supermomentum

$\widehat{\mathcal{D}}(\vec{N})$ on functionals of the gravitational degrees of freedom. If we spell out the action on a functional, we see that the differential operator $\mathbf{X}_\mu^{\gamma_1\dots\gamma_L}$ naturally appears as the derivative operator in front of the derivatives of the shift, since

$$\begin{aligned}
\left\{f[\varphi(\cdot)],\widehat{\mathcal{D}}(\vec{N})\right\} &= \int_\Sigma \mathrm{d}^3y\,\frac{\delta f[\varphi(\cdot)]}{\delta\varphi^A(y)}\left(N^\mu\varphi^A{}_{,\mu}-(\partial_\gamma N^\mu)\mathrm{F}^A{}_\mu{}^\gamma\right)(y)\\
&= \sum_{K=0}^\infty\sum_{L=0}^\infty\binom{K+L}{K} f_{:A}{}^{\alpha_1\dots\alpha_K\beta_1\dots\beta_L}\left((\partial_{\alpha_1\dots\alpha_K}N^\mu)\varphi^A{}_{,\mu\beta_1\dots\beta_L}-(\partial_{\gamma\alpha_1\dots\alpha_K}N^\mu)\mathrm{F}^A{}_\mu{}^\gamma{}_{,\beta_1\dots\beta_L}\right)\\
&= \sum_{L=0}^\infty N^\mu f_{:A}{}^{\beta_1\dots\beta_L}\,\varphi^A{}_{,\mu\beta_1\dots\beta_L}+\\
&\quad+\sum_{L=1}^\infty(\partial_{\alpha_1\dots\alpha_L}N^\mu)\left[\sum_{K=0}^\infty\left(\binom{K+L}{K}f_{:A}{}^{\alpha_1\dots\alpha_L\beta_1\dots\beta_K}\varphi^A{}_{,\mu\beta_1\dots\beta_K}\right.\right.\\
&\qquad\left.\left.-\binom{K+L-1}{K}f_{:A}{}^{\beta_1\dots\beta_K(\alpha_1\dots\alpha_{L-1}|}\mathrm{F}^A{}_\mu{}^{|\alpha_L)}{}_{,\beta_1\dots\beta_K}\right)\right]\\
&= \underbrace{\sum_{L=0}^\infty N^\mu f_{:A}{}^{\beta_1\dots\beta_L}\varphi^A{}_{,\mu\beta_1\dots\beta_L}}_{=:N^\mu\partial_\mu^{(\varphi)}f}+\underbrace{\sum_{L=1}^\infty(\partial_{\gamma_1\dots\gamma_L}N^\mu)\,\mathbf{X}_\mu^{\gamma_1\dots\gamma_L}f}_{:=\mathbf{X}(\vec{N})\,f}\;. \qquad (3.186)
\end{aligned}$$

This shows that the differential operator $\mathbf{X}(\vec{N})$ is linked to the supermomentum and, as a result, we expect that the algebra relation of the supermomentum, given in form of the Poisson algebra relation $\left\{\vec{\mathcal{D}}(\vec{N}),\vec{\mathcal{D}}(\vec{M})\right\}=\widehat{\mathcal{D}}(\mathcal{L}_{\vec{N}}\vec{M})$, to translate into some commutator relation for the differential operator $\mathbf{X}(\vec{N})$.

We can use the Jacobi identity to obtain the repeated application of the supermomentum, i.e.

$$\left\{\left\{f[\varphi(\cdot)],\widehat{\mathcal{D}}(\vec{N})\right\},\widehat{\mathcal{D}}(\vec{M})\right\}-\left\{\left\{f[\varphi(\cdot)],\widehat{\mathcal{D}}(\vec{M})\right\},\widehat{\mathcal{D}}(\vec{N})\right\}=\left\{f[\varphi(\cdot)],\widehat{\mathcal{D}}(\mathcal{L}_{\vec{N}}\vec{M})\right\}\,. \qquad (3.187)$$

Then spelling out the Poisson bracket and repeating the steps from (3.186) we find that it can be written as commutators of the two differential operators $N^\mu\partial_\mu^{(\varphi)}$ and $\mathbf{X}(\vec{N})$. The Jacobi identity then gives that

$$\begin{aligned}
\left[\mathbf{X}(\vec{N}),\mathbf{X}(\vec{M})\right]f[\varphi(x)] \;&=\; \left(\mathcal{L}_{\vec{N}}\vec{M}\right)^\mu\partial_\mu^{(\varphi)}f[\varphi(x)]+\mathbf{X}(\mathcal{L}_{\vec{N}}\vec{M})f[\varphi(x)]+\left[N^\mu\partial_\mu^{(\varphi)},M^\nu\partial_\nu^{(\varphi)}\right]f[\varphi(x)]\\
&\quad+\left(\left[N^\mu\partial_\mu^{(\varphi)},\mathbf{X}(\vec{M})\right]+\left[M^\mu\partial_\mu^{(\varphi)},\mathbf{X}(\vec{N})\right]\right)f[\varphi(x)]\,. \qquad (3.188)
\end{aligned}$$

The $N^\mu\partial_\mu^{(\varphi)}M^\nu\partial_\nu^{(\varphi)}$ terms are symmetric in N and M and thus cancel each other due to the anti-symmetry of the commutator. The other two terms are more complicated to calculate explicitly, however, it turns out that they do not contribute anyways. In order to see this, observe that in these terms, one of the shift vector fields appears without a spatial derivative, the other one with a single derivative, i.e.

$$\left(\mathcal{L}_{\vec{N}}\vec{M}\right)^\mu\partial_\mu^{(\varphi)}f \;\propto\; N\partial M-M\partial N \quad , \qquad (3.189a)$$

$$\left(\left[N^\mu\partial_\mu^{(\varphi)},\mathbf{X}(\vec{M})\right]-\left[M^\mu\partial_\mu^{(\varphi)},\mathbf{X}(\vec{N})\right]\right)f[\varphi(x)] \;\propto\; N\partial M-M\partial N \quad . \qquad (3.189b)$$

On the left-hand side, that is in the commutator of $\mathbf{X}(\vec{N})$, the terms have at least one derivative acting on each of the shift vector fields. As a result, once we single out the different derivative orders of the shift on the left-hand side, we find out that on the right-hand side, all contributions come from the $\mathbf{X}(\mathcal{L}_{\vec{N}}\vec{M})$ term. This leaves us with

$$[\mathbf{X}_\mu^{\alpha_1\ldots\alpha_N},\mathbf{X}_\nu^{\beta_1\ldots\beta_M}]f[\varphi(x)] = \binom{N+M-1}{M}\delta_\nu^{(\alpha_1}\mathbf{X}_\mu^{\alpha_2\ldots\alpha_N)\beta_1\ldots\beta_M}\,f[\varphi(x)] - \binom{N+M-1}{N}\delta_\mu^{(\beta_1}\mathbf{X}_\nu^{\beta_2\ldots\beta_M)\alpha_1\ldots\alpha_N}\,f[\varphi(x)]\,, \tag{3.190}$$

which indeed proves the claim. □

Thus, we can directly conclude that no integrability condition arises for the scalar output coefficients. However, for the other output coefficients, the situation is more involved since we also need to account for the inhomogeneous part in the $L=1$ equation. In principle, it would be possible for this part to produce a lower order equation once we eliminate the first-order derivatives with the help of the commutator relation above.

Our last step now is to show that this is, in fact, not the case. We start by mixing equation (3.182b), for $L \geq 2$, with equation (3.182a) for a single index β. In this case we easily find that no additional information appears since

$$\begin{aligned}
0 &= \left[\mathbf{X}_\mu^{\alpha_1\ldots\alpha_L},\mathbf{X}_\nu^{\beta}\right]\widetilde{C}_{B_1\ldots B_N} - N\cdot\left(\mathbf{X}_\mu^{\alpha_1\ldots\alpha_L}\widetilde{C}_{A(B_1\ldots B_{N-1}}\right)F^A{}_\nu{}^\beta{}_{:|B_N)} - N\cdot\widetilde{C}_{A(B_1\ldots B_{N-1}}\underbrace{\left(\mathbf{X}_\mu^{\alpha_1\ldots\alpha_L}F^A{}_\nu{}^\beta{}_{:|B_N)}\right)}_{\text{vanishes}}\\
&= L\cdot\delta_\nu^{(\alpha_1}\left(\mathbf{X}_\mu^{\alpha_2\ldots\alpha_L)\beta}\widetilde{C}_{B_1\ldots B_N}\right) - \delta_\mu^\beta\left(\mathbf{X}_\nu^{\alpha_1\ldots\alpha_L}\widetilde{C}_{B_1\ldots B_N}\right) - N\cdot\left(\mathbf{X}_\mu^{\alpha_1\ldots\alpha_L}\widetilde{C}_{A(B_1\ldots B_{N-1}}\right)F^A{}_\nu{}^\beta{}_{:|B_N)}\\
&\equiv 0\,,
\end{aligned}$$

where we used the commutator relation we derived (3.190), that the input coefficient does not contain any spatial derivatives of the degrees of freedom and, in the last line, that all terms are a linear combination of the L^{th} equation. What remains to be checked is the case $L=1$, that is, the commutator of (3.182a) for two spatial indices α,β. Here, we obtain the following expression

$$\begin{aligned}
0 &= \left[\mathbf{X}_\mu^\alpha,\mathbf{X}_\nu^\beta\right]\widetilde{C}_{B_1\ldots B_N} - N\left(\mathbf{X}_\mu^\alpha\widetilde{C}_{A(B_1\ldots B_{N-1}|}\right)F^A{}_\nu{}^\beta{}_{:|B_N)} - N\,\widetilde{C}_{A(B_1\ldots B_{N-1}|}\left(\mathbf{X}_\mu^\alpha F^A{}_\nu{}^\beta{}_{:|B_N)}\right)\\
&\quad +N\left(\mathbf{X}_\nu^\beta\widetilde{C}_{A(B_1\ldots B_{N-1}|}\right)F^A{}_\mu{}^\alpha{}_{:|B_N)} + N\,\widetilde{C}_{A(B_1\ldots B_{N-1}|}\left(\mathbf{X}_\nu^\beta F^A{}_\mu{}^\alpha{}_{:|B_N)}\right)\\
&= \delta_\nu^\alpha\left(\mathbf{X}_\mu^\beta\widetilde{C}_{B_1\ldots B_N}\right) - N\left(\mathbf{X}_\mu^\alpha\widetilde{C}_{A(B_1\ldots B_{N-1}|}\right)F^A{}_\nu{}^\beta{}_{:|B_N)} - N\,\widetilde{C}_{A(B_1\ldots B_{N-1}|}\left(\mathbf{X}_\mu^\alpha F^A{}_\nu{}^\beta{}_{:|B_N)}\right) - \xleftrightarrow{\mu\alpha\quad\nu\beta}\\
&= \delta_\nu^\alpha\left(\mathbf{X}_\mu^\beta\widetilde{C}_{B_1\ldots B_N}\right) - N\left(\mathbf{X}_\mu^\alpha\widetilde{C}_{A(B_1\ldots B_{N-1}|}\right)F^A{}_\nu{}^\beta{}_{:|B_N)} + N\,\widetilde{C}_{A(B_1\ldots B_{N-1}|}F^A{}_\nu{}^\beta{}_{:|B_N)M}F^M{}_\mu{}^\alpha - \xleftrightarrow{\mu\alpha\quad\nu\beta}\,.
\end{aligned} \tag{3.191}$$

Due to the inhomogeneity in the differential equation the terms in the brackets do not directly vanish. However, we can employ the first input coefficient identity (3.120a) again to simplify the expression. By calculating the derivative of the identity, we obtain

$$F^A{}_\nu{}^\beta{}_{:B_N M}F^M{}_\mu{}^\alpha - \xleftrightarrow{\mu\alpha\quad\nu\beta} \quad = \quad -F^A{}_\mu{}^\beta{}_{:B_N}\delta_\nu^\alpha - F^A{}_\nu{}^\beta{}_{:B}F^B{}_\mu{}^\alpha{}_{:B_N} - \xleftrightarrow{\mu\alpha\quad\nu\beta}\,. \tag{3.192}$$

Inserting this into the equation we obtain additional contributions for the first and second term in (3.191)

$$\begin{aligned}
0 \;=\; & \delta^\alpha_\nu \left(\mathbf{X}^\beta_\mu \widetilde{\mathrm{C}}_{B_1\ldots B_N} - N\cdot\widetilde{\mathrm{C}}_{A(B_1\ldots B_{N-1}|}\mathrm{F}^A{}_\mu{}^\beta{}_{:|B_N)}\right)\\
& -N\left(\left(\mathbf{X}^\alpha_\mu \widetilde{\mathrm{C}}_{A(B_1\ldots B_{N-1}|}\right) - \widetilde{\mathrm{C}}_{B(B_1\ldots B_{N-1}|}\mathrm{F}^B{}_\mu{}^\alpha{}_{:A} - (N-1)\widetilde{\mathrm{C}}_{AB(B_1\ldots B_{N-2}|}\mathrm{F}^B{}_\mu{}^\alpha{}_{:|B_{N-1}|}\right)\mathrm{F}^A{}_\nu{}^\beta{}_{:|B_N)}\\
& - \overset{\mu\alpha \quad \nu\beta}{\longleftrightarrow}\\
\equiv\; & 0\,,
\end{aligned}$$

which is, again, a linear combination of the existing covariance part of the closure equations. Note that we introduced the term $(N-1)\widetilde{\mathrm{C}}_{AB(B_1\ldots B_{N-2}|}\mathrm{F}^B{}_\mu{}^\alpha{}_{:|B_{N-1}|}\mathrm{F}^A{}_\nu{}^\beta{}_{:|B_N)}$ that is symmetric under the exchange of the index pairs μ,α and ν,β and is, thus, eliminated by the antisymmetry of the equation. As a result, no new first (or lower) order equation can be obtained from the equations, and we can finally conclude that the covariance part of the closure equations is indeed involutive:

THEOREM

The covariance part of the closure equations is an involutive system of first-order linear partial differential equations.

Curvature invariants

Now, since we know that we deal with an involutive system, we can again play some combinatorial games and consider the formal solutions of the differential equations. Let us start with the analysis of the scalar output coefficient.

As we have seen before in the discussion of scalar densities, for the first-order equations, the number of functionally independent solutions will be given by the number of independent variables minus the number of equations we have in the covariance part of the closure equations. The number of independent variables reads if we assume that we have at most q spatial derivatives of the degrees of freedom:

$$\#(\text{independent variables})_q = \frac{F}{6}(q+1)(q+2)(q+3)\,. \tag{3.193}$$

For the number of independent equations we need to make sure that we can solve in each equation form the covariance part for a series expansion coefficient of a separate class. If we inspect the geometric symbol we find that we can always solve for

$$\varphi^A{}_{,\mu}\delta^{(\gamma_1}_{(\lambda_1}\cdots\delta^{\gamma_L)}_{\lambda_L)}\,, \tag{3.194}$$

that is, for the coefficient of class $^A{}_{(\lambda_1\ldots\lambda_L)}$. As a result, we can count the number of coefficients that are eliminated from the covariance part of the closure equations. Taking the sum from $L = 1$ to $q+1$ we get

$$
\begin{aligned}
\#(\text{equations from the covariance part})_q &= \#(\mathbf{X}_\mu^{\beta_1}) + \#(\mathbf{X}_\mu^{\beta_1\beta_2}) + \#(\mathbf{X}_\mu^{\beta_1\beta_2\beta_3})) + \dots \\
&= 3\cdot 3 + 3\cdot 6 + 3\cdot 10 + \dots \\
&= 3\cdot\sum_{k=1}^{q+1}\binom{k+2}{k} \\
&= \frac{1}{2}(q+2)(q+3)(q+4) - 3\,,
\end{aligned}
\tag{3.195}
$$

principal coefficients. The remaining coefficients are free in the sense that they need to be supplied as initial data to the differential equations. Combining this, that is subtracting equation (3.195) from equation (3.193), leads to the total number of curvature invariants, i.e functionally independent solutions to the covariance part of the closure equations

$$
N_q := \#(\text{curvature invariants})_q = \frac{1}{6}(q+2)(q+3)\left((F-3)q + F - 12\right) + 3\,. \tag{3.196}
$$

In particular this tells us that every additional gravitational degree of freedom adds

$$
\frac{1}{6}(q+1)(q+2)(q+3) \tag{3.197}
$$

many new curvature invariants to the system. The general solution of the covariance part of the closure equations for the scalar output coefficient is then an *arbitrary* function of the curvature invariants we obtained. This means that the scalar output coefficient is of the form

$$
\mathrm{C}[\varphi] := \frac{\Psi_{(0)}\left(\sigma^{(1)}(\varphi),\dots,\sigma^{(R)}(\varphi)\right)}{\sqrt{-\det \mathrm{p}^{\cdot\cdot}(\varphi)}}\widetilde{\mathrm{C}}\left(\Gamma_1[\varphi],\dots,\Gamma_{N_q}[\varphi]\right)\,. \tag{3.198}
$$

We can now repeat the same calculation for the other output coefficients. The number of curvature invariants is then simply multiplied by the number of components in each output coefficient, i.e. for the output coefficient $\mathrm{C}_{A_1\dots A_N}[\varphi]$ by

$$
\binom{F+N-1}{N}\,. \tag{3.199}
$$

In particular, for $\mathrm{C}_{AB}, \mathrm{C}_{ABC}, \dots$ we know that $q = 2$ from the derivation of the closure equations. This means that for $N \geq 2$ we have

$$
\#(\text{curvature invariants})_{\mathrm{C}_{A_1\dots A_N}, q=2} = \binom{F+N-1}{N}(10\,F - 57) \qquad \text{for } N \geq 2\,. \tag{3.200}
$$

How does any of this help for the gravitational closure equations? It turns out that when solving the equations it is often simpler to write down the functionally independent solutions than to obtain an expression by integrating the partial differential equations.

Take for example general relativity: Here, from these combinatorial considerations we immediately find for $q = 2$ and the six degrees of freedom of the metric that the scalar output coefficient must be constructed from three functionally independent curvature invariants. These invariants are then identified to be the three traces of the Ricci curvature tensor one can construct

$$\Gamma_1[\varphi] := \mathcal{R}[\varphi]\,, \tag{3.201a}$$

$$\Gamma_2[\varphi] := \mathcal{R}^{\mu\nu}[\varphi]\mathcal{R}_{\mu\nu}[\varphi]\,, \tag{3.201b}$$

$$\Gamma_3[\varphi] := \mathcal{R}^{\mu}{}_{\nu}[\varphi]\mathcal{R}^{\nu}{}_{\lambda}[\varphi]\mathcal{R}^{\lambda}{}_{\mu}[\varphi]\,. \tag{3.201c}$$

Although higher-order traces can be defined in the same fashion, they can all be written as functionals of the three invariants described above due to the Cayley-Hamilton theorem.

Thus, the output coefficient must be of the form

$$\mathrm{C}[\varphi] = \frac{1}{\sqrt{-\det \mathrm{p}^{\cdot\cdot}(\varphi)}}\widetilde{\mathrm{C}}(\Gamma_1[\varphi], \Gamma_2[\varphi], \Gamma_3[\varphi])\,. \tag{3.202}$$

Once we insert this into the selective part of the closure equations, this will simplify the partial differential equations. We will see this in more detail in chapter 4.1 in the gravitational closure of Maxwellian electrodynamics. In general, the situation can be seen as follows: the covariance part of the closure equations identifies the covariant building blocks we can construct, whereas the selective part of the closure equations will then relate to these building blocks.

3.5.2 Analysis of the selective part of the closure equations

Now that we analysed the covariance part of the closure equations in more detail, we will take a closer look at the remaining gravitational closure equations, that is, the collection that make up the selective part. In contrast to the equations in the covariance part which are differential equations for each output coefficient on their own, the selective part of the closure equations connects the separate output coefficients. This is schematically illustrated in figure 3.5.

Being able to express the higher-order output coefficients in terms of the lower-order ones is, of course, essential if we want to obtain a solution of the closure equations parametrized in terms of finitely many *constants* of integration. If each order would be independent of the others – as in the covariance part of the closure equations – this would mean that we get undetermined constants (or in the worst case functions) at each order. Since we have a series expansion in terms of infinitely many output coefficients, this would, in any case, lead to an infinite number of parameters that need to be determined by experiments.

To prevent the "combinatorial explosion" of the output coefficients, it turns out that we have two types of equations in the selective part of the closure equations that do determine many of the components. These are the equations ($\mathbf{C16_N}$) (and its analogues ($\mathbf{C3}$) and ($\mathbf{C6}$)), as well as the equations ($\mathbf{C17_N}$) (and ($\mathbf{C4}$)). The remaining equations either relate the scalar output coefficients C and C_A, or are equations that only concern a single output coefficient.

Again, one further complication is that we do not have a collapse to finitely many spatial derivatives for the first two output coefficients. This has the effect that the selective part of the closure equations is of higher differential order, in contrast to the first-order covariance part of the closure equations. These higher orders can all be traced back to spatial derivatives acting on the output coefficients.

Take for example the following term in the closure equation ($\mathbf{C4}$), where we need to expand all the

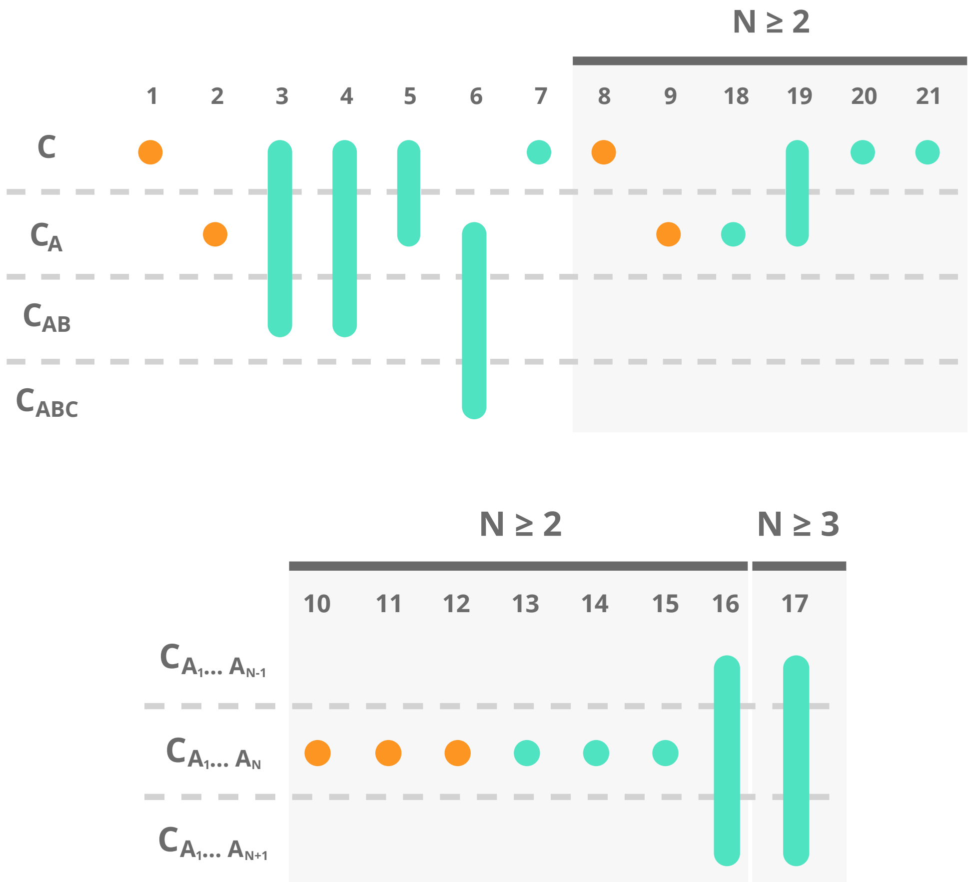


Figure 3.5 The structure of the gravitational closure equations and how the output coefficients are related by the equations. Equations from the covariance part of the closure equations are drawn in orange, the selective part of the closure equations in green. Only equations from the selective part couple separate output coefficients to each other.

spatial derivatives

$$\sum_{K=0}^{\infty}(-1)^K(K+1)\left(\partial_{\alpha_1\ldots\alpha_K}C_{:B}{}^{\alpha_1\ldots\alpha_K\mu}\right) = C_{:B}{}^{\mu} - 2\sum_{L=0}^{\infty} C_{:B}{}^{\mu(\alpha}{}_{C}{}^{\beta_1\ldots\beta_L)}\,\varphi^{C}{}_{,,\alpha\beta_1\ldots\beta_L} + \ldots \,. \tag{3.203}$$

The terms become increasingly complicated since we need to distribute the spatial derivatives on more and more terms that arise from the chain rule. This makes an analysis of the involutivity almost impossible, as we presented for the covariance part of the closure equations. For practical purposes, one needs to cut off artificially at some spatial derivative order if we want to end up with finitely many constants in the solution.

Solution strategy

Tackling such an enormous and complicated system in practice, thus, requires a clear strategy for how to, and for which objects to solve for, in the differential equations in order to keep track of the extracted information. Motivated by the lessons from the formal study of involutive differential equations, we decide on the following guideline: We always

- solve for as many of the higher-order output coefficients in terms of coefficients of lower order as possible.
- solve for dependencies of higher spatial derivatives of the degrees of freedom in terms of dependencies with less or no spatial derivatives of the degrees of freedom.

With this, we can describe a general solution strategy that is advisable to comply with when solving for a specific gravitational theory.

Step 1 As the first step, we solve the covariance part of the closure equations for the scalar output coefficient C. For this, we construct the functionally independent curvature invariants up to a *chosen* maximal derivative order q_0 of spatial derivatives of the gravitational degrees of freedom.

Step 2 We insert the scalar coefficient C we obtained in the previous step into all equations from the selective part of the equations that only depend on C. These are the equations $(\mathbf{C7})$, $(\mathbf{C20}_{\mathbf{N}\text{ even}})$ and $(\mathbf{C21}_{\mathbf{N}\text{ odd}})$ that will further restrict the dependencies of the general solution of the covariance part of the closure equations with the help of the normal deformation input coefficient $\mathrm{M}^{A\gamma}$.

Additionally, since we assumed C to depend at most to order q_0 on the derivatives of φ, we also get further information from the prolongations of equation $(\mathbf{C3})$ and $(\mathbf{C4})$ since the output coefficient C_A depends at most to order q_0 on our jet variables, however, due to the spatial derivatives acting on the scalar output coefficients we obtain higher-order derivatives of the φ. These prolongations must all vanish separately.

Step 3 With all the equations that only concern C, solved, we continue by constructing the curvature invariants for the output coefficient C_A by solving the covariance part of the closure equations. Again, we need to *choose* a maximal derivative order q_1 for the output coefficient. Note that due to $(\mathbf{C19_N})$ we have that $q_0 \leq q_1 + 1$.

Step 4 Use the curl equation $(\mathbf{C18_N})$ to eliminate the dependencies of some of these curvature invariants.

Step 5 Use $(\mathbf{C5})$ and $(\mathbf{C19_N})$ to relate components of the output coefficient C_A to the scalar output coefficient

Step 6 Use the prolongations obtained from equations $(\mathbf{C3})$ and $(\mathbf{C4})$ from $\varphi^A{}_{,,\mu\nu\lambda\alpha_4\ldots\alpha_q}$ to further express components from C_A in terms of C.

Step $7_\mathbf{N}$ Implement the covariance part of the closure equations for the higher-order output coefficients. Luckily, we now know that the coefficients depend at most on $\varphi^A{}_{,,\mu\nu}$ such that the complexity is limited to some degree. Still, we get increasingly more complicated curvature invariants with each order.

Step $8_{\mathbf{N}}$ Due to $(\mathbf{C17_N})$ we find the condition

$$0 = C_{B_1 \dots B_n M}{}^{\alpha(\beta}{}_N{}^{\gamma\delta)} \qquad \text{for } n \geq 2\,, \tag{3.204}$$

that will restrict the dependencies on our curvature invariants. We will see in the following section that this forces the output coefficients to be polynomials of (at most) degree three in the second spatial derivatives of the degrees of freedom (for a three-dimensional screen manifold).

Step $9_{\mathbf{N}}$ Use $(\mathbf{C16_N})$ and $(\mathbf{C17_N})$, and similarly $(\mathbf{C3})$ and $(\mathbf{C4})$ for the output coefficient C_{AB} and $(\mathbf{C6})$ for C_{ABC}, to express as many components of the current output coefficient in terms of the lower-order ones. For all the components we can solve for, one recursively finds that they can be expressed in terms of C and C_A.

Step $10_{\mathbf{N}}$ Use $(\mathbf{C14_N})$ to express some of the undetermined components of the current output coefficient in terms of the determined components from the previous step.

Step $11_{\mathbf{N}}$ Use $(\mathbf{C12_N})$ and the curl condition $(\mathbf{C15_N})$ to further eliminate some of the $\varphi^A{}_{,\mu\nu}$ terms.

Step $12_{\mathbf{N}}$ In principle, this means that we now have employed all of the equations from the covariance part. Since it is unclear if the selective part of the closure equations is involutive, it is, in principle, possible to combine some of the prolongations of the selective part of the closure equations such that the highest derivative term disappears. This may yield further restrictions.

From there, we have to repeat the steps 7_N to 12_N for the output coefficient until, in the best case, one of the output coefficients vanishes at some order N. By the recurrence equations, this will further restrict the subsequent output coefficient until the series expansion terminates. This is precisely what we will observe for the derivation of general relativity in chapter 4.1.

What can also happen is that we, at least, recover a recurrence relation on how to express the output coefficients at higher-order by the lower order ones. While this produces a Lagrangian of infinitely many velocity terms, the solution still consists of at least finitely many constants of integration or finitely many undetermined functions.

The worst-case scenario occurs if we have residual undetermined parts at each order. These enter as undetermined functions *at each order*. The resulting theory is experimentally clearly of limited use since we would have infinitely many parameters that need to be fixed by measurements.

We will, in the following, present a dimensionally dependent argument to derive that the output coefficients are polynomials in the second derivative terms – a fact that becomes especially useful when one wants to construct the curvature invariants from the covariance part of the closure equations. Afterwards, we present some additional points that can be derived for theories with vanishing normal input coefficient $M^{A\gamma}$.

Cubic polynomial in second-order derivative terms

Let us now further investigate the equation we encountered in step 8 of the algorithm laid out in the previous section. As stated before, we consider the closure equation $(\mathbf{C17_N})$. Since the last term in the equation contains a spatial derivative acting on the output coefficient, we get

$$2\,C_{B_1\ldots B_N:B_{N+1}}{}^{\gamma\mu}{}_{:M}\varphi^M{}_{,\gamma} + 2\,C_{B_1\ldots B_N:B_{N+1}}{}^{\gamma\mu}{}_{:M}{}^{\alpha}\varphi^M{}_{,\alpha\gamma} + 2\,C_{B_1\ldots B_N:B_{N+1}}{}^{\gamma\mu}{}_{:M}{}^{\alpha\beta}\varphi^M{}_{,\alpha\beta\gamma}\,. \quad (3.205)$$

The output coefficients that appear in this equation do not depend on third derivatives of the degrees of freedom. This means that the last term must vanish separately, which yields the equation

$$0 = C_{B_1\ldots B_N:B_{N+1}}{}^{\mu(\alpha}{}_{:M}{}^{\beta\gamma)}\,. \quad (3.206)$$

Using this equation, we can show the following:

THEOREM

In three dimensions, all output coefficients $C_{B_1\ldots B_N}$ for $N \geq 2$ are polynomials of degree 3 in $\varphi^A{}_{,\mu\nu}$, i.e.

$$C_{B_1\ldots B_N:A_1}{}^{\alpha\beta}{}_{:A_2}{}^{\gamma\delta}{}_{:A_3}{}^{\mu\nu}{}_{:A_4}{}^{\lambda\kappa} = 0\,.$$

In order to show this we first introduce the following symbol

$$\Lambda^{\alpha\beta\gamma\delta\mu\nu\lambda\kappa} := C_{B_1\ldots B_N:A_1}{}^{\alpha\beta}{}_{:A_2}{}^{\gamma\delta}{}_{:A_3}{}^{\mu\nu}{}_{:A_4}{}^{\lambda\kappa}\,, \quad (3.207)$$

where we omit the Φ indices in Λ to simplify notation, since they are irrelevant for the argument. The tensor Λ has the following properties:

- It is symmetric in each block of two indices.
- It is also symmetric under the exchange of blocks.
- The tensor vanishes if we symmetrize over three subsequent indices, i.e. $\Lambda^{(\alpha\beta\gamma)\delta\mu\nu\lambda\kappa} = 0$.

By spelling out the symmetrization over the first three indices, we observe that we can swap an index from one index to the other since

$$0 = 3\Lambda^{(\alpha\beta\gamma)\delta\mu\nu\lambda\kappa} = \Lambda^{\alpha\beta\gamma\delta\mu\nu\lambda\kappa} + 2\Lambda^{\gamma(\alpha\beta)\delta\mu\nu\lambda\kappa}\,. \quad (3.208)$$

This allows us to generalise an argument made by Lovelock (see Lovelock (1972) for further details). This will tell us that the tensor Λ is vanishing in three dimensions since at least three of our eight indices have to be the same.

For these, two different situations can now occur. First, two of these three equal indices can appear at a common block. In this case, the symmetries of Λ directly imply that the component is zero. Second, the three indices can appear on three separate blocks. But also in this case, it follows that it is zero. For example, for the following component, we find due to equation (3.208) that

$$\Lambda^{13121223} = -\frac{1}{2}\Lambda^{11231223} = -\frac{1}{2}\Lambda^{(11|23|1)223} = 0\,. \tag{3.209}$$

As a result, this tensor is identically zero in three dimensions and we find

$$C_{B_1 \dots B_N : A_1}{}^{\alpha\beta}{}_{:A_2}{}^{\gamma\delta}{}_{:A_3}{}^{\mu\nu}{}_{:A_4}{}^{\lambda\kappa} = 0 \qquad \text{for } N \geq 2\,, \tag{3.210}$$

or, in other words, the output coefficients $C_{A_1 \dots A_N}$ are at most cubic in the second derivatives of the degrees of freedom[10]. $\square$

This result is especially helpful when coming up with the curvature invariants to the covariance part of the closure equations, since we can start by constructing the separate orders in $\varphi^A{}_{,\mu\nu}$ independently from each other. We will see this in further detail in our discussion of Maxwellian electrodynamics in chapter 4.1.

For C and C_A the situation is, as before, more involved since we do not know the maximal order of spatial derivatives they contain. Assuming that there is *some* highest derivative q for the output coefficients we can use $(\mathbf{C17_N})$ to derive an equation similar to (3.206) for those higher derivatives since the equation contains up to $\varphi^A{}_{,\alpha_1 \dots \alpha_{2q-1}}$ terms that all need to vanish separately. Similarly, as described in Edgar and Höglund (2002) (and in chapter 5.2.2), such dimensionally dependent identities can be obtained by non-trivial over-antisymmetrizations. These are, in d dimensions, antisymmetrization over $d+1$ indices and can be used to derive helpful identities. We expect that for $q \geq 2$, we can eliminate many components for the first two output coefficients due to such identities. We leave this for future research.

At this abstract level, with the input coefficients unspecified, the rest of the gravitational closure equations is shrouded in mystery. Some further simplifications can be made given that the input coefficient $M^{A\gamma}$ is vanishing. We will dedicate the following section to a presentation of these.

3.5.3 Vanishing normal input coefficient

In case the normal input coefficient $M^{A\gamma}$ vanishes – as it does in the gravitational closure of Maxwellian electrodynamics that we will discuss in chapter 4.1 – the gravitational closure equations simplify significantly. Four out of the 21 equations are solved trivially, which concretely are equations $(\mathbf{C7})$, $(\mathbf{C13_N})$, $(\mathbf{C20_{N\,\text{even}}})$ und $(\mathbf{C21_{N\,\text{odd}}})$. And also for the remaining equations, multiple simplifications can be made that allow us to restrict the solution space further. We will present these results in the following.

The even and odd output coefficients decouple

A minor, but nonetheless quite valuable, fact is that in the selective part of the closure equations, all output coefficients $C_{A_1 \dots A_N}$ for N even decouple from the odd ones. This is due to the fact that the odd or even terms, respectively, in the closure equations come with the non-local normal deformation coefficient $M^{A\gamma}$. This allows discussing the two sectors on their own.

[10] In general the argument can be repeated to show that actually the coefficients are a polynomial of degree $\dim \Sigma$ in $\varphi^A{}_{,\mu\nu}$.

Dependency of C on up to 2nd derivatives of the degrees of freedom

Given that $\mathrm{M}^{A\gamma}$ vanishes it is possible to show that C depends on up to the 2nd derivatives of the geometric degrees of freedom φ, exactly as is the case for the coefficients $\mathrm{C}_{A_1 \ldots A_N}$ for $N \geq 2$. The weak assumption that we must make is that it does depend on finitely many derivatives q, i.e.

$$\mathrm{C}_{:A}{}^{\alpha_1 \ldots \alpha_K} = 0 \qquad \text{for} \qquad K > q\,. \tag{3.211}$$

But once this assumption is made, we can consider the closure equation $\left(\mathbf{C19_{q-1}}\right)$ to find that

$$\mathrm{C}_{:A}{}^{\alpha_1 \ldots \alpha_q} = 0\,. \tag{3.212}$$

We can then use $(\mathbf{C19_N})$ recursively until we reach $N = 2$ to find that

$$\mathrm{C}_{:A}{}^{\alpha\beta\gamma} = 0\,, \tag{3.213}$$

which proves the claim. As a result all output coefficients but C_A depend on φ, $\partial\varphi$ and $\partial\partial\varphi$.

$\mathbf{C_A}$ is a functional gradient

While we still have no collapse to finitely many derivatives of the degrees of freedom for C_A, we can use the gravitational closure equation to show that

$$\mathrm{C}_A[\varphi(x)] = \frac{\delta \Lambda[\varphi]}{\delta \varphi^A(x)}\,, \tag{3.214}$$

for some functional of the degrees of freedom. In order to see this, we first remark that the condition for C_A to be of this form is that the following functional curl condition is fulfilled:

$$\frac{\delta \mathrm{C}_A[\varphi(x)]}{\delta \varphi^B(y)} - \frac{\delta \mathrm{C}_B[\varphi(y)]}{\delta \varphi^A(x)} = 0 \tag{3.215}$$

This condition can be turned into an equivalent system of partial differential equation by the same techniques that led to the gravitational closure equations. We then find that (3.215) can be written as

$$0 = \mathrm{C}_{A:B}{}^{\mu_1 \ldots \mu_N} - \sum_{K=0}^{\infty} (-1)^{K+N} \binom{K+N}{K} \left(\partial_{\alpha_1 \ldots \alpha_K} \mathrm{C}_{B:A}{}^{\alpha_1 \ldots \alpha_K \mu_1 \ldots \mu_N}\right) \qquad \text{for} \qquad N \geq 0\,. \tag{3.216}$$

Careful inspection of the closure equations for a similar equation yields that, for $N \geq 1$ we find these in the closure equations $(\mathbf{C18_N})$ and $(\mathbf{C6})$. However, the $N = 0$ equation is entirely absent from our system.

It turns out, however, that we can generalize an elegant argument due to Hojman (Kuchar, 1974) to show that the functional curl (3.215) does indeed vanish. For this we consider the integral

$$\mathcal{I} := \int \mathrm{d}^3 y \left(\frac{\delta \mathrm{C}_A[\varphi(x)]}{\delta \varphi^B(y)} - \frac{\delta \mathrm{C}_B[\varphi(y)]}{\delta \varphi^A(x)} \right) \left(\mathcal{L}_{\vec{N}} \varphi\right)^B(y)\,, \tag{3.217}$$

and show that this, given that C_A solves $(\mathbf{C5})$, vanishes for any vector $\vec{N}$.

We first observe that the first term in equation (3.217) corresponds precisely to the action of the supermomentum on the scalar density C_A of weight 1. Then we use the functional product rule to rewrite the second part, i.e.

$$\begin{aligned}\mathcal{I} = &\left\{\mathrm{C}_A, \widehat{\mathcal{D}}(\vec{N})\right\} + \int \mathrm{d}^3y\, \mathrm{C}_B[\varphi(y)] \frac{\delta \left(\mathcal{L}_{\vec{N}}\varphi\right)^B(y)}{\delta\varphi^A(x)} \\ &- \frac{\delta}{\delta\varphi^A(x)}\left[\int \mathrm{d}^3y\, \mathrm{C}_B \left(\mathcal{L}_{\vec{N}}\varphi\right)^B\right] .\end{aligned} \tag{3.218}$$

Using the definition of the kinematic coefficient $\mathrm{F}^A{}_\mu{}^\gamma$, we get

$$\frac{\delta \left(\mathcal{L}_{\vec{N}}\varphi\right)^B(y)}{\delta\varphi^A(x)} = N^\mu(y)\, \delta^B_A \left(\partial_\mu \delta_x\right)(y) - \left(\partial_\gamma N^\mu\right)(y)\, \mathrm{F}^B{}_\mu{}^\gamma{}_{:A}(y)\, \delta_x(y) . \tag{3.219}$$

Inserting this into the integral and integrating out the distributions, we find that this equals the Lie derivative of the density C_A.

$$\begin{aligned}\mathcal{I} = &\left\{\mathrm{C}_A, \widehat{\mathcal{D}}(\vec{N})\right\} - N^\mu \left(\partial_\mu \mathrm{C}_A\right) - \left(\partial_\mu N^\mu\right) \mathrm{C}_A - \left(\partial_\gamma N^\mu\right) \mathrm{C}_B \mathrm{F}^B{}_\mu{}^\gamma{}_{:A} \\ &- \frac{\delta}{\delta\varphi^A(x)}\left[\int \mathrm{d}^3y\, \mathrm{C}_B \left(\mathcal{L}_{\vec{N}}\varphi\right)^B\right] .\end{aligned} \tag{3.220}$$

By expanding the Poisson bracket of the first term and separating by orders of derivatives on the shift, we observe that (**C9**) eliminates all terms with more than two derivatives on the shift. Furthermore, (**C2**) can be used to obtain the term containing $\mathrm{F}^A{}_\mu{}^\gamma{}_{:B}$ and the density term in the Lie derivative. Finally, one easily sees that the term containing no derivative terms on the shift is just a spelled out spatial derivative acting on C_A by the chain rule. Collecting all terms we find

$$\left\{\mathrm{C}_A, \widehat{\mathcal{D}}(\vec{N})\right\} = N^\mu \left(\partial_\mu \mathrm{C}_A\right) + \left(\partial_\gamma N^\mu\right) \mathrm{C}_B \mathrm{F}^B{}_\mu{}^\gamma{}_{:A} + \left(\partial_\mu N^\mu\right) \mathrm{C}_A , \tag{3.221}$$

which tells us that the terms in the first line of (3.220) cancel each other. The only remaining part is to show that the functional derivative of the integral vanishes.

But for this, we can contract (**C5**), using that $\mathrm{M}^{A\gamma}$ vanishes, with a shift $\vec{N}$ to show that the integrand is actually a divergence term

$$\mathrm{C}_B \left(\mathcal{L}_{\vec{N}}\varphi\right)^B = -\partial_\gamma \left(N^\mu \mathrm{C}_A \mathrm{F}^A{}_\mu{}^\gamma\right) . \tag{3.222}$$

As a result, the remaining term vanishes, and we find $\mathcal{I} = 0$. But, by expanding the functional derivatives in the definition of $\mathcal{I}$ and by the fact that the integral vanishes for any $\vec{N}$ we can conclude from this that the remaining equations from (3.216) are also fulfilled, and we find that C_A is a functional gradient.

Using the exact form of the equations of motion (3.151) we furthermore see that C_A solves the evolutionary equation identically and, thus, does not contribute to the dynamics of the gravitational degrees of freedom. Even further, due to (3.222) we can show that C_A is a boundary term in the Lagrangian that will disappear from the equations of motion, since

$$\begin{aligned}\int \mathrm{d}t \int \mathrm{d}^3x \frac{\delta\Lambda}{\delta\varphi^A(t,x)}\left(\dot{\varphi}^A(t,x) - \left(\mathcal{L}_{\vec{N}}\varphi\right)^A(t,x)\right) &= \int \mathrm{d}t \left(\dot{\Lambda} + \int \mathrm{d}^3x\, \partial_\gamma \left(N^\mu \frac{\delta\Lambda}{\delta\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma\right)(t,x)\right) \\ &= \text{boundary term} .\end{aligned} \tag{3.223}$$

Consequently, we can ignore C_A in our discussions and leave it undetermined when solving the closure equations for $M^{A\gamma} = 0$. We will use this when we perform the gravitational closure of Maxwellian electrodynamics.

Note that for $M^{A\gamma} \neq 0$, even if C_A is a functional gradient and disappears from the evolutionary equations, equation (3.222) is not valid anymore and we *cannot* conclude directly whether C_A is a boundary term that can be ignored in the constraints.

(Partial) linear dependency on the 2$^{\text{nd}}$ derivatives of the degrees of freedom

For some theories, we can generalise the dimensionally dependent argument that allowed us to derive that the output coefficients $C_{A_1 \ldots A_N}$ are polynomials of degree 3 in $\varphi^A{}_{,\mu\nu}$ and now show that they are, in fact, linear. To end up at this conclusion, we observe that we can again construct a tensor of rank 9 with the help of the input coefficient

$$\mathcal{U}^{A\mu\nu} := p^{\mu\sigma} F^A{}_\sigma{}^\nu\,, \tag{3.224}$$

by defining

$$\Lambda^{\alpha\beta\gamma\delta\mu\nu\lambda\kappa} := C_{B_1 \ldots B_N : A}{}^{\alpha\beta}{}_B{}^{\gamma\delta}\, \mathcal{U}^{A\mu\nu}\, \mathcal{U}^{B\lambda\kappa}\,, \tag{3.225}$$

for $N \neq 1$. Due to $(\mathbf{C17_N})$ (as well as $(\mathbf{C4})$) we again have that the following symmetrizations vanish

$$\Lambda^{(\alpha\beta\gamma)\delta\mu\nu\lambda\kappa} = \Lambda^{\alpha(\beta\gamma\delta)\mu\nu\lambda\kappa}\,. \tag{3.226}$$

Similarly, we have from the covariance part of the closure equations $(\mathbf{C12_{N\geq 2}})$ and $(\mathbf{C8_3})$ that

$$0 = C_{B_1 \ldots B_N : A}{}^{(\alpha\beta|}\mathcal{U}^{A\mu|\gamma)}\,, \tag{3.227}$$

which tells us that also the symmetrizations over two indices at the first two blocks and one index at one of the last two blocks vanishes.

A priori, we do not know if the coefficient $\mathcal{U}^{A\mu\nu}$ is symmetric in its indices $\mu\nu$. Luckily, since $M^{A\gamma}$ is vanishing, we can employ the closure equation $(\mathbf{C14_N})$ to get

$$\Lambda^{\alpha\beta\gamma\delta[\mu\nu]\lambda\kappa} = \Lambda^{\alpha\beta\gamma\delta\mu\nu[\lambda\kappa]} = 0\,. \tag{3.228}$$

We can then repeat the argument made in equation (3.208) and show that we can exchange an index from one of the blocks that belongs to the derivative of the output coefficient, and one index from the input coefficient, i.e.

$$\Lambda^{\alpha\beta\gamma\delta\mu\nu\lambda\kappa} = -2\Lambda^{\alpha\beta\mu(\gamma\delta)\nu\lambda\kappa}\,. \tag{3.229}$$

Since three of the eight indices must be the same, we can again check all the possibilities of how to distribute the indices to the four blocks, as in (3.209), and ultimately find that.

$$C_{B_1 \ldots B_N : A}{}^{\alpha\beta}{}_B{}^{\beta\gamma}\, \mathcal{U}^{A\mu\nu}\, \mathcal{U}^{B\gamma\delta} = 0 \qquad \text{for} \qquad N \neq 1\,. \tag{3.230}$$

If we have a right inverse $\mathcal{U}^{-1}$ to $\mathcal{U}$, in the sense that

$$\left(\mathcal{U}^{-1}\right)_{B\mu\nu}\mathcal{U}^{A\mu\nu} = \delta^A_B\,, \tag{3.231}$$

we can conclude that the output coefficient depends *linearly* on the second derivatives of the degrees of freedom. In general, as we have seen in our discussion of the covariance part of the closure equations, the rank will be less than 9 such that we can draw that conclusion only for some of the components. Since we can always reparametrize our degrees of freedom, it is possible to separate the kernel of $\mathrm{F}^A{}_\mu{}^\gamma$ from the image to obtain these linear components. We will take a closer look at such reparametrizations in the following section of this chapter.

Note that this result is not crucially dependent on $\mathrm{M}^{A\gamma} = 0$: the only requirement one needs is that the coefficient is symmetric in the two index blocks belonging to the $\mathrm{p}^{\rho(\mu|}\mathrm{F}^A{}_\rho{}^{|\nu)}$ terms.

3.5.4 Field reparametrization

Before concluding this general chapter about gravitational closure, let us consider one last point. In general relativity, it is often favourable to pick a system of coordinates adapted to the particular problem at hand. The same applies when considering the gravitational closure equations, where we can make use of the fact that many different parametrizations of our gravitational degrees of freedom exist, and none is favoured over the others. We can use this freedom to try to simplify our setup.

Under a reparametrization, that is a coordinate change on the space Φ, one finds that the input coefficients transform as Φ tensors. In particular, this means that the tangential deformation coefficient fulfills

$$\mathrm{F}^A{}_\mu{}^\gamma(\varphi) \longrightarrow \frac{\partial\psi^A}{\partial\varphi^B}\mathrm{F}^B{}_\mu{}^\gamma\left(\psi(\varphi)\right)\,, \tag{3.232}$$

as can easily be verified by inserting the reparametrization into equation (3.86).

Writing the coefficient down as a matrix, with rows arranged by the nine components for μ and γ and the columns by the F components for A, it is obvious that the coefficient has *at most* a rank of 9, as we saw before in the discussion of the covariance part of the closure equations. We can use this to our advantage to construct a reparametrization such that the kinematical coefficient is brought into a form that simplifies the gravitational closure equations. We can, in particular, reparametrize such that we can separate the kernel of $\mathrm{F}^A{}_\mu{}^\gamma$ from its image.

For example, we could aim to construct a parametrization $(\psi^a, \nu^\alpha, \sigma^{(i)})$, with the indices $a = 1,\ldots,6$, $\alpha = 1,\ldots,3$ and $(i) = 1,\ldots,F-9$. For these functions we write down the system of linear first-order partial differential equations

$$\frac{\partial\psi^a}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma = 2\mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_b\psi^b(\varphi)\,, \tag{3.233a}$$

$$\frac{\partial\nu^\alpha}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma = \delta^\alpha_\mu\nu^\gamma(\varphi)\,, \tag{3.233b}$$

$$\frac{\partial\sigma^{(i)}}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma = 0\,, \tag{3.233c}$$

where the right hand side terms correspond to $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ in our new parametrization of the degrees of freedom. The symbol $\mathcal{I}^{a}{}_{\mu\nu}$ is defined as

$$\mathcal{I}^{\alpha\beta}{}_{a} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{\alpha\beta}{}_{A} , \tag{3.234}$$

and has the inverse

$$\mathcal{I}^{a}{}_{\alpha\beta} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{a}{}_{\alpha\beta} . \tag{3.235}$$

The expressions in equations (3.233) are, of course, far from random but chosen such that the transformed input coefficients $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ have the interpretation of six metric-, three vector- and $F-9$ scalar degrees of freedom. Given that such a reparametrization exists and is a diffeomorphism, i.e. that the determinant of the Jacobian matrix is non-vanishing

$$\det\left(\frac{\partial\psi^{a}}{\partial\varphi^{A}}, \frac{\partial\nu^{\alpha}}{\partial\varphi^{A}}, \frac{\partial\sigma^{(i)}}{\partial\varphi^{A}}\right) \neq 0 , \tag{3.236}$$

we can, at least locally, express our original degrees of freedom by the new coordinates $\varphi^{A}(\psi^{a}, \nu^{\alpha}, \sigma^{(i)})$. The obvious question is now if such – or rather under which conditions – a solution exists and if it is unique.

The latter part can be answered rather quickly: one immediately finds that the solutions of this system (3.233) are clearly not unique, since *any* scalar, vector or inverse metric constructed from the geometry solve the equation. In particular, we find for any scalar function $f(\sigma^{(i)})$ that

$$\frac{\partial f}{\partial\varphi^{A}}\mathrm{F}^{A}{}_{\mu}{}^{\gamma} = \frac{\partial f}{\partial\sigma^{(i)}}\frac{\partial\sigma^{(i)}}{\partial\varphi^{A}}\mathrm{F}^{A}{}_{\mu}{}^{\gamma} = 0 , \tag{3.237}$$

which tells us that f is also a solution of the differential equation.

The existence of a solution is more subtle. Clearly, we can use the screen manifold metric obtained from the principal polynomial as the metric degrees of freedom. From this object, we can then always project the six independent components with the help of the constant intertwiner $\mathcal{I}^{\alpha\beta}{}_{a}$, as described in section 3.2.1 and in the following in 4.1, and write

$$\psi^a(\varphi) := \mathcal{I}^a{}_{\mu\nu}\mathrm{p}^{\mu\nu}(\varphi)\,.$$

It is then easy to verify that this is a solution of the differential equation (3.233a).

For the scalar degrees of freedom, we can inspect equation (3.233c) more closely: it clearly states that the gradient $\frac{\partial\sigma^{(i)}}{\partial\phi^A}$ lies in the kernel of the coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ which must be *at least* of dimension $F-9$, as we have already seen in the discussion of the covariance part of the closure equations. We can construct a basis $n^{(i)}_A$ with the typical algorithms from linear algebra, the result being dependent on the jet space point[11].

We can write these basis objects as derivative of a scalar function if the condition

$$n^{(i)}_{A:B} - n^{(i)}_{B:A} = 0$$

is fulfilled. Conversely, as long as the basis vectors are sufficiently differentiable, we can at least in a formal series expansion solve the expansion coefficients in terms of derivatives of our basis vectors. Typically, however, it is easier to directly construct the scalar functions from the geometric fields, as we will see in the discussion of birefringent electrodynamics in chapter 4.2. We expect this to be the case in any physically suitable theory.

The most interesting equation thus is (3.233b) that introduces an additional hypersurface vector[12]. Let us raise the index μ with $\mathrm{p}^{\cdot\cdot}$ and anti-symmetrize in both spatial indices. Given that the anti-symmetric part vanishes, i.e.

$$\mathrm{p}^{\sigma[\mu|}\mathrm{F}^A{}_\sigma{}^{|\nu]} = 0\,, \tag{3.238}$$

we see directly from equation (3.233b) that this implies that the vector has to vanish. This instructs us that, while the symmetric part of the coefficient contributes six degrees of freedom in the form of a metric, the antisymmetric part of $\mathrm{F}^A{}_\mu{}^\gamma$ is linked to three degrees of freedom that can be put into the form of a vector on the screen manifold.

We will assume now that for a theory, we *can* make such a reparametrization and consider some of the implications. Before we can enter the gravitational closure equation for this parametrization, we need to calculate the three input coefficients. The inverse metric $\mathrm{p}^{\cdot\cdot}$ becomes, by construction, the neat linear expression

$$\mathrm{p}^{\alpha\beta} \longrightarrow \mathcal{I}^{\alpha\beta}{}_m\psi^m\,. \tag{3.239}$$

Similarly, the coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ can be read off directly from the construction of our reparametrization

$$\mathrm{F}^A{}_\mu{}^\gamma \longrightarrow \begin{cases} 2\,\mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_b\psi^b & \qquad A = a \\ \delta^\alpha_\mu\,\nu^\gamma & \text{for} \quad A = \alpha \\ 0 & \qquad A = (i) \end{cases}\,. \tag{3.240}$$

[11] The rank of the input coefficient may even differ at points. For example, at $\varphi = 0$, one usually finds that the rank is zero. However, apart from a perturbative parametrization, this typically means that the geometric fields are trivial themselves, which are of limited use. It thus makes sense to restrict ourselves to only the relevant subspace, and we will, therefore, not consider such rank defects.

[12] Equivalently, it would be possible to use a covector ν_α by lowering the index with $(\mathrm{p}^{\cdot\cdot -1})$.

For the non-local normal deformation coefficient $\mathrm{M}^{A\gamma}$, no a priori statement can be made about the form the coefficient will take once we perform the reparametrization. Given that we have an explicit expression for the reparametrization, as well as its inverse, we can calculate it directly via

$$\mathrm{M}^{A\gamma} \quad \longrightarrow \quad \begin{cases} \frac{\partial \psi^a}{\partial \varphi^A}\left(\varphi(\psi,\nu,\sigma^{(i)})\right)\mathrm{M}^{A\gamma}\left(\varphi(\psi,\nu,\sigma^{(i)})\right) & A = a\,, \\ \frac{\partial \nu^\alpha}{\partial \varphi^A}\left(\varphi(\psi,\nu,\sigma^{(i)})\right)\mathrm{M}^{A\gamma}\left(\varphi(\psi,\nu,\sigma^{(i)})\right) \qquad \text{for} & A = \alpha\,, \\ \frac{\partial \sigma^{(i)}}{\partial \varphi^A}\left(\varphi(\psi,\nu,\sigma^{(i)})\right)\mathrm{M}^{A\gamma}\left(\varphi(\psi,\nu,\sigma^{(i)})\right) & A = (i)\,. \end{cases} \tag{3.241a}$$

Further simplification can be made with the help of the chain rule (3.56) for the coefficient to derive the explicit expression once the vector and scalar functions are set up. However, no normal form that makes its interpretation conceptually clear can be read off.

This is not surprising, since the $\mathrm{M}^{A\gamma}$ contains information about the *spacetime* geometry, whereas $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ only depends on the projected fields on the hypersurface and how they transform under spatial diffeomorphisms. Only for the metric sector, we can verify that the coefficient reads

$$\mathrm{M}^{a\gamma}(\psi^a,\nu^\alpha,\sigma^{(i)}) = -(\deg P - 2)\,\mathcal{I}^a{}_{\alpha\beta}\,\mathrm{p}^{\alpha\beta\gamma}(\psi^a,\nu^\alpha,\sigma^{(i)})\,, \tag{3.242}$$

which tells us that the non-local contribution in the equations of motion of the metric degrees of freedom is related to the degree of the polynomial and the projection $\mathrm{p}^{\alpha\beta\gamma}$ – for any theory with degree 4 or higher. Note that we can only arrive at this result because we know how the projection of the principal polynomial is defined, such that it is straightforward to derive the coefficient. For the scalars and the vector, this is not the case, so that, a priori, we need to leave the coefficient a remaining input coefficient to the gravitational closure equations.

This is interesting for the following reason: let us forget for a moment about the spacetime theory we started with and take the (limited) perspective of the human view (compare figure 3.3) with fields that are prescribed on the three-dimensional screen manifold. There, we start with initial data values for the principal polynomial $\mathrm{p}^{\alpha\beta}$, the vector field and *some* number of scalars. Then two gravitational theories, with the same number of degrees of freedom, will only differ in their coefficients $\mathrm{M}^{A\gamma}$. Moreover, their coefficients in the equations of motion, as a solution of the gravitational closure equations, are parametrized by the kinematical coefficient. As a result, once all the lengthy and highly non-trivial calculations are carried through, and one ends up with the interaction of matter with these gravitational theories, their difference in predictions are parametrized by the input coefficient $\mathrm{M}^{A\gamma}$. This means that doing gravitational experiments to probe the nature of spacetime geometry is equivalent to probing the kinematical $\mathrm{M}^{A\gamma}$ coefficient.

Before we conclude this chapter, we will now further explore how one can work towards a solution of the gravitational closure equations for such a setup and what constructions we can make.

Dual variables

When constructing expressions for the output coefficients, it is clear that some inverse objects need to show up that allow us to contract the vector field and the principal polynomial. In general relativity, the situation is obvious as we formulate the coefficients in terms of the metric *and* its inverse.

Similarly we can also define dual objects in our setting for the principal polynomial $\mathrm{p}^{\cdot\cdot}$ and our vector components $\nu^{\cdot}$ by the following constructions

$$\omega_a(\psi) = \frac{1}{\det \mathrm{p}^{\cdot\cdot}(\psi)}\,\mathcal{I}^{\alpha\beta}{}_a\,\mathrm{adj}\left(\mathrm{p}^{\cdot\cdot}(\psi)\right)_{\alpha\beta} = \mathcal{I}^{\alpha\beta}{}_a\left(\mathrm{p}^{\cdot\cdot -1}\right)_{\alpha\beta}(\psi)\,, \tag{3.243a}$$

$$\mu_\alpha(\nu,\psi) = \frac{(\mathrm{p}^{\cdot\cdot -1})_{\alpha\beta}(\psi)\nu^\beta}{(\mathrm{p}^{\cdot\cdot -1})_{\sigma\rho}(\psi)\nu^\sigma\nu^\rho}\,, \tag{3.243b}$$

where ω are the six components of a metric – the inverse of $\mathrm{p}^{\cdot\cdot}$ – and the covector μ_α. By contracting the objects with our degrees of freedom we find that

$$\mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\nu}{}_b\,\omega_a(\psi)\,\psi^b = \delta^\nu_\mu\,, \tag{3.244a}$$

$$\mu_\alpha(\psi,\nu)\,\nu^\alpha = 1\,. \tag{3.244b}$$

Both these objects will, in some form, show up in any solution of the covariance part of the closure equations. For simplicity we will denote $\mathrm{p}^{\cdot\cdot -1}$ simply as $\mathrm{p}_{\cdot\cdot}$ in the following.

The derivatives of our dual objects are straight forward to calculate and one finds that

$$\frac{\partial\omega_a}{\partial\psi^b} = -\mathcal{I}^{\alpha\beta}{}_a\mathcal{I}^{\gamma\delta}{}_b\mathrm{p}_{\alpha\gamma}\mathrm{p}_{\beta\delta}\,, \tag{3.245a}$$

$$\frac{\partial\mu_\alpha}{\partial\nu^\beta} = \frac{\mathrm{p}_{\alpha\beta}}{\mathrm{p}_{\sigma\rho}\nu^\sigma\nu^\rho} - 2\,\mu_\alpha\mu_\beta\,, \tag{3.245b}$$

$$\frac{\partial\mu_\alpha}{\partial\psi^a} = \frac{\mathcal{I}^{\gamma\delta}{}_a\mu_\alpha\mu_\gamma\mu_\delta}{\mathrm{p}^{\sigma\rho}\mu_\sigma\mu_\rho} - \mathcal{I}^{\gamma\delta}{}_a\mu_\gamma\mathrm{p}_{\alpha\delta}\,. \tag{3.245c}$$

One input coefficient that is essential to determine the form of the output coefficient in terms of the lower order coefficients is the tangential deformation coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ with the index raised by $\mathrm{p}^{\cdot\cdot}$. Take for example (**C3**) that relates C_{AB} to C and C_A via

$$\begin{aligned} 0 = {} & 2\left(\deg P - 1\right)\mathrm{C}_{AB}\,\mathrm{p}^{\sigma(\mu|}\mathrm{F}^A{}_\sigma{}^{|\nu)} + \sum_{K=0}^{\infty}(K+1)\,\mathrm{C}_{B:A}{}^{\alpha_1\ldots\alpha_K(\mu|}\mathrm{M}^{A|\nu)}{}_{,\alpha_1\ldots\alpha_K} \\ & - \sum_{K=0}^{\infty}(-1)^K\binom{K+2}{K}\left(\partial_{\alpha_1\ldots\alpha_K}\mathrm{C}_{:B}{}^{\alpha_1\ldots\alpha_K\mu\nu}\right)\,. \end{aligned} \tag{3.246}$$

In our normal form parametrization this coefficient takes the form

$$\mathcal{U}^{A\mu\nu} := \mathrm{p}^{\mu\sigma}\mathrm{F}^A{}_\sigma{}^\nu = \begin{cases} 2\,\mathcal{I}^a{}_{\sigma\rho}\mathcal{I}^{\sigma(\mu}{}_m\mathcal{I}^{\nu)\rho}{}_n\psi^m\psi^n & \text{for} \quad A = a \\ \mathcal{I}^{\mu\alpha}{}_m\psi^m\nu^\nu & \qquad\quad A = \alpha \\ 0 & \qquad\quad A = (i) \end{cases}. \tag{3.247}$$

In order to solve the equation (**C3**) for C_{AB} – at least for as many components as possible – we need to be able to invert $\mathcal{U}$ from the equation. Clearly, it is impossible for more than nine degrees of freedom to construct a matrix inverse. However, if we make the educated guess

$$(\mathcal{U}^{-1})_{A\mu\nu} := \begin{cases} \frac{1}{2}\,\mathcal{I}^{\sigma\rho}{}_a\mathcal{I}^m{}_{\sigma(\mu}\mathcal{I}^n{}_{\nu)\rho}\omega_m\omega_n & \qquad\quad A = a \\ \mathcal{I}^m{}_{\mu\alpha}\omega_m\mu_\nu & \text{for} \quad A = \alpha \\ 0 & \qquad\quad A = (i) \end{cases}, \tag{3.248}$$

it is then easy to check that contraction over μ and ν gives

$$\mathcal{U}^{A\mu\nu}(\mathcal{U}^{-1})_{B\mu\nu} = \left(\begin{array}{c|c|c} \delta^a_b & 2\left(\mathrm{p}^{\sigma\rho}\mu_\sigma\mu_\rho\right)\,\mathcal{I}^a{}_{\alpha\beta}\nu^\alpha & 0 \\ \hline \frac{1}{2\left(\mathrm{p}^{\sigma\rho}\mu_\sigma\mu_\rho\right)}\,\mathcal{I}^{\alpha\beta}{}_b\mu_\beta & \delta^\alpha_\beta & 0 \\ \hline 0 & 0 & 0 \end{array}\right)^A{}_B \,. \tag{3.249}$$

This will allow us to simplify the selective part of the closure equations significantly in the following. While a complete evaluation will be out of reach, alone due to the fact that we do not have an explicit expression for the input coefficient $\mathrm{M}^{A\gamma}$, some results can be obtained that will simplify the evaluation in the future.

Application to the covariance part of the closure equations

Due to the involutivity of the covariance part of the closure equations, we already know that we can construct finitely many curvature invariants from the degrees of freedom that will parametrize the solution space of the gravitational closure equations. Depending on the maximal number of spatial derivatives of the degrees of freedom, we can even calculate the number of these invariants. For example, for at most second derivatives, we have that the scalar output coefficient can be written in terms of

$$10 \cdot F - 57 \tag{3.250}$$

invariants (see equation (3.200)). For general relativity, as we will see in the next chapter 4.1, one ends up with three curvature invariants built from the Ricci curvature tensor. Clearly, if we construct these scalars from $\mathrm{p}^{\cdot\cdot}(\psi)$, we obtain a solution to the covariance part of the closure equations such that three of the curvature invariants can be attributed to the metric sector of the degrees of freedom.

Using the principal polynomial we can simplify the derivation of the curvature invariants: we can construct the Christoffel symbols

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\mathrm{p}^{\alpha\lambda}\left(2\,\mathrm{p}_{\lambda(\beta,\gamma)} - \mathrm{p}_{\beta\gamma,\lambda}\right) \tag{3.251}$$

for the degrees of freedom to define a covariant derivative and to change coordinates on our jet space. The first derivatives $\psi^a{}_{,\mu}$ can then be replaced with

$$\Gamma^a{}_\mu := \left(\frac{1}{2}\delta^a_b\delta^\nu_\mu - \mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\nu}{}_b\right)\psi^b{}_{,\nu}\,. \tag{3.252}$$

Similarly we can then replace *some* of the 2nd derivatives by the Riemann tensor constructed from $\mathrm{p}^{\cdot\cdot}$ in three dimension. In three dimensions we find that its components are given by

$$\begin{aligned}\mathcal{R}^a[\psi] = {} & 2\,\mathcal{I}^a{}_{\alpha\beta}\epsilon^{\alpha\mu\lambda}\epsilon^{\beta\nu\kappa}\mathrm{p}_{\lambda\sigma}\mathrm{p}_{\rho\kappa}\mathcal{I}^{\sigma\rho}{}_b\psi^b{}_{,\mu\nu} \\ & + \mathcal{I}^a{}_{\alpha\beta}\epsilon^{\alpha\lambda_1\lambda_2}\epsilon^{\beta\kappa_1\kappa_2}\Big[\delta^\mu_{\sigma_1}\mathrm{p}_{\sigma_2\lambda_1}\mathrm{p}_{\lambda_2\rho_1}\mathrm{p}_{\rho_2\kappa_2}\delta^\nu_{\kappa_1} + \delta^\mu_{\sigma_1}\mathrm{p}_{\sigma_2\kappa_1}\mathrm{p}_{\kappa_2\rho_1}\mathrm{p}_{\rho_2\lambda_2}\delta^\nu_{\lambda_1} + \delta^\mu_{\lambda_1}\mathrm{p}_{\lambda_2\sigma_1}\mathrm{p}_{\sigma_2\rho_1}\mathrm{p}_{\rho_2\kappa_2}\delta^\nu_{\kappa_1} \\ & \quad + 2\,\delta^\mu_{\lambda_1}\mathrm{p}_{\lambda_2\rho_1}\mathrm{p}_{\rho_2\sigma_1}\mathrm{p}_{\sigma_2\kappa_2}\delta^\nu_{\kappa_1} + \frac{1}{2}\mathrm{p}_{\sigma_1\lambda_1}\mathrm{p}_{\lambda_2\rho_1}\mathrm{p}_{\rho_2\sigma_2}\delta^\mu_{\kappa_2}\delta^\nu_{\kappa_1} \\ & \quad + \frac{1}{2}\mathrm{p}_{\sigma_1\kappa_1}\mathrm{p}_{\kappa_2\rho_1}\mathrm{p}_{\rho_2\sigma_2}\delta^\mu_{\lambda_2}\delta^\nu_{\lambda_1} + \frac{1}{2}\mathrm{p}^{\mu\nu}\mathrm{p}_{\sigma_1\lambda_1}\mathrm{p}_{\lambda_2\rho_1}\mathrm{p}_{\rho_2\kappa_1}\mathrm{p}_{\kappa_2\sigma_2}\Big]\mathcal{I}^{\sigma_1\sigma_2}{}_m\mathcal{I}^{\rho_1\rho_2}{}_n\psi^m{}_{,\mu}\psi^n{}_{,\nu}\,,\end{aligned} \tag{3.253}$$

where we dualized with the help of $\epsilon^{\cdots}$ and the constant intertwiner. This, however, only gives six of the 36 second-order jet variables. For the remaining variables we introduce the object $\mathcal{S}_{\lambda(\alpha\beta\gamma)}$ in the following fashion

$$\begin{aligned}\mathcal{S}_{\lambda(\alpha\beta\gamma)}[\psi] = &\left[\frac{1}{2}\delta^{(\mu}_{\lambda}\delta^{\nu)}_{(\alpha|}\mathrm{p}_{|\beta|(\sigma}\mathrm{p}_{\rho)|\gamma)} - \delta^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta}\mathrm{p}_{\gamma)(\sigma}\mathrm{p}_{\rho)\lambda}\right]\mathcal{I}^{\sigma\rho}{}_{a}\psi^{a}{}_{,\mu\nu}\\ &+\left[2\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta}\mathrm{p}_{\gamma)(\sigma_1}\mathrm{p}_{\rho_1)\lambda} - \delta^{\mu}_{\lambda}\delta^{\nu}_{(\alpha}\mathrm{p}_{\beta|(\sigma_1}\mathrm{p}_{\rho_1)|\gamma)}\right]\\ &\times \mathrm{p}_{\sigma_2\rho_2}\mathcal{I}^{\sigma_1\sigma_2}{}_{a}\mathcal{I}^{\rho_1\rho_2}{}_{b}\psi^{a}{}_{,\mu}\psi^{b}{}_{,\nu}\,. \end{aligned}\tag{3.254}$$

It is then possible to express all $\psi^{a}{}_{,\mu\nu}$ terms by the Riemann tensor and the object $\mathcal{S}_{\mu(\alpha\beta\gamma)}$. Note that the latter does *not* transform as a tensor under coordinate changes.

Once we have three vector degrees of freedom, the number of curvature invariants increases to 33, i.e. we get additional thirty expressions that can be formed. For these we can also change the coordinates to the covariant derivatives

$$v^{\alpha}{}_{;\mu} = v^{\alpha}{}_{,\mu} + \Gamma^{\alpha}{}_{\mu\lambda}v^{\lambda}\,, \tag{3.255a}$$

$$\begin{aligned}v^{\alpha}{}_{;(\mu\nu)} = &\, v^{\alpha}{}_{,\mu\nu} + \Gamma^{\alpha}{}_{(\mu|\lambda,|\nu)}v^{\lambda} + 2\,\Gamma^{\alpha}{}_{(\mu|\lambda}v^{\lambda}{}_{,|\nu)} - \Gamma^{\lambda}{}_{\mu\nu}v^{\alpha}{}_{,\lambda}\\ &\left(\Gamma^{\alpha}{}_{(\mu|\lambda}\Gamma^{\lambda}{}_{|\nu)\kappa} - \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\alpha}{}_{\lambda\kappa}\right)v^{\kappa}\,,\end{aligned}\tag{3.255b}$$

where it suffices to take the the symmetric part of the second order covariant derivatives since the anti-symmetric part can be expressed in terms of the Riemann tensor. With these we can then try to come up with the thirty functionally independent invariants such as

$$v^{\alpha}{}_{;\alpha}\quad,\quad \mathrm{p}^{\mu\nu}\mathrm{p}_{\alpha\beta}\,v^{\alpha}{}_{;\mu}v^{\beta}{}_{;\nu}\quad,\quad \mathrm{p}^{\mu\nu}\mu_{\alpha}\mu_{\beta}\,v^{\alpha}{}_{;\mu}v^{\beta}{}_{;\nu}\quad,\ldots,$$

where one can verify that each of these is a solution of the covariance part of the closure equations $(\mathbf{C1})$ and $(\mathbf{C8_N})$ for the scalar coefficient.

Last but not least, each additional scalar degree of freedom introduces further ten curvature invariants. We can again construct the covariant derivatives

$$\sigma^{(i)}_{;\mu} = \sigma^{(i)}_{,\mu}\,, \tag{3.256}$$

$$\sigma^{(i)}{}_{;(\mu\nu)} = \sigma^{(i)}{}_{,\mu\nu} - \Gamma^{\lambda}{}_{(\mu\nu)}\sigma^{(i)}{}_{,\lambda}\,, \tag{3.257}$$

and, in principle, come up with the functionally independent terms. Since the exact number of invariants depends on the chosen number of spatial derivatives and the output coefficient one considers, we will not further pursue this here and simply note that ultimately one ends up with finitely many terms. The obtained expressions can then be used in the next step, the evaluation of the selective part of the closure equations, to determine the dependency of the output coefficient on the constructed curvature invariants, as explained before.

Application to the selective part of the closure equations

Similarly, we will sketch now how the selective part of the closure equations can be used to reduce many of the output coefficients and express them by C and C_A. For the output coefficients C_{ABC}, C_{ABCD}, ..., we have essentially five equations from the selective part of the same form, of which only two relate equations of different orders.

If we consider the closure equation $(\mathbf{C16_N})$, we find the relation between the output coefficient $\mathrm{C}_{B_1\dots B_{N+1}}$ and the lower order coefficients $\mathrm{C}_{B_1\dots B_N}$ and $\mathrm{C}_{B_1\dots B_{N-1}}$ for $N \geq 2$

$$
\begin{aligned}
0 = {} & N(N+1)(\deg P - 1)\,\mathrm{C}_{AB_1\dots B_N}\mathcal{U}^{A(\mu\nu)} + N\,\mathrm{C}_{B_1\dots B_N:A}{}^{(\mu|}\mathrm{M}^{A|\nu)} \\
& + 2N\,\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha(\mu|}\mathrm{M}^{A|\nu)}{}_{,\alpha} + (N-2)\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\mu\nu}\,.
\end{aligned} \tag{3.258}
$$

We can contract this with $\mathcal{U}^{-1}_{A\mu\nu}$ that we introduced in equation (3.248) to solve for the highest order output coefficient in the equation. For simplicity, we introduce the symbol

$$
\begin{aligned}
X^{\mu\nu}_{B_1\dots B_N}[\varphi] = {} & \frac{1}{(N+1)(\deg P - 1)}\Bigg[\sum_{K=0}^{\infty}(K+1)\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_K(\mu|}\mathrm{M}^{A|\nu)}{}_{,\alpha_1\dots\alpha_K} \\
& + \frac{N-2}{2}\sum_{K=0}^{\infty}(-1)^K\binom{K+2}{K}\left(\partial_{\alpha_1\dots\alpha_K}\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu\nu}\right)\Bigg]\,.
\end{aligned} \tag{3.259}
$$

Combined, we thus find for $N \geq 1$ due to $(\mathbf{C3})$ that we can solve for the components

$$
\mathrm{C}_{aB_1\dots B_N} = -\frac{1}{2}\mathcal{I}^{\alpha\beta}{}_a\mu_\alpha\mu_\beta\,\nu^\lambda\mathrm{C}_{\lambda B_1\dots B_N} - \frac{1}{2}\mathcal{I}^{\alpha\beta}{}_a\mathrm{p}_{\alpha\mu}\mathrm{p}_{\beta\nu}X^{\mu\nu}_{B_1\dots B_N}[\psi]\,, \tag{3.260a}
$$

$$
0 = \left(\delta^\alpha_\beta - \mu_\beta\nu^\alpha\right)\mathrm{C}_{\alpha B_1\dots B_N}\,. \tag{3.260b}
$$

Similarly, the equation from the selective part $(\mathbf{C14_N})$ can be brought into the form such that we can solve for three of the output coefficients by solving

$$
0 = \left(\mu_\alpha Y^\alpha{}_\beta(\varphi) - \frac{1}{2}\mathrm{p}_{\sigma\rho}\nu^\sigma\nu^\rho\,\mathcal{I}^{\gamma\delta}{}_a\mu_\gamma\mu_\delta Y^a{}_\beta(\varphi)\right)\nu^\lambda\mathrm{C}_{\lambda B_1\dots B_N} + \mathrm{C}_{(i)B_1\dots B_N}Y^{(i)}{}_\beta\,, \tag{3.261}
$$

for $\beta = 1, \dots, 3$, $N \geq 1$, where the symbol

$$
Y^A{}_\beta = \mathcal{U}^{-1}_{\beta\mu\nu}\,\mathrm{M}^{B[\mu|}\mathrm{M}^{A|\nu]}{}_{:B}(\varphi) \tag{3.262}
$$

is obtained by the input coefficient $\mathrm{M}^{A\gamma}$. Last but not least, we can bring the closure equations $(\mathbf{C4})$, $(\mathbf{C6})$ and $(\mathbf{C17_N})$ in the form that further components of the coefficient can be solved for

$$
0 = \left(\mu_\alpha\nu^\alpha{}_{;\mu}\right)\nu^\lambda\mathrm{C}_{\lambda B_1\dots B_N} + \sigma^{(i)}{}_{;\mu}\mathrm{C}_{(i)B_1\dots B_N} + Z_{\mu B_1\dots B_N}[\varphi]\,, \tag{3.263}
$$

with the covariant derivative of the vector and scalar constructed with $\mathrm{p}^{\cdot\cdot}$, as well as the coefficient

$$
\begin{aligned}
Z_{\mu B_1\dots B_N}[\varphi] = {} & -\frac{\mathrm{p}_{\mu\nu}}{(N+1)N(\deg P - 1)}\Bigg[N^2\,\mathrm{C}_{A(B_1\dots B_{N-1}|}\mathrm{M}^{A\nu}{}_{:|B_N)} + N\sum_{K=0}^{\infty}\mathrm{C}_{B_1\dots B_N:A}{}^{\alpha_1\dots\alpha_K}\mathrm{M}^{A\nu}{}_{,\alpha_1\dots\alpha_K} \\
& + \sum_{K=1}^{N}\mathrm{C}_{B_1\dots\widetilde{B_K}\dots B_N:B_K}{}^{\nu} + \sum_{K=1}^{\infty}(-1)^K(K+1)\left(\partial_{\alpha_1\dots\alpha_K}\mathrm{C}_{B_1\dots B_{N-1}:B_N}{}^{\alpha_1\dots\alpha_K\mu}\right)\Bigg] \\
& - \mathcal{U}^{-1}_{\alpha\sigma\rho}\,X^{\sigma\rho}_{B_1\dots B_N}[\varphi](\psi^a{}_{,\mu} - \mathrm{p}_{\mu\nu}\mathrm{p}^{\lambda\nu}{}_{,\gamma}\mathrm{F}^a{}_\lambda{}^\gamma)\,.
\end{aligned} \tag{3.264}
$$

Since the output coefficients are totally symmetric in its indices, and due to the equations (3.260) to (3.263), we can express any of the components that contain at least one metric index or vector index in terms of the lower order output coefficient and the output coefficient with only scalar indices. As a result, if we would further insert the expressions for the coefficients iteratively into the coefficients X and Z, we would ultimately find that the coefficients can be expressed solely by the scalar output coefficient C, C_A and the remaining scalar coefficients $\mathrm{C}_{(i_1)\ldots(i_N)}$ and the input coefficients, i.e.

$$\mathrm{C}_{B_1\ldots B_N} = f\left(\mathrm{C}, \mathrm{C}_A, \mathrm{C}_{(i)(j)}, \ldots, \mathrm{C}_{(i_1)\ldots(i_N)}\right) \qquad \text{for} \qquad N \geq 2\,. \tag{3.265}$$

The remaining questions that then need to be solved for the remaining coefficients are equations (**C5**), (**C7**), ($\mathbf{C13_N}$), ($\mathbf{C15_N}$) and ($\mathbf{C18_N}$) to ($\mathbf{C21_{N\,odd}}$), as well as the covariance part of the closure equations, as described in the previous section.

We will not pursue this any further by deriving an explicit recursion relation for the output coefficient and, instead, leave this up for future research. The special case for a single scalar and Lagrangians limited such that $\mathrm{C}_{\mu B_1\ldots B_N} = 0$ was derived in Witte (2014). This assumption simplifies the relations (3.261), (3.260) and (3.263) such that the recurrence relation between the output coefficients and, ultimately, the Lagrangian can indeed be given for all orders. With the results we sketched here in this section, we are confident that it is possible to generalise this result in the future.

CHAPTER 4
EXACT SOLUTIONS

Now that we derived the gravitational closure equations in the previous chapter and discussed parts of their general properties, it is about time to put everything we learned to good use for some specific gravitational theories. Although we already presented some aspects of the setups as examples during the derivations in the previous chapter, we will repeat them in the following for an as self-complete discussion as possible.

The chapter is structured in the following fashion: we will start with the most well-known and well-studied examples and perform gravitational closure for Maxwellian electrodynamics which results in Einstein's theory of general relativity. Afterwards, we present the setup for general linear electrodynamics, which aims for the dynamics of the area metric.

The kinematic setup and parametrization of the gravitational degrees of freedom for both theories discussed in this chapter have already been published as

M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational Closure of Matter Field Equations
Phys. Rev. D97 (2018), 084036

The solution of the gravitational closure equations for the Lorentzian metric in the following section expands upon joint work with Maximilian Düll that was presented in

M. Düll
Gravitational Closure of Matter Field Equations: General Theory and Symmetrization
PhD thesis, Universität Heidelberg (2020)

4.1 MAXWELLIAN ELECTRODYNAMICS: GENERAL RELATIVITY REGAINED

It is long known that if one wants to give dynamics to the causal coefficients appearing in Maxwell's equations there are almost no options besides Einstein's theory of relativity, formulated in the Einstein-Hilbert action

$$\mathcal{S}_{\mathrm{EH}}[g] = \frac{1}{\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{-g} \left(R[g] - 2\Lambda \right) \, . \tag{4.1}$$

However, as we will discuss now, we can also arrive at this result constructively by solving the gravitational closure equations for Maxwellian electrodynamics – or, by extension, the whole standard model of particle physics that is in fact tailored to be causally compatible and consistent with electrodynamics.

4.1.1 Kinematic setup

The first step in performing gravitational closure of a matter field is to calculate the principal polynomial of the matter field equations. As electrodynamics is equipped with an $\mathrm{U}(1)$ gauge symmetry, we need to put some additional effort in to extract the polynomial, as laid out in 2.2.1, but one ultimately finds that the principal polynomial of Maxwellian electrodynamics, and the whole standard model, is given by

$$P(x,k) = g^{ab}(x)\, k_a(x)k_b(x)\,, \tag{4.2}$$

for the inverse metric $g^{\cdot\cdot}$. The next step is then to impose the three matter conditions. The first matter condition reveals that $g^{\cdot\cdot}$ must have Lorentzian signature. We further make the assumption, without loss of generality, that for a future-directed covector n we have $P(n) > 0$, i.e. we agree on positive signature convention.

For the second matter conditions we need to derive the dual polynomial. As was shown in Rivera (2012), this polynomial is given by the metric $g_{\cdot\cdot}$

$$P^\sharp(x,v) = g_{ab}(x)v^a(x)v^b(x)\,. \tag{4.3}$$

As a result, one finds that the massless point particle action takes the familiar form

$$\mathcal{S}_{\text{massless}}[x,\lambda] = \int \mathrm{d}\tau\, \lambda g_{ab}\dot{x}^a\dot{x}^b\,. \tag{4.4}$$

In this case, the second matter condition, which demands the hyperbolicity of the dual polynomial, gives no further restrictions since $g_{\cdot\cdot}$, being the inverse of $g^{\cdot\cdot}$, is already of Lorentzian signature. Also, the last matter condition, being energy-distinguishing, can be shown to give no further restrictions (Raetzel et al., 2011).

Using the principal polynomial and its dual, we can write down our two possible definitions for a Legendre map

$$\ell(k)^a := \frac{g^{ab}k_b}{g^{mn}k_mk_n}\,, \tag{4.5a}$$

$$\widetilde{\ell}(v)_a := \frac{g_{ab}v^b}{g_{mn}v^mv^n}\,, \tag{4.5b}$$

where it is easy to verify that $\widetilde{\ell} = \ell^{-1}$. The action of a massive point particle with the dispersion relation $g^{ab}k_ak_b = m^2$ is then given by the well known expression

$$\mathcal{S}_{\text{massive}}[x] = m\int \mathrm{d}\tau\,\sqrt{g_{ab}\dot{x}^a\dot{x}^b}\,, \tag{4.6}$$

and the Finsler metric constructed from the polynomial $P^\star$ coincides with the inverse metric. With all matter conditions implemented, and equipped with the Legendre map, we can move on to construct the observer frame and project the spacetime geometry to the screen manifold Σ.

Recall that for the observer frame, constructed with the help of the Legendre map ℓ, to be unique we make the choice that in this frame

$$1 = P(\epsilon^0, \epsilon^0) = g^{ab}\epsilon^0_a\epsilon^0_b\,, \tag{4.7a}$$

$$0 = P(\epsilon^\alpha, \epsilon^0) = g^{ab}\epsilon^0_a\epsilon^\alpha_b\,. \tag{4.7b}$$

As a result, the only projection of the inverse metric to the screen manifold that is not fixed by the annihilation and normalisation condition in the construction of the observer frame is the pull-back given by

$$g^{\alpha\beta} = g^{ab}\,\epsilon^\alpha_a\,\epsilon^\beta_b\,. \tag{4.8}$$

The annihilation and normalisation conditions will then be trivially conserved under time evolution in our canonical formulation, as the four components eliminated by the conditions are not carried over in the phase space construction in the first place. However, as explained in the previous chapter, implementing the conditions by simply omitting the two projections is only possible due to the linearity of the conditions of the metric setup. In more complex geometries, for instance, an area metric that is seen by general linear electrodynamics, this is not possible anymore.

4.1.2 Parametrization

Since the annihilation and normalisation conditions have no impact on the symmetric inverse metric $g^{\cdot\cdot}$ on the screen manifold, we can parametrize the six degrees of freedom linearly as

$$\widehat{g}^{\alpha\beta}(\varphi^1, \dots, \varphi^6) = \mathcal{I}^{\alpha\beta}{}_A\varphi^A\,, \tag{4.9a}$$

$$\widehat{\varphi}^A(g^{\alpha\beta}) = \mathcal{I}^A{}_{\alpha\beta}g^{\alpha\beta}\,, \tag{4.9b}$$

with the constant intertwiner

$$\mathcal{I}^{\alpha\beta}{}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{\alpha\beta}_{\quad A}\,, \tag{4.10}$$

and its inverse

$$\mathcal{I}^{A}{}_{\alpha\beta} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{A}{}_{\alpha\beta} . \tag{4.11}$$

It is easily verified that the construction above provides a suitable parametrization. Note that this linear parametrization is chosen so that the inverse of the constant intertwiners is obtained by taking the transpose.

Before we can enter the gravitational closure equations, we still need to calculate the three input coefficients. With the principal polynomial being the geometry itself, one immediately finds that

$$\mathrm{p}^{\mu\nu} = \mathcal{I}^{\mu\nu}{}_{A}\, \varphi^{A} . \tag{4.12}$$

By decomposing the Lie derivative of $g^{\cdot\cdot}$ into its two parts we can furthermore read off that

$$\mathrm{F}^{A}{}_{\mu}{}^{\gamma} = 2\,\mathcal{I}^{A}{}_{\mu\rho}\, \mathcal{I}^{\rho\gamma}{}_{B}\, \varphi^{B} . \tag{4.13}$$

For the last input coefficient, the non-local normal deformation coefficient, we find

$$\begin{aligned} \mathrm{M}^{A\gamma} &= \frac{\partial \widehat{\varphi}^{A}}{\partial g^{\alpha\beta}} \left(\widehat{g}^{\cdot\cdot}\right)\, e^{a}_{0} \frac{\partial g^{\alpha\beta}}{\partial(\partial_\gamma X^{a})} \\ &= 0 . \end{aligned} \tag{4.14}$$

As we have seen in the previous chapter, this simplifies the gravitational closure equations significantly, and we can use many of the results derived in section 3.5.3. With all input coefficients calculated, this means we can move on to the next step and solve the gravitational closure equations.

4.1.3 Solving the gravitational closure equations

Since $\mathrm{M}^{A\gamma} = 0$, we can immediately state the following facts about the system and its solution:

- C_A is a functional gradient that appears as a boundary term in the gravitational action. We can therefore ignore it in our derivation.
- C also depends on finitely many derivatives of the geometrical degrees of freedom. In fact $\mathrm{C} = \mathrm{C}(\varphi, \partial\varphi, \partial\partial\varphi)$.
- $\mathrm{C}_{B_1 \dots B_N}$ only depends linearly on $\partial\partial\varphi$ for $N \geq 2$.
- The equations for odd and even coefficients decouple.

We now continue to evaluate the remaining equations along the lines of our solution algorithm presented in section 3.5.2. This means that we start with the derivation of the scalar output coefficient $\mathrm{C}[\varphi]$.

Curvature invariants of C

As described in the previous chapter, we start by constructing the curvature invariants for the scalar output coefficient with the help of the covariance part before moving to the selective part of the gravitational closure equations. With the components input coefficient $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ we identified above, we find that

$$\operatorname{rank} \mathrm{F}^{A}{}_{\mu}{}^{\gamma} = 6\,. \tag{4.15}$$

In particular this means, since the kernel of $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ is trivial, that we cannot construct any scalar fields from the degrees of freedom. As a result, the unique density term of weight 1, up to multiplication by a constant, is given by

$$\chi(\varphi) = \frac{1}{\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}}\,. \tag{4.16}$$

For this density we find by straight forward calculation that its derivative reads

$$\chi_{:A} = -\frac{1}{2}\chi(\varphi)\widehat{g}_{\alpha\beta}(\varphi)\mathcal{I}^{\alpha\beta}{}_{A}\,, \tag{4.17}$$

where $g_{\alpha\beta}(\varphi)$ is the inverse of $g^{\alpha\beta}(\varphi)$ that can be obtained in terms of the adjoint and determinant, i.e.

$$\widehat{g}_{\alpha\beta}(\varphi) = \chi(\varphi)^2 \cdot \begin{pmatrix} \frac{(\varphi^5)^2}{2} - \varphi^4\varphi^6 & \frac{\varphi^2\varphi^6}{\sqrt{2}} - \frac{\varphi^3\varphi^5}{2} & \frac{\varphi^3\varphi^4}{\sqrt{2}} - \frac{\varphi^2\varphi^5}{2} \\ \frac{\varphi^2\varphi^6}{\sqrt{2}} - \frac{\varphi^3\varphi^5}{2} & \frac{(\varphi^3)^2}{2} - \varphi^1\varphi^6 & \frac{\varphi^1\varphi^5}{\sqrt{2}} - \frac{\varphi^2\varphi^3}{2} \\ \frac{\varphi^3\varphi^4}{\sqrt{2}} - \frac{\varphi^2\varphi^5}{2} & \frac{\varphi^1\varphi^5}{\sqrt{2}} - \frac{\varphi^2\varphi^3}{2} & \frac{(\varphi^2)^2}{2} - \varphi^1\varphi^4 \end{pmatrix}_{\alpha\beta}\,. \tag{4.18}$$

Observe that, due to the determinant factor in front, the expression becomes non-polynomial in the degrees of freedom. From this we can then verify that

$$\chi_{:A}\mathrm{F}^{A}{}_{\mu}{}^{\gamma} + \chi\delta^{\gamma}_{\mu} = 0\,. \tag{4.19}$$

Since we know that C depends at most on the 2nd derivatives of the degrees of freedom, we can repeat the Lovelock-type argument from section 3.5.3 to conclude that the scalar output coefficient is at most a polynomial of degree 3 in $\partial\partial\varphi$. Furthermore, from the combinatorial considerations of the covariance part of the closure equations, we know that

$$\#\,(\text{curvature invariants})_{\mathrm{C}} = 10\cdot 6 - 57 = 3\,, \tag{4.20}$$

which indicates that we can create the relevant curvature invariants by considering the different orders in $\partial\partial\varphi$ separately.

We start with the ansatz for the linear term, i.e. an ansatz of the form

$$\Gamma_1[\varphi] = \lambda_{A}{}^{\mu\nu}(\varphi,\partial\varphi)\varphi^{A}{}_{,\mu\nu} + \mu(\varphi,\partial\varphi)\,. \tag{4.21}$$

From $(\mathbf{C8_3})$ we then get the restriction

$$0 = \lambda_{A}{}^{(\alpha\beta|}\mathrm{F}^{A}{}_{\mu}{}^{|\gamma)}\,, \tag{4.22}$$

and from $(\mathbf{C8_2})$ we find the two equations

$$0 = \lambda_A{}^{\beta_1\beta_2}\varphi^A{}_{,\mu} - \frac{\partial\mu}{\partial\varphi^A{}_{,(\beta_1|}}\mathrm{F}^A{}_\mu{}^{|\beta_2)} - -2\,\lambda_A{}^{\alpha(\beta_1|}\mathrm{F}^A{}_\mu{}^{|\beta_2)}{}_{,\alpha}\,, \tag{4.23a}$$

$$0 = \frac{\partial\lambda_B{}^{\mu\nu}}{\partial\varphi^A{}_{,(\beta_1|}}\mathrm{F}^A{}_\mu{}^{|\beta_2)}\,, \tag{4.23b}$$

where we separated the equations by orders of $\partial\partial\varphi$. In the second equation we can invert the input coefficient $\mathrm{F}^A{}_\mu{}^\gamma$ and obtain that the coefficient λ does not depend on the first derivatives of the field and is, thus, built only by the metric and inverse metric. Using this, we can only construct the coefficient λ in terms of the available structure as

$$\lambda_A{}^{\mu\nu}(\varphi) = c_1\cdot\chi(\varphi)\,\mathcal{I}^{\mu\nu}{}_A + c_2\cdot\chi(\varphi)\,\mathcal{I}^{\alpha\beta}{}_A\widehat{g}^{\mu\nu}(\varphi)\widehat{g}_{\alpha\beta}(\varphi)\,. \tag{4.24}$$

Once we insert this into equation (4.22) this tells us that

$$\lambda_A{}^{\mu\nu}(\varphi) = c_1\cdot\chi(\varphi)\left(\mathcal{I}^{\mu\nu}{}_A - \mathcal{I}^{\alpha\beta}{}_A\widehat{g}^{\mu\nu}(\varphi)\widehat{g}_{\alpha\beta}(\varphi)\right)\,. \tag{4.25}$$

Inserting this into (4.23a) we furthermore find that the function μ reads

$$\mu(\varphi,\partial\varphi) = \overline{\mu}_A{}^\mu{}_B{}^\nu(\varphi)\,\varphi^A{}_{,\mu}\varphi^B{}_{,\nu} + \widetilde{\mu}(\varphi)\,, \tag{4.26}$$

with

$$\begin{aligned}\overline{\mu}_A{}^\mu{}_B{}^\nu(\varphi) = \frac{c_1}{4}\chi(\varphi)\Big(&5\,\mathcal{I}^{\alpha\beta}{}_A\mathcal{I}^{\gamma\delta}{}_B\widehat{g}_{\alpha\gamma}(\varphi)\widehat{g}_{\beta\delta}(\varphi)\widehat{g}^{\mu\nu}(\varphi) + \mathcal{I}^{\alpha\beta}{}_A\mathcal{I}^{\gamma\delta}{}_B\widehat{g}_{\alpha\beta}(\varphi)\widehat{g}_{\gamma\delta}(\varphi)\widehat{g}^{\mu\nu}(\varphi)\\ &- 2\,\mathcal{I}^{\alpha\nu}{}_A\mathcal{I}^{\beta\mu}{}_A\widehat{g}_{\alpha\beta}(\varphi) - 4\,\mathcal{I}^{\mu\nu}{}_A\mathcal{I}^{\alpha\beta}{}_A\widehat{g}_{\alpha\beta}(\varphi)\Big)\,.\end{aligned} \tag{4.27}$$

From the remaining eqation from covariance part of the closure equations for the scalar output coefficient $(\mathbf{C1})$ we then obtain that the function $\widetilde{\mu}(\varphi)$ is zero. Inspecting the resulting expression closely we can in fact identify it with the Ricci scalar in three dimensions, namely

$$\Gamma_1[\varphi] = c_1\mathcal{R}[\varphi]\,. \tag{4.28}$$

This tells us that all curvature invariants that are linear in the 2$^{\text{nd}}$ derivatives are proportional to the Ricci scalar and we can set $c_1 = 1$ in the following.

One then proceeds in the same fashion for the ansatz that is quadratic in the 2$^{\text{nd}}$ derivatives. We first make the ansatz

$$\Gamma_2\left[\varphi\right] = \sigma_A{}^{\mu\nu}{}_B{}^{\lambda\kappa}\left(\varphi,\partial\varphi\right)\varphi^A{}_{,\mu\nu}\varphi^B{}_{,\lambda\kappa} + \rho\left(\varphi,\partial\varphi\right)\,, \tag{4.29}$$

where we then again find, thanks to the closure equation $(\mathbf{C8_2})$, that the coefficient σ does not depend on $\partial\varphi$. Since we are only interested in functionally independent curvature invariants we then need to exclude all the ansätze for which

$$\sigma_A{}^{\mu\nu}{}_B{}^{\lambda\kappa}\left(\varphi\right) \propto \lambda_A{}^{\mu\nu}\left(\varphi\right)\lambda_A{}^{\lambda\kappa}\left(\varphi\right)\,. \tag{4.30}$$

While $(\mathbf{C8_3})$ will then fix the coefficient up to a single constant, the equations $(\mathbf{C8_2})$ and $(\mathbf{C1})$ will then again fix the $\partial\varphi$ dependencies. Once we have finished the straightforward but tedious calculations, one ultimately finds that the three curvature invariants are given by

$$\Gamma_1\left[\varphi\right] = \mathcal{R}\left[\varphi\right] , \tag{4.31a}$$
$$\Gamma_2\left[\varphi\right] = \mathcal{R}^m{}_n\left[\varphi\right]\mathcal{R}^n{}_m\left[\varphi\right] , \tag{4.31b}$$
$$\Gamma_3\left[\varphi\right] = \mathcal{R}^m{}_n\left[\varphi\right]\mathcal{R}^n{}_p\left[\varphi\right]\mathcal{R}^p{}_m\left[\varphi\right] . \tag{4.31c}$$

Note that if we would have started with an ansatz that does not contain any 2$^{\text{nd}}$ derivatives at all the equations would have eliminated the ansätze altogether. As a result one can convince themselves that these are, indeed the only possible terms. This means that after having solved the covariance part of the closure equations, the general solution for C is

$$\mathrm{C}(\varphi,\partial\varphi,\partial\partial\varphi) = \frac{1}{\sqrt{-\det\widehat{g}^{\cdot\cdot}(\varphi)}}\widetilde{\mathrm{C}}\left(\Gamma_1[\varphi],\Gamma_1[\varphi],\Gamma_1[\varphi]\right) , \tag{4.32}$$

for the, as of now, undetermined function $\widetilde{\mathrm{C}}$ of the three curvature invariants.

C depends linearly on 2$^{\text{nd}}$ derivatives of the degrees of freedom

It turns out that also the coefficient C only depends linearly on the highest derivative order of the degrees of freedom. To see this, we look at the coefficient $\mathcal{U}$, as defined in the previous chapter, in more detail

$$\begin{aligned}\mathcal{U}^{A\mu\nu} &= \mathrm{p}^{\rho\mu}\,\mathrm{F}^A{}_\rho{}^\nu \\ &= 2\,\mathcal{I}^A{}_{\sigma\rho}\,\widehat{g}^{\rho\mu}(\varphi)\,\widehat{g}^{\sigma\nu}(\varphi) \\ &= 2\,\mathcal{I}^{\rho\mu}{}_M\mathcal{I}^A{}_{\rho\sigma}\mathcal{I}^{\sigma\nu}{}_N\varphi^M\varphi^N ,\end{aligned} \tag{4.33}$$

which is easily seen to be symmetric in $\mu\nu$. As a result, we can repeat the derivation from section 3.5.3 again and obtain that C is linear in $\partial\partial\varphi$.

We also observe that $\mathcal{U}$ is invertible and that its inverse is given by the analogous expression of section 3.5.4

$$\left(\mathcal{U}^{-1}\right)_{A\mu\nu} = \frac{1}{2}\mathcal{I}_A{}^{\sigma\rho}\,\widehat{g}_{\sigma\mu}(\varphi)\,\widehat{g}_{\rho\nu}(\varphi) , \tag{4.34}$$

where $\widehat{g}_{\cdot\cdot}$ is the metric in terms of the degrees of freedom φ. It can be explicitly obtained in terms of the adjugate matrix of $g^{\cdot\cdot}$, but we will not need its explicit definition in the following.

Since we know that the scalar output coefficient is written in terms of the three curvature invariants we find from this that

$$\frac{\partial \mathrm{C}}{\partial\mathcal{R}\partial\mathcal{R}} = 0 , \tag{4.35a}$$
$$\frac{\partial \mathrm{C}}{\partial\Gamma_2} = 0 , \tag{4.35b}$$
$$\frac{\partial \mathrm{C}}{\partial\Gamma_3} = 0 . \tag{4.35c}$$

This already fixes the scalar output coefficient completely to

$$\mathrm{C}(\varphi, \partial\varphi, \partial\partial\varphi) = \frac{1}{\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}} \left(a_1 \mathcal{R}[\varphi] + a_2\right) , \tag{4.36}$$

with the two constants a_1 and a_2. The remaining equations for C, that is $(\mathbf{C7})$, $(\mathbf{C20_{N\ even}})$ and $(\mathbf{C21_{N\ odd}})$ do not give any further restrictions since $\mathrm{M}^{A\gamma}$ is vanishing.
Since we already know that C_A can be written as a functional gradient and solves the equations of motion exactly, we will continue with the output coefficient C_{AB} and deal with C_A later.

$\mathbf{C_{AB}}$ regained

In principle, we would now start by again setting up the covariance part of the closure equations to determine the curvature invariants for the output coefficients. As indicated by equation (3.200), this becomes an increasingly complex endeavor. Luckily, we here find that the coefficient C_{AB} can be obtained directly from the derivatives of C. In order to see this, let us consider $(\mathbf{C3})$ which directly gives

$$2\,\mathrm{C}_{AB}\,\mathcal{U}^{A\mu\nu} = \mathrm{C}_{:B}{}^{\mu\nu} , \tag{4.37}$$

and by application of the inverse $\mathcal{U}^{-1}_{A\mu\nu}$ that

$$\begin{aligned} \mathrm{C}_{AB}(\varphi, \partial\varphi, \partial\partial\varphi) &= \frac{1}{2}\left(\mathcal{U}^{-1}\right)_{A\mu\nu} \mathrm{C}_{:B}{}^{\mu\nu} \\ &= \frac{a_0}{2}\left(\mathcal{U}^{-1}\right)_{A\mu\nu} \frac{\partial\mathcal{R}}{\partial\varphi^A{}_{,\mu\nu}} . \end{aligned} \tag{4.38}$$

If we moreover calculate the derivative of the Ricci scalar we find that the output coefficient reads

$$\mathrm{C}_{AB}(\varphi, \partial\varphi, \partial\partial\varphi) = \frac{a_1}{4\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}} \mathcal{I}^{\alpha\beta}{}_A \mathcal{I}^{\gamma\delta}{}_B \left(\widehat{g}_{\alpha\gamma}(\varphi)\widehat{g}_{\delta\beta}(\varphi) - \widehat{g}_{\alpha\beta}(\varphi)\widehat{g}_{\gamma\delta}(\varphi)\right) . \tag{4.39}$$

We can then verify by plugging this into the remaining equations that no further restrictions appear. This is, of course, essential since any remaining restriction would set C_{AB} to zero and C would only contain the cosmological constant term.

Since the coefficient does not depend on derivatives of the degrees of freedom the covariance part of the closure equations $(\mathbf{C11_2})$ and $(\mathbf{C12_2})$ can be seen to vanish directly. For $(\mathbf{C10_2})$ we have

$$0 = \mathrm{C}_{B_1 B_2 : A} \mathrm{F}^A{}_\mu{}^\gamma + \mathrm{C}_{B_1 B_2} \delta^\gamma_\mu + 2\,\mathrm{C}_{A(B_1} \mathrm{F}^A{}_\mu{}^\gamma{}_{:B_2)} . \tag{4.40}$$

With the output coefficient C_{AB} containing the metric $g_{\cdot\cdot}(\varphi)$, it proves useful to calculate its derivative by the degrees of freedom. This can be easily obtained by taking the derivative of $\widehat{g}^{\alpha\mu}\widehat{g}_{\mu\beta} = \delta^\alpha_\mu$ from which we end up with

$$\frac{\partial \widehat{g}_{\alpha\beta}}{\partial\varphi^A} = -\mathcal{I}^{\mu\nu}{}_A\, \widehat{g}_{\alpha\mu}(\varphi)\widehat{g}_{\beta\nu}(\varphi) . \tag{4.41}$$

Together with the derivative of the density term we then indeed find that the equation (4.40) is fulfilled by (4.39). The last non-trivial equation we need to check is $(\mathbf{C4})$

$$0 = 2\,\mathrm{C}_{AB}\left(\mathrm{p}^{\mu\nu}\varphi^A{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\mathrm{F}^A{}_\nu{}^\gamma\right) - \mathrm{C}_{:B}{}^\mu + 2\,\left(\partial_\alpha \mathrm{C}_{:B}{}^{\alpha\mu}\right) . \tag{4.42}$$

Inserting both our expressions for the output coefficients and the Ricci scalar one finds that this is fulfilled.

$C_{A_1 \ldots A_N}$ vanishes for N > 3

Just as before, we can use the equation of the type $(\mathbf{C16_N})$ to express the output coefficients in terms of the lower order ones. In fact, already for the case $N = 2$ we find

$$0 = C_{ABC}\,\mathcal{U}^{A\mu\nu}\,. \tag{4.43}$$

But since $\mathcal{U}$ is invertible, we immediately see that C_{ABC} vanishes. We can then use $(\mathbf{C16_N})$, for N even, and see that all odd coefficients are zero, except C_A.

Similarly, we can look at the same closure equation for the even output coefficients. So far, we evaluated the even N of $(\mathbf{C16_N})$ to see that the odd coefficients vanish. Looking into $(\mathbf{C16_3})$ we find

$$0 = 12\,C_{AMNP}\,\mathcal{U}^{A\mu\nu} + C_{MN:P}{}^{\mu\nu}\,. \tag{4.44}$$

But since we have that C_{AB} does not depend on the 2nd derivatives of the degrees of freedom, this means that C_{ABCD} must be zero. Then by again repeating this argument for the odd $(\mathbf{C16_N})$ iteratively, we find that all $C_{A_1 \ldots A_N}$ vanish for $N \geq 3$. As a result, our gravitational Lagrangian is quadratic in the velocities, and the only remaining undetermined output coefficient is C_A.

Towards C_A

In principle, it is also possible to solve the equations for the coefficient C_A with the same techniques. Next to the covariance part of the closure equations $(\mathbf{C2})$ and $(\mathbf{C9})$, we have to evaluate $(\mathbf{C5})$ and the curl condition $(\mathbf{C18_N})$, for $N \geq 1$ (here due to $(\mathbf{C6})$). From the curl condition (and by the argument from section 3.5.3), we know that we can write the coefficient as the functional derivative of a scalar density, i.e.

$$C_A[\varphi(x)] = \frac{\delta\Lambda}{\delta\varphi^A(x)}\,. \tag{4.45}$$

As a result, the covariance part of the closure equations turns into the condition that Λ is a scalar density of weight 1 which has the same differential equations we found for the scalar output coefficients, i.e. the closure equations $(\mathbf{C1})$ and $(\mathbf{C8_N})$.

From the remaining closure equation $(\mathbf{C5})$ we obtain

$$0 = C_A\varphi^A{}_{,\mu} + \partial_\gamma\left(C_A F^A{}_\mu{}^\gamma\right)\,, \tag{4.46}$$

which, once we insert the definition of the input coefficient, can be recognized as the condition that C_A be divergence free, i.e.

$$0 = \mathcal{I}^A{}_{\mu\nu}\left(\nabla^\nu C_A\right)\,, \tag{4.47}$$

with the metric compatible covariant derivative ∇_μ.

If we now restrict to solutions that only contain second derivatives of the degrees of freedom one finds, again by the Lovelock-like argument, that C_A must be linear in $\partial\partial\varphi$. By everything we learned for the scalar output coefficient, this immediately tells us that such a scalar density must be of the form

$$\Lambda[\varphi] = \frac{1}{\sqrt{-\det\widehat{g}^{\cdot\cdot}(\varphi)}}\left(b_1\mathcal{R}[\varphi] + b_2\right)\,. \tag{4.48}$$

From this we obtain that the output coefficient is given in terms of the metric and the three-dimensional Einstein tensor $G_{\alpha\beta}[\varphi]$

$$\mathrm{C}_A = \frac{1}{\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}} \mathcal{I}^{\alpha\beta}{}_A \left(b_1 \cdot G_{\alpha\beta}[\varphi] + b_2 \cdot \widehat{g}_{\alpha\beta}(\varphi) \right) . \tag{4.49}$$

Note, however, that this result is crucially dependent on the assumption that we have no higher derivatives of the degrees of freedom. Although tedious, this could be extended to higher derivatives. We then need to construct new curvature invariants and ensure that the result is divergence-free. As the output coefficient is a boundary term, it will not contribute to the equations of motion in any case.

This means that we have now solved all of the gravitational closure equations and determined all the required output coefficients. In particular we found that the two coefficients that appear in the equations of motion remarkably only contain two single parameters a_0 and a_1 that can be, of course, identified with the gravitational constant and the cosmological constant by setting

$$a_1 := -\frac{1}{2\kappa} , \tag{4.50}$$

$$a_2 := \frac{\Lambda}{\kappa} . \tag{4.51}$$

Suppose we perform the Legendre transformation of the Lagrangian back to our original Hamiltonian formulation. In that case, we see that the Hamiltonian is equivalent to the superhamiltonian found in the ADM formulation of general relativity, as expected (Witte, 2014). As a result, we, as promised, were able to derive general relativity entirely with the help of the gravitational closure equations. The most general gravitational action for the ten degrees of freedom of a metric is thus indeed given by the expression in the following box:

THEOREM GRAVITATIONAL CLOSURE OF MAXWELLIAN ELECTRODYNAMICS

Performing gravitational closure of Maxwellian electrodynamics for the inverse metric gives the action

$$\mathcal{S}_{\text{gravity}}[\varphi] = \int \mathrm{d}t \int \mathrm{d}^3x\, N(t,x) \Bigg(\mathrm{C}[\varphi] + \frac{1}{N^2} \mathrm{C}_{AB}(\varphi) \left(\dot{\varphi}^A - N^\mu \varphi^A{}_{,\mu} + (\partial_\gamma N^\mu) \mathrm{F}^A{}_\mu{}^\gamma \right) \\ \times \left(\dot{\varphi}^B - N^\nu \varphi^B{}_{,\nu} + (\partial_\delta N^\nu) \mathrm{F}^B{}_\nu{}^\delta \right) \Bigg) ,$$

with the coefficients

$$\mathrm{C} = -\frac{1}{2\kappa} \frac{1}{\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}} \left(\mathcal{R}\,[\varphi] - 2\,\Lambda \right) ,$$

$$\mathrm{C}_{AB} = \frac{1}{8\kappa} \frac{1}{\sqrt{-\det \widehat{g}^{\cdot\cdot}(\varphi)}} \mathcal{I}^{\alpha\beta}{}_A \mathcal{I}^{\mu\nu}{}_B \left(\widehat{g}_{\alpha\mu}(\varphi)\, \widehat{g}_{\beta\nu}(\varphi) - \widehat{g}_{\alpha\beta}(\varphi)\, \widehat{g}_{\mu\nu}(\varphi) \right) .$$

4.2 GENERAL LINEAR ELECTRODYNAMICS

As our next example, we will again pick up our theory of interest we introduced in chapter 1 and discuss the gravitational closure of general linear electrodynamics. The underlying geometry is the area metric, i.e. the rank 4 tensor field G^{abcd}.

Of course, we already extensively discussed aspects of such a theory in the previous chapters in the presentation of the kinematical aspects of matter theories and the derivation of the gravitational closure equations. For an (almost) self-complete treatment, we will nonetheless follow the necessary steps in the algorithm and present the constructions in the setup of birefringent electrodynamics.

4.2.1 Kinematic setup

We start by considering the kinematic setup of the geometry of general linear electrodynamics. From the equations of motion

$$0 \quad = \partial_n \left(\omega(G) G^{abmn} F_{ab} \right) , \tag{4.52}$$

we can derive the principal polynomial by fixing the $\mathrm{U}(1)$ gauge freedom or treating it as described in section 2.2.1. The symbol $\omega(G)$ denotes the de-densitization factor that we will fix in the following. Once the gauge freedom is carefully dealt with, one obtains the polynomial of degree four

$$P(k) = -\frac{1}{24}\omega(G)^2 \epsilon_{mnpq}\epsilon_{rstu} G^{mnri} G^{jpsk} G^{lqtu} k_i k_j k_k k_l \,. \tag{4.53}$$

Given that the cyclic part of the area metric is non-zero, i.e. $\epsilon_{mnpq} G^{mnpq} \neq 0$, we can use it to construct a de-densitization from it. While the possible density factors are far from unique (as we have seen before), they all differ by a scalar function built from the gravitational degrees of freedom. For the following discussion we will make the choice $\omega(G) = (\frac{1}{24}\epsilon_{mnpq} G^{mnpq})^{-1}$ to de-densitize our polynomial.

With the principal polynomial obtained, one can then move on to derive the dual polynomial $P^\sharp$, which is given by the degree 4 polynomial

$$P^\sharp(v) = -\frac{1}{24}\frac{1}{\omega(G)^2} \epsilon^{mnpq}\epsilon^{rstu} G_{mnri} G_{jpsk} G_{lqtu} v^i v^j v^k v^l \,, \tag{4.54}$$

in terms of the inverse object G_{abcd} of the same symmetries as the area metric, for which we have that

$$G^{abmn} G_{mncd} = 4\, \delta^{[a}_c \delta^{b]}_d \,. \tag{4.55}$$

This then immediately gives the Lagrange action for a massless particle, in the geometrical optical limit, coupled to the area metric

$$\mathcal{S}_{\text{massless}}[x, \mu] = -\frac{1}{24} \int \mathrm{d}\tau \, \lambda \frac{1}{\omega(G)^2} \epsilon^{mnpq}\epsilon^{rstu} G_{mnri} G_{jpsk} G_{lqtu} \dot{x}^i \dot{x}^j \dot{x}^k \dot{x}^l \,. \tag{4.56}$$

We now need to make sure that the three mater conditions are implemented, i.e. that both polynomials are hyperbolic and the setup is energy-distinguishing, in order to be able to construct the Legendre map that allows us to construct the observer frame for gravitational closure. Requiring these three conditions restricts the area metric to one of seven algebraic classes (of possible 23 classes in total) (Witte, 2009; Schuller et al., 2010). Once the conditions are implemented we can construct the Legendre map via

$$\ell(k)^a = -\frac{1}{6}\frac{1}{P(k)}\omega(G)^2\epsilon_{mnpq}\epsilon_{rstu}G^{mnr(a|}G^{j|ps|k}G^{l)qtu}k_jk_kk_l\,. \tag{4.57}$$

For the inverse Legendre map there is no known closed form expression. In fact, it can be shown that the map is non-polynomial (Rivera and Schuller, 2011). Luckily, we do not need the map explicitly for performing gravitational closure and it suffices to know that the map does exist. Its existence is guaranteed due to the three matter conditions.

Next, we project the components of the spacetime area metric to fields on the screen manifold using the orthonormal frame. We construct the following three fields

$$\bar{g}^{\alpha\beta} = -G(\epsilon^0,\epsilon^\alpha,\epsilon^0,\epsilon^\beta)\,, \tag{4.58a}$$

$$\bar{\bar{g}}_{\alpha\beta} = \frac{1}{4}\frac{1}{\det\bar{g}^{\cdot\cdot}}\epsilon_{\alpha\mu\nu}\epsilon_{\beta\lambda\kappa}G(\epsilon^\mu,\epsilon^\nu,\epsilon^\lambda,\epsilon^\kappa)\,, \tag{4.58b}$$

$$\bar{\bar{\bar{g}}}_{\alpha\beta} = \frac{1}{2}\frac{1}{\sqrt{\det\bar{g}^{\cdot\cdot}}}(\bar{g}^{-1})_{\alpha\mu}\epsilon_{\beta\lambda\kappa}G(\epsilon^0,\epsilon^\mu,\epsilon^\lambda,\epsilon^\kappa) - (\bar{g}^{-1})_{\alpha\beta}\,. \tag{4.58c}$$

Note that we slightly changed the third field in comparison to the one we presented in section 3.1.2. This is due to the parametrization we will present in the following. Ultimately, the choice is arbitrary, as long as we can reconstruct the whole spacetime area metric.

With the chosen hypersurface fields, we can, once a foliation and the corresponding observer frame (e,ϵ) are given, indeed reconstruct the area metric via

$$\begin{aligned} G^{abcd}\left(X_t(x)\right) &= 4\,\bar{g}^{\alpha\beta}\,e_0^{[a}e_\alpha^{b]}e_0^{[c}e_\beta^{d]} + (\det\bar{g}^{\cdot\cdot})\cdot\bar{\bar{g}}_{\mu\nu}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\gamma\delta}\,e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d \\ &\quad + 2\sqrt{\det\bar{g}^{\cdot\cdot}}\left(\bar{g}^{\alpha\nu}\bar{\bar{\bar{g}}}_{\mu\nu} + \delta_\mu^\alpha\right)\epsilon^{\mu\beta\gamma}\,e_0^{[a}e_\alpha^{b]}e_\beta^c e_\gamma^d\,. \end{aligned} \tag{4.59}$$

When we implement the four frame conditions we find for the normalisation condition that

$$\begin{aligned} 1 &= P(\epsilon^0,\epsilon^0,\epsilon^0,\epsilon^0) \\ &= \omega(G)^2\det\bar{g}^{\cdot\cdot}\,, \end{aligned} \tag{4.60}$$

which relates the de-densitization of the polynomial – in our case the cyclic part of the area metric – to the determinant of the first screen manifold tensor field. But from this, we can calculate the trace of the third hypersurface field and find

$$\bar{g}^{\alpha\beta}\bar{\bar{\bar{g}}}_{\alpha\beta} = \frac{1}{2}\frac{1}{\sqrt{\det\bar{g}^{\cdot\cdot}}}\underbrace{\epsilon_{\alpha\beta\gamma}G(\epsilon^0,\epsilon^\alpha,\epsilon^\beta,\epsilon^\gamma)}_{\frac{1}{4}\epsilon_{mnpq}G^{mnpq}=\frac{6}{\omega(G)}} - 3 = 0\,. \tag{4.61}$$

Clearly, this imposes a quadratic condition on the degrees of freedom of the area metric that we need to take care of in the next section. From the annihilation condition we find that the anti-symmetric part of the third screen manifold projection vanishes, i.e.

$$\bar{\bar{\bar{g}}}_{[\alpha\beta]} = 0\,. \tag{4.62}$$

This antisymmetry condition eliminates three of the nine components in the field, and another component is eliminated by the trace condition (4.61). Together with the unconstrained six degrees of the first two hypersurface fields, respectively, we thus obtain the 17 degrees of freedom of the area metric we need to parametrize.

4.2.2 Parametrization

Constructing a parametrization for the two frame conditions is a non-trivial endeavour. One possible solution can be obtained by introducing a vector t^a such that $\mathcal{I}^{\alpha\beta}{}_a t^a$ is a positive definitive matrix, where the index a runs from 1 to 6 and with the constant intertwiner $\mathcal{I}^{\alpha\beta}{}_a$ we already used for general relativity. We can then complete this into a complete basis of $\mathbb{R}^6$ by the introduction of five orthogonal elements $e^{(m)a}$ and their dual objects n_a and $\epsilon_{(m)a}$, respectively. The objects are orthogonal with respect to a standard inner product Δ_{ab} and the index m runs from $m = 1, \ldots, 5$.

For example, one such choice can be made with the vector

$$t^a = \left(\frac{1}{\sqrt{3}}, 0, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}} \right)^t . \tag{4.63}$$

Contracting with the intertwiner we obtain the positive definitive matrix

$$\mathcal{I}^{\alpha\beta}{}_a t^a = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\alpha\beta} . \tag{4.64}$$

Next we can complete this, by performing Gram-Schmidt orthonormalisation on the collection $(1,0,0,0,0,0)^t$, $(1,1,0,0,0,0)^t$ ... $(1,1,1,1,1,1)^t$ and t^a, into an orthonormal basis. The $\epsilon_{(m)a}$ we find are given by

$$\begin{aligned}
\epsilon_{(1)a} &= \left(\sqrt{\frac{2}{3}}, 0, 0, -\frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}} \right)^t , \\
\epsilon_{(2)a} &= (0,1,0,0,0,0)^t , \\
\epsilon_{(3)a} &= \left(0, 0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}} \right)^t , \\
\epsilon_{(4)a} &= (0,0,0,0,1,0)^t , \\
\epsilon_{(5)a} &= \left(0, 0, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}} \right)^t .
\end{aligned} \tag{4.65}$$

The dual basis is numerically equivalent to the components of the vectors presented above, where the a and m indices are raised / lowered with the standard inner product $\Delta_{ab} = \mathrm{diag}(1,1,1,1,1,1)$ and $\overline{\Delta}_{mn} = \mathrm{diag}(1,1,1,1,1)$, respectively:

$$n_a = \Delta_{ab} t^b \qquad \text{and} \qquad \epsilon_{(m)a} = \Delta_{ab} \overline{\Delta}_{mn} e^{(n)b} . \tag{4.66}$$

Once this choice is fixed, we can employ them to construct the following parametrization via

$$
\begin{aligned}
\widehat{\overline{g}}^{\alpha\beta}(\varphi) &= \mathcal{I}^{\alpha\beta}{}_a \overline{\varphi}^a\,, &(4.67a)\\
\widehat{\overline{\overline{g}}}_{\alpha\beta}(\varphi) &= \mathcal{I}^a{}_{\alpha\beta}\Delta_{ab}\overline{\overline{\varphi}}^b\,, &(4.67b)\\
\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}(\varphi) &= \mathcal{I}^a{}_{\alpha\beta}\left(\delta^b_a - \frac{n_a\overline{\varphi}^b}{n_c\overline{\varphi}^c}\right)\epsilon_{(m)b}\overline{\overline{\overline{\varphi}}}^m\,, &(4.67c)
\end{aligned}
$$

and the inverse map of the parametrization given by

$$
\begin{aligned}
\widehat{\overline{\varphi}}^a(g) &= \mathcal{I}^a{}_{\alpha\beta}\overline{g}^{\alpha\beta}\,, &(4.68a)\\
\widehat{\overline{\overline{\varphi}}}^a(g) &= \mathcal{I}^{\alpha\beta}{}_b\Delta^{ab}\overline{\overline{g}}_{\alpha\beta}\,, &(4.68b)\\
\widehat{\overline{\overline{\overline{\varphi}}}}^m(g) &= \mathcal{I}^{\alpha\beta}{}_a e^{(m)a}\overline{\overline{\overline{g}}}_{\alpha\beta}\,. &(4.68c)
\end{aligned}
$$

All three fields are clearly symmetric matrices, as all their free indices are located at the standard constant intertwiner $\mathcal{I}^{\alpha\beta}{}_a$ and its inverse. In order for this construction to be a proper parametrization it is, of course, necessary that the frame condition (4.61) is implemented. Indeed, one finds by direct calculation that

$$
\widehat{\overline{g}}^{\alpha\beta}(\varphi)\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}(\varphi) = \overline{\varphi}^a\left(\delta^b_a - \frac{n_a\overline{\varphi}^b}{n_c\overline{\varphi}^c}\right)\epsilon_{(m)b}\overline{\overline{\overline{\varphi}}}^m = \epsilon_{(m)a}\overline{\varphi}^a\overline{\overline{\overline{\varphi}}}^m - \epsilon_{(m)a}\overline{\varphi}^a\overline{\overline{\overline{\varphi}}}^m \equiv 0\,. \qquad (4.69)
$$

We can then move on to calculate the intertwiners by taking the derivative with respect to the degrees of freedom. This gives the following expressions, where we omit all vanishing components

$$
\frac{\partial\widehat{\overline{g}}^{\alpha\beta}}{\partial\overline{\varphi}^a} = \mathcal{I}^{\alpha\beta}{}_a \quad , \quad \frac{\partial\widehat{\overline{\overline{g}}}_{\alpha\beta}}{\partial\overline{\overline{\varphi}}^a} = \mathcal{I}^b{}_{\alpha\beta}\Delta_{ab} \quad , \quad \frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}}{\partial\overline{\overline{\overline{\varphi}}}^m} = \mathcal{I}^a{}_{\alpha\beta}\left(\delta^b_a - \frac{n_a\overline{\varphi}^b}{n_c\overline{\varphi}^c}\right)\epsilon_{(m)b}\,,
$$

$$
\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}}{\partial\overline{\varphi}^m} = \mathcal{I}^b{}_{\alpha\beta}n_b\left(\frac{n_a\overline{\varphi}^c\epsilon_{(m)c}\overline{\overline{\overline{\varphi}}}^m}{(n_d\overline{\varphi}^d)^2} - \frac{\epsilon_{(m)a}\overline{\overline{\overline{\varphi}}}^m}{n_d\overline{\varphi}^d}\right)\,. \qquad (4.70)
$$

For the inverse maps we obtain the constant expressions

$$
\frac{\partial\widehat{\overline{\varphi}}^a}{\partial\overline{g}^{\alpha\beta}} = \mathcal{I}^a{}_{\alpha\beta} \quad , \quad \frac{\partial\widehat{\overline{\overline{\varphi}}}^a}{\partial\overline{\overline{g}}_{\alpha\beta}} = \mathcal{I}^{\alpha\beta}{}_b\Delta^{ab} \quad , \quad \frac{\partial\widehat{\overline{\overline{\overline{\varphi}}}}^m}{\partial\overline{\overline{\overline{g}}}_{\alpha\beta}} = \mathcal{I}^{\alpha\beta}{}_a e^{(m)a}\,, \qquad (4.71)
$$

where, again, the derivatives that are not displayed above are vanishing.

Input coefficients

The next step is to calculate the three input coefficients with our chosen parametrization. We start with the tangential deformation coefficient $\mathrm{F}^A{}_\mu{}^\gamma$. By decomposing the Lie derivative of the hyperfields, one finds the three functions, one for each of the three sectors

$$
\begin{aligned}
\mathrm{F}^{\overline{a}}{}_\mu{}^\gamma(\varphi) &= 2\,\mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_b\overline{\varphi}^b\,, &(4.72a)\\
\mathrm{F}^{\overline{\overline{a}}}{}_\mu{}^\gamma(\varphi) &= -2\,\Delta^{am}\Delta_{bn}\mathcal{I}^b{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_m\overline{\overline{\varphi}}^n\,, &(4.72b)\\
\mathrm{F}^{\overline{\overline{\overline{m}}}}{}_\mu{}^\gamma(\varphi) &= -2\,\frac{\partial\widehat{\overline{\overline{\overline{\varphi}}}}^m}{\partial\overline{\overline{\overline{g}}}_{\sigma\gamma}}\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\mu\sigma}}{\partial\overline{\overline{\overline{\varphi}}}^n}\overline{\overline{\overline{\varphi}}}^n\,. &(4.72c)
\end{aligned}
$$

Note that the coefficient here has, in contrast to general relativity, a rank of 9 and is thus not of full rank. This means that no inverse coefficient exists. Furthermore, it already hints that it is possible to construct scalar fields out of the three screen manifold fields.

For the input coefficient $\mathrm{p}^{\alpha\beta}$ we calculate the screen-manifold projection $P(\epsilon^{\alpha},\epsilon^{\beta},\epsilon^{0},\epsilon^{0})$ and insert the parametrization. We find that the coefficient reads

$$\mathrm{p}^{\alpha\beta}(\varphi)=\frac{1}{6}\left(\widehat{g}^{\alpha\gamma}\widehat{g}^{\beta\delta}\widehat{\overline{\overline{g}}}_{\gamma\delta}-\widehat{g}^{\alpha\beta}\widehat{g}^{\gamma\delta}\widehat{\overline{\overline{g}}}_{\gamma\delta}-2\,\widehat{g}^{\alpha\beta}\widehat{g}^{\delta\mu}\widehat{g}^{\gamma\nu}\widehat{\overline{\overline{g}}}_{\mu\gamma}\widehat{\overline{\overline{\overline{g}}}}_{\nu\delta}+3\,\widehat{g}^{\gamma\delta}\widehat{g}^{\alpha\mu}\widehat{g}^{\beta\nu}\widehat{\overline{\overline{g}}}_{\mu\gamma}\widehat{\overline{\overline{\overline{g}}}}_{\nu\delta}\right)(\varphi)\,. \tag{4.73}$$

Last but not least, the third input coefficient can be calculated with the usual procedure. One finds the following objects in terms of our screen manifold projections:

$$\overline{\mathrm{M}}^{\alpha\beta\gamma}=2\left(\det\overline{g}^{\cdot\cdot})\right)^{\frac{1}{2}}\epsilon^{\mu\gamma(\alpha}\overline{g}^{\beta)\nu}\overline{\overline{\overline{g}}}_{\mu\nu}\,, \tag{4.74a}$$

$$\overline{\overline{\mathrm{M}}}_{\alpha\beta}{}^{\gamma}=6\left(\det\overline{g}^{\cdot\cdot})\right)^{-\frac{1}{2}}\epsilon_{\mu\nu(\alpha}\overline{g}^{\lambda\nu}\overline{\overline{\overline{g}}}_{\beta)\lambda}\mathrm{p}^{\mu\gamma}\,, \tag{4.74b}$$

$$\overline{\overline{\overline{\mathrm{M}}}}_{\alpha\beta}{}^{\gamma}=-\left(\det\overline{g}^{\cdot\cdot})\right)^{\frac{1}{2}}\epsilon^{\mu\nu\gamma}\left(\widehat{g}^{-1}\right)_{\mu(\alpha|}\left(\overline{g}^{\lambda\kappa}\overline{\overline{\overline{g}}}_{|\beta)\lambda}\overline{\overline{\overline{g}}}_{\kappa\nu}+\overline{\overline{g}}_{|\beta)\nu}\right)\,. \tag{4.74c}$$

Once we insert the parametrization, we find the coefficient as it needs to be plugged into the gravitational closure equations

$$\mathrm{M}^{\bar{a}\gamma}(\varphi)=2\left(\det\widehat{g}^{\cdot\cdot}(\varphi)\right)^{\frac{1}{2}}\mathcal{I}^{a}{}_{\alpha\beta}\mathcal{I}^{\mu\alpha}{}_{b}\epsilon^{\beta\nu\gamma}\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\mu\nu}}{\partial\overline{\overline{\overline{\varphi}}}^{m}}(\varphi)\overline{\varphi}^{b}\overline{\overline{\overline{\varphi}}}^{m}\,, \tag{4.75a}$$

$$\mathrm{M}^{\bar{\bar{a}}\gamma}(\varphi)=6\left(\det\widehat{g}^{\cdot\cdot}(\varphi))\right)^{-\frac{1}{2}}\epsilon_{\alpha\mu\nu}\Delta^{ab}\mathcal{I}^{\alpha\beta}{}_{b}\mathcal{I}^{\lambda\nu}{}_{c}\mathrm{p}^{\mu\gamma}(\varphi)\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\beta\lambda}}{\partial\overline{\overline{\overline{\varphi}}}^{m}}(\varphi)\overline{\varphi}^{c}\overline{\overline{\overline{\varphi}}}^{m}\,, \tag{4.75b}$$

$$\mathrm{M}^{\bar{\bar{\bar{m}}}\gamma}(\varphi)=-\left(\det\widehat{g}^{\cdot\cdot}(\varphi)\right)^{\frac{1}{2}}\epsilon^{\mu\nu\gamma}\left(\widehat{g}^{-1}\right)_{\mu\alpha}(\varphi)\frac{\partial\widehat{\overline{\overline{\overline{\varphi}}}}^{m}}{\partial\overline{\overline{\overline{g}}}_{\alpha\beta}}(\varphi)\left(\mathcal{I}^{\kappa\lambda}{}_{b}\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\beta\lambda}}{\partial\overline{\overline{\overline{\varphi}}}^{n}}(\varphi)\frac{\partial\widehat{\overline{\overline{\overline{g}}}}_{\kappa\nu}}{\partial\overline{\overline{\overline{\varphi}}}^{l}}(\varphi)\overline{\varphi}^{b}\overline{\overline{\overline{\varphi}}}^{n}\overline{\overline{\overline{\varphi}}}^{l}+\mathcal{I}^{b}{}_{\beta\nu}\Delta_{bc}\overline{\overline{\varphi}}^{c}\right)\,. \tag{4.75c}$$

This is clearly more complicated than the vanishing coefficient of general relativity that we treated in the previous section.

4.2.3 Gravitational closure equations

With the setup finished, we are in principle suitably equipped to derive the Lagrangian for the 17 degrees of freedom of the area metric. As we will see, constructing an exact solution is, however, complicated for two reasons.

First, as shown above, the input coefficient $\mathrm{M}^{A\gamma}$ does not vanish, which leads to multiple complications. For one, this means that the odd and even output coefficients do not decouple anymore. Moreover, we cannot conclude anymore that the scalar output coefficient depends at most on the second derivatives of the degrees of freedom. Practically, this is a problem since we deal with both infinitely many equations for an infinite number of dependent functions of infinitely many independent variables.

Second, we found that the input coefficient $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ is of rank 9 and is thus *not* invertible. This means that we cannot establish a recursion relation via the selective part of the closure equations (**C3**), (**C4**), (**C6**),

$(\mathbf{C16_N})$ and $(\mathbf{C17_N})$ between *all* higher-order output coefficients in terms of the lower order coefficients. Such a sequence is, however, essential to obtain an expression in terms of finitely many constants of integration. So if it becomes increasingly complicated to derive the output coefficients for a given geometry, what can we do?

One possible simplification for an exact solution of the area metric may be to employ the reparametrization we described in section 3.5.4. For this, we observe that we can indeed define (at least) eight scalar functions $\sigma^{(i)}$ such that they locally constitute a basis of the kernel of $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$, i.e.

$$\sigma^{(i)}_{:A}\mathrm{F}^{A}{}_{\mu}{}^{\gamma} = 0\,. \tag{4.76}$$

These scalars occur since we can build non-trivial endomorphisms out of our gravitational fields by raising and lowering indices with $\overline{g}^{\cdot\cdot}$ and its inverse. For example, we can construct the endomorphisms

$$\overline{\overline{h}}^{\alpha}{}_{\beta} := \widehat{\overline{g}}^{\alpha\sigma}\widehat{\overline{\overline{g}}}_{\sigma\beta}\,, \tag{4.77a}$$

$$\overline{\overline{\overline{h}}}^{\alpha}{}_{\beta} := \widehat{\overline{g}}^{\alpha\sigma}\widehat{\overline{\overline{\overline{g}}}}_{\sigma\beta}\,, \tag{4.77b}$$

whose eigenvalues can then be expressed in terms of the three scalars

$$\mathrm{tr}\left(\overline{\overline{h}}\right) \quad , \quad \mathrm{tr}\left(\overline{\overline{h}}^{2}\right) \quad , \quad \det\overline{\overline{h}}\,, \tag{4.78}$$

and in the same fashion for $\overline{\overline{\overline{h}}}$. Due to the Cayley-Hamilton theorem we can express the traces of the third power and all higher powers in terms of these three scalars, since

$$\overline{\overline{h}}^{3} = \mathrm{tr}(\overline{\overline{h}})\cdot\overline{\overline{h}}^{2} + \frac{1}{2}\left(\mathrm{tr}\left(\overline{\overline{h}}^{2}\right) - \left(\mathrm{tr}(\overline{\overline{h}})\right)^{2}\right)\cdot\overline{\overline{h}} + \det\overline{\overline{h}}\cdot\mathrm{id}\,. \tag{4.79}$$

Taking into account that $\overline{\overline{\overline{h}}}$ is traceless, we can define the following eight scalars by taking traces or determinants of the endomorphisms

$$\sigma^{(1)}(\varphi) := \overline{\overline{h}}^{\alpha}{}_{\alpha}(\varphi)\,, \tag{4.80a}$$

$$\sigma^{(2)}(\varphi) := \overline{\overline{h}}^{\alpha}{}_{\beta}(\varphi)\overline{\overline{h}}^{\beta}{}_{\alpha}(\varphi)\,, \tag{4.80b}$$

$$\sigma^{(3)}(\varphi) := \overline{\overline{h}}^{\alpha}{}_{\beta}(\varphi)\overline{\overline{\overline{h}}}^{\beta}{}_{\alpha}(\varphi)\,, \tag{4.80c}$$

$$\sigma^{(4)}(\varphi) := \overline{\overline{\overline{h}}}^{\alpha}{}_{\beta}(\varphi)\overline{\overline{\overline{h}}}^{\beta}{}_{\alpha}(\varphi)\,, \tag{4.80d}$$

$$\sigma^{(5)}(\varphi) := \overline{\overline{h}}^{\alpha}{}_{\beta}(\varphi)\overline{\overline{\overline{h}}}^{\beta}{}_{\gamma}(\varphi)\overline{\overline{\overline{h}}}^{\gamma}{}_{\alpha}(\varphi)\,, \tag{4.80e}$$

$$\sigma^{(6)}(\varphi) := \overline{\overline{h}}^{\alpha}{}_{\beta}(\varphi)\overline{\overline{h}}^{\beta}{}_{\gamma}(\varphi)\overline{\overline{\overline{h}}}^{\gamma}{}_{\alpha}(\varphi)\,, \tag{4.80f}$$

$$\sigma^{(7)}(\varphi) := \det\overline{g}^{\cdot\cdot}(\varphi)\cdot\det\overline{\overline{g}}_{\cdot\cdot}(\varphi)\,, \tag{4.80g}$$

$$\sigma^{(8)}(\varphi) := \det\widehat{\overline{g}}^{\cdot\cdot}(\varphi)\cdot\det\widehat{\overline{\overline{\overline{g}}}}_{\cdot\cdot}(\varphi)\,, \tag{4.80h}$$

which are functionally independent and whose derivatives with respect to the degrees of freedom are linearly independent. Furthermore, it is possible to define the vector field

$$\nu^{\alpha}(\varphi) := \sqrt{\det\overline{g}^{\cdot\cdot}(\varphi)}\,\epsilon^{\alpha\mu\nu}\widehat{\overline{g}}^{\lambda\kappa}(\varphi)\widehat{\overline{\overline{g}}}_{\mu\lambda}(\varphi)\widehat{\overline{\overline{\overline{g}}}}_{\kappa\nu}(\varphi)\,. \tag{4.81}$$

Together with the six components of the input coefficient $\psi^a(\varphi) := \mathcal{I}^a{}_{\alpha\beta}\mathrm{p}^{\alpha\beta}(\varphi)$ we have then 17 functions of the degrees of freedom that fulfill the partial differential equations

$$\frac{\partial\psi^a}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma(\varphi) = 2\,\mathcal{I}^a{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_b\psi^b(\varphi)\,, \tag{4.82a}$$

$$\frac{\partial\nu^\alpha}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma(\varphi) = \delta^\alpha_\mu\nu^\gamma(\varphi)\,, \tag{4.82b}$$

$$\frac{\partial\sigma^{(i)}}{\partial\varphi^A}\mathrm{F}^A{}_\mu{}^\gamma(\varphi) = 0\,. \tag{4.82c}$$

The reparametrization of the area metric degrees of freedom to the normal form has the advantage that it turns the rather involved structure of the gravitational closure equations into a form where the sectors are disentangled, such that we can easily identify the output coefficients we can solve for.

However, for the area metric, this reparametrization is non-linear and employs higher orders of the degrees of freedom such that an analytical expression for $\varphi^A(\psi^a,\nu^\alpha,\sigma^{(i)})$ is unfortunately out of reach – if it does exist at all globally. Locally, however, the maps exist if the Jacobian matrix has full rank. Although singular points exist, such as the obvious choice $\varphi \equiv 0$, one can check that we can find this reparametrization for most points. This hints that the presented reparametrization may be a suitable road to pursue in the future in order to derive an exact solution for the area metric.

Instead, we will dedicate the remainder of this thesis to an alternative path to extract physically meaningful statements from the gravitational closure equations: Instead of solving the involved structure of the system of partial differential equations, we transform them into a collection of linear algebraic equations that give the Lagrangian of the gravitational degrees of freedom in a perturbative system with respect to a given background solution.

CHAPTER 5
PERTURBATIVE CLOSURE

The practical use of the gravitational closure equations was, so far, rather limited due to their vast complexity. The situation is comparable to Einstein's equations: Due to their non-linearity, the equations of motion required techniques that were way beyond the familiar terrain of linear equations. Nonetheless, almost directly after publication of the field equations (Einstein, 1915b), they could be employed to explain the perihelion precession of Mercury by considering a situation where the gravitational fields have *small* deviations from the solution in vacuo (Einstein, 1915a). In this case, we find ourselves in a perturbative regime that allows us to derive the dynamics for these small deviations in an iterative fashion, increasing the precision with each order.

In the last decades of dealing with the general theory of relativity, this idea was extended and put on a more rigorous level to consider gravitational interactions perturbatively. One famous example is the binary pulsar, where two orbiting sources have an increasing orbital frequency due to the energy loss created by gravitational radiation. Since this increase in frequency can be experimentally measured, it serves as an indirect high precision test of general relativity.

There is also another option to solve Einstein's equations that was employed shortly after the introduction of general relativity to derive exact solutions. The clever trick is to consider situations with a certain symmetry, thereby drastically reducing the solution space's complexity. This led, for example, to the discovery of the Schwarzschild solution, which describes the physical situation of an uncharged, non-rotating black hole (Schwarzschild, 1916).

The same methods can be employed for the gravitational closure equations. The latter, i.e. the consideration under symmetry conditions, for example can be used to derive gravitational Lagrangians for cosmological scales (see Düll et al. (2020) and Düll (2020) for further details).

This chapter is dedicated to the first of the described methods. We will present a perturbative treatment of the gravitational closure equations that allows us to derive the Lagrangians for small gravitational fields and propagation of gravitational waves. Note that an early treatment of perturbative gravitational closure was published in the pre-print

J. Schneider, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational Closure of Weakly Birefringent Electrodynamics
arXiv: 1708.03870 (2017)

in which the linear field equations were first derived for general linear electrodynamics. The methodology, however, was extended throughout this thesis to allow for a proper framework that can be employed to (at least in theory) arbitrary order.

5.1 PERTURBATIVE GRAVITATIONAL DYNAMICS

We start our discussion of a perturbative treatment of the gravitational closure equations by revisiting some of the constructions that we made in the previous chapters. Once we explore those from a perturbative perspective, it turns out that some simplifications can be made.

The starting point is, as in every perturbative setting, that we are equipped with some *known* background solution to the equations of motion. Such a background solution can be given in the form of a collection of screen manifold tensor fields $g_{\text{background}}$. In the very same fashion, we can also provide a section $\varphi_{\text{background}}$ on our space Φ, since we can always employ the parametrization to map one onto the other. In the following, we will directly use the latter approach and, for simplicity, denote the background solution by $n^A(x,t)$. Note that we, at least for now, make no assumption on this background solution to be constant everywhere or static. In addition, the background solution needs to contain an expression for both lapse and shift. Typically, this corresponds to a constant lapse $N_{\text{background}} = 1$ and vanishing shift $\vec{N}_{\text{background}} = 0$[1].

Employed with the background solution, we aim to reformulate the generally non-linear equations of motion by introducing an embedding parameter $\varepsilon \in [0,1]$ that allows us to continuously deform our background solution into a full solution. For this, we can introduce the homotopy $\varphi(x,t;\varepsilon)$, with $\varphi^A(x,t,0) = n^A(x,t)$, and analogously for the other fields in our setup, namely the lapse $N(x,t;\varepsilon)$ and shift $\vec{N}(x,t;\varepsilon)$. The parameter ε will also play an additional role as the coupling between the matter action $\mathcal{S}_{\text{matter}}$, formulated for the matter field ψ, and the gravitational action, i.e.

$$\begin{aligned}\mathcal{S}_{\text{total}}\left[\varphi(x,t;\varepsilon),N(x,t;\varepsilon),\vec{N}(x,t;\varepsilon),\psi\right] &= \mathcal{S}_{\text{gravity}}\left[\varphi(x,t;\varepsilon),N(x,t;\varepsilon),\vec{N}(x,t;\varepsilon)\right]\\ &\quad+\varepsilon\cdot\mathcal{S}_{\text{matter}}\left[\varphi(x,t;\varepsilon),N(x,t;\varepsilon),\vec{N}(x,t;\varepsilon),\psi(x,t)\right],\end{aligned} \tag{5.1}$$

which, as a theoretical tool, ensures that both actions have the same units and, phenomenologically, implements that the deviations from the background solution will be sourced from the matter sector.

We now assume that this homotopy is a solution to the *exact* gravitational equations of motion obtained from the action by variation with respect to the degrees of freedom φ, i.e.

$$\frac{\delta\mathcal{S}_{\text{gravity}}}{\delta\varphi^A}[\varphi(x,t;\varepsilon),N(x,t;\varepsilon),\vec{N}(x,t;\varepsilon)] \quad = \quad -\varepsilon\frac{\delta\mathcal{S}_{\text{matter}}}{\delta\varphi^A}[\varphi(x,t;\varepsilon),N(x,t;\varepsilon),\vec{N}(x,t;\varepsilon),\psi(x,t;\varepsilon)]\,,$$

[1] Since for a globally hyperbolic spacetime we have that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}\times\Sigma$ (Bernal and Sánchez, 2003) we can always find a gauge that effectively removes the shift vector field. However, it is not necessarily true that the lapse is constant everywhere.

as well as the four constraints obtained by varying with respect to both lapse and shift

$$\begin{aligned}\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N}[\varphi(x,t;\varepsilon), N(x,t;\varepsilon), \vec{N}(x,t;\varepsilon)] &= -\varepsilon \frac{\delta \mathcal{S}_{\text{matter}}}{\delta N}[\varphi(x,t;\varepsilon), N(x,t;\varepsilon), \vec{N}(x,t;\varepsilon), \psi(x,t;\varepsilon)]\,,\\ \frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N^\alpha}[\varphi(x,t;\varepsilon), N(x,t;\varepsilon), \vec{N}(x,t;\varepsilon)] &= -\varepsilon \frac{\delta \mathcal{S}_{\text{matter}}}{\delta N^\alpha}[\varphi(x,t;\varepsilon), N(x,t;\varepsilon), \vec{N}(x,t;\varepsilon), \psi(x,t;\varepsilon)]\,,\end{aligned}$$

and the matter field equations of motion

$$0 = \frac{\delta \mathcal{S}_{\text{matter}}}{\delta \psi^a(x,t)} \left[\psi(x,t), \varphi(x,t;\varepsilon), N(x,t;\varepsilon), \vec{N}(x,t;\varepsilon)\right]\,.$$

By solving the latter equation, we couple the matter field to the geometric fields, which makes them, indirectly, dependent on the embedding parameter ε.

At this point, it still seems that no progress was made since we have neither a solution to the equations of motion nor the coefficients appearing in the equations of motion. However, since our homotopies depend smoothly on the embedding parameter ε, we can calculate the homotopy-Maclaurin expansion of all fields and obtain

$$\varphi^A(x,t;\varepsilon) = n^A(x,t) + \sum_{k=1}^{\infty} \frac{1}{k!} h^A_{(k)}\, \varepsilon^k\,, \tag{5.2a}$$

$$N(x,t;\varepsilon) = N_{\text{background}}(x,t) + \sum_{k=1}^{\infty} \frac{1}{k!} A_{(k)}\, \varepsilon^k\,, \tag{5.2b}$$

$$N^\mu(x,t;\varepsilon) = N^\mu \text{background}(x,t) + \sum_{k=1}^{\infty} \frac{1}{k!} B^\mu_{(k)}\, \varepsilon^k\,, \tag{5.2c}$$

in terms of the perturbations $h_{(k)}$, $A_{(k)}$ and $\vec{B}_{(k)}$ of order k. Inserting this expansion into the equation of motion displayed above, we obtain a polynomial in the embedding parameter. In order to be able to *continuously* deform the known solution into the real solution, we find that each order must vanish separately. This allows us to generate a perturbative solution to the equations of motion to, at least in principle, arbitrary precision.

The 0^{th} order contribution does not reveal any new information but enforces that the background objects n^A, $N_{\text{background}}$ and $\vec{N}_{\text{background}}$ be solutions of the equations of motion:

$$\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta \varphi^A}\left[n, N_{\text{background}}, \vec{N}_{\text{background}}\right] = 0\,, \tag{5.3}$$

$$\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N}\left[n, N_{\text{background}}, \vec{N}_{\text{background}}\right] = 0\,, \tag{5.4}$$

$$\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N^\alpha}\left[n, N_{\text{background}}, \vec{N}_{\text{background}}\right] = 0\,. \tag{5.5}$$

Since the embedding parameter appears in front of the matter action, we obtain no contribution from the matter sector. This is, of course, not surprising but merely motivates the name *background solution*.

From the 1^{st} order terms we find linear partial differential equations for the linear perturbations, i.e.

$$\left.\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta \varphi^A}\right|_{(1)} \left[h_{(1)}, A_{(1)}, B_{(1)}\right] = -\frac{\delta \mathcal{S}_{\text{matter}}}{\delta \varphi^A}\left[\psi|_{(0)}\right]\,, \tag{5.6a}$$

$$\left.\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N}\right|_{(1)} \left[h_{(1)}, A_{(1)}, B_{(1)}\right] = -\frac{\delta \mathcal{S}_{\text{matter}}}{\delta N}\left[\psi|_{(0)}\right]\,, \tag{5.6b}$$

$$\left.\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta N^\alpha}\right|_{(1)} \left[h_{(1)}, A_{(1)}, B_{(1)}\right] = -\frac{\delta \mathcal{S}_{\text{matter}}}{\delta N^\alpha}\left[\psi|_{(0)}\right]\,, \tag{5.6c}$$

where the subscript (1) on the functional derivatives on the left hand side indicates that we collect all the linear terms from the equations. On the right hand side we see that, since the matter action itself is multiplied by ε, we take from the source term that is evaluated on the solution of the matter equations only the terms containing the background geometry. This will source the linear deviations from the dynamics of a matter theory that lives on the background geometry. This interplay is, of course, exactly what John A. Wheelers famously summarized for general relativity as "matter tells spacetime how to curve, spacetime tells matter how to move".

Moving to the next iteration, we again decompose the equations of motion into the separate orders of ε. Doing so one obtains equations in the form

$$\begin{aligned} \left.\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta \varphi^A}\right|_{(1)} \left[h_{(2)}, A_{(2)}, \vec{B}_{(2)}\right] &= -\left.\frac{\delta \mathcal{S}_{\text{gravity}}}{\delta \varphi^A}\right|_{(2)} \left[h_{(1)}, A_{(1)}, \vec{B}_{(1)}\right] \\ &\quad - \frac{\delta \mathcal{S}_{\text{matter}}}{\delta \varphi^A}\left[\psi|_{(1)}\right], \end{aligned} \tag{5.7}$$

and in the same fashion for the constraint equations. From the matter action we need to extract all terms that are quadratic in ε, which are obtained from the background geometry and the linear perturbations. In addition we obtain another source term by the linear perturbations.

For the 2nd order perturbations, we find that they appear with the same differential operator as for the linear perturbations, which means that we need to solve *linear* partial differential equations. This can be done with the usual methods and the *same* Green's functions as for the linear perturbations. This makes this iterative method an incredibly helpful tool to generate a solution to the equations of motion.

We can repeat this step iteratively to some maximal perturbation order $k_{\max}$. At each order k, one finds linear partial differential equations for the k^{th} perturbations that are sourced in an increasingly complex manner by the lower order perturbations as well as the matter fields. We then obtain our final, approximate, solution to the equations of motion by truncating the homotopies at order $k_{\max}$ and evaluating at $\varepsilon = 1$, i.e.

$$\varphi^A(x,t) \approx n^A(x,t) + \sum_{k=1}^{k_{\max}} \frac{1}{k!} h^A_{(k)}(x,t), \tag{5.8a}$$

$$N(x,t) \approx N_{\text{background}}(x,t) + \sum_{k=1}^{k_{\max}} \frac{1}{k!} A_{(k)}(x,t), \tag{5.8b}$$

$$N^\alpha(x,t) \approx N^\alpha_{\text{background}}(x,t) + \sum_{k=1}^{k_{\max}} \frac{1}{k!} B^\alpha_{(k)}. \tag{5.8c}$$

This, at least in principle, allows us to generate arbitrarily precise solutions.

This discussion was, so far, at a quite abstract level where none of the coefficients in the equations of motion are explicitly obtained. Moreover, they still require the knowledge of the exact output coefficients $C, C_A, C_{AB}, \ldots$ to obtain the exact expressions of these coefficients in the equations of motion.

In order to obtain these, one expands the exact equations of motion (see equation (3.151)) in the different orders in the embedding parameter ε. This means that we expand the output coefficients as

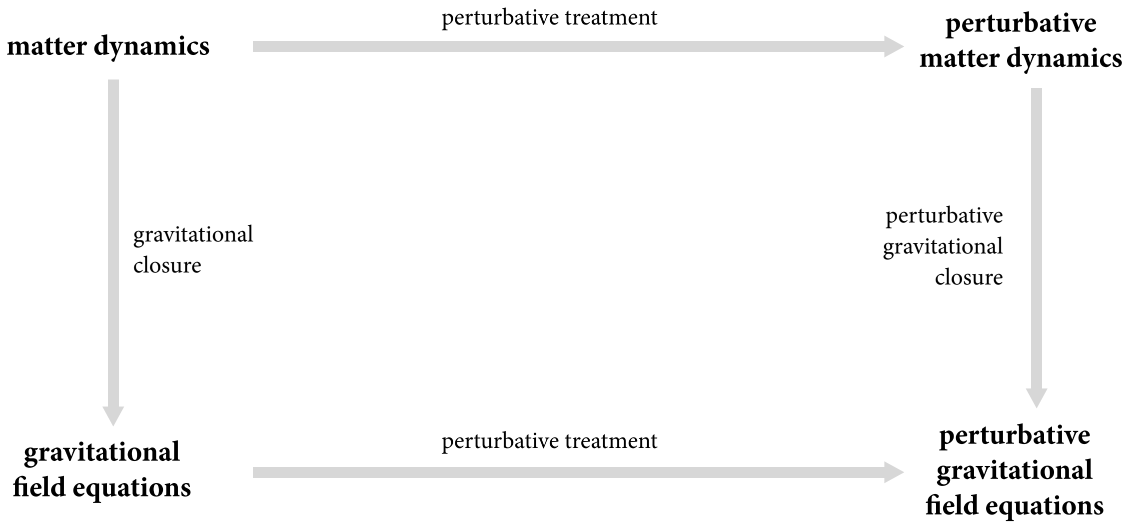


Figure 5.1 Perturbative gravitational closure can be performed by a series expansion of the output coefficients around some background solution and carefully evaluating the linear algebraic equations for the constant coefficients from the series expansion.

follows

$$
\begin{aligned}
\mathrm{C}_{A_1\ldots A_N}\left[\varphi(x,t;\varepsilon)\right] &= \sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{\mathrm{d}^k\mathrm{C}_{A_1\ldots A_N}\left[\varphi(x,t;\varepsilon)\right]}{\mathrm{d}\varepsilon^k}\right|_{\varepsilon=0}\varepsilon^k \\
&= \mathrm{C}_{A_1\ldots A_N}[n]\,\varepsilon^0 + \int_\Sigma \mathrm{d}^3y\,\left.\frac{\delta\mathrm{C}_{A_1\ldots A_N}(x)}{\delta\varphi^M(y)}\right|_{\varphi=n} h^M_{(1)}(y)\,\varepsilon^1 + \\
&\quad +\frac{1}{2}\left(\int_\Sigma \mathrm{d}^3y\,\left.\frac{\delta\mathrm{C}_{A_1\ldots A_N}(x)}{\delta\varphi^M(y)}\right|_{\varphi=n} h^M_{(2)}(y) + \right. \\
&\quad \left. +\frac{1}{2}\int_\Sigma \mathrm{d}^3y_1\int_\Sigma \mathrm{d}^3y_2\,\left.\frac{\delta^2\mathrm{C}_{A_1\ldots A_N}(x)}{\delta\varphi^{M_1}(y_1)\delta\varphi^{M_2}(y_2)}\right|_{\varphi=n} h^{M_1}_{(1)}(y_1)h^{M_2}_{(1)}(y_2)\right)\varepsilon^2 \\
&\quad +\ldots\,,
\end{aligned}
\tag{5.9}
$$

and similar for the functional derivatives of the output coefficients appearing in the equations of motion. Afterwards, we can then collect the separate orders in ε as described above. The important realization is that we can obtain these Taylor expansion coefficients – all evaluated on the background solution n – from the gravitational closure equations: we first perform a linear reparametrization, such that all output coefficients are evaluated at the jet space point

$$
\mathrm{C}_{B_1\ldots B_N}\left[n+\varepsilon\varphi\right]\,, \tag{5.10}
$$

and afterwards we expand the closure equations again in ε. In the equations we obtain, just as before for the equations of motion, a polynomial in ε for which each order must vanish separately. The expansion coefficients appearing are simply the Taylor coefficients that appeared on the right hand side in (5.9) – exactly the coefficients we need for our equations of motion. We will refer to these Taylor expansion coefficients as *constant output coefficients* in the following[2]. As a result, we obtain *linear algebraic* equations for these constant output coefficients that we need solve, once we have expanded the gravitational closure equations.

This opens up an alternative, but much more approachable, path to deriving the gravitational field equations. The idea is illustrated in the diagram in figure 5.1. The important point is that, in general, the resulting predictions are independent of the concrete path taken.

Special case: linear dynamics for flat backgrounds

Although the whole procedure we will discuss can be conducted to any order, let us illustrate the steps laid out in the previous section in more detail for linear dynamics around flat backgrounds. A prototypical example for such a scenario is the perturbation around a background geometry induced by a flat Minkowski metric, as we will consider in section 5.4 for a weakly birefringent geometry obtained from an area metric.

In this setting, we have that the background fulfills the following equations

$$
\begin{aligned}
\dot{n}^A(x,t) &= 0\,, & \left(\partial_\mu n^A\right)(x,t) &= 0\,,\\
\dot{A}_{(0)}(x,t) &= 0\,, & \left(\partial_\mu A_{(0)}\right)(x,t) &= 0\,,\\
\dot{B}^\alpha_{(0)}(x,t) &= 0\,, & \left(\partial_\mu B^\alpha_{(0)}\right)(x,t) &= 0\,,\\
k^A_{(0)} &= 0\,, & \mathrm{M}^{A\gamma}|_{\varphi=n} &= 0
\end{aligned}
$$

everywere. Let us start with the 0^{th} order contribution of the equations of motion. Using the expansion of the scalar constraint (see equation (3.152)) we find that only a single term survives on the right hand side

$$0 = -\,\mathrm{C}|_{\varphi=n}\ . \tag{5.11}$$

This tells us that the constant $\mathrm{C}|_{\varphi=n}$ needs to vanish. Similarly, we can analyze the remaining equations. While the vector constraint is fulfilled trivially one finds from the evolution equations that

$$0 = \mathrm{C}_{:A}|_{\varphi=n}\ . \tag{5.12}$$

As for the scalar constraint, we conclude that $\mathrm{C}_{:A}|_{\varphi=n}$ needs to vanish if we want the background to be a solution of the gravitational equations of motion.

Next, we can derive the equations of motion of the linear modes. For this, we collect all terms linear

[2]This is for historic reasons. Clearly, they are only constant for a constant background solution.

in ε on both sides. Careful calculation gives

$$-\left.\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N(x)}\right|_{(1)}\left[\psi|_{(0)}\right] = \left.\mathrm{C}_{:A}{}^{\mathcal{A}}\right|_{\varphi=n} h^{A}_{(1),\mathcal{A}} + \mathrm{C}_A|_{\varphi=n}\,\left.\mathrm{M}^{A\gamma}{}_{:B}\right|_{\varphi=n} h^{B}_{(1),\gamma}\,, \tag{5.13a}$$

$$\begin{aligned}-\left.\frac{\delta\mathcal{S}_{\text{matter}}}{\delta N^{\mu}(x)}\right|_{(1)}\left[\psi|_{(0)}\right] = &+\mathrm{C}_A|_{\varphi=n}\left(\delta^A_B\delta^\gamma_\mu + \left.\mathrm{F}^A{}_\mu{}^\gamma{}_{:B}\right|_{\varphi=n}\right)h^B_{(1),\gamma} + \left.\mathrm{C}_{A:B}{}^{\mathcal{A}}\right|_{\varphi=n}\left.\mathrm{F}^A{}_\mu{}^\gamma\right|_{\varphi=n} h^B_{(1),\gamma\mathcal{A}}\\ &+\frac{2}{A_{(0)}}\,\mathrm{C}_{AB}|_{\varphi=n}\left.\mathrm{F}^A{}_\mu{}^\gamma\right|_{\varphi=n}\left(\dot h^B_{(1),\gamma} - B^\nu_{(0)}h^B_{(1),\nu\gamma} + B^\nu_{(1),\epsilon\gamma}\left.\mathrm{F}^B{}_\nu{}^\epsilon\right|_{\varphi=n}\right)\,,\end{aligned} \tag{5.13b}$$

$$\begin{aligned}\left.\frac{\delta\mathcal{S}_{\text{matter}}}{\delta\varphi^A(x)}\right|_{(1)}\left[\psi|_{(0)}\right] = &\frac{2}{A_{(0)}}\,\mathrm{C}_{AB}|_{\varphi=n}\Bigg(\ddot h^B_{(1)} - B^\mu_{(0)}\dot h^B_{(1),\mu} + \dot B^\nu_{(1),\epsilon}\left.\mathrm{F}^B{}_\nu{}^\epsilon\right|_{\varphi=n} +\\ &- B^\mu_{(0)}\dot h^B_{(1),\mu} + B^\mu_{(0)}B^\nu_{(0)}h^B_{(1),\mu\nu} - B^\mu_{(0)}\dot B^\nu_{(1),\mu\epsilon}\left.\mathrm{F}^B{}_\nu{}^\epsilon\right|_{\varphi=n}\Bigg)\\ &+\left(\left.\mathrm{C}_{A:B}{}^{\mathcal{A}}\right|_{\varphi=n} - (-1)^{|\mathcal{A}|}\frac{1}{A_{(0)}}\left.\mathrm{C}_{B:A}{}^{\mathcal{A}}\right|_{\varphi=n}\right)\left(\dot h^B_{(1),\mathcal{A}} - B^\mu_{(0)}h^B_{(1),\mu\mathcal{A}}\right)\\ &-(-1)^{|\mathcal{A}|}\frac{1}{A_{(0)}}\left.\mathrm{C}_{B:A}{}^{\mathcal{A}}\right|_{\varphi=n} B^\mu_{(1),\gamma\mathcal{A}}\left.\mathrm{F}^B{}_\mu{}^\gamma\right|_{\varphi=n}\\ &+\mathrm{C}_B|_{\varphi=n}\left(A_{(1),\gamma}\left.\mathrm{M}^{B\gamma}{}_{:A}\right|_{\varphi=n} - B^\mu_{(1),\gamma}\left(\delta^B_A\delta^\gamma_\mu + \left.\mathrm{F}^B{}_\mu{}^\gamma{}_{:A}\right|_{\varphi=n}\right)\right)\\ &-(-1)^{|\mathcal{A}|}\left.\mathrm{C}_{:A}{}^{\mathcal{A}}\right|_{\varphi=n}A_{(1),\mathcal{A}} - (-1)^{|\mathcal{A}|}A_{(0)}\left.\mathrm{C}_{:A}{}^{\mathcal{A}}{}_B{}^{\mathcal{B}}\right|_{\varphi=n}h^B_{(1),\mathcal{A}\mathcal{B}}\,,\end{aligned} \tag{5.13c}$$

where the left hand side denotes all contributions of the source part that are linear in ε.

From these three equations, we can now read off two things: First, it immediately shows us that the only output coefficients that contribute for linear dynamics are C, C_A and C_{AB}. This is, of course, not so surprising since one typically needs a quadratic Lagrangian to obtain linear equations of motion. Second, it tells us precisely which contributions from those three output coefficients, evaluated on our flat background, we need to obtain from the gravitational closure equations:

$$\mathrm{C} \longrightarrow \left.\mathrm{C}_{:A}{}^{\mathcal{A}}\right|_{\varphi=n}\quad,\quad \left.\mathrm{C}_{:A}{}^{\mathcal{A}}{}_B{}^{\mathcal{B}}\right|_{\varphi=n}\,, \tag{5.14a}$$

$$\mathrm{C}_A \longrightarrow \mathrm{C}_A|_{\varphi=n}\quad,\quad \left.\mathrm{C}_{A:B}{}^{\mathcal{A}}\right|_{\varphi=n}\,, \tag{5.14b}$$

$$\mathrm{C}_{AB} \longrightarrow \mathrm{C}_{AB}|_{\varphi=n}\,. \tag{5.14c}$$

All other coefficients will not appear in the equations of motion anyway. We will see in the following section how these coefficients can be obtained from the gravitational closure equations.

One may imagine the procedure by this analogy: By calculating the perturbative equations of motion of the abstract Lagrangian to 1st order, we only see a vague shadow cast by the exact theory. We can reduce the vast complexity in the gravitational closure equations by looking at all the terms that cast precisely those shadows – and ignoring the rest.

If one considers perturbations around a different background, for instance, one that is not flat or has a contribution of the $\mathrm{M}^{A\gamma}$ coefficient evaluated at the background, one proceeds similarly by collecting the terms in ε. However, the resulting expressions typically become more involved as additional contributions appear.

5.2 PERTURBATIVE SOLUTIONS TO THE GRAVITATIONAL CLOSURE EQUATIONS

We will dedicate this section to a presentation of the required technical steps for the perturbative evaluation of the gravitational closure equations. This will include a quick review of parametrizations, the inverse intertwiners, as well as the three input coefficients. Afterwards, we expand on the series expansion of the output coefficients and how the system of linear partial differential equations will be translated into a linear algebraic system for finitely many parameters. From a physical point of view, these parameters constitute the *constants of nature* that we need to fix by experiments. Once all technical aspects have been laid out in detail, we will apply these techniques to the concrete example of the area metric geometry obtained from birefringent electrodynamics.

5.2.1 Perturbative parametrizations

Just as in the case of exact solutions, we will again need a parametrization of the frame conditions for our gravitational fields on the screen manifold Σ, in terms of the geometric degrees of freedom. While, in principle, we could simply use an existing parametrization, it has proven beneficial to first perform a linear parametrization around the background n^A. Afterwards, we series expand in the degrees of freedom.

The reason for this is the following: We ultimately want to read off the output coefficients – and their derivatives – evaluated on the background geometry, as they appear in the perturbative equations of motion. With an exact parametrization, as they have been presented before, this will require us to evaluate the closure equations at the jet space point $\varphi = n$. Once the linear reparametrization has been performed, we can simply evaluate at 0. This will simplify the calculations in practice. Since such a linear reparametrization constitutes a canonical transformation (compare section 3.2.1), the results are independent of the particular parametrization chosen.

Let us make this more precise for the two examples we considered throughout this thesis: In the case of Maxwellian electrodynamics, we parametrized the pull-back of the metric to the screen manifold in terms of six geometric degrees of freedom via

$$\widehat{g}^{\alpha\beta}(\varphi) = \mathcal{I}^{\alpha\beta}{}_A \varphi^A \,. \tag{5.15}$$

If we want to perform a perturbative expansion around the Minkowski solution, we identify the background point n via

$$n^A = -\mathcal{I}^A{}_{\alpha\beta}\gamma^{\alpha\beta} \,, \tag{5.16}$$

in terms of the constant inverse intertwiner and the flat Riemannian metric $\gamma^{\cdot\cdot}$ on the screen manifold Σ. By performing a linear reparametrization we obtain

$$\widehat{g}^{\alpha\beta}_{\text{perturbative}}(\varphi) := \widehat{g}^{\alpha\beta}(n+\varphi) = -\gamma^{\alpha\beta} + \mathcal{I}^{\alpha\beta}{}_A \varphi^A \,. \tag{5.17}$$

This is, of course, equivalent to the typical decomposition of the inverse metric $g^{\alpha\beta} = -\gamma^{\alpha\beta} + h^{\alpha\beta}$, where $h^{\alpha\beta}$ captures the first-order perturbation.

In the case of general linear electrodynamics, we start with the parametrization we constructed in section 4.2: The background solution, again given by a Minkowskian background, reads

$$G^{abcd} = \eta^{a[c}\eta^{d]b} - \epsilon^{abcd}\,. \tag{5.18}$$

From this, we can identify the background point $n^A = (\overline{N}^a, \overline{\overline{N}}^a, \overline{\overline{\overline{N}}}^m)$ from

$$\begin{aligned}
\widehat{g}^{\alpha\beta}(N) &= \gamma^{\alpha\beta} =: \mathcal{I}^{\alpha\beta}{}_a\overline{N}^a\,, && (5.19)\\
\widehat{\overline{\overline{g}}}_{\alpha\beta}(N) &= \gamma_{\alpha\beta} =: \mathcal{I}^a{}_{\alpha\beta}\Delta_{ab}\overline{\overline{N}}^b\,, && (5.20)\\
\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}(N) &= 0 =: \mathcal{I}^a{}_{\alpha\beta}\left(\delta^b_a - \frac{n_a\overline{N}^b}{n_c\overline{N}^c}\right)\epsilon_{(m)b}\overline{\overline{\overline{N}}}^m\,. && (5.21)
\end{aligned}$$

In the perturbative setting we can always choose the basis vector t^a (compare equation 5.81) to be $\overline{N}^a$ since $\gamma^{\alpha\beta}$ is a symmetric positive definite matrix. As a result we find that $n_a\overline{N}^a = 1$, $\overline{N}^a\epsilon_{(m)a} = 0$ and that $\mathcal{I}^a{}_{\alpha\beta}\epsilon_{(m)a}\overline{\overline{\overline{N}}}^m = 0$. We moreover expand the denominator in the field $\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta}$ and find the perturbative parametrization

$$\begin{aligned}
\widehat{g}^{\alpha\beta}_{\text{perturbative}}(\varphi) &= \gamma^{\alpha\beta} + \mathcal{I}^{\alpha\beta}{}_a\overline{\varphi}^a\,, && (5.22a)\\
\widehat{\overline{\overline{g}}}_{\alpha\beta,\text{perturbative}}(\varphi) &= \gamma_{\alpha\beta} + \mathcal{I}^a{}_{\alpha\beta}\Delta_{ab}\overline{\overline{\varphi}}^b\,, && (5.22b)\\
\widehat{\overline{\overline{\overline{g}}}}_{\alpha\beta,\text{perturbative}}(\varphi) &= \mathcal{I}^a{}_{\alpha\beta}\left(\delta^b_a - \sum_{k=0}^{\infty}(-1)^k\left(n_c\overline{\varphi}^c\right)^k n_a\overline{\varphi}^b\right)\epsilon_{(m)b}\overline{\overline{\overline{\varphi}}}^m\\
&= 0 + \mathcal{I}^a{}_{\alpha\beta}\epsilon_{(m)a}\overline{\overline{\overline{\varphi}}}^m - \frac{1}{3}\gamma_{\alpha\beta}\epsilon_{(m)a}\overline{\varphi}^a\overline{\overline{\overline{\varphi}}}^m + \mathcal{O}(3)\,. && (5.22c)
\end{aligned}$$

Since we expressed all fields as polynomials in φ, it will be a simple task to separate the different polynomial orders of the jet variables when evaluating the gravitational closure equations.

Inverse intertwiner

Due to the non-linearity of the frame conditions, in general, it may become practically hard to explicitly spell out the inverse map $\widehat{\varphi}$ in terms of the fields. This is especially true for all theories where we construct the parametrization of the fields directly by a perturbative ansatz: Here, we cannot perform a well-defined series expansion of the map $\widehat{\varphi}$ in terms of the field components. Any expansion order will contribute to the calculations once the background fields are part of the game.

Luckily, it turns out that the explicit knowledge of the inverse map will not be required. The only appearance of the map occurs in the form of the inverse intertwiner $\frac{\partial\widehat{\varphi}^A}{\partial g^A}(\widehat{g}(\varphi))$, which is, after all, a map in terms of the degrees of freedom φ. This intertwiner can then, in practice, be obtained iteratively from the completeness relation

$$\frac{\partial\widehat{\varphi}^A}{\partial g^A}(\widehat{g}(\varphi))\,\frac{\partial\widehat{g}^A}{\partial\varphi^B}(\varphi) = \delta^A_B\,, \tag{5.23}$$

given that the 0^{th} order of the intertwiner $\frac{\partial \widehat{g}^{\mathcal{A}}}{\partial \varphi^{B}}(\varphi)$ is non-zero. For this, we first expand both intertwiners in φ, i.e.

$$\frac{\partial \widehat{g}^{\mathcal{A}}}{\partial \varphi^{A}}(\varphi) = \mathcal{I}|_0{}^{\mathcal{A}}{}_{A} + \mathcal{I}|_1{}^{\mathcal{A}}{}_{AB}\,\varphi^{B} + \mathcal{I}|_2{}^{\mathcal{A}}{}_{AB_1B_2}\,\varphi^{B_1}\,\varphi^{B_2} + \dots\,, \tag{5.24a}$$

$$\frac{\partial \widehat{\varphi}^{A}}{\partial g^{\mathcal{A}}}(\widehat{g}(\varphi)) = \mathcal{J}|_0{}^{A}{}_{\mathcal{A}} + \mathcal{J}|_1{}^{A}{}_{\mathcal{A}B}\,\varphi^{B} + \mathcal{J}|_2{}^{\mathcal{A}}{}_{AB_1B_2}\,\varphi^{B_1}\,\varphi^{B_2} + \dots\,, \tag{5.24b}$$

where in the first line, all expansions coefficients are known from our chosen parametrization of the gravitational screen manifold tensor fields. In the second line, all coefficients are unknown and have to be determined.

We can then evaluate the completeness relation for the different orders in φ: At 0^{th} order we find the requirement that

$$\mathcal{J}|_0^{A}{}_{\mathcal{A}}\,\mathcal{I}|_0^{\mathcal{A}}{}_{B} = \delta^{A}_{B}\,. \tag{5.25}$$

This tells us that the Moore-Penrose pseudoinverse of the matrix $\mathcal{I}|_0^{\mathcal{A}}{}_{B}$ gives the 0^{th} order coefficient $\mathcal{J}|_0^{A}{}_{\mathcal{A}}$ of the inverse intertwiner, in case the former has full rank F. Physically, this requirement corresponds to the statement that we need to be able to determine *all* F coordinates φ for the background solution n^{A} from the background tensor components *and vice-versa*.

When moving to higher-order contributions, we then find that the remaining terms that contain the degrees of freedom φ must cancel, as the completeness relation has to be valid for any jet space point. As a result, to first order in φ, we find the relation

$$0 = \mathcal{J}|_0^{A}{}_{\mathcal{A}}\,\mathcal{I}|_1^{\mathcal{A}}{}_{BM} + \mathcal{J}|_1^{A}{}_{\mathcal{A}M}\,\mathcal{I}|_0^{\mathcal{A}}{}_{B}\,, \tag{5.26}$$

where we can use the pseudoinverse again to find the 1^{st} order contribution, i.e.

$$\mathcal{J}|_1^{A}{}_{\mathcal{A}M} = -\,\mathcal{J}|_0^{A}{}_{\mathcal{B}}\,\mathcal{I}|_1^{\mathcal{B}}{}_{BM}\,\mathcal{J}|_0^{B}{}_{\mathcal{A}}\,. \tag{5.27}$$

The same can be repeated iteratively for the higher orders, so we will not spell the results out explicitly. If an exact parametrization is known, we can, of course, directly obtain the expansion coefficients. This procedure, however, becomes particularly useful when turning the perturbative closure programme into a computer algebra problem that can be solved algorithmically for any theory with the parametrization of the fields and the input coefficient as its sole input.

Input coefficients

Once equipped with a perturbative parametrization, one proceeds as usual by calculating the three input coefficients $\mathrm{p}^{\alpha\beta}$, $\mathrm{M}^{A\gamma}$ and $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ with the chosen parametrization. Afterwards, we can expand the expressions again in terms of φ since we need to separate the different orders in the closure equations by orders in φ later. To which order one must expand the input coefficients is crucially dependent on the order one wants to calculate in the equations of motion. For instance, given that we want to calculate the equations of motion to linear order, one typically finds from the gravitational closure equations that we need to calculate the input coefficients to the following orders

$$\begin{aligned} &\mathrm{p}^{\mu\nu} && \text{order } 2\,, \\ &\mathrm{M}^{A\gamma} && \text{order } 2\,, \\ &\mathrm{F}^{A}{}_{\mu}{}^{\gamma} && \text{order } 1\,. \end{aligned} \tag{5.28}$$

In section 5.2.3, once we finally consider the evaluation of the gravitational closure equations in this perturbative treatment, we will make more precise to which order we need to calculate the input coefficients and how to derive which orders are necessary.

5.2.2 Output coefficients

As in the previous section, we can now consider the output coefficients in our perturbative setting. For this, we use our linear reparametrization and functionally Taylor expand the output coefficients around the background solution in the following fashion:

$$\begin{aligned} \mathrm{C}_{A_1\dots A_N}[n+\varphi] &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_\Sigma \mathrm{d}^3 y_1 \cdots \int_\Sigma \mathrm{d}^3 y_k \; \left.\frac{\delta^k \mathrm{C}_{A_1\dots A_N}}{\delta\varphi^{B_1}(y_1)\cdots\delta\varphi^{B_k}(y_k)}\right|_{\varphi=n} \varphi^{B_1}(y_1)\cdots\varphi^{B_k}(y_k) \\ &=: \Lambda_{A_1\dots A_N} + \sum_{m=0}^{\max} \Lambda_{A_1\dots A_N|B}{}^{\alpha_1\dots\alpha_m}\varphi^{B}{}_{,\alpha_1\dots\alpha_m} + \\ &\quad + \frac{1}{2}\sum_{p=0}^{\max}\sum_{q=0}^{\max} \Lambda_{A_1\dots A_N|B_1}{}^{\alpha_1\dots\alpha_p}{}_{B_2}{}^{\beta_1\dots\beta_q}\varphi^{B_1}{}_{,\alpha_1\dots\alpha_p}\varphi^{B_2}{}_{,\beta_1\dots\beta_q} \\ &\quad \dots \end{aligned} \tag{5.29}$$

where the notation $\Lambda_{A_1\dots A_N|B}{}^{\alpha_1\dots\alpha_M} := \left(\mathrm{C}_{A_1\dots A_N:B}{}^{\alpha_1\dots\alpha_M}\right)(0)$ is introduced for the expansion coefficients. This provides the missing link to our discussion of the equations of motion: The coefficients Λ that appear in the expansion of the output coefficient are required, for a consistent theory, to have the same expression as the coefficients that appear in the equations of motion. By making a list of all the coefficients we need in the equations of motion (up to the required order), we find to which order we must expand the output coefficients and evaluate the closure equations.

In the linear case, we have found that we only see the constant parts in C_{AB}, up to linear parts in C_A and up to quadratic terms in C. For 2^{nd} order equations of motion, as is required for the analysis of gravitational waves that are generated by matter that is subject to linear gravitational interaction[3], we would need the constant part from C_{ABC} and for the lower N coefficients everything to one order higher than in the linear case.

Given the decomposition of the output coefficient, it turns out that we can go even further to limit our solution space. As the Λ are all evaluated on n, we know that they can only be given in terms of the available background structure that we have on the screen manifold Σ, that is

- the identity δ^α_β.
- the totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma}$ and its inverse $\epsilon^{\alpha\beta\gamma}$.

[3] A prominemt example for this would be the generation of gravitational waves by a gravitationaly bound binary pulsar.

- the background inverse metric $\mathrm{p}^{\alpha\beta}(n) := \gamma^{\alpha\beta}$ and the metric we obtain by matrix inversion, as well as its spatial derivatives.
- the background fields $\widehat{g}^{\mathcal{A}}(n)$ and their spatial derivatives.

The totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma}$ is available in case Σ is orientable, which we assume since otherwise, we would not have been able to derive the gravitational closure equations in the first place. Constructing the constant output coefficients from the available structure is, of course, precisely what we described for the curvature invariants in our analysis of the covariance part of the closure equations.

Since we are equipped with the constant intertwiners, we can isomorphically transform the tensors with Φ indices into objects with only spatial indices. But equipped with these three tensors, we know that the resulting object must be generated by all the possible background tensor combinations one can write down.

For example, let us assume that we want to generate the coefficient $\Lambda_{A|B}{}^{\alpha\beta}$ in terms of the background structure, in this case given by a Minkowskian background. Then, by introduction of the intertwiner, and pulling all indices up with $\gamma^{\cdot\cdot}$, we find that any such coefficient can be written as

$$\Lambda_{A|B}{}^{\alpha\beta} = \mathcal{I}\big|_0^{\rho\sigma}{}_A\, \mathcal{I}\big|_0^{\lambda\kappa}{}_B\, \gamma_{\rho\gamma}\gamma_{\sigma\delta}\gamma_{\lambda\mu}\gamma_{\kappa\nu}\Lambda^{\alpha\beta\gamma\delta\mu\nu}\,, \tag{5.30}$$

that is, in terms of the rank 6 tensor on Σ with symmetries in $(\alpha\beta)(\gamma\delta)(\mu\nu)$. One finds that the most general ansatz for tensors with these symmetries, built only with the available background structure, takes the form

$$\begin{aligned}\Lambda^{(\alpha\beta)(\gamma\delta)(\mu\nu)} = \quad & e_1 \cdot \gamma^{\alpha\beta}\gamma^{\gamma\delta}\gamma^{\mu\nu} + e_2 \cdot \gamma^{\alpha(\gamma}\gamma^{\delta)\beta}\gamma^{\mu\nu} + e_3 \cdot \gamma^{\alpha\beta}\gamma^{\gamma(\mu}\gamma^{\nu)\delta} \\ & + e_4 \cdot \gamma^{\alpha(\mu}\gamma^{\nu)\beta}\gamma^{\gamma\delta} + \frac{1}{2}\cdot e_5\left(\gamma^{\alpha(\gamma}\gamma^{\delta)(\mu}\gamma^{\nu)\beta} + \gamma^{\beta(\gamma}\gamma^{\delta)(\mu}\gamma^{\nu)\alpha}\right)\,. \end{aligned} \tag{5.31}$$

Note that no terms containing $\epsilon^{\cdot\cdot\cdot}$ need to be considered: The only possible combination that has six indices one could come up with are terms in the form $\epsilon^{\cdot\cdot\cdot}\epsilon^{\cdot\cdot\cdot}$. However, this is equivalent to a linear combination of $\gamma^{\cdot\cdot}\gamma^{\cdot\cdot}\gamma^{\cdot\cdot}$ terms and is thus already contained in all the terms written down above. Solving the gravitational closure equations perturbatively for the coefficient $\Lambda_{A|B}{}^{\alpha\beta}$, as a result, means that we need to determine the five scalar coefficients $e_1, \ldots, e_5$.

Since we always have the inverse background metric $\gamma^{\cdot\cdot} = \mathrm{p}^{\cdot\cdot}(n)$ available, we can always use it to pull the indices up. It thus suffices to generate contravariant tensorial ansätze. This means that the only building blocks we need to consider reduce to $\gamma^{\cdot\cdot}$ and $\epsilon^{\cdot\cdot\cdot}$.

The same procedure can be applied for any background we perturb around: For example, the perturbation around a known cosmological solution of the equations of motion can, in the very same fashion, be calculated, with the only difference that we may need to consider the available screen manifold fields as the building blocks for the constant output coefficients ($\Lambda_{A|B}{}^{\alpha\beta}$ in the example above).

Tensor canonicalization

Let us illuminate the methods to obtain the ansätze in general in some more detail. It turns out to be, in most parts, a group theory problem, with the surprising exception that after one has obtained an ansatz where symmetries are implemented, one *still* needs to check if all basis terms are linearly independent.

This is due to the fact that there may still be some dimensional dependent identities present. Afterwards, we will show how to use these ansätze to efficiently solve the gravitational closure equations for the finitely many scalar coefficients $e_1, \dots, e_{\text{finite}}$ obtained from the construction.

The first step towards obtaining such a list of all tensorial ansätze is to read off the rank of the screen manifold tensors and their symmetries. Since we can always pull all spatial indices down in the constant intertwiner, we will assume that all indices are already pulled down, i.e.

$$\mathcal{I}|_{0\ A\mathcal{A}} = \mathcal{I}|_{0\ A\alpha_1\dots\alpha_m}\,, \tag{5.32}$$

where m is the rank of the geometric field $g^{\mathcal{A}}$ that belongs to the multi-index $\mathcal{A}$. Note that in case we have more than one field, we need to use all the constant intertwiners separately to generate the ansätze. For instance, for an area metric that has three screen manifold tensors, we need to use all three constant intertwiners in the construction and then sum up the separate ansätze in the end.

Using these constant intertwiners we can, in general, move to the following screen manifold ansätze

$$\begin{aligned}\Lambda_{A_1\dots A_N|B_1}{}^{\mathcal{M}_1}{}_{\dots B_M}{}^{\mathcal{M}_M} = {} & \mathcal{I}|_{0\ A_1\mathcal{A}_1}\cdots\mathcal{I}|_{0\ A_N\mathcal{A}_N}\cdot\mathcal{I}|_{0\ B_1\mathcal{B}_1}\cdots\mathcal{I}|_{0\ B_N\mathcal{B}_N}\\ & \times\Lambda^{\mathcal{A}_1\dots\mathcal{A}_N\mathcal{B}_1\mathcal{M}_1\dots\mathcal{B}_M\mathcal{M}_M}\,.\end{aligned} \tag{5.33}$$

We can immediately read off some symmetries of these constant coefficients. First of all, since the output coefficient is totally symmetric in its indices $A_1, \dots, A_N$, we have an exchange symmetry between the different $\mathcal{A}_i$ blocks. Moreover, due to the symmetries of the derivative, it follows that we have an exchange symmetry of the $\mathcal{B}_i\mathcal{M}_i$ blocks. Since the $\mathcal{M}_i$ multi-indices stem from spatial derivatives, these blocks are totally symmetric in their spatial indices.

Further symmetry conditions are then imposed by the screen manifold fields: Since all algebraic symmetries of the field $g^{\mathcal{A}}$ are inherited by the intertwiner, i.e.

$$\mathcal{S}^{\mathcal{A}}{}_{\mathcal{B}}\,\mathcal{I}|_{0\ A\mathcal{A}} = \mathcal{I}|_{0\ A\mathcal{B}}\,, \tag{5.34}$$

the corresponding symmetrizer is applied to the multi-indices on our constant output coefficient and, thus, introduces additional symmetry conditions on the coefficients.

Acting with the symmetries on the indices of the tensorial ansätze is, as claimed above, a group-theoretic problem. These (mono-term) symmetries of the output expression can be represented as a subgroup $\mathcal{L}$ of the signed symmetric group, defined as $\{-1, 1\} \times S_N$ with S_N being the symmetric group since for a tensor T of rank N we have the following symmetry

$$T^{\alpha_1\dots\alpha_N} = \varepsilon T^{\pi(\alpha_1\dots\alpha_N)}\,. \tag{5.35}$$

For example, if a constant coefficient $\Lambda^{\alpha\beta\gamma\delta}$ is symmetric in the indices α, β and γ, δ then its symmetries are represented by the three elements

$$\mathcal{L} = \left\{(1, \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}), (1, \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}), (1, \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 4 & 3\end{pmatrix})\right\}\,. \tag{5.36}$$

In the very same fashion, one finds that each possible assignment of indices $\alpha_1, \dots, \alpha_N$ to the tensor slots of our $\gamma^{\cdot\cdot}\cdots\gamma^{\cdot\cdot}$ terms for N even, and $\epsilon^{\cdot\cdot\cdot}\gamma^{\cdot\cdot}\cdots\gamma^{\cdot\cdot}$ for N odd (and additional tensors, depending on the available structure from the background geometry) is represented by a permutation. For example, in case of the constant coefficient $\Lambda^{\alpha\beta\gamma\delta}$ we have two $\gamma^{\cdot\cdot}$ to generate the coefficient. All the possible index assignments, ignoring the symmetries of $\gamma^{\cdot\cdot}$ itself for now, are the $4! = 24$ permutations

$$\mathcal{C} = \left\{ \overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}}^{\gamma^{\alpha\beta}\gamma^{\gamma\delta}} , \overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}}^{\gamma^{\beta\alpha}\gamma^{\gamma\delta}} , \overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}}^{\gamma^{\gamma\beta}\gamma^{\alpha\delta}} , \ldots \right\} . \tag{5.37}$$

However, we do have internal symmetries of our two building blocks themselves. Due to those symmetries, it is clear that not all of the permutations are linearly independent and we must take care of these as well. At this point it is no surprise that, also here, we deal with another subgroup of the signed symmetric group of slots N: In our example from above this subgroup is given by the eight element group

$$\begin{aligned} \mathcal{S} = \Big\{ & (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}), \\ & (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}), (1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}) \Big\} , \end{aligned} \tag{5.38}$$

and similarly for terms containing an ε, where the use of the *signed* symmetric group finally becomes important.

Both symmetries of the output slots and the internal symmetries of the building blocks must be ultimately considered to identify all linearly independent index assignments, i.e. all assignments that cannot be obtained by employing any of the stated symmetry operations. As a result, it turns out that those assignments of interest lie in the double coset

$$\mathcal{L} \setminus \mathcal{C} / \mathcal{S} \, . \tag{5.39}$$

Finding a basis is then a mere problem of finding a representative for each orbit, and luckily algorithms exist from computational group theory that allow us to obtain such a traversal of the double coset in practice, given a subgroup $\mathcal{L}$ and $\mathcal{S}$.

DEFINITION TENSORIAL ANSÄTZE

The tensorial ansätze for a constant coefficient of rank N are double-coset representatives of the double coset $\mathcal{L} \setminus \mathcal{C} / \mathcal{S}$, with

- $\mathcal{L}$ being the subgroup of the signed symmetric group with N symbols that describes the symmetries of the output coefficient,
- $\mathcal{C}$ being the symmetric group with N elements and represents the space of all possible assignments of the N index labels to the N slots of the building blocks,
- $\mathcal{S}$ being the subset of the signed symmetric group that describes the internal symmetries of the building blocks themselves.

The canonical form can be obtained by traversal algorithms that select one representative per orbit.

The original, rather naïve implementation by the author (see Wolz (2018)) that generates such a traversal of the double-coset for Minkowskian backgrounds starts by spelling out all the possible terms and then iteratively applying the symmetries. Afterwards, it brings the indices into lexicographic order and uses Gaussian elimination to eliminate all remaining linearly dependent objects from the same orbit. Luckily, more elegant solutions exist that can be used to obtain the representatives in practice.

One particularly efficient algorithm that gives a canonical representative per orbit was given by Butler (1982) (and further expanded since then, see for example Slattery (2001)) for generic groups $\mathcal{L}$ and $\mathcal{S}$. Many different publicly available implementations of the algorithm exist that can be used to obtain the tensorial ansätze[4].

Dimensionally dependent identities

Given the double coset representative for our constant output coefficient, our job of reducing terms is, however, not over. It turns out that we still find linear dependencies between the expressions. Take for example a constant output coefficient of rank 7, where we find the following nine canonical representatives of the double coset constructed with $\epsilon^{\cdots}$ and $\gamma^{\cdot\cdot}$:

$$\begin{gathered}
\epsilon^{\mu(\alpha|(\gamma}\gamma^{\delta)|\beta)}\gamma^{\lambda\kappa}\,, \qquad \epsilon^{\mu(\alpha|(\lambda}\gamma^{\kappa)|\beta)}\gamma^{\gamma\delta}\,, \qquad \epsilon^{\mu(\lambda|(\gamma}\gamma^{\delta)|\kappa)}\gamma^{\alpha\beta}\,, \\
\epsilon^{\mu(\gamma|(\alpha}\gamma^{\beta)(\lambda}\gamma^{\kappa)|\delta)}\,, \qquad \epsilon^{\mu(\lambda|(\alpha}\gamma^{\beta)(\gamma}\gamma^{\delta)|\kappa)}\,, \qquad \epsilon^{\mu(\alpha|(\gamma}\gamma^{\delta)(\lambda}\gamma^{\kappa)|\beta)}\,, \\
\epsilon^{(\alpha|(\lambda|(\gamma}\gamma^{\delta)|\kappa)}\gamma^{|\beta)\mu}\,, \qquad \epsilon^{(\gamma|(\lambda|(\alpha}\gamma^{\beta)|\kappa)}\gamma^{|\delta)\mu}\,, \qquad \epsilon^{(\lambda|(\alpha|(\gamma}\gamma^{\delta)|\beta)}\gamma^{|\kappa)\mu}\,.
\end{gathered} \tag{5.40}$$

[4] One notable example is the open-source Python package `sympy`.

However, if we assign to each of the nine tensor monomials from above a vector of its components, i.e.

$$\epsilon^{\mu(\alpha|(\gamma}\gamma^{\delta)|\beta)}\gamma^{\lambda\kappa} \longmapsto \begin{pmatrix} \epsilon^{1(0|(1}\gamma^{1)|1)}\gamma^{11} \\ \epsilon^{1(0|(1}\gamma^{1)|1)}\gamma^{12} \\ \epsilon^{1(0|(1}\gamma^{1)|1)}\gamma^{13} \\ \epsilon^{1(0|(1}\gamma^{1)|1)}\gamma^{21} \\ \epsilon^{1(0|(1}\gamma^{1)|1)}\gamma^{22} \\ \cdots \end{pmatrix}, \tag{5.41}$$

and perform Gaussian elimination, we find that only six of these monomials are indeed linearly independent. As a result, we need to eliminate the linearly dependent ones from our list of ansätze. Similar observations can be made for constant coefficients of rank larger than 7.

It turns out that these identities are dependent on the dimension of the screen manifold Σ, and we would find other linear dependencies if we would consider the ansätze in different dimensions. This is not the first time that we have encountered such identities that are dependent on the dimension of the screen manifold: We have observed such dimensionally dependent identities before in the previous chapters by repeating arguments similar to the ones made by Lovelock to derive useful identities (Lovelock, 1970). Such identities can, in practice, also be obtained by considering an over-antisymmetrization (Edgar and Höglund, 2002). These over-antisymmetrizations in d dimensions vanish identically, i.e.

$$(d+1)!\,\delta^{b_1}_{[a_1}\cdots\delta^{b_{d+1}}_{a_{d+1}]} = 0\,. \tag{5.42}$$

By contraction of this with tensors of interest it is possible to derive non-trivial identities for tensors that are, however, dependent on the dimension. For instance, following a discussion of Edgar and Höglund (2002), we can use this over-antisymmetrization to show that the Weyl tensor vanishes in three dimensions: First one contracts two of its four indices with the over-antisymmetrized $\delta\cdots\delta$ tensor defined above, naturally yielding zero

$$\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta}\delta^{\mu}_{\gamma}\delta^{\nu}_{\delta]}W_{\mu\nu}{}^{\lambda\kappa} = \delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta}W_{\gamma\delta]}{}^{\lambda\kappa} = 0\,. \tag{5.43}$$

Contracting κ with δ we find

$$\delta^{[\rho}_{[\alpha}W_{\beta\gamma]}{}^{\sigma]\lambda} = 0\,, \tag{5.44}$$

and by further contracting λ and γ we finally find that, indeed, the Weyl tensor vanishes in three dimensions.

Another interesting example can be derived for an endomorphism $M^a{}_b$ in dimension N. Here, the over-antisymmetrization

$$M^{c_1}{}_{[c_1}M^{c_2}{}_{c_2}\ldots M^{c_N}{}_{c_N}\delta^{a}_{b]} = 0\,, \tag{5.45}$$

gives the Cayley-Hamilton theorem (Edgar and Höglund, 2002). This can further be generalized to derive syzygies between different endomorphisms (see for example Ouchterlony (1997); Sneddon (1996) and Sneddon (1998)).

However, using this method to derive identities is highly dependent on the tensor one considers. It requires that there are non-trivial contractions between the $\delta \cdots \delta$ operator and the tensor under consideration. One can generate all these non-trivial contractions by taking into account the standard Young tableaux of the $\delta \cdots \delta$ operator and its property that it is traceless, and then eliminate all linear dependent terms.

To illustrate this, we again consider the example above: Due to the symmetrization in the $(\alpha\beta)$, $(\gamma\delta)$ and $(\lambda\kappa)$ blocks, any non-trivial over-antisymmetrization must be performed over two indices on the $\epsilon^{\cdots}$ block and over one index at each of the γ blocks. One example is the expression

$$\begin{aligned} 0 &= \epsilon^{[\mu\alpha|\gamma}\gamma^{\delta|\beta}\gamma^{|\lambda]\kappa} \\ &= \frac{1}{4}\left(\epsilon^{\mu[\alpha|\gamma}\gamma^{|\delta|\beta}\gamma^{|\lambda]\kappa} - \epsilon^{\alpha[\mu|\gamma}\gamma^{|\delta|\beta}\gamma^{|\lambda]\kappa} + \epsilon^{\delta[\mu|\gamma}\gamma^{|\alpha|\beta}\gamma^{|\lambda]\kappa} - \epsilon^{\lambda[\mu|\gamma}\gamma^{|\alpha|\beta}\gamma^{|\delta]\kappa}\right) = \ldots\,, \end{aligned}$$

where we took the representative of the first term in (5.40) and ignored the symmetrizations in the three index blocks on purpose. We can then read off the linear dependencies between our basis terms above, by reinstating the symmetries in the $(\alpha\beta)$, $(\gamma\delta)$ and $(\lambda\kappa)$ blocks again. Three such over-antisymmetrizations exist which confirms the brute-force result we obtained by Gaussian elimination.

When performing perturbative gravitational closure to obtain the coefficients appearing in the linear perturbations, it suffices to simply eliminate the linearly dependent terms by performing Gaussian elimination. The matrices one obtains are often small enough not to be a calculational bottleneck. However, moving to higher-order perturbations, one usually deals with constant output coefficients of ranks higher than twelve[5]. In those cases, the matrices become large enough (with thousands of rows and hundreds of columns) such that having an alternative option to discover the linear dependencies becomes increasingly valuable.

Since we are finally equipped with a method to generate all the constant output coefficients that can appear in the gravitational closure equations, we can start to analyse how the equations have to be evaluated and how this presents us with a significant linear algebraic problem for the scalar coefficients e_i that appear from the tensorial ansätze we generated.

5.2.3 Solution algorithm

In order to solve the gravitational closure equations perturbatively, that is, to determine the scalar coefficients e_i in the output coefficients, we again reparametrize by transforming to the jet space point $n + \varphi$, such that the degrees of freedom parametrize the deviation from the background solution. Suppose we now insert all the expressions we obtained in the previous sections, both the input coefficients and the series expansion of the output coefficients. In that case, we obtain a polynomial in the jet space variables. Since the gravitational closure equations are linear homogeneous partial differential equations and must be fulfilled at each jet space point, i.e. for all values for the degrees of freedom and their derivatives, we

[5]The tensor of rank 12 have $3^{12} = 531441$ components, which corresponds to the columns of the matrix we need to diagonalise. In the same fashion, the number of double coset representatives grows with the rank of the tensor, which makes the brute-force Gaussian elimination impractical at some point.

find that all coefficients appearing in front of the jet variables need to vanish separately. This gives us linear *algebraic* equations for the constant output coefficients that appear from the expansion of the output coefficients.

Let us make that more clear by considering a specific closure equation. If we expand all the expressions in (**C3**) around n, we obtain the following polynomial in the perturbative degrees of freedom φ, i.e.

$$\begin{aligned}
0 = &\left[2(\deg P - 1)\, \Lambda_{AB}\, \mathrm{p}^{(\mu|\rho}\Big|_0 \, \mathrm{F}^{A}{}_{\rho}{}^{|\nu)}\Big|_0 + \Lambda_{B|A}{}^{(\mu|}\, \mathrm{M}^{A|\nu)}\Big|_0 - \Lambda_{|B}{}^{\mu\nu}\right] + \\
&+ \Bigg[2(\deg P - 1)\left(\Lambda_{AB|M}\, \mathrm{p}^{(\mu|\rho}\Big|_0 \, \mathrm{F}^{A}{}_{\rho}{}^{|\nu)}\Big|_0 + \Lambda_{AB}\left(\mathrm{p}^{(\mu|\rho}{}_{:M}\Big|_0 \, \mathrm{F}^{A}{}_{\rho}{}^{|\nu)}\Big|_0 + \mathrm{p}^{(\mu|\rho}\Big|_0 \, \mathrm{F}^{A}{}_{\rho}{}^{|\nu)}{}_{:M}\Big|_0\right)\right) \\
&\qquad + \Lambda_{B|MA}{}^{(\mu|}\, \mathrm{M}^{A|\nu)}\Big|_0 + \Lambda_{B|A}{}^{(\mu|}\, \mathrm{M}^{A|\nu)}{}_{M}\Big|_0 - \Lambda_{|MB}{}^{\mu\nu}\Bigg]\varphi^M \\
&+ \Bigg[2(\deg P - 1)\, \Lambda_{AB|M}{}^{\alpha}\, \mathrm{p}^{(\mu|\rho}\Big|_0 \, \mathrm{F}^{A}{}_{\rho}{}^{|\nu)}\Big|_0 + \Lambda_{B|A}{}^{(\mu|}{}_{M}{}^{\alpha}\, \mathrm{M}^{A|\nu)}\Big|_0 + 2\, \Lambda_{B|A}{}^{\alpha(\mu|}\, \mathrm{M}^{A|\nu)}{}_{:M}\Big|_0 \\
&\qquad - \Lambda_{|B}{}^{\mu\nu}{}_{M}{}^{\alpha} + 3\Lambda_{|MB}{}^{\alpha\mu\nu}\Bigg]\varphi^{M}{}_{,\alpha} \\
&+ \dots,
\end{aligned} \tag{5.46}$$

where we naturally omitted the remaining (infinitely many) terms that arise from the expansion. Since these equation must be fulfilled for all values of φ and its derivatives, as described above, we can read off several independent relations for the constant output coefficients. But, at this point, we still have infinitely many constant coefficients.

However, it turns out we do not need all the coefficients but only the ones that appear in the perturbative equations of motion to the desired order since we will discard the remaining coefficients anyway. From the equations of motion of the linear perturbations, we know that we need

$$\mathrm{C} \longrightarrow \Lambda \quad , \quad \Lambda_{|\mathcal{A}}{}^{\mathcal{A}} \quad , \quad \Lambda_{|\mathcal{A}}{}^{\mathcal{A}}{}_{\mathcal{B}}{}^{\mathcal{B}}\,, \tag{5.47a}$$

$$\mathrm{C}_{A} \longrightarrow \Lambda_{A} \quad , \quad \Lambda_{A|B}{}^{\mathcal{B}}\,, \tag{5.47b}$$

$$\mathrm{C}_{AB} \longrightarrow \Lambda_{AB}\,. \tag{5.47c}$$

All the other coefficients do not contribute to the equations of motion of the linear perturbation modes. In the same fashion, we can associate to each of the constant output coefficients the corresponding *lowest* perturbation order where they appear in the equations of motion. Let us now look at each of the relations in equation (5.46) separately. We see that the 0^{th} order contribution gives us a relation between three of the constant output coefficients Λ_{AB}, $\Lambda_{A|B}{}^{\alpha}$ and $\Lambda_{|B}{}^{\mu\nu}$, all of which *do* make their appearance in the linear equations of motion. As a result, this is a relation we want to evaluate.

In the case of the next coefficient, the one in front of the φ^M term, the situation is already different. Here, we see that the coefficient $\Lambda_{AB|M}$, $\Lambda_{A|M}{}^{\alpha}$ and $\Lambda_{|MN}{}^{\alpha\beta}$ occur, all three of which do not appear in the equations of motion of the linear perturbations but the ones of the 2^{nd} order perturbations. As a result, ideally, we would use this relation to express the scalar coefficients from the tensorial ansätze of the highest perturbation order constant output coefficients in the equation in terms of the lower perturbation order coefficients. But then, no restriction on the coefficients appearing in the linear equations of motion arises, and we do not need to consider the equation when deriving linear field equations.

One complication could arise in the relations that needs to be treated with great care: It may be possible to combine relations that seemingly determine higher perturbation order coefficients algebraically such that we eliminate the highest perturbation order coefficients. In this case, we would have drawn the wrong conclusion because the relations *do* contribute to the solution. This is analogous to how hidden integrability conditions may arise in the Cartan-Kuranishi algorithm and must be considered to properly analyse the solution space of a system of partial differential equations.

In the example of $(\mathbf{C3})$, this means that if we are interested in linear dynamics, one can check that no such combination can be found and, thus, we only need to evaluate the 0^{th} order term in the series expanded equation. In the same fashion, we then analyse all closure equations and find the relations that contribute to the relations.

Once all the relations are identified, we can also read off which orders of the input coefficients are required. Solving a relation is then just an exercise of generating the tensorial ansätze, as described in the previous section, inserting the definition of the 0^{th} order of the intertwiners and all input coefficients and their derivates, and afterwards solving the resulting linear equations for the scalar coefficients – either with the help of computer algebra or by hand.

DEFINITION PERTURBATIVELY SOLVING THE GRAVITATIONAL CLOSURE EQUATIONS

A perturbative solution of order k to the gravitational closure equations, giving the equations of motion to perturbations of orders up to k, is obtained by performing the following algorithm:

1. Formally transform each closure equation from φ to $n + \varphi$ and expand around n.
2. To each constant output coefficient appearing in the series expansion, associate the lowest perturbation order for which the coefficient appears in the equations of motion.
3. To each coefficient appearing in front of the φ (and derivatives) polynomials (called a *relation*), associate the highest perturbation order of a constant output coefficient appearing in this relation.
4. Check for all relations of perturbation order $k + 1$ if one can obtain a relation of perturbation order k by algebraical means.
5. Solve all relations of order $0, \ldots, k$.

"The formal theory of system of partial differential equations" perspective

The fact that we need to check if higher perturbation order relations can be manipulated such that they give restrictions on the constant output coefficients of interest bears a striking resemblance to the hidden integrability conditions that may appear in the Cartan-Kuranishi algorithm. This similarity is, in fact, not by accident:

Remember that in the framework of the formal theory of differential equations, one makes a formal series expansion of the dependent variables $u^{\alpha}(x)$ of the differential equations. Each of the expansion coefficients u^{α}_{μ} is then coordinates in a jet bundle and is of some order given $|\mu|$. The aim in this theory is to ultimately be able to construct a solution to the partial differential equation order by order, i.e. such that the coefficient $u^{\alpha}_{\mu_1\ldots\mu_k}$ is either

- given as functions of the lower order coefficients $u^{\alpha}_{\mu_1\ldots\mu_l}$ for $l < k$, or
- has to be provided as initial data .

In order to judge this correctly, it is clear that we need to make sure that all the integrability conditions are present. This is decided upon the highest derivative coefficient in each equation. One looks at the geometric symbol, where the coefficients in front of the highest order u^{α}_{μ} are ordered by some class respecting order.

Our aim is, of course, almost entirely the same: We formally expand the gravitational closure equations first – not around zero but around the background solution – and then want to be able to iteratively construct a solution to the system, order by order, in the sense that we try to express as many of the higher perturbation order coefficients in terms of the lower perturbation order ones. Since φ is arbitrary, we find the equivalent collection of algebraic relations that must be fulfilled. Where our algorithm now does deviate is that, instead of classifying the expansion coefficients Λ in the relations by their differential order, we classify them by their perturbation order. This order is determined by the equations of motion of the perturbations. The remaining constructions stay the same: At each perturbation order, we can build a symbol that collects all the coefficients appearing in front of the highest perturbative order coefficient to check for hidden integrability conditions – that is, hidden lower perturbation order relations.

Note that in our setting, we cannot obtain any new information by considering prolongations of our series expanded system, i.e. derivatives of the equations by $\varphi^{A}{}_{,\mathcal{A}}$. If one were to calculate this, one would obtain a polynomial φ, where we still get all of the relations we found in the original equation, except the 0^{th} order. As a result, all the information we need is already present in our formal expansion. Integrability conditions, in this case, do arise with respect to perturbation order, not differential order. Still, the procedure remains the same: Take all of the next-order relations and check if we can generate a lower-order relation from it.

We are now perfectly equipped to perturbatively solve the gravitational closure equations from a theoretical point. In practice, however, it is far from trivial to actually follow all of the steps required towards obtaining the solution. It is the 4^{th} step of the algorithm, in particular, the identification of hidden relations, that proves to be incredibly complicated[6].

But also from a computational perspective, one quickly finds in practice that the matrices and tensorial ansätze involved drastically increase in complexity with each perturbation order. In some cases already the

[6] The cheap trick is to solve the relations up to one order higher than we would like to and getting rid of all the coefficients of higher perturbation order in the end. Here we are guaranteed that we indeed implemented all the hidden relations – given that they exist.

linear dynamics presents challenges. For instance, if one solves the equations for the linear dynamics of a general principal polynomial of degree 4, the tensorial ansätze involved are of ranks up to 12 (Schneider, 2017).

REMARK

All of the steps in our algorithm may sound like a somewhat bloated approach, and one may argue that much simpler steps may suffice to arrive at the same conclusions: For example, for linear field equations, one may argue that, since we need the Lagrangian to 2nd order, it suffices to require the hypersurface deformation algebra to 2nd order and solve this directly. However, it was shown (Schneider, 2017) that this does *not* give all the relations necessary to determine the constant output coefficients.

One may also argue that we can expand C to 2nd order, C_A to 1st order and C_{AB} to 0th order directly and, in each closure equation, trace the occurrences of $\mathcal{O}(\varphi^n)$ terms to decide if a relation needs to be evaluated or not. If done carefully, the conclusions would be the same in almost all of the steps of our algorithm, but the fourth: Since the only information we are left with about the higher orders are the $\mathcal{O}(\varphi^n)$ expressions, it is impossible to decide if we can algebraically combine higher perturbation order relations into relevant relations. As a result, the obtained gravitational dynamics may still contain hidden relations between its gravitational constants.

Required evaluation order (up to hidden relations)

Of course, in practice we will not burden ourselves with calculating arbitrarily many expansion coefficients to read off relations that we will ultimately not need. Luckily, we can analyze each of the gravitational closure equations separately and analyze to which perturbation order they can contribute – up to hidden relations that can appear in the next perturbation order relations. Assuming we want the equations of motion of the perturbation up to order k, then we found that in general we need the output coefficients up to order $k+1$. Even further, one finds that

$$\mathrm{C} \longrightarrow \Lambda \quad , \quad \Lambda_{|A}{}^{\mathcal{A}} \quad , \quad , \dots , \Lambda_{|A_1}{}^{\mathcal{A}_1}{}_{\dots A_{k+1}}{}^{\mathcal{A}_{k+1}} \, , \tag{5.48a}$$

$$\mathrm{C}_A \longrightarrow \Lambda_A \quad , \quad \Lambda_{A|A_1}{}^{\mathcal{A}_1} \quad , \dots , \Lambda_{A|A_1}{}^{\mathcal{A}_1}{}_{\dots A_k}{}^{\mathcal{A}_k} \, , \tag{5.48b}$$

$$\dots ,$$

$$\mathrm{C}_{A_1 \dots A_{k+1}} \longrightarrow \Lambda_{A_1 \dots A_{k+1}} \, . \tag{5.48c}$$

If we enter this into the gravitational closure equations, we can read off for each equation to which order we need the relations. The results are summarized in table 5.1.

One notable closure equation, where one easily can miss a relation is (**C5**): Here, due to the derivative acting on the first term and the spatial derivatives in the second term, a relation contributing to our constant output coefficient follows from the relation belonging to $\varphi^{A_1} \cdots \varphi^{A_{k+1}}$, although all other $k+1$ perturbation order relations do not need to be evaluated anymore (up to hidden relations, of course).

We still have not specified to which order we need the input coefficients. In general, it is dictated by all the relevant relations one identified – including the hidden relations. Typically, we find that the orders,

Closure Equation	*Minimal* Evaluation Orders	
	$\mathrm{M}^{A\gamma}\big\vert_0 = 0$	$\mathrm{M}^{A\gamma}\big\vert_0 \neq 0$
$(\mathbf{C1})$	$0,\dots,k$	$0,\dots,k$
$(\mathbf{C2})$	$0,\dots,k-1$	$0,\dots,k-1$
$(\mathbf{C3})$	$0,\dots,k-1$	$0,\dots,k-1$
$(\mathbf{C4})$	$0,\dots,k$	$0,\dots,k-1$
$(\mathbf{C5})$	$0,\dots,k,\ \ (\varphi)^{k+1}$	$0,\dots,k,\ \ (\varphi)^{k+1}$
$(\mathbf{C6})$	$0,\dots,k-1$	$0,\dots,k-2$
$(\mathbf{C7})$	$0,\dots,k-2$	$0,\dots,k-3$
$(\mathbf{C8_N})$	$0,\dots,k$	$0,\dots,k$
$(\mathbf{C9_N})$	$0,\dots,k-1$	$0,\dots,k-1$
$(\mathbf{C10_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N$
$(\mathbf{C11_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N$
$(\mathbf{C12_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N$
$(\mathbf{C13_{N=2\dots k}})$	$0,\dots,k-N+1$	$0,\dots,k-N$
$(\mathbf{C14_{N=2\dots k+1}})$	$0,\dots,k-N+1$	$0,\dots,k-N+1$
$(\mathbf{C15_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N$
$(\mathbf{C16_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N-1$
$(\mathbf{C17_{N=2\dots k}})$	$0,\dots,k-N$	$0,\dots,k-N-1$
$(\mathbf{C18_N})$	$0,\dots,k-1$	$0,\dots,k-1$
$(\mathbf{C19_N})$	$0,\dots,k$	$0,\dots,k-1$
$(\mathbf{C20_{N\ even}})$	$0,\dots,k+1$	$0,\dots,k$
$(\mathbf{C21_{N\ odd}})$	$0,\dots,k+1$	$0,\dots,k$

Table 5.1 The *minimal* required evaluation order of each closure equations. Due to hidden relations it is possible that we need one more perturbation order for each closure equation.

as presented in table 5.2, give a good indication to which order the intertwiners and input coefficients have to be calculated. Note that, luckily, the highest perturbation order in each relation will always couple to the 0^{th} orders of the input coefficient. As a result, it can be decided if hidden relations exist even without having the higher orders of the input coefficients. Even if it turns out that we need to calculate a higher-order term in an input coefficient, once all hidden relations are revealed and added to our system, no further hidden relations can arise.

This concludes the abstract discussion of solving the gravitational closure equations perturbatively. This algorithm can, in principle, be performed completely by hand (and impressively, has been in the past, see Stritzelberger (2016)), or aided by computer algebra system, see for example (Schneider, 2017; Wierzba, 2018; Beier, 2018; Mansuroglu, 2018) where some of the tedious steps have already been auto-

Coefficient	*Minimal* required perturbation orders
$\frac{\partial \widehat{g}^A}{\partial \varphi^A}(\varphi)$	k
$\frac{\partial \widehat{\varphi}^A}{\partial g^A}(\widehat{g}(\varphi))$	k
$\mathrm{F}^A{}_\mu{}^\gamma)(\varphi)$	k
$\mathrm{p}^{\mu\nu}(\varphi)$	$k+1$
$\mathrm{M}^{A\gamma}(\varphi)$	$k+1$

Table 5.2 The typical *minimal* required evaluation order of the intertwiners and input coefficients. Hidden relations can, again, increase the required order. In some cases properties of the fields, of a particular theory under investigation, can also introduce some simplifications but the table still provides an useful indicator.

mated to some degrees. Initial efforts were already made[7] to create a completely automated system that can perform perturbative closure to arbitrary order[8], depending on the available computational resources. The parametrization $\widehat{g}$ and the input coefficients $\mathrm{M}^{A\gamma}$ and $\mathrm{p}^{\mu\nu}$ are its sole input. Its output is all the constant output coefficients that can then be inserted into the perturbative equations of motion.

5.3 THE SPACETIME PICTURE AND GAUGE INVARIANTS

Once we have solved all the gravitational closure equations perturbatively, with the procedures described in the previous section, we have all the constant coefficients appearing in the perturbative equations of motion. These are expressed with some, finitely many, gravitational constants $g_1, \ldots, g_{\text{finite}}$ that need to be obtained by experiments.

Still, there is the subtlety that not all gravitational degrees of freedom are physical but still contain gauge degrees of freedom. In order to obtain physical predictions, one first needs to identify the gauge-invariant quantities and formulate the predictions in terms of these.

In the following section, we will describe how the identification of gauge invariants can be dealt with. It will, however, turn out to be a rather open question for the second-order perturbation theory (and higher).

5.3.1 Point identification maps

If we want to understand how gauge transformations will act on our geometric degrees of freedom, we need to move back to the spacetime picture, where our geometry is a tensor field (density) on $\mathcal{M}$. This stems from the fact that for gravity, the underlying symmetry group is given by the diffeomorphism group of $\mathcal{M}$.

Obtaining the spacetime geometry is, of course, a simple task with all the constructions we have at hand in the constructive gravity framework: Using the degrees of freedom φ, we can create the hyper-

[7] The source code is publicly available at `https://github.com/florianwolz/prime`.
[8] Performance-wise it is roundabout the level of a well-educated master student.

surface fields with the help of the parametrization $\hat{g}(\varphi)$. These fields can then be lifted to the spacetime tensor field G with the help of the observer frame.

The important realization is that, since we have the (arbitrarily precise) approximate solutions to the equations of motion labelled by the homotopy parameter ε, i.e.

$$\varphi(t,x;\varepsilon) = n^A(t,x) + \varepsilon\, h^A_{(1)}(t,x) + \frac{\varepsilon^2}{2}\, h^A_{(2)}(t,x) + \frac{\varepsilon^3}{6}\, h^A_{(3)}(t,x) + \dots\,, \tag{5.49}$$

we obtain an ε-family of spacetime geometries that each live *on their own* spacetime $\mathcal{M}_\varepsilon$. Each of these are, however, related by diffeomorphisms. This allows one to introduce a family of diffeomorphisms ϕ_ε called a *point identification map* from $\mathcal{M}_0$, that is the spacetime of the background solution, and $\mathcal{M}_\varepsilon$. We can compare all the different geometries to each other by pulling them back to $\mathcal{M}_0$, i.e.

$$\phi^*_\varepsilon(G_\epsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left.\frac{\partial^n(\phi^*_\varepsilon(G_\varepsilon))}{\partial\varepsilon^n}\right|_{\varepsilon=0} =: N + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \left(\delta H_{(n)}\right)\,, \tag{5.50}$$

where N is the background geometry, and we have the different perturbations $\left(\delta H_{(n)}\right)$, which are all tensor fields on $\mathcal{M}_0$. Pulling back the geometry to a common manifold is, in fact, a necessity: Only this makes it possible to compare the separate perturbations and legitimately state that they are *small deviations*. Note that the perturbations are dependent on the particular choice of point identification map. See figure 5.2 for an illustration of the ideas of point identification maps (Bruni et al., 1997).

However, which point identification map is chosen is completely arbitrary. But having two different point identification maps φ_ε and ψ_ε, this induces a one-parameter diffeomorphism on $\mathcal{M}_0$ via

$$\Phi_\varepsilon := \varphi_{-\varepsilon} \circ \psi_\varepsilon\,, \tag{5.51}$$

which is notably *not* a one-parameter group diffeomorphism. Still, it turns out that these diffeomorphisms are exactly where gauge transformation in our perturbative setup come in again.

5.3.2 Gauge transformations

The diffeomorphism constructed in the previous section is easily seen to fail to be a one-parameter group diffeomorphism. Since we have

$$\Phi_\varepsilon \circ \Phi_\delta \neq \Phi_{\varepsilon+\delta} \qquad \text{and} \qquad \Phi_{-\varepsilon} \neq \Phi_\varepsilon^{-1}\,,$$

it can not be formulated in terms of the flows of a vector field ξ. However, it turns out that one can create arbitrarily complex diffeomorphisms by combining the flows of the different vector fields.

For example, for two vector fields $\xi_{(1)}$ and $\xi_{(2)}$ we can displace a point p in a manifold $\mathcal{M}$ first for a parameter interval of ϵ along the integral curves of $\xi_{(1)}$, and afterwards by an interval of $\epsilon^2/2$ along the integral curves $\xi_{(2)}$. Diffeomorphisms constructed in that fashion are termed *knight diffeomorphisms*[9].

[9]The name is inspired from the moves of a knight in chess, where one first makes a large step, followed by a smaller step in another direction.

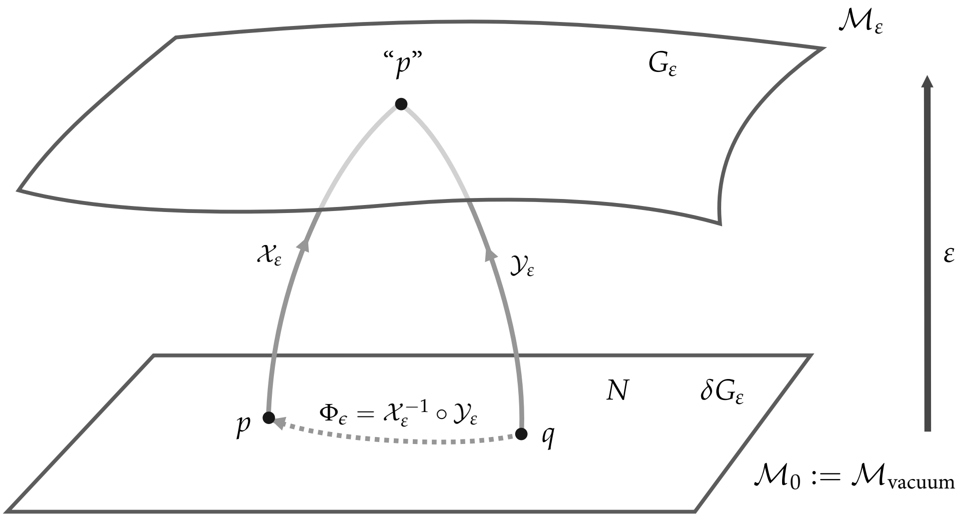


Figure 5.2 Point identification maps identify points of the vacuum spacetime $(\mathcal{M}_{\text{vacuum}}, N)$ with points in the spacetime $(\mathcal{M}_\varepsilon, G_\varepsilon)$. While the geometry G_ε is a tensor field (density) on the spacetime $\mathcal{M}_\varepsilon$, the vacuum geometry and the perturbation δG_ε are objects in vacuum spacetime. Two different choices of point identification maps induce a gauge transformation Φ_ε in the vacuum spacetime.

DEFINITION KNIGHT DIFFEOMORPHISMS

A *knight diffeomorphism* of rank n is a one-parameter diffeomorphism $\mathbb{R} \times \mathcal{M} \longrightarrow \mathcal{M}$ that is defined in terms of n flows $\phi^{(1)}, \ldots, \phi^{(n)}$, generated by vector fields $\xi_{(1)}, \ldots, \xi_{(n)}$ (called the *generators* of Φ) via

$$\Phi_\varepsilon := \phi^{(n)}_{\varepsilon^n/n!} \circ \cdots \circ \phi^{(2)}_{\varepsilon^2/2} \circ \phi^{(1)}_{\varepsilon} .$$

This arcane construction turns out to be incredibly useful, as one can show that a one-parameter family of diffeomorphisms – which includes the ones obtained from moving from one point identification map to another one – can be approximated to order ε^n by a knight diffeomorphism of rank n (Bruni et al., 1997). This makes them a helpful tool for analysing the action of gauge transformation in perturbation theory.

One can, moreover, expand the pull-back of the a field by the gauge transformation Φ_ε in terms of ε, and find (see Bruni et al. (1997)) that it is given by applications of Lie derivatives in the direction of its generators via

$$(\Phi_\varepsilon^* G) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots \frac{\varepsilon^{l_1+2l_2+\cdots+kl_k+\cdots}}{2^{l_2} \cdots (k!)^{l_k} \cdots l_1! l_2! \cdots l_k! \cdots} \mathcal{L}^{l_1}_{\xi_{(1)}} \cdots \mathcal{L}^{l_k}_{\xi_{(k)}} \cdots G . \tag{5.52}$$

If we spell out the expression by orders in ε, this gives us

$$\begin{aligned}(\Phi_\varepsilon^* G) =& G \,+\, \varepsilon \left(\mathcal{L}_{\xi_{(1)}} G\right) \,+\, \frac{\varepsilon^2}{2}\left(\mathcal{L}^2_{\xi_{(1)}} + \mathcal{L}_{\xi_{(2)}}\right) G \\ &+\, \frac{\varepsilon^3}{6}\left(\mathcal{L}^3_{\xi_{(1)}} + 3\,\mathcal{L}_{\xi_{(1)}}\mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(3)}}\right) G \,+\, \dots\,. \end{aligned} \tag{5.53}$$

We can now combine these results to analyze how the different perturbation modes on $\mathcal{M}_0$ transform under a gauge transformation. For two gauge choices ψ and φ we find that the perturbations are related via

$$\left(\delta H_{(1)}\right)^\psi - \left(\delta H_{(1)}\right)^\varphi = \left(\mathcal{L}_{\xi_{(1)}} N\right)\,, \tag{5.54a}$$

$$\left(\delta H_{(2)}\right)^\psi - \left(\delta H_{(2)}\right)^\varphi = \left(\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}}\right) N + 2\,\mathcal{L}_{\xi_{(1)}}\left(\delta H_{(1)}\right)^\varphi\,, \tag{5.54b}$$

$$\begin{aligned}\left(\delta H_{(3)}\right)^\psi - \left(\delta H_{(3)}\right)^\varphi =& \left(\mathcal{L}^3_{\xi_{(1)}} + 3\,\mathcal{L}_{\xi_{(1)}}\mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(3)}}\right) N+\,, \\ &+3\left(\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}}\right)\left(\delta H_{(1)}\right)^\varphi + 3\,\mathcal{L}_{\xi_{(1)}}\left(\delta H_{(2)}\right)^\varphi\,, \end{aligned} \tag{5.54c}$$

$$\dots\,.$$

One then typically proceeds by writing down the spacetime geometry G_ε in coordinates adapted to the foliation, generated by the solutions φ_ε, N_ε and $\vec{N}_\varepsilon$, and in terms of a Helmholtz-Hodge decomposition[10]. Using the different transformation rules of the perturbation orders, see equation (5.54), we obtain partial differential equations. One then needs to, at each perturbation order k, choose the vector field $\xi_{(k)}$ in such a fashion that we eliminate four degrees of freedom from the different modes of the Helmholtz-Hodge decomposition in such a fashion that the equations of motion are formulated in terms of gauge-invariant quantities. This becomes a "computational *tour de force*" (Bruni et al., 1997) already at the second order and, unfortunately, remains practically a rather open problem in general[11].

Once we succeeded with the Herculean task of deriving the gauge-invariant quantities at each of our required perturbation orders, we can, however, finally move to consider various physically relevant setups. For instance, it would be possible to couple to two orbiting point sources and derive, with the help of the second-order perturbations, how their orbital frequency changes due to the energy emitted as gravitational waves. This setup provides a high precision test of gravitational theories and can, even without the direct measurements of any gravitational waves, reveal much about a gravitational model (Taylor and Weisberg, 1982, 1989; Weisberg and Taylor, 2005).

Gauge fixing by harmonic coordinates

Before we move on and apply everything presented so far in this chapter to a concrete example, the perturbative gravitational closure of general linear electrodynamics, we will quickly comment on an alternative possibility to fix the gauge within the constructive gravity programme.

[10] The precise method is highly dependent on the actual geometry under consideration. The procedure is standard for vector and tensor modes of rank 2. For fields of higher rank, more work is required for the decomposition but can be done. See for example Schneider (2017) for the decomposition in the setting of perturbative gravitational closure of a generic rank 4 principal polynomial.

[11] See for example Nakamura (2007) and Nakamura (2012) for further details for a higher-order treatment of gauge invariants.

In general relativity, a popular choice of gauge is the De-Donder gauge defined by the four equations

$$0 = \partial_a \left(\sqrt{-\det g} g^{ab} \right) . \tag{5.55}$$

This choice, equivalently, corresponds to the particular coordinate choice of harmonic coordinates x^a defined via

$$\Delta^g x^a = 0 , \tag{5.56}$$

with the Laplace-Beltrami operator Δ^g. It turns out that we can always perform a gauge transformation into such a coordinate system and, thus, can use this equation to fix four of the degrees of freedom.

In general, as laid out in great detail in the previous chapters, we do not have a metric. We can, however, always obtain the Finsler metric $g^{\cdot\cdot}(x, \dot{X}_t)$ from the homogeneous function $P^\star$, constructed from the principal polynomial and the inverse Legendre map, namely

$$g_{mn}(x, \dot{X}_t) v^m w^n := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left(P^\star(\dot{X}_t + s \cdot u + t \cdot w) \right)^{2/\deg P} \Big|_{s=0, t=0} . \tag{5.57}$$

With this object, we can then write down the analogue of the De-Donder gauge condition, i.e.

$$0 = \partial_a \left(\sqrt{-\det g^{\cdot\cdot}(\dot{X}_t)} g^{ab}(\partial_t) \right) . \tag{5.58}$$

This condition, regarding whether exact or perturbative, can then fix four degrees of freedom.

What must be checked, of course, is if such a coordinate system exists and if we always can find a gauge transformation to move into this system. It turns out that both are true (Caponio and Masiello, 2019). This yields another method to fix the gauge freedom in the equations of motion that may be useful in practice.

This concludes our theoretical discussion of perturbative closure. In the following section, we will look at the dynamics of an area metric again and derive its linear dynamics with all of the techniques laid out in this chapter. This would allow us, for instance, to study the propagation of gravitational waves in such a theory and draw some conclusions compared to the "standard" theory of a Lorentzian metric obeying the dynamics dictated by the Einstein-Hilbert action.

5.4 WEAKLY BIREFRINGENT ELECTRODYNAMICS

We have seen in chapter 4.2 that obtaining an exact solution to the gravitational closure equations for general linear electrodynamics is somewhat involved. Still, area metric geometry and its ability to describe the effect of gravitational birefringence is intriguing and provides a perfect testing ground for perturbative gravitational closure (and has played an incredibly important rôle in its derivation and development of all the techniques required for it). If followed carefully, one obtains its diffeomorphism invariant dynamics and can study its interaction with (classical) matter fields[12].

We will now present the results we obtain by performing the steps from the abstract discussion in the previous section to obtain the dynamics of linear perturbations when expanded around a Minkowskian area metric background. Physically, this corresponds to a setting of weakly birefringent electrodynamics.

[12] Quantum matter is another story, since it turns out the notion of the Dirac algebra must change radically.

5.4.1 Minkowskian background solution

The discussion of weakly birefringent electrodynamics starts with choosing a suitable background solution. Physically, this corresponds to a setting where the gravitational effects are switched off and the electrodynamical interaction, thus, reduces to Maxwellian electrodynamics. As a result, we find that our background area metric $N^{\cdots\cdot}$ is induced by the Minkowski metric η such that

$$N^{abcd} = 2\,\eta^{a[c}\eta^{d]b} - \epsilon^{abcd}\,. \tag{5.59}$$

When projecting the flat area metric to the screen manifold, as described in our exact discussion of general linear electrodynamics in section 4.2, we find that

$$\begin{aligned} g^{\alpha\beta}(N) &= \gamma^{\alpha\beta}\,, && (5.60a)\\ g_{\alpha\beta}(N) &= \gamma_{\alpha\beta}\,, && (5.60b)\\ g^{\alpha}{}_{\beta}(N) &= 0\,, && (5.60c) \end{aligned}$$

with the flat Riemannian metric $\gamma^{\alpha\beta}$ on the screen manifold. Additionally, in this setting we have the following background values for lapse and shift

$$\begin{aligned} A_{(0)} &= 1\,, && (5.61)\\ B^{\mu}_{(0)} &= 0\,. && (5.62) \end{aligned}$$

Note that this background solution is not derived, i.e. *shown* to be a solution of the equations of motion, but the other way around: We impose that the background fields, described above, pose a solution to the equations of motion and derive the coefficients in the equations of motion for an area metric. In this case, to 0^{th} order in the equations of motion, we find that, in order for the Minkowskian background to be a solution in vacuo, the following output coefficients need to vanish

$$\mathrm{C}\big|_{\varphi=n} \qquad , \qquad \mathrm{C}_{:A}\big|_{\varphi=n}\,. \tag{5.63}$$

With this set-up, we are equipped to read off which output coefficients we need to calculate with the help of the gravitational closure equations.

5.4.2 Required constant output coefficients

The next step involves identifying all the constant output coefficients from our series expansion that appear in the perturbative equations of motion to 1^{st} order. For linear dynamics, we already spelled this out in section 5.1, so we directly summarize the result that of all the constant output coefficients we need

$$\begin{aligned} \mathrm{C} &\longrightarrow \lambda_{|A}{}^{\mathcal{A}} \quad , \quad \lambda_{|A}{}^{\mathcal{A}}{}_{B}{}^{\mathcal{B}}\,, && (5.64a)\\ \mathrm{C}_{A} &\longrightarrow \xi_{A} \quad , \quad \xi_{A|B}{}^{\mathcal{B}}\,, && (5.64b)\\ \mathrm{C}_{AB} &\longrightarrow \theta_{AB}\,. && (5.64c) \end{aligned}$$

Note that we introduced different greek symbols for each output coefficient evaluated at the background. All remaining coefficients do not appear in the equations of motion. Thus, we will only look for relations from the gravitational closure equations that relate these coefficients with each other.

Observe that for the output coefficients C and C_A we have *a priori* infinitely many constant output coefficients since they can depend on infinitely many spatial derivatives of the degrees of freedom. We will assume that we have at most second-order derivatives and leave the higher-order derivative theories up for future research. As none of the presented techniques depends on the truncation, one can proceed in the same fashion as we do in the following.

5.4.3 Parametrization

The next step is to set up the parametrization of the three projected fields from an area metric around the background geometry. In section 5.2.1 we already presented a perturbative expansion of our exact parametrization of the area metric. For historic reasons, we will use the endomorphism $\overline{\overline{\overline{g}}}^{\cdot}{}_{\cdot}$ instead of the metric $\overline{\overline{\overline{g}}}_{\cdot\cdot}$. Both are related via

$$\overline{\overline{\overline{g}}}^{\alpha}{}_{\beta} = \overline{g}^{\alpha\sigma}\,\overline{\overline{\overline{g}}}_{\sigma\beta}\,. \tag{5.65}$$

We can then, in our perturbative setting, parametrize the three screen manifold fields by the following constructions

$$\widehat{\overline{g}}^{\alpha\beta}(n+\varphi) = \gamma^{\alpha\beta} + \mathcal{I}^{\alpha\beta}{}_{\overline{A}}\varphi^{\overline{A}}\,, \tag{5.66a}$$

$$\widehat{\overline{\overline{g}}}_{\alpha\beta}(n+\varphi) = \gamma_{\alpha\beta} + \mathcal{I}_{\alpha\beta\overline{\overline{A}}}\varphi^{\overline{\overline{A}}}\,, \tag{5.66b}$$

$$\widehat{\overline{\overline{\overline{g}}}}^{\alpha}{}_{\beta}(n+\varphi) = \mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}}\varphi^{\overline{\overline{\overline{A}}}} + f^{\alpha}{}_{\beta}(\varphi)\,, \tag{5.66c}$$

with the γ antisymmetric and traceless endomorphism f defined by

$$f^{\alpha}{}_{\beta}(\varphi) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^n}\underbrace{\Big\{\overline{\varphi},\Big\{\overline{\varphi},\Big\{\ldots,\Big\{}_{n-1\text{ curly brackets}}\overline{\varphi},\Big[\overline{\varphi},\overline{\overline{\overline{\varphi}}}\Big]\Big\}\Big\}\Big\}\Big\}^{\alpha}{}_{\beta}\,. \tag{5.67}$$

In the commutator $[\cdot,\cdot]$ and anti-commutators $\{\cdot,\cdot\}$ brackets we use the endomorphisms created from the degrees of freedom and the constant intertwiners via

$$\overline{\varphi}^{\alpha}{}_{\beta} := \gamma_{\beta\mu}\,\mathcal{I}^{\alpha\mu}{}_{\overline{A}}\varphi^{\overline{A}}\,, \tag{5.68a}$$

$$\overline{\overline{\overline{\varphi}}}^{\alpha}{}_{\beta} := \mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}}\varphi^{\overline{\overline{\overline{A}}}}\,. \tag{5.68b}$$

For the sake of completeness we also spell out the three constant intertwiners. These are given by

$$\mathcal{I}^{\alpha\beta}{}_{\overline{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{\alpha\beta}_{\overline{A}}\,,\quad \mathcal{I}^{\alpha\beta}{}_{\overline{\overline{A}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}^{\alpha\beta}_{\overline{\overline{A}}}\,, \tag{5.69}$$

and[13]

$$\mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}^{\alpha\beta}_{\quad\overline{\overline{\overline{A}}}} . \tag{5.70}$$

The barred index labels $\overline{A}, \overline{\overline{A}}, \overline{\overline{\overline{A}}}$ run from $\overline{A} = 1, \ldots, 6$, $\overline{\overline{A}} = 7, \ldots, 12$ and $\overline{\overline{\overline{A}}} = 13, \ldots, 17$, respectively. The inverses of the constant intertwiners are given by their transpose. One can check that the projectors one obtains by contraction read

$$\mathcal{I}^{\alpha\beta}{}_{\overline{A}}\mathcal{I}^{\overline{A}}{}_{\mu\nu} = \delta^{(\alpha}_{\mu}\delta^{\beta)}_{\nu} , \tag{5.71a}$$

$$\mathcal{I}_{\mu\nu\overline{\overline{A}}}\mathcal{I}^{\overline{\overline{A}}\alpha\beta} = \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} , \tag{5.71b}$$

$$\mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\nu}{}^{\mu} = \frac{1}{2}\left(\delta^{\alpha}_{\nu}\delta^{\mu}_{\beta} + \gamma^{\alpha\mu}\gamma_{\beta\nu}\right) - \frac{1}{3}\delta^{\alpha}_{\beta}\delta^{\mu}_{\nu} , \tag{5.71c}$$

and that the completeness relation

$$\mathcal{I}^{A}{}_{\mathcal{A}}\mathcal{I}^{\mathcal{A}}{}_{B} = \delta^{A}_{B} \tag{5.72}$$

is indeed fulfilled.

For this perturbative parametrization, we have no inverse map $\widehat{\varphi}$ at hand. As described in section 5.2.1, however, we only need the inverse intertwiners. For the first and second subranges, labelled by the degrees of freedom $\overline{A} = 1, \ldots, 6$ and $\overline{\overline{A}} = 7, \ldots, 12$ they are given by

$$\frac{\partial \widehat{\varphi}^{\overline{A}}}{\partial g^{\alpha\beta}} = \mathcal{I}^{\overline{A}}{}_{\alpha\beta} , \quad \frac{\partial \widehat{\varphi}^{\overline{A}}}{\partial g_{\alpha\beta}} = 0 \quad , \quad \frac{\partial \widehat{\varphi}^{\overline{A}}}{\partial g^{\alpha}{}_{\beta}} = 0 , \tag{5.73a}$$

$$\frac{\partial \widehat{\varphi}^{\overline{\overline{A}}}}{\partial g^{\alpha\beta}} = 0 \quad , \quad \frac{\partial \widehat{\varphi}^{\overline{\overline{A}}}}{\partial g_{\alpha\beta}} = \mathcal{I}^{\overline{\overline{A}}\alpha\beta} \quad , \quad \frac{\partial \widehat{\varphi}^{\overline{\overline{A}}}}{\partial g^{\alpha}{}_{\beta}} = 0 . \tag{5.73b}$$

For the 3rd subrange, $\overline{\overline{\overline{A}}} = 13, \ldots, 17$, we also obtain higher orders due to the non-linear terms in the map $\widehat{\overline{\overline{\overline{g}}}}^{\alpha}{}_{\beta}$. Using the iterative procedure we can obtain the inverse intertwiners for our parametrization and find that they read

$$\frac{\partial \widehat{\varphi}^{\overline{\overline{\overline{A}}}}}{\partial g^{\alpha\beta}} = 0 + \mathcal{O}(2) , \tag{5.74a}$$

$$\frac{\partial \widehat{\varphi}^{\overline{\overline{\overline{A}}}}}{\partial g_{\alpha\beta}} = 0 , \tag{5.74b}$$

$$\frac{\partial \widehat{\varphi}^{\overline{\overline{\overline{A}}}}}{\partial g^{\alpha}{}_{\beta}} = \mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha}{}^{\beta} + \frac{1}{2}\left(\gamma_{\sigma\rho}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha}{}^{\sigma}\mathcal{I}^{\rho\beta}{}_{\overline{M}} - \gamma_{\alpha\sigma}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\rho}{}^{\beta}\mathcal{I}^{\sigma\rho}{}_{\overline{M}}\right)\varphi^{\overline{M}} + \mathcal{O}(2) . \tag{5.74c}$$

[13] Note that this corresponds to the intertwiner $\gamma^{\alpha\sigma}\mathcal{I}^{a}{}_{\sigma\beta}\epsilon_{(m)a}$ in the parametrization we setup in 5.2.1.

Here we already truncated the series after the 1st order since an inspection of the gravitational closure equations, as well as the relations one reads off by virtue of the presented solution algorithm, tells us that we only need the linear part.

This provides a viable parametrization of the area metric degrees of freedom to any perturbation order, as one can check (Stritzelberger, 2016; Schneider et al., 2017; Mansuroglu, 2018). Using this, we can now calculate the series expansion of the three input coefficients.

5.4.4 Input coefficients

Now the input coefficients can be calculated by simply inserting the parametrization obtained in the previous section into the expressions given in section 4.2. We will thus keep the discussion as brief as possible and mainly state the results directly.

Non-local normal deformation coefficient

The exact non-local normal deformation coefficients for the three hypersurface fields read

$$\overline{\mathrm{M}}^{\alpha\beta\gamma} = 2\sqrt{\det g^{\cdot\cdot}}\,\epsilon^{\sigma\gamma(\alpha}\overline{\overline{g}}^{\beta)}{}_{\sigma}\,, \tag{5.75a}$$

$$\overline{\overline{\mathrm{M}}}_{\alpha\beta}{}^{\gamma} = \frac{6}{\sqrt{\det g^{\cdot\cdot}}}\epsilon_{\sigma\tau(\alpha}\overline{\overline{\overline{g}}}^{\tau}{}_{\beta)}p^{\sigma\gamma}\,, \tag{5.75b}$$

$$\overline{\overline{\overline{\mathrm{M}}}}^{\alpha}{}_{\beta}{}^{\gamma} = -\sqrt{\det \overline{g}^{\cdot\cdot}}\,\epsilon^{\gamma\alpha\sigma}\overline{\overline{g}}_{\sigma\beta} + \frac{3}{\sqrt{\det\overline{g}^{\cdot\cdot}}}\epsilon_{\beta\sigma\tau}\overline{g}^{\alpha\sigma}p^{\tau\gamma}\,. \tag{5.75c}$$

In the last expression we used the fact that the coefficient $\mathrm{M}^{A\gamma}$ obeys the chain rule (3.56) that allows to calculate the expression for the third screen manifold field from the one we used in section 4.2 via

$$\overline{\overline{\overline{\mathrm{M}}}}^{\alpha}{}_{\beta}{}^{\gamma} = \overline{g}^{\alpha\sigma}\,\overline{\overline{\overline{\mathrm{M}}}}_{\sigma\beta}{}^{\gamma} + \overline{\overline{\overline{g}}}_{\sigma\beta}\overline{\mathrm{M}}^{\alpha\sigma\gamma}\,. \tag{5.76}$$

Inserting the parametrization from equation (5.66), expanding the de-densitization and contracting with the inverse intertwiner, we find the following expressions for the non-local normal deformation input coefficient

$$\begin{aligned}
\mathrm{M}^{\overline{A}\gamma} = {}& 2\,\epsilon^{\mu\gamma\alpha}\,\mathcal{I}^{\overline{A}}{}_{\alpha\beta}\,\mathcal{I}^{\beta}{}_{\mu\overline{\overline{\overline{M}}}}\,\varphi^{\overline{\overline{\overline{M}}}} \\
& + \epsilon^{\mu\gamma\alpha}\mathcal{I}^{\overline{A}}{}_{\alpha\beta}\left(\gamma_{\sigma\mu}\mathcal{I}^{\beta\tau}{}_{\overline{M}}\mathcal{I}^{\sigma}{}_{\tau\overline{\overline{\overline{N}}}} - \gamma_{\sigma\mu}\mathcal{I}^{\sigma\tau}{}_{\overline{M}}\mathcal{I}^{\beta}{}_{\sigma\overline{\overline{\overline{N}}}} + \gamma_{\sigma\tau}\mathcal{I}^{\sigma\tau}{}_{\overline{M}}\mathcal{I}^{\beta}{}_{\mu\overline{\overline{\overline{N}}}}\right)\varphi^{\overline{M}}\varphi^{\overline{\overline{\overline{N}}}} + \mathcal{O}(\varphi^3)\,,
\end{aligned} \tag{5.77a}$$

$$\begin{aligned}
\mathrm{M}^{\overline{\overline{A}}\gamma} = {}& 2\,\epsilon_{\mu\nu\alpha}\,\gamma^{\nu\gamma}\,\mathcal{I}^{\overline{\overline{A}}\alpha\beta}\,\mathcal{I}^{\mu}{}_{\beta\overline{\overline{\overline{M}}}}\,\varphi^{\overline{\overline{\overline{M}}}} \\
& + \epsilon_{\mu\nu\alpha}\mathcal{I}^{\overline{\overline{A}}\alpha\beta}\Big(-\mathcal{I}^{\mu\gamma}{}_{M}\mathcal{I}^{\nu}{}_{\beta\overline{\overline{\overline{N}}}} + \gamma^{\mu\sigma}\gamma^{\gamma\tau}\mathcal{I}_{\sigma\tau M}\mathcal{I}^{\nu}{}_{\beta\overline{\overline{\overline{N}}}} - \gamma^{\mu\gamma}\gamma^{\sigma\tau}\mathcal{I}_{\sigma\tau M}\mathcal{I}^{\nu}{}_{\beta\overline{\overline{\overline{N}}}} \\
& - \gamma^{\mu\gamma}\gamma_{\tau\beta}\mathcal{I}^{\nu\sigma}{}_{M}\mathcal{I}^{\tau}{}_{\sigma\overline{\overline{\overline{N}}}} + \gamma^{\mu\gamma}\gamma_{\tau\beta}\mathcal{I}^{\sigma\tau}{}_{N}\mathcal{I}^{\nu}{}_{\sigma\overline{\overline{\overline{N}}}}\Big)\varphi^{M}\varphi^{\overline{\overline{\overline{N}}}} + \mathcal{O}(\varphi^3)\,,
\end{aligned} \tag{5.77b}$$

$$\begin{aligned}
\mathrm{M}^{\overline{\overline{\overline{A}}}\gamma} = {}& -\epsilon_{\mu\nu\beta}\gamma^{\nu\gamma}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha}{}^{\beta}\mathcal{I}^{\alpha\mu}{}_{\overline{M}}\varphi^{\overline{M}} - \epsilon^{\mu\gamma\alpha}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha}{}^{\beta}\mathcal{I}_{\mu\beta\overline{\overline{M}}}\varphi^{\overline{\overline{M}}} \\
& + \frac{1}{2}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\sigma}{}^{\rho}\left(\epsilon^{\gamma}{}_{\alpha\beta}\gamma_{\rho\mu}\mathcal{I}^{\alpha\mu}{}_{\overline{M}}\mathcal{I}^{\sigma\beta}{}_{\overline{N}} - \epsilon^{\gamma}{}_{\alpha\rho}\gamma_{\beta\mu}\mathcal{I}^{\alpha\mu}{}_{\overline{M}}\mathcal{I}^{\sigma\beta}{}_{\overline{N}} + \epsilon^{\gamma}{}_{\mu\rho}\gamma_{\alpha\beta}\mathcal{I}^{\alpha\beta}{}_{\overline{M}}\mathcal{I}^{\sigma\mu}{}_{\overline{N}}\right)\varphi^{\overline{M}}\varphi^{\overline{N}}
\end{aligned}$$

$$+\frac{1}{2}\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\sigma}{}^{\rho}\left(\epsilon^{\gamma\alpha}{}_{\rho}\mathcal{I}^{\sigma\beta}{}_{\overline{M}}\mathcal{I}_{\alpha\beta\overline{N}}+\epsilon^{\gamma}{}_{\alpha}{}^{\beta}\mathcal{I}^{\sigma\alpha}{}_{\overline{M}}\mathcal{I}_{\rho\beta\overline{N}}+\epsilon^{\gamma\alpha\sigma}\gamma_{\mu\nu}\mathcal{I}^{\mu\nu}{}_{\overline{M}}\mathcal{I}_{\rho\alpha\overline{N}}\right)\varphi^{\overline{M}}\varphi^{\overline{N}}+\mathcal{O}(\varphi^{3}),\tag{5.77c}$$

where we displayed the three subranges $\overline{A}=1,\ldots,6$, $\overline{\overline{A}}=7,\ldots,12$ and $\overline{\overline{\overline{A}}}=13,\ldots,17$ separately.

Also, observe that the $\mathrm{M}^{A\gamma}$ coefficient has no 0th order. This is not surprising, as this corresponds to the metric induced sector for the flat Minkowski metric. As a result, we reproduce the fact that the metric, for a 2nd degree principal polynomial, has a vanishing non-local normal deformation coefficient. The same observation was made in all of the perturbative gravitational closure calculations so far around a Minkowskian background (Wierzba, 2018; Beier, 2018).

Tangential deformation coefficient

Expanding the Lie derivative of each field separately, inserting the parametrization and contracting with the inverse intertwiner, gives the tangential deformation input coefficients. We again display it for the different barred subranges

$$\mathrm{F}^{\overline{A}}{}_{\mu}{}^{\gamma}=2\,\mathcal{I}^{\overline{A}}{}_{\mu\sigma}\,\gamma^{\sigma\gamma}+2\,\mathcal{I}^{\overline{A}}{}_{\mu\sigma}\mathcal{I}^{\sigma\gamma}{}_{\overline{M}}\,\varphi^{\overline{M}},\tag{5.78a}$$

$$\mathrm{F}^{\overline{\overline{A}}}{}_{\mu}{}^{\gamma}=-2\,\mathcal{I}^{\overline{\overline{A}}\gamma\sigma}\,\gamma_{\sigma\mu}-2\,\mathcal{I}^{\overline{\overline{A}}\gamma\sigma}\mathcal{I}_{\sigma\mu\overline{\overline{M}}}\,\varphi^{\overline{\overline{M}}},\tag{5.78b}$$

$$\mathrm{F}^{\overline{\overline{\overline{A}}}}{}_{\mu}{}^{\gamma}=\left(\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\mu}{}^{\sigma}\,\mathcal{I}^{\gamma}{}_{\sigma\overline{\overline{\overline{M}}}}-\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\sigma}{}^{\gamma}\,\mathcal{I}^{\sigma}{}_{\mu\overline{\overline{\overline{M}}}}\right)\varphi^{\overline{\overline{\overline{M}}}}+\mathcal{O}(\varphi^{3}).\tag{5.78c}$$

Note that the first two coefficients here are exact for any perturbation order considered. Only in the third one a series truncation was made to the required order.

Metric from the principal polynomial

The last input coefficient is obtained by inserting the parametrization into the exact expression for the coefficient $\mathrm{p}^{\alpha\beta}$ one obtains from the principal polynomial

$$\mathrm{p}^{\alpha\beta}=\frac{1}{6}\left(\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu}\overline{\overline{g}}_{\mu\nu}-\overline{g}^{\alpha\beta}\overline{g}^{\mu\nu}\overline{\overline{g}}_{\mu\nu}-2\,\overline{g}^{\alpha\beta}\overline{\overline{\overline{g}}}^{\mu}{}_{\nu}\overline{\overline{\overline{g}}}^{\nu}{}_{\mu}+3\,\overline{g}^{\mu\nu}\overline{\overline{\overline{g}}}^{\alpha}{}_{\mu}\overline{\overline{\overline{g}}}^{\beta}{}_{\nu}\right).\tag{5.79}$$

By inserting all expressions we find, to 2nd order in the degrees of freedom, that

$$\begin{aligned}\mathrm{p}^{\alpha\beta}=&-\frac{1}{3}\gamma^{\alpha\beta}+\\&+\left(\gamma^{\alpha\gamma}\gamma^{\beta\delta}\mathcal{I}_{\gamma\delta M}-\mathcal{I}^{\alpha\beta}{}_{M}-\gamma^{\alpha\beta}\gamma_{\gamma\delta}\mathcal{I}^{\gamma\delta}{}_{M}-\gamma^{\alpha\beta}\gamma^{\gamma\delta}\mathcal{I}_{\gamma\delta M}\right)\varphi^{M}\\&+\Big(\gamma_{\gamma\delta}\mathcal{I}^{\alpha\gamma}{}_{M}\mathcal{I}^{\delta\beta}{}_{N}+\gamma^{\alpha\delta}\mathcal{I}^{\beta\gamma}{}_{M}\mathcal{I}_{\gamma\delta N}+\gamma^{\beta\delta}\mathcal{I}^{\alpha\gamma}{}_{M}\mathcal{I}_{\gamma\delta N}+3\,\gamma^{\gamma\delta}\mathcal{I}^{\alpha}{}_{\gamma M}\mathcal{I}^{\beta}{}_{\delta N}\\&\quad-\gamma^{\gamma\delta}\mathcal{I}^{\alpha\beta}{}_{M}\mathcal{I}_{\gamma\delta N}-\gamma_{\gamma\delta}\mathcal{I}^{\alpha\beta}{}_{M}\mathcal{I}^{\gamma\delta}{}_{N}-\gamma^{\alpha\beta}\mathcal{I}^{\gamma\delta}{}_{M}\mathcal{I}_{\gamma\delta N}-2\,\gamma^{\alpha\beta}\mathcal{I}^{\gamma}{}_{\delta M}\,\mathcal{I}^{\delta}{}_{\gamma N}\Big)\varphi^{M}\varphi^{N}\\&+\mathcal{O}(\varphi^{3}).\end{aligned}\tag{5.80}$$

As we now have all of the input coefficients, we could move on to deriving the relations from the gravitational closure equations.

5.4.5 Perturbative gravitational closure

In order to derive the relations for the constant output coefficients of our weakly birefringent theory, one proceeds as described in the previous section: We first expand all the equations around n and separate the polynomials in φ. Some simplifications can be made in this case since the input coefficient $\mathrm{M}^{A\gamma}$ has no 0^{th} order term. Doing so carefully reveals – as an intermediate result, where we naïvely forgot about the hidden relations for a moment – that indeed we need to evaluate each of the closure equations to the orders presented in table 5.1.

We also find that $(\mathbf{C14_N})$ becomes irrelevant: by calculation of the coefficient appearing in the equation we find that it is (at least) of second order, i.e.

$$\mathrm{M}^{B[\mu|}\mathrm{M}^{A|\nu]}{}_{:B} + \left(\deg P - 1\right) \mathrm{p}^{\rho[\mu|}\mathrm{F}^{A}{}_{\rho}{}^{|\nu]} = 0 + \mathcal{O}(2)\,. \tag{5.81}$$

Since we can only evaluate the equation to 0^{th} order, no information can be extracted from this.

The intermediate state, after collecting the relations from the closure equations, is summarised in table 5.3. Before we can proceed, we, however, must return to the issue of identifying potential hidden relations. Doing so requires us to look at each closure equation to one perturbation order higher and check if we can eliminate the highest perturbation order coefficients by algebraically combining the relations.

Hidden relations

As we have seen in (5.81) above, there is indeed a combination of input coefficients that we can employ to *potentially* eliminate the highest order constant output coefficient of the equation: All we need to do is to make a list of the relations from the closure equations that appear with the output coefficient contracted with $\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$ – which is most likely one from the covariance part of the closure equations – and one with the same coefficient appearing with $\mathrm{M}^{A\gamma}$. Contracting the latter with the derivative $\mathrm{M}^{A\mu}{}_{:B}$ and subtracting both equations and anti-symmetrizing in the free spatial indices μ and γ we may end up with a hidden integrability condition if the constant output coefficient of perturbation order 2 remains in the equation.

Evaluating this careful for the constant output coefficient $\lambda_{B_1}{}^{\mathcal{B}_1}{}_{B_2}{}^{\mathcal{B}_2}{}_{B_3}{}^{\mathcal{B}_3}$, $\xi_{AB_1}{}^{\mathcal{B}_1}{}_{B_2}{}^{\mathcal{B}_2}$, $\theta_{A_1A_2B_1}{}^{\mathcal{B}_1}$ and $\chi_{A_1A_2A_3}$ (describing the output coefficient $\mathrm{C}_{A_1A_2A_3}$), we find that no such combination can be constructed, because either second perturbation order coefficients remain in the resulting equation or the equation vanishes identically. As a result, we can conclude that no hidden relations need to be considered for the area metric.

5.4.6 Gravitational Lagrangian leading to linear dynamics

With all the relations at hand that need to be solved to determine all the coefficients in the equation of motion (and equivalently in the gravitational Lagrangian), the remaining task is now simply a matter of linear algebra.

The resulting expression, as we will quickly see, is rather complicated. For the sake of simplicity we will introduce the following projected screen-manifold fields for the degrees of freedom and their local

Closure Equation	Allowed Orders	Partial Differential Equation
$(\mathbf{C1})$	$0, 1$	$0 = \mathrm{C}\,\delta^{\gamma}_{\mu} - \mathrm{C}_{:A}{}^{\gamma}\varphi^{A}{}_{,\mu} - 2\,\mathrm{C}_{:A}{}^{\alpha\gamma}\varphi^{A}{}_{,\alpha\mu} + \mathrm{C}_{:A}\mathrm{F}^{A}{}_{\mu}{}^{\gamma} + \mathrm{C}_{:A}{}^{\alpha}\mathrm{F}^{A}{}_{\mu}{}^{\gamma}{}_{,\alpha} + \mathrm{C}_{:A}{}^{\alpha\beta}\mathrm{F}^{A}{}_{\mu}{}^{\gamma}{}_{,\alpha\beta}$
$(\mathbf{C2})$	0	$0 = \mathrm{C}_{A}\left(\delta^{A}_{B}\delta^{\gamma}_{\mu} + \mathrm{F}^{A}{}_{\mu}{}^{\gamma}{}_{:B}\right) + \mathrm{C}_{B:A}\mathrm{F}^{A}{}_{\mu}{}^{\gamma}$
$(\mathbf{C3})$	0	$0 = 6\,\mathrm{C}_{AB}\,\mathrm{p}^{\rho(\mu\vert}\mathrm{F}^{B}{}_{\rho}{}^{\vert\nu)} - \mathrm{C}_{:A}{}^{\mu\nu}$
$(\mathbf{C4})$	$0, 1$	$0 = 6\,\mathrm{C}_{AB}\left(\mathrm{p}^{\mu\nu}\varphi^{A}{}_{,\nu} - \mathrm{p}^{\mu\nu}{}_{,\gamma}\mathrm{F}^{A}{}_{\nu}{}^{\gamma}\right) - \mathrm{C}_{:A}\mathrm{M}^{A\mu}{}_{:B} - \mathrm{C}_{B:A}\mathrm{M}^{A\mu} - \mathrm{C}_{B:A}{}^{\alpha}\mathrm{M}^{A\mu}{}_{,\alpha} - \mathrm{C}_{:B}{}^{\mu} + 2\,(\partial_{\alpha}\mathrm{C}_{:B}{}^{\alpha\mu})$
$(\mathbf{C5})$	$0, 1, \varphi\varphi$	$0 = -6\,\mathrm{p}^{\rho\gamma}\left(\partial_{\mu}\mathrm{C}_{A}\right)\mathrm{F}^{A}{}_{\rho}{}^{\mu} + 2\,\mathrm{C}_{:A}\mathrm{M}^{A\gamma} + \mathrm{C}_{:A}{}^{\alpha\beta}\mathrm{M}^{A\gamma}{}_{,\alpha\beta}$
$(\mathbf{C6})$	0	$0 = 4\,\mathrm{C}_{A(B_1\vert}\mathrm{M}^{A\gamma}{}_{:B_2)} + 2\,\mathrm{C}_{(B_1:B_2)}{}^{\gamma}$
$(\mathbf{C8_2})$	$0, 1$	$0 = \mathrm{C}_{:A}{}^{\alpha\beta}\varphi^{A}{}_{,\mu} - \mathrm{C}_{:A}{}^{(\alpha\vert}\mathrm{F}^{A}{}_{\mu}{}^{\vert\beta)} - 2\,\mathrm{C}_{:A}{}^{\gamma(\alpha\vert}\mathrm{F}^{A}{}_{\mu}{}^{\vert\beta)}{}_{,\gamma}$
$(\mathbf{C8_3})$	$0, 1$	$0 = \mathrm{C}_{:A}{}^{(\alpha\beta\vert}\mathrm{F}^{A}{}_{\mu}{}^{\vert\gamma)}$
$(\mathbf{C9_2})$	0	$0 = \mathrm{C}_{A:B}{}^{(\alpha\vert}\mathrm{F}^{B}{}_{\mu}{}^{\vert\beta)}$
$(\mathbf{C21_3})$	$1, 2$	$0 = \mathrm{C}_{:A}{}^{(\alpha\beta\vert}\mathrm{M}^{A\vert\gamma)}$

Table 5.3 Closure equations to derive the coefficients in the equations of motion of the geometry of weakly birenfringent electrodynamics.

velocities

$$\overline{\varphi}^{\alpha\beta} := \mathcal{I}^{\alpha\beta}{}_{\overline{A}}\varphi^{\overline{A}}\,, \qquad \overline{\overline{\varphi}}_{\alpha\beta} := \mathcal{I}_{\alpha\beta\overline{\overline{A}}}\varphi^{\overline{\overline{A}}}\,, \qquad \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta} := \mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}}\varphi^{\overline{\overline{\overline{A}}}}\,, \tag{5.82a}$$

$$\overline{k}^{\alpha\beta} := \mathcal{I}^{\alpha\beta}{}_{\overline{A}}k^{\overline{A}}\,, \qquad \overline{\overline{k}}_{\alpha\beta} := \mathcal{I}_{\alpha\beta\overline{\overline{A}}}k^{\overline{\overline{A}}}\,, \qquad \overline{\overline{\overline{k}}}{}^{\alpha}{}_{\beta} := \mathcal{I}^{\alpha}{}_{\beta\overline{\overline{\overline{A}}}}k^{\overline{\overline{\overline{A}}}}\,. \tag{5.82b}$$

Putting everything carefully together, we obtain the following expression for the gravitational Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{birefringent,linear}}(\varphi, \partial\varphi, \partial\partial\varphi) &= 4\left(g_{18} - 2\,g_{19} + g_{20}\right) \\
&+ 2\left(g_{19} - g_{18}\right)\gamma_{\alpha\beta}\cdot\overline{\varphi}^{\alpha\beta} + 2\left(g_{20} - g_{19}\right)\gamma^{\alpha\beta}\cdot\overline{\overline{\varphi}}_{\alpha\beta} \\
&+ \left[\frac{1}{3}\left(g_4 - 4\,g_2 - g_3 + g_6 - 2\,g_7\right)\gamma_{\alpha\beta}\gamma^{\mu\nu} + \left(g_3 + g_6 - 6\,g_7 - 2\,g_9\right)\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\right]\cdot\overline{\varphi}^{\alpha\beta}{}_{,\mu\nu} \\
&+ \left[\frac{1}{3}\left(8\,g_3 - 4\,g_2 + g_4 + 4\,g_6 - 44\,g_7 + 12\,g_8 - 12\,g_9\right)\gamma^{\alpha\beta}\gamma^{\mu\nu}\right. \\
&\left. + 2\left(4\,g_7 - g_3 - 2\,g_8 + g_9\right)\gamma^{\mu\alpha}\gamma^{\beta\nu}\right]\cdot\overline{\overline{\varphi}}_{\alpha\beta,\mu\nu} \\
&+ \left[g_{18}\gamma_{\alpha\beta}\gamma_{\mu\nu} + \left(2\,g_{18} - 4\,g_{19} + 2\,g_{20} + g_{21}\right)\gamma_{\alpha\mu}\gamma_{\beta\nu}\right]\cdot\overline{\varphi}^{\alpha\beta}\overline{\varphi}^{\mu\nu} \\
&+ \left[g_{20}\gamma^{\alpha\beta}\gamma^{\mu\nu} + g_{21}\gamma^{\alpha\mu}\gamma^{\nu\beta}\right]\cdot\overline{\overline{\varphi}}_{\alpha\beta}\overline{\overline{\varphi}}_{\mu\nu} + \left[g_{19}\gamma_{\alpha\beta}\gamma^{\mu\nu} + \left(2\,g_{20} - g_{19} + g_{21}\right)\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\right]\cdot\overline{\varphi}^{\alpha\beta}\overline{\overline{\varphi}}_{\mu\nu} \\
&+ g_{22}\gamma_{\mu\alpha}\delta^{\nu}_{\beta}\cdot\overline{\varphi}^{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu} + g_{22}\gamma^{\nu\alpha}\delta^{\beta}_{\mu}\cdot\overline{\overline{\varphi}}_{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu} + 4\left(g_{19} - g_{20} - g_{21}\right)\cdot\overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu} \\
&+ g_{17}\,\epsilon_{\alpha\mu}{}^{\lambda}\gamma_{\beta\nu}\cdot\overline{\varphi}^{\alpha\beta}\overline{\varphi}^{\mu\nu}{}_{,\lambda} + g_{17}\,\epsilon_{\alpha}{}^{\mu\lambda}\delta^{\nu}_{\beta}\cdot\overline{\varphi}^{\alpha\beta}\overline{\overline{\varphi}}_{\mu\nu,\lambda} + g_{17}\,\epsilon^{\alpha}{}_{\mu}{}^{\lambda}\delta^{\beta}_{\nu}\cdot\overline{\overline{\varphi}}_{\alpha\beta}\overline{\varphi}^{\mu\nu}{}_{,\lambda}
\end{aligned}$$

$$+ g_{17}\, \epsilon^{\alpha\mu\lambda} \gamma^{\beta\nu} \cdot \overline{\overline{\varphi}}_{\alpha\beta} \overline{\overline{\varphi}}_{\mu\nu,\lambda} + 4\, g_{17}\, \epsilon_{\alpha\mu}{}^{\lambda} \gamma^{\beta\nu} \cdot \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta} \overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda}$$

$$+ \Bigg[\frac{1}{6} \left(6\, g_1 + 4\, g_2 + g_3 - g_4 - g_6 + 2\, g_7\right) \gamma_{\alpha\beta} \gamma_{\mu\nu} \gamma^{\lambda\kappa}$$

$$+ \frac{1}{3} \left(4\, g_2 + g_3 - g_4 - g_6 + 2\, g_7 + 3\, g_{10}\right) \gamma_{\alpha\mu} \gamma_{\nu\beta} \gamma^{\lambda\kappa}$$

$$+ \frac{1}{4} \left(g_4 + 2\, g_6 - 8\, g_7 - 2\, g_9\right) \gamma_{\alpha\beta} \delta^{\lambda}_{\mu} \delta^{\kappa}_{\nu} + g_3 \gamma_{\alpha\mu} \delta^{\lambda}_{\nu} \delta^{\kappa}_{\beta} + \frac{1}{3} \left(g_4 - g_2 - g_3 + g_6 - 2\, g_7\right) \gamma_{\mu\nu} \delta^{\lambda}_{\alpha} \delta^{\kappa}_{\beta} \Bigg]$$

$$\cdot \overline{\varphi}^{\alpha\beta} \overline{\varphi}^{\mu\nu}{}_{,\lambda\kappa} + \Bigg[\frac{1}{2} \left(2\, g_1 + g_3 + g_6 - 6\, g_7 - 2\, g_9\right) \gamma_{\alpha\beta} \gamma^{\mu\nu} \gamma^{\lambda\kappa}$$

$$+ \frac{1}{12} \left(8\, g_2 - 4\, g_3 + g_4 - 2\, g_6 + 16\, g_7 + 6\, g_9\right) \gamma_{\alpha\beta} \gamma^{\mu\lambda} \gamma^{\kappa\nu}$$

$$+ \frac{1}{2} \left(g_3 - g_6 - 2\, g_7 + 4\, g_8 + 2\, g_{10}\right) \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} \gamma^{\lambda\kappa}$$

$$+ \frac{1}{6} \left(7\, g_3 - 2\, g_2 + 2\, g_4 + 5\, g_6 - 46\, g_7 + 12\, g_8 - 12\, g_9\right) \gamma^{\mu\nu} \delta^{\lambda}_{\alpha} \delta^{\kappa}_{\beta}$$

$$+ \left(g_6 + 2\, g_7 - 4\, g_8\right) \delta^{\mu}_{\alpha} \gamma^{\nu\lambda} \delta^{\kappa}_{\beta} \Bigg] \cdot \overline{\varphi}^{\alpha\beta} \overline{\overline{\varphi}}_{\mu\nu,\lambda\kappa} + \Bigg[\frac{1}{2} g_{14} \gamma_{\alpha\beta} \gamma^{\nu\lambda} \delta^{\kappa}_{\mu} + \frac{1}{2} \left(2\, g_{12} + g_{13}\right) \gamma_{\alpha\mu} \gamma^{\lambda\kappa} \delta^{\nu}_{\beta}$$

$$+ g_{15} \gamma_{\alpha\mu} \gamma^{\nu\lambda} \delta^{\kappa}_{\beta} \Bigg] \cdot \overline{\varphi}^{\alpha\beta} \overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda\kappa} + \Bigg[g_1 \gamma^{\alpha\beta} \gamma_{\mu\nu} \gamma^{\lambda\kappa} + \frac{1}{4} \left(2\, g_3 + g_4 + 4\, g_6 - 20\, g_7 - 6\, g_9\right) \gamma^{\alpha\beta} \delta^{\lambda}_{\mu} \delta^{\kappa}_{\nu}$$

$$+ g_{10} \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} \gamma^{\lambda\kappa} + g_3 \gamma^{\alpha\lambda} \delta^{\kappa}_{\mu} \delta^{\beta}_{\nu} + g_2 \gamma^{\alpha\lambda} \gamma_{\mu\nu} \gamma^{\kappa\beta} \Bigg] \cdot \overline{\overline{\varphi}}_{\alpha\beta} \overline{\varphi}^{\mu\nu}{}_{,\lambda\kappa}$$

$$+ \Bigg[\frac{1}{6} \left(6\, g_1 - 4\, g_2 + 11\, g_3 + g_4 + 7\, g_6 - 62\, g_7 + 12\, g_8 - 18\, g_9\right) \gamma^{\alpha\beta} \gamma^{\mu\nu} \gamma^{\lambda\kappa}$$

$$+ \frac{1}{12} \left(8\, g_2 - 16\, g_3 + g_4 - 2\, g_6 + 64\, g_7 - 24\, g_8 + 18\, g_9\right) \gamma^{\alpha\beta} \gamma^{\mu\lambda} \gamma^{\kappa\nu}$$

$$+ \frac{1}{6} \left(8\, g_2 - 13\, g_3 - 2\, g_4 - 11\, g_6 + 82\, g_7 - 12\, g_8 + 24\, g_9 + 6\, g_{10}\right) \gamma^{\lambda\kappa} \gamma^{\alpha\mu} \gamma^{\nu\beta}$$

$$+ \left(4\, g_3 + g_6 - 14\, g_7 + 4\, g_8 - 4\, g_9\right) \gamma^{\lambda\alpha} \gamma^{\beta\mu} \gamma^{\nu\kappa}$$

$$+ \frac{1}{2} \left(2\, g_2 - 3\, g_3 - g_6 + 14\, g_7 - 4\, g_8 + 4\, g_9\right) \gamma^{\mu\nu} \gamma^{\alpha\lambda} \gamma^{\kappa\beta} \Bigg] \cdot \overline{\overline{\varphi}}_{\alpha\beta} \overline{\overline{\varphi}}_{\mu\nu,\lambda\kappa} + \Bigg[\frac{1}{2} g_{14} \gamma^{\alpha\beta} \gamma^{\nu\lambda} \delta^{\kappa}_{\mu}$$

$$+ \frac{1}{2} \left(2\, g_{12} + g_{13}\right) \gamma^{\lambda\kappa} \gamma^{\nu\alpha} \delta^{\beta}_{\mu} + g_{15} \gamma^{\alpha\lambda} \gamma^{\kappa\nu} \delta^{\beta}_{\mu} \Bigg] \cdot \overline{\overline{\varphi}}_{\alpha\beta} \overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda\kappa} + \Bigg[\frac{1}{2} \left(2\, g_{12} + g_{13}\right) \gamma_{\alpha\mu} \gamma^{\lambda\kappa} \delta^{\beta}_{\nu}$$

$$+ g_{15} \gamma_{\alpha\mu} \gamma^{\beta\lambda} \delta^{\kappa}_{\nu} + \frac{1}{2} g_{14} \gamma_{\mu\nu} \gamma^{\beta\lambda} \delta^{\kappa}_{\alpha} \Bigg] \cdot \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta} \overline{\varphi}^{\mu\nu}{}_{,\lambda\kappa} + \Bigg[\frac{1}{2} \left(2\, g_{12} + g_{13}\right) \gamma^{\beta\mu} \gamma^{\lambda\kappa} \delta^{\nu}_{\alpha} + g_{15} \gamma^{\beta\lambda} \gamma^{\kappa\mu} \delta^{\nu}_{\alpha}$$

$$+ \frac{1}{2} g_{14} \gamma^{\mu\nu} \gamma^{\beta\lambda} \delta^{\kappa}_{\alpha} \Bigg] \cdot \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta} \overline{\overline{\varphi}}_{\mu\nu,\lambda\kappa}$$

$$+ \Bigg[\frac{1}{3} \left(13\, g_3 - 8\, g_2 + 2\, g_4 + 11\, g_6 - 82\, g_7 + 12\, g_8 - 24\, g_9 - 12\, g_{10}\right) \gamma_{\alpha\mu} \gamma^{\beta\nu} \gamma^{\lambda\kappa}$$

$$- 2 \left(3\, g_3 + g_6 - 6\, g_7 - 2\, g_9\right) \gamma_{\alpha\mu} \gamma^{\beta\lambda} \gamma^{\kappa\nu} \Bigg] \cdot \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta} \overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda\kappa} + \Bigg[\frac{1}{2} g_4 \left(\gamma_{\alpha\beta} \delta^{\lambda}_{\mu} \delta^{\kappa}_{\nu} + \gamma_{\mu\nu} \delta^{\lambda}_{\alpha} \delta^{\kappa}_{\beta} \right)$$

$$
\begin{aligned}
&+ g_5\gamma_{\alpha\beta}\gamma_{\mu\nu}\gamma^{\lambda\kappa} + g_6\gamma_{\alpha\mu}\delta^{\lambda}_{\beta}\delta^{\kappa}_{\nu} + g_{11}\gamma_{\alpha\mu}\gamma_{\beta\nu}\gamma^{\lambda\kappa} + g_9\gamma_{\alpha\mu}\delta^{\lambda}_{\nu}\delta^{\kappa}_{\beta}\bigg] \cdot \overline{\varphi}^{\alpha\beta}{}_{,\lambda}\overline{\varphi}^{\mu\nu}{}_{,\kappa} \\
&+ \bigg[\frac{1}{6}\left(g_4 - 4\,g_2 - g_3 + 6\,g_5 + g_6 - 2\,g_7\right)\gamma_{\alpha\beta}\gamma^{\mu\nu}\gamma^{\lambda\kappa} \\
&+ \frac{1}{6}\left(8\,g_2 + 2\,g_3 + g_4 - 2\,g_6 + 4\,g_7\right)\gamma_{\alpha\beta}\gamma^{\mu\lambda}\gamma^{\kappa\nu} + \frac{1}{2}\left(2\,g_3 + g_4 + 2\,g_6 - 12\,g_7 - 4\,g_9\right)\gamma^{\mu\nu}\delta^{\lambda}_{\alpha}\delta^{\kappa}_{\beta} \\
&+ \left(g_3 + g_6 - 6\,g_7 - g_9\right)\gamma^{\mu\lambda}\delta^{\kappa}_{\alpha}\delta^{\nu}_{\beta} \\
&+ \frac{1}{6}\left(4\,g_4 - 16\,g_2 - 7\,g_3 + g_6 + 10\,g_7 + 6\,g_9 + 6\,g_{11}\right)\gamma^{\lambda\kappa}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + g_6\gamma^{\mu\kappa}\delta^{\lambda}_{\alpha}\delta^{\nu}_{\beta}\bigg] \cdot \overline{\varphi}^{\alpha\beta}{}_{,\lambda}\overline{\overline{\varphi}}_{\mu\nu,\kappa} \\
&+ \bigg[g_{14}\gamma_{\alpha\beta}\gamma^{\nu\lambda}\delta^{\kappa}_{\mu} + g_{15}\gamma_{\alpha\mu}\left(\gamma^{\nu\lambda}\delta^{\kappa}_{\beta} + \gamma^{\nu\kappa}\delta^{\lambda}_{\beta}\right) + g_{13}\gamma_{\alpha\mu}\gamma^{\lambda\kappa}\delta^{\nu}_{\beta}\bigg] \cdot \overline{\varphi}^{\alpha\beta}{}_{,\lambda}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\kappa} \\
&+ \bigg[\frac{1}{6}\left(8\,g_2 - 10\,g_3 + g_4 - 2\,g_6 + 52\,g_7 - 24\,g_8 + 12\,g_9\right)\left(\gamma^{\alpha\beta}\gamma^{\lambda\mu}\gamma^{\nu\kappa} + \gamma^{\mu\nu}\gamma^{\lambda\alpha}\gamma^{\beta\kappa}\right) \\
&+ \frac{1}{6}\left(7\,g_3 - 8\,g_2 + 2\,g_4 + 6\,g_5 + 5\,g_6 - 46\,g_7 + 12\,g_8 - 12\,g_9\right)\gamma^{\alpha\beta}\gamma^{\mu\nu}\gamma^{\lambda\kappa} \\
&+ \left(4\,g_3 + g_6 - 16\,g_7 + 8\,g_8 - 4\,g_9\right)\gamma^{\lambda\alpha}\gamma^{\beta\mu}\gamma^{\nu\kappa} \\
&+ \frac{1}{2}\left(54\,g_7 11\,g_3 - 5\,g_6 - 12\,g_8 + 16\,g_9 + 2\,g_{11}\right)\gamma^{\alpha\mu}\gamma^{\nu\beta}\gamma^{\kappa\lambda} \\
&+ \left(3\,g_3 + g_6 - 14\,g_7 + 4\,g_8 - 3\,g_9\right)\gamma^{\lambda\mu}\gamma^{\nu\alpha}\gamma^{\beta\kappa}\bigg] \cdot \overline{\overline{\varphi}}_{\alpha\beta,\lambda}\overline{\overline{\varphi}}_{\mu\nu,\kappa} + \bigg[\frac{1}{2}g_{14}\gamma^{\alpha\beta}\left(\gamma^{\nu\kappa}\delta^{\lambda}_{\mu} + \gamma^{\nu\lambda}\delta^{\kappa}_{\mu}\right) \\
&+ g_{15}\left(\gamma^{\alpha\kappa}\gamma^{\lambda\nu} + \gamma^{\alpha\lambda}\gamma^{\kappa\nu}\right)\delta^{\beta}_{\mu} + g_{13}\gamma^{\lambda\kappa}\gamma^{\alpha\nu}\delta^{\beta}_{\mu}\bigg] \cdot \overline{\overline{\varphi}}_{\alpha\beta,\lambda}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\kappa} \\
&+ \bigg[\left(4\,g_8 - 7\,g_3 - g_6 - 2\,g_7 + 4\,g_9\right)\gamma^{\nu\kappa}\delta^{\lambda}_{\alpha}\delta^{\beta}_{\mu} \\
&+ \frac{2}{3}\left(8\,g_2 + 26\,g_3 - 2\,g_4 + g_6 - 56\,g_7 - 27\,g_9 - 6\,g_{11}\right)\gamma_{\alpha\mu}\gamma^{\beta\nu}\gamma^{\lambda\kappa} \\
&+ 4\left(2\,g_7 - 2\,g_3 + 2\,g_8 + g_9\right)\gamma_{\alpha\mu}\gamma^{\beta\kappa}\gamma^{\lambda\nu}\bigg] \cdot \overline{\overline{\overline{\varphi}}}{}^{\alpha}{}_{\beta,\lambda}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\kappa} + g_{16}\gamma_{\alpha\mu}\delta^{\nu}_{\beta} \cdot \overline{k}^{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu} \\
&+ g_{16}\gamma^{\alpha\nu}\delta^{\beta}_{\mu} \cdot \overline{\overline{k}}_{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu} + \left(g_{16} + g_{17}\right)\gamma_{\mu\alpha}\delta^{\beta}_{\nu} \cdot \overline{\overline{\overline{k}}}{}^{\alpha}{}_{\beta}\overline{\varphi}^{\mu\nu} + \left(g_{16} + g_{17}\right)\gamma^{\mu\beta}\delta^{\nu}_{\alpha} \cdot \overline{\overline{\overline{k}}}{}^{\alpha}{}_{\beta}\overline{\overline{\varphi}}_{\mu\nu} \\
&+ \left(g_6 - g_3 + 2\,g_7\right)\epsilon_{\alpha\mu}{}^{\lambda}\delta^{\nu}_{\beta} \cdot \overline{k}^{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda} + 4\,g_8\epsilon^{\alpha}{}_{\mu}{}^{\lambda}\gamma^{\beta\nu} \cdot \overline{\overline{k}}_{\alpha\beta}\overline{\overline{\overline{\varphi}}}{}^{\mu}{}_{\nu,\lambda} + 4\,g_7\epsilon_{\alpha\mu}{}^{\lambda}\delta^{\beta}_{\nu} \cdot \overline{\overline{\overline{k}}}{}^{\alpha}{}_{\beta}\overline{\varphi}^{\mu\nu}{}_{,\lambda} \\
&+ 4\,g_7\epsilon_{\alpha}{}^{\mu\lambda}\gamma^{\beta\nu} \cdot \overline{\overline{\overline{k}}}{}^{\alpha}{}_{\beta}\overline{\overline{\varphi}}_{\mu\nu,\lambda} + g_{12}\gamma_{\alpha\mu}\delta^{\nu}_{\beta} \cdot \overline{k}^{\alpha\beta}\overline{\overline{\overline{k}}}{}^{\mu}{}_{\nu} + g_{12}\gamma^{\alpha\nu}\delta^{\beta}_{\mu} \cdot \overline{\overline{k}}_{\alpha\beta}\overline{\overline{\overline{k}}}{}^{\mu}{}_{\nu} \\
&+ \bigg[\frac{1}{4}\left(4\,g_1 + 4\,g_2 - g_4 - 2\,g_5\right)\gamma_{\alpha\beta}\gamma_{\mu\nu} \\
&+ \frac{1}{6}\left(8\,g_2 + 5\,g_3 - 2\,g_4 - 5\,g_6 + 10\,g_7 + 6\,g_{10} - 3\,g_{11}\right)\gamma_{\alpha\mu}\gamma_{\nu\beta}\bigg] \cdot \overline{k}^{\alpha\beta}\overline{k}^{\mu\nu} \\
&+ \bigg[\frac{1}{12}\left(12\,g_1 + 8\,g_2 - g_3 - 2\,g_4 - 6\,g_5 + g_6 - 2\,g_7\right)\gamma_{\alpha\beta}\gamma^{\mu\nu}
\end{aligned}
$$

$$+\frac{1}{12}\left(16\,g_2+13\,g_3-4\,g_4-7\,g_6+2\,g_7-6\,g_9+12\,g_{10}-6\,g_{11}\right)\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\Bigg]\cdot\overline{k}^{\alpha\beta}\overline{\overline{k}}_{\mu\nu}$$
$$+\frac{1}{3}\left(4\,g_4-16\,g_2-10\,g_3+7\,g_6-8\,g_7+3\,g_9-12\,g_{10}+6\,g_{11}\right)\cdot\overline{\overline{k}}^{\alpha}{}_{\beta}\overline{\overline{k}}^{\beta}{}_{\alpha}$$
$$+\Bigg[\frac{1}{12}\left(12\,g_1+4\,g_2+7\,g_3-g_4-6\,g_5+5\,g_6-46\,g_7+12\,g_8-12\,g_9\right)\gamma^{\alpha\beta}\gamma^{\mu\nu}$$
$$+\frac{1}{12}\left(16\,g_2+7\,g_3-4\,g_4-7\,g_6+26\,g_7-12\,g_8+12\,g_{10}-6\,g_{11}\right)\gamma^{\alpha\mu}\gamma^{\nu\beta}\Bigg]\cdot\overline{\overline{k}}_{\alpha\beta}\overline{\overline{k}}_{\mu\nu}+\mathcal{O}(3)\,.$$

The complexity of this result again shows the power and usefulness of the gravitational closure mechanism: If we would have "simply" written down a linear combination of all possible contractions between the projected hypersurface fields – employing the available background geometry on the hypersurface and the volume form – the obtained expression would contain over 100 parameters, that is gravitational constants that need to be fixed by experiments. By solving the gravitational closure equations, we found out that these constants are not constant at all: Instead, they are functions that are parametrized by 22 parameters $g_1,\ldots,g_{22}$ that remain after solving the closure equations.

The total number of independent factors further reduces once we calculate the equations of motion. When evaluating how many independent linear combinations appear in front of the terms in the equations of motion, one finds that we can reduce the number to eleven constants $\kappa_1,\ldots,\kappa_{11}$, given in terms of the g_i (compare table 5.4).

This indicates that eleven of the constants g_i are related to the boundary terms in the gravitational action[14] and worthwhile to further investigate in the future.

We could, in principle, now derive the equations of motion for the degrees of freedom by variation of the action. We will refrain from this for the following two reasons: First, as the Lagrangian already suggests, the corresponding expressions are quite lengthy. The second (far better) reason is that one usually considers the propagation of the separate modes in the degrees of freedom by a Helmholtz-Hodge decomposition. Also, the interesting physical degrees of freedom are only obtained once we identified the gauge-invariants and fixed a gauge. For that reason, we will directly discuss these first and then present the equations of motion directly for the obtained scalar, vector and tensor modes of the area metric.

Helmholtz-Hodge decomposition

Since the area metric is projected into three hypersurface tensors of rank 2, it is straightforward to perform a Helmholtz-Hodge decomposition of the three fields into their scalar, solenoidal vector, and transverse and γ traceless tensor modes. With the help of our constant intertwiner, we can pull this decomposition on Φ and obtain the following expressions:

$$\varphi^{\overline{A}}=\mathcal{I}^{\overline{A}}{}_{\alpha\beta}\left(\widetilde{F}\gamma^{\alpha\beta}+\Delta^{\alpha\beta}F+2\,\partial^{(\alpha}F^{\beta)}+F^{\alpha\beta}\right), \tag{5.83a}$$
$$\varphi^{\overline{\overline{A}}}=\mathcal{I}^{\overline{\overline{A}}\alpha\beta}\left(\widetilde{E}\gamma_{\alpha\beta}+\Delta_{\alpha\beta}E+2\,\partial_{(\alpha}E_{\beta)}+E_{\alpha\beta}\right), \tag{5.83b}$$
$$\varphi^{\overline{\overline{\overline{A}}}}=\mathcal{I}^{\overline{\overline{A}}}{}_{\alpha}{}^{\beta}\left(\Delta^{\alpha}{}_{\beta}C+\partial_{\beta}C^{\alpha}+\partial^{\alpha}C_{\beta}+C^{\alpha}{}_{\beta}\right)+\mathcal{O}(2)\,, \tag{5.83c}$$

[14]To be more precise: One decomposes the linear space of the coefficients g_i into the subspace spanned by the gravitational constants κ_i and the eleven-dimensional orthogonal subspace.

Gravitational Constant	Combination of Coefficients in the Lagrangian
κ_1	$-\frac{8}{3}g_2 + \frac{2}{3}g_4 + \frac{7}{6}g_6 + g_9 - 2g_{10} + g_{11} - \frac{1}{3}g_7 - \frac{13}{6}g_3$
κ_2	$\frac{8}{3}g_2 + \frac{5}{3}g_3 + \frac{8}{3}g_7 + 2g_{10} - g_{11} - \frac{2}{3}g_4 - \frac{4}{3}g_6 - \frac{2}{3}g_8 - \frac{1}{3}g_9$
κ_3	$-2g_{12}$
κ_4	g_{17}
κ_5	g_{21}
κ_6	g_{22}
κ_7	$-g_2 + g_3 + \frac{1}{4}g_4 + \frac{3}{2}g_6 - 8g_7 - \frac{5}{2}g_9$
κ_8	$-g_2 + 2g_3 + \frac{1}{4}g_4 + \frac{3}{2}g_6 - 12g_7 + 2g_8 - \frac{7}{2}g_9$
κ_9	$\frac{32}{3}g_2 + \frac{20}{3}g_3 + \frac{16}{3}g_7 - 2g_9 + 8g_{10} - 4g_{11} - \frac{8}{3}g_4 - \frac{14}{3}g_6$
κ_{10}	$-6g_1 + \frac{13}{6}g_4 + 3g_5 + \frac{5}{3}g_6 - 2g_{10} + g_{11} - \frac{26}{3}g_2 - \frac{5}{3}g_3 - \frac{10}{3}g_7$
κ_{11}	$3g_{18} + g_{21}$

Table 5.4 The definition of the 11 gravitational constants that appear in the gravitational equations of motion of the linear perturbations, in terms of the 22 constants appearing in the Lagrangian.

with the five scalar modes $\widetilde{F}, F, \widetilde{E}, E$ and C, the three solenoidal vectors F^α, E^α and C^α, and the three transverse, γ-traceless tensor modes $F^{\alpha\beta}$, $E^{\alpha\beta}$ and $C^{\alpha\beta}$. We used the trace-removed Hesse operator $\Delta^{\alpha\beta} := \partial^\alpha\partial^\beta - \frac{1}{3}\gamma^{\alpha\beta}\Delta$ and indices are pulled and raised with the flat metric γ. In the same fashion, we can decompose the shift $N^\alpha = \partial^\alpha B + B^\alpha$, while the lapse naturally is simply a scalar mode $N = 1 + A$.

Since our equations of motion are linear, the equations can be decomposed in the same fashion, and the different sectors – scalars, solenoidal vector and transverse-traceless vectors – can be studied independently from each other. Note that this is only the case for linear perturbations: While we can also decompose the degrees of freedom just as displayed in (5.83) for higher-order perturbations, the different modes will interact due to the non-linearity in the source terms.

We can then insert this into the first order perturbations (compare section 5.3.1) and find that

$$\left(\delta H_{(1)}\right)^{0\alpha0\beta} = (2A - \widetilde{F})\gamma^{\alpha\beta} - \Delta^{\alpha\beta}F - 2\partial^{(\alpha}F^{\beta)} - F^{\alpha\beta}, \tag{5.84a}$$

$$\left(\delta H_{(1)}\right)^{0\alpha\beta\gamma} = \epsilon^{\beta\gamma}{}_\mu\left((\widetilde{F} - A)\gamma^{\alpha\mu} + \epsilon^{\alpha\mu\nu}(\partial_\nu B + B_\nu) + \Delta^{\alpha\mu}C + 2\partial^{(\alpha}C^{\mu)} + C^{\alpha\mu}\right), \tag{5.84b}$$

$$\left(\delta H_{(1)}\right)^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\mu}\epsilon^{\gamma\delta\nu}\left((3\widetilde{F} + \widetilde{E})\gamma_{\mu\nu} + \Delta_{\mu\nu}E + 2\partial_{(\mu}E_{\nu)} + E_{\mu\nu}\right). \tag{5.84c}$$

The next step is to identify the gauge-invariant quantities by calculating the action of a gauge transformation on the perturbations.

Gauge invariants

As laid out in section 5.3.1, a gauge transformation, generated by a spacetime vector field ξ, induces a change in the perturbations to first order via

$$\left(\Delta_\xi H_{(1)}\right)^{abcd} = \left(\mathcal{L}_\xi N\right)^{abcd}. \tag{5.85}$$

Using that $N^{abcd} = 2\,\eta^{a[c}\eta^{d]b} - \epsilon^{abcd}$, and by decomposing the generator ξ into $\xi^0 =: T$ and $\xi^\alpha := \partial^\alpha L + L^\alpha$, with L^α being a solenoidal vector, we can express this change in terms of the scalar, vector and tensor modes:

$$\left(\Delta_\xi H_{(1)}\right)^{0\alpha 0\beta} = 2\,\partial^\alpha\partial^\beta L + 2\,\partial^{(\alpha}L^{\beta)} + 2\,\dot{T}\gamma^{\alpha\beta}\,, \tag{5.86a}$$

$$\left(\Delta_\xi H_{(1)}\right)^{0\alpha\beta\gamma} = \epsilon^{\beta\gamma}{}_\mu \left(\left(-\dot{T} - \Delta L\right)\gamma^{\alpha\mu} + \left(\dot{L}_\nu + \partial_\nu \dot{L} - \partial_\nu T\right)\epsilon^{\alpha\mu\nu}\right)\,, \tag{5.86b}$$

$$\left(\Delta_\xi H_{(1)}\right)^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\mu}\epsilon^{\gamma\delta\nu}\left(2\,\partial_\mu\partial_\nu L + 2\,\partial_{(\mu}L_{\nu)} - 2\,\gamma_{\mu\nu}\Delta L\right)\,. \tag{5.86c}$$

By comparing with the decomposition (5.84), we see that we obtain the following induced changes for the different modes. For all the scalar modes we find

$$\Delta_\xi\widetilde{F} = -\frac{2}{3}\Delta L\,, \qquad \Delta_\xi A = \dot{T}\,, \qquad \Delta_\xi\widetilde{E} = \frac{2}{3}\Delta L\,, \qquad \Delta_\xi B = \dot{L} - T$$
$$\Delta_\xi F = -2\,L\,, \qquad \Delta_\xi E = 2\,L\,, \qquad \Delta_\xi C = 0\,,$$

and for the solenoidal vector modes

$$\Delta_\xi B^\alpha = \dot{L}^\alpha\,, \qquad \Delta_\xi F^\alpha = -L^\alpha\,, \qquad \Delta_\xi E^\alpha = L^\alpha\,, \qquad \Delta_\xi C^\alpha = 0\,.$$

In case of the transverse-traceless tensor modes, we find that they do not change under a gauge transformation, i.e.

$$\Delta_\xi F^{\alpha\beta} = 0\,, \qquad \Delta_\xi E_{\alpha\beta} = 0\,, \qquad \Delta_\xi C^\alpha{}_\beta = 0\,.$$

Using all of this information, we can search for combinations of those modes that will not change under a gauge transformation, that is they are gauge invariant. For example, we see that since the two scalar modes E and F transform with $\pm 2L$ under a gauge transformation that their sum will remain invariant under the transformation. In the same fashion, we can identify in total eleven gauge invariants that are displayed in the following box:

DEFINITION GAUGE INVARIANTS FOR LINEAR PERTURBATION

In the equations of motion of linear perturbations of an area metric, the following eleven quantities are invariant under gauge transformations:

$$\mathcal{J}_1 = E + F\,, \qquad \mathcal{J}_2 = \widetilde{E} + \widetilde{F}\,, \qquad \mathcal{J}_3 = C\,, \qquad \mathcal{J}_4 = A + \widetilde{B} + \frac{1}{2}\ddot{F}\,, \qquad \mathcal{J}_5 = \widetilde{E} - \widetilde{F} + \frac{2}{3}\Delta F\,,$$
$$\mathcal{J}_6^\alpha = F^\alpha + E^\alpha\,, \qquad \mathcal{J}_7^\alpha = B^\alpha - \dot{E}^\alpha\,, \qquad \mathcal{J}_8^\alpha = C^\alpha\,,$$
$$\mathcal{J}_9^{\alpha\beta} = F^{\alpha\beta}\,, \qquad \mathcal{J}_{10}^{\alpha\beta} = E^{\alpha\beta}\,, \qquad \mathcal{J}_{11}^{\alpha\beta} = C^{\alpha\beta}\,,$$

for scalar, solenoidal vector and transverse-traceless tensor perturbations, respectively. They contain 17 degrees of freedom in total.

Using only those gauge invariant quantities ensures that we obtain physical relevant statements from the equations of motion. Using a gauge transformation, we can effectively set four of our components to 0. For example, by the choice

$$T := B + \frac{1}{2}\dot{F}\,, \qquad L := \frac{1}{2}F\,, \qquad L^{\alpha} := -E^{\alpha}\,, \tag{5.87}$$

we remove the four modes E^{α}, F, and B. The particular choice of a gauge is, however, completely arbitrary and up to the user to choose since, if done right, the results will not depend on the choice made.

REMARK

If we inspect the gauge invariants more closely, we see that in almost all of the invariants we obtain a linear combination of the modes from $\varphi^{\overline{A}}$ and $\varphi^{\overline{\overline{A}}}$. In fact, if we would have chosen the rather arcane decomposition of the degrees of freedom into

$$\begin{aligned}
\varphi^{\overline{A}} =& \mathcal{I}^{\overline{A}}{}_{\alpha\beta}\left(\left(\widetilde{\mathcal{U}}+\widetilde{\mathcal{V}}\right)\gamma^{\alpha\beta}+\Delta^{\alpha\beta}\left(\mathcal{U}+\mathcal{V}\right)+2\,\partial^{(\alpha|}\left(\mathcal{U}^{|\beta)}+\mathcal{V}^{|\beta)}\right)+\mathcal{U}^{\alpha\beta}+\mathcal{V}^{\alpha\beta}\right),\\
\varphi^{\overline{\overline{A}}} =& \mathcal{I}^{\overline{\overline{A}}\alpha\beta}\left(\left(\widetilde{\mathcal{U}}-\widetilde{\mathcal{V}}\right)\gamma_{\alpha\beta}+\Delta_{\alpha\beta}\left(\mathcal{U}-\mathcal{V}\right)+2\,\partial_{(\alpha|}\left(\mathcal{U}_{|\beta)}-\mathcal{V}_{|\beta)}\right)+\mathcal{U}_{\alpha\beta}-\mathcal{V}_{\alpha\beta}\right),\\
\varphi^{\overline{\overline{\overline{A}}}} =& \mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha}{}^{\beta}\left(\Delta^{\alpha}{}_{\beta}C+\partial_{\beta}C^{\alpha}+\partial^{\alpha}C_{\beta}+C^{\alpha}{}_{\beta}\right)+\mathcal{O}(2)\,,
\end{aligned}$$

the gauge invariants would simplify drastically since almost all of the invariants are directly given by components of either $\mathcal{U}$ or $\mathcal{V}$ terms.

Furthermore, this is conceptually incredibly useful since, as we will show in the following, the tensor mode $\mathcal{V}^{\alpha\beta}$ correspond to the massless spin-2 metric degrees of freedom from general relativity. In the end, the results do not depend on how we chose to perform the decomposition.

Equations of motion

Now, with the Helmholtz-Hodge decomposition performed and a gauge chosen, we can finally spell out the equations of motion. Deriving these, while tedious, is a straightforward task: We vary the total action, i.e. the sum of the gravitational and the matter actions, with respect to lapse, shift and the degrees of freedom. For the contributions from the matter sector, we Helmholtz-Hodge decompose the functional derivatives, i.e.

$$\mathcal{I}^{\overline{A}}{}_{\alpha\beta}\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\varphi^{\overline{A}}} =: \left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(TT)}_{\alpha\beta}+2\,\partial_{(\alpha}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(V)}_{\beta)}+\gamma_{\alpha\beta}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(S_{\text{tr}})}+\Delta_{\alpha\beta}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(S_{\text{tr-free}})}, \tag{5.88a}$$

$$\mathcal{I}^{\overline{\overline{A}}}{}_{\alpha\beta}\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\varphi^{\overline{\overline{A}}}} =: \left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(TT)}_{\alpha\beta}+2\,\partial_{(\alpha}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(V)}_{\beta)}+\gamma_{\alpha\beta}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(S_{\text{tr}})}+\Delta_{\alpha\beta}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(S_{\text{tr-free}})}, \tag{5.88b}$$

$$\mathcal{I}^{\overline{\overline{\overline{A}}}}{}_{\alpha\beta}\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\varphi^{\overline{\overline{\overline{A}}}}} =: \left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\overline{\varphi}}}}\right]^{(TT)}_{\alpha\beta} + 2\,\partial_{(\alpha}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\overline{\varphi}}}}\right]^{(V)}_{\beta)} + \Delta_{\alpha\beta}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\overline{\varphi}}}}\right]^{(S_{\text{tr-free}})}, \tag{5.88c}$$

$$\frac{\delta\mathcal{H}_{\text{matter}}}{\delta N^{\alpha}} =: \left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\vec{N}}\right]^{(V)}_{\alpha} + \partial_{\alpha}\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\vec{N}}\right]^{(S)}, \tag{5.88d}$$

$$\frac{\delta\mathcal{H}_{\text{matter}}}{\delta N} =: \left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta N}\right]^{(S)}, \tag{5.88e}$$

where we conveniently lowered all indices with γ. Observe that the transverse-traceless tensor modes are only seen by the three evolution equations and are not constrained. We can now separately discuss the different modes.

Transverse-traceless tensor modes We start the discussion with the transverse-traceless tensor modes. As stated above, one only finds three evolution equations for the three tensors $E^{\alpha\beta}$, $F^{\alpha\beta}$, $C^{\alpha\beta}$, with the separate terms parametrized by the gravitational constants κ that need to be fixed by experiments:

$$\begin{aligned}
\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(TT)}_{\alpha\beta} = &-(2\,\kappa_1+3\,\kappa_2-\kappa_7+\kappa_8)\,\ddot{F}_{\alpha\beta} - (3\,\kappa_2-\kappa_7+\kappa_8-\kappa_9)\,\Delta F_{\alpha\beta} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\,F_{\beta)\gamma,\mu}\\
&+\kappa_5\,F_{\alpha\beta} + \kappa_1\,\ddot{E}_{\alpha\beta} + (\kappa_9-3\,\kappa_2)\,\Delta E_{\alpha\beta} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}E_{\beta)\gamma,\mu} + \kappa_5\,E_{\alpha\beta}\\
&+(2\,\kappa_1+6\,\kappa_2-2\,\kappa_7+2\,\kappa_8-\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\dot{C}_{\beta)\gamma,\mu}\\
&+\kappa_3\ddot{C}_{\alpha\beta} - \kappa_3\,\Delta C_{\alpha\beta} + \kappa_4\,\dot{C}_{\alpha\beta} + \kappa_6 C_{\alpha\beta}\,,
\end{aligned} \tag{5.89a}$$

$$\begin{aligned}
\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(TT)}_{\alpha\beta} = &\,\kappa_1\,\ddot{F}_{\alpha\beta} + (\kappa_9-3\,\kappa_2)\,\Delta F_{\alpha\beta} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}F_{\beta)\gamma,\mu} + \kappa_5\,F_{\alpha\beta} + (\kappa_1-\kappa_7+\kappa_8)\,\ddot{E}_{\alpha\beta}\\
&+(3\,\kappa_1-\kappa_7+\kappa_8+\kappa_9)\,\Delta E_{\alpha\beta} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\,E_{\beta)\gamma,\mu} + \kappa_5\,E_{\alpha\beta} + \kappa_3\,\ddot{C}_{\alpha\beta} - \kappa_3\Delta C_{\alpha\beta}\\
&-(4\,\kappa_1-2\,\kappa_7+2\,\kappa_8+\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\dot{C}_{\beta)\gamma,\mu} + \kappa_4\,\dot{C}_{\alpha\beta} + \kappa_6\,C_{\alpha\beta}\,,
\end{aligned} \tag{5.89b}$$

$$\begin{aligned}
\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\overline{\varphi}}}}\right]^{(TT)}_{\alpha\beta} = &\,\kappa_3\,\ddot{F}_{\alpha\beta} - \kappa_3\,\Delta F_{\alpha\beta} - (2\,\kappa_1+6\,\kappa_2-2\,\kappa_7+2\,\kappa_8-\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\dot{F}_{\beta)\gamma,\mu} - \kappa_4\,\dot{F}_{\alpha\beta} + \kappa_6\,F_{\alpha\beta}\\
&+\kappa_3\,\ddot{E}_{\alpha\beta} - \kappa_3\,\Delta E_{\alpha\beta} + (4\,\kappa_1-2\,\kappa_7+2\,\kappa_8+\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\dot{E}_{\beta)\gamma,\mu} - \kappa_4\,\dot{E}_{\alpha\beta} + \kappa_6\,E_{\alpha\beta}\\
&+\kappa_9\,\ddot{C}_{\alpha\beta} + (4\,\kappa_1-2\,\kappa_7+2\,\kappa_8+\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\dot{E}_{\beta)\gamma,\mu} - \kappa_4\,\dot{E}_{\alpha\beta} + \kappa_6\,E_{\alpha\beta} + \kappa_9\,\ddot{C}_{\alpha\beta}\\
&+(4\,\kappa_1-12\,\kappa_2+3\,\kappa_9)\,\Delta C_{\alpha\beta} + 8\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\,C_{\beta)\gamma,\mu} - 4\,\kappa_5\,C_{\alpha\beta}\,.
\end{aligned} \tag{5.89c}$$

Solenoidal vector modes We continue with the equations of motion of the three solenoidal vectors (two degrees of freedom each). Similar to the transverse-traceless modes above, we get

$$\begin{aligned}
2\,\partial_{(\alpha}\left[\frac{\partial\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(V)}_{\beta)} = &\,(-4\,\kappa_1-6\,\kappa_2+2\,\kappa_7-2\,\kappa_8)\,\partial_{(\alpha}\ddot{F}_{\beta)} + \frac{\kappa_9}{2}\Delta\partial_{(\alpha}F_{\beta)} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\partial_{\beta)}F_{\gamma,\mu}\\
&+2\,\kappa_5\,\partial_{(\alpha}F_{\beta)} + 2\,\kappa_3\,\partial_{(\alpha}\ddot{C}_{\beta)} - 2\,\kappa_3\,\Delta\partial_{(\alpha}C_{\beta)}\\
&+(2\,\kappa_1+6\,\kappa_2-2\,\kappa_7+2\,\kappa_8-\kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\partial_{\beta)}\dot{C}_{\gamma,\mu} + 2\,\kappa_4\,\partial_{(\alpha}\dot{C}_{\beta)} + 2\,\kappa_6\,\partial_{(\alpha}C_{\beta)}\\
&+(-6\,\kappa_1-6\,\kappa_2+2\,\kappa_7-2\,\kappa_8)\,\partial_{(\alpha}\dot{B}_{\beta)}\,,
\end{aligned} \tag{5.90a}$$

$$2\,\partial_{(\alpha}\left[\frac{\partial\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(V)}_{\beta)} = 2\,\kappa_1\,\partial_{(\alpha}\ddot{F}_{\beta)} + \frac{\kappa_9}{2}\Delta\partial_{(\alpha}F_{\beta)} + 2\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\partial_{\beta)}F_{\gamma,\mu} + 2\,\kappa_5\,\partial_{(\alpha}F_{\beta)} + 2\,\kappa_3\,\partial_{(\alpha}\ddot{C}_{\beta)}$$
$$- 2\,\kappa_3\,\Delta\partial_{(\alpha}C_{\beta)} + (-4\,\kappa_1 + 2\,\kappa_7 - 2\,\kappa_8 - \kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\partial_{\beta)}\dot{C}_{\gamma,\mu} + 2\,\kappa_4\,\partial_{(\alpha}\dot{C}_{\beta)}$$
$$+ 2\,\kappa_6\,\partial_{(\alpha}C_{\beta)} + (2\kappa_7 - 2\kappa_8)\,\partial_{(\alpha}\dot{B}_{\beta)}\,, \tag{5.90b}$$

$$2\,\partial_{(\alpha}\left[\frac{\partial\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(V)}_{\beta)} = 2\,\kappa_3\,\partial_{(\alpha}\ddot{F}_{\beta)} - 2\,\kappa_3\Delta\partial_{(\alpha}F_{\beta)} + (-2\,\kappa_1 - 6\,\kappa_2 + 2\,\kappa_7 - 2\,\kappa_8 + \kappa_9)\,\epsilon_{(\alpha}{}^{\gamma\mu}\,\partial_{\beta)}\dot{F}_{\gamma,\mu}$$
$$- 2\,\kappa_4\,\partial_{(\alpha}\dot{F}_{\beta)} + 2\,\kappa_6\,\partial_{(\alpha}F_{\beta)} + 2\,\kappa_9\,\partial_{(\alpha}\ddot{C}_{\beta)} + (2\,\kappa_1 - 6\,\kappa_2)\,\Delta\partial_{(\alpha}C_{\beta)}$$
$$+ 8\,\kappa_4\,\epsilon_{(\alpha}{}^{\gamma\mu}\partial_{\beta)}C_{\gamma,\mu} - 8\,\kappa_5\,\partial_{(\alpha}C_{\beta)} - (6\kappa_1 + 6\kappa_2 - 4\kappa_7 + 4\kappa_8)\,\epsilon_{(\alpha}{}^{\gamma\mu}\partial_{\beta)}B_{\gamma,\mu}\,. \tag{5.90c}$$

Additionally we get contributions from the constraint equations in this case. From the vector constraint we obtain the equation

$$\left[\frac{\partial\mathcal{H}_{\text{matter}}}{\delta\vec{N}}\right]^{(V)}_{\alpha} = -2\,(3\,\kappa_1 + 3\,\kappa_2 - \kappa_7 + \kappa_8)\,\Delta\dot{F}_\alpha - 6\,(\kappa_1 + \kappa_2)\,\Delta B_\alpha$$
$$+ 2\,(3\,\kappa_1 + 3\,\kappa_2 - 2\,\kappa_7 + 2\,\kappa_8)\,\epsilon_\alpha{}^{\gamma\mu}\Delta C_{\gamma,\mu}\,. \tag{5.91}$$

Scalar modes Last but not least, we present the equations of motion for the scalar modes. This corresponds to five separate evolution equations from the trace and trace-free part of each of our three hypersurface fields. In addition, we get further restrictions from both constraint equations.

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(S_{\text{tr-free}})} = \kappa_1\,\ddot{E} + \kappa_2\,\Delta E + \kappa_3\,\ddot{C} - \kappa_3\,\Delta C + \kappa_4\,\dot{C} + \kappa_5\,E + \kappa_6\,C + \kappa_7\,\widetilde{F} + \kappa_8\,\widetilde{E}$$
$$+ 2\,(\kappa_7 - 3\,\kappa_1 - 3\,\kappa_2 - \kappa_8)\,A\,, \tag{5.92a}$$

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\varphi}}\right]^{(S_{\text{tr}})} = \kappa_{10}\,\ddot{\widetilde{F}} + \left(16\,\kappa_1 + 16\,\kappa_2 - \frac{8}{3}\,\kappa_7 + \frac{16}{3}\,\kappa_8 - \kappa_{10}\right)\Delta\widetilde{F} + \kappa_{11}\widetilde{F}$$
$$+ (2\,\kappa_7 - 12\,\kappa_1 - 12\,\kappa_2 - 4\,\kappa_8 + \kappa_{10})\,\ddot{\widetilde{E}}$$
$$+ (20\,\kappa_1 + 20\,\kappa_2 - 4\,\kappa_7 + 8\,\kappa_8 - \kappa_{10})\,\Delta\widetilde{E} + \kappa_{11}\widetilde{E}$$
$$+ \left(8\,\kappa_1 + 8\,\kappa_2 - \frac{4}{3}\,\kappa_7 + \frac{8}{3}\,\kappa_8\right)\Delta A - \frac{2}{3}\left(2\,\kappa_1 + 2\,\kappa_2 - \kappa_7 + \frac{4}{3}\kappa_8\right)\Delta\Delta E\,, \tag{5.92b}$$

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(S_{\text{tr-free}})} = (\kappa_1 - \kappa_7 + \kappa_8)\,\ddot{E} + \frac{1}{3}\,(\kappa_7 - 3\,\kappa_1 - \kappa_8)\,\Delta E + \kappa_5\,E + \kappa_3\,\ddot{C} - \kappa_3\,\Delta C + \kappa_4\,\dot{C}$$
$$+ \kappa_6\,C + (3\,\kappa_7 - 6\,\kappa_1 - 6\,\kappa_2 - 4\,\kappa_8)\,\widetilde{F} + (2\,\kappa_7 - 3\,\kappa_1 - 3\,\kappa_2 - 3\,\kappa_8)\,\widetilde{E}$$
$$+ 2\,(\kappa_7 - \kappa_8)\,A\,, \tag{5.92c}$$

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\varphi}}}\right]^{(S_{\text{tr}})} = (2\,\kappa_7 - 12\,\kappa_1 - 12\,\kappa_2 - 4\,\kappa_8 + \kappa_{10})\,\ddot{\widetilde{F}} + (20\,\kappa_1 + 20\,\kappa_2 - 4\,\kappa_7 + 8\,\kappa_8 - \kappa_{10})\,\Delta\widetilde{F}$$

$$+\left(4\,\kappa_7 - 18\,\kappa_1 - 18\,\kappa_2 - 8\,\kappa_8 + \kappa_{10}\right)\ddot{\widetilde{E}} - \frac{2}{3}\left(\kappa_1 + \kappa_2 - \frac{2}{3}\,\kappa_7 + \kappa_8\right)\Delta\Delta E$$
$$+\left(22\,\kappa_1 + 22\,\kappa_2 - \frac{16}{3}\,\kappa_7 + \frac{32}{3}\,\kappa_8 - \kappa_{10}\right)\Delta\widetilde{E}$$
$$+4\left(\kappa_1 + \kappa_2 - \frac{1}{3}\,\kappa_7 + \frac{2}{3}\kappa_8\right)\Delta A + \kappa_{11}\widetilde{F} + \kappa_{11}\widetilde{E}\,, \tag{5.92d}$$

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\overline{\overline{\overline{\varphi}}}}\right]^{(S_{\text{tr-free}})} = \kappa_3\,\ddot{E} - \kappa_3\,\Delta E - \kappa_4\,\dot{E} + \kappa_6\,E + \kappa_9\,\ddot{C} - \kappa_9\,\Delta C - 4\,\kappa_5\,C\,. \tag{5.92e}$$

From the vector constraint we obtain the equation

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta\vec{N}}\right]^{(S)} = 4\,\left(6\,\kappa_1 + 6\,\kappa_2 - \kappa_7 + 2\,\kappa_8\right)\dot{\widetilde{F}} + 4\,\left(3\,\kappa_1 + 3\,\kappa_2 - \kappa_7 + 2\,\kappa_8\right)\dot{\widetilde{E}} + \frac{4}{3}\left(\kappa_7 - \kappa_8\right)\Delta\dot{E}\,, \tag{5.93}$$

and, last but not least, from the scalar constraint

$$\left[\frac{\delta\mathcal{H}_{\text{matter}}}{\delta N}\right]^{(S)} = 4\,\left(6\,\kappa_1 + 6\,\kappa_2 - \kappa_7 + 2\,\kappa_8\right)\Delta\widetilde{F} + 4\,\left(3\,\kappa_1 + 3\,\kappa_2 - \kappa_7 + 2\,\kappa_8\right)\Delta\widetilde{E}$$
$$+\frac{4}{3}\left(\kappa_7 - \kappa_8\right)\Delta\Delta E\,. \tag{5.94}$$

Metric induced modes

Since an area metric can be induced by a Lorentzian metric, it is interesting to identify the corresponding modes in our decomposition above. For this we observe that to first order, i.e. for $g^{ab} = \eta^{ab} + h^{ab} + \mathcal{O}(2)$ an induced area metric takes the form

$$G^{abcd}(g) = g^{ac}g^{bd} - g^{ad}g^{bc} - \sqrt{-\det g^{\cdot\cdot}}\,\epsilon^{abcd} \tag{5.95}$$
$$= 2\,\eta^{a[c}\eta^{d]b} + 2\left(\eta^{a[c}h^{d]b} - \eta^{b[c}h^{d]a}\right) + \left(1 - \frac{1}{2}\eta_{mn}h^{mn}\right)\epsilon^{abcd} + \mathcal{O}(2)\,. \tag{5.96}$$

Projecting these fields to the screen manifold we find that

$$\overline{g}^{\alpha\beta} = \gamma^{\alpha\beta} + h^{\alpha\beta} + \mathcal{O}(2)\,, \tag{5.97a}$$
$$\overline{\overline{g}}_{\alpha\beta} = \gamma_{\alpha\beta} - h_{\alpha\beta} + \mathcal{O}(2)\,, \tag{5.97b}$$
$$\overline{\overline{\overline{g}}}^{\alpha}{}_{\beta} = 0 + \mathcal{O}(2)\,. \tag{5.97c}$$

This tells us that the metric sub-sector of weakly birefringent electrodynamics correspond to the case

$$E_{\alpha\beta} = -F_{\alpha\beta}\,, \qquad E_\alpha = -F_\alpha\,, \qquad F = -E\,, \qquad \widetilde{F} = -\widetilde{E}\,, \tag{5.98a}$$
$$C^{\alpha}{}_{\beta} = 0\,, \qquad C_\beta = 0\,, \qquad C = 0\,. \tag{5.98b}$$

This can be used to further massage the Lagrangian and equations of motion into a form that separates the metric from the non-metric modes. We will refer the interested reader to Alex (2020, 2021) for further details on the propagation of the gravitational modes in vacuo and sourced by a point particle.

The equations of motion obtained for an area metric with the help of gravitational closure – at least to linear order – can, and already have been, put to good use to analyse interesting phenomenological effects. Grosse-Holz et al. (2017) calculated effects that appear in birefringent quantum electrodynamics. For instance, Licklederer (2017) considered planetary motions in the weakly birefringent spacetime. In Schuller and Werner (2017) a derivation of the Etherington distance duality relation that connects redshift, luminosity distance and angular diameter distance was performed. One finds a Yukawa-type correction to the relation. Rieser (2020) discusses stellar models, galaxy rotation curves and structure formations in area metric spacetimes. With this, we conclude our discussion of a perturbative treatment of the gravitational closure equations.

CHAPTER 6
CONCLUSION

We started this thesis with a very specific task: obtain the dynamics of a more refined geometry than the spacetime metric. However, it quickly turned out that this is far from trivial since the naïve approach, as presented in chapter 1, to simply spell out all possible terms that may appear in the gravitational Lagrangian leads to infinitely many viable options. This makes such a theory unfeasible in practice: it is impossible to make any prediction if we need infinitely many experiments to obtain values for the parameters (constants of nature) in the Lagrangian. We thus concluded that we need a more fundamental set-up to derive gravitational dynamcis for refined spacetime geometries, which goes beyond simply constructing and adding building blocks that transform in a covariant way.

Thus, the necessity to construct gravitational theories in a more systematic way lead to the central result we were then able to derive in the following chapters 2 and 3: Suppose we restrict ourselves to a well-behaved class of matter field theories that satisfy three mild conditions that ensure that the matter fields are predictive and canonically quantisable. Then the prescribed matter dynamics already contains all the necessary information to derive the action functional of the gravitational degrees of freedom. This is due to the fact that we can always set up a canonical phase space for the tensorial geometric degrees of freedom and mimic an algebra that describes deformations of initial data hypersurfaces. Even further, one then finds that the expansion coefficients of the gravitational Lagrangian fulfil a system of linear partial differential equations called the gravitational closure equations. This is an extension of results by Kuchar (1974) and Hojman et al. (1976).

Arriving at this result required a careful analysis of the kinematical concepts involved. We analysed the causal structure of test matter coupled to a geometry we are interested in. Requiring that the equations of motion are predictive restricts the central object of these considerations, the principal polynomial, to be hyperbolic. In the derivation of this polynomial we could furthermore show in chapter 2 how to deal gauge symmetries, namely how to separate any arbitrariness arising from gauge orbits from physical null vectors. We also discussed how to detect hidden integrability condition in the equations of motion under consideration and what these mean for the according principal polynomial.

Once one has derived the polynomial that fulfils the three matter conditions, we are rewarded with all objects required to perform gravitational closure to derive the Lagrangian of a geometry. The necessary steps can be described as follows: Firstly, one needs to calculate the three input coefficients $\mathrm{p}^{\alpha\beta}$, $\mathrm{M}^{A\gamma}$

and $F^A{}_\mu{}^\gamma$ for the geometric fields on a three-dimensional space called the screen manifold Σ. Afterwards, one solves all the gravitational closure equations for the series expansion coefficients, which constitute the gravitational Lagrangian. One can then finally construct the gravitational action, derive the equations of motion and consider its interaction with matter sectors.

Suppose we, for example, start with the principal polynomial of the standard model of particle physics, i.e. a Lorentzian spacetime metric, and follow the steps laid out in this thesis. In that case, one indeed obtains general relativity, as demonstrated in chapter 4.1. Once we consider theories beyond general relativity, as was presented for the area metric of general linear electrodynamics, the complexity increases drastically. This is not only due to the non-linearity of the frame conditions – that can and were dealt with by the introduction of a parametrization of the gravitational degrees of freedom –, but mainly because of the complexity in the input coefficients one encounters. This prevents us, practically, as of now and to the best of our knowledge, from deriving an exact solution.

This is no reason to despair, though, since alternative routes are available: In particular, if we are interested in perturbative solutions around a Minkowskian background, it is possible to derive the output coefficients, now as Taylor expansion coefficients around the background solution, by solving linear *algebraic* equations. Besides developing the general theory on how to conduct such a calculation and how to obtain the general ansätze for this theory, we developed a computer program to "crunch the numbers" and successfully carried out the gravitational closure of weakly birefringent electrodynamics. The resulting linear equations of motion contain eleven gravitational constants that must be fixed by experiments.

FURTHER RESEARCH

The pool of topics from the constructive gravity programme is, of course, far from exhausted, and there is much left and worthwhile to explore. Although it will certainly not be a complete list, let us consider some of these topics that may be helpful in the future.

Collapse via causal compatibility

One of the most significant complications one deals with when solving the gravitational closure equations is the missing collapse to a finite derivative order of the degrees of freedom for the two output coefficients C and C_A (at least for theories where the input coefficient $M^{A\gamma}$ is non-vanishing). This means that *a priori* we expect the solution of these two output coefficients to contain infinitely many curvature invariants. Even further, the equations of motion then contain infinitely many spatial derivatives of our degrees of freedom.

But possibly the situation is not as bad as it seems at first sight. As described in our short discussion of the equations of motion in section 3.4.1, we saw that the principal polynomial of the gravitational sector, as obtained by gravitational closure, is not necessarily equal to the principal polynomial of the matter sector. In order to make this right, we need to impose additional conditions on the output coefficients to allow for a common canonical description of both matter and gravitational fields.

If we look at the equations of motion, we realise that due to our phase space formulation, we deal with at most *second time derivatives* but generally higher orders in *spatial directions*. This tells us that the covector $\mathrm{d}t$ is a characteristic covector of our equations of motion. This is not what matter fields see: In the chosen frame, the principal polynomial has the value

$$P\,(\mathrm{d}t) = \frac{1}{N^{\deg P}} \geq 0\,, \tag{6.1}$$

and is thus, as expected, an element from the cone C_x. This indicates that we need to decrease the maximum allowed derivative order by adding the condition that the geometric symbol of the highest derivative order vanishes to the closure equations until

$$P_G(\mathrm{d}t) \neq 0\,. \tag{6.2}$$

This certainly makes the causal analysis of the equations of motion obtained from gravitational closure worthwhile to analyse in further detail in the future.

Normal form parametrization

We have already touched upon using the parametrization invariance to bring any input setup into a normal form. The advantage is obvious: If we can, indeed, bring any theory into a form such that it is described by a metric $\mathrm{p}^{\alpha\beta}$, a vector ν^α and multiple scalar fields $\sigma^{(i)}$ on the screen manifold then solving the gravitational closure equations will consist of the very same steps for any case; all of the complexity we once had to deal with is absorbed into the diffeomorphism that maps the components of our normal form fields into the geometric fields we originally started with, as well as the single remaining input coefficient $\mathrm{M}^{A\gamma}$ that distinguishes separate spacetime phenomenologies.

Moreover, we have a great understanding of the geometrical properties of the three types of objects that appear in such a parametrization: from the projection of the principal polynomial $\mathrm{p}^{\cdot\cdot}$ we can always set up a metric compatible connection $\Gamma^{\cdot}{}_{\cdot\cdot}$ and a covariant derivative $\nabla_{\cdot}$. Furthermore, we naturally expect its Riemann tensor to always show up in the curvature invariants we construct and the final solution. This will significantly simplify the analysis of the gravitational closure equations.

Thus, further investigating these parametrizations, their perturbative dynamics, the refined Friedmann equations obtained by symmetry reduction (as laid out in Düll (2020)), as well as their possible exact solutions seems quite appealing.

Construction of curvature invariants

As we have seen in our derivation of general relativity in chapter 4.1 and the analysis of the covariance part, the solution of the selective part of the closure equations is given in terms of the curvature invariants, i.e. particular functionally independent solutions that span the space of initial data.

The advantage of this, over simply solving all of the 21 closure equations simultaneously until we end up with a solution, is that it is simpler to come up with particular solutions than integrating the vast set of differential equations. As we saw in chapter 4.1 this was in particular entirely algorithmic, which begs the question if one can, similar to the constructions from the tensorial ansätze in chapter 5, find a solution algorithm that is guaranteed to construct all of the curvature invariants.

This can be investigated quite naturally for the normal form parametrization: Restricting to second derivative order, we can count the number of curvature invariants that the separate sectors add, for instance, to the scalar output coefficient. From the metric and vector sector, we obtain three and thirty invariants, respectively. Each additional scalar degree of freedom then constitutes ten further invariants. It should be possible to construct these once and for all, which reduces the gravitational closure equations to their remaining selective part.

Gravitational radiation from a binary pulsar for area metric gravity

One of the high precision tests of general relativity is the orbital frequency shift obtained in binary pulsars such as the Hulse-Taylor PSR B1913+16. Just as in Einstein's theory, one can generally derive this effect in any theories obtained by gravitational closure. The perturbative treatment we presented in chapter 5 provides the perfect framework to, at least in principle, tackle this topic.

However, one quickly finds that we need to evaluate the closure equations up to 2nd perturbation order if we want to derive the shift in orbital frequency. Practically, this proves rather difficult as the complexity increases tremendously compared to the linear case we presented in the last chapter. Finding an expression for the orbital frequency shift in an area metric spacetime will, nonetheless, be quite rewarding and should be pursued in the future.

Parametrized post-Newtonian parameters

Last but not least, one powerful tool is the parametrized post-Newtonian (PPN) formalism that allows to capture and compare a large class of relativistic gravitational theories. It uses multiple experimental observations to constrain theories with the help of a general test metric of the form

$$\begin{aligned} g_{00} &= -1 + 2\,U - 2\beta\,U^2 - 2\,\xi\,\Phi_W + (2 + 2\,\gamma + \alpha_3 + \zeta_1 - 2\,\xi)\,\Phi_1 \\ &\quad + 2\,(1 + 3\,\gamma - 2\,\beta + \zeta_2 + \xi)\,\Phi_2 + 2\,(1 + \zeta_3)\,\Phi_3 + 2\,(3\,\gamma + 3\,\zeta_4 - 2\,\xi)\,\Phi_4 \\ &\quad - (\zeta_1 - 2\,\xi)\,A - (\alpha_1 - \alpha_2 - \alpha_3)\,w^2 U - \alpha_2\, w^\alpha w^\beta U_{\alpha\beta} + (2\,\alpha_3 - \alpha_1)\,w^\alpha V_\alpha + \mathcal{O}(\varepsilon^3)\,, \qquad (6.3a)\\ g_{0\alpha} &= -\frac{1}{2}\,(3 + 4\,\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\,\xi)\,V_\alpha - \frac{1}{2}\,(1 + \alpha_2 - \zeta_1 + 2\,\xi)\,W_\alpha \\ &\quad - \frac{1}{2}\,(\alpha_1 - 2\,\alpha_2)\,w_\alpha U + \alpha_2\,w^\beta U_{\alpha\beta} + \mathcal{O}(\varepsilon^{\frac{5}{2}})\,, \qquad (6.3b)\\ g_{\alpha\beta} &= (1 + 2\,\gamma\,U)\,\delta_{\alpha\beta} + \mathcal{O}(\varepsilon^2)\,, \qquad (6.3c) \end{aligned}$$

with the ten metric potentials U, $U_{\alpha\beta}$, Φ_W, Φ_1, Φ_2, Φ_3, Φ_4, V_α, W_α and A, the velocity vector w^α of the PPN coordinate system, relative to the mean rest-frame of the universe, and the ten PPN parameters $\gamma, \beta, \alpha_1, \ldots, \alpha_3, \zeta_1, \ldots, \zeta_4$ and ξ. General relativity is, of course, also contained in this formalism for $\beta = \gamma = 1$ and the remaining parameters being zero. For further details, see for example Will (1993).

In chapter 2 we saw that the equations of motion of a massive particle in the geometrical optical limit is a geodesics equation for the Finsler metric constructed as

$$g_{mn}(x,v)u^m w^n := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\,\left(P^\star(x, v + s\cdot u + t\cdot w)\right)^{2/\deg P}\Big|_{s=0,t=0}\,. \qquad (6.4)$$

Parameter	Bound	Effects	Experiment
$\gamma - 1$	2.3×10^{-5}	Time delay, light deflection	Cassini tracking
$\beta - 1$	8×10^{-5}	Perihelion shift	Perihelion shift
	2.3×10^{-4}	Nordtvedt effect with assumption $\eta_N = 4\beta - \gamma - 3$	Nordtvedt effect
ξ	4×10^{-9}	Spin precession	Millisecond pulsars
α_1	1×10^{-4}	Orbital polarization	Lunar laser ranging
	4×10^{-5}	Orbital polarization	PSR J1738+0333
α_2	2×10^{-9}	Spin precession	Millisecond pulsars
α_3	4×10^{-20}	Self-acceleration	Pulsar spin-down statistics
η_N	9×10^{-4}	Nordtvedt effect	Lunar laser ranging
ζ_1	0.02	Combined PPN bounds	–
ζ_2	4×10^{-5}	Binary-pulsar acceleration	PSR 1913+16
ζ_3	1×10^{-8}	Newton's 3rd law	Lunar acceleration
ζ_4	0.006	–	Kreuzer experiment

Table 6.1 Experimental bounds on parametrized post-Newtonian parameters

Similarly, we expect that it is possible to extend the pressureless dust that was presented for any principal polynomial of even degree in Witte (2014) and derive the canonical Gotay-Marsden energy-momentum tensor of a perfect fluid. Also, efforts have already been made to derive cosmological equations of motion with the help of gravitational closure that, once combined, should allow one to calculate the parametrized post-Newtonian parameters in the constructive gravity framework.

This is intriguing as it would allow us to immediately relate theories obtained by gravitational closure, such as the area metric, to observational bounds obtained for the ten PPN parameters, such as the ones presented in table 6.1).

BIBLIOGRAPHY

Alex, N. (2020). Solutions of gravitational field equations for weakly birefringent spacetimes. In *15th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories*.

Alex, N. (2021). PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg. in preparation.

Arnowitt, R., Deser, S., and Misner, C. W. (1959). Dynamical structure and definition of energy in general relativity. *Phys. Rev.*, 116:1322–1330.

Beier, U. (2018). Perturbative gravitational closure of a bi-metric matter theory. Master's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

Bernal, A. N. and Sánchez, M. (2003). On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem. *Communications in Mathematical Physics*, 243(3):461–470.

Bouillot, V. R., Alimi, J.-M., Corasaniti, P.-S., and Rasera, Y. (2015). Probing dark energy models with extreme pairwise velocities of galaxy clusters from the DEUS-FUR simulations. *Mon. Not. Roy. Astron. Soc.*, 450:145–159.

Bruni, M., Matarrese, S., Mollerach, S., and Sonego, S. (1997). Perturbations of space-time: Gauge transformations and gauge invariance at second order and beyond. *Class. Quant. Grav.*, 14:2585–2606.

Bullock, J. S. and Boylan-Kolchin, M. (2017). Small-Scale Challenges to the ΛCDM Paradigm. *Ann. Rev. Astron. Astrophys.*, 55(1):343–387.

Butler, G. (1982). *On computing double coset representatives in permutation groups*. Basser Dept. of Computer Science, University of Sydney Sydney.

Caponio, E. and Masiello, A. (2019). Harmonic Coordinates for the Nonlinear Finsler Laplacian and Some Regularity Results for Berwald Metrics. *arXiv e-prints*, page arXiv:1907.09826.

Chae, K.-H., Desmond, H., Lelli, F., McGaugh, S. S., and Schombert, J. M. (2021). Testing the Strong Equivalence Principle. II. Relating the External Field Effect in Galaxy Rotation Curves to the Large-Scale Structure of the Universe.

Chae, K.-H., Lelli, F., Desmond, H., McGaugh, S. S., Li, P., and Schombert, J. M. (2020). Testing the Strong Equivalence Principle: Detection of the External Field Effect in Rotationally Supported Galaxies. *Astrophys. J.*, 904(1):51. [Erratum: Astrophys.J. 910, 81 (2021)].

Dirac, P. (1964). *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, monograph series. Dover Publications.

Dirac, P. A. M. (1950). Generalized hamiltonian dynamics. *Canadian Journal of Mathematics*, 2:129–148.

Dirac, P. A. M. (1958). Generalized hamiltonian dynamics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 246(1246):326–332.

Douglas, J. (1939). Solution of the Inverse Problem of the Calculus of Variations. *Proceedings of the National Academy of Science*, 25(12):631–637.

Drummond, I. T. and Hathrell, S. J. (1980). Qed vacuum polarization in a background gravitational field and its effect on the velocity of photons. *Phys. Rev. D*, 22:343–355.

Düll, M. (2020). *Gravitational closure of matter field equations: General theory & symmetrization*. PhD thesis, Universität Heidelberg.

Düll, M., Fischer, N. L., Schaefer, B. M., and Schuller, F. P. (2020). Symmetric gravitational closure.

Edgar, S. B. and Höglund, A. (2002). Dimensionally dependent tensor identities by double antisymmetrization. *Journal of Mathematical Physics*, 43(1):659–677.

Einstein, A. (1915a). Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1915:831–839.

Einstein, A. (1915b). Zur Allgemeinen Relativitätstheorie. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1915:778–786. [Addendum: Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.) 1915, 799–801 (1915)].

Giulini, D. (2009). The Superspace of Geometrodynamics. *Gen. Rel. Grav.*, 41:785–815.

Gotay, M. J. and Marsden, J. E. (2001). Stress-energy-momentum tensors and the belifante-resenfeld formula. pages 367–392.

Grosse-Holz, S., Schuller, F. P., and Tanzi, R. (2017). Gravitational effects in birefringent quantum electrodynamics. *arXiv e-prints*, page arXiv:1703.07183.

Hassan, S. F. and Rosen, R. A. (2012). Bimetric Gravity from Ghost-free Massive Gravity. *JHEP*, 02:126.

Hehl, F. and Obukhov, Y. (2003). *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*. Progress in Mathematical Physics. Birkhäuser Boston.

Hehl, F. W., Obukhov, Y. N., and Rubilar, G. F. (1999). Space-time metric from linear electrodynamics. 2. In *International European Conference on Gravitation: Journees Relativistes 99*.

Hojman, S. A., Kuchar, K., and Teitelboim, C. (1976). Geometrodynamics regained. *Annals of Physics*, 96(1):88 – 135.

Holst, S. (1996). Barbero's hamiltonian derived from a generalized hilbert-palatini action. *Phys. Rev. D*, 53:5966–5969.

Hörmander, L. (1955). On the theory of general partial differential operators. *Acta Mathematica*, 94(1):161–248.

Isham, C. J. and Kuchar, K. V. (1985a). Representations of Space-time Diffeomorphisms. 1. Canonical Parametrized Field Theories. *Annals Phys.*, 164:288.

Isham, C. J. and Kuchar, K. V. (1985b). Representations of Space-time Diffeomorphisms. 2. Canonical Geometrodynamics. *Annals Phys.*, 164:316.

Itin, Y. (2009). On light propagation in premetric electrodynamics: the covariant dispersion relation. *Journal of Physics A: Mathematical and Theoretical*, 42(47):475402.

Klusoň, J. (2014). Hamiltonian Formalism of Bimetric Gravity In Vierbein Formulation. *Eur. Phys. J.*, C74(8):2985.

Kostelecký, V. A. and Mewes, M. (2002). Signals for lorentz violation in electrodynamics. *Phys. Rev. D*, 66:056005.

Kostelecký, V. A. and Mewes, M. (2006). Sensitive polarimetric search for relativity violations in gamma-ray bursts. *Phys. Rev. Lett.*, 97:140401.

Kroupa, P. (2015). Galaxies as simple dynamical systems: observational data disfavor dark matter and stochastic star formation. *Canadian Journal of Physics*, 93(2):169–202.

Kuchar, K. (1974). Geometrodynamics regained: A Lagrangian approach. *Journal of Mathematical Physics*, 15(6):708–715.

Kuranishi, M. (1957). On E. Cartan's Prolongation Theorem of Exterior Differential Systems. *American Journal of Mathematics*, 79(1):1–47.

Lämmerzahl, C. and Hehl, F. W. (2004). Riemannian light cone from vanishing birefringence in premetric vacuum electrodynamics. *Phys. Rev. D*, 70:105022.

Licklederer, S. (2017). Planetary motion in a weakly birefringent spacetime. Bachelor's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

Lovelock, D. (1970). Dimensionally dependent identities. *Mathematical Proceedings of the Cambridge Philosophical Society*, 68(2):345–350.

Lovelock, D. (1972). The four-dimensionality of space and the Einstein tensor. *J. Math. Phys.*, 13:874–876.

Mansuroglu, R. (2018). Second order perturbative solution to the gravitational closure equations of general linear electrodynamics. Bachelor's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

McGaugh, S. S., Lelli, F., and Schombert, J. M. (2016). Radial acceleration relation in rotationally supported galaxies. *Phys. Rev. Lett.*, 117:201101.

Milgrom, M. (1994). Dynamics with a nonstandard inertia-acceleration relation: An alternative to dark matter in galactic systems. *Annals of Physics*, 229(2):384–415.

Nakamura, K. (2007). Second-order gauge invariant cosmological perturbation theory: Einstein equations in terms of gauge invariant variables. *Prog. Theor. Phys.*, 117:17–74.

Nakamura, K. (2012). Gauge-invariant variables in general-relativistic perturbations: Globalization and zero-mode problem. *Int. J. Mod. Phys. D*, 21:1242004.

Ostrogradsky, M. (1850). Mémoires sur les équations différentielles, relatives au problème des isopérimètres. *Mem. Acad. St. Petersbourg*, 6(4):385–517.

Ouchterlony, E. (1997). A systematic approach to matrix invariants. Master's thesis, Matematiska institutionen Linköpings universitet, Sweden.

Palatini, A. (1919). Deduzione invariantiva delle equazioni gravitazionali dal principio di hamilton. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 43:203–212.

Pawlowski, M. S. (2018). The Planes of Satellite Galaxies Problem, Suggested Solutions, and Open Questions. *Mod. Phys. Lett. A*, 33(06):1830004.

Perivolaropoulos, L. and Skara, F. (2021). Challenges for ΛCDM: An update.

Piña, P. E. M., Fraternali, F., Oosterloo, T., Adams, E. A. K., Oman, K. A., and Leisman, L. (2021). No need for dark matter: resolved kinematics of the ultra-diffuse galaxy AGC 114905.

Piña, P. E. M., Fraternali, F., Adams, E. A. K., Marasco, A., Oosterloo, T., Oman, K. A., Leisman, L., di Teodoro, E. M., Posti, L., Battipaglia, M., Cannon, J. M., Gault, L., Haynes, M. P., Janowiecki, S., McAllan, E., Pagel, H. J., Reiter, K., Rhode, K. L., Salzer, J. J., and Smith, N. J. (2019). Off the baryonic tully–fisher relation: A population of baryon-dominated ultra-diffuse galaxies. *The Astrophysical Journal*, 883(2):L33.

Raetzel, D., Rivera, S., and Schuller, F. P. (2011). Geometry of physical dispersion relations. *Phys. Rev. D*, 83:044047.

Reiß, D. (2014). Covariance equations for tensors with symmetries: a general and practical approach via intertwiners. Bachelor's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

Rieser, H.-M. (2020). *Cosmological and Astrophysical Tests of Constructive Gravity*. PhD thesis, Universität Heidelberg.

Rivera, S. (2012). *Tensorial spacetime geometries carrying predictive, interpretable and quantizable matter dynamics*. PhD thesis, Universität Potsdam.

Rivera, S. and Schuller, F. P. (2011). Quantization of general linear electrodynamics. *Phys. Rev. D*, 83:064036.

Rubilar, G. (2002). Linear pre-metric electrodynamics and deduction of the light cone. *Annalen der Physik*, 11(10-11):717–782.

Rubilar, G. F., Obukhov, Y. N., and Hehl, F. W. (2002). Generally covariant fresnel equation and the emergence of the light cone structure in linear pre-metric electrodynamics. *International Journal of Modern Physics D*, 11(08):1227–1242.

Schneider, J. (2017). Linearized gravitational dynamics for geometries modifying etherington's distance duality relation. Master's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

Schneider, J., Schuller, F. P., Stritzelberger, N., and Wolz, F. (2017). Gravitational closure of weakly birefringent electrodynamics.

Schuller, F. and Witte, C. (2014). How quantizable matter gravitates: A practitioner's guide. *Physical Review D*, 89.

Schuller, F. P. and Werner, M. C. (2017). Etherington's distance duality with birefringence. *Universe*, 3(3).

Schuller, F. P., Witte, C., and Wohlfarth, M. N. (2010). Causal structure and algebraic classification of area metric spacetimes in four dimensions. *Annals Phys.*, 325:1853–1883.

Schwarzschild, K. (1916). Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin*, pages 189–196.

Seiler, W. (1995). On the arbitrariness of the general solution of an involutive partial differential equation. *Journal of Mathematical Physics*, 35.

Seiler, W. (2009). *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*. Algorithms and Computation in Mathematics. Springer Berlin Heidelberg.

Seiler, W. and Tucker, R. (1999). Involution and constrained dynamics i: The dirac approach. *Journal of Physics A: Mathematical and General*, 28:4431.

Seiler, W. M. (1994). *Analysis and Application of the Formal Theory of Partial Differential Equations*. PhD thesis, Lancaster University.

Slattery, M. C. (2001). Computing double cosets in soluble groups. *J. Symb. Comput.*, 31(1):179–192.

Sneddon, G. E. (1996). The identities of the algebraic invariants of the four-dimensional riemann tensor. *Journal of Mathematical Physics*, 37(2):1059–1075.

Sneddon, G. E. (1998). The identities of the algebraic invariants of the four-dimensional riemann tensor. ii. *Journal of Mathematical Physics*, 39(3):1659–1679.

Stritzelberger, N. (2016). Perturbative canonical gravitational dynamics for birefringent spacetimes. Master's thesis, Friedrich-Alexander-Universität Erlangen Nürnberg.

Taylor, J. H. and Weisberg, J. M. (1982). A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16. *Astrophysical Journal*, 253:908–920.

Taylor, J. H. and Weisberg, J. M. (1989). Further Experimental Tests of Relativistic Gravity Using the Binary Pulsar PSR 1913+16. *Astrophysical Journal*, 345:434.

Teitelboim, C. (1973). How commutators of constraints reflect the spacetime structure. *Annals of Physics*, 79(2):542–557.

Velo, G. and Zwanzinger, D. (1969). Noncausality and other defects of interaction lagrangians for particles with spin one and higher. *Phys. Rev.*, 188:2218–2222.

Weisberg, J. M. and Taylor, J. H. (2005). Relativistic binary pulsar B1913+16: Thirty years of observations and analysis. *ASP Conf. Ser.*, 328:25.

Wierzba, A. (2018). Parametrodynamics. Master's thesis, Ludwig-Maximilians-Universität München.

Will, C. M. (1993). *Theory and Experiment in Gravitational Physics*.

Witte, C. (2009). Algebraic classification and causal structure of area metric spacetimes in four dimensions. Bachelor's thesis, Humboldt Universität zu Berlin.

Witte, C. (2014). *Gravity actions from matter actions*. PhD thesis, Humboldt-Universität zu Berlin.

Wolz, F. (2018). Construct. `https://github.com/constructivegravity/construct`.

Wolz, F. (2021). Causal Structure of Matter Field Equations (to appear). *Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity*.

Woodard, R. P. (2015). Ostrogradsky's theorem on Hamiltonian instability. *Scholarpedia*, 10(8):32243.

LIST OF PUBLICATIONS

Parts of the results presented in this thesis have already been published in the following articles and conference proceedings

M. Düll, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational closure of matter field equations
Phys. Rev. D97 (2018), 084036

J. Schneider, F. P. Schuller, N. Stritzelberger and F. Wolz
Gravitational Closure of Weakly Birefringent Electrodynamics
arXiv: 1708.03870 (2017)

F. Wolz
Causal Structure of Matter Field Equations
to appear in: Proceedings of the 15th Marcel Grossmann Meeting on General Relativity

Parts of chapter 3.2, in particular the derivation of the gravitational closure equations in section 3.3 are joint work with Maximilian Düll. The results were also presented, next to the publications named above, in his PhD thesis

M. Düll
Gravitational Closure of Matter Field Equations: General Theory & Symmetrization
Universität Heidelberg

It was indicated in each chapter if its content is subject to one of the listed publications.

CURRICULUM VITAE

EDUCATION

- **PhD candidate** *since Oct. 2015*
  - Topic: General Solutions to Gravitational Closure Equations
  - Supervisors: Prof. Dr. Domenico Giulini, Prof. Dr. Frederic P. Schuller
- **Master of Science (M.Sc. hon.)** *Apr. 2014 – Sep. 2015*
  Friedrich-Alexander University Erlangen-Nuremberg
  - Course of studies: Elite Graduate Program “Physics with integrated Doctorate Program” (Elite Network of Bavaria)
  - Master Thesis: On spatially diffeomorphism invariant representations of the bosonic string
  - Supervisor: Prof. Dr. Hanno Sahlmann
- **Bachelor of Science (B.Sc.)** *May 2011 – Apr. 2014*
  Friedrich-Alexander University Erlangen-Nuremberg
  - Course of studies: Elite Graduate Program “Physics with integrated Doctorate Program” (Elite Network of Bavaria)
  - Bachelor Thesis: Geometric Meaning of the Penrose Metric
  - Supervisor: Prof. Dr. Hanno Sahlmann
- **Abitur** *Sep. 2001 – May 2015*
  Sigmund-Schuckert-Gymnasium Nuremberg

WORK EXPERIENCE

- **Siemens Healthineers AG** Forchheim
  Detector Physicist *since May 2019*
- **Siemens Healthineers AG** Forchheim
  Working Student *Sep. 2017 – Apr. 2019*
- **Statistical Physics and Thermodynamics for teaching students** Friedrich-Alexander University
  Teaching Assistent *Mar. 2017 – Jul. 2017*
- **Statistical Physics and Thermodynamics, Theoretical Physics** Friedrich-Alexander University
  Teaching Assistent *Oct. 2016 – Feb. 2017*
- **Theoretical Quantum Mechanics for teaching students** Friedrich-Alexander University
  Teaching Assistent *Oct. 2015 – Feb. 2016, Oct. 2014 – Feb. 2015*

FELLOWSHIPS

- Elite Network of Bavaria *since Mar. 2012*
- Leonardo-Kolleg Erlangen *Sep. 2014 – Mar. 2016*

TALKS

- **Causal structure of matter field equations** *Jul. 2018*
  Marcel-Grossmann Meeting 2018, University of Rome La-Sapienza
- **Pre-metric 2-form gravity** *Mar. 2018*
  DPG Tagung 2018, University of Würzburg
- **Gravitational dynamics beyond the standard model – a case study** *Aug. 2016*
  ICPS 2016, University of Malta
- **Deriving Gravitational Lagrangians from Matter Dynamics** *Feb. 2016*
  Shanghai Jiaotong University, China
- **The Relativistic Light Transport equation: Rendering a black-hole** *Jan. 2015*
  University of Regensburg, Germany

ACKNOWLEDGEMENTS

First of all, dear Nico, I want to thank you for your support and for making it possible to bring this thesis to an end!

Dear Frederic, thank you for so many invaluable lessons: how to thoroughly approach and realise a project, how to consider everything from different, and maybe sometimes unusual, angles and the importance of telling a story. I will forever cherish our collaboration, this research project and your supervision. *In aequationibus veritas*.

Dear Max, it is crazy that we have already known each other for over ten years! Thank you for all of our discussions, the many study trips and conferences we visited, and the time we shared during our theses; it has always been a pleasure.

Dear Eileen, thank you for your thorough and valuable comments on my drafts and interest in this research program.

It was a delight to work with all my fellow partners in crime – Roberto, Nadine, Alexander, Simon, Moritz, Jonas, Refik, Udo, Stefan, Nils F. Hans-Martin and Nils A. – at the constructive gravity group. Our group meetings, seminars, coffee breaks, all our discussions, as well as our occasional trips to the monastery have always been enormously productive and educational.

I also want to thank Prof. Dr. Björn-Malte Schäfer for being a referee of my thesis and Prof. Dr. Elmar Schrohe for being the head of my examination committee.

Great thanks also to my colleagues at Siemens Healthineers for their support and guidance in switching from the purely theoretical field of differential geometry to the almost entirely hands-on integral geometry of computer tomography systems. The last three years of work have been challenging, intriguing and incredibly instructive.

To all my fellow iGEMers from the years 2017 – 2019, most notably, Franzi, Selina, Alex, Moritz, Verena, David and Sirka: thank you for the rollercoaster ride of emotions that were our sometimes exhausting but also tremendously informative projects and the incredible fun we had in the past years.

Last but not least, I want to thank my family for their endless support, patience and adamant belief in me. None of this would have been possible without you.