TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 1135-1144

SOME MEAN SQUARE INTEGRAL INEQUALITES INVOLVING THE BETA FUNCTION AND GENERALIZED CONVEX STOCHATIC PROCESSES

J. E. H. HERNÁNDEZ[1], J. F. GÓMEZ[2], §

Abstract. In the present work some integral inequalities which involve the Beta function for stochastic processes whose absolute values posses the property of convexity, or $P-$convexity or $s-$convexity in the second sense are established. In the same way some others integral inequalities for stochastic processes whose $k-$th powers of its absolute values posses these kind of generalized convexity are established making use of the Hölder's inequality and power mean inequality.

Keywords: Mean square integral inequalities, convex stochastic processes, beta function.

AMS Subject Classification: 35A23, 60G20, 52A01.

1. Introduction

The study of convex functions has been of interest for mathematical analysis because of the properties that are deduced from this concept.

Due to generalization requirements of the convexity concept in order to obtain new applications, in the last years great efforts have been made in the study and investigation of this topic.

A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds.

Numerous research works extending results on inequalities for convex functions towards others much more generalized have been realized, using new concepts such as quasi-convexity [17], $s-$convexity [3], logarithmically convexity [1], $\eta-$convexity [26], $m-$convexity

[1] Universidad Centroccidental Lisandro Alvarado, Decanato de Ciencias Económicas y Empresariales, Departamento de Técnicas Cuantitativas, Barquisimeto, Venezuela.
e-mail: jorgehernandez@ucla.edu.ve; ORCID: https://orcid.org/0000-0002-4406-5469.

[2] Universidad Centroccidental Lisandro Alvarado, Decanato de Ciencias Económicas y Empresariales, Dirección de Investigación, Barquisimeto, Venezuela.
e-mail: juanfranciscogomez@ucla.edu.ve; ORCID: https://orcid.org/0000-0001-9939-9602.

§ Manuscript received: June 17, 2020; accepted: October 28, 2020.

Both authors are supported by the “Consejo de Desarrollo de las Ciencias, Humanidades y Tecnología (CDCHT)” from Universidad Centroccidental Lisandro Alvarado, Venezuela.

[15], $(s, \eta)-$convexity [24], $(s, m)-$convexity in the second sense [25], convexity in coordinates [11], $p-$harmonic log-convexity [27], $h-$convexity [28], etc.

The study on convex stochastic processes began in 1974 when B. Nagy applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation [13]. In 1980 Nikodem considered convex stochastic processes [14]. In 1995 A. Skowronski obtained some further results on convex stochastic processes, which generalize some known properties of convex functions [21]. From that moment many researchers began to merge the properties of generalized convexity with the stochastic processes. By example, in the year 2014, E. Set et. al. investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense [16], in 2015 M. Tomar et. al. worked on log-convex stochastic processes [23], recently, in 2018, the authors introduced the concept of $(m, h_1, h_2)-$convex stochastic processes and they related it to some inequalities for fractional integrals [8]. For other results related to stochastic processes see [4],[6],[7],[12],[18] and [19], where further references are given.

Following this line of research, in the present work we propose to find some integral inequalities that involve the Beta function and the stochastic processes which absolute value are covex, $P-$convex or $s-$convex in the second sense.

2. Preliminaries

Definition 2.1. *Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. Let $I \subset \mathbb{R}$ be time. A collection of random variable $X(t, w), t \in I$ with values in $\mathbb{R}$ is called a stochastic processes.*

(1) *If $X(t, w)$ takes values in $S = \mathbb{R}^d$, it is called vector-valued stochastic process.*
(2) *If the time I is a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process.*
(3) *If the time I is an interval in $\mathbb{R}$, it is called a stochastic process with continuous time.*

Definition 2.2. *Let (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called*

(1) *Continuous in probability in interval I if for all $t_0 \in I$ we have*
$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$
where $P - \lim$ denotes the limit in probability.
(2) *Mean-square continuous in the interval I if for all $t_0 \in I$*
$$P - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$
where $\mathbb{E}(X(t, \cdot))$ denotes the expectation value of the random variable $X(t, \cdot)$.
(3) *Increasing (decreasing) if for all $u, v \in I$ such that $t < s$,*
$$X(u, \cdot) \leq X(v, \cdot), \qquad (X(u, \cdot) \geq X(v, \cdot)).$$
(4) *Monotonic if it's increasing or decreasing.*
(5) *Differentiable at a point $t \in I$ if there exists a random variable $X'(t, \cdot) : I \times \Omega \to \mathbb{R}$, such that*
$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I (See [9], [12],[21]).

Definition 2.3. *Let (Ω, A, P) be a probability space $I \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in I$. Let $[a,b] \subset I, a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a,b]$ if the following identity holds:*

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{\infty} X(\theta_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right]^2 = 0,$$

then we can write

$$\int_a^b X(t, \cdot)dt = Y(\cdot) \qquad (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot)dt \le \int_a^b Z(t, \cdot)dt \qquad (a.e.)$$

where $X(t, \cdot) \le Z(t, \cdot)$ in $[a,b]$.

For further reading on stochastic calculus, reader may refer to [19] and [22].

The following definition can be found in the works of D. Kotrys [10], E. Set [16] and A. Skowronski [20].

Definition 2.4. *Set $(\Omega, \mathcal{A}, P)$ be a probability space and $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is*

(1) *Convex if the inequality*

$$X(\lambda u + (1-\lambda)v, \cdot) \le \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) \tag{1}$$

holds almost everywhere for all $u, v \in I$ and $\lambda \in [0,1]$.

(2) *$P-$convex if the inequality*

$$X(\lambda u + (1-\lambda)v, \cdot) \le X(u, \cdot) + X(v, \cdot) \tag{2}$$

holds almost everywhere for all $u, v \in I$ and $\lambda \in [0,1]$.

(3) *$s-$convex in the second sense if the inequality*

$$X(\lambda u + (1-\lambda)v, \cdot) \le \lambda^s X(u, \cdot) + (1-\lambda)^s X(v, \cdot) \tag{3}$$

holds almost everywhere for all $u, v \in I$ and $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Also, in the development of this work we use the Beta function wich is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dy, \qquad Re(x) > 0, Re(y) > 0.$$

3. Main Results

Lemma 3.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Then the equality*

$$\int_a^b (u-a)^p (b-u)^q X(u, \cdot)du = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q X(ta + (1-t)b, \cdot)\, dt$$

holds for some fixed $p, q > 0$.

Proof. Let $u = ta + (1-t)b$. Then $t = (b-u)/(b-a)$, $1-t = (u-a)/(b-a)$ and $dt = -du/(b-a)$, so

$$\int_0^1 (1-t)^p t^q X(ta + (1-t)b, \cdot)\, dt = \frac{1}{(b-a)^{p+q+1}} \int_a^b (u-a)^p (b-u)^q X(u, \cdot)\, du.$$

The proof is complete. □

The following results are established for convex stochastic processes.

Theorem 3.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|$ is convex on $[a,b]$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^p (b-u)^q X(u,\cdot) du \\ &\quad \le (b-a)^{p+q+1} \left(B(p+1, q+2) |X(a,\cdot)| + B(p+2, q+1) |X(b,\cdot)| \right). \end{aligned} \tag{4}$$

Proof. Using Lemma 3.1, the definition of the Beta function and the convexity of $|X|$, we have

$$\begin{aligned} &\int_a^b (u-a)^p (b-u)^q X(u,\cdot) du \\ \le\ & (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta + (1-t)b, \cdot)|\, dt \\ \le\ & (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q (t |X(a,\cdot)| + (1-t) |X(b,\cdot)|)\, dt \\ \le\ & (b-a)^{p+q+1} \left(B(p+1, q+2) |X(a,\cdot)| + B(p+2, q+1) |X(b,\cdot)| \right). \end{aligned}$$

The proof is complete □

Theorem 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|^l$ is convex on $[a,b]$ for $l > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^p (b-u)^q X(u,\cdot) du \\ &\le (2)^{-1/l} (b-a)^{p+q+1} [B(kp+1, kq+1)]^{1/k} \left(|X(a,\cdot)|^l + |X(b,\cdot)|^l \right)^{1/l}, \end{aligned} \tag{5}$$

where $(1/k) + (1/l) = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\begin{aligned} &\int_a^b (u-a)^p (b-u)^q X(u,\cdot) du \\ &\quad \le (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta + (1-t)b, \cdot)|\, dt \\ &\le (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{kp} t^{kq} dt \right)^{1/k} \left(\int_0^1 |X(ta + (1-t)b, \cdot)|^l\, dt \right)^{1/l}. \end{aligned} \tag{6}$$

Since $|X|^l$ is a convex stochastic process then

$$\int_0^1 |X(ta + (1-t)b, \cdot)|^l\, dt \le \int_0^1 t |X(a,\cdot)|^l + (1-t) |X(b,\cdot)|^l\, dt \tag{7}$$

$$= \frac{|X(a,\cdot)|^l + |X(b,\cdot)|^l}{2},$$

and using the definition of the Beta function we get

$$\int_0^1 (1-t)^{kp}\, t^{kq} dt = B(kp+1, kq+1).$$

So replacing (7) and (8) in (6) it is attained the required inequality (5).
The proof is complete. $\square$

Theorem 3.3. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|^r$ is convex on $[a,b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^p\,(b-u)^q\, X(u,\cdot)du \\ &\quad \le (b-a)^{p+q+1}\,[B(p+1,q+1)]^{1-1/r} \times \\ &\qquad \left(B(p+1,q+2)\,|X(a,\cdot)|^r + B(p+2,q+1)\,|X\,(b,\cdot)|^r\right)^{1/r}. \end{aligned} \tag{8}$$

Proof. From Lemma 3.1 and using the power mean inequality for $r \ge 1$ we have

$$\begin{aligned} &\int_a^b (u-a)^p\,(b-u)^q\, X(u,\cdot)du \\ &\le (b-a)^{p+q+1}\int_0^1 (1-t)^p\, t^q\, |X\,(ta+(1-t)b,\cdot)|\, dt \\ &\le (b-a)^{p+q+1}\left(\int_0^1 (1-t)^p\, t^q dt\right)^{1-1/r} \times \\ &\qquad \left(\int_0^1 (1-t)^p\, t^q\, |X\,(ta+(1-t)b,\cdot)|^r\, dt\right)^{1/r}. \end{aligned}$$

Making use of the convexity of the stochastic process $|X|^r$ and the definition the Beta function, we get

$$\begin{aligned} &\int_0^1 (1-t)^p\, t^q\, |X\,(ta+(1-t)b,\cdot)|^r\, dt \\ &\quad \le \int_0^1 (1-t)^p\, t^q\,(t\,|X\,(a,\cdot)|^r + (1-t)\,|X\,(b,\cdot)|^r)\, dt \\ &\quad = B(p+1,q+2)\,|X(a,\cdot)|^r + B(p+2,q+1)\,|X\,(b,\cdot)|^r. \end{aligned} \tag{9}$$

Replacing (9) in the previous inequality it is attained the desired inequality (8).
The proof is complete. $\square$

The following results are established for $P-$convex stochastic processes.

Theorem 3.4. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|$ is $P-$convex on $[a,b]$ where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^p\,(b-u)^q\, X(u,\cdot)du \\ &\quad \le (b-a)^{p+q+1}\, B(p+1,q+1)\,(|X(a,\cdot)| + |X\,(b,\cdot)|). \end{aligned} \tag{10}$$

Proof. Using Lemma 3.1 , the definition of the Beta function and the $P-$convexity of $|X|$, we have

$$
\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left|X\left(ta+(1-t)b,\cdot\right)\right| dt \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left(\left|X\left(a,\cdot\right)\right| + \left|X\left(b,\cdot\right)\right|\right) dt \\
&= \left(\left|X\left(a,\cdot\right)\right| + \left|X\left(b,\cdot\right)\right|\right)(b-a)^{p+q+1} \int_0^1 (1-t)^p t^q dt \\
&= (b-a)^{p+q+1} B(p+1,q+1)\left(\left|X(a,\cdot)\right| + \left|X\left(b,\cdot\right)\right|\right).
\end{aligned}
$$

The proof is complete □

Theorem 3.5. *Let $X : I\times\Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q > 0$, if $|X|^r$ is $P-$convex on $[a,b]$ for $r > 1$, where $a,b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$
\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} \left[B(lp+1,lq+1)\right]^{1/l} \left(\left|X(a,\cdot)\right|^r + \left|X\left(b,\cdot\right)\right|^r\right)^{1/r},
\end{aligned}
\tag{11}
$$

where $(1/r)+(1/l)=1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$
\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left|X\left(ta+(1-t)b,\cdot\right)\right| dt \\
&\leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{lp} t^{lq} dt\right)^{1/l} \left(\int_0^1 \left|X\left(ta+(1-t)b,\cdot\right)\right|^r dt\right)^{1/r}.
\end{aligned}
\tag{12}
$$

Since $|X|^r$ is $P-$convex Stochastic process then

$$
\int_0^1 \left|X\left(ta+(1-t)b,\cdot\right)\right|^r dt \leq \left|X(a,\cdot)\right|^r + \left|X\left(b,\cdot\right)\right|^r, \tag{13}
$$

and using the definition of the Beta function we get

$$
\int_0^1 (1-t)^{lp} t^{lq} dt = B(lp+1,lq+1). \tag{14}
$$

So replacing (13) and (14) in (12) it is attained the required inequality (11).

The proof is complete. □

Theorem 3.6. *Let $X : I\times\Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q > 0$, if $|X|^r$ is $P-$convex on $[a,b]$ for $r > 1$, where $a,b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$
\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} B(sp+1,sq+1) \left(\left|X(a,\cdot)\right|^r + \left|X\left(b,\cdot\right)\right|^r\right)^{1/r}.
\end{aligned}
\tag{15}
$$

Proof. From Lemma 3.1 and using the power mean inequality for $r \geq 1$ and the $P-$convexity of $|X|^r$ we have

$$\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)| dt \\
&\leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^p t^q \right)^{1-1/r} \left(\int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)|^r dt \right)^{1/r} \\
&\leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^p t^q \right)^{1-1/r} \times \\
&\qquad (|X(a,\cdot)|^r + |X(b,\cdot)|^r)^{1/r} \left(\int_0^1 (1-t)^p t^q dt \right)^{1/r} \\
&= (b-a)^{p+q+1} B(sp+1,sq+1) (|X(a,\cdot)|^r + |X(b,\cdot)|^r)^{1/r}.
\end{aligned}$$

The proof is complete. □

The following results are established for $s-$convex stochastic processes.

Theorem 3.7. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|$ is $s-$convex in the second sense on $[a,b]$ for some $s \in (0,1]$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \qquad (16)\\
&\leq (b-a)^{p+q+1} (|X(a,\cdot)| B(p+1,q+s+1) + |X(b,\cdot)| B(p+s+1,q+1)).
\end{aligned}$$

Proof. Using Lemma 3.1, the definition of the Beta function and the $s-$convexity of $|X|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)| dt \\
&\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q (t^s |X(a,\cdot)| + (1-t)^s |X(b,\cdot)|) dt \\
&\leq (b-a)^{p+q+1} \left(|X(a,\cdot)| \int_0^1 (1-t)^p t^{q+s} dt + |X(b,\cdot)| \int_0^1 (1-t)^{p+s} t^q dt \right) \\
&\leq (b-a)^{p+q+1} (|X(a,\cdot)| B(p+1,q+s+1) + |X(b,\cdot)| B(p+s+1,q+1)).
\end{aligned}$$

The proof is complete □

Theorem 3.8. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|^r$ is $s-$convex in the second sense on $[a,b]$ for $r > 1$ and $s \in (0,1]$, where $a, b \in I$ and $a < b$, then the following inequality holds almost every where*

$$\begin{aligned}
&\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \qquad (17)\\
&\leq (b-a)^{p+q+1} (s+1)^{-1/r} [B(lp+1,lq+1)]^{1/l} (|X(a,\cdot)|^r + |X(b,\cdot)|^r)^{1/r}
\end{aligned}$$

where $(1/r) + (1/l) = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\begin{aligned}\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\ \leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)| dt \\ \leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{lp} t^{lq} dt\right)^{1/l} \left(\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt\right)^{1/r}.\end{aligned} \tag{18}$$

Since $|X|^r$ is $s-$convex stochastic process in the second sense then

$$\begin{aligned}\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt \\ \leq \quad |X(a,\cdot)|^r \int_0^1 t^s dt + |X(b,\cdot)|^r \int_0^1 (1-t)^s dt \\ = \quad \frac{|X(a,\cdot)|^r + |X(b,\cdot)|^r}{s+1},\end{aligned} \tag{19}$$

and using the definition of the Beta function we get

$$\int_0^1 (1-t)^{lp} t^{lq} dt = B(lp+1, lq+1). \tag{20}$$

So, replacing (19) and (20) in (18) it is attained the desired inequality (17).
The proof is complete. □

Theorem 3.9. *Let $X : I\times\Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q > 0$, if $|X|^r$ is $s-$convex in the second sense on $[a, b]$ for $r > 1$ and $s \in (0, 1]$, where $a, b \in I$ with $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned}\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\ \leq (b-a)^{p+q+1} (B(p+1,q+1))^{1-1/r} \times \\ (|X(a,\cdot)|^r B(p+1,q+s+1) + |X(b,\cdot)|^r B(p+s+1,q+1))^{1/r}\end{aligned}$$

Proof. From Lemma 3.1 and using the power mean inequality for $k \geq 1$ we have

$$\begin{aligned}\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du \\ \leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)| dt \\ \leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^p t^q dt\right)^{1-1/r} \times \\ \left(\int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)|^r dt\right)^{1/r}.\end{aligned} \tag{21}$$

Since $|X|^r$ is $s-$convex in the second sense and using the definition of the Beta function we get

$$\begin{aligned}\int_0^1 (1-t)^p t^q |X(ta+(1-t)b,\cdot)|^r dt \\ \leq \int_0^1 (1-t)^p t^q (t^s |X(a,\cdot)|^r + (1-t)^s |X(b,\cdot)|^r) dt \\ = |X(a,\cdot)|^r B(p+1,q+s+1) + |X(b,\cdot)|^r B(p+s+1,q+1).\end{aligned}$$

With this last result and again using the definition of the Beta function in the inequality (21) we obtain

$$\int_a^b (u-a)^p (b-u)^q X(u,\cdot)du$$
$$\leq (b-a)^{p+q+1} \left(B(p+1,q+1)\right)^{1-1/r} \times$$
$$\left(|X(a,\cdot)|^r B(p+1,q+s+1) + |X(b,\cdot)|^r B(p+s+1,q+1)\right)^{1/r}$$

The proof is complete. □

4. Conclusion

In the development of this work were found some integral inequalities that involve the Beta function and stochastic processes, in particular those whose absolute values are convex, $P-$convex or $s-$convex in the second sense. Similar inequalities were also found for $k-$th powers of the same type of processes stochastic using the Hölder's inequality and power mean inequality. We hope that the results shown in this paper will serve as a basis for the study of integral inequalities involving stochastic processes with other types of generalized convexity.

Acknowledgement

The authors thank the Council for Scientific, Humanistic and Technological Development (Consejo de Desarrollo Científico, Humanístico y Tecnológico, CDCHT) of the Centroccidental University Lisandro Alvarado (Universidad Centroccidental Lisandro Alvarado, UCLA) for the technical support provided in the preparation of this article belonging to the project RAC-2018-1. They also thank to Dr. Miguel Vivas (from Pontificia Universidad Católica del Ecuador), for his valuable collaboration, and the arbitrators appointed for the evaluation of this article.

References

[1] Alomari, M. and Darus, M. , (2009), On The Hadamard's Inequality for Log-Convex Functions on the Coordinates, J. Ineq. Appl., 2009, 13 pp.

[2] Alomari, M., Darus, M., Dragomir, S.S. and Cerone, P., (2010), Otrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl.Math. Lett., 23, pp. 1071-1076.

[3] Alomari, M., Darus, M. and Kirmaci, U., (2011), Some Inequalities of Hermite-Hadamard type for $s-$Convex Funstions, Acta Mathematica Scientia, 31B(4), pp. 1643 - 1652.

[4] Bain, A. and Crisan, D., (2009), Fundamentals of Stochastic Filtering. Stochastic Modelling and Applied Probability, 60. Springer, New York.

[5] Barani, A., Barani, S. and Dragomir, S. S., (2012), Refinements of Hermite-Hadamard Inequalities for Functions When a Power of the Absolute Value of the Second Derivative Is P-Convex, J. Appl. Math.,2012, 10 pp.

[6] Devolder, P., Janssen, J. and Manca, R., (2015) , Basic stochastic processes, Mathematics and Statistics Series. ISTE, London, John Wiley and Sons, Inc.

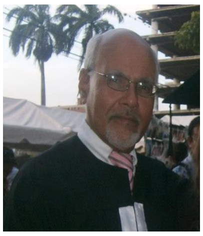
[7] Hernández Hernández, J.E. and Gómez, J.F., (2018), Hermite-Hadamard type inequalities, convex stochastic processes and Katugampola fractional integral, Rev. Integración (Univ. de Santander, 36(2) , pp. 133-149.

[8] Hernández Hernández, J.E. and Gómez, J.F., (2018), Hermite Hadamard type inequalities for Stochastic Processes whose Second Derivatives are $(m, h_1, h_2)-$Convex using Riemann-Liouville Fractional Integral, Rev. Matua, Univ. del Atlántico, 5(1) , pp. 13-28.

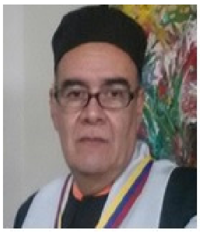
[9] Kotrys, D., (2012), Hermite-Hadamard inequality for convex stochastic processes, Aequat. Math., 83, pp. 143-151

[10] Kotrys, D., (2013), Remarks on strongly convex stochastic processes, Aequat. Math., 86, pp. 91–98.

[11] Medina Viloria, J. and Vivas-Cortez M.,(2018), Jensen's Inequality for Convex Functions on N-Coordinates, Appl. Math. Inf. Sci., 12(5), pp. 931 – 935
[12] Mikosch, T., (2010), Elementary stochastic calculus with finance in view, Advanced Series on Statistical Science and Applied Probability, 6. World Scientific Publishing Co., Inc.
[13] Nagy, B., (1974), On a generalization of the Cauchy equation, Aequat. Math., 11, pp. 165–171.
[14] Nikodem, K., (1980), On convex stochastic processes, Aequat. Math., 20(2-3), pp. 184–197.
[15] Pavić, Z. and Avci Ardic, M., (2017), The most important inequalities for $m-$convex functions, Turk J. Math., 41, pp. 625-635.
[16] Set, E. , Tomar, M. and Maden, S., (2014), Hermite Hadamard Type Inequalities for $s-$Convex Stochastic Processes in the Second Sense, Turk. J. Anal. Numb. Theory, 2(6), pp. 202-207.
[17] Set, E., Akdemir, A. and Uygun, N., (2016), On New Simpson Type Inequalities for Generalized Quasi-Convex Mappings, In: Xth International Statistics Days Conference, Giresun, Turkey.
[18] Shaked, M. and Shantikumar, J., (1985), Stochastic Convexity and its Applications, Arizona Univ. Tuncson.
[19] Shynk, J. J., (2013), Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications, Wiley, 2013
[20] Skowronski, A., (1992), On some properties of $J-$convex stochastic processes, Aequat. Math., 44, pp. 249-258.
[21] Skowronski, A., (1995), On Wright-Convex Stochastic Processes, Ann. Math. Sil., 9, pp. 29-32.
[22] Sobczyk, K., (1991), Stochastic differential equations with applications to physics and engineering, Kluwer Academic Publishers B.V.
[23] Tomar, M., Set, E. and Maden, S., (2015), Hermite-Hadamard Type Inequalities For Log-Convex Stochastic Processes, J. New Theory, 2015(2), pp. 23-32.
[24] Vivas-Cortez, M. and Rangel Oliveros, Y. C., (2018), Ostrowski Type Inequalities for Functions Whose Second Derivatives are Convex Generalized, App. Math. Inf. Sci., 12(6), pp. 1117 – 1126.
[25] Vivas-Cortez, M.,(2016), Féjer Type Inequalities for $(s, m)-$Convex Functions in Second Sense, App. Math. Inf. Sci., 10(5), pp. 1689 – 1696.
[26] Vivas-Cortez, M.J., Kashuri, A., García, C. and Hernández Hernández, J. E., (2020), Hermite – Hadamard Type Mean Square Integral Inequalities for Stochastic Processes whose Twice Mean Square Derivative are Generalized $\eta-$convex, Appl. Math. Inf. Sci., 14(3), pp. 493 – 502
[27] Vivas-Cortez, M.J. and Hernández Hernández, J. E., (2018), Some Inequalities via Strongly p-Harmonic Log-Convex Stochastic Processes, Appl. Math. Inf. Sci., 12(3), pp. 1-8
[28] Vivas-Cortez, M.J., Hernández Hernández, J. E. and Azócar, A., (2017), Some New Generalized Jensen and Hermite-Hadamard Inequalities for Operator h-Convex Functions, Appl. Math. Inf. Sci., 11(2), pp. 383 –392



Jorge Eliecer Hernández Hernández earned his M.Sc. degree from Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Estado Lara (2001) in the field Pure Mathematics (Harmonic Analysis). He is currently an associated professor in Decanato de Ciencias Económicas y Empresariales of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela.



Juan Francisco Gómez earned his specialist in planning and evaluation of learning and Ms. degree in educational sciences from Universidad Santa María, PhD from the National Polytechnic Experimental University of the Bolivarian National Armed Forces, doctoral degree in Economics from the University of Seville, Spain, and Postdoctoral Course in Public Policies and Education at the Universidad Nacional Experimental Del Yaracuy (Venezuela). He is currently a teacher in the area of econometrics and coordinator of the Research Center of the Dean of Economic and Business Sciences of the Universidad Centroccidental Lisandro Alvarado.