TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 1061-1068

NUMERICAL SOLUTIONS OF NONLINEAR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS VIA A NEW INTEGRAL TRANSFORM

S. CETINKAYA[1], A. DEMIR[1], §

Abstract. In this study, solutions of system of time-space fractional partial differential equations (FPDEs) are obtained by utilizing the Shehu transform iterative method. The utility of the technique is shown by getting numerical solutions to a large number of system of FPDEs.

Keywords: Shehu transform, system of time-space fractional partial differential equations, Caputo fractional derivative, iterative method.

AMS Subject Classification: 26A33, 34K37.

1. Introduction

Mathematical models by fractional differential equations for various physical phenomena play important roles in all applied sciences such as mathematics physics, biology, dynamical systems, control systems, engineering and soon [1],[2], [3], [4], [5], [6], [7], [8], [16], [17], [18], [19], [20], [21]. Moreover, nonlinear fractional partial differential equations (FPDEs) are employed in modeling of various nonlinear phenomena, mainly dealing with memory, they present a crucial role in technology and science. Taking physical knowledge and physical properties of the nonlinear problem into account the exact solution of nonlinear FPDEs can be obtained. It is well known that the linear FPDEs can be solved by means of integral transform techniques [9],[10]. This knowledge provides us profound insights that numerical solutions of system of the nonlinear FPDEs can be constructed by the combination of Daftardar-Jafari method (DJM) and Shehu transform. Shehu transform, introduced by Shehu Maitama and Weidong Zhao [11], is an integral transformation converting the ordinary and partial differential equations into simpler equations. It is obtained by generalizing Laplace transformation. Moreover it is a linear transformation like Laplace and Sumudu transformations. Laplace and Yang integral transformations are obtained from Shehu transformation by taking $q = 1$ and $p = 1$ respectively. From this point of view, it could be better to use Shehu transform instead of Laplace or Yang transforms [12]. In this study, Shehu Transform iterative method (STIM) is extended to obtain

[1] Kocaeli University, Faculty of Arts and Sciences, Umuttepe Campus, 41380, Kocaeli, Turkey.
e-mail: suleyman.cetinkaya@kocaeli.edu.tr; ORCID: https://orcid.org/0000-0002-8214-5099.
e-mail: ademir@kocaeli.edu.tr; ORCID: https://orcid.org/0000-0003-3425-1812.
§ Manuscript received: April 30, 2020; accepted: October 02, 2020.

solutions for system of time-space FPDEs. The STIM method is employed to solve a variety system of linear and nonlinear FPDEs. STIM generally generates an accurate solution of FPDEs, which can be represented in terms of the fractional trigonometric functions or Mittag-Leffler functions. Moreover, it has been shown that semi-analytical methods with Shehu transform need fewer CPU time to compute the solutions of nonlinear fractional models, which are utilized in engineering and applied science. STIM is a robust method to obtain solutions for distinct types of nonlinear and linear system of FPDEs. STIM can decrease the time of calculation as well as error margin of the approximate solution.

2. Preliminaries

In this section, preliminaries, notations and features of the fractional calculus are given [2], [4]. Riemann-Liouville time-fractional integral of a real valued function $u(x,t)$ is defined as

$$I_t^{\alpha} u\left(x,t\right) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(t-s\right)^{\alpha-1} u(x,s)ds \tag{1}$$

where $\alpha > 0$ denotes the order of the integral.

α^{th} order the Caputo time-fractional derivative operator of $u(x,t)$ is defined as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = I_t^{m-\alpha}\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \left(t-y\right)^{m-\alpha-1} \frac{\partial^m u(x,y)}{\partial y^m} dy, & m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \end{cases} \tag{2}$$

Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ Re\left(\alpha\right) > 0,\ z,\beta \in \mathbb{C} \tag{3}$$

where α and β are parameters.

The following set of functions has Shehu transformation

$$\left\{\ f(t) |\, \exists P, \tau_1,\ \tau_2 > 0, |f\left(t\right)| < Pe^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty) \right\}$$

and it is defined as

$$\mathbb{S}\left[f(t)\right] = F\left(p,q\right) = \int_0^{\infty} e^{-\frac{p}{q}t} f\left(t\right) dt \tag{4}$$

which has the following property

$$\mathbb{S}\left[t^{\alpha}\right] = \int_0^{\infty} e^{-\frac{pt}{q}} t^{\alpha} dt = \Gamma\left(\alpha+1\right)\left(\frac{q}{p}\right)^{\alpha+1},\ Re\left(\alpha\right) > 0 \tag{5}$$

inverse Shehu inverse transform of $\left(\frac{q}{p}\right)^{n\alpha+1}$ is defined as

$$\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{n\alpha+1}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\ Re\left(\alpha\right) > 0 \tag{6}$$

where $n > 0$ [12]. For the further properties of Shehu transform, we refer the reader to [11]

For α^{th} order the Caputo time-fractional derivative of $f(x,t)$, the Shehu transformation has the following form [12]:

$$\mathbb{S}\left[\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}\right]=\left(\frac{p}{q}\right)^{\alpha}\mathbb{S}\left[f(x,t)\right]-\sum_{k=0}^{n-1}\left[\left(\frac{p}{q}\right)^{\alpha-k-1}\frac{\partial^{k} f(x,0)}{\partial t^{k}}\right], n-1<\alpha\leq n, n\in\mathbb{N} \quad (7)$$

3. Methodology

In this section, we take the general system of time and space FPDE

$$\frac{\partial^{\zeta} f_a}{\partial t^{\zeta_a}}=\mathbb{F}_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right), j_a-1<\zeta_a\leq j_a, a=1,2,\ldots,a, i-1<\eta\leq i, l, j_a, i, a\in\mathbb{N} \quad (8)$$

along with the initial conditions

$$\frac{\partial^{m} f_a(x,0)}{\partial t^{m}}=h_{am}(x), m=0,1,2,\ldots,j_a-1, \quad (9)$$

into account where $\mathbb{F}\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right)$ could be linear or nonlinear and the function $\bar{f}=(f_1,f_2,\ldots,f_w)$ is unknown.

Step 1: Appyling the Shehu transform to both sides of Eq. (8) and rearranging leads to

$$\mathbb{S}\left[f_a(x,t)\right]=\sum_{m=0}^{j_a-1}\left[\left(\frac{q}{p}\right)^{m+1}\frac{\partial^{m} f_a(x,0)}{\partial t^{m}}\right]+\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right)\right]. \quad (10)$$

Step 2: Employing the inverse Shehu transform of Eq. (10), we obtain

$$f\left(x,t\right)=\mathbb{S}^{-1}\left[\sum_{m=0}^{j-1}\left[\left(\frac{q}{p}\right)^{m+1}h_{am}(x)\right]\right]+\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right)\right]\right]. \quad (11)$$

Equation (11) can be rearranged as

$$f_a\left(x,t\right)=g_a\left(x,t\right)+G_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right), a=1,2,\ldots,w \quad (12)$$

where

$$\begin{cases} g_a\left(x,t\right)=\mathbb{S}^{-1}\left[\sum_{m=0}^{j_a-1}\left[\left(\frac{q}{p}\right)^{m+1}h_{am}(x)\right]\right] \\ G_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right)=\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\bar{f},\frac{\partial^{\eta}\bar{f}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}}{\partial x^{l\eta}}\right)\right]\right] \end{cases} \quad (13)$$

Here G_a is a nonlinear / linear operator and g_a is known function. The solution of Eq. (12) can be obtained by the DJM introduced by Daftardar-Gejji and Jafari [13].

Step 3: The solution is represented as an infinite series:

$$f_a=\sum_{m=0}^{\infty} f_a^{(m)}, 1\leq a\leq w, \quad (14)$$

where the terms $f_a^{(m)}$ are recursively computed. Note that henceforth we use the following abbreviations:

$$\bar{f}^{(m)} = (f_1^{(m)}, f_2^{(m)}, \ldots, f_w^{(m)}),$$

$$\sum_{m=0}^{b} \bar{f}^{(m)} = \left(\sum_{m=0}^{b} f_1^{(m)}, \sum_{m=0}^{b} f_2^{(m)}, \ldots, \sum_{m=0}^{b} f_w^{(m)} \right), b \in \mathbb{N} \cup \infty,$$

$$\frac{\partial^{k\eta}\left(\sum_{m=0}^{b} \bar{f}^{(m)}\right)}{\partial x^{k\eta}} = \left(\frac{\partial^{k\eta}\left(\sum_{m=0}^{b} f_1^{(m)}\right)}{\partial x^{k\eta}}, \frac{\partial^{k\eta}\left(\sum_{m=0}^{b} f_2^{(m)}\right)}{\partial x^{k\eta}}, \ldots, \frac{\partial^{k\eta}\left(\sum_{m=0}^{b} f_w^{(m)}\right)}{\partial x^{k\eta}} \right), k \in \mathbb{N}.$$

Step 4: Decomposing the operator G_a leads to

$$\begin{aligned} &G_a\left(x, \textstyle\sum_{m=0}^{\infty} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{\infty} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{\infty} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right) = \\ &G_a\left(x, \bar{f}^{(0)}, \frac{\partial^{\eta} \bar{f}^{(0)}}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta} \bar{f}^{(0)}}{\partial x^{l\eta}}\right) \\ &+ \textstyle\sum_{c=1}^{\infty}\left(G_a\left(x, \sum_{m=0}^{c} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right) \\ &- \textstyle\sum_{c=1}^{\infty}\left(G\left(x, \sum_{m=0}^{c-1} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right) \end{aligned} \tag{15}$$

Therefore

$$\begin{aligned} &\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \textstyle\sum_{m=0}^{\infty} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{\infty} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{\infty} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right] = \\ &\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \bar{f}^{(0)}, \frac{\partial^{\eta} \bar{f}^{(0)}}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta} \bar{f}^{(0)}}{\partial x^{l\eta}}\right)\right]\right] \\ &+ \textstyle\sum_{c=1}^{\infty} \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \sum_{m=0}^{c} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right] \\ &- \textstyle\sum_{c=1}^{\infty} \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \sum_{m=0}^{c-1} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right]. \end{aligned} \tag{16}$$

Step 5: Using Eqs. (14), (16) in Eq. (12), we get

$$\begin{aligned} \sum_{m=0}^{\infty} f_a^{(m)} &= \mathbb{S}^{-1}\left[\sum_{m=0}^{j_a-1}\left[\left(\frac{q}{p}\right)^{m+1} h_{am}(x)\right]\right] \\ &+ \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \bar{f}^{(0)}, \frac{\partial^{\eta} \bar{f}^{(0)}}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta} \bar{f}^{(0)}}{\partial x^{l\eta}}\right)\right]\right] \\ &+ \sum_{c=1}^{\infty}\left(\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}_a\left(x, \sum_{m=0}^{c} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right]\right) \\ &- \sum_{c=1}^{\infty}\left(-\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1} \mathbb{S}\left[\mathbb{F}\left(x, \sum_{m=0}^{c-1} \bar{f}^{(m)}, \frac{\partial^{\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{\eta}}, \ldots, \frac{\partial^{l\eta}\left(\sum_{m=0}^{c-1} \bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right]\right). \end{aligned}$$

Step 6: The recurrence relation is defined by as follows:

$$
\begin{aligned}
&f_a^{(0)} = \mathbb{S}^{-1}\left[\sum_{m=0}^{j_a-1}\left[\left(\frac{q}{p}\right)^{m+1} h_{am}(x)\right]\right]\\
&f_a^{(1)} = \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\bar{f}^{(0)},\frac{\partial^{\eta}\bar{f}^{(0)}}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\bar{f}^{(0)}}{\partial x^{l\eta}}\right)\right]\right]\\
&f_a^{(j+1)} = \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\sum_{m=0}^{c}\bar{f}^{(m)},\frac{\partial^{\eta}\left(\sum_{m=0}^{c}\bar{f}^{(m)}\right)}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\left(\sum_{m=0}^{c}\bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right]\\
&-\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_a+1}\mathbb{S}\left[\mathbb{F}_a\left(x,\sum_{m=0}^{c-1}\bar{f}^{(m)},\frac{\partial^{\eta}\left(\sum_{m=0}^{c-1}\bar{f}^{(m)}\right)}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}\left(\sum_{m=0}^{c-1}\bar{f}^{(m)}\right)}{\partial x^{l\eta}}\right)\right]\right]
\end{aligned}
\tag{17}
$$

for $j \geq 1$.

The j- term truncated solution of Eqs. (8), (9) is constructed as $f_a \approx f_a^{(0)} + f_a^{(1)} + \ldots + f_a^{(j-1)}$ or $u_a \approx u_{a0} + u_{a1} + \ldots + u_{a(j-1)}$. For the convergence of DJM, we refer the reader to [14]. It is clear from Eq. (17) and [14] that the parameters influences the rate of convergence as well as the space of solution.

4. Illustratives Example

Consider system of time and space fractional equations which is called system of time and space fractional Boussinesq PDEs:

$$
\begin{aligned}
\frac{\partial^{\zeta_1} f_1}{\partial t^{\zeta_1}} &= -\frac{\partial^{\eta} f_2}{\partial x^{\eta}},\\
\frac{\partial^{\zeta_2} f_1}{\partial t^{\zeta_2}} &= -m_1\frac{\partial^{\eta} f_1}{\partial x^{\eta}} + 3f_1\left(\frac{\partial^{\eta} f_1}{\partial x^{\eta}}\right) + m_2\frac{\partial^{3\eta} f_1}{\partial x^{3\eta}}, t >, 0\zeta_1,\ \zeta_2, \eta \in (0,1],
\end{aligned}
\tag{18}
$$

along with the initial conditions

$$
f_1(x,0) = a + bx^{\eta}\ , f_2(x,0) =,\ c, a, b, c \in \mathbb{R}. \tag{19}
$$

Let's apply the Shehu transform on both sides of (18).

$$
\mathbb{S}\left[\frac{\partial^{\zeta} f}{\partial t^{\zeta}}\right] = \mathbb{S}\left[\left(\frac{\partial^{\eta} f}{\partial x^{\eta}}\right)^2 - f\left(\frac{\partial^{\eta} f}{\partial x^{\eta}}\right)\right].
$$

By means of the property (7), we obtain

$$
\begin{aligned}
\mathbb{S}\left[\frac{\partial^{\zeta_1} f_1}{\partial t^{\zeta_1}}\right] &= \mathbb{S}\left[-\frac{\partial^{\eta} f_2}{\partial x^{\eta}}\right],\\
\mathbb{S}\left[\frac{\partial^{\zeta_2} f_1}{\partial t^{\zeta_2}}\right] &= \mathbb{S}\left[-m_1\frac{\partial^{\eta} f_1}{\partial x^{\eta}} + 3f_1\left(\frac{\partial^{\eta} f_1}{\partial x^{\eta}}\right) + m_2\frac{\partial^{3\eta} f_1}{\partial x^{3\eta}}\right].
\end{aligned}
$$

In view of (7), we obtain

$$
\begin{aligned}
\mathbb{S}\left[f_1(x,t)\right] &= \left(\frac{q}{p}\right) f_1(x,0) + \left(\frac{q}{p}\right)^{\zeta_1+1}\left(\mathbb{S}\left[-\frac{\partial^{\eta} f_2}{\partial x^{\eta}}\right]\right),\\
\mathbb{S}\left[f_2(x,t)\right] &= \left(\frac{q}{p}\right) f_2(x,0) + \left(\frac{q}{p}\right)^{\zeta_2+1}\left(\mathbb{S}\left[-m_1\frac{\partial^{\eta} f_1}{\partial x^{\eta}} + 3f_1\left(\frac{\partial^{\eta} f_1}{\partial x^{\eta}}\right) + m_2\frac{\partial^{3\eta} f_1}{\partial x^{3\eta}}\right]\right).
\end{aligned}
\tag{20}
$$

Applying the inverse Shehu transform to both sides of Eq. (20). For the uniqueness of Shehu transform, we refer the reader to [22].

$$
\begin{aligned}
f_1(x,t) &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right) f_1(x,0)\right] + \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_1+1}\left(\mathbb{S}\left[-\frac{\partial^\eta f_2}{\partial x^\eta}\right]\right)\right] \\
f_2(x,t) &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right) f_2(x,0)\right] + \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_2+1}\left(\mathbb{S}\left[-m_1\frac{\partial^\eta f_1}{\partial x^\eta} + 3f_1\left(\frac{\partial^\eta f_1}{\partial x^\eta}\right) + m_2\frac{\partial^{3\eta} f_1}{\partial x^{3\eta}}\right]\right)\right]
\end{aligned}
$$

are obtained. Using the recurrence relation (17)

$$
\begin{aligned}
f_{10} &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right) f_1(x,0)\right] = a + bx^\eta, \\
f_{20} &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right) f_2(x,0)\right] = c, \\
f_{11} &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_1+1}\left(\mathbb{S}\left[-\frac{\partial^\eta f_{20}}{\partial x^\eta}\right]\right)\right] = 0, \\
f_{21} &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\zeta_2+1}\left(\mathbb{S}\left[-m_1\frac{\partial^\eta f_{10}}{\partial x^\eta} + 3f_{10}\left(\frac{\partial^\eta f_{10}}{\partial x^\eta}\right) + m_2\frac{\partial^{3\eta} f_{10}}{\partial x^{3\eta}}\right]\right)\right], \\
f_{21} &= \frac{3ab\Gamma(\eta+1)t^{\zeta_2}}{\Gamma(\zeta_2+1)} + \frac{3b^2\Gamma(\eta+1)t^{\zeta_2}x^\eta}{\Gamma(\zeta_2+1)} - \frac{bm_1\Gamma(\eta+1)t^{\zeta_2}}{\Gamma(\zeta_2+1)}, \\
f_{12} &= -\frac{3b^2\Gamma(\eta+1)^2 t^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1+\zeta_2+1)}, \\
f_{22} &= 0, \\
f_{13} &= 0, \\
f_{23} &= -\frac{9b^3\Gamma(\eta+1)^3 t^{\zeta_1+2\zeta_2}}{\Gamma(\zeta_1+2\zeta_2+1)}, \\
f_{1j} &= 0, j \geq 4, \\
f_{2j} &= 0, j \geq 4.
\end{aligned}
$$

As a result, the series solution of the problem (18)-(19) are obtained by

$$
\begin{aligned}
f_1(x,t) &= f_{10} + f_{11} + f_{12} + f_{13} + f_{14} = a - \frac{3b^2\Gamma(\eta+1)^2 t^{\zeta_1+\zeta_2}}{\Gamma(\zeta_1+\zeta_2+1)} + bx^\eta \\
f_2(x,t) &= f_{20} + f_{21} + f_{22} + f_{23} + f_{24}, \\
f_2(x,t) &= c + \frac{3ab\Gamma(\eta+1)t^{\zeta_2}}{\Gamma(\zeta_2+1)} + \frac{3b^2\Gamma(\eta+1)t^{\zeta_2}x^\eta}{\Gamma(\zeta_2+1)} - \frac{bm_1\Gamma(\eta+1)t^{\zeta_2}}{\Gamma(\zeta_2+1)} - \frac{9b^3\Gamma(\eta+1)^3 t^{\zeta_1+2\zeta_2}}{\Gamma(\zeta_1+2\zeta_2+1)}.
\end{aligned}
$$

If $a = e$, $b = 2$, and $c = \frac{3}{2}$, it is the same as reached in [15].

5. Conclusion

STIM is developed by taking the combination of DJM [13] and Shehu transform. This new approach is convenient for acquiring numerical solutions of system of time and space FPDEs. Its appicability is illustrated by an example in this study. As a result the combination of DJM with Shehu transform provides a better and more effective approach than combination Laplace transformation and homotopy, Sumudu or Adomian polynomials.

6. Nomenclature

x : space variable
t : time variable
$I_t^\alpha u(x,t)$: Riemann-Liouville time-fractional integral of order α
$\frac{\partial^\zeta f_a}{\partial t^{\zeta_a}}$: The Caputo time-fractional derivative operator of order ζ
$\frac{\partial^\eta \bar{f}}{\partial x^\eta}$: The Caputo space-fractional derivative operator of order η
$f(x,t)$: unknown function
$h_{am}(x)$: The initial conditions
$\mathbb{S}$: The Shehu transform

$\mathbb{S}^{-1}$: The inverse Shehu transform
$E_{\alpha,\beta}(z)$: Mittag-Leffler function with two parameters α and β

References

[1] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., (2006), Theory and applications of fractional differential equations, Elsevier, Amsterdam.

[2] Podlubny, I., (1999), Fractional differential equation, Academic Press, San Diego, CA.

[3] Sabatier, J., Agarwal, O. P., and Machado, J. A. T., (2007), Advances in fractional calculus: theoretical developments and applications in physics and engineering, Springer, Dordrecht.

[4] Samko, S. G., Kilbas, A. A., and Marichev, O. I., (1993), Fractional integrals and derivatives theory and applications, Gordon and Breach, Amsterdam.

[5] Cetinkaya, S., Demir, A., and Kodal Sevindir, H., (2020), The analytic solution of initial boundary value problem including time-fractional diffusion equation, Facta Universitatis Ser. Math. Inform, 35(1), pp. 243-252. DOI Number: https://doi.org/10.22190/FUMI2001243C.

[6] Cetinkaya, S., Demir, A., and Kodal Sevindir, H., (2020), The analytic solution of sequential space-time fractional diffusion equation including periodic boundary conditions, Journal of Mathematical Analysis, 11(1), pp. 17-26.

[7] Cetinkaya, S., and Demir, A., (2019), The Analytic Solution of Time-Space Fractional Diffusion Equation via New Inner Product with Weighted Function, Communications in Mathematics and Applications, 10(4), pp. 865-873, DOI:http://dx.doi.org/10.26713

[8] Cetinkaya, S., Demir, A. and Kodal Sevindir, H., (2020), The Analytic Solution of Initial Periodic Boundary Value Problem Including Sequential Time Fractional Diffusion Equation, Communications in Mathematics and Applications, 11(1), pp. 173-179, DOI: 10.26713/cma.v11i1.1293

[9] Podlubny, I., (1998), Fractional Differential Equations:An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198. Academic Press, New York.

[10] Debnath, L., and Bhatta, D. D., (2004), Solutions to few linear fractional inhomogeneous partial differential equations in fluid mechanics. Fract. Calc. Appl. Anal., 7(1), pp. 21–36.

[11] Maitama, S., and Zhao, W., (2019), New Integral Transform: Shehu Transform a Generalization of Sumudu and Laplace Transform for Solving Differential Equations, Int. J. Anal. Appl., 17(2), pp. 167-190.

[12] Belgacem, R., Baleanu, D., and Bokhari, A., (2019), Shehu Transform and applications to Caputo-fractional differential equations, International Journal of Analysis and Applications, 17(6), pp. 917-927, DOI: 10.28924/2291-8639-17-2019-917

[13] Daftardar-Gejji, V., and Jafari, H., (2006), An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316(2), pp. 753–763.

[14] Bhalekar, S., and Daftardar-Gejji, V., (2011), Convergence of the new iterative method, Int. J. Differ. Equ.

[15] Choudhary, S., and Daftardar-Gejji, V., (2019), Solving systems of multi-term fractional PDEs: Invariant subspace approach, Int. J. Model. Simul. Sci. Comput., 10(1).

[16] Arqub, O. A. and Shawagfeh, N., (2019), Application of Reproducing Kernel Algorithm for Solving Dirichlet Time-Fractional Diffusion-Gordon Types Equations in Porous Media, Journal of Porous Media, 22(4).

[17] Arqub, O. A., (2019), Application of Residual Power Series Method for the Solution of Time-fractional Schrödinger Equations in One-dimensional Space, Fundamenta Informaticae, 166(2), pp. 87-110.

[18] Arqub, O. A., (2018), Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm, Calcolo, 55.

[19] Arqub, O. A., (2019), Numerical Algorithm for the Solutions of Fractional Order Systems of Dirichlet Function Types with Comparative Analysis, Fundamenta Informaticae, 166(2), pp. 111-137.

[20] Arqub, O. A. and Al-Smadi, M., (2020), Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions, Soft Computing, 24, pp. 12501–12522

[21] Arqub, O. A., Al-Smadi, M., Momani, S. and Hayat, T., (2016), Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method, Soft Computing, 20, pp. 3283–3302.

[22] ALbukhuttar, A. N., Ridha, Z. D. and Kadhim H. N., (2020), Applications of A Shehu Transform to the Heat and Transport Equations, International Journal of Psychosocial Rehabilitation, 24(5), pp. 4254-4263.

Suleyman Cetinkaya obtained his M.Sc. degree in Applied Mathematics at Kocaeli University in 2015. He earned his B.Sc. degree in Mathematics department from Bulent Ecevit University in 2012. His main research interests are fractional derivative, machine learning and wavelet transform. He has been working for Kocaeli University since 2013.

Ali Demir obtained his Ph.D. degree in Applied Mathematics from Kocaeli University in 2007. He obtained his M.Sc. degrees in Mathematics from the University of Illinois at Urbana-Champaing (USA) in 1997. He earned his B.Sc. degree in Mathematical Engineering from Istanbul Technical University in 1993. His main research interests are dynamical systems, inverse problems and fractional differential equations. He has been working for Kocaeli University since 2003.