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CERTAIN EXPANSION FORMULAE OF INCOMPLETE H-FUNCTIONS ASSOCIATED WITH LEIBNIZ RULE

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Abstract. In this article, we have derived some expansion formulae of the incomplete H-functions by the use of the Leibniz rule for the Riemann-Liouville type derivatives. Further, expansion formulae of the incomplete Meijer's G-function, incomplete Fox-Wright function and incomplete generalized hypergeometric function are derived as special cases of our main results.

Keywords: Incomplete Gamma functions; Incomplete H-function; Riemann-Liouville fractional derivatives; Leibniz rule.

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1. Introduction, Definitions and Preliminaries

In the year 1961, Charles Fox [2] investigated and defined a new function, known as Fox's H-function. This function involved Mellin-Barnes integrals with symmetrical Fourier kernel and defined as:

$$H^{m,\,n}_{p,\,q}(z) = H^{m,\,n}_{p,\,q}\left[z \left| \begin{array}{c} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{array}\right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(s)\ z^{-s}\ ds, \tag{1}$$

here,

$$\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)}, \tag{2}$$

and $m, n, p, q \in N_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_j(j = 1, .., p)$, $B_j(j = 1, ..., q) \in \mathbb{R}^+$ and a_j, $b_j \in \mathbb{C}$. Also, $\mathcal{L}$ is a suitable contour which separates the poles. The H-function converges absolutely under the set of conditions defined in [2] (see also [5, 7, 8, 14]).

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We next recall and defined the familiar lower and upper incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, respectively, as:

$$\gamma(s,x) = \int_0^x y^{s-1}\, e^{-y}\, dy, \qquad \big(\Re(s) > 0;\ x \geqq 0\big), \tag{3}$$

and

$$\Gamma(s,x) = \int_x^\infty y^{s-1}\, e^{-y}\, dy, \qquad \big(x \geqq 0;\ \Re(s) > 0 \quad \text{if} \quad x = 0\big). \tag{4}$$

These functions fulfill the following relation (known as decomposition formula):

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s), \qquad (\Re(s) > 0). \tag{5}$$

By the use of above defined incomplete gamma functions, Srivastava et al. [15] defined the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ given below. The incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ are widely used in science and engineering problems (see [2, 4, 5, 7, 8]).

$$\begin{aligned} {}_p\gamma_q\left[\begin{matrix}(a_1,x), a_2, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix}\ z\right] &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{\ell=0}^\infty \frac{\gamma(a_1+\ell,x)\prod_{j=2}^p \Gamma(a_j+\ell)}{\prod_{j=1}^q \Gamma(b_j+\ell)} \frac{z^\ell}{\ell!} \\ &= \frac{1}{2\pi i} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \int_L \frac{\gamma(a_1+s,x)\prod_{j=2}^p \Gamma(a_j+s)}{\prod_{j=1}^q \Gamma(b_j+s)} \Gamma(-s)(-z)^s ds, \quad (|\arg(-z)| < \pi), \end{aligned} \tag{6}$$

and

$$\begin{aligned} {}_p\Gamma_q\left[\begin{matrix}(a_1,x), a_2, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix}\ z\right] &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{\ell=0}^\infty \frac{\Gamma(a_1+\ell,x)\prod_{j=2}^p \Gamma(a_j+\ell)}{\prod_{j=1}^q \Gamma(b_j+\ell)} \frac{z^\ell}{\ell!} \\ &= \frac{1}{2\pi i} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \int_L \frac{\Gamma(a_1+s,x)\prod_{j=2}^p \Gamma(a_j+s)}{\prod_{j=1}^q \Gamma(b_j+s)} \Gamma(-s)(-z)^s ds, \quad (|\arg(-z)| < \pi), \end{aligned} \tag{7}$$

with the existence and convergence conditions setout in [15].

In terms of the incomplete gamma functions defined in (3) and (4), the incomplete Pochhammer symbols $(a;x)_n$ and $[a;x]_n$ are defined as follows (see [15]):

$$(a;x)_n = \frac{\gamma(a+n,x)}{\Gamma(a)}, \qquad (a,\, n \in \mathbb{C};\ x \geq 0), \tag{8}$$

and

$$[a;x]_n = \frac{\Gamma(a+n,x)}{\Gamma(a)}, \qquad (a,\, n \in \mathbb{C};\ x \geq 0). \tag{9}$$

Inspired by the applications of ${}_p\gamma_q$ and ${}_p\Gamma_q$ functions defined above and their representation as Mellin-Barnes contour integrals, Srivastava et al. [16] presented and researched the incomplete H-functions as follows:

$$\gamma_{p,\,q}^{m,\,n}(z) = \gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p} \\ (b_j,B_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} g(s,x)\, z^{-s}\, ds, \tag{10}$$

and

$$\Gamma_{p,\,q}^{m,\,n}(z) = \Gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p} \\ (b_j,B_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} G(s,x)\, z^{-s}\, ds, \tag{11}$$

where

$$g(s,x) = \frac{\gamma(1-a_1-A_1 s, x) \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=2}^{n} \Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)}, \tag{12}$$

and

$$G(s,x) = \frac{\Gamma(1-a_1-A_1 s, x) \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=2}^{n} \Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)}, \tag{13}$$

with the set of conditions setout in [16].

These incomplete H-functions fulfill the following relation (known as decomposition formula):

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \tag{14}$$

1.1. **Riemann-Liouville Fractional Operators.** The Riemann-Liouville fractional operators of order ν for the function $f(z)$ are defined as follows (see [5]):

$$I^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} f(t) dt, \tag{15}$$

where, the integral is well defined provided f is a locally integrable function and ν is a complex number in the half plane $\Re(\nu) > 0$.

$$D_z^{\nu} f(z) = \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{(-\nu-1)} f(t) dt, \quad (\Re(\nu) < 0), \tag{16}$$

if $\Re(\nu) \geq 0$ and $m \in \mathbb{N}$ is the smallest integer with $m-1 \leq \Re(\nu) < m$, then

$$D_z^{\nu} f(z) = \frac{d^m}{dz^m} D_z^{\nu-m} f(z) = \frac{d^m}{dz^m} \left[\frac{1}{\Gamma(-\nu+m)} \int_0^z (z-t)^{-\nu+m-1} f(t) dt \right]. \tag{17}$$

1.2. **Leibniz Formula for Fractional Derivative.** The classical Leibniz rule for two differentiable functions f and g is defined as follows:

$$D^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)].$$

This Leibniz rule can be extended for the Riemann-Liouville type derivatives. If f and g are two functions of class C, then the fractional generalization of the Leibniz rule is defined as (see [11])

$$D^{\mu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k g(t)][D^{\mu-k} f(t)], \qquad \mu > 0, \quad k \in \mathbb{N}. \tag{18}$$

In particular, if f is function of class C, then

$$D^{\mu}[t^p f(t)] = \sum_{r=0}^{p} \binom{\mu}{r} [D^r t^p][D^{\mu-r} f(t)], \qquad \mu > 0.$$

The Leibniz rule, which generalizes the differentiation law of the product, may be used to extract a mechanism that calculates the composition representation of the differential operators. In principle, it would be helpful to infer some fascinating transformations, summations, generating functions and expansions concerning the different special functions (including q-functions) of one and sometimes more variables, see papers [1, 3, 6, 9, 10, 12, 13, 17] and references within. The key aim of this paper is to extract several expansion formulas from incomplete H-functions by implementing the Leibniz rule for fractional derivatives of the Riemann-Liouville type. There are also several fascinating special cases and implementations with the primary outcome.

2. Main Results

We have established some expansion formulae of the incomplete H-functions by the use of Leibniz rule.

Theorem 2.1. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\Gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(a_1,A_1,x),(1-\lambda,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(1-\lambda+\mu,\sigma)\end{array}\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\Gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(a_1,A_1,x),(0,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(k,\sigma)\end{array}\right]. \quad (19)$$

Proof. To prove the result (19), let us consider $f(z)=z^{\lambda-1}$ and

$$g(z)=\Gamma^{m,n}_{p,q}(az^{\sigma})=\Gamma^{m,n}_{p,q}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(a_1,A_1,x),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\end{array}\right].$$

Now, substituting the values of $f(z)$ and $g(z)$ in (18), we obtain

$$D^{\mu}\left[z^{\lambda-1}\Gamma^{m,n}_{p,q}(az^{\sigma})\right]=\sum_{k=0}^{\infty}\binom{\mu}{k}\left[D^{k}\Gamma^{m,n}_{p,q}(az^{\sigma})\right]\left[D^{\mu-k}z^{\lambda-1}\right]. \quad (20)$$

On taking L.H.S of equation (20), we obtain

$$\begin{aligned}D^{\mu}\left[z^{\lambda-1}\Gamma^{m,n}_{p,q}(az^{\sigma})\right] &= D^{\mu}\left[z^{\lambda-1}\frac{1}{2\pi i}\int_{\mathcal{L}}G(s,x)a^{-s}z^{-\sigma s}ds\right]\\ &= \frac{1}{2\pi i}\int_{\mathcal{L}}G(s,x)a^{-s}D^{\mu}\left[z^{\lambda-\sigma s-1}\right]ds\\ &= \frac{1}{2\pi i}\int_{\mathcal{L}}G(s,x)a^{-s}\frac{\Gamma(\lambda-\sigma s)}{\Gamma(\lambda-\mu-\sigma s)}z^{\lambda-\mu-\sigma s-1}\,ds,\end{aligned}$$

using the definition (11), we obtain

$$D^{\mu}\left[z^{\lambda-1}\Gamma^{m,n}_{p,q}(az^{\sigma})\right]=z^{\lambda-\mu-1}\Gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(a_1,A_1,x),(1-\lambda,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(1-\lambda+\mu,\sigma)\end{array}\right]. \quad (21)$$

Similarly, the R.H.S. of equation (20) is the immediate consequences of the definitions (11) and (18), we obtain

$$\sum_{k=0}^{\infty}\binom{\mu}{k}\left[D^{k}\Gamma^{m,n}_{p,q}(az^{\sigma})\right]\left[D^{\mu-k}z^{\lambda-1}\right]$$
$$=z^{\lambda-\mu-1}\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\Gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(a_1,A_1,x),(0,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(k,\sigma)\end{array}\right]. \quad (22)$$

Substituting the equation (21) and (22) into (20), we get the required result (19). □

Below theorem are the immediate consequences of the definitions (10), (11) and (18) and hence they are given without proof here.

Theorem 2.2. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\Gamma^{m+1,\,n}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\,,(\lambda-\mu,\sigma)\\(\lambda,\sigma),(b_j,B_j)_{1,q}\end{matrix}\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\Gamma^{m+1,\,n}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\,,(1-k,\sigma)\\(1,\sigma),(b_j,B_j)_{1,q}\end{matrix}\right]. \quad (23)$$

Theorem 2.3. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(1-\lambda,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(1-\lambda+\mu,\sigma)\end{matrix}\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\gamma^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(0,\sigma),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\,,(k,\sigma)\end{matrix}\right]. \quad (24)$$

Theorem 2.4. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\gamma^{m+1,\,n}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\,,(\lambda-\mu,\sigma)\\(\lambda,\sigma),(b_j,B_j)_{1,q}\end{matrix}\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\gamma^{m+1,\,n}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\,,(1-k,\sigma)\\(1,\sigma),(b_j,B_j)_{1,q}\end{matrix}\right]. \quad (25)$$

3. Special Cases

In this part, we derive the expansion formulae of the incomplete Meijer's ${}^{(\Gamma)}G$-function, incomplete Fox-Wright ${}_p\Psi_q^{(\Gamma)}$-function and incomplete generalized hypergeometric ${}_p\Gamma_q$ function, as special cases of Theorem 2.1 and Theorem 2.2. For the definitions of classical; Meijer's G-function, Fox-Wright Ψ-function and generalized hypergeometric ${}_pF_q$ function one could see Mathai et al. [8]. To illustrate the applications of main results, if we take particular values to the parameters, such as $A_j = 1$ $(j = 1, \cdots, p)$, $B_j = 1$ $(j = 1, \cdots, q)$, $\sigma = 1$, and using the relation, namely

$$\Gamma^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(a_1,1,x),(a_j,1)_{2,p}\\(b_j,1)_{1,q}\end{matrix}\right]={}^{(\Gamma)}G^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(a_1,x),(a_j)_{2,p}\\(b_j)_{1,q}\end{matrix}\right], \quad (26)$$

in (19) and (23), respectively, then we obtain the following corollaries:

Corollary 3.1. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$${}^{(\Gamma)}G^{m,\,n+1}_{p+1,\,q+1}\left[az\,\middle|\,\begin{matrix}(a_1,x),1-\lambda,a_2,\cdots,a_p\\b_1,\cdots,b_q\,,1-\lambda+\mu\end{matrix}\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{(\Gamma)}G^{m,\,n+1}_{p+1,\,q+1}\left[az\,\middle|\,\begin{matrix}(a_1,x),0,a_2,\cdots,a_p\\b_1,\cdots,b_q\,,k\end{matrix}\right]. \quad (27)$$

Corollary 3.2. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$ {}^{(\Gamma)}G^{m+1,\,n}_{p+1,\,q+1}\left[az^{-1}\,\middle|\,\begin{array}{c}(a_1,x),a_2,\cdots,a_p,\lambda-\mu\\ \lambda,b_1,\cdots,b_q\end{array}\right] = \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{(\Gamma)}G^{m+1,\,n}_{p+1,\,q+1}\left[az^{-1}\,\middle|\,\begin{array}{c}(a_1,x),a_2,\cdots,a_p,1-k\\ 1,b_1,\cdots,b_q\end{array}\right]. \quad (28)$$

Further, if we take the substitution $a=-a$, $m=1$, $n=p$, $q=q+1$, $a_j \to (1-a_j)\,(j=1,...,p)$ and $b_j \to (1-b_j)\,(j=1,...,q)$ in (19) and (23), and making use of the following functional relation (see [16])

$$\Gamma^{1,\,p}_{p,\,q+1}\left[-z\,\middle|\,\begin{array}{c}(1-a_1,A_1,x),(1-a_j,A_j)_{2,p}\\ (0,1),(1-b_j,B_j)_{1,q}\end{array}\right] = {}_p\Psi^{(\Gamma)}_q\left[\begin{array}{c}(a_1,A_1,x),(a_j,A_j)_{2,p}\,;\\ (b_j,B_j)_{1,q}\,;\end{array}\; z\right], \quad (29)$$

we get the subsequent corollaries:

Corollary 3.3. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$${}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{array}{c}(a_1,A_1,x),(1-\lambda,\sigma),(a_j,A_j)_{2,p}\,;\\ (b_j,B_j)_{1,q}\,,(1-\lambda+\mu,\sigma)\,;\end{array}\; az^{\sigma}\right] = \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{array}{c}(a_1,A_1,x),(0,\sigma),(a_j,A_j)_{2,p}\,;\\ (b_j,B_j)_{1,q}\,,(k,\sigma)\,;\end{array}\; az^{\sigma}\right]. \quad (30)$$

Corollary 3.4. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$${}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{array}{c}(a_1,A_1,x),(\lambda-\mu,\sigma),(a_j,A_j)_{2,p}\,;\\ (b_j,B_j)_{1,q}\,,(\lambda,\sigma);\end{array}\; az^{\sigma}\right] = \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}_{p+1}\Psi^{(\Gamma)}_{q+1}\left[\begin{array}{c}(a_1,A_1,x),(1-k,\sigma),(a_j,A_j)_{2,p}\,;\\ (b_j,B_j)_{1,q}\,,(1,\sigma);\end{array}\; az^{\sigma}\right]. \quad (31)$$

Again, if we take the following substitution $\sigma = 1$, $A_j = 1\,(j=1,...,p)$ and $B_j = 1\,(1=2,...,q)$ in (30) and (31), and making use of the relations (29) and (see [16])

$$\Gamma^{1,\,p}_{p,\,q+1}\left[-z\,\middle|\,\begin{array}{c}(1-a_1,1,x),(1-a_j,1)_{2,p}\\ (0,1),(1-b_j,1)_{1,q}\end{array}\right] = \mathcal{C}^p_q\,{}_p\Gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array}\; z\right], \quad (32)$$

where, $\mathcal{C}^p_q = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}$, then we get the following corollaries:

Corollary 3.5. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\mu)}\,{}_{p+1}\Gamma_{q+1}\left[\begin{array}{c}(a_1,x),1-\lambda,a_2,\cdots,a_p;\\ b_1,\cdots,b_q,1-\lambda+\mu;\end{array}\; az\right] = \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)\Gamma(k)}\,{}_{p+1}\Gamma_{q+1}\left[\begin{array}{c}(a_1,x),0,a_2,\cdots,a_p;\\ b_1,\cdots,b_q,k;\end{array}\; az\right] \quad (33)$$

Corollary 3.6. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\,{}_{p+1}\Gamma_{q+1}\left[\begin{matrix}(a_1,x),\lambda-\mu,a_2,\cdots,a_p;\\ b_1,\cdots,b_q,\lambda;\end{matrix}\; az\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)\Gamma(1-k)}{\Gamma(\lambda-\mu+k)}\,{}_{p+1}\Gamma_{q+1}\left[\begin{matrix}(a_1,x),1-k,a_2,\cdots,a_p;\\ b_1,\cdots,b_q,1;\end{matrix}\; az\right]. \tag{34}$$

Particularly, if we take the following substitution $p=2, q=1$ and $p=q=1$ in (33) and (34), we get the subsequent results for the incomplete Gauss's and Kummer's hypergeometric functions:

Corollary 3.7. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\mu)}\,{}_3\Gamma_2\left[\begin{matrix}(a_1,x),1-\lambda,a_2;\\ b_1,1-\lambda+\mu;\end{matrix}\; az\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)\Gamma(k)}\,{}_3\Gamma_2\left[\begin{matrix}(a_1,x),0,a_2;\\ b_1,k;\end{matrix}\; az\right]. \tag{35}$$

Corollary 3.8. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\,{}_3\Gamma_2\left[\begin{matrix}(a_1,x),\lambda-\mu,a_2;\\ b_1,\lambda;\end{matrix}\; az\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)\Gamma(1-k)}{\Gamma(\lambda-\mu+k)}\,{}_3\Gamma_2\left[\begin{matrix}(a_1,x),1-k,a_2;\\ b_1,1;\end{matrix}\; az\right]. \tag{36}$$

Corollary 3.9. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\mu)}\,{}_2\Gamma_2\left[\begin{matrix}(a_1,x),1-\lambda;\\ b_1,1-\lambda+\mu;\end{matrix}\; az\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)\Gamma(k)}\,{}_2\Gamma_2\left[\begin{matrix}(a_1,x),0;\\ b_1,k;\end{matrix}\; az\right]. \tag{37}$$

Corollary 3.10. *Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\,{}_2\Gamma_2\left[\begin{matrix}(a_1,x),\lambda-\mu;\\ b_1,\lambda;\end{matrix}\; az\right]=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)\Gamma(1-k)}{\Gamma(\lambda-\mu+k)}\,{}_2\Gamma_2\left[\begin{matrix}(a_1,x),1-k;\\ b_1,1;\end{matrix}\; az\right]. \tag{38}$$

Remark: Similarly, special cases for the Theorem 2.3 and Theorem 2.4 may be derived.

4. Conclusions

We have derived some expansion formulae for the incomplete H-functions by the use of the Leibniz rule for the Riemann-Liouville type fractional derivatives in this paper. Special cases of our main results are derived in the Section 3. It is important to note that the particular cases of our results for $x=0$ (or using the decomposition formula (14)) would give the corresponding new or known expansion formulas involving classical Fox's H-functions, G-functions, Fox-Wright functions, hypergeometric functions including

Gauss's and Kummer's hypergeometric functions. Therefore, we summarize with the remark that the findings discussed here seem to be of general nature and can give rise to various expansions formulas for a certain type of special functions, which we left for interested readers.

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