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A SEMI–ANALYTICAL STUDY OF DIFFUSION TYPE MULTI–TERM TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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Abstract. This work suggested algorithm for the solution of multi–term time fractional partial differential equation by the application of homotopy analysis fractional Sumudu transform method based on iterative process. The method is cumulation of Sumudu transform and homotopy analysis method. Also, the multi-term time fractional partial differential equation represented in the form of system of fractional partial differential equations as per certain conditions of fractional derivatives. The Caputo fractional order derivatives are taken for the multi–term time fractional partial differential equations. Numerical examples are discussed for the support of theory and the approximate solution compared with exact solutions at the integer value of derivatives.

Keywords: Caputo derivative, Diffusion equation, Homotopy Analysis Fractional Sumudu Transform Method, Multi–term time fractional partial differential equations.

AMS Subject Classification: 26A33, 34A08, 65Bxx, 65R10.

1. Introduction

Fractional calculus is old as classical calculus, today it plays significant role in various fields of science and engineering including mathematical modeling astrophysics biology etc. Recently many researchers and mathematicians give valuable contributions to enhance the knowledge in this field [1, 2, 3].

For explaining dynamical systems, the integer – order system of differential equations are significant tool up to recent era. Unfortunately modern studied have depicts that integer-order derivatives are not reasonably explaining the multifaceted and typical nature of various types of non dynamical system. Currently differential equations of fractional order are popularly used by many researchers in all over world to form various scientific models. Importantly fractional derivatives introduce for understanding of real life phenomena to reduce shortcoming of classical calculus and also for the explanation Brownian nature of

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particle in non dynamic system. Some classical example can be observed in the study of ground water system in-heterogeneous media.

In [4], authors discussed a generalized fractional derivative which produced different kinds of singular and nonsingular fractional derivatives based on different types of kernels. Kumar et al. [5], solve the multidimensional heat equations of arbitrary order with new Yang-Abdel-Aty-Cattani (YAC) fractional operator by using two approaches homotopy perturbation transform method and residual power series method. Ghanbari et al. [6], solve fractional immunogenetic tumour model using the Adam Bashforth's Moulton method, where the fractional Atangana-Baleanu derivative has been utilized in the structure of model. Kumar et al. [7], applied the Bernstein wavelet and Euler methods for the solution of nonlinear fractional predator- prey biological model of two species. Alshabanat et al. [8], generalizes the fractional operator with non-singular derivatives as a special case of Caputo-Fabrizio fractional derivative. Later discussed the application in electrical circuits. Kumar et al. [9, 10] applied the operational matrix based on Bernstein wavelets method, Adams–Bashforth predictor correcter method in SIR model and Haar wavelet and Adams-Bashforth-Moulton methods in Lotka-Volterra (LV) system. Veeresha et al. [11] applied the q-homotopy analysis transform method (q-HATM) for the solution of fractional generalized nonlinear Schrödinger (FGNS) equation. Bansal et al. [12] discussed the solutions for fractional differential equations involving the generalized composite fractional derivative and integral operator associated with the incomplete H-function with various special cases. In [13] Singh et al. presented q-local fractional homotopy analysis transform method (q-LFHATM) and applied it on the solution of local fractional linear transport equations (LFLTE) in fractal porous media. Authors of [14] presented solution of systems of nonlinear fractional differential equations by the application of homotopy asymptotic method.

The multi–term time fractional order partial differential equations played significant role to explain many physical and non-physical phenomenon's such as the non-Markovian diffusion process with memory, propagation of mechanical waves in viscoelastic media, transport in amorphous semiconductors [15, 16, 17, 18]. The variable order differential operators may better describe the behaviour of various time varying processes instead of time varying coefficients [19, 20, 21].

Variable order and distributed order fractional operators are also discussed by Lorenzo and Hartley [22]. Then many authors proposed the physical meaning of variable operators, see references therein [23, 24, 25].

Many methods applied to solve system of fractional partial differential equations namely Adomian decomposition method (ADM)[26, 27, 28, 29, 30, 31], homotopy perturbation method (HPM) [32], homotopy analysis method (HAM) [33, 34], Predict, Evaluate, Correct, Evaluate (PECE) [35], Chebyshev spectral methods [36], Variational Iteration Method [37], Spectral method [38]. These methods have been proposed to obtain exact and approximate analytical solutions of multi–term fractional partial differential equations.

In this communication, we are interested to solve multi–term time fractional nonlinear fractional order partial differential equation. Using applicability of HAFSTM [39, 40], we transform it into system of fractional order partial differential equations [28], some numerical experiments of linear and nonlinear systems of fractional PDE's will be presented.

The paper is organize as follows. In Sec. 2 some basics definitions of applicable terms are given. The multi term fractional partial differential equations transformation as a system of fractional partial differential equations has been discussed in Sec. 3. The algorithm of method HAFSTM for the solution of system of fractional PDE's are introduced in Sec. 4.

The convergence analysis of problem is given in Sec. 5. Next, application of the discussed algorithm and numerical comparisons with graphical analysis are given in Sec. 6. Finally, conclusions are drawn in Sec. 7.

2. Some Basic definitions

Definition 1 Let the function $f(t), t>0$, be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p\,(>\mu)$, such that $f(t)=t^p f_1(t)$, where$f_1(t)\in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)}\in C_\mu$, $m\in N$.

Definition 2 The left sided Liouville Fractional integral operator of order $\alpha \geq 0$, of a function $f(t)\in C_\mu$, and $\mu \geq -1$ is defined as [41, 42]
$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ \alpha>0,\ x>0$ and $J^0 f(t)=f(t)$.

Definition 3 The left sided Riemann–Liouville fractional differential operator of order $\alpha \geq 0$, [1]

$$D^\alpha f(t)=\frac{d^m}{dt^m} I^{m-\alpha} f(t),\ m-1<\alpha\leq m,\ m\in\mathbb{N}.$$

Definition 4 The left sided caputo of $f(t)$ derivative is defined as [1]

$$D_t^\alpha f(t)=\begin{cases} J^{m-\alpha} D^n f(t), \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-T)^{m-\alpha-1} f^{(m)}(\tau)\,d\tau, \end{cases}$$

where $m-1<\alpha\leq m,\ m\in N,\ t>0$.

Definition 5 In early 90's, Watugala [43] introduced an incipient integral transforms. The Sumudu transform is defined over the set of functions

$$A=\left\{ f(t) \,\middle|\, \exists M,\ \tau_1,\tau_2>0,\ |f(t)|<Me^{\frac{|t|}{\tau_j}},\ if\ t\in(-1)^j\times[0,\infty) \right\},$$

by the following formula

$$\bar{f}(u)=\mathbb{S}[f(t)]=\int_0^\infty f(ut)\ e^{-t}\,dt,\ u\in(-\tau_1,\tau_2).$$

Definition 6 The Sumudu transform of $f(t)=t^\alpha$ is defined as [44]

$$\mathbb{S}[t^\alpha]=\int_0^\infty e^{-t}t^\alpha\,dt=\Gamma(\alpha+1)\ u^\alpha,\ R(\alpha)>0.$$

Definition 7 The Sumudu transform $\mathbb{S}[f(t)]$ of the Riemann–Liouville fractional integral is defined as [44]

$$\mathbb{S}[I^\alpha f(t)]=u^{-\alpha}F(u).$$

Definition 8 The Sumudu transform $\mathbb{S}\left[f\left(t\right)\right]$ of the Caputo fractional derivative is defined as [44]

$\mathbb{S}\left[D_t^{\alpha} f\left(t\right)\right] = u^{-\alpha}\mathbb{S}\left[f\left(t\right)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}\left(0^{+}\right),$ where $m-1<\alpha\leq m$.

3. Multi-Term FPDE as a system of FPDE

Here, we consider the following time fractional diffusion wave equations of multi–term [45]

$$\sum_{i=1}^{n} c_i \, {}_0^c D_t^{\alpha_i} U\left(x,t\right) = K_{\alpha_1} U_{xx}\left(x,t\right) + f\left(x,t\right), \tag{1}$$

where $0<\alpha_1<...,<\alpha_n<1$ or $0<\alpha_1<...,<\alpha_n<2$ and K_{α_1}, c_i are constants, ${}_0^c D_*^{\alpha_i}$ denotes the caputo derivative of arbitrary order $\forall\,\alpha_i\in\mathbb{Q}$, $\alpha_i-\alpha_{i-1}\leq 1$, $\forall i$ and $0\leq\alpha_i\leq 1$.

We translate equation (1) as a system of FPDE, using the algorithm proposed in [46].

4. Analysis of the homotopy analysis fractional Sumudu transform method

We apply the homotopy analysis fractional Sumudu transform method to solve the fractional multi –term diffusion equations

$$\begin{array}{l} D_t^{\alpha_i} U_i\left(x,t\right) = U_{i+1}, \quad i=n-1,\, n-2, ...,\, 1. \\ D_t^{\alpha_n} U_i\left(x,t\right) = f\left(x,t,U_1,U_2,....,U_n\right), \\ U^{(k)}\left(x,0\right) = C_k^j,\ 0\leq k\leq m_j,\ m_j<\alpha_i\leq m_{j+1},\ 1\leq j\leq n. \end{array} \tag{2}$$

Now, applying the Sumudu transform in equation (2), we get

$$\begin{array}{l} \mathbb{S}\left[D_t^{\alpha_i} U_i\left(x,t\right)\right] = \mathbb{S}\left[U_{i+1}\right], \quad i=n-1,\, n-2, ...,\, 1. \\ \mathbb{S}\left[D_t^{\alpha_n} U_i\left(x,t\right)\right] = \mathbb{S}\left[f\left(x,t,U_1,U_2,....,U_n\right)\right],\ \ 0\leq k\leq m_j,\ m_j<\alpha_i\leq m_{j+1},\ 1\leq j\leq n. \end{array}$$

Using the differentiation property of the Sumudu transform

$$\begin{array}{l} \frac{\mathbb{S}[U(\mathrm{x},t)]}{u^{\alpha_i}} - \sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{(\alpha_i-k)}} = \mathbb{S}\left[U_{i+1}\right], \quad i=n-1,\, n-2, ...,\, 1. \\ \frac{\mathbb{S}[U(\mathrm{x},t)]}{u^{\alpha_n}} - \sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{(\alpha_n-k)}} = \mathbb{S}\left[f\left(x,t,U_1,U_2,....,U_n\right)\right], \end{array} \tag{3}$$

we define nonlinear operator as

$$\begin{array}{l} N_i\left[\varphi_i\left(\mathrm{x,t;q}\right)\right] = \mathbb{S}\left[\varphi_i\left(\mathrm{x,t;q}\right)\right] - \sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{(-k)}} - u^{\alpha_i}\mathbb{S}\left[\varphi_{i+1}\left(\mathrm{x,t;q}\right)\right], \quad i=1,2,...,n-1, \\ N_n\left[\varphi_n\left(\mathrm{x,t;q}\right)\right] = \mathbb{S}\left[\varphi_n\left(\mathrm{x,t;q}\right)\right] - \sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{(-k)}} - u^{\alpha_n}\mathbb{S}\left[f\left(x,t,\varphi_1,\varphi_2,....,\varphi_n\right)\right], \end{array} \tag{4}$$

where $q\in[0,1]$ be an embedding parameter and $\varphi\left(\mathrm{x,t;q}\right)$ is a real function of x, t and q. we construct the homotopies are as follow:

$$\begin{array}{l} \left(1-\mathrm{q}\right)\mathbb{S}\left[\varphi_i\left(\mathrm{x,t;q}\right) - \mathrm{U}_{i0}\left(\mathrm{x,t}\right)\right] = \hbar_i \mathrm{q}\mathrm{H}_i\left(\mathrm{x,t}\right) N\left[\varphi_i\left(\mathrm{x,t;q}\right)\right], \\ \left(1-\mathrm{q}\right)\mathbb{S}\left[\varphi_n\left(\mathrm{x,t;q}\right) - \mathrm{U}_{n0}\left(\mathrm{x,t}\right)\right] = \hbar_n \mathrm{q}\mathrm{H}_n\left(\mathrm{x,t}\right) N\left[\varphi_n\left(\mathrm{x,t;q}\right)\right]. \end{array} \tag{5}$$

$\hbar_i\neq 0$ and $\mathrm{H}_i\left(\mathrm{x,t}\right)\neq 0$, $i=1,2,3,...,n$ are nonzero auxiliary functions, $\mathrm{U}_{i0}\left(\mathrm{x,t}\right)$ are initial guess of $\mathrm{U}_i\left(\mathrm{x,t}\right)$and $\varphi_i\left(\mathrm{x,t;q}\right)$ is unknown function. It is important that one has great freedom to choose auxiliary parameter in HAFSTM. Obviously, when $q=0$ and $q=1$ it holds

$$\varphi_i\left(\mathrm{x,t;0}\right) = U_{i0}\left(\mathrm{x,t}\right), \qquad \varphi_i\left(\mathrm{x,t;1}\right) = U_i\left(\mathrm{x,t}\right),\ \ i=1,2,3,...,n. \tag{6}$$

Thus as q increases from 0 to 1,then the solution varies from initial guess $U_{i0}\left(\mathrm{x,t}\right)$ to $U_i\left(\mathrm{x,t}\right)$ Now, expanding $\varphi\left(\mathrm{x,t;q}\right)$ on Taylor's series with respect to q, we get

$$\varphi_i\left(x,t;q\right)=U_{i0}\left(x,t\right)+\sum_{m=1}^{\infty}q^m\,U_{im}\left(x,t\right)\ , \tag{7}$$

where

$$U_{im}\left(x,t\right)=\frac{1}{\underline{|m}}\left.\frac{\partial^m\varphi_i\left(x,t;q\right)}{\partial q^m}\right|_{q=0}. \tag{8}$$

The convergence of the series solution (7) is controlled by $\hbar$. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are properly chosen, the series (7) converges at $q=1$. Hence we obtain

$$U_i\left(x,t\right)=U_{i0}\left(x,t\right)+\sum_{m=1}^{\infty}U_{im}\left(x,t\right)\ \ , \tag{9}$$

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess $U_{i0}\left(x,t\right)$ and the exact solution $U\left(x,t\right)$ by means of the terms $U_{i\,m}\left(x,t\right)$ $\left(m=1,2,3,...\right)$, which are still to be determined. Define the vectors

$$\overrightarrow{U}\ =\left\{U_{i\,0}\left(x,t\right),U_{i\,1}\left(x,t\right),U_{i\,2}\left(x,t\right),...,U_{i\,m}\left(x,t\right)\right\}. \tag{10}$$

Differentiating the eq. (5) m times with respect to embedding parameter q and then setting $q=0$, and finally dividing them by $m!$, we obtain the m^{th} order deformation equation as follows:

$$\begin{aligned}&\mathbb{S}\left[U_{im}\left(\mathrm{x,t}\right)-\chi_m\mathrm{U}_{i(m-1)}\left(\mathrm{x,t}\right)\right]=\hbar_i\mathrm{H}_i\left(\mathrm{x,t}\right)N_i\left[U_i\left(\mathrm{x,t}\right)\right],\\&\mathbb{S}\left[U_{nm}\left(\mathrm{x,t}\right)-\chi_m\mathrm{U}_{n(m-1)}\left(\mathrm{x,t}\right)\right]=\hbar_n\mathrm{H}_n\left(\mathrm{x,t}\right)N_m\left[U_n\left(\mathrm{x,t}\right)\right].\end{aligned} \tag{11}$$

Operating the inverse Sumudu transform of both sides, we get

$$\begin{aligned}&U_{im}\left(x,t\right)=\chi_mU_{i(m-1)}\left(x,t\right)+\hbar_i\mathbb{S}^{-1}\left[H_i\left(x,t\right)R_{im}\left(\overrightarrow{U}_{i(m-1)},x,t\right)\right],\\&U_{nm}\left(x,t\right)=\chi_mU_{n(m-1)}\left(x,t\right)+\hbar_n\mathbb{S}^{-1}\left[H_n\left(x,t\right)R_{nm}\left(\overrightarrow{U}_{n(m-1)},x,t\right)\right],\end{aligned} \tag{12}$$

where

$$R_{im}\left(\overrightarrow{U}_{i(m-1)},x,t\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\varphi_i\left(x,t;q\right)}{\partial q^{m-1}}\right|_{q=0}. \tag{13}$$

and

$$\chi_m=\left\{\begin{array}{ll}0, & m\leq 1,\\ 1 & m>1.\end{array}\right.$$

In this way, it is easy to obtain $U_{i\,m}\left(x,t\right)$ for $m\geq 1$, at M^{th} order, we have

$$U_i\left(x,t\right)=\sum_{m=0}^{M}U_{im}\left(x,t\right), \tag{14}$$

where $M\rightarrow\infty$, we obtain an accurate approximation of the original equation (2).

5. Convergence Analysis

Theorem 5.1 Let $T : \Omega \rightarrow \Omega$ is nonlinear mapping defined on Banach space $(\Omega, \|.\|)$. The series solution (14) of problem (1) using HAFSTM converges if $|A_{i1}| < \gamma$ where $\gamma > 0$, using the Banach's fixed point theory [47].

Proof: Consider a Banach space $(C(X), \|.\|)$ with all continuous functions on $X$with norm $\|V(t)\| = \max_{t\in X} |V(t)|$.

Let the sequence $\{A_{ri}\}$ be defined as $A_{ri} = \sum\limits_{m=0}^{r} U_{im}(x,t)$, which is a part of solution (14),

Since,

$$D_t^{\alpha_n} U_i(x,t) = f(x,t,U_1,U_2,....,U_n) = \sum_{m=0}^{r} U_{im}(x,t) \tag{15}$$

Where f satisfies the Lipschitz condition with Lipschitz constant τ such as,

$$|f(x,t,A_{r1},A_{r2},....,A_{rn}) - f(x,t,A_{t1},A_{t2},....,A_{tn})| = \tau \sum_{j=0}^{n} \left|A_{j(i(r-1))} - A_{j(i(t-1))}\right| \tag{16}$$

From (4.2) and ,

$$U_{im} = (\chi_m + \hbar_i) U_{i(m-1)} - \hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[f(x,t,U_1,U_2,....,U_n)\right]\right] \tag{17}$$

Assuming A_{ri} and A_{ti} are the two arbitrary partial sums where $ri > ti$,

$$A_{ri} = (\chi_m + \hbar_i) A_{i(r-1)} - \hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[f(x,t,A_{r1},A_{r2},....,A_{rn})\right]\right] \tag{18}$$

and

$$A_{ti} = (\chi_m + \hbar_i) A_{i(t-1)} - \hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[f(x,t,A_{t1},A_{t2},....,A_{tn})\right]\right] \tag{19}$$

Now, we can show that sequence $\{A_{ri}\}$ is Cauchy sequence in Banach space $(\Omega, \|.\|)$

$$\begin{aligned}
A_{ri} - A_{ti} = &\ (\chi_m + \hbar_i)\left(A_{i(r-1)} - A_{i(t-1)}\right) \\
&-\hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[f(x,t,A_{r1},A_{r2},....,A_{rn}) - f(x,t,A_{t1},A_{t2},....,A_{tn})\right]\right] \\
|A_{ri} - A_{ti}| = &\ \Big|(\chi_m + \hbar_i)\left(A_{i(r-1)} - A_{i(t-1)}\right) \\
&-\hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[f(x,t,A_{r1},A_{r2},....,A_{rn}) - f(x,t,A_{t1},A_{t2},....,A_{tn})\right]\right]\Big| \\
\le &\ (\chi_m + \hbar_i)\left|A_{i(r-1)} - A_{i(t-1)}\right| \\
&-\hbar_i H_i(x,t) \mathbb{S}^{-1}\left[u^{-\alpha_i} \mathbb{S}\left[|f(x,t,A_{r1},A_{r2},....,A_{rn}) - f(x,t,A_{t1},A_{t2},....,A_{tn})|\right]\right]\Big|
\end{aligned}$$

Applying the convolution theorem of Sumudu transform [44]

$$\begin{aligned}
|A_{ri} - A_{ti}| \le &\ (\chi_m + \hbar_i)\left|A_{i(r-1)} - A_{i(t-1)}\right| \\
-\hbar_i H_i(x,t) \int_0^t &\left[|f(x,t,A_{r1},A_{r2},....,A_{rn}) - f(x,t,A_{t1},A_{t2},....,A_{tn})|\right] \frac{(t-\theta)^{\alpha_i}}{\Gamma(\alpha_i+1)}
\end{aligned}$$

Using (22)

$$|A_{ri}-A_{ti}|\leq\ (\chi_m+\hbar_i)\left|A_{i(r-1)}-A_{i(t-1)}\right|-\hbar_i H_i\left(x,t\right)\tau\sum_{j=0}^{n}\left|A_{j(i(r-1))}-A_{j(i(t-1))}\right|$$

Taking maximum value

$$\|A_{ri}-A_{ti}\|\leq\ \lambda\left\|A_{i(r-1)}-A_{i(t-1)}\right\| \tag{20}$$

Replacing $ri=ti+1$ in (26) then,

$$\|A_{ti+1}-A_{ti}\|\leq\ \lambda\|A_{ti}-A_{ti-1}\|\leq\lambda^2\|A_{ti-1}-A_{ti-2}\|\leq.....\leq\lambda^{ti}\|A_{0i}-A_{1i}\|$$

Using the triangle inequality

$$\begin{aligned}\|A_{ri}-A_{ti}\|&\leq\ \|A_{ti+1}-A_{ti}\|+\|A_{ti}-A_{ti-1}\|+\|A_{ti-1}-A_{ti-2}\|+\ \ldots\ +\|A_{ri}-A_{ri-1}\|\\&\leq\ \left[\lambda^{ti}+\lambda^{ti+1}+\ \ldots\ +\lambda^{ri-1}\right]\|A_{0i}-A_{1i}\|\\&\leq\ \lambda^{ti}\left[1+\lambda+\lambda^2+\ \ldots\ +\lambda^{ri-ti-1}\right]\|A_{0i}-A_{1i}\|\\&\leq\ \lambda^{ti}\left[\frac{1-\lambda^{ri-ti-1}}{1-\lambda}\right]\|A_{0i}-A_{1i}\|\ .\end{aligned}$$

Where $0<\lambda<1$, then

$$1-\lambda^{ri-ti-1}<1$$

$$\|A_{ri}-A_{ti}\|\leq\ \frac{\lambda^{ti}}{1-\lambda}\|A_{0i}-A_{1i}\|$$

$$\|A_{ri}-A_{ti}\|\leq\ \frac{\lambda^{ti}}{1-\lambda}\max_{t\in X}|A_{i1}|$$

Given that

$$|A_{i1}|<\gamma$$

and as $ti\to\infty$ then $\|A_{ri}-A_{ti}\|\to 0$ and
hence the sequence $\{A_{ri}\}$ is a Cauchy sequence in this Banach space $(\Omega,\ \|\,.\,\|)$. Therefore (14) is converges.

Remark: Since the function satisfies the Lipschitz condition (22) (5.2) then the (1) posses the unique solution in $(C\left(X\right),\ \|\,.\,\|)$.

6. Illustrative Examples

To illustrate the efficiency and accuracy of above discussed method, we consider some multi –term time fractional diffusion equations. We transform the MTTFDE as a system of FPDE and evaluate it using the HAFSTM.

Example 1 we consider the following two –term time fractional diffusion equation [45]

$$\begin{cases}{}_0^cD_t^{\alpha_1}U\left(x,t\right)+{}_0^cD_t^{\alpha_2}U\left(x,t\right)=\partial_{xx}U\left(x,t\right)+F\left(x,t\right),\\U\left(x,0\right)=0,\ x\in\left(0,1\right),\\U\left(0,t\right)=U\left(1,t\right)=0,\ t\in\left(0,1\right],\end{cases} \tag{21}$$

where

$$\begin{aligned}F\left(x,t\right)=&\ \frac{6}{\Gamma\left(4-\alpha_1\right)}t^{3-\alpha_1}\sin\,\pi x\ +\ \frac{6}{\Gamma\left(4-\alpha_2\right)}t^{3-\alpha_2}\sin\,\pi x\\&+\pi^2\,t^3\sin\ \pi x.\end{aligned}$$

The exact solution of Eq. (21) is $U(x,t) = t^3 \sin \pi x$.
We can convert the Eq. (21) in following system of time fractional partial differential equation

$$\begin{aligned} &D_t^{\alpha_2} U(x,t) = V(x,t), \quad U(x,0) = 0, \\ &D_t^{\alpha_1-\alpha_2} V(x,t) = -V(x,t) + \partial_{xx} U(x,t) + F(x,t). \end{aligned} \tag{22}$$

Applying the Sumudu transform of Eq. (22)

$$\begin{aligned} &\frac{\mathbb{S}[U(x,t)]}{u^{\alpha_2}} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha_2-k)}} - \mathbb{S}[V(x,t)] = 0, \\ &\frac{\mathbb{S}[V(x,t)]}{u^{\alpha_1-\alpha_2}} - \sum_{l=0}^{n-1} \frac{V^{(l)}(0)}{u^{(\alpha_1-\alpha_2-l)}} + \mathbb{S}[V(x,t) - \partial_{xx} U(x,t) - F(x,t)] = 0, \end{aligned}$$

$$\begin{aligned} &\mathbb{S}[U(x,t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(-k)}} - u^{\alpha_2}\, \mathbb{S}[V(x,t)] = 0, \\ &\mathbb{S}[V(x,t)] - \sum_{l=0}^{n-1} \frac{V^{(l)}(0)}{u^{(-l)}} + u^{\alpha_1-\alpha_2} \mathbb{S}[V(x,t) - \partial_{xx} U(x,t) - F(x,t)] = 0. \end{aligned} \tag{23}$$

Now, the nonlinear operator is defined as

$$\begin{aligned} &N[\phi_1(\text{x,t;q})] = \mathbb{S}[\phi_1(\text{x,t;q})] - \sum_{k=0}^{n-1} \frac{{\phi_1}^{(k)}(0)}{u^{(-k)}} - u^{\alpha_2}\, \mathbb{S}[\phi_1(\text{x,t;q})], \\ &N[\phi_2(\text{x,t;q})] = \mathbb{S}[\phi_2(\text{x,t;q})] - \sum_{l=0}^{n-1} \frac{{\phi_2}^{(l)}(0)}{u^{(-l)}} + u^{\alpha_1-\alpha_2} \mathbb{S}[\phi_2(\text{x,t;q}) - \partial_{xx}\phi_1(\text{x,t;q}) - F(x,t)]. \end{aligned} \tag{24}$$

In the view of discussion, we can construct the zeroth –order deformation equation

$$\begin{aligned} &(1-\mathrm{q})\, \mathbb{S}[\varphi_1(\mathrm{x,t;q}) - \mathrm{U}_0(\mathrm{x,t})] = \hbar_1 \mathrm{q} \mathrm{H}_1(\mathrm{x,t})\, N[\varphi_1(\mathrm{x,t;q})], \\ &(1-\mathrm{q})\, \mathbb{S}[\varphi_2(\mathrm{x,t;q}) - V_0(\mathrm{x,t})] = \hbar_2 \mathrm{q} \mathrm{H}_2(\mathrm{x,t})\, N[\varphi_2(\mathrm{x,t;q})]. \end{aligned} \tag{25}$$

The $m^{th}-$ order deformation equation is given by

$$\begin{aligned} &U_m(x,t) = \chi_m U_{m-1}(x,t) + \hbar_1 \mathbb{S}^{-1}\left[H_1(x,t)\, R_{1m}\left(\overrightarrow{U}_{(m-1)}, x, t\right)\right], \\ &V_m(x,t) = \chi_m V_{m-1}(x,t) + \hbar_2 \mathbb{S}^{-1}\left[H_2(x,t)\, R_{2m}\left(\overrightarrow{V}_{m-1}, x, t\right)\right], \end{aligned} \tag{26}$$

where

$$\begin{aligned} R_{1m}\left(\overrightarrow{U}_{m-1}\right) &= \mathbb{S}[U_{m-1}(x,t)] - u^{\alpha_2}\, \mathbb{S}[U_{m-1}(x,t)], \\ R_{2m}\left(\overrightarrow{V}_{m-1}\right) &= \mathbb{S}[V_{m-1}(x,t)] + u^{\alpha_1-\alpha_2} \mathbb{S}[V_{m-1}(x,t) \\ &\quad -\partial_{xx} U_{m-1}(x,t) - (1-\chi_m)\, F(x,t)]. \end{aligned} \tag{27}$$

On solving above equation from $m = 1, 2, ...$, we get

$$U_1(x,t) = 0,$$

$$V_1(x,t) = -\hbar_2 6t^{3-2\alpha_1} \sin \pi x \left(\frac{t^{\alpha_1}}{\Gamma(4-\alpha_1)} + \frac{t^{\alpha_2}}{\Gamma(4-2\alpha_1+\alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4-2\alpha_1+\alpha_2)} \right),$$

$$U_2(x,t) = \hbar_1\hbar_2 6t^{3-2\alpha_1+\alpha_2} \sin \pi x \left(\frac{t^{\alpha_1}}{\Gamma(4-\alpha_1+\alpha_2)} + \frac{t^{\alpha_2}}{\Gamma(4-2\alpha_1+2\alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4-\alpha_1+2\alpha_2)} \right),$$

$$V_2(x,t) = -\hbar_2 6t^{3-2\alpha_1}(1+\hbar_2) \sin \pi x \left(\frac{t^{\alpha_1}}{\Gamma(4-\alpha_1)} + \frac{t^{\alpha_2}}{\Gamma(4-2\alpha_1+2\alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4-\alpha_1+2\alpha_2)} \right) - \hbar_2^2 6t^{3-2\alpha_1+\alpha_2} \sin \pi x \left(\frac{t^{\alpha_1}}{\Gamma(4-2\alpha_1+\alpha_2)} + \frac{t^{\alpha_2}}{\Gamma(4-3\alpha_1+2\alpha_2)} + \frac{\pi^2 t^{\alpha_1+\alpha_2}}{\Gamma(4-2\alpha_1+2\alpha_2)} \right),$$

$$\begin{aligned} U_3(x,t) = {} & \frac{12\hbar_1\hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} + \frac{12\hbar_1\hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} \\ & + \frac{12\pi^2\hbar_1\hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} + \frac{6\,\hbar_1^2\hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} \\ & + \frac{6\,\hbar_1^2\hbar_2 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+2\alpha_2)} + \frac{6\pi^2\,\hbar_1^2\hbar_2 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+2\alpha_2)} \\ & + \frac{6\,\hbar_2^2\hbar_1 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} + \frac{12\,\hbar_2^2\hbar_1 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+2\alpha_2)} \\ & + \frac{6\pi^2\,\hbar_2^2\hbar_1 t^{3-\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+2\alpha_2)} + \frac{6\,\pi^2\hbar_2^2\hbar_1 t^{3-2\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+3\alpha_2)}, \end{aligned}$$

$$\begin{aligned} V_3(x,t) = {} & \frac{-6\hbar_2 t^{3-\alpha_1} \sin \pi x}{\Gamma(4-\alpha_1)} - \frac{6\hbar_2 t^{3-2\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+\alpha_2)} \\ & - \frac{6\pi^2\hbar_2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} - \frac{12\hbar_2^2 t^{3-\alpha_1} \sin \pi x}{\Gamma(4-\alpha_1)} \\ & - \frac{24\,\hbar_2^2 t^{3-2\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+\alpha_2)} - \frac{12\,\pi^2\hbar_2^2 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} \\ & - \frac{12\,\hbar_2^2 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-3\alpha_1+2\alpha_2)} - \frac{12\,\pi^2\hbar_2^2 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+2\alpha_2)} \\ & + \frac{6\,\pi^2\hbar_1\hbar_2^2 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+2\alpha_2)} + \frac{6\,\pi^2\hbar_1\hbar_2^2 t^{3-3\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4-3\alpha_1+3\alpha_2)} \\ & + \frac{6\,\pi^4\hbar_1\hbar_2^2 t^{3-2\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+3\alpha_2)} - \frac{6\,\hbar_2^3 t^{3-\alpha_1} \sin \pi x}{\Gamma(4-\alpha_1)} \\ & - \frac{18\,\hbar_2^3 t^{3-2\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+\alpha_2)} - \frac{6\,\pi^2\hbar_2^3 t^{3-\alpha_1+\alpha_2} \sin \pi x}{\Gamma(4-\alpha_1+\alpha_2)} \\ & - \frac{18\,\hbar_2^3 t^{3-3\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-3\alpha_1+2\alpha_2)} - \frac{12\,\pi^2\hbar_2^3 t^{3-2\alpha_1+2\alpha_2} \sin \pi x}{\Gamma(4-2\alpha_1+2\alpha_2)} \\ & - \frac{6\,\hbar_2^3 t^{3-4\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4-4\alpha_1+3\alpha_2)} - \frac{6\,\pi^2\hbar_2^3 t^{3-2\alpha_1+3\alpha_2} \sin \pi x}{\Gamma(4-3\alpha_1+3\alpha_2)}, \end{aligned}$$

$$\vdots$$

etc.

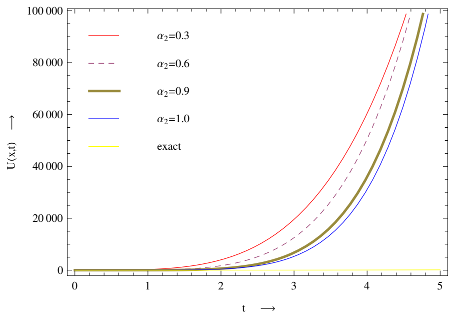

Figure 1: Plot of $U(x,t)$ w.r.t t at $x = 0.5$ for $\alpha =$ 0.3, 0.6, 0.9, 1.0 and exact solution.

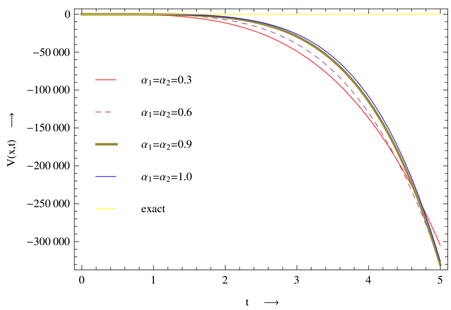

Figure 2: Plot of $V(x,t)$ w.r.t t at $x = 0.5$ for $\alpha =$ 0.3, 0.6, 0.9, 1.0 and exact solution.

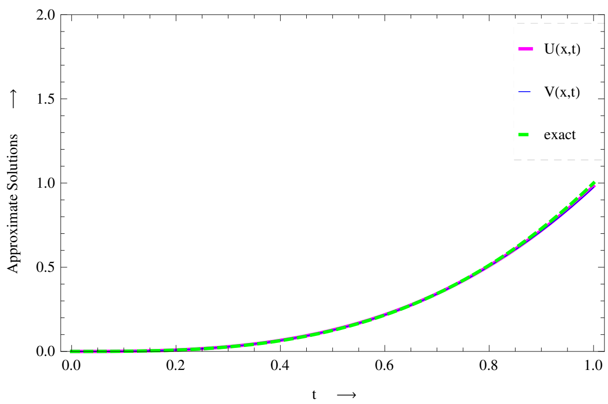

Figure 3: Plot of approximate solutions and exact solution w.r.t. t.

Figs. 1, 2 reflects the association of Brownian variation of variable order multi–term time fractional advection differential equations at $\hbar_1 = \hbar_2 = -1$, for $x = 0.5$ with respect to time variable t. The figure 3 shows the he comparative study of exact and approximate solutions of certain value which can be controlled by parameters $\hbar_1$, $\hbar_2$, the values are taken $\alpha_1 = 0.248,\ \alpha_2 = 0.074,\ \hbar_1 = -2,\ \hbar_2 = -0.023.$

Example 2 we consider the following two –term time fractional diffusion equation [45]

$$\begin{cases} {}_0^cD_t^{\alpha_1}U(x,t) + {}_0^cD_t^{\alpha_2}U(x,t) + {}_0^cD_t^{\alpha_3}U(x,t) = \partial_{xx}U(x,t) + F(x,t), \\ U(x,0) = 0,\ x \in (0,1), \\ U(0,t) = U(1,t) = 0,\ t \in (0,1], \end{cases} \tag{28}$$

where

$$F(x,t) = \pi^2 t^3 \sin\ \pi x + \frac{6}{\Gamma(4-\alpha_1)} t^{3-\alpha_1} \sin \pi x \ + \frac{6}{\Gamma(4-\alpha_2)} t^{3-\alpha_2} \sin \pi x + \frac{6}{\Gamma(4-\alpha_3)} t^{3-\alpha_3} \sin \pi x.$$

The exact solution of Eq. (28) is $U(x,t) = t^3 \sin \pi x$.
We can convert the Eq. (28) in following system of time fractional partial differential equation

$$\begin{aligned} D_t^{\alpha_3}U(x,t) &= V(x,t), \quad U(x,0) = 0, \\ D_t^{\alpha_2-\alpha_3}V(x,t) &= W(x,t), \quad V(x,0) = 0, \\ D_t^{\alpha_1-\alpha_2}W(x,t) &= -W(x,t) - V(x,t) + \partial_{xx}U(x,t) + F(x,t). \end{aligned} \tag{29}$$

Applying the Sumudu transform of Eq. (29)

$$\begin{aligned} &\frac{\mathbb{S}[U(x,t)]}{u^{\alpha_3}} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha_3-k)}} - \mathbb{S}[V(x,t)] = 0, \\ &\frac{\mathbb{S}[V(x,t)]}{u^{\alpha_2-\alpha_3}} - \sum_{l=0}^{n-1} \frac{V^{(l)}(0)}{u^{(\alpha_2-\alpha_3-l)}} - \mathbb{S}[W(x,t)] = 0, \\ &\frac{\mathbb{S}[W(x,t)]}{u^{\alpha_1-\alpha_2-\alpha_3}} - \sum_{m=0}^{n-1} \frac{W^{(m)}(0)}{u^{(\alpha_1-\alpha_2-m)}} + \mathbb{S}[V(x,t) + W(x,t) - \partial_{xx}U(x,t) - F(x,t)] = 0, \end{aligned}$$

$$\begin{aligned} &\mathbb{S}[U(x,t)] - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(-k)}} - u^{\alpha_3}\,\mathbb{S}[V(x,t)] = 0, \\ &\mathbb{S}[V(x,t)] - \sum_{l=0}^{n-1} \frac{V^{(l)}(0)}{u^{(-l)}} - u^{\alpha_2-\alpha_3}\,\mathbb{S}[W(x,t)] = 0, \\ &\mathbb{S}[W(x,t)] - \sum_{m=0}^{n-1} \frac{W^{(m)}(0)}{u^{(-m)}} + u^{\alpha_1-\alpha_2}\mathbb{S}[V(x,t) + W(x,t) - \partial_{xx}U(x,t) - F(x,t)] = 0. \end{aligned} \tag{30}$$

Now, the nonlinear operator is defined as

$$
\begin{aligned}
N\left[\phi_1\left(\mathrm{x,t;q}\right)\right] &= \mathbb{S}\left[\phi_1\left(\mathrm{x,t;q}\right)\right] - \sum_{k=0}^{n-1}\frac{\phi_1{}^{(k)}\left(0\right)}{u^{(-k)}} - u^{\alpha_3}\,\mathbb{S}\left[\phi_1\left(\mathrm{x,t;q}\right)\right],\\
N\left[\phi_2\left(\mathrm{x,t;q}\right)\right] &= \mathbb{S}\left[\phi_2\left(\mathrm{x,t;q}\right)\right] - \sum_{l=0}^{n-1}\frac{\phi_2{}^{(l)}\left(0\right)}{u^{(-l)}} - u^{\alpha_2-\alpha_3}\,\mathbb{S}\left[\phi_2\left(\mathrm{x,t;q}\right)\right],\\
N\left[\phi_3\left(\mathrm{x,t;q}\right)\right] &= \mathbb{S}\left[\phi_3\left(\mathrm{x,t;q}\right)\right] - \sum_{m=0}^{n-1}\frac{\phi_3{}^{(m)}\left(0\right)}{u^{(-m)}} + u^{\alpha_1-\alpha_2}\mathbb{S}\left[\phi_2\left(\mathrm{x,t;q}\right)\right.\\
&\left.+\phi_3\left(\mathrm{x,t;q}\right) - \partial_{xx}\phi_3\left(\mathrm{x,t;q}\right) - F\left(x,t\right)\right].
\end{aligned}
\tag{31}
$$

In the view of discussion, we can construct the zeroth –order deformation equation

$$
\begin{aligned}
&(1-\mathrm{q})\;\mathbb{S}\left[\varphi_1\left(\mathrm{x,t;q}\right)-\mathrm{U}_0\left(\mathrm{x,t}\right)\right] = \hbar_1\mathrm{q}\mathrm{H}_1\left(\mathrm{x,t}\right)N\left[\varphi_1\left(\mathrm{x,t;q}\right)\right],\\
&(1-\mathrm{q})\;\mathbb{S}\left[\varphi_2\left(\mathrm{x,t;q}\right)-V_0\left(\mathrm{x,t}\right)\right] = \hbar_2\mathrm{q}\mathrm{H}_2\left(\mathrm{x,t}\right)N\left[\varphi_2\left(\mathrm{x,t;q}\right)\right],\\
&(1-\mathrm{q})\;\mathbb{S}\left[\varphi_3\left(\mathrm{x,t;q}\right)-W_0\left(\mathrm{x,t}\right)\right] = \hbar_3\mathrm{q}\mathrm{H}_3\left(\mathrm{x,t}\right)N\left[\varphi_3\left(\mathrm{x,t;q}\right)\right].
\end{aligned}
\tag{32}
$$

The $m^{th}-$ order deformation equation is given by

$$
\begin{aligned}
&U_m\left(x,t\right) = \chi_m U_{m-1}\left(x,t\right) + \hbar_1\mathbb{S}^{-1}\left[H_1\left(x,t\right)R_{1m}\left(\overrightarrow{U}_{(m-1)},x,t\right)\right],\\
&V_m\left(x,t\right) = \chi_m V_{m-1}\left(x,t\right) + \hbar_2\mathbb{S}^{-1}\left[H_2\left(x,t\right)R_{2m}\left(\overrightarrow{V}_{(m-1)},x,t\right)\right],\\
&W_m\left(x,t\right) = \chi_m W_{m-1}\left(x,t\right) + \hbar_1\mathbb{S}^{-1}\left[H_3\left(x,t\right)R_{3m}\left(\overrightarrow{W}_{(m-1)},x,t\right)\right],
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
R_{1m}\left(\overrightarrow{U}_{m-1}\right) &= \mathbb{S}\left[U_{m-1}\left(x,t\right)\right] - u^{\alpha_3}\,\mathbb{S}\left[U_{m-1}\left(x,t\right)\right],\\
R_{2m}\left(\overrightarrow{V}_{m-1}\right) &= \mathbb{S}\left[V_{m-1}\left(x,t\right)\right] - u^{\alpha_3-\alpha_2}\,\mathbb{S}\left[V_{m-1}\left(x,t\right)\right],\\
R_{3m}\left(\overrightarrow{W}_{m-1}\right) &= \mathbb{S}\left[W_{m-1}\left(x,t\right)\right] + u^{\alpha_1-\alpha_2}\mathbb{S}\left[W_{m-1}\left(x,t\right) - V_{m-1}\left(x,t\right)\right.\\
&\left.-\partial_{xx}U_{m-1}\left(x,t\right) - \left(1-\chi_m\right)F\left(x,t\right)\right].
\end{aligned}
\tag{34}
$$

On solving above equation from $m = 1, 2, ...,$ we get

$$
\begin{aligned}
U_1\left[x,t\right] &= 0,\\
V_1\left[x,t\right] &= 0,\\
W_1\left[x,t\right] &= \frac{-6\hbar_3 t^{3+\alpha_1-2\alpha_2}\,\sin\,\pi x}{\Gamma\left(4+\alpha_1-2\alpha_2\right)} - \frac{6\hbar_3 t^{3-\alpha_2}\,\sin\,\pi x}{\Gamma\left(4-\alpha_2\right)}\\
&-\frac{6\hbar_3\pi^2 t^{3-\alpha_2}\,\sin\,\pi x}{\Gamma\left(4-\alpha_2\right)} - \frac{6\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\,\sin\,\pi x}{\Gamma\left(4+\alpha_1-\alpha_2-\alpha_3\right)},
\end{aligned}
$$

$$U_2\left[x,t\right] = 0,$$

$$
\begin{aligned}
V_2\left[x,t\right] &= \frac{6\hbar_2\hbar_3 t^{3+\alpha_1-2\alpha_3}\,\sin\,\pi x}{\Gamma\left(4+\alpha_1-2\alpha_2\right)} + \frac{6\hbar_2\hbar_3 t^{3-\alpha_3}\,\sin\,\pi x}{\Gamma\left(4-\alpha_3\right)}\\
&+\frac{6\pi^2\hbar_2\hbar_3\pi^2 t^{3+\alpha_1-\alpha_3}\,\sin\,\pi x}{\Gamma\left(4+\alpha_1-\alpha_3\right)} + \frac{6\hbar_2\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\,\sin\,\pi x}{\Gamma\left(4+\alpha_1-\alpha_2-\alpha_3\right)},
\end{aligned}
$$

$$W_2[x,t]=\frac{-6\hbar_3 t^{3+\alpha_1-2\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}-\frac{6\hbar_3 t^{3-\alpha_2}\sin\pi x}{\Gamma(4-\alpha_2)}$$
$$-\frac{6\hbar_3\pi^2 t^{3+\alpha_1-\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2)}-\frac{6\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}$$
$$-\frac{6\hbar_3^2 t^{3+\alpha_1-2\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}-\frac{6\hbar_3^2 t^{3-\alpha_2}\sin\pi x}{\Gamma(4-\alpha_2)}$$
$$-\frac{6\pi^2\hbar_3^2 t^{3+\alpha_1-\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2)}-\frac{6\hbar_3^2 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)},$$

$$U_3[x,t]=-6\hbar_1\hbar_2\hbar_3\sin\pi x\left(\frac{t^3}{6}+\frac{\pi^2 t^{3+\alpha_1}}{\Gamma(4+\alpha_1)}\right.$$
$$\left.+\frac{t^{3+\alpha_1-\alpha_2}}{\Gamma(4+\alpha_1-\alpha_2)}+\frac{t^{3+\alpha_1-\alpha_3}}{\Gamma(4+\alpha_1-\alpha_2)}\right),$$

$$V_3[x,t]=\frac{12\,\hbar_2\hbar_3 t^{3+\alpha_1-2\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}+\frac{12\hbar_2\hbar_3 t^{3-\alpha_3}\sin\pi x}{\Gamma(4-\alpha_3)}$$
$$+\frac{12\pi^2\hbar_2\hbar_3\pi^2 t^{3+\alpha_1-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_3)}+\frac{12\hbar_2\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}$$
$$+\frac{6\,\hbar_2^2\hbar_3 t^{3+\alpha_1-2\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}+\frac{6\,\hbar_2^2\hbar_3 t^{3-\alpha_3}\sin\pi x}{\Gamma(4-\alpha_3)}$$
$$+\frac{6\pi^2\,\hbar_2^2\hbar_3 t^{3+\alpha_1-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_3)}+\frac{6\,\hbar_2^2\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_3)}$$
$$+\frac{6\,\hbar_2^2\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}+\frac{6\,\hbar_3^2\hbar_2 t^{3+\alpha_1-2\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_3)}$$
$$+\frac{6\,\pi^2\hbar_3^2\hbar_2 t^{3+\alpha_1-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_3)}+\frac{6\,\hbar_3^2\hbar_2 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)},$$

$$W_3[x,t]=\frac{-6\hbar_3 t^{3+\alpha_1-2\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}-\frac{6\hbar_3 t^{3-\alpha_2}\sin\pi x}{\Gamma(4-\alpha_2)}$$
$$-\frac{6\hbar_3\pi^2 t^{3+\alpha_1-\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2)}-\frac{6\hbar_3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}$$
$$-\frac{12\hbar_3^2 t^{3+\alpha_1-2\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}-\frac{12\hbar_3^2 t^{3-\alpha_2}\sin\pi x}{\Gamma(4-\alpha_2)}$$
$$-\frac{12\pi^2\hbar_3^2 t^{3+\alpha_1-\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2)}-\frac{12\hbar_3^2 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}$$
$$+\frac{6\hbar_2\hbar_3^2 t^{3+2\alpha_1-\alpha_2-2\alpha_3}\sin\pi x}{\Gamma(4+2\alpha_1-\alpha_2-2\alpha_3)}+\frac{6\hbar_2\hbar_3^2 t^{3+2\alpha_1-2\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+2\alpha_1-2\alpha_2-\alpha_3)}$$
$$+\frac{6\hbar_2\hbar_3^2 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)}+\frac{6\pi^2\hbar_2\hbar_3^2 t^{3+2\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+2\alpha_1-\alpha_2-\alpha_3)}$$
$$-\frac{6\hbar_3^3 t^{3+\alpha_1-2\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-2\alpha_2)}-\frac{6\hbar_3^3 t^{3-\alpha_2}\sin\pi x}{\Gamma(4-\alpha_2)}$$
$$-\frac{6\pi^2\hbar_3^3 t^{3+\alpha_1-\alpha_2}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2)}-\frac{6\hbar_3^3 t^{3+\alpha_1-\alpha_2-\alpha_3}\sin\pi x}{\Gamma(4+\alpha_1-\alpha_2-\alpha_3)},$$

.
.
.

etc.

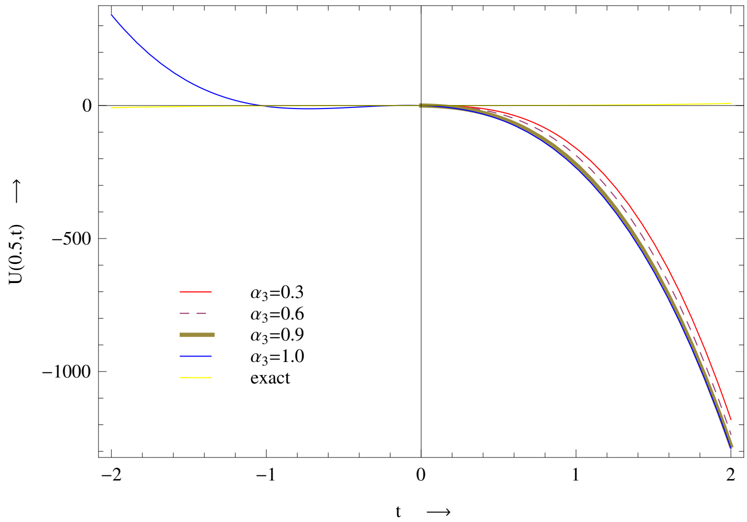

Figure 4: Plot of $U(x,t)$ w.r.t t at $x = 0.5$ for $\alpha =$ 0.3, 0.6, 0.9, 1.0 and exact solution.

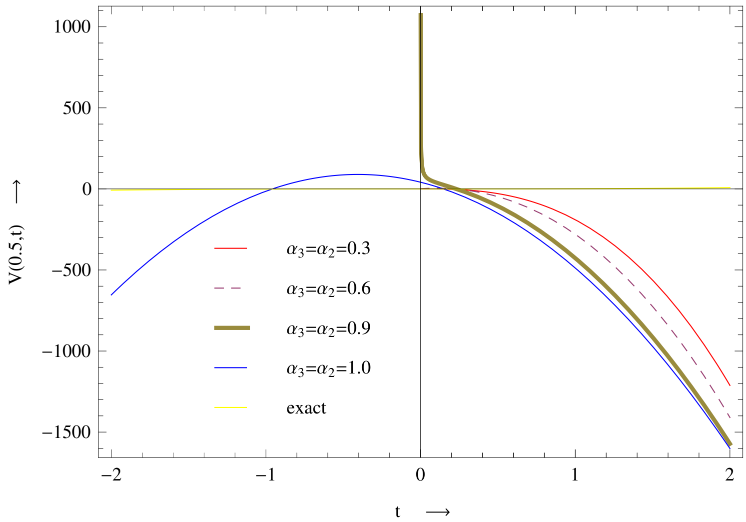

Figure 5: Plot of $V(x,t)$ w.r.t t at $x = 0.5$ for $\alpha =$ 0.3, 0.6, 0.9, 1.0 and exact solution.

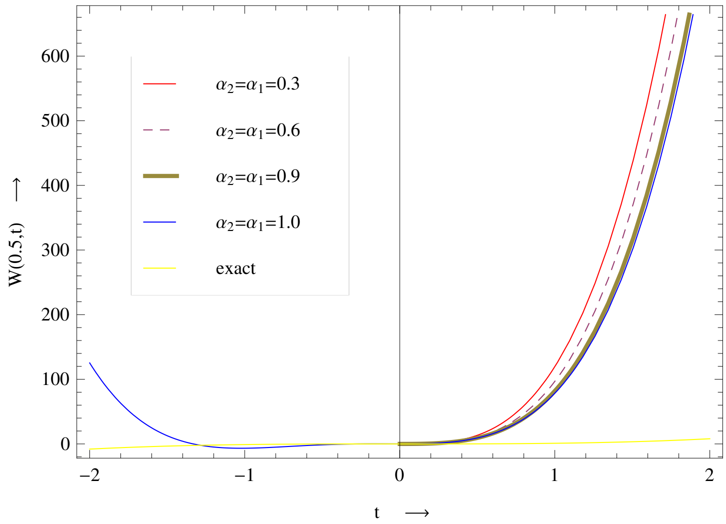

Figure 6: Plot of $W(x,t)$ w.r.t t at $x = 0.5$ for $\alpha =$ 0.3, 0.6, 0.9, 1.0 and exact solution.

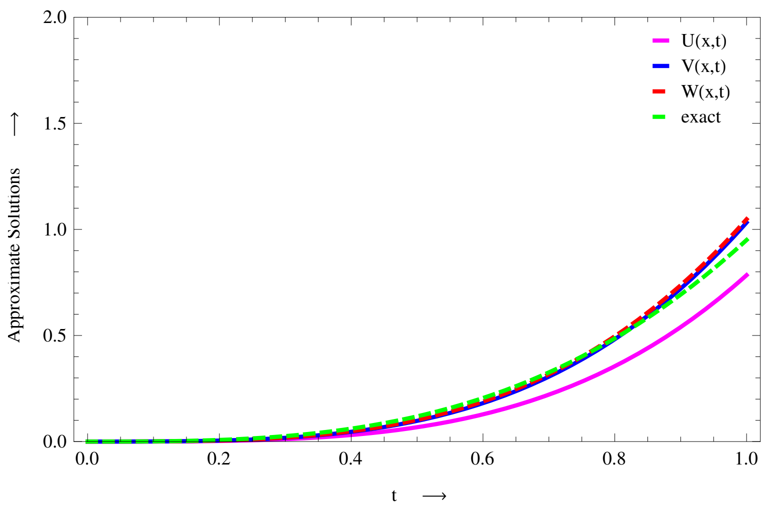

Figure 7: Plot of approximate solutions and exact solution w.r.t. t.

Figs. 4, 5 and 6 shows the association of Brownian variation of variable order multi − term time fractional advection differential equations at $\hbar_1 = \hbar_2 = \hbar_3 = -1$, for $x = 0.5$ with respect to time variable t. The figure 7 shows the he comparative study of exact and approximate solutions of certain value which can be controlled by parameters $\hbar_1$, $\hbar_2, \hbar_3$, the values are taken $\alpha_1 = 0.939$, $\alpha_2 = 0.105$, $\alpha_3 = 0.112$, $\hbar_1 = -1.235$, $\hbar_2 = -1.485$, $\hbar_3 = -0.055$, which adjust the convergence region appropriately for exact and $W(x,t)$.

7. Conclusion

This work presents effective semi-analytic method for the solution of the multi–order fractional partial differential equations. These are firstly transformed into the system of PDE's and then the HAFSTM method has been applied with the transformation of domain change using Sumudu transform, which reduces the he complexity without loss of generality. The obtained results are compared with existing exact solutions at integer value of fractional differential equations. The results gained are eloquent in understanding and free of rounding off errors, which are mostly occurring in mess method or perturbation methods. This method can be generalized to solve any kind of multi–order fractional partial differential equations.

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