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CONVERGENCE ANALYSIS OF PICARD-S HYBRID ITERATION SCHEME FOR MULTI-VALUED MAP HAVING A FIXED POINT

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Abstract. In this paper, we define Picard-S hybrid iteration for a multi-valued mapping of T with an invariant point η along with explanation that under certain conditions, this iteration gets converged to an invariant point ζ belonging to T. However, it is essential, to note that this invariant point ζ may be different from η. In this process, several results are generalized.

Keywords: Multi-valued Map, Picard-S hybrid iteration scheme, Fixed point analysis.

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1. Introduction

Let (X, d) be a complete metric space, having a subset K, which is said to be proximinal if for every $x \in X$, there exists an element $k \in K$ that

$$d(x, k) = d(x, K) = \inf\{d(x, y) : y \in K\}.$$

Every closed convex subset of X is proximinal if X is a Hilbert space. The families of all bounded proximinal subsets of K in X, and those of nonempty bounded and closed subsets of X are denoted by $P(K)$ and $CB(X)$ respectively.

Let A, B be two bounded subsets of X. The Hausdorff distance between A and B is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\right\}.$$

Transformation from single valued map to multi valued map, thereby extending the convergence results of single valued mapping with the aid of Picard-S hybrid iteration scheme shall be the focal point of this paper. We will denote the set of all natural numbers by $\mathbb{N}$ over the course of this paper. Also, throughout the paper let X be a Hilbert space and K

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be a compact and convex subset of X.

The most popular and the simplest iteration method which is commonly used to approximate fixed point is known as Picard iteration [1], which is formulated for every $\jmath_0 \in K$, as

$$\jmath_{n+1} = f^n x, \quad n \in \mathbb{N}.$$

But this iteration scheme does not converge with reference to nonexpansive mapping. E.g., the iteration sequence $\jmath_{n+1} = f^n x$ which maps $f : [-1, 1] \to [-1, 1]$ and is defined by $fx = -x$ is not convergent to 0 for every non initial point (being non zero) which is, as a matter of fact, the invariant point of f. Mann [2] introduced an iteration scheme for non expansive mapping, which was convergent iteration sequence for arbitrary $\jmath_1 \in K$ as follows:

$$\jmath_{n+1} = (1 - \vartheta_n)\jmath_n + \vartheta_n f \jmath_n, \quad n \in \mathbb{N},$$

where $\vartheta_n \in (0, 1)$.

In 1974, with a view to appoximate fixed point of pseudo-contractive compact mappings in Hilbert spaces, Ishikawa [3] formulated new iteration procedue for $\jmath_1 \in K$ is as follows:

$$\begin{cases} \ell_n = (1 - \vartheta_n)\jmath_n + \vartheta_n f \jmath_n, \\ \jmath_{n+1} = (1 - \varsigma_n)\jmath_n + \varsigma_n f \ell_n, \quad n \in \mathbb{N}, \end{cases}$$

where ϑ_n, $\varsigma_n \in (0, 1)$.

In order to compare two iteration schemes in one dimension, the scholar has referred to Rhoades [4]. Herein, Ishikawa Iteration convergence rate is shown to better than that of Mann Iteration procedure under favorable conditions. Nadler [5] and Markin [6] studied invariant points for multi-valued nonexpansive mappings and it is for their efforts that now, there is an extensive and vast literature on multi-valued invariant point theory having wide range of applications in diverse areas, be it optimization, or be it differential inclusion [2]. It is because of Lim [7], that the existence of invariant points belonging to mappings which are multi-valued nonexpansive, in Banach Spaces (characteristically uniformly convex), could be proved. In order to approximate the invariant points of multi-valued nonexpansive mappings, a number of iteration schemes processes have been used for the last few years. Among these, noteworthy generalizations of iteration processes given by Mann and Ishikawa, notably in cases of multi-valued mapping can be seen in the iteration processes of Sastry and Babu [8], Panyanak [9], Song and Wang [10] and Shahzad and Zegeye [11].

It's not been long that a single valued iterate scheme known as Picard-S hybrid was introduced by Gürsoy and Karakaya [12] which provided for an iteration convergence rate, which was faster than that developed by Mann [2], Ishikawa [3], Noor [13], SP [14] and S [4, 15] which itself was faster than the one introduced earlier by Picard. Also, there is reference made to carve out a detailed analysis and review of literature with respect to Picard-S hybrid iterates by taking recourse to the Gürsoy and Karakaya [12]. The multi-valued Picard-S hybrid iteration scheme is as follows:

$$\begin{cases} \jmath_0 \in X, \\ \jmath_{n+1} = T\ell_n, \\ \ell_n = (1-\vartheta_n)T\jmath_n + \vartheta_n T z_n, \\ z_n = (1-\varsigma_n)\jmath_n + \varsigma_n T\jmath_n, \end{cases}$$

where real sequences $\{\vartheta_n\}$, $\{\varsigma_n\}$ satisfy $0 < \vartheta_n,\, \varsigma_n < 1$ and , $n \in \mathbb{N}$.

Different spaces having different mappings have been the subjects of various studies undertaken by several reputed authors [16, 17, 18, 19, 15, 20, 21, 22, 23] as a part of the schemes followed by them. As, the Picard-S hybrid iteration scheme is for single valued mapping, we define Picard-S hybrid iterates for a multivalued map T with a fixed point η under certain conditions. For such definition, let a mapping T defined from X to $P(X)$ and consider η as an invariant point belonging to T. The Picard-S hybrid iteration scheme for multi-valued mapping is defined as

$$\begin{cases} \jmath_0 \in X, \\ z_n = (1-\varsigma_n)\jmath_n + \varsigma_n z_n'' \\ \ell_n = (1-\vartheta_n)z_n'' + \vartheta_n z_n''' \\ \jmath_{n+1} = z_n', \end{cases} \tag{A}$$

where $z_n' \in T\ell_n,\ z_n'' \in T\jmath_n$ and $z_n''' \in Tz_n$ such that, $||z' - \eta|| = d(T\ell_n, \eta),\ ||z'' - \eta|| = d(T\jmath_n, \eta)$ and $||z_n''' - \eta|| = d(Tz_n, \eta),\ \forall\ n \in \mathbb{N}$. Also, $\{\vartheta_n\}$ and $\{\varsigma_n\}$ being real sequences such that

$$0 = \vartheta_n,\ \varsigma_n < 1,\ \varsigma_n \to 0 \text{ and } \sum \vartheta_n \varsigma_n = \infty.$$

In this paper, we extend the convergence results for various mappings, such as nonexpansive, quasi-nonexpansive and quasi-contractive and it'll be shown that the sequence of Picard-S hybrid iteration converges to a fixed point for all dissimilar mappings.

2. Preliminaries

The proof of main Theorems are studied by us using some Lemma and Definitions, so here we are mentioning all relevant results to make this article self contained.

Definition 2.1. [8] A mapping T satisfying different inequalities shall have different names according to the satisfaction thereby achieved. Hence, the mapping is known as:

(1) *Multi-valued nonexpansive*, whereby

$$H(Tx, Ty) \leq ||x - y|| \quad \text{for all } x, y \in K.$$

(2) *Multi-valued generalized nonexpansive*, whereby

$$H(Tx, Ty) \leq \alpha||x - y|| + \beta\{d(x, Tx) + d(y, Ty)\} + \gamma\{d(x, Ty) + d(y, Tx)\}$$

for all $x, y \in X$ where $\alpha + 2\beta + 2\gamma \leq 1$.

(3) *Multi-valued quasi-contractive*, wherein for some $0 \leq k < 1$,

$$H(Tx, Ty) \leq \max\{||x - y||, d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

The following Lemmas are useful in our subsequent discussion and are easy to establish.

Lemma 2.1. [8] Considering $\{\vartheta_n\}$, $\{\varsigma_n\}$ being real sequences, wherein

(1) $0 \leq \vartheta_n,\, \varsigma_n < 1$,
(2) $\varsigma_n \to 0$ as $n \to \infty$ and
(3) $\sum \vartheta_n \varsigma_n = \infty$.

Let there be some real sequence $\{\gamma_n\}$ which is non negative and exists in such a manner that $\sum \vartheta_n \varsigma_n (1 - \varsigma_n)\gamma_n$ is bounded, then γ_n has a sub sequence which converges to 0.

Lemma 2.2. [23] If there is a real sequence $\{\jmath_n\}$ satisfying

$$\jmath_{n+1} = \vartheta_n \jmath_n + \varsigma_n$$

where $\jmath_n = 0$, $\varsigma_n = 0$ and $\lim\limits_{n\to\infty} \varsigma_n = 0$, $0 \leq \vartheta_n < 1$, Then $\lim\limits_{n\to\infty} \jmath_n = 0$.

Lemma 2.3. [3] Let $\Theta \in [0, 1]$. Let x, y in a Hilbert space X. Then for any $x, y \in X$, we have

$$||(1 - \Theta)x + \Theta y||^2 = (1 - \Theta)||x||^2 + \Theta||y||^2 - \Theta(1 - \Theta)||x - y||^2.$$

3. Main Results

Theorem 3.1. Suppose that there is a Hilbert space X, having a subset K, which is compact and convex and also that there is a non expansive mapping T, defined from K to $P(K)$, has an invariant point. Assume that

(1) $0 \leq \vartheta_n,\, \varsigma_n < 1$,
(2) $\varsigma_n \to 0$, and
(3) $\sum \vartheta_n \varsigma_n = \infty$.

Then convergence of Picard-S hybrid iteration scheme which is defined as (A) takes place to a fixed point ζ of T.

Proof. From Lemma 2.3 it follows that,

$$\begin{aligned} ||\jmath_{n+1} - \eta||^2 &= ||z'_n - \eta||^2 \\ &\leq H^2(T\ell_n, T\eta) \qquad (1) \\ &\leq ||\ell_n - \eta||^2, \qquad (2) \end{aligned}$$

Also,

$$\begin{aligned} ||\ell_n - \eta||^2 &= ||\,(1 - \vartheta_n)\, z''_n + \vartheta_n z'''_n - \eta||^2 \\ &= (1 - \vartheta_n)||z''_n - \eta||^2 + \vartheta_n||z'''_n - \eta||^2 - \vartheta_n(1 - \vartheta_n)||z''_n - z'''_n||^2 \\ &\leq (1 - \vartheta_n)H^2(T\jmath_n, T\eta) + \vartheta_n H^2(Tz_n, T\eta) - \vartheta_n(1 - \vartheta_n)||z''_n - z'''_n||^2 \\ &\leq (1 - \vartheta_n)||\jmath_n - \eta||^2 + \vartheta_n||z_n - \eta||^2 - \vartheta_n(1 - \vartheta_n)||z''_n - z'''_n||^2, \qquad (3) \end{aligned}$$

which implies

$$||z_n - \eta||^2 = ||\,(1 - \varsigma_n)\,\jmath_n + \varsigma_n z''_n - \eta||^2$$

$$
\begin{aligned}
&= (1-\varsigma_n)||\jmath_n-\eta||^2+\varsigma_n||z''_n-\eta||^2-\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\\
&\le (1-\varsigma_n)||\jmath_n-\eta||^2+\varsigma_n\ H^2(T\jmath_n,T\eta)-\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\\
&\le (1-\varsigma_n)\,||\jmath_n-\eta||^2+\varsigma_n||\jmath_n-\eta||^2-\varsigma_n\,(1-\varsigma_n)\,||\jmath_n-z''_n||^2\\
&\le ||\jmath_n-\eta||^2-\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2. \qquad (4)
\end{aligned}
$$

From (4) and (3), we have

$$
\begin{aligned}
||\ell_n-\eta||^2 &\le (1-\vartheta_n)||\jmath_n-\eta||^2+\vartheta_n[||\jmath_n-\eta||^2-\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2]\\
&\quad -\vartheta_n(1-\vartheta_n)||z''_n-z'''_n||^2\,,\\
||\ell_n-\eta||^2 &\le ||\jmath_n-\eta||^2-\vartheta_n(1-\vartheta_n)||z''_n-z'''_n||^2-\vartheta_n\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\,. \qquad (5)
\end{aligned}
$$

From (5) and (2), we have

$$||\jmath_{n+1}-\eta||^2\le ||\jmath_n-\eta||^2-\vartheta_n(1-\vartheta_n)||z''_n-z'''_n||^2-\vartheta_n\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2.$$

Therefore,

$$\vartheta_n\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\le ||\jmath_n-\eta||^2-||\jmath_{n+1}-\eta||^2-\vartheta_n(1-\vartheta_n)||z''_n-z'''_n||^2.$$

Also,

$$\vartheta_n\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\le ||\jmath_n-\eta||^2-||\jmath_{n+1}-\eta||^2$$

which implies that

$$\sum_{n=1}^{\infty}\vartheta_n\varsigma_n(1-\varsigma_n)||\jmath_n-z''_n||^2\le ||\jmath_n-\eta||^2<\infty.$$

Considering $\{\gamma_n\}$ being a real sequence, which is non negative, so much so that the series $\sum\vartheta_n\varsigma_n(1-\varsigma_n)\gamma_n$ being bounded, wherein convergence of sub-sequence of γ_n to 0 takes place, as is suggested through Lemma 2.1. As such, whenever there is an approach by l towards infinity, there is an approach towards 0 by $||\jmath_{n_l}-z''_{n_l}||$ wherein $\{\jmath_n-z''_n\}$ bears a sub-sequence $||\jmath_{n_l}-z''_{n_l}||$. It is shown that $z''_{n_l}\in T\jmath_{n_l}$, therefore,

$$d(T\jmath_{n_l},\jmath_{n_l})\le ||\jmath_{n_l}-z''_{n_l}||\to 0.$$

Since, $||\jmath_{n_l}-z''_{n_l}||\to 0$ as $l\to\infty$ and $\{\jmath_{n_l}\}\subset K$, where K being compact and assumption can be made that $\jmath_{n_l}\to\zeta$ whenever ∞ is approached by l. Now,

$$d(T\jmath_{n_l},\zeta)\le d(T\jmath_{n_l},\jmath_{n_l})+||\jmath_{n_l}-\zeta||\to 0$$

as $l\to\infty$. Also, $H(d(T\jmath_{n_l},T\zeta))\to 0$ as $l\to\infty$.
Consequently,

$$d(T\zeta,\zeta)\le d(\zeta,T\jmath_{n_l})+H(T\jmath_{n_l},T\zeta)\to 0$$

as $l\to\infty$. Thereby, it can be seen that $\zeta\in T\zeta$. And, thus follows the Theorem 3.1.

□

Theorem 3.2. Suppose that while X being a Hilbert space having K as a subset, which is compact and convex, the generalized nonexpansive mapping T, defined from K to $P(K)$, having an invariant point η. Assume that

(4) $0 \leq \vartheta_n$, $\varsigma_n < 1$,
(5) $\varsigma_n \to 0$, and
(6) $\sum \vartheta_n \varsigma_n = \infty$.

Then, the Picard-S hybrid iteration scheme characterized by (A) gets converged to an invariant point ζ belonging to T.

Proof. Since, we have

$$||\jmath_{n+1} - \eta||^2 \leq ||z'_n - \eta||^2 \leq H^2(T\ell_n, T\eta). \tag{6}$$

Also, T is generalized nonexpansive mapping, we have

$$\begin{aligned} H(Tp, T\ell_n) &\leq \alpha||\ell_n - \eta|| + \beta d(\ell_n, T\ell_n) + \gamma\{d(\eta, T\ell_n) + d(\ell_n, T\eta)\} \\ &\leq \alpha||\ell_n - \eta|| + \beta\{||\ell_n - \eta|| + d(\eta, T\ell_n)\} + \gamma\{d(\eta, T\ell_n) + d(\ell_n, T\eta)\} \\ &\leq (\alpha + \beta + \gamma)||\ell_n - \eta|| + (\beta + \gamma)d(\eta, T\ell_n) \\ &\leq (\alpha + \beta + \gamma)||\ell_n - \eta|| + (\beta + \gamma)H(T\eta, T\ell_n). \end{aligned}$$

Hence

$$H(T\eta, T\ell_n) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}||\ell_n - \eta||. \tag{7}$$

Since

$$\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \leq 1,$$

we have

$$H(T\eta, T\ell_n) \leq ||\ell_n - \eta||.$$

Now, from equations (6) and (7), we have

$$||\jmath_{n+1} - \eta||^2 \leq ||\ell_n - \eta||^2,$$

which is the inequality (2). In the same way, it is of very little significance to show that from inequality (3) and (4), we have

$$||\ell_n - \eta||^2 \leq (1 - \vartheta_n)||\jmath_n - \eta||^2 + \vartheta_n||z_n - \eta||^2 - \vartheta_n(1 - \vartheta_n)||z''_n - z'''_n||^2$$

and

$$||z_n - \eta||^2 \leq ||\jmath_n - \eta||^2 - \varsigma_n(1 - \varsigma_n)||\jmath_n - z''_n||^2.$$

□

Now, proceeding as we did with Theorem 3.1, the aforementioned theorem necessarily follows.

Theorem 3.3. Suppose, X is a Hilbert space having a subset K which is closed as well as convex and bounded, and that a quasi-contractive mapping T is a mapping defined from K to $P(K)$ is a mapping and has an invariant point η. Suppose real sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ in such a manner, that

(1) $0 \leq \vartheta_n$, $\varsigma_n < 1$ for all n,
(2) $\varsigma_n \to 0$ as $n \to \infty$ with $\delta \leq \vartheta_n \leq 1 - k^2$ for some $\delta > 0$.

Thereby, Picard-S iteration sequence as is defined by (A), gets converged to η of T.

Proof. Since, we have

$$||\jmath_{n+1} - \eta||^2 = ||z'_n - \eta||^2, \tag{8}$$
$$||z'_n - \eta|| = d(\eta, T\ell_n) \leq H(T\eta, T\ell_n).$$

Therefore

$$||z'_n - \eta||^2 \leq H^2(T\eta, T\ell_n) \leq k^2 \max\{||\ell_n - \eta||^2, d^2(\ell_n, T\ell_n), d^2(\eta, T\ell_n)\} \tag{9}$$

Also,

$$d^2(\ell_n, T\ell_n) \leq ||\ell_n - \eta||^2.$$

Let $d(\eta, T\ell_n)$ is maximum, we have

$$H^2(T\eta, T\ell_n) \leq k^2 d^2(\eta, T\ell_n) \leq k^2 \max d^2(z, T\ell_n) \leq k^2 H^2(T\eta, T\ell_n)$$

where

$$0 \leq ||z'_n - \eta||^2 \leq H^2(T\eta, T\ell_n) = 0.$$

Hence from (9), we have

$$\begin{aligned} ||\jmath_{n+1} - \eta||^2 &= ||z'_n - \eta||^2 \\ &\leq H^2(T\eta, T\ell_n) \\ &\leq k^2 \max\{||\ell_n - \eta||^2, d^2(\ell_n, T\ell_n)\} \\ &\leq k^2[||\ell_n - \eta||^2 + d^2(\ell_n, T\ell_n)]. \end{aligned} \tag{10}$$

Considering

$$\begin{aligned} ||\ell_n - \eta||^2 &= (1-\vartheta_n)||z''_n - \eta||^2 + \vartheta_n||z'''_n - \eta||^2 - \vartheta_n(1-\vartheta_n)||z''_n - z'''_n||^2 \\ d^2(\ell_n, T\ell_n) &\leq ||\ell_n - z'_n||^2 = ||(1-\vartheta_n)z''_n + \vartheta_n z'''_n - z'_n||^2 \\ &= (1-\vartheta_n)||z''_n - z'_n||^2 + \vartheta_n||z'''_n - z'_n||^2 - \vartheta_n(1-\vartheta_n)||z''_n - z'''_n||^2 \end{aligned} \tag{11}$$

Also, it is to note that

$$||z''_n - \eta||^2 = d^2(\eta, T\jmath_n) \leq H^2(T\eta, T\jmath_n)$$

Therefore

$$||z''_n - \eta||^2 \leq H(T\eta, T\jmath_n) \leq k^2 \max\{||\jmath_n - \eta||^2, d^2(\jmath_n, T\jmath_n), d^2(\eta, T\jmath_n)\}$$

Also,

$$d^2(\jmath_n, T\jmath_n) \leq ||\jmath_n - \eta||^2.$$

Now, on considering $d(\eta, T\jmath_n)$ as maximum, we have

$$H^2(T\eta, T\jmath_n) \leq k^2 d^2(\eta, T\jmath_n).$$

So that

$$0 \leq ||z''_n - \eta||^2 \leq H^2(T\eta, T\jmath_n) = 0,$$

which implies that

$$\begin{aligned} ||z''_n - \eta||^2 &\leq H(T\eta, T\jmath_n) \leq k^2 \max\{||\jmath_n - \eta||^2, d^2(\jmath_n, T\jmath_n)\} \\ &\leq k^2[||\jmath_n - \eta||^2 + d^2(\jmath_n, T\jmath_n)]. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}||z_n''' - \eta||^2 &\le H(Tp, Tz_n) \le k^2 \max\{||z_n - \eta||^2, d^2(z_n, Tz_n)\} \\ &\le k^2[||z_n - \eta||^2 + d^2(z_n, Tz_n)]\,. \end{aligned} \tag{13}$$

Also,

$$||z_n - \eta||^2 = ||(1-\varsigma_n)\jmath_n + \varsigma_n z_n'' - \eta||^2$$
$$= (1-\varsigma_n)||\jmath_n - \eta||^2 + \varsigma_n||z_n'' - \eta||^2 - \varsigma_n(1-\varsigma_n)||\jmath_n - z_n''||^2\,, \tag{14}$$
$$d^2(z_n, Tz_n) \le ||z_n - z_n'''||^2$$
$$= ||(1-\varsigma_n)\jmath_n + \varsigma_n z_n'' - z_n'''||^2$$
$$= (1-\varsigma_n)||\jmath_n - z_n'''||^2 + \varsigma_n||z_n'' - z_n'''||^2 - \varsigma_n(1-\varsigma_n)||\jmath_n - z_n''||^2, \tag{15}$$

which on combining with above mentioned equations, gives

$$\begin{aligned}||\ell_n - \eta||^2 &= (1-\vartheta_n)k^2[||\jmath_n - \eta||^2 + d^2(\jmath_n, T\jmath_n)] + \vartheta_n k^2[||z_n - \eta||^2 + d^2(z_n, Tz_n)] \\ &\quad - \vartheta_n(1-\vartheta_n)||z_n'' - z_n'''||^2\,, \\ ||\ell_n - \eta||^2 &= ||\jmath_n - \eta||^2[(1-\vartheta_n)k^2 + \vartheta_n k^4(1-\varsigma_n)] + d^2(\jmath_n, T\jmath_n)[(1-\vartheta_n)k^2] \\ &\quad + \vartheta_n k^4 \varsigma_n||z_n'' - p||^2 - (1-k^2)\vartheta_n k^4\varsigma_n(1-\varsigma_n)||\jmath_n - z_n''||^2 \\ &\quad + \vartheta_n k^2(1-\varsigma_n)||\jmath_n - z_n'''||^2 + [\vartheta_n k^2\varsigma_n - \vartheta_n(1-\vartheta_n)]||z_n'' - z_n'''||^2. \end{aligned} \tag{16}$$

Consequently from (12), we get

$$\begin{aligned}||\ell_n - \eta||^2 &= ||\jmath_n - \eta||^2[(1-\vartheta_n)k^2 + \vartheta_n k^4(1-\varsigma_n) + \vartheta_n k^6\varsigma_n] \\ &\quad + d^2(\jmath_n, T\jmath_n)[(1-\vartheta_n)k^2 + \vartheta_n k^6\varsigma_n] - (1-k^2)\vartheta_n k^4\varsigma_n(1-\varsigma_n)||\jmath_n - z_n''||^2 \\ &\quad + \vartheta_n k^2(1-\varsigma_n)||\jmath_n - z_n'''||^2 + [\vartheta_n k^2\varsigma_n - \vartheta_n(1-\vartheta_n)]||z_n'' - z_n'''||^2. \end{aligned} \tag{17}$$

Consequently from (10), we have

$$\begin{aligned}&||\jmath_{n+1} - \eta||^2 \\ &= ||\jmath_n - \eta||^2[(1-\vartheta_n)k^4 + \vartheta_n k^6(1-\varsigma_n) + \vartheta_n k^8\varsigma_n] + d^2(\jmath_n, T\jmath_n)[(1-\vartheta_n)k^4 + \vartheta_n k^8\varsigma_n] \\ &\quad - (1-k^2)\vartheta_n k^6\varsigma_n(1-\varsigma_n)||\jmath_n - z_n''||^2 + \vartheta_n k^4(1-\varsigma_n)||\jmath_n - z_n'''||^2 \\ &\quad + [\vartheta_n k^4\varsigma_n - 2k^2\vartheta_n(1-\vartheta_n)]||z_n'' - z_n'''||^2 + (1-\vartheta_n)k^2||z_n'' - z_n'||^2 + k^2\vartheta_n||z_n''' - z_n'||^2 \end{aligned} \tag{18}$$

we have

$$(1-k^2)\vartheta_n k^6\varsigma_n(1-\varsigma_n) \ge (1-k^2)^2k^6\varsigma_n(1-\varsigma_n) \ge 0 \;\; \forall\; n.$$

Also, As $\delta \le \vartheta_n \le 1-k^2$, there exists a positive integer $n \ge N_1$ for which,

$$\begin{aligned}(1-\vartheta_n)k^4 + \vartheta_n k^6(1-\varsigma_n) + \vartheta_n k^8\varsigma_n &\le k^6 + (1-k^2)k^6(1-\varsigma_n) + (1-k^2)k^8\varsigma_n \\ &= \gamma \text{ (say)}\end{aligned}$$

and $0 < \gamma < 1$. In a similar manner, it is easy to choose sufficiently large n for which

$$\begin{aligned}||\jmath_{n+1} - \eta||^2 &= \gamma||\jmath_n - \eta||^2 + [(1-\vartheta_n)k^4 + \vartheta_n k^8\varsigma_n]D_1 \\ &\quad + [\vartheta_n k^4\varsigma_n - 2k^2\vartheta_n(1-\vartheta_n)]D_2 + (1-\vartheta_n)k^2D_3 + k^2\vartheta_n D_4\end{aligned}$$

if $D = \max\{D_1, D_2, D_3, D_4\}$,

$$||\jmath_{n+1} - \eta||^2 = \alpha||\jmath_n - \eta||^2 + [\vartheta_n k^2(k^2\varsigma_n - 2(1 - \vartheta_n)] + 1) + (1 - \vartheta_n)k^2]D$$

with K having diameter D. The convergence of sequence $\{\jmath_n\}$ to η takes place whereby there is an approach by n towards infinity. The same follows from Lemma 2.2, and so does the theorem.

In order to support theory, we will use a numerical example provided by Shahzad and Zegeye [20].

Example 3.1. Let $X = [0, \infty)$ be equipped with the usual metric $d(x, y) = |x - y|$, defining a multivalued mapping T from K to $CB(K)$ as

$$Tx = \begin{cases} [x - \frac{3}{4}, x - \frac{1}{3}] & \text{if } x > 1, \\ \{0\} & \text{if } x \leq 1. \end{cases}$$

Then T is generalized nonexpansitve mapping; however, T is not nonexpansive.

Remark. A well illustrated example ([8], page 826) proved that the limit of the sequence of Ishikawa iterates depends on the choice of the invariant point ζ, and the initial choice of $\jmath_0$ and the invariant point may be different from ζ, same does for Picard-S hybrid iterates.

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References

[1] Picard, E., (1890), Memoire sur la theorie des equatiees aux derives partielles et la method des approximatins successive, J. Math. Pures et Appl. 6, pp. 145-210.

[2] Mann, W. R., (1953), Mean value methods in iteration, Proc. Amer. Math. Soc., 6, pp. 506-510.

[3] Ishikawa, S., (1974), Fixed points by a new iteration method, Proc. Amer. Math. Soc., 44, pp. 147-150.

[4] Rhoades, B. E., (1976), Comments on two fixed point iteration method, J. Math. Soc., 56, pp. 741-750.

[5] Nadler Jr., S. B., (1969), Multi-valued contraction mappings, Pacific. J. Math. Anal. Appl., 30, pp. 475-488.

[6] Markin, J. T., (1973), Continuous dependence of fixed point sets, Proc. Amer. Math. Soc., 38, pp. 545-547

[7] Lim, T. C., (1974), A fixed point theorem for multi-valued nonexpansive mappings in a uniformly convex Banach space, Bulletin of the American Mathematical Society, 80, pp. 1123-1126.

[8] Sasty, K. P. R. and Babu, G. V. R., (2005), Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, Czechoslovak Mathematical Journal, 55, pp. 817-826.

[9] Panyanak, B., (2007), Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces, Comput. Math. Anal. Appl., 54, pp. 872-877.

[10] Panyanak, B., (2007), Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces, Comput. Math. Anal. Appl., 54, pp. 872-877.

[11] Shahzad, N. and Zegeye, H., (2009), On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Anal., 71, pp. 838-844.

[12] Gursoy, F. and Karakaya, V., (2014), A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, ArXiv preprint arXiv:1403.2546.

[13] Noor, M. A., (2000), New approximation schemes for general variational inequalities, Journal of Mathematical Analysis and Applications, 251, pp. 217-229.

[14] Phuengrattana, W. and Suantai, S., (2011), On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, Journal of Computational and Applied Mathematics, 235, pp. 3006-3014.

[15] Sahu, D. R., (2011), Applications of the S-iteration process to constrained minimization problems and split feasibility problems, Fixed Point Theory, 12, pp. 187-204.

[16] Abbas, H., Mebawondu, A. A. and Mewomo, O., (2018), Some results for a new three steps iteration scheme in Banach spaces, Bull. of Tran. Uni. of Brasov., 11, pp. 1-18.

[17] Abed, S. S., Abas, R. F., (2017), S-iteration for general quasi multi-valued contraction mapping, Int. J. Adv. Math. Stud., 79, pp. 9-22.

[18] Agarwal, R., O'Regan, D. and Sahu, D. R., (2007), Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, Journal of Nonlinear and Convex Analysis, 8, pp. 61-79.

[19] Gorniewicz, L., (1999), Topological fixed point theory of multi-valued mappings, Kluwer Academia Pub., Dordrecht, Netherlands.

[20] Okeke, G. A. and Abbas, M., (2017), A solution of delay differential equations via Picard Krasnoselskii hybrid iterative process, Arab. J. Math., pp. 1-5.

[21] Uddin, I., Abdou, A. and Imdad, M., (2014), A new iteration scheme for a hybrid pair of generalized nonexpansive mapping, Fixed Point Theory Appl., 1, pp. 2-13.

[22] Uddin, I., Imdad, M. and Ali, J., (2015), Convergence theorems for a hybrid pair of generalized nonexpansive mappings in Banach spaces, Bulletin of the Malaysian Mathematical Sciences, 2, pp. 695-705.

[23] Qihou, L., (1990), A convergence theorem of the sequence of ishikawa iterates for quasi contractive mappings, J. Math. Anal. Appl., 146, pp. 301-305.

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