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ON SOME RESULTS OF PERFECT DOMINATIONS OF SOME GRAPHS

M. L. CAAY[1], S. R. PALAHANG[2], §

Abstract. A dominating set $D \subseteq V(G)$ of a simple graph G is the set of all u such that for every $v \in V(G) \backslash D$, $uv \in E(G)$. An independent set $I \subseteq V(G)$ is a set of non-adjacent vertices in G. An independent dominating set $D_i \subseteq V(G)$ is a subset of $V(G)$ that is both independent set and dominating set. A subset S of $V(G)$ is called a perfect dominating set of S if for each v belongs to $V(G) \backslash S$, there exists a unique element $u \in S$, such that v and u are adjacent. Define an independent perfect dominating set D_{ip} of G to be a dominating set that is both independent dominating set and perfect dominating set. The minimum cardinality of an independent perfect dominating set of G is called an independent perfect domination number of G, denoted by $\gamma_{ip}(G)$. If a graph has a perfect dominating set, we say that the graph G is γ_{ip}-graph. In this study, we determine some bounds and parameters of the graph as well as the existence existence of this invariant to some graphs and graphs formed by some binary operations.

Keywords: perfect dominating set, independent dominating set, independent perfect dominating set, independent perfect domination number.

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1. Introduction

Let G be a connected simple graph. Suppose $v \in V$, the neighborhood of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G).\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v \in D} N_G(v)$ and the set $N_G[D] = N[D] = D \bigcup N(D)$ are the open neighborhood and the closed neighborhood of D respectively. In this paper, we denote $\Delta(G)$ and $\delta(G)$ to be the minimum and maximum degree of G, respectively. That is,

$$\Delta(G) = \max_{v \in V} \text{ deg } (v)$$

and

$$\delta(G) = \min_{v \in V} \text{ deg } (v),$$

[1] Institute of Mathematics, University of the Philippines, Diliman, Quezon City.
e-mail: mlcaay@up.edu.ph; ORCID: https://orcid.org/0000-0002-7477-2957.
[2] School of Statistics, University of the Philippines, Diliman, Quezon City.
e-mail: srpalahang@up.edu.ph; ORCID: https://orcid.org/0000-0001-8145-778X.
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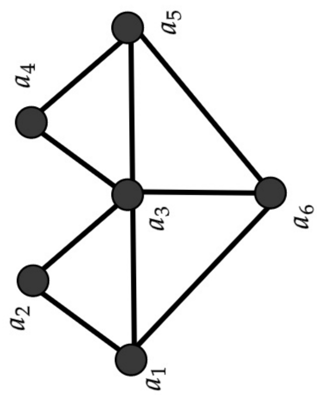
respectively.

The concept of domination has been so popular in Graph Theory and its applications anchored to some applications on computer engineering, finance and other allied sciences in mathematics. We say that D is the dominating set of G if for every $v \in V(G)\backslash D$, there exists $u \in D$ such that $uv \in E(G)$, that is, u is said to dominate v. Thus, $N[D] = V$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

One of the concepts of independent set was used by DeMaio and Jacobson [4] in 2014, in which they used the independent set to show the fibonacci and lucas relationship present in a Tadpole graph. A set $I \subseteq V(G)$ is an independent set of vertices if no two vertices in I are adjacent. The concept of the independent domination was introduced by Allan and Laskar in [3]. A subset $D_i \in V(G)$ is an independent dominating set of G if D_i is both an independent set and dominating set. The minimum cardinality of an independent dominating set of G is called an independent domination number of G, denoted by $\gamma_i(G)$. The concept of perfect dominating set was established by Livingston and Stout in [8] on their work involving the resource allocation and placement in parallel computers. Another concept of the perfect dominating set was used by Caay and Arugay in [2] when they introduced the perfect equitable dominating set. A subset S of $V(G)$ is called a *perfect dominating set* of S if for each v belongs to $V(G)\backslash S$, there exists a unique element $u \in S$, such that v and u are adjacent. The minimum cardinality of the perfect dominating set of G is called a *perfect domination number* of G, denoted by $\gamma_p(G)$.

The independent perfect domination was introduced by Jaeun Lee in July 2001 [7]. The authors propose to study the independent perfect domination number for some standard graphs and some graphs operations in particular join graph and corona product. Define an independent perfect dominating set D_{ip} of G to be a dominating set that is both independent dominating set and perfect dominating set. The minimum cardinality of an independent perfect dominating set of G is called an independent perfect domination number of G, denoted by $\gamma_{ip}(G)$. If a graph has a perfect dominating set, we say that the graph G is γ_{ip}-graph.

Consider the graph G below.



Observe that a_3 is adjacent to all vertices of G. This is a singleton set which implies that a_3 is an element of an independent set and so is an element of an independent perfect dominating set. Thus, $\gamma_{ip}(G) = 1$.

The graphs we consider here are simple and connected. For further terminologies, you may refer to [6].

By the definition of the independent perfect domination, we have an immediate results:

Remark 1.1. *Given a graph G*

i. *If $a, b \in D_{ip}$, then $ab \notin E(G)$.*
ii. *An independent perfect dominating set is an independent dominating set but the converse does not necessarily hold.*
iii. *An independent perfect dominating set is a perfect dominating set but the converse does not necessarily hold.*

The study has been motivated by a real-life scenario: Consider the fact that there are many food suppliers in the area with the same supply of food. The rule is they have to acquire many customers as they can such that they would not have conflict with other suppliers. That is if one has gotten a customer, such customer will not be dealt anymore by another supplier. This scenario can be applied in many business networking that requires many retailers retailing products in the area.

2. Some Results on Some Graphs

The following are the results of the independent perfect domination of graphs.

Proposition 2.1. *For a path graph P_n, $\gamma_{ip}(G) = \left\lceil \frac{n}{3} \right\rceil$. That is, P_n is γ_{ip}-graph.*

Proof. We will consider the following cases:

- Case 1: If $n \equiv 0 \mod 3$.

  We divide the vertices in a group of 3. Without loss of generality, suppose the first group of three vertices are labeled in consecutive manner as v_1, v_2, v_3. If v_1 dominates v_2, v_3 is not dominated. If v_3 dominates v_2, v_1 is not dominated. If v_2 dominates v_1, it also dominates v_3. Simlarly, for the second group of vertices v_4, v_5, v_6, we have v_5 that dominates v_4 and v_6. Continuing the process, we obtain vertices place in the middle of every group that can dominate the rest. Since 3 divides n and the number of vertices are divided by 3. We have $\left\lceil \frac{n}{3} \right\rceil$.

- Case 2: If $n \equiv 1 \mod 3$.

  Dividing the vertices in a group of 3 vertices per group, we have one group of one vertex. Without a loss of generality, we label such vertex as v_1 and the vertices next to it are v_2, v_3, v_4. Further suppose v_1 dominates v_2. Then we may assume v_4 dominates v_3. Suppose that the next group of vertices are labeled v_5, v_6, v_7. Then v_4 from the previous group dominate v_5 and so we may assume v_7 dominates v_6. Continuing this process, we can find out that the dominating vertices are the third vertex in each group. There are $\frac{n-1}{3}$ groups of three vertices. Since v_1 is also a dominating vertex, it follows that the dominating vertices are $\frac{n-1}{3}+1$ or in this case is $\left\lceil \frac{n}{3} \right\rceil$.

- Case 3: If $n \equiv 2 \mod 3$.

  Dividing the vertices in a group of 3 vertices per group, we have one group of one vertex. Without a loss of generality, we label such verteices as v_1 and v_2, and the vertices next to it are v_3, v_4, v_5. Further suppose v_2 dominates v_1. Then following the same processwith case 2, we arrive at the same answer as $\frac{n-1}{3}+1$ or in this case is $\left\lceil \frac{n}{3} \right\rceil$.

□

Proposition 2.2. *For all integer $n \geq 1$, $\gamma_{ip}(C_{3n}) = n$. That is, C_n is γ_{ip}-graph if n is divisible by 3.*

Proof. Let $a_i \in V(C_{3n})$ such that a_i dominates a_{i+1} and a_{i-1}. Suppose further that the the vertices are labeled orderly such that a_{i+1} is next to a_i and a_{i+2} is next to a_{i+1} and so on. This means that since a_i dominates a_{i+1} and a_{i-1}, we can partition the vertices in a partition of 3. That is, in every 3 consecutive vertices, there is a dominating vertex placed at the center. Diving $3n$ by 3, we have n number of vertices dominating the others. This proves the claim. □

Corollary 2.1. *For any integer n not divisible by 3, C_n is not a γ_{ip}-graph.*

Proposition 2.3. *The following are graphs of any order having $\gamma_{ip}(G) = 1$.*

i. *Complete graph, K_n*
ii. *Star graph, S_n*
iii. *Wheel graph, W_n*

3. Some Results on Corona and Join of Graphs

Definition 3.1. *The* ***join*** *$G + H$ of two graphs G and H is the graph with vertex set*

$$V(G+H) = V(G) \bigcup V(H)$$

and edge set

$$E(G+H) + E(G) \bigcup E(H) \bigcup \{uv : u \in V(G), v \in V(H)\}.$$

Definition 3.2. *The* ***corona*** *$G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

Theorem 3.1. *Given graphs G and H of order m and n, respectively, the $\gamma_{ip}(G+H) = 1$ if and only if either $\Delta(G) = m-1$ or $\Delta(H) = n-1$.*

Proof. Supppose $\gamma_{pi}(G + H) = 1$. Then there exists a vertex, say a in $D_{ip}(G + H)$ that is either in $V(G)$ or $V(H)$ but not both. If $a \in V(G)$, then for all $b_i \in V(G)$, $i = \{1, 2, \cdots, m-1\}$, $ab_i \in E(G)$. This means that $\deg(a) = m-1$. Hence, $\Delta(G) = m-1$. If $a \in V(H)$, then for all $c_j \in V(G)$, $j = \{1, 2, \cdots, n-1\}$, $ac_j \in E(G)$. This means that $\deg(a) = n - 1$. Hence, $\Delta(H) = n - 1$.

Conversely, assume that either $\Delta(G) = m - 1$ or $\Delta(H) = n - 1$. Without loss of generality, take $\Delta(G) = m - 1$. Then there exists a vertex say $v \in V(G)$ such that $\deg(a) = m - 1$. Since $a \in V(G + H)$, it follows that a is adjacent to all vertices of $V(G + H)$. Thus, a dominates all vertices of $G + H$. That is, $\gamma_{ip}(G + H) = 1$. □

Theorem 3.2. *Let G_n be any graph and K_p be a complete graph. Then $G_n \circ K_p$ is γ_{ip}-graph if $G_n \cong (P_n$ or $C_n)$. Then $\gamma_{ip}(G_n \circ K_p) = n$.*

Proof. Let $G_n \cong P_n$. By Proposition 2.1, P_n is γ_{ip}-graph. Thus, it follows that $G_n \circ K_p$ is also a γ_{ip}-graph. Suppose $G_n \cong C_n$. If 3 divides n, then by Proposition 2.2, C_n is a γ_{ip}-graph and and so $G_n \circ K_p$ is also a γ_{ip}-graph. If 3 does not divide n, then C_n is not a $G_n \circ K_p$ is also a γ_{ip}-graph. However, by Proposition 2.3, K_p is γ_{ip}-graph. Thus, $G_n \circ K_p$ is also a γ_{ip}-graph. Moreover, since K_p is complete, there are n copies of K_p. Since all vertices of K_p is adjacent to all vertices including the vertex from G_n incident to it, it follows that if we pick one vertex from each K_p to be a dominating set, it also dominates each vertex from G_n incident to it. Thus, it is enough to pick one vertex from each K_p. Hence, $\gamma_{ip}(G_n \circ K_p) = n$. □

Theorem 3.3. *Let K_p be any complete graph of order $p \geq 2$ and G_n be any graph isomorphic to P_n or C_n. Then $K_p \circ G_n$ is not γ_{ip}-graph if $n > 3$.*

Proof. Suppose on the contrary that $K_p \circ G_n$ is a γ_{ip}-graph. Then either we can pick an element of D_{ip} that is in $V(K_p)$ or $V(G_n)$. If $a \in V(K_p)$ be an element of D_{ip}, then a dominates $u_j \in V(G_n^1)$ where G_n^1 is the graph isomorphic to P_n or C_n incident to a. Also, a dominates $b_i \in V(K_p)$, $i = 1, \cdots, p-1$. But $c_j \in V(G_n^t)$, $t = 1, \cdots, p-1$, $j = 1, 2, \cdots, n$ is not dominated by a. If we pick an element from $V(G_n^t)$, $t = 1, \cdots, p-1$, $j = 1, 2, \cdots, n$, it also dominates $b_i \in V(K_p), i = 1, 2, \cdots, p-1$ that is already dominated by a. Hence, we cannot pick a vertex from K_p to be in D_{ip}. Let $v^t \in V(G_n^t)$, $t = 1, 2, \cdots, p$ be an element of D_{ip} instead. Since $n > 3$, it follows that there must be possible 2 vertices of each G_n^t to be in dominating set. This means that two possible vertices dominate the vertex from K_p incident to each G_n^t. Hence, we cannot pick a vertex from G_n^t to be in D_{ip}. The two cases show contradiction. Therefore, $K_p \circ G_n$ is not a γ_{ip}-graph. □

The following are consequences of Theorem 3.3 and the proofs are obvious.

Corollary 3.1. *Let K_p be any complete graph of order $p \geq 2$ and G_n be any graph isomorphic to P_n or C_n. Then $K_p \circ G_n$ is a γ_{ip}-graph if $n \leq 3$. In fact, $\gamma_{ip}(K_p \circ G_n) = p$.*

Corollary 3.2. *Let G be any graph. Then $K_1 \circ G$ is a γ_{ip}-graph. In fact, $\gamma_{ip}(K_p \circ G) = 1$.*

Corollary 3.3. *Let G be any graph. Then $G \circ K_1$ is a γ_{ip}-graph. In fact, $\gamma_{ip}(G \circ K_1) = |V(G)|$.*

4. Some Results on Cartesian Product and Tensor Product of Graphs

Definition 4.1. *The **cartesian product** $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and e is an edge of $G \times H$ if and only if $e = (u_i, v_j)\,(u_k, v_l)$ where either*

i. *$i = k$ and $v_j v_l \in E(H)$*
ii. *$j = l$ and $u_i u_k \in E(G)$.*

Definition 4.2. *The **tensor product** $G \times H$ of two graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and and edge set*

$$E(G \times H) = \{(a,b)(c,d) \mid ac \in E(G) \text{ and } bd \in E(H)\}.$$

Proposition 4.1. *For every integer $n \geq 2$, $K_n \,\square\, K_n$ is non-γ_{ip}-graph.*

Proof. If n is the number of vertices of a complete graph K_n, then $K_n \,\square\, K_n$ has n^2 vertices. Suppose $K_n \,\square\, K_n$ is γ_{ip}-graph and let $(a,b) \in V(K_n \,\square\, K_n)$ where a is the vertex of the first K_n and b is the vertex of the second K_n. Then (a,b) is adjacent to $n-1$ vertices of the form (a, b_i) where b_i are $n-1$ vertices of the second K_n. Thus, (a,b) has dominated $2(n-1)$ vertices of $K_n \,\square\, K_n$ implying that $(a,b) \in D_{ip}$. Thus, we will pick another elements for D_{ip} from the remaining $n^2 - 2(n-1)$ vertices. Without loss of generality, suppose $(\overline{a}, \overline{b})$ is one of the $n^2 - 2(n-1)$ vertices which are not adjacent to (a,b). Then we can pick $(\overline{a}, \overline{b})$ to be element of D_{ip}. But, $(\overline{a}, \overline{b})$ is adjacent to one of the vertices adjacent to (a,b). This is a contradiction to the perfect domination. Thus, there is not independent perfect dominating set in $K_n \,\square\, K_n$. That is, $K_n \,\square\, K_n$ is non-γ_{ip}-graph for $n \geq 2$. $\square$

Proposition 4.2. *Let G be any graph. $K_1 \,\square\, G$ and $G \,\square\, K_1$ is γ_{ip}-graph. Moreover, $\gamma_{ip}(K_1 \,\square\, G) = \gamma_{ip}(G \,\square\, K_1) = 1$.*

Proof. Let $V(K_1) = \{a\}$. Then a dominates all vertices in G. Thus, $a \in D_{ip}$ and so $K_1 \,\square\, G$ is γ_{ip}-graph. Moreover, $\gamma_{ip}(K_1 \,\square\, G) = 1$. The same goes with $G \,\square\, K_1$. $\square$

Theorem 4.1. *For an integer n and m with $n \geq 3, m \geq 2$, $K_n \,\square\, P_m$ is non-γ_{ip}-graph.*

Proof. Let $V(K_n) = \{k_1, k_2, \cdots, k_n\}$ and $V(P_m) = \{p_1, p_2, \cdots, p_m\}$. Then there nm number of vertices. Suppose that $K_n \,\square\, P_m$ is γ_{ip}-graph. Without loss of generality, let $(k_1, p_1) \in D_{ip}$. Then (k_1, p_1) dominates (k_1, p_2) and (k_j, p_1), for some $j = 2, 3, \cdots, n$. Note that $(k_i, p_2) \notin D_{ip}$ since $(k_i, p_1)(k_i, p_2) \in E(K_n \,\square\, P_M)$, for some $i = 1, 2, \cdots, n$. Thus, (k_i, p_3) can dominate (k_i, p_12) but (k_1, p_2) is already dominated by (k_1, p_1). This is a contradiction to the definition of perfect domination. Thus, $K_n \,\square\, P_m$ is non-γ_{ip}-graph. $\square$

Theorem 4.2. *For every integer odd n, $K_2 \square P_n$ is γ_{ip}-graph. In particular,*

$$\gamma_{ip}(K_2 \square P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Corollary 4.1. *$K_2 \square P_n$ is non-γ_{ip}-graph if n is even.*

Proposition 4.3. *Given any integer $n \geq 2$, $P_n \times P_n$ is a γ_{ip}-graph. In fact,*

$$\gamma_{ip}(P_n \times P_n) = n \left\lceil \frac{n}{3} \right\rceil$$

Proof. Let $V(P_n) = \{a_1, a_2, \cdots, a_n\}$. Then we can arrange the vertices of $P_n \times P_n$ in the form

$$\begin{array}{cccc} (a_1, b_1) & (a_2, b_1) & \cdots & (a_n, b_1) \\ (a_1, b_2) & (a_2, b_2) & \cdots & (a_n, b_2) \\ & \vdots & & \\ & \vdots & & \\ (a_1, b_n) & (a_2, b_n) & \cdots & (a_n, b_n). \end{array}$$

Without of loss of generality, if we take (a_2, b_2) dominates (a_1, b_1), it also dominates (a_3, b_3) but not on (a_1, b_2) and (a_3, b_2). Thus, (a_1, b_2) and (a_3, b_2) can also be members of the dominating set. Hence, the importance to check is the path connecting the vertices of the form (a_i, b_i), $i = 1, 2, \cdots, n$. By Proposition 2.1, there are $\left\lceil \frac{n}{3} \right\rceil$ vertices to choose from a path connecting the vertices of the form (a_i, b_i), $i = 1, 2, \cdots, n$. Since there n vertices not adjacent in each selected vertices in the path connecting the vertices of the form (a_i, b_i), it follows that there are $n \left\lceil \frac{n}{3} \right\rceil$ members of the dominating set. □

Proposition 4.4. *For every integer $n \geq 3, m \geq 3$, $K_n \times P_m$ is non-γ_{ip}-graph.*

5. Conclusions

In this paper, we have evaluated the parameters of independent perfect dominations to some graphs and some graphs resulting to binary operations such as join, corona, cartesian product and tensor product. Further, we examine some graphs and operations with the existence of independent perfect domination because not all graphs have this set. The study is beyond helpful in the application of real-life scenario. Fact is, it has been mentioned in the introduction of this paper on how this work has been motivated.

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Mark Lantaca Caay is a graduate from Mindanao State University, Iligan Institute of Technology, Iligan City, Philippines on his Bachelor of Science in Mathematics and Master of Science in Mathematics in 2012 and 2018, respectively. He was a former instructor at the University of the Visayas, Main Campus, Cebu City. He is currently a Ph.D. student at the Institute of Mathematics, University of the Philippines - Diliman, Quezon City, Philippines.

Saturnino R. Palahang Jr. graduated from the Mindanao State University, Iligan Institute of Technology in 2016 on his Bachelor of Science in Mathematics and he is currently a M.Sc. student at the School of Statistics, University of the Philippines, Diliman.