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A STUDY ON FINITE DIFFERENCE METHOD USING EXPLICIT AND MONOTONE SCHEME HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. All the problems almost in science and technology can be expressed mathematically in the form of partial differential equation. Mainly all the types of partial differential equation has a specific characters. Specially hyperbolic equation is associated with vibrations and sounds especially problems related to time, heat, diffusion and elasticity. In this paper, the author discussed the explicit and monotone scheme based on finite difference method to find the numerical solution of hyperbolic partial differential equation with linear advection equation and also discussed the upwind difference scheme which was extended to an monotone scheme.

Keywords: Explicit scheme, monotone scheme, numerical, hyperbolic PDE.

AMS Subject Classification: 39A10, 39A12.

1. Introduction

A partial derivatives of a dependent variable is a differential equation with one or more than one independent variable which is referred as partial differential equation. If such a dependent variable with partial derivatives occurs linearly in any partial differential equation is called linear partial differential equation, otherwise it is referred as nonlinear partial differential equation. A partial differential equation is an equation which contains the partial derivatives and unknown multivariable functions. Hyperbolic equation is the wave equation which is sustainable contemporary interest. A hyperbolic partial differential contains a well proposed initial value problem with n derivatives. A fine example of hyperbolic equation is the hydrodynamics equation. Hyperbolicity is a fundamental qualitative kind of differential equation. Explicit methods are those which focuses the status of later time from the status of current time. Explicit is a finite difference scheme with conservation form. Also finite difference scheme is monotonicity preserving with finite

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difference operator L, if v is a monotone mesh function is $L \cdot v$. Finite difference scheme is a monotone scheme if H is a monotone nondecreasing function of each of its $2k+1$ arguments. Already in [1-6] the hyperbolic equation condition of finite difference using explicit scheme has been discussed earlier, whereas in this paper, we discuss the basic definition of numerical,monotone, explicit partial differential equation and in next section we deeply discussed the hyperbolic explicit monotonic equation using upwind finite difference method.

2. Basic Definition

Definition 2.1. *The numerical fluxes is defined by time averages of the boundary data*

$$f_{-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta t^{n+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} f\left(v\left(t\right)\right) dt \ , \forall\ 0 \le n < N. \tag{1}$$

Definition 2.2. *The numerical domain of dependence contain the physical domain of dependence iff the timestep satisfies the Courant Friedrichs Levy condition (CFL condition) as*

$$c\Delta t^{n+\frac{1}{2}} \le min_i \left\{\Delta x_i\right\}. \tag{2}$$

The explicit upwind difference scheme with CFL condition, then the dimensionless Courant Friedrichs Levy number (CFL number) as

$$\gamma_i^{n+\frac{1}{2}} = \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}. \tag{3}$$

If the timesteps is chosen, the CFL number satisfies $\gamma_i^{n+\frac{1}{2}} \le 1, \forall i$.

Definition 2.3. *In numerical approximation to linear advection equation is explicit upwind difference method*

$$y_i^{n+1} = y_i^n - \left[y_i^n - y_{i-1}^n\right] \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i} \forall\ 0 < i < I. \tag{4}$$

It is a conservation difference scheme in the numerical fluxes is

$$f_{i+\frac{1}{2}}^{n+\frac{1}{2}} = cy_i^n \ , \forall\ 0 < i < I.$$

In a finite difference approximation, the explicit upwind difference scheme in linear advection equation

$$\frac{y_i^{n+1} - y_i^n}{\Delta t^{n+\frac{1}{2}}} + \frac{cy_i^n - cy_{i-1}^n}{\Delta x_i} = 0 \ , \forall\ 0 < i < I.$$

The spatial difference is a first order approximation to $\partial_x cy$.

Definition 2.4. *In numerical approximation to linear advection equation is explicit downwind difference method*

$$y_i^{n+1} = y_i^n - \left[y_{i+1}^n - y_i^n\right] \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i} \ , \forall\ 0 < i < I. \tag{5}$$

It is a conservation difference scheme in the numerical fluxes is

$$f_{i+\frac{1}{2}}^{n+\frac{1}{2}} = cy_{i+1}^n.$$

In a finite difference approximation, the explicit downwind difference scheme in linear advection equation

$$\frac{y_i^{n+1}-y_i^n}{\Delta t^{n+\frac{1}{2}}}+\frac{cy_{i+1}^n-cy_i^n}{\Delta x_i}=0\ ,\forall\ 0<i<I.$$

It uses a different first order approximation to the spatial derivative in the linear advection equation

Definition 2.5. *In numerical approximation to linear advection equation is explicit centered difference method*

$$y_i^{n+1}=y_i^n-\left[y_{i+1}^n-y_{i-1}^n\right]\frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}\ ,\forall\ 0\leq i<I. \tag{6}$$

On a uniform grid, the numerical fluxes is

$$f_{i+\frac{1}{2}}^{n+\frac{1}{2}}=\frac{c\left(y_i^n+y_{i+1}^n\right)}{2}.$$

In a finite difference approximation, the explicit centered difference scheme in linear advection equation

$$\frac{y_i^{n+1}-y_i^n}{\Delta t^{n+\frac{1}{2}}}+\frac{cy_{i+1}^n-cy_{i-1}^n}{\Delta x_i}=0\ ,\ \forall\ 0<i<I.$$

The second order spatial approximation with cell averages y_i^n to form a quadratic approximation to the integral of y and differentiate the quadratic at a cell side.

Definition 2.6. *A weak solution scalar initial value problem*

$$\begin{gathered}\partial_t y+\partial_x f\left(y\right)=y_\epsilon+a\left(y\right)\partial_x y=0,\\ a\left(y\right)=d_y f,\quad y\left(x,0\right)=\phi\left(x\right),\ -\infty,x<\infty.\end{gathered} \tag{7}$$

where $\Phi\left(x\right)$ is to be of bounded total variation. It has the following monotonicity property as a function of t.

1. *No new local extrema in x is created.*
2. *The value of a local minimum is nondecreasing, the value of a local maximum is nonincreasing.*

It follows from this monotonicity property that the total variation is x, denoted $TV\left(y\left(t\right)\right)$ of $y\left(x,t\right)$ is nonincreasing in t, such that

$$TV\left(y\left(t_2\right)\right)=TV\left(y\left(t_1\right)\right),\ \forall\ t_2=t_1. \tag{8}$$

Definition 2.7. *In conservation form of weak solution are obtained by $(2k+1)-$ point explicit schemes*

$$v_i^{n+1}=v_i^n-\lambda\left(\overline{f}_{i+\frac{1}{2}}^n-\overline{f}_{i-\frac{1}{2}}^n\right), \tag{9a}$$

$$where\,\overline{f}_{i+\frac{1}{2}}^n=\overline{f}\left(v_{i-k+1}^n,\cdots,v_{i+k}^n\right). \tag{9b}$$

$$\overline{f}\left(y,\cdots,y\right)=f\left(y\right). \tag{9c}$$

Here,$v_i^n=v\left(i\Delta x,\ n\Delta t\right)$ and $\overline{f}$ is a numerical flux function. This flux to be consistent with the flux $f\left(y\right)$ in the following sense (9c).

Definition 2.8. *Explicit* $(2k+1)$*- point finite difference scheme in conservation from* (9) *approximating* (7) *as*

$$v_i^{n+1} = H\left(v_{i-k}^n, v_{i-k+1}^n, \cdots, v_{i+k}^n\right).$$

$$v_i^{n+1} = v_i^n - \lambda\left\{\overline{f}\left(v_{i-k+1}^n, \cdots, v_{i+k}^n\right) - \overline{f}\left(v_{i-k}^n, \cdots, v_{i+k-1}^n\right)\right\} \tag{10a}$$

$$V^{n+1} = L \cdot V^n. \tag{10b}$$

And denote $(10a)$ *in a operator form as* $(10b)$.

Definition 2.9. *If* $y(x,t)$ *and* $w(x,t)$ *are two solutions of*

$$\begin{aligned} \partial_t y + \partial_x f(y) = 0, \\ y(x,0) = \phi(x), \quad \forall -\infty < x < \infty. \end{aligned} \tag{11}$$

and $y(x,0) - w(x,0) \in L_1$ *and if it forms an* L_1- *contractive semigroup such that*

$$\|y(\cdot,t_2) - w(\cdot,t_2)\| \leq \|y(\cdot,t_1) - w(\cdot,t_1)\|.$$

For all $t_2 \geq t_1 \geq 0$, *here* $\|\ \|$ *denotes the* L_1- *norm in the space variables.*

Definition 2.10. *Finite difference scheme* (10) *is total variation nonincreasing(TVNI) if for all* v *of bounded total variation*

$$TV(L \cdot v) = TV(v), \tag{12a}$$

$$where\, TV(y) = \sum_{i=-\infty}^{\infty} \left|_{i+\frac{1}{2}} y\right|. \tag{12b}$$

and $\Delta_{i+\frac{1}{2}} y = y_{i+1} - y_i$.

3. Main Result

In the main result, we discussed the hyperbolic explicit equation by using discrete value which satisfies the explicit upwind difference scheme.

Theorem 3.1. *Consider the discrete value* y_i^n *as*

$$y_i^n = \tilde{y}(i\Delta x, n\Delta t) + O\left(\Delta t^2\right) + O\left(\Delta x^2\right) + O(\Delta t \Delta x)$$

which satisfy the explicit upwind difference scheme

$$y_i^{n+1} = y_i^n - \frac{c\Delta t}{\Delta x}\left[y_{i+1}^n - y_i^n\right], \tag{13}$$

where $\tilde{y}$ *is twice continuously differentiable in* x *and* t *and it satisfies a modified equation of the form*

$$\partial_t \tilde{y} + \partial_x c\tilde{y} = e \equiv O(\Delta t) + O(\Delta x). \tag{14}$$

Then the modified e *satisfies*

$$e = -\frac{c\Delta x}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right)\partial_{xx}^2 \tilde{y} + O(\Delta t) + O(\Delta x).$$

Proof: Given, $\tilde{y}$ *is twice continuously differentiable*

$$y_{i+1}^n = \tilde{y}(x+\Delta x, t) + O\left(\Delta t^2\right) + O\left(\Delta x^2\right) + O(\Delta t \Delta x),$$

$$y_{i+1}^n = \tilde{y}(x,t) + \partial_x \tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2 \tilde{y}\Delta x^2 + O\left(\Delta t^2\right) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

Since $\tilde{y}$ *satisfies modified equation* (14), *it gives*

$$\partial_t \tilde{y} = e - \partial_x c\tilde{y}, \partial_{tt}^2 \tilde{y} = \partial_t e - c\partial_x e + c^2 \partial_{xx}^2 \tilde{y}. \tag{15}$$

Substitute the approximation into explicit upwind difference scheme, we have (13)

$$y_i^{n+1} - y_i^n = -\frac{c\Delta t}{\Delta x}\left[y_{i+1}^n - y_i^n\right].$$

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} = -\frac{c}{\Delta x}\left[y_{i+1}^n - y_i^n\right].$$

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} + c\frac{y_{i+1}^n - y_i^n}{\Delta x} = 0. \tag{16}$$

Take,

$$y_{i+1}^n - y_i^n = \tilde{y}(x,t) + \partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2 + O\left(\Delta t^2\right) + O\left(\Delta x^2\right)$$

$$+O(\Delta t\Delta x) - \tilde{y}(x,t) - O\left(\Delta t^2\right) - O\left(\Delta x^2\right) - O(\Delta t\Delta x).$$

$$y_{i+1}^n - y_i^n = \partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2. \tag{17}$$

We know that,

$$y_i^{n+1} - y_i^n = \partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2. \tag{18}$$

Substitute (19) *and* (18) *in* (17)*, we get*

$$0 = \frac{\partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2}{\Delta t} + c\frac{\partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2}{\Delta x}.$$

$$0 = \partial_t\tilde{y} + \partial_{tt}^2\tilde{y}\frac{\Delta t}{2} + O(\Delta t) + c\left[\partial_x\tilde{y} + \partial_{xx}^2\tilde{y}\frac{\Delta x}{2}\right] + O(\Delta x).$$

$$0 = \partial_t\tilde{y} + c\partial_x\tilde{y} + c\partial_{xx}^2\tilde{y}\frac{\Delta x}{2} + \partial_{tt}^2\tilde{y}\frac{\Delta t}{2} + O(\Delta t) + O(\Delta x).$$

By (16)*, we get*

$$0 = \partial_t\tilde{y} + c\partial_x\tilde{y} + \frac{c\Delta x}{2}\partial_{xx}^2\tilde{y} + \frac{\Delta t}{2}\left[\partial_t e - c\partial_x e + c^2\partial_{xx}^2\tilde{y}\right] + O(\Delta t) + O(\Delta x).$$

$$0 = \partial_t\tilde{y} + c\partial_x\tilde{y} + \frac{c\Delta x}{2}\partial_{xx}^2\tilde{y} + \frac{\Delta t}{2}\partial_t e - \frac{c\Delta t}{2}\partial_x e + \frac{c^2\Delta t}{2}\partial_{xx}^2\tilde{y} + O(\Delta t) + O(\Delta x).$$

$$0 = \partial_t\tilde{y} + \partial_x c\tilde{y} + \frac{c\Delta x}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right)\partial_{xx}^2\tilde{y} + \frac{\Delta t}{2}\left[\partial_t e - c\partial_x e\right] + O(\Delta t) + O(\Delta x).$$

Since e is some arbitrary, we have

$$\partial_t\tilde{y} + \partial_x c\tilde{y} + \frac{c\Delta x}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right)\partial_{xx}^2\tilde{y} + O(\Delta t) + O(\Delta x) = 0.$$

$$\partial_t\tilde{y} + \partial_x c\tilde{y} = -\frac{c\Delta x}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right)\partial_{xx}^2\tilde{y} + O(\Delta t) + O(\Delta x).$$

By given (14)*,*

$$e = -\frac{c\Delta x}{2}\left(1 + \frac{c\Delta t}{\Delta x}\right)\partial_{xx}^2\tilde{y} + O(\Delta t) + O(\Delta x).$$

The proof is complete.

Theorem 3.2. *Consider the discrete value* y_i^n *as*

$$y_i^n = \tilde{y}\left(i\Delta x, n\Delta t\right) + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O(\Delta t \Delta x^2),$$

which satisfy the explicit upwind difference scheme

$$y_i^{n+1} = y_i^n - \frac{c\Delta t}{2\Delta x}\left[y_{i+1}^n - y_{i-1}^n\right]. \tag{19}$$

where $\tilde{y}$ *is twice continuously differentiable in* x *and* t *and it satisfies a modified equation of the form* (14)*. Then the modified* e *satisfies*

$$e = -\frac{c^2\Delta t}{2}\partial_{xx}^2\tilde{y} + O\left(\Delta t\right) + O\left(\Delta x^2\right) + O\left(\Delta t\Delta x\right).$$

Proof: Given, $\tilde{y}$ *is twice continuously differentiable*

$$y_{i+1}^n = \tilde{y}\left(x + \Delta x, t\right) + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

$$y_{i+1}^n = \tilde{y}\left(x,t\right) + \partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2 + \frac{1}{3}\partial_{xxx}^3\tilde{y}\Delta x^3 + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

Similarly,

$$y_{i-1}^n = \tilde{y}\left(x - \Delta x, t\right) + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

$$y_{i-1}^n = \tilde{y}\left(x,t\right) - \partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2 - \frac{1}{3}\partial_{xxx}^3\tilde{y}\Delta x^3 + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

And

$$y_i^{n+1} = \tilde{y}\left(x, t + \Delta t\right) + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

$$y_i^{n+1} = \tilde{y}\left(x,t\right) + \partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2 + O\left(\Delta t^2\right) + O\left(\Delta x^3\right) + O\left(\Delta t\Delta x^2\right).$$

Since $\tilde{y}$ *satisfies modified equation* (14) *, it gives* (15) *and* (16)*. Take,*

$$y_{i+1}^n - y_{i-1}^n = \tilde{y}\left(x,t\right) + \partial_x\tilde{y}\Delta x + \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2 + \frac{1}{3}\partial_{xxx}^3\tilde{y}\Delta x^3 + O\left(\Delta t^2\right) + O\left(\Delta x^3\right).$$

$$+O\left(\Delta t\Delta x^2\right) - \tilde{y}\left(x,t\right) + \partial_x\tilde{y}\Delta x - \frac{1}{2}\partial_{xx}^2\tilde{y}\Delta x^2 + \frac{1}{3}\partial_{xxx}^3\tilde{y}\Delta x^3 - O\left(\Delta t^2\right) - O\left(\Delta x^3\right) - O\left(\Delta t\Delta x^2\right) l.$$

$$y_{i+1}^n - y_{i-1}^n = 2\partial_x\tilde{y}\Delta x + \frac{2\Delta x^3}{3}\partial_{xxx}^3\tilde{y}. \tag{20}$$

And

$$y_i^{n+1} - y_i^n = \tilde{y}\left(x,t\right) + \partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2 + O\left(\Delta t^2\right) + O\left(\Delta x^3\right).$$

$$+O\left(\Delta t\Delta x^2\right) - \tilde{y}\left(x,t\right) - O\left(\Delta t^2\right) - O\left(\Delta x^3\right) - O\left(\Delta t\Delta x^2\right).$$

$$y_i^{n+1} - y_i^n = \partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2. \tag{21}$$

Substitute the approximation into explicit upwind difference scheme, we have (20)

$$y_i^{n+1} - y_i^n = -\frac{c\Delta t}{2\Delta x}\left[y_{i+1}^n - y_{i-1}^n\right].$$

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} = -\frac{c}{2\Delta x}\left[y_{i+1}^n - y_{i-1}^n\right].$$

$$\frac{y_i^{n+1} - y_i^n}{\Delta t} + c\frac{y_{i+1}^n - y_{i-1}^n}{2\Delta x} = 0.$$

Using (21) *and* (22)*, we get*

$$0 = \frac{\partial_t\tilde{y}\Delta t + \frac{1}{2}\partial_{tt}^2\tilde{y}\Delta t^2}{\Delta t} + c\frac{2\partial_x\tilde{y}\Delta x + \frac{2\Delta x^3}{3}\partial_{xxx}^3\tilde{y}}{2\Delta x}.$$

$$0 = \partial_t \tilde{y} + \partial_{tt}^2 \tilde{y} \frac{\Delta t}{2} + O(\Delta t) + c \left[\partial_x \tilde{y} + \partial_{xxx}^3 \tilde{y} \frac{\Delta x^2}{3} \right] + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

$$0 = \partial_t \tilde{y} + \partial_{tt}^2 \tilde{y} \frac{\Delta t}{2} + c \partial_x \tilde{y} + \frac{c \Delta x^2}{3} \partial_{xxx}^3 \tilde{y} + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

The maximum order of x is 2. So, the order of 3 is eliminated.

$$0 = \partial_t \tilde{y} + c \partial_x \tilde{y} + \frac{\Delta t}{2} \left[\partial_t e - c \partial_x e + c^2 \partial_{xx}^2 \tilde{y} \right] + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

$$0 = \partial_t \tilde{y} + c \partial_x \tilde{y} + \frac{\Delta t}{2} \left[\partial_t e - c \partial_x e \right] + \frac{c^2 \Delta t}{2} \partial_{xx}^2 \tilde{y} + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

Since e is some arbitrary, we have

$$\partial_t \tilde{y} + \partial_x c \tilde{y} + \frac{c^2 \Delta t}{2} \partial_{xx}^2 \tilde{y} + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x) = 0.$$

$$\partial_t \tilde{y} + \partial_x c \tilde{y} = -\frac{c^2 \Delta t}{2} \partial_{xx}^2 \tilde{y} + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

By given (14),

$$e = -\frac{c^2 \Delta t}{2} \partial_{xx}^2 \tilde{y} + O(\Delta t) + O\left(\Delta x^2\right) + O(\Delta t \Delta x).$$

Hence proved.

Lemma 3.1. *If y_i^n is a grid function, then the interpolation operator $I_{\Delta x} : L^2(Z\Delta x) \to L^2(\mathbb{R})$,*

$$(I_{\Delta x} v)(x) = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{\iota x \xi} \hat{v}^n(\xi)\, d\xi. \tag{22}$$

satisfies $\|I_{\Delta x} y^n\| = \|y^n\|_{\Delta x}$.
Proof: Using Parseval's identities

$$\frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |\hat{y}^n(\xi)|^2 d\xi = \sum_{i=-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |y_i^n|^2 \Delta x \equiv \|y^n\|_{\Delta x}^2. \tag{23}$$

And

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{y}^n(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |y(x)|^2 dx. \tag{24}$$

We have,

$$\|I_{\Delta x} y^n\|^2 = \int_{-\infty}^{\infty} |I_{\Delta x} y^n(x)|^2 dx.$$

Using (23),

$$\|I_{\Delta x} y^n\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{I_{\Delta x} y}^n(\xi) \right|^2 d\xi.$$

Given the interpolation operator (23) *satisfies where*

$$\widehat{I_{\Delta x} v}(\xi) = \begin{cases} \hat{v}(\xi) & |\xi| \le \frac{\pi}{\Delta x} \\ & |\xi| > \frac{\pi}{\Delta x} \end{cases}$$

Thus,

$$\|I_{\Delta x} y^n\|^2 = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |\hat{y}(\xi)|^2 d\xi.$$

By (24)*, we get*

$$\|I_{\Delta x} y^n\|^2 = \sum_{i=-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |y_i|^2 \Delta x.$$

$$\|I_{\Delta x} y^n\|^2 \equiv \|y\|_{\Delta x}^2.$$

Taking square-root on both side, finally we get the results

$$\therefore \|I_{\Delta x} y^n\| = \|y^n\|_{\Delta x}.$$

The proof is completed.

Lemma 3.2. *If* v_i^n *is a grid function and* $y \in L^2(\mathbb{R})$*. Then the truncation operator* $T_{\Delta x} : L^2(\mathbb{R}) \to L^2(Z\Delta x)$ *by*

$$(T_{\Delta x} y)_i = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{\iota i \Delta x \xi} \hat{y}(\xi)\, d\xi. \tag{25}$$

satisfies for all $\Delta x > 0$ *such that* $\|T_{\Delta x} y - v\|_{\Delta x} \leq \|y - I_{\Delta x} v\|$*.*
Proof: Using Parseval's identities of (23) *and* (24)*. Now,*

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \sum_{i=-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |(T_{\Delta x} y)_i - v_i|^2 \Delta x.$$

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} \left|\widehat{T_{\Delta x} y}(\xi) - \hat{v}(\xi)\right|^2 d\xi.$$

Given the truncation operator (25) *it satisfies where*

$$\widehat{T_{\Delta x} y}(\xi) = \hat{y}(\xi) \ , \quad \forall |\xi| \leq \frac{\pi}{\Delta x}.$$

Thus,

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} |\hat{y}(\xi) - \hat{v}(\xi)|^2 d\xi.$$

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 \leq \frac{1}{2\pi} \int_{|\xi| \leq \frac{\pi}{\Delta x}} |\hat{y}(\xi) - \hat{v}(\xi)|^2 d\xi + \frac{1}{2\pi} \int_{|\xi| > \frac{\pi}{\Delta x}} |\hat{y}(\xi) - \hat{v}(\xi)|^2 d\xi.$$

The given equation it satisfies the truncation operator. So the second term is zero. We know that, by (23)

$$\widehat{I_{\Delta x} v}(\xi) = \begin{cases} \hat{v}(\xi) & |\xi| \leq \frac{\pi}{\Delta x} \\ 0 & |\xi| > \frac{\pi}{\Delta x} \end{cases}$$

Finally we get,

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{y}(\xi) - \widehat{I_{\Delta x} v}(\xi)\right|^2 d\xi.$$

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \int_{-\infty}^{\infty} |y(x) - I_{\Delta x} v(x)|^2 dx \,[by\ \ (25)]\,.$$

$$\|T_{\Delta x} y - v\|_{\Delta x}^2 = \|y - I_{\Delta x} v\|_{\Delta x}^2.$$

Taking square-root on the both side, we get

$$\therefore \|T_{\Delta x} y - v\|_{\Delta x} \leq \|y - I_{\Delta x} v\|\,.$$

Hence its claimed.

Theorem 3.3. *A monotone scheme is total variation nonincreasing (TVNI).*
Proof: The monotone schemes from an L_1- contractive semigroup,

$$\|L \cdot v - L \cdot Z\|_{L_1} \leq \|v - Z\|_{L_1}. \tag{26}$$

From all L_1- summable v and Z. Take,

$$\|u\|_{L_1} = \sum_{i=-\infty}^{\infty} |u_i|.$$

Equation (12) follows immediately from applying (26) to v and

$$Z = T.v,$$

$$Z_i = v_{i+1} \quad \forall\ i$$

$therefore$ It is TVNI.
Hence, a monotone scheme is total variation nonincreasing.

Theorem 3.4. *A TVNI scheme is monotonicity preserving.*
Proof: Let finite difference scheme (10) be a TVNI scheme. Let v be a monotone mesh function of bounded total variation. Consider,

$$w = L \cdot v. \tag{27}$$

Since L has a finite support of $2k+1$ points.

Claim:*w is a monotone for all v of the form*

$$v = \begin{cases} constant = v_L & if\ j < J_- \\ monotone & if\ J_- \leq j \leq J_+\ ,\ J_+ \geq J_- \\ constant = v_R & if\ j \geq J_+ \end{cases}$$

$$i.e., \quad TV(v) = |v_R - v_L|.$$

Suppose w is not monotone, then it has atleast one local minimum and one local maximum. Denote v_m and v_M be the values of the first two successive local extrema, then

$$TV(w) \geq |v_R - v_L| + |v_m - v_M|.$$

$$TV(w) \geq TV(v).$$

$$TV(L \cdot v) \geq TV(v).$$

This is contradicts to TVNI scheme. $therefore$ w is a monotone Hence, TVNI scheme is monotonicity preserving.

4. Finding

Hyperbolic equations retain any discontinuous of functions or derivatives in the initial data. Finite difference method and finite element method are the two important classes of numerical method of partial differential equation. Hence we find out the solution of hyperbolic explicit and monotone equation using upwind finite difference method.

5. Conclusion

Finite difference method in partial differential equation contains the approximate derivatives by combining nearby function values using a set of weights. Several different algorithms are available for calculating such weights. To use a finite difference method to approximate the solution to a problem, one must discretize the problem's domain which is usually done by dividing the domain into uniform grid. This means the finite difference method produce sets of discrete numerical approximations of the derivative often in a time stepping manner. Thus we use explicit and monotone scheme methods to obtain a classical numerical method of partial differential equation. This paper can be extended not only for the further studies for implicit monotone scheme but also for explicit hyperbolic partial differential equation.

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