TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1410-1418

SPLIT AND NON-SPLIT HUB NUMBER OF GRAPHS

M. PAVITHRA[1*], B. SHARADA[2], §

ABSTRACT. A split hub set S in a graph G is a hub set such that the induced subgraph $\langle V \setminus S \rangle$ is disconnected. The split hub number of G, denoted by $h_s(G)$ is the minimum size of a split hub set in G. In this paper, the split hub number for several classes of graphs is computed and we investigate the relationship of $h_s(G)$ with other known parameters of G. Also the concept of non-split hub number is introduced and its exact values for some standard graphs are computed.

Keywords: Hub set, A split hub set, A non-split hub set.

AMS Subject Classification: 05C40, 05C99.

1. Introduction

All graphs considered here are finite, undirected without loops or multiple edges. As usual p and q denote the number of vertices and edges of a graph G. Any undefined term or notation in this paper can be found in [1, 3]. The degree of a vertex v in a graph G denoted by $degv$ is the number of edges of G incident with v. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)(\delta(G))$. We denote the minimum number of edges in independent set of G (i.e., edge independence number) by $\beta_1(G)$. A vertex of degree one is called a pendant vertex. The symbols $\alpha(G)$ and $\kappa(G)$ denote the vertex cover number and the connectivity of G, respectively.
The distance between vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest $v_i - v_j$ path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

[1] Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, 570 006, India.
e-mail: sampavi08@gmail.com; ORCID: https://orcid.org/0000-0002-1489-1911.
[*] Corresponding author.
[2] Department of Studies in Computer Science, University of Mysore, Manasagangotri, Mysuru, 570 006, India.
sharadab21@gmail.com; ORCID: https://orcid.org/0000-0001-5580-4354.
§ Manuscript received: June 30, 2020; accepted: June 16, 2021.

The graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G - v$. In a tree, a leaf is a vertex of degree one.

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$. The connected hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G [17].
In 2018, the authors in [6] introduced the concept of hubtic number of graphs. In 2018, Shadi and Veena [7] introduced the restrained hub number. The authors in [8, 9], studied the edge hub number in graphs and edge hubtic number in graphs. For more details on the hub studies we refer to [5, 13]. With this motivation, we define and introduce the concept of split hub number of graphs, also we introduce the concept of non-split hub number of graphs.
In this paper, the split and non-split hub numbers of some graphs are obtained. The relations between split and non-split hub number and other parameters are determined. Also, a split and non-split hub numbers of join and corona of graphs are computed.
A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V \setminus S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [4]. We need the following to prove main results.

Theorem 1.1. [16] *For any graph G, $\lceil \frac{p}{1+\Delta(G)} \rceil \leq \gamma(G)$, where $\lceil x \rceil$ is the least integer not less than x.*

Theorem 1.2. [17] *Let T be a tree with p vertices and l leaves. Then $h(T) = h_c(T) = p - l$.*

Theorem 1.3. [16] *For any (p, q) graph G,*
(1) $p - q \leq \gamma(G)$.
(2) $\lceil \frac{diam(G)+1}{3} \rceil \leq \gamma(G)$.

Lemma 1.1. [17] *Let $diam(G)$ denote the diameter of G. Then $h(G) \geq diam(G) - 1$ and the inequality is sharp.*

Lemma 1.2. [14] *For any graph G, $\beta_1(G) \leq \alpha(G)$.*

MAIN RESULTS

2. Split hub number of graphs

Definition 2.1. *A split hub set S in a graph G is a hub set such that the induced subgraph $\langle V \setminus S \rangle$ is disconnected. The split hub number of G, denoted by $h_s(G)$ is the minimum size of a split hub set in G.*

A split hub set S of minimum cardinality is called h_s- set of G. We note that h_s-sets exist if the graph is not complete and either contains a non-complete component or contains at least two non-trivial components. In this section, we will assume that G is a non-complete graph. It is clear that $h_s(G)$ is well-defined for any graph G, since $V(G)$ is a split hub set. In all situations of interest, we will assume G to be connected, if G is a disconnected graph then any split hub set must contain union of the set of vertices

from all the components except one chosen component and the split hub set of the chosen component. It is obvious that any split hub set in a graph G is also a hub set and thus $h_s(G) \geq h(G)$.
We now proceed to compute $h_s(G)$ for some standard graphs.

Proposition 2.1.

(1) For any path P_p with $p \geq 3$, $h_s(P_p) = p - 2$.
(2) For any cycle C_p,

$$h_s(C_p) = \begin{cases} p-2, & \text{if } p = 4 \ ; \\ p-3, & \text{if } p \geq 5. \end{cases}$$

(3) For the star $K_{1,p-1}$, $h_s(K_{1,p-1}) = 1$.
(4) For the double star $S_{n,m}$, $h_s(S_{n,m}) = 2$.
(5) For the complete bipartite graph $K_{n,m}, h_s(K_{n,m}) = \min\{n, m\}$.
(6) For the wheel graph $W_{1,p-1}, h_s(W_{1,p-1}) = 3$.
(7) For the complete k-partite graph $K_{n_1,n_2,\dots\dots,n_k}$,

$$h_s(K_{n_1,n_2,\dots\dots,n_k}) = \sum_{i=1}^{k} n_i - \max_{i=1}^{k} n_i.$$

Proposition 2.2. *If a hub set H of G is also a split hub set, then there exist two vertices v_1, v_2 in $V - H$ such that $d(v_1, v_2) \geq 2$.*

Proof. Suppose for any two vertices v_1, v_2 in $V \backslash H, d(v_1, v_2) = 1$. Thus, $\langle V \backslash H \rangle$ is connected which is a contradiction to the fact that H is a split hub set of G and this completes the proof. □

Observation 2.1.

(i): For any graph G, $1 \leq h_s \leq p - 2$.
(ii): $h_s(\overline{nK_2}) = 2n - 2, n > 2$, where nK_2 is the n copies of K_2.

Definition 2.2. [12] *A firefly graph $F_{s,t,p-2s-2t-1}$ ($s \geq 0, t \geq 0$ and $p-2s-2t-1 \geq 0$) is a graph of order p that consists of s triangles, t pendant paths of length 2 and $p-2s-2t-1$ pendant edges sharing a common vertex.*

Proposition 2.3. $h_s(G) = 1$ *if* $G = F_{s,0,p-2s-1}$.

Theorem 2.1. *For any connected graph G, $h_s(G) = 1$ if and only if there exists only one cut vertex in G with degree $p - 1$.*

Proof. Let $h_s(G) = 1$ and $H_s = \{v\}$ is a split hub set of G. It is clear, $\langle G \setminus \{v\} \rangle$ is disconnected. Since v is adjacent to all vertices and $\langle G \setminus \{v\} \rangle$ is disconnected. Then v is a cut vertex with degree $p - 1$. The converse is obvious. □

This result has the following immediate corollary.

Corollary 2.1. *If the graph G has no cut vertices, then $h_s(G) \geq 2$.*

Observation 2.2. *In general, the inequality $h_s(G') \leq h_s(G)$ is not true for a subgraph G' of G. For example, consider the graph G and a subgraph G' of G shown in Figure 1.*

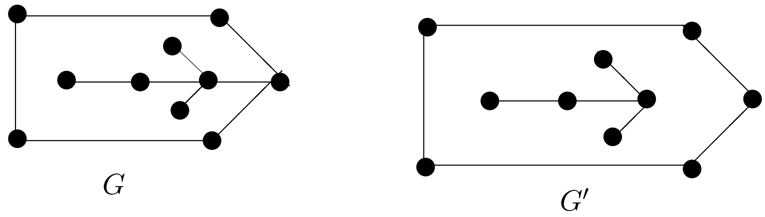

Figure 1

We note that $h_s(G) = 5$ and $h_s(G') = 7$.

Theorem 2.2. *Let T be a tree with p vertices and l leaves. Then $h_s(T) = p - l$.*

Proof. H_s the set of all internal vertices of T which is a split hub set of T. So $h_s(T) \leq p - l$. Since $h(T) = p - l$ by Theorem 1.2 and since $h_s \geq h$, we get $h_s(T) \geq p - l$. Therefore $h_s(T) = p - l$. □

Theorem 2.3. *For any graph $G \neq K_p$, $h_s(G) \geq \gamma(G)$.*

Proof. Since every split hub set of G is a dominating set, the required result is obtained. □

Theorem 2.4. *For any tree T, $h_s(T) \geq \alpha(T)$.*

Proof. Since a split hub set of T is a vertex covering set, this completes the proof. □

Lemma 2.1. *For any tree T, $h_s(T) \geq \beta_1(T)$.*

Proof. By Lemma 1.2 and Theorem 2.4 we get the result. □

By Lemma 1.1 and by using the inequality $h_s \geq h$ for any graph G, we get the following theorem.

Theorem 2.5. *For any connected graph $G \neq K_p$, $h_s(G) \geq diam(G) - 1$.*

Theorem 2.6. *For any graph $G \neq K_p$, $h_s(G) \geq p - q$.*

Proof. By Theorem 1.3 and Theorem 2.3 we get the result. □

Theorem 2.7. *For any graph $G \neq K_p$, $h_s(G) \geq \lceil \frac{p}{1+\Delta(G)} \rceil$.*

Proof. The proof follows from Theorem 1.1 and Theorem 2.3. □

Theorem 2.8. *For any graph $G \neq K_p$, $h_s(G) \geq \lceil \frac{diam(G)+1}{3} \rceil$.*

Proof. By Theorem 1.3 and Theorem 2.3 the result follows. □

Definition 2.3. [3] *Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2, respectively. Then their join $G_1 + G_2$ is the graph consisting of $G_1 \cup G_2$ with all edges joining V_1 with V_2.*

Theorem 2.9. *For any connected graphs G_1 and G_2, $h_s(G_1 + G_2) = min\{|V(G_1)| + \kappa(G_2), |V(G_2)| + \kappa(G_1)\}$, if G_1 and G_2 are both non-complete.*

Proof. Let $V(G_1) = \{v_1, v_2, v_3, ..., v_p\}$ and $V(G_2) = \{u_1, u_2, u_3, ..., u_p\}$.
Case 1: If $|V(G_1)| \leq |V(G_2)|$ and $\kappa(G_2) < \lceil \frac{p}{2} \rceil$ consider $H_s = \{v_1, v_2, v_3, ..., v_p\}$ is a split hub set of $G_1 + G_2$ and $|H_s| = |V(G_1)|$. Since $v_i, 1 \leq i \leq p$ is adjacent to all the vertices of graph G_2 then there exists H_s-path between any two vertices of G_2, but $(G_1 + G_2) \setminus H_s$ is connected graph and $(G_1 + G_2) \setminus H_s = G_2$. To complete the proof, we should remove some vertices of G_2 to get disconnected. So the connectivity set is the smallest set that

will be obtained by removal of minimum number of vertices from G_2 such that it gets disconnected. Therefore, $|H_s| = |V(G_1)| + \kappa(G_2)$.
Case 2: If $|V(G_1)| \leq |V(G_2)|$ and $\kappa(G_2) \geq \lceil \frac{p}{2} \rceil$, consider $H_s = \{u_1, u_2, u_3, ..., u_p\}$ is a split hub set of $G_1 + G_2$, and $|H_s| = |V(G_2)|$. The proof is similar to case 1. So $|H_s| = |V(G_2)| + \kappa(G_1)$. Hence the result.
Case 3: If $|V(G_2)| \leq |V(G_1)|$ and $\kappa(G_1) < \lceil \frac{p}{2} \rceil$ consider $H_s = \{u_1, u_2, u_3, ..., u_p\}$ is a split hub set of $G_1 + G_2$, and $|H_s| = |V(G_2)|$. The proof is similar to case 1. So $|H_s| = |V(G_2)| + \kappa(G_1)$. Hence the result.
Case 4: If $|V(G_2)| \leq |V(G_1)|$ and $\kappa(G_2) \geq \lceil \frac{p}{2} \rceil$ consider $H_s = \{v_1, v_2, v_3, ..., v_p\}$ is a split hub set of $G_1 + G_2$, and $|H_s| = |V(G_1)|$. The proof is similar to case 1. So $|H_s| = |V(G_1)| + \kappa(G_2)$. Hence the result. □

Proposition 2.4. *Let G be disconnected graph and $K_1, K_2, ..., K_s$ be its components. Then*

$$h_s(G) = min_{1 \leq r \leq s}\{w_r\}, \text{ where } w_r = h_s(K_r) + \sum_{j=1, j \neq r}^{s} |K_j|.$$

Proof. From the definition, the split hub set H_s of a disconnected graph must contain union of vertices from all the components, except vertices from the r^{th} component and split hub set of the r^{th} component. It remains to show that H_s is minimum. By taking the arbitrary union of the order of all components except r^{th} component, we count h_s of r^{th} component that gives all possible value of $h_s(G)$. Thus, $w_r = h_s(K_r) + \sum_{j=1, j \neq r}^{s} |K_j|$. Then $h_s(G) = min_{1 \leq r \leq s}\{w_r\}$. □

Definition 2.4. [2] *The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Theorem 2.10. *For any connected graphs G_1 and G_2,*

$$h_s(G_1 \circ G_2) = \begin{cases} 1 + \kappa(G_2), \ \text{if } |V(G_1)| = 1 \text{ and } G_2 \text{ is non-complete ;} \\ |V(G_1)|, \ \text{if } |V(G_1)| \geq 2. \end{cases}$$

Proof. Suppose $|V(G_1)| = 1$ and G_2 is non-complete, this means that $G_1 \circ G_2$ is $G_1 + G_2$ and $h_s(G_1 + G_2) = 1 + \kappa(G_2)$ by Theorem 2.9 .
Now, suppose $|V(G_1)| \geq 2$,
Consider $H_s = V(G_1)$, and let $x, y \in V(G_1 \circ G_2) \setminus H_s$, we discuss the following cases:
Case 1: Suppose $x, y \in V(G_{2a})$ for some $a \in V(G_1)$. Then $\{x, a, y\}$ is a path in $G_1 \circ G_2$ and $a \in G_1$. Therefore, there is an H_s-path between x and y in $G_1 \circ G_2$.
Case 2: If $x \in V(G_{2a})$ and $y \in V(G_{2b})$ for some $a, b \in V(G_1)$. Note that there is a path $\{x, a, v_1, v_2, ..., b, y\}$ in $G_1 \circ G_2$. Thus, there is an H_s-path between x and y in $G_1 \circ G_2$. Then H_s is a split hub set of $G_1 \circ G_2$ and $h(G_1 \circ G_2) \leq |V(G_1)|$. To show that $V(G_1)$ is the minimum split hub set, if $v_i \in G_1$, for $1 \leq i \leq p$, is removed from set H_s, then there do not exist path between x and y. Thus H_s is a minimum split hub set, hence $h(G_1 \circ G_2) = |V(G_1)|$. □

Definition 2.5. [15] *The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other.*

Theorem 2.11. *Let $n \geq 2$ be a positive integer. Then $h_s(B_n) = |V(B_{n-1})|$.*

Proof. Let H_s be a minimum hub set of B_n and let r be the number of internal vertices in B_n, so $|H_s| = r$ because the internal vertices in B_n form a minimal split hub set of B_n. Since the number of internal vertices of B_n from a minimum split hub is equal to the number of vertices in B_{n-1}, it follows that $|H_s| = |V(B_{n-1})|$ and $V \setminus H_s$ is totally disconnected graph. Hence split hub number of B_n is $|V(B_{n-1})|$. □

3. Non-split hub number of graphs

Definition 3.1. *A non-split hub set H_{ns} in a graph G is a hub set such that the induced subgraph by $\langle V \setminus H_{ns}\rangle$ is connected. The non-split hub number of G, denoted by $h_{ns}(G)$, is the minimum size of a non-split hub set in G.*

Proposition 3.1.

(1) For any complete graph K_p, $h_{ns}(K_p) = 0$.
(2) For any path P_p, with $p \geq 4$, $h_{ns}(P_p) = p - 2$.
(3) For any cycle C_p, $p \geq 4$, $h_{ns}(C_p) = p - 3$.
(4) For the star $K_{1,p-1}$, $h_{ns}(K_{1,p-1}) = p - 2$.
(5) For the double star $S_{n,m}$, $h_{ns}(S_{n,m}) = n + m - 2$.
(6) For the complete bipartite graph $K_{n,m}, h_{ns}(K_{n,m}) = 2$.
(7) For the wheel graph $W_{1,p-1}$,$h_{ns}(W_{1,p-1}) = 1$.

Theorem 3.1. *Let $G = K_{n_1,n_2,\ldots,n_k}$ with $k \geq 3$. Then*

$$h_{ns}(G) = \begin{cases} 1, & \text{if } n_i \geq 2,\ n_l = 2 \ for\ some\ l,\ 1 \leq l \leq k\ ; \\ 2, & \text{if } n_i \geq 3 \ for\ all\ i, 1 \leq i \leq k\ . \end{cases}$$

Proof. Two cases are considered:
Case 1: $n_i \geq 2$,and $n_l = 2$ for some l ,$1 \leq l \leq k$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of $V(G)$. Consider $v_1, v_2 \in V_l$. Then $H_{ns} = \{v_1\}$ is a non-split hub set of G. Now, v_1 are adjacent to all vertices in the other partite sets of G, so there exists an H_{ns}-path between any two vertices of other partite sets. Also, v_2 is adjacent to all vertices of $V(G) \setminus v$. Since, G is non-complete graph, H_{ns} is a minimum non-split hub set of G and hence, $h_{ns}(G) = 1$.

Case 2: $n_i \geq 3$ for all i, $1 \leq i \leq k$. Let $H_{ns} = u, v$ be a set of any two vertices of G taken from two different partite sets of G. Then $G \setminus u, v$ is connected, and there exists a path between any two vertices of the other partite sets of G. We must show that H_{ns} is a minimum non-split hub set of G. If u is removed from set H_{ns}, then there do not exist H_{ns}-path between the remaining vertices of the same partite sets that contains v. Similarly if we remove v, then the resulting set will not be a non-split hub set. Therefore, $h_{ns}(G) = 2$. □

Proposition 3.2. *For any graph G, $0 \leq h_{ns}(G) \leq p - 2$.*

Theorem 3.2. *Let T be any tree of order $p \geq 3$. Then $h_{ns}(T) = p - 2$.*

Proof. Let H_{ns} be the set of all vertices of T except two vertices u, v such u is adjacent to v, this implies that $V \setminus \{u, v\}$ is non-split hub set of T. It is clear that H_{ns} is minimum, because if H_{ns} consists of all vertices of T except three vertices u, v and w such that uvw is a connected path, so u and w is not adjacent, then there do not exist H-path between them and hence H_{ns} is not hub set of T. □

Proposition 3.3. *For any tree T, $h_{ns}(T) \geq h_s(T)$.*

Theorem 3.3. *For any connected spanning subgraph H of G, $h_{ns}(G) \leq h_{ns}(H)$.*

Since their proofs are trivial, we omit the same.

Theorem 3.4. *For any graph G,*

(1) $h_{ns}(G) \geq \gamma(G) - 1$.
(2) $h(G) = min\{h_s(G), h_{ns}(G)\}$.

Theorem 3.5. *For any connected graphs G_1 and G_2,*

$$h_{ns}(G_1 + G_2) = \begin{cases} 0, & \text{if } G_1 \text{ and } G_2 \text{ are complete ;} \\ 1, & \text{if } G_1 \text{ or } G_2 \text{ is complete,} \end{cases}$$

and $h_{ns}(G_1 + G_2) \leq 2$, if G_1 and G_2 are both non-complete.

Proof. Suppose G_1 and G_2 are both complete. Then $G_1 + G_2$ is complete and the proof follows from Proposition 3.1.
Suppose G_1 is complete. Let $x \in V(G_1)$ and $H_{ns} = \{x\}$. Let $y, z \in V(G_1 + G_2) \backslash H_{ns}$. The following cases are considered:
Case 1: $y, z \in V(G_1)$. Since G_1 is complete, there is a path $\{y, x, z\}$ in G_1. Hence, there is an H_{ns}-path between y and z in $G_1 + G_2$.
Case 2: $y \in V(G_1)$, $z \in V(G_2)$, y and x are adjacent and x is adjacent to z, (by definition of $G_1 + G_2$). Thus, there is $\{y, x, z\}$-path in $G_1 + G_2$.
Case 3: Suppose $y, z \in V(G_2)$. Clearly, x is adjacent to both y and z. So there is $\{y, x, z\}$-path in $G_1 + G_2$ and $G_1 + G_2 - \{x\}$ is connected graph. Therefore, H_{ns} is a non-split hub set of $G_1 + G_2$ and $h_{ns}(G_1 + G_2) = 1$.
If G_2 is complete, the proof is similar.
Now, suppose G_1 and G_2 are both non-complete. Consider the following cases:

Case 1: $h_{ns}(G_1) = 1$. Consider $x \in V(G_1)$ and let $H = \{x\}$ be a minimum non-split hub set of G_1. Let $y, z \in V(G_1 + G_2) \backslash \{x\}$. Consider the following subcases:
Subcase 1.1: $y, z \in V(G_1) \backslash \{x\}$. It is clear, there is an H_{ns}-path between y and z in $G_1 + G_2$.
Subcase 1.2: $y \in V(G_1) \backslash \{x\}$, $z \in V(G_1)$ and y is adjacent to x, because H_{ns} is a non-split hub set of G_1 and also x is adjacent to z, by definition of $G_1 + G_2$. So $\{y, x, z\}$ is an H_{ns}-path in $G_1 + G_2$. Then $h_{ns}(G_1 + G_2) \leq 1$.
Subcase 1.3: $y, z \in V(G_2)$. From the definition of $G_1 + G_2$, x is adjacent to y and z, $\{y, x, z\}$ is an H_{ns}-path in $G_1 + G_2$. Therefore, $h_{ns}(G_1 + G_2) \leq 1$.
Since G_1 and G_2 are both non-complete, $G_1 + G_2$ is non-complete. So $h_{ns}(G_1 + G_2) \neq 0$. Then $h_{ns}(G_1 + G_2) = 1$.
Case 2: Suppose $h_{ns}(G_2) = 1$. The proof is similar to Case 1.
Case 3: $h_{ns}(G_1) = h_{ns}(G_2) \geq 2$. Let $q \in V(G_1), r \in V(G_2)$ and $H_{ns} = \{q, r\}$. We have the following cases:
Subcase 3.1: $x, y \in V(G_1) \setminus \{q\}$. Note that x and y are adjacent to r, this means that there is $\{x, r, y\}$-path in $G_1 + G_2$. Thus, H_{ns} is a non-split hub set of $G_1 + G_2$ and $h_{ns}(G_1 + G_2) \leq 2$.
Subcase 3.2: Let $x, y \in V(G_2) \setminus \{r\}$. The proof is similar to subcase 1.3.
Subcase 3.3: Consider $x \in V(G_1) \setminus \{q\}$, $y \in V(G_2) \setminus \{r\}$, it is clear that x is adjacent to r, r is adjacent to q and q is adjacent to y. That is $\{x, r, q, y\}$ is an H_{ns}-path in $G_1 + G_2$. Then H_{ns} is a non-split hub set of $G_1 + G_2$. Therefore, $h_{ns}(G_1 + G_2) \leq 2$. □

Proposition 3.4. *Let G_1 be a graph of order 1 and G_2 be non-complete graph. Then $h_{ns}(G_1 \circ G_2) = 1$.*

Theorem 3.6. *Let G_1 be a connected graph of order $p_1 > 2$ and $G_1 \neq K_p$ and G_2 be any connected graph of order p_2. Then $h_{ns}(G_1 \circ G_2) = p_1 + (p_1 - 1)p_2$.*

Proof. Let $V(G_1 \circ G_2) = V(G_1) \cup (\cup_{i=1}^{p_1} V_i)$ where, $V_i = \{u_{i1}, u_{i2}, u_{i3}, ..., u_{ip_2}\}$ and $\langle V_i \rangle \cong G_2$ for $i = 1, 2, ..., p_1$ and $V(G_1) = \{v_1, v_2, ..., v_{p_1}\}$
Let $H_{ns} = V(G_1) \cup (\cup_{i=1}^{p_1-1} V_i)$ is non-split hub set of $G_1 \circ G_1$. Then clearly, H_{ns} is a minimum non-split hub set and the induced subgraph $\langle V(G_1 \circ G_2) \backslash H_{ns} \rangle$ is connected graph and $h_{ns}(G_1 \circ G_2) = |V(G_1) \cup (\cup_{i=1}^{p_1-1} V_i)| = p_1 + (p_1 - 1)p_2$. □

4. Conclusions

In this paper, we introduced the concept of split and non-split hub numbers of a graph and there is still much to be discovered in this concept. Among the questions raised by this research, the following are of particular interest to the authors:

(1) Characterize graphs G for which $h_s(G) = h(G)$.
(2) Characterize graphs G for which $h_{ns}(G) = h(G)$.
(3) Characterize graphs G for which $h_{ns}(G) = h_s(G)$.
(4) Characterize graphs G for which $h_s(G) = 2$.

Acknowledgement: Our thanks are due to the anonymous referee(s) for careful reading and constructive suggestions for the improvement in the first draft of this paper.

References

[1] Bondy, J. A., Murty, U. S. R., (1976), Graph Theory with Applications, The Macmillan Press Ltd.
[2] Frucht, R., Harary, F., (1970), On the corona of two graphs, Aequat. Math., 4, pp. 322-325.
[3] Harary, F. (1969), Graph Theory, Addison Wesley, Reading Mass.
[4] Haynes, T. W., Hedetniemi, S. T., Slater, P. J., (1998), Fundamentals of Domination in graphs, Marce Dckker, New York.
[5] Johnson, P., Slater, P., Walsh, M., (2011), The connected hub number and the connected domination number, Wiley Online Library, pp. 232-237.
[6] Khalaf, S. I., Mathad, V., Mahde, S. S., (2018), Hubtic number in graphs, Opuscula Math., 38, pp. 841-847.
[7] Khalaf, S. I., Mathad, V., (2019), Restrained hub numbr in graphs, Bull. Int. Math. Virtual Inst., 9, pp. 103-109.
[8] Khalaf, S. I., Mathad, V., Mahde, S. S., (2019), Edge hub number in graphs, Online journal of analytic combinatorics, 14.
[9] Khalaf, S. I., Mathad, V., Mahde, S. S., (2018), Edge hubtic number in graphs, International J. Math. Combin., 3, pp. 140-145.
[10] Kulli, V. R., (2010), Theory of domination in graphs, Vishwa International Puplications, Gulbarga, India.
[11] Kulli, V. R., (2012), College Graph Theory, Vishwa International Publications, Gulbarga, India.
[12] Li, J. X., Guo, J. M., Shiu, W. C., (2013), On the second largest laplacian eigenvalues, Linear algebra appl., 438, pp. 2438-2446.
[13] Mathad, V., Sahal, A. M. and Kiran, S., (2014), The total hub number of graphs, Bulletin of the International Mathematical Virtual Institute, 4, pp. 61-67.
[14] Sultan Senan, M. and Mathad, V., (2017), Pure Edge-Neighbor-Integrity of Graphs, International Journal of Mathematical Combinatorics, 1, pp. 111-126.
[15] Sundareswaran, R., Swaminathan, V., (2012), Domination Integrity in Trees, Bulletin of International Mathematical Virtual Institute, 2, pp. 153-161.
[16] Walikar, H. B., Acharya, B. D., Sampathkumar, E., (1997), Recent developments in the theory of domination in graphs, MRI Lecture Notes in Mathematics, 1.
[17] Walsh, M., (2006), The hub number of graphs, International Journal of Mathematics and Computer Science, 1, pp. 117-124.

M. Pavithra was born in India. She got her M. Sc. and M. Phil degrees from University of Mysore. She is a research scholar in the Department of Studies in Mathematics, University of Mysore, Mansagangotri, Mysuru.

Dr. B. Sharada was born in India. She has been working as an assistant professor for 14 years in the Department of Studies in Computer Science, University of Mysore, Manasagangotri, Mysuru.