Prime ideals and Noetherian properties in vector lattices

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Positivity

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Abstract

In this paper we study the set of prime ideals in vector lattices and how the properties of the prime ideals structure the vector lattice in question. The different properties that will be considered are firstly, that all or none of the prime ideals are order dense, secondly, that there are only finitely many prime ideals, thirdly, that every prime ideal is principal, and lastly, that every ascending chain of prime ideals is stationary (a property that we refer to as prime Noetherian). We also completely characterize the prime ideals in vector lattices of piecewise polynomials, which turns out to be an interesting class of vector lattices for studying principal prime ideals and ascending chains of prime ideals.

1 Introduction

Prime ideals play an important role in the study of vector lattices. These ideals have been studied extensively in [10] and the analogy between the prime ideal structure in commutative rings and vector lattices was studied in some detail in [6]. The prime ideals are precisely the ideals for which the quotient vector lattices are linearly ordered, and every vector lattice contains an abundance of prime ideals. Indeed, for any element

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f in a vector lattice, the collection of ideals that do not contain f has a maximal element with respect to set inclusion by Zorn's lemma, and this maximal element is a prime ideal. Furthermore, any ideal containing a prime ideal is again a prime ideal, and the set of prime ideals containing a fixed prime ideal is linearly ordered. The set of prime ideals in a vector lattice, and certain subsets herein, also yield various representations of vector lattices. For instance, Yosida proved in [13] that an Archimedean vector lattice is isomorphic to the vector lattice of extended functions on the set of prime ideals that avoid a maximal disjoint set in the underlying vector lattice. Johnson and Kist proved a more general result in [7] which states that every Archimedean vector lattice is isomorphic to the extended functions on a subset of prime ideals that have zero intersection and avoid a maximal disjoint set in the underlying vector lattice. Properties of the prime ideals in a vector lattice can also encode the structure of the vector lattice. For example, in $C(K)$, the continuous functions on a compact Hausdorff space K, it follows that every prime ideal is maximal if and only if K is finite, and more generally, in a uniformly complete vector lattice every prime ideal is maximal if and only if the vector lattice is isomorphic to $c_{00}(\Omega)$ for some set Ω. See [10] for details. The goal of this paper is to further study the structure of vector lattices in terms of the properties of its prime ideals. We characterize the vector lattices for which the set of prime ideals is specialized to having the following properties. Firstly, that all or none of the prime ideals are order dense, secondly, that there are only finitely many prime ideals, thirdly, that every prime ideal is principal, and lastly, that ascending chains of prime ideals are stationary.

The connection between order dense prime ideals and atoms in vector lattices is a dichotomous one. Namely, every prime ideal is order dense if and only if the vector lattice is atomless, and that none of the prime ideals are order dense if and only if the vector lattice is of the form $c_{00}(\Omega)$ for some set Ω.

As for vector lattices with only finitely many prime ideals, we prove that these vector lattices must be finite-dimensional.

The specialization to all prime ideals being principal is inspired by Cohen's theorem for commutative rings, which says that a commutative ring is Noetherian if and only if every prime ideal is finitely generated, see [5,Theorem 2]. Since in vector lattices an ideal is finitely generated if and only if it is principal, the restriction to principal prime ideals is taken for this reason. Cohen's theorem is connected to another classical result from commutative ring theory, Kaplansky's theorem. Kaplansky proved in [9,Theorem 12.3] that in a Noetherian commutative ring every ideal is principal if and only if every maximal ideal is principal. Combining the two theorems yields that a commutative ring is a principal ideal ring if and only if every prime ideal is principal, which is referred to as the Cohen–Kaplansky theorem. It turns out that an Archimedean vector lattice is Noetherian if and only if it is finite-dimensional. Hence, we prove the vector lattice analogue of the Cohen–Kaplansky theorem stating that every prime ideal in an Archimedean vector lattice is principal if and only if it is finite-dimensional, which is further equivalent with all ideals being principal. We prove that in a uniformly complete Archimedean vector lattice there are no non-maximal principal prime ideals, and we use this result to prove a vector lattice analogue of the Cohen–Kaplansky theorem for uniformly complete vector lattices. More precisely, every ideal in a uniformly complete Archimedean vector lattice is principal if and

only if every maximal ideal is principal, which in turn is equivalent with the vector lattice being finite-dimensional.

Since Noetherian vector lattices are finite-dimensional and, as mentioned above, there are naturally occurring chains of prime ideals in vector lattices, studying vector lattices in which every ascending chain of prime ideals $P_1 \subseteq P_2 \subseteq \ldots$ is stationary is more interesting. We propose to call these vector lattices *prime Noetherian*. The uniformly complete Archimedean prime Noetherian vector lattices are completely characterized, and these characterizations depend on the existence of a strong unit. It turns out that a prime Noetherian vector lattice with a strong unit is finite-dimensional and in general, a prime Noetherian vector lattice is isomorphic to $c_{00}(\Omega)$ for some set Ω.

The vector lattices of piecewise polynomials are a resourceful class of non-uniformly complete vector lattices when studying principal prime ideals and prime Noetherian properties. In these vector lattices we can explicitly construct prime Noetherian vector lattices with prescribed finite maximal lengths of ascending chains of prime ideals. Furthermore, we can construct chains of non-maximal principal prime ideals of prescribed length in these vector lattices, which further shows the significant role uniform completeness plays when studying principal prime ideals in vector lattices.

The structure of our paper is as follows. Section 2 is the preliminary section where the basics about vector lattices are covered. In Sect. 3 we investigate the relation between order dense prime ideals and atoms in vector lattices. The main result of this section is the dichotomous connection between the two. In Sect. 4 we characterize vector lattices with only finitely many prime ideals. Vector lattices in which every prime ideal is principal are studied in Sect. 5 and the main results are Theorem 5.2, the vector lattice analogue of the Cohen–Kaplansky theorem for commutative rings, and Theorem 5.6, the uniformly complete vector lattice analogue of the Cohen–Kaplansky theorem for commutative rings. Prime Noetherian vector lattices are studied in Sect. 6. The main result of this section characterizes the uniformly complete Archimedean vector lattices with this property in Theorem 6.9. In Sect. 7 we study vector lattices of piecewise polynomials.

2 Preliminaries

Let E be a vector lattice. We say that E is *Archimedean* whenever it follows from $0 \leq nx \leq y$ for each $n \in \mathbb{N}$ that $x = 0$. A vector subspace I of E is called an *order ideal* or just an *ideal* whenever $0 \leq |x| \leq |y|$ and $y \in I$ imply $x \in I$. Given a non-empty subset A consisting of positive vectors in E, there is the smallest ideal I_S in E containing the set S. It is a standard fact that

$$I_S = \left\{ y \in E : 0 \leq |y| \leq \sum_{k=1}^{n} \lambda_k x_k \text{ for some } n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \geq 0, \text{ and } x_1, \ldots, x_n \in S \right\}.$$

The ideal I_S is called the *ideal generated by* S. If $S = \{x\}$, then I_S is said to be *principal* and we write I_x instead of $I_{\{x\}}$. It should be clear that the ideal I_S is principal if and only if S is finite. If $I_x = E$ for some positive vector x, then x is called a *strong unit*.

A proper ideal M of E is called *maximal* whenever for each ideal J satisfying $M \subseteq J \subseteq E$ it follows that either $J = M$ or $J = E$. If the co-dimension of an ideal I in E is one, then I is a maximal ideal and conversely, all maximal ideals have co-dimension one by [11,Corollary p.66]. If a vector lattice contains a strong unit, then every proper ideal is contained in a maximal ideal (see [10,Theorem 27.4]). If E does not contain a strong unit, then it may happen that there are no maximal ideals in E (see [10,Example 27.8]). A vector lattice contains a strong unit whenever it contains a principal maximal ideal. In fact, we prove a more general statement below.

Lemma 2.1 *Let I be a principal ideal in a vector lattice E such that the quotient vector lattice has a strong unit. Then E has a strong unit.*

Proof Suppose that there exists a positive vector $x \in E$ such that $I = I_x$ and let $y + I$ be a strong unit for E/I for some positive vector $y \in E$. Pick a vector $u \in E$. Then there exists a $\lambda > 0$ such that $|u| + I \leq \lambda y + I$. From the definition of the order on a quotient vector lattice, it follows that there exists a positive vector $w \in I_x$ such that $|u| \leq \lambda y + w$. Hence, there exists a $\mu > 0$ such that $|u| \leq \lambda y + \mu x$ which proves that $x + y$ is a strong unit for E. $\qquad\square$

The set of all maximal ideals is completely characterized in the vector lattice $C(K)$ of continuous functions on a compact Hausdorff space K. It is well-known that an ideal I in $C(K)$ is maximal if and only if I is of the form

$$M_x := \{f \in C(K): \ f(x) = 0\}$$

for some $x \in K$. If E is a sublattice of $C(K)$, then for $x \in K$ we denote the ideal $\{f \in E: f(x) = 0\}$ by M_x^E. The norm convergence in the Banach space $C(K)$ is sometimes referred to as uniform convergence. This notion can be generalized to vector lattices as follows. We say that a net $(x_\alpha)_\alpha$ in a vector lattice E converges *relatively uniformly* to some $x \in E$ if there exists a positive vector y, the regulator, such that for each $\epsilon > 0$ there exists an index α_ϵ such that for all $\alpha \geq \alpha_\epsilon$ we have $|x_\alpha - x| \leq \epsilon y$. In general, a sequence can converge relatively uniformly to more than one vector. If every relatively uniform Cauchy net converges relatively uniformly in E with respect to the same regulator, then E is called *uniformly complete*. If E is Archimedean, then the relative uniform limit is uniquely determined. In Archimedean vector lattices the general notion of relative uniform convergence is not far away from the classical one for $C(K)$ spaces. Given a positive vector x in an Archimedean vector lattice E, the principal ideal I_x equipped with

$$\|y\|_x := \inf\{\lambda > 0 : \ |y| \leq \lambda x\}$$

is a normed space. The norm completion of $(I_x, \|\cdot\|_x)$ is an AM-space with a strong unit x which by the Kakutani representation theorem (see [2,Theorem 4.29]) is lattice isometric to the Banach lattice $C(K)$ for some compact Hausdorff space K where the strong unit x is mapped to the constant one function. Hence, if the normed lattice $(I_x, \|\cdot\|_x)$ is norm complete, then I_x itself is already lattice isometric to $C(K)$. It is

quite easy to see that E is relatively uniformly complete if and only if $(I_x, \|\cdot\|_x)$ is a Banach lattice for each positive vector $x \in E$.

An ideal $P \subsetneq E$ is said to be *prime* whenever $x \wedge y \in P$ implies $x \in P$ or $y \in P$. By [10,Theorem 33.3] every maximal ideal is prime. For the proof of the following useful characterization of prime ideals we refer the reader to [10,Theorem 33.2].

Theorem 2.2 *For an ideal P in a vector lattice E the following conditions are equivalent.*

  (i) *P is prime.*
 (ii) *If $x \wedge y = 0$, then $x \in P$ or $y \in P$.*
(iii) *The quotient vector lattice E/P is linearly ordered.*
 (iv) *For any ideals I and J satisfying $I \cap J \subseteq P$ we have $I \subseteq P$ or $J \subseteq P$.*

If P is a prime ideal, then by [10,Theorem 33.3] every ideal containing P is also prime, and furthermore, the family of all ideals containing P is linearly ordered.

A non-zero positive vector $a \in E$ is said to be an *atom* whenever it follows from $0 \leq x \leq a$, $0 \leq y \leq a$ and $x \wedge y = 0$ that $x = 0$ or $y = 0$. If the principal ideal I_a generated by a equals the linear span of a, then a is a *discrete element*. While every discrete element is an atom, both classes coincide in Archimedean vector lattices. Examples of atomic vector lattices are $\mathbb{R}^n$ ordered coordinatewise and the vector lattice $c_{00}(\Omega)$ of all functions with finite support ordered pointwise on a non-empty set Ω. The atoms in $c_{00}(\Omega)$ are precisely the positive multiples of characteristic functions of singleton sets. In an Archimedean vector lattice the principal ideal I_a generated by an atom a is a projection band and so $M_a := \{a\}^d$ is a maximal ideal in E. The linear span A_0 of all atoms in a vector lattice E is always an ideal. The *atomic* part A of E is the band generated by A_0. The disjoint complement $C = A^d$ is called the *continuous* or *atomless* part of E. If $C = 0$ or equivalently $A^{dd} = E$, then E is an *atomic* vector lattice. An ideal I in E with the property $I^d = \{0\}$ is said to be *order dense*. It turns out that I is order dense in E if and only if for each non-zero positive vector $x \in E$ there is a non-zero positive vector $y \in I$ such that $0 < y \leq x$. We refer the reader to the standard texts [2, 10, 11] and [1] for any unexplained terminology about vector lattices.

3 Order dense prime ideals in vector lattices

It turns out that order dense prime ideals are closely related to the existence of atoms, in fact, their dichotomous connection is proved in Theorem 3.2. For example, if an Archimedean vector lattice E contains an atom a, then its disjoint complement $M_a = \{a\}^d$ is of co-dimension one, so it is a maximal ideal in E. Since a is disjoint with M_a, it follows that the prime ideal M_a is not order dense in E. The canonical Archimedean vector lattice which does not contain any order dense prime ideals is of the form $c_{00}(\Omega)$ for some set Ω as stated in Theorem 3.2. The equivalent statements in the proposition below follow from a combination of [10,p. 430] and [10,Theorem 61.4].

Proposition 3.1 *Let Ω be a non-empty set. Then an ideal $I \subseteq c_{00}(\Omega)$ is maximal if and only if I is of the form*

$$M_w := \{x \in c_{00}(\Omega) \colon x(w) = 0\}$$

for some $w \in \Omega$. Moreover, for a proper ideal I in $c_{00}(\Omega)$ the following statements are equivalent.

  (i) *I is a minimal prime ideal.*
  (ii) *I is a prime ideal.*
 (iii) *I is a maximal ideal.*

Proof For $w \in \Omega$ the ideal M_w is maximal since it has co-dimension one. If I is maximal in $c_{00}(\Omega)$, then it follows that $I \subseteq M_w$ for some $w \in \Omega$. Indeed, if for every $w \in \Omega$ there is a function $f \in I$ such that $f(w) > 0$, this would imply $I = c_{00}(\Omega)$. Hence, there must exist $w \in \Omega$ such that $f(w) = 0$ for all $f \in I$. Since I is a maximal ideal, we conclude that $I = M_w$. ☐

The dichotomous connection between the existence of order dense prime ideals and the existence of atoms is made precise in the following theorem.

Theorem 3.2 *Let E be an Archimedean vector lattice.*

  (i) *E is atomless if and only if every prime ideal in E is order dense in E.*
  (ii) *None of the prime ideals are order dense in E if and only if E is atomic and $E = A_0$.*

Proof (i) Suppose that E contains a prime ideal P that is not order dense. Then $P^d \neq \{0\}$. If $\dim P^d \geq 2$, then as E is Archimedean, there must exist two vectors x_1 and x_2 in P^d that are incomparable. It follows that $x := x_1 - x_1 \wedge x_2$ and $y := x_2 - x_1 \wedge x_2$ are now two non-zero disjoint vectors in P^d. Since neither x nor y is in P, this contradicts the fact that it is a prime ideal. Hence $\dim P^d = 1$, so that P^{dd} is necessarily a maximal band in E. By [10,Theorem 26.7] we conclude that there exists an atom $a \in E$ such that $P^d = \{a\}^d$.

To prove the converse implication, assume that E contains an atom. Then M_a is a maximal ideal in E. Since there is no positive vector $x \in M_a$ such that $0 < x \leq a$, the prime ideal M_a is not order dense in E.

(ii) Suppose that E does not contain order dense prime ideals. We first show that E has the projection property. To see this, pick a band B in E and consider the sum $J := B \oplus B^d$. Then J is an order dense ideal in E. If $J \neq E$, then there exists a vector $x \in E \setminus J$, so that by [10,Theorem 33.4] there exists a proper prime ideal P containing J which does not contain x. Since P is necessarily order dense, this contradicts our assumption. It follows that $E = B \oplus B^d$ and so B is a projection band.

To prove that E is atomic, note that by (i) the vector lattice E contains an atom. We need to prove that $E = A$ or equivalently $C = \{0\}$. Suppose that $C \neq \{0\}$. Then C is a non-trivial atomless band in E. By (i), there is an order dense prime ideal P in C. The set $A \oplus P$ is clearly an ideal in E. Since $(A \oplus P)^d = A^d \cap P^d = C \cap P^d$ and since P is order dense in C, we conclude that $A \oplus P$ is order dense in E.

We claim that $A \oplus P$ is a prime ideal in E. To see this, pick vectors x and $y \in E$ with $x \wedge y = 0$. Since $E = A \oplus C$, x and y can be decomposed as $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in A$ and $x_2, y_2 \in C$. From

$$x \wedge y = x_1 \wedge y_1 + x_2 \wedge y_2 = 0$$

we conclude that $x_2 \wedge y_2 = 0$. Since P is a prime ideal in C, we conclude that $x_2 \in P$ or $y_2 \in P$. This proves that $x \in A \oplus P$ or $y \in A \oplus P$, and so $A \oplus P$ is an order dense prime ideal in E which is impossible. Hence $C = \{0\}$, which proves that E is atomic. If $A_0 \neq E$, then there exists a proper prime ideal P in E which contains A_0. Since E is atomic, it follows that A_0, and therefore also P, is order dense in E which is impossible by assumption, hence $E = A_0$.

To conclude the proof note that by Proposition 3.1 the class of prime ideals coincides with the class of maximal ideals in A_0 and that none of the maximal ideals in A_0 are order dense. $\qquad \square$

Remark 3.3 Note that statement (i) in Theorem 3.2 can be reformulated so that it characterizes when non-order dense prime ideals exist in E. That is, an Archimedean vector lattice E contains a prime ideal that is not order dense if and only if E contains an atom.

4 Vector lattices with finitely many prime ideals

In this section we consider vector lattices which have finitely many (minimal) prime ideals and prove in Theorem 4.3 that these vector lattices must be finite-dimensional. We start with a general lemma that locates prime ideals and characterizes maximal ideals of norm dense sublattices of $C(K)$-spaces.

Lemma 4.1 *Let K be a compact Hausdorff space and let E be a sublattice of $C(K)$.*

(i) *For every proper ideal J in E there exists a point $x \in K$ such that J is contained in M_x^E.*

(ii) *If E is norm dense, then an ideal J in E is maximal if and only if it is of the form M_x^E for a unique point $x \in K$.*

(iii) *If E is norm dense, then for every prime ideal P in E there exists a unique point $x \in K$ such that P is contained in M_x^E.*

Proof (i) Suppose that for each $x \in K$ there exists $f_x \in J$ such that $f_x(x) > 0$. By continuity, for each $f_x \in J$ one can find an open neighborhood U_x of x such that f_x is strictly positive on U_x. Since we have an open covering $\{U_x : x \in K\}$ of K, there exists a finite subcover $U_{x_1}, \ldots, U_{x_n}$. The function $f := f_{x_1} + \cdots + f_{x_n}$ is strictly positive on K, so it is a strong unit in E and therefore, we have $J = E$. This contradiction shows that there exists $x \in K$ such that $J \subseteq M_x^E$.

(ii) Let $x \in K$ and consider the ideal M_x^E in E. It follows that M_x^E is proper, as otherwise $f(x) = 0$ for all $f \in C(K)$ since E is norm dense in $C(K)$. Suppose that J is a proper ideal in E containing M_x^E. By (i) there is a point $y \in K$ such that $J \subseteq M_y^E$.

If there would exist a function $f \in M_y^E \setminus M_x^E$, then for $h \in E \setminus M_y^E$ we can define $g \in E$ by

$$g(z) := f(z) - \frac{f(x)}{h(x)} h(z).$$

But then $g(x) = 0$ and $g(y) \neq 0$, which contradicts the fact that $M_x^E \subseteq M_y^E$. Hence $M_x^E = M_y^E$, and so $J = M_x^E$ showing that M_x^E is a maximal ideal in E.

On the other hand, if J is a maximal ideal in E, then there is a point $x \in K$ such that $J \subseteq M_x^E$ by (i) and so $J = M_x^E$.

For the uniqueness of the point $x \in K$, suppose that $M_x^E = M_y^E$ for $x \neq y$. We claim that for any $f, g \in E$ it follows that $f(x)g(y) = f(y)g(x)$. Indeed, if $f(x) = 0$, then $f \in M_x^E = M_y^E$, so $f(y) = 0$ and $f(x)g(y) = f(y)g(x)$. If $f(x) \neq 0$, then $h \in E$ defined by

$$h(z) := g(z) - \frac{g(x)}{f(x)} f(z)$$

satisfies $h(x) = 0$, so $h \in M_x^E = M_y^E$ and $h(y) = 0$ implies that $f(x)g(y) = f(y)g(x)$, proving the claim. Since E is norm dense in $C(K)$, it follows that $f(x)g(y) = f(y)g(x)$ for all $f, g \in C(K)$, but this contradicts Urysohn's lemma. We conclude that $x = y$ and the maximal ideals in E are therefore uniquely determined by the points in K.

(iii) Let P be any prime ideal in E. Then by (i) there is a point $x \in K$ such that $P \subseteq M_x^E$. Suppose that there is another point $y \in K$ such that $P \subseteq M_y^E$. Since P is a prime ideal, it follows that we must have $M_x^E \subseteq M_y^E$ or $M_y^E \subseteq M_x^E$ and so $M_x^E = M_y^E$. Hence, $x = y$ by (ii). □

Although Lemma 4.1 is stated for sublattices of $C(K)$-spaces, it is useful in a more general setting when studying prime ideals in Archimedean vector lattices. Indeed, if E is an Archimedean vector lattice and $x \in E$ is a non-zero positive vector, then by the Kakutani representation theorem, the normed lattice $(I_x, \|\cdot\|_x)$ is lattice isometric to a dense sublattice of $C(K)$ for some compact Hausdorff space K. Since any prime ideal P in E yields a prime ideal $P \cap I_x$ in I_x, this allows us to study the properties of the principal ideals in E and the topological properties of K given the restrictions on the set of prime ideals in question.

Lemma 4.2 *Let K be a compact Hausdorff space. If there is a norm dense sublattice E of $C(K)$ that contains only finitely many minimal prime ideals, then K is finite and $E = C(K)$.*

Proof Suppose K contains a sequence of distinct points $(x_n)_{n \in \mathbb{N}}$ all of which correspond to maximal ideals $M_{x_n}^E$ by Lemma 4.1. Since every maximal ideal contains a minimal prime ideal by [10,Theorem 33.7] and distinct maximal ideals do not contain the same minimal prime ideal by [10,Theorem 33.3], it follows that E must contain infinitely many minimal prime ideals, contradicting the assumption. Hence K is finite and so $E = C(K)$ as it is norm dense. □

The main result of this section is proved by using Lemma 4.1 since Archimedean vector lattices are saturated with copies of dense sublattices of $C(K)$-spaces.

Theorem 4.3 *If an Archimedean vector lattice E contains finitely many minimal prime ideals, then it is finite-dimensional.*

Proof Pick any positive vector $x \in E$ and consider the principal ideal I_x. We claim that I_x contains only finitely many minimal prime ideals. Suppose that I_x contains infinitely many minimal prime ideals $\{P_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ there exists a minimal prime ideal Q_n in E such that $P_n = Q_n \cap I_x$ by [10,Theorem 52.3]. Since $P_n \neq P_m$ for $n \neq m$, the family $\{Q_n : n \in \mathbb{N}\}$ of minimal prime ideals of E is infinite. Thus, I_x contains only finitely many minimal prime ideals.

The Banach lattice completion of the normed space $(I_x, \|\cdot\|_x)$ is lattice isometric to $C(K)$ for some compact Hausdorff space K by the Kakutani representation theorem. Hence, $C(K)$ contains a norm dense sublattice with only finitely many minimal prime ideals. By Lemma 4.2 we conclude that K is finite, so that I_x is finite-dimensional. Hence, by [10,Theorem 61.4] we have that E is lattice isomorphic to the space $c_{00}(\Omega)$ for some set Ω. By Proposition 3.1 we have that Ω is finite, making E finite-dimensional. □

5 Cohen's and Kaplansky's theorem in vector lattices

In this section we prove vector lattice analogues of two well-known results in the theory of commutative rings that involve prime ideals. Cohen's theorem (see [5,Theorem 2]) states that a commutative ring is Noetherian if and only if every prime ideal is finitely generated, and Kaplansky's theorem (see [9,Theorem 12.3]) states that in a Noetherian commutative ring every ideal is principal if and only if every maximal ideal is principal. Combining Cohen's theorem and Kaplansky's theorem yields that a commutative ring is a principal ideal ring if and only if every prime ideal is principal, which is referred to as the Cohen–Kaplansky theorem. In vector lattices an ideal is finitely generated if and only if it is principal, so Cohen's theorem becomes a statement about principal prime ideals in vector lattices. We prove the vector lattice analogue of the Cohen–Kaplansky theorem stating that every prime ideal in an Archimedean vector lattice is principal if and only if it is finite-dimensional, which is further equivalent with all ideals being principal. Furthermore, we prove that in a uniformly complete Archimedean vector lattice there are no non-maximal principal prime ideals, and we use this result to prove a vector lattice analogue of the Cohen–Kaplansky theorem for uniformly complete Archimedean vector lattices. That is, every ideal is principal if and only if every maximal ideal is principal, which in turn is equivalent with the vector lattice being finite-dimensional.

We say that a vector lattice E is *Noetherian* if every ascending chain of ideals in E is stationary. This notion is completely analogous to that of Noetherian rings or modules. Since Noetherian vector spaces are precisely the finite-dimensional ones, the following result should not be too surprising.

Proposition 5.1 *For an Archimedean vector lattice E the following statements are equivalent.*

(i) E is finite-dimensional.
(ii) E is Noetherian.

Proof Since every finite-dimensional vector lattice can only have finitely many ideals, it is clear that (i) implies (ii). Assume that E is infinite-dimensional. By [10,Theorem 26.10] E contains an infinite sequence $(e_n)_{n\in\mathbb{N}}$ of pairwise disjoint positive vectors. If for each $n \in \mathbb{N}$ we define the ideal J_n generated by the set $\{e_1, \ldots, e_n\}$, then the ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of ideals in E is not stationary.   □

In view of Proposition 5.1 the vector lattice analogue of Cohen's theorem characterizes the finite-dimensional vector lattices among those for which all prime ideals are principal. Furthermore, a well-known alternative definition of a commutative ring being Noetherian is that every ideal is finitely generated, which leads to the following theorem. This result can be thought of as a combination of Cohen's theorem and the Cohen–Kaplansky theorem for vector lattices.

Theorem 5.2 (Cohen–Kaplansky theorem for vector lattices) *Let E be a vector lattice, and consider the following statements.*

(i) E is finite-dimensional.
(ii) Every proper ideal in E is principal.
(iii) Every prime ideal in E is principal.

Then (i) implies (ii), (i) implies (iii), and (iii) implies (ii). Moreover, in the case where E is Archimedean, we have that (ii) implies (i), so all statements are equivalent.

Proof Since every finite-dimensional vector lattice has a strong unit, the fact that (i) implies (ii) and (iii) is clear. We proceed to prove that (iii) implies (ii). To this end, suppose there is a proper ideal I in E which is not principal. Let $z \in E \setminus I$ be positive and define

$$\mathscr{I} := \{J \subseteq E \colon J \text{ is a non-principal ideal with } z \notin J\}.$$

Then $\mathscr{I}$ is non-empty an can be partially ordered by set inclusion. For any chain $(J_i)_i$ in $\mathscr{I}$ it follows that its union $J_0 := \bigcup_i J_i$ is an ideal which is not principal. Indeed, if $J_0 = I_x$ for some positive vector $x \in E$, then $x \in J_i$ for some i and so $J_0 \subseteq I_x \subseteq J_i \subseteq J_0$ hence $J_i = I_x$, which is impossible by definition of $\mathscr{I}$. Furthermore, J_0 does not contain z making it a proper ideal in E. By Zorn's lemma $\mathscr{I}$ has a maximal element, say P.

We claim that P is a prime ideal in E. Suppose $x, y \in E$ are so that $x \wedge y = 0$ and neither x nor y are in P. Then the prime ideals $P + I_x$ and $P + I_y$ are principal, so there are positive vectors $v, w \in E$ such that $P + I_x = I_v$ and $P + I_y = I_w$. It follows that $(P + I_x) \cap (P + I_y) = I_v \cap I_w = I_{v \wedge w}$ and by [11,Proposition II.2.3] we find

$$(P + I_x) \cap (P + I_y) = P + P \cap I_x + P \cap I_y + I_x \cap I_y = P + I_{x \wedge y} = P.$$

Hence, $P = I_{v \wedge w}$, but that is impossible as $P \in \mathscr{I}$. Thus, P must be a prime ideal, but then by assumption it must be principal contradicting $P \in \mathscr{I}$ once more. We conclude that every proper ideal in E must be principal.

Let E be Archimedean such that every proper ideal in E is principal. To prove that (ii) implies (i) by contradiction, suppose that E is not finite-dimensional. Then by [10,Theorem 26.10] there exists an infinite sequence $(x_n)_{n\in\mathbb{N}}$ of non-zero positive pairwise disjoint vectors in E. Let I be the ideal generated by $\{x_{n+1}: n \in \mathbb{N}\}$ in E. By construction I is proper, so it is principal and there exists $x \in E$ such that $I = I_x$. Hence there are $x_{k_1}, \ldots, x_{k_m}$ and positive scalars $\lambda_1, \ldots, \lambda_m$ such that

$$0 < x \leq \lambda_1 x_{k_1} + \lambda_2 x_{k_2} + \cdots + \lambda_m x_{k_m}.$$

Thus, there is an $n > 1$ such that $x \perp x_n$. Since $x_n \in I_x$, we conclude that $x_n = 0$ which is impossible. This contradiction shows that E is finite-dimensional. $\square$

By Theorem 5.2 it is clear that every infinite-dimensional Archimedean vector lattice contains a non-principal prime ideal. The following example shows that there even exists an atomic infinite-dimensional Archimedean vector lattice whose maximal ideals are all principal, yet none of the remaining prime ideals are principal.

Example 5.3 Let $x := (x_n)_{n\in\mathbb{N}}$ be an element of c_0 and suppose $x_n > 0$ for all $n \geq 1$. By adjoining the constant sequence $\mathbf{1}$ to the ideal I_x generated by x in c we obtain the atomic vector lattice $E := I_x + \mathbb{R}\mathbf{1}$ that is not uniformly complete, and its uniform completion equals c. Since c is lattice isometric to $C(\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ denotes the one-point compactification of $\mathbb{N}$, the maximal ideals in E are of the form $M_n^E = M_n \cap E$ for a maximal ideal M_n in $C(\mathbb{N}\cup\{\infty\})$ by Lemma 4.1. For $n \in \mathbb{N}$ the maximal ideal M_n^E is principal with generator $\mathbf{1} - e_n \in E$ and for the adjoined point it follows that $M_\infty = c_0$ and so $M_\infty^E = I_x$ which is principal by construction.

We claim that no non-maximal prime ideal P can be contained in M_n^E for some $n \in \mathbb{N}$. Indeed, if there is an $n \in \mathbb{N}$ such that $P \subsetneq M_n^E$, then it follows from the fact that $e_n \notin M_n^E$ and $e_n \wedge (\mathbf{1} - e_n) = 0$ that $\mathbf{1} - e_n \in P$, but this would imply that $P = M_n^E$. Thus, there are no proper non-maximal prime ideals contained in M_n^E for all $n \in \mathbb{N}$.

Suppose now that there exists a principal prime ideal P which is not maximal. Since P is not a subset of M_n^E for $n \in \mathbb{N}$, we have $P \subseteq I_x$. If $y \in I_x$ is positive such that $P = I_y$, then $y_n > 0$ for all $n \geq 1$. Moreover, for all $k \in \mathbb{N}$ there are x_{n_k} and y_{n_k} such that $x_{n_k} > k y_{n_k}$ since I_y is properly contained in I_x. Define $v := (v_m)_{m\in\mathbb{N}}$ and $w := (w_m)_{m\in\mathbb{N}}$ as follows. If $m = n_{2k}$ for some $k \in \mathbb{N}$, then put $v_m = x_{n_{2k}}$ otherwise put $v_m = 0$. Similarly, if $m = n_{2k-1}$ for some $k \in \mathbb{N}$, then put $w_m = x_{n_{2k-1}}$ otherwise put $w_m = 0$. Then $v, w \in I_x$ and $v \wedge w = 0$, but there is no multiple of y that dominates v or w. We conclude that P is not principal and E is an infinite-dimensional atomic Archimedean vector lattice that is not uniformly complete in which all maximal ideals are principal and all non-maximal prime ideals are not principal. Note again that by Theorem 5.2 such non-maximal prime ideals must exist in E.

Hence, by Example 5.3 it is not true in general that an Archimedean vector lattice in which all maximal ideals are principal must be finite-dimensional. However, when passing to uniformly complete Archimedean vector lattices, it follows that the Cohen–Kaplansky theorem reduces to the vector lattice analogue involving only principal maximal ideals.

Theorem 5.4 *Let E be an Archimedean uniformly complete vector lattice.*

(i) *If E has a strong unit, then every principal prime ideal is maximal and equals the disjoint complement of an atom.*

(ii) *If E contains a principal prime ideal, then E has a strong unit.*

Proof (*i*) Suppose that E has a strong unit e. Since E is uniformly complete it is lattice isometric to $C(K)$ for some compact Hausdorff space K by the Kakutani representation theorem. Hence, we may assume without loss of generality that $E = C(K)$. Let $P = I_f$ be a principal prime ideal in E for some non-negative function $f \in E$. By [10,Theorem 34.1] there exists a unique point x_0 such that each function in P vanishes at x_0. Since P is principal, the point x_0 is the unique zero of f.

We will show that x_0 must be an isolated point in K. Indeed, suppose that x_0 is not isolated in K. Then we can inductively construct a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $(f(x_n))_{n\in\mathbb{N}}$ is a strictly decreasing sequence with $f(x_n) \to 0$. For $n \geq 1$ define the open neighbourhoods of x_0 by

$$U_n := \{x \in K : f(x) < f(x_n)\}.$$

As $x_{n+1} \in U_n \setminus U_{n+1}$, it follows that $U_{n+1} \subsetneq U_n$ for all $k \geq 1$, and since $\overline{U}_n \subseteq \{x \in K : f(x) \leq f(x_n)\}$, it follows that $\overline{U}_{n+1} \subseteq U_n$ for all $n \geq 1$. Furthermore, the closure $\overline{U}_{n+2}$ is properly contained in U_n for all $n \geq 1$. Hence, the open neighborhoods $V_n := U_{2n-1}$ of x_0 satisfy $\overline{V}_{n+1} \subsetneq V_n$ for $n \geq 1$ and we define the non-empty closed sets $F_n := \overline{V}_{2n-1} \setminus V_{2n}$ for $n \geq 1$. Note that for $n < m$ we have

$$F_n \cap F_m = \overline{V}_{2n-1} \cap \overline{V}_{2m-1} \cap V_{2n}^c \cap V_{2m}^c = \overline{V}_{2m-1} \cap V_{2n}^c \subseteq V_{2n} \cap V_{2n}^c = \emptyset,$$

so the closed sets F_n are pairwise disjoint. Define for $n \geq 2$ the continuous function f_n on the closed set $F_n \cup \overline{V}_{2n+1} \cup V_{2n-2}^c$ by

$$f_n(x) := \begin{cases} \sqrt{f(x)} & \text{if } x \in F_n \\ 0 & \text{if } x \in \overline{V}_{2n+1} \cup V_{2n-2}^c, \end{cases}$$

which by the Tietze extension theorem can be extended to a positive continuous function on K that will also be denoted by f_n such that $f_n \leq \sqrt{f}$. The functions f_{2n} and f_{2m} are disjoint for all $m \neq n$ and we define the sequences of partial sums $g_n := \sum_{k=1}^{n} f_{4k}$ and $h_n := \sum_{k=1}^{n} f_{4k-2}$. Note that for n and m the functions g_n and h_m are disjoint and that for $m < n$ we have $(g_n - g_m)(x) \leq \sqrt{f(x)}$ on V_{8m+6} and $(g_n - g_m)(x) = 0$ on V_{8m+6}^c, so that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(K)$ for $\|\cdot\|_\infty$. Similarly, we find that $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(K)$ for $\|\cdot\|_\infty$. Let g and h be the uniform limits of $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$, respectively, in $C(K)$. Since $g_n \wedge h_n = 0$ for all $n \geq 1$, it follows that $g \wedge h = 0$ and so either $g \in P$ or $h \in P$ as P is a prime ideal. Suppose that $g \in P$. Then there is a $\lambda > 0$ such that $f_{4n} \leq g_n \leq g \leq \lambda f$ for all $n \geq 1$. Hence, for all $x \in F_{4n}$ we have that $\sqrt{f(x)} \leq \lambda f(x)$ and since $x_{16n-1} \in F_{4n}$, it follows that $1 \leq \lambda\sqrt{f(x_{16n-1})}$ for all $n \geq 1$ which contradicts the fact that $f(x_n) \to 0$. Hence, $g \notin P$ and similarly we have $h \notin P$, so we conclude that x_0 must be an isolated point.

The set $K \setminus \{x_0\}$ is therefore closed in K and since f is strictly positive on $K \setminus \{x_0\}$, it attains a strictly positive minimum on $K \setminus \{x_0\}$. Thus, for each function $g \in M_{x_0}$ there exists a $\lambda > 0$ such that $g \leq \lambda f$ which implies that $P = I_f = M_{x_0}$ is a maximal ideal. Furthermore, the characteristic function χ_{x_0} is an atom in E and $M_{x_0} = \{\chi_{x_0}\}^d$.

(ii) Let $P = I_x$ be a prime ideal in E, and let $y \in E \setminus P$ be positive. Then $Q := P + I_y = I_{x+y}$ is an ideal in E and suppose that $Q \subsetneq E$. Then for a positive vector $z \in E \setminus Q$ it follows from the fact that ideals in uniformly complete vector lattices are uniformly complete that $E' := I_{x+y+z}$ is a uniformly complete vector lattice with a strong unit that contains a principal prime ideal that is not maximal. This contradicts (i), so $Q = E$ and we conclude that E has $x + y$ as a strong unit. $\qquad \square$

It will be shown in Sect. 7 that there are vector lattices containing non-maximal principal prime ideals. In view of Theorem 5.4, these vector lattices cannot be uniformly complete.

Corollary 5.5 *Let K be a compact Hausdorff space. For $x \in K$ the maximal ideal M_x in $C(K)$ is a principal ideal if and only if x is an isolated point of K.*

Proof It was shown in the proof of Theorem 5.4 that x must be an isolated point of K whenever M_x is principal. On the other hand, if x is an isolated point of K, then the characteristic function χ_x is continuous on K and it follows that $M_x = I_f$ for $f := \mathbf{1} - \chi_x$. $\qquad \square$

The vector lattice analogue of the Cohen–Kaplansky theorem for uniformly complete Archimedean vector lattices is the following.

Theorem 5.6 (Cohen–Kaplansky theorem for uniformly complete vector lattices) *The following statements are equivalent for a uniformly complete Archimedean vector lattice E.*

(i) E is finite-dimensional.
(ii) Every proper ideal in E is principal.
(iii) E contains maximal ideals, and every maximal ideal in E is principal.

Proof Statements (i) and (ii) are equivalent by Theorem 5.2, and the fact that (i) implies (iii) is clear. Suppose E contains maximal ideals and all of them are principal. By Lemma 2.1 the vector lattice E has a strong unit, so it is lattice isometric to a $C(K)$-space. It follows from Corollary 5.5 that every point in K is isolated. We conclude that K must be finite in order to remain compact, so E must be finite-dimensional. $\square$

6 Prime Noetherian vector lattices

As was proved in Proposition 5.1, all Noetherian Archimedean vector lattices are finite-dimensional. When considering chains of ideals in vector lattices, the prime ideals are also a natural class of ideals to study. For example, all ideals that contain a fixed prime ideal are prime ideals and this set is linearly ordered by set inclusion. The vector lattices for which the ascending chains of prime ideals are finite will be studied in this

section, and we propose the following definition. A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals $P_1 \subseteq P_2 \subseteq \ldots$ in E is stationary. Uniformly complete Archimedean prime Noetherian vector lattices are completely characterized in Proposition 6.8 and Theorem 6.9, and these characterizations depend on the existence of a strong unit. More specifically, if E has a strong unit, then E is finite-dimensional and in general E must be lattice isomorphic to $c_{00}(\Omega)$ for some set Ω. We start this section by studying how the prime Noetherian property transfers between sublattices, ideals, and the whole vector lattice.

Proposition 6.1 *Let E be a vector lattice and let F be a vector sublattice. If E is prime Noetherian, then F is prime Noetherian.*

Proof Suppose that $Q_1 \subseteq Q_2 \subseteq \ldots$ is an ascending chain of distinct proper prime ideals in F. We can now create an ascending chain of distinct prime ideals $P_1 \subseteq P_2 \subseteq \ldots$ in E such that $Q_n = P_n \cap F$ for all $n \in \mathbb{N}$ by induction. Indeed, for $Q_1 \subseteq Q_2$ there are distinct prime ideals $P_1 \subseteq P_2$ in E such that $Q_1 = P_1 \cap F$ and $Q_2 = P_2 \cap F$ by [10,Theorem 52.4]. Suppose that for $Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_k$ there are distinct prime ideals $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_k$ in E such that $Q_i = P_i \cap F$ for $i = 1, \ldots, k$. Consider the canonical Riesz homomorphism $\pi \colon E \to E/P_k$ and note that $\pi(F)$ is a sublattice of $\pi(E)$. The zero ideal is prime in both quotient vector lattices, so that $\pi(Q_{k+1})$ is a prime ideal in $\pi(F)$ by [10,Theorem 33.3(iii)]. Furthermore, by [10,Theorem 52.2] there is a prime ideal P' in $\pi(E)$ such that $\pi(Q_{k+1}) = P' \cap \pi(F)$, and define $P_{k+1} := \pi^{-1}(P')$. It is readily verified that P_{k+1} is a prime ideal in E that contains P_k. Furthermore, if $x \in P_{k+1} \cap F$ then there is a $p \in P_k$ such that $x - p \in Q_{k+1}$, hence $p \in P_k \cap F = Q_k$ so that $x \in Q_{k+1}$. Since for $x \in Q_{k+1}$ it follows that $\pi(x) \in P'$ we also have $x \in P_{k+1}$, so $Q_{k+1} = P_{k+1} \cap F$. Note that since Q_k and Q_{k+1} are distinct, we must have that P_k and P_{k+1} are distinct, which concludes the induction argument. Since we assumed E to be prime Noetherian, the chain $Q_1 \subseteq Q_2 \subseteq \ldots$ must be stationary. Hence, F is prime Noetherian. □

If on the other hand we have a vector lattice E with a prime Noetherian sublattice F, then it is not true in general that E is prime Noetherian even if F is an order dense ideal. See the paragraph preceding Corollary 6.12. In the case where F is a projection band we can prove the following.

Proposition 6.2 *Let B be a projection band in a vector lattice E. Then E is prime Noetherian if and only if B and B^d are prime Noetherian.*

Proof If E is prime Noetherian, then B and B^d are prime Noetherian by Proposition 6.1. Assume now that B and B^d are prime Noetherian and let $P_1 \subseteq P_2 \subseteq \ldots$ be a chain of prime ideals in E. Then

$$P_1 \cap B \subseteq P_2 \cap B \subseteq \ldots \quad \text{and} \quad P_1 \cap B^d \subseteq P_2 \cap B^d \subseteq \ldots$$

are increasing chains of prime ideals in B and B^d, respectively. Hence, there exists $n \in \mathbb{N}$ such that for all $m \geq n$ we have

$$P_m \cap B = P_n \cap B \quad \text{and} \quad P_m \cap B^d = P_n \cap B^d.$$

Since the lattice of ideals in a vector lattice is distributive, for each $m \geq n$ we obtain

$$P_m = (P_m \cap B) \oplus (P_m \cap B^d) = (P_n \cap B) \oplus (P_n \cap B^d) = P_n.$$

$\square$

In a prime Noetherian vector lattice there are large prime ideals in the sense that they have finite co-dimension and every prime ideal is always contained in a large prime ideal.

Proposition 6.3 *In a prime Noetherian vector lattice every prime ideal is contained in a prime ideal of finite co-dimension.*

Proof Pick a prime ideal P in a prime Noetherian vector lattice E. Suppose that for each ideal Q containing P the vector lattice E/Q is infinite-dimensional. Let $P_1 := P$. Let $P \subsetneq P_2$ be any proper ideal in E. By assumption E/P_2 is infinite-dimensional. Pick any proper ideal $P_2 \subsetneq P_3$ in E. By assumption the vector lattice E/P_3 is infinite-dimensional. Inductively we can construct an ascending chain $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \ldots$ of distinct prime ideals which contradicts the assumption that E is prime Noetherian.
$\square$

It is not true in general that vector lattices contain maximal ideals, however, prime Noetherian vector lattices always do.

Proposition 6.4 *Let E be an at least two-dimensional prime Noetherian vector lattice. Then every proper ideal of E is contained in a maximal ideal.*

Proof Let I be a proper ideal in E. By [10,Theorem 33.5] the ideal I is contained in a non-trivial prime ideal P. By Proposition 6.3 the ideal P is contained in a prime ideal Q such that the dimension of E/Q is finite. Let

$$Q = Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_{n-1} \subseteq Q_n = E$$

be the maximal chain $\mathcal{C}$ of ideals between Q and E. Clearly, Q_{n-1} is then a maximal ideal in E which contains I. $\square$

In the proposition below we show that maximal lattice ideals in $C_0(X)$, where X is a locally compact Hausdorff space, also consist of functions that are zero at a fixed point in X.

Proposition 6.5 *Let X be a locally compact Hausdorff space and let M be a maximal ideal in $C_0(X)$. Then there exists an $x \in X$ such that $M = \{f \in C_0(X) : f(x) = 0\}$.*

Proof Let $C_0(X)_c$ be the complexification of $C_0(X)$ and let M_c be the complexification of M in $C_0(X)_c$. By [14,Theorem 13.8] it follows that M_c is an ideal in the vector lattice $C_0(X)_c$ and we claim that it is maximal. If this is not the case, then there exists a proper order ideal N in $C_0(X)_c$ such that $M_c \subsetneq N$. From this it follows that the real parts satisfy $M \subseteq N_r$ and since M is maximal, we have that either $N_r = M$ or

$N_r = C_0(X)$ as N_r is an ideal in $C_0(X)$ by [14,Theorem 13.8]. The former yields that $M = N$ and the latter yields that $N = C_0(X)_c$ which is impossible. Hence, M_c is maximal in $C_0(X)_c$ and has co-dimension one as M has co-dimension one in $C_0(X)$. Since M_c is an order ideal, it is also a maximal algebra ideal. By [4,Theorem 2.4(i)] we conclude that either $C_0(X)_c^2 \subseteq M_c$ or M_c is closed. We claim that the former is impossible. To this end, assume that $C_0(X)_c^2 \subseteq M_c$ and pick any non-negative function $f \in C_0(X)_c$. Then $f = (\sqrt{f})^2 \in M_c$, hence, M_c contains all non-negative functions of $C_0(X)_c$. Since M_c is an order ideal, we conclude that $M_c = C_0(X)_c$ which is impossible. By [8,Theorem 1.4.6] there exists a unique point $x \in X$ such that $M_c = \{f \in C_0(X)_c : f(x) = 0\}$. To finish the proof note that $M = (M_c)_r$.   □

The following proposition will be useful for proving the main results of this section. It characterizes the maximal ideals in a uniformly complete vector lattice with a strong unit among the prime ideals with finite co-dimension.

Proposition 6.6 *Let E be an Archimedean uniformly complete vector lattice with a strong unit. Then a prime ideal P is a maximal ideal if and only if $\dim E/P < \infty$.*

Proof If $\dim E = 1$, there is nothing to prove. So we may assume that the dimension of E is at least two. Furthermore, since E is uniformly complete with a strong unit, by the Kakutani representation theorem we may assume that $E = C(K)$ for some compact Hausdorff space K where K contains at least two points.

Pick a prime ideal P in $C(K)$ such that $\dim C(K)/P < \infty$. Then there exists an $x \in K$ such that P is contained in the maximal ideal M_x. Let

$$P = P_1 \subseteq P_2 \subseteq \ldots \subseteq P_{n-1} \subseteq P_n = M_x$$

be the maximal chain $\mathcal{C}$ of ideals, that are necessarily prime, in E between P and M_x. Since $\mathcal{C}$ is maximal, the ideal P_{n-1} is a maximal order ideal in P_n.

We claim that P_{n-1} is a maximal algebra ideal in P_n. To see that P_{n-1} is also an algebra ideal in M_x, pick $f \in P_{n-1}$ and $g \in M_x$ and note that the inequality $|fg| \leq \|g\|_\infty |f|$ together with the fact that P_{n-1} is an order ideal in M_x yields that $fg \in P_{n-1}$. Since P_{n-1} is a maximal order ideal in P_n, the co-dimension of P_{n-1} in P_n is one, so that P_{n-1} is also maximal as an algebra ideal in P_n.

We claim that there exists $y \in K \setminus \{x\}$ such that

$$P_{n-1} = \{f \in C(K) : f(x) = f(y) = 0\}.$$

To see this, consider the locally compact Hausdorff space $K \setminus \{x\}$. It is a standard fact from general topology that the one-point compactification of $K \setminus \{x\}$ is homeomorphic to K and that the mapping $\Phi : C_0(K \setminus \{x\}) \to M_x \subseteq C(K)$ defined by

$$\Phi(f)(t) := \begin{cases} f(t) & \text{if } t \neq x, \\ 0 & \text{if } t = x, \end{cases}$$

is an isometric lattice and algebra isomorphism. Since P_{n-1} is a maximal algebra ideal in M_x, it follows that $\Phi^{-1}(P_{n-1})$ is a maximal algebra ideal in $C_0(K \setminus \{x\})$. By

Proposition 6.5 there exists $y \in K \setminus \{x\}$ such that

$$\Phi^{-1}(P_{n-1}) = \{f \in C_0(K \setminus \{x\}) \colon f(y) = 0\},$$

so that

$$\begin{aligned} P_{n-1} &= \Phi(\Phi^{-1}(P_{n-1})) = \{\Phi(f) \colon f \in \Phi^{-1}(P_{n-1})\} \\ &= \{f \in C(K) \colon f(x) = f(y) = 0\}. \end{aligned}$$

We claim that P_{n-1} is not a prime ideal. To this end, note that Urysohn's lemma yields functions f and g in $C(K)$ such that $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$. Then $f \wedge g$ belongs to the ideal P_{n-1}, yet neither f nor g belongs to P_{n-1}. This contradiction shows that $n = 1$ so that $P = M_x$ is a maximal ideal. Since maximal ideals have co-dimension one, this concludes the proof.   □

Corollary 6.7 *A uniformly complete Archimedean vector lattice E with a strong unit is prime Noetherian if and only if every prime ideal in E is maximal.*

Proof If every prime ideal in E is maximal, then E is necessarily prime Noetherian. Suppose now that E is a prime Noetherian vector lattice and pick a prime ideal P in E. We will prove that E/P is finite-dimensional since Proposition 6.6 will yield then that P is a maximal ideal in E.

By way of contradiction, assume that E/P is infinite-dimensional and let $P_1 := P$. Pick any non-maximal ideal Q in E that properly contains P_1. By Proposition 6.6 we have that E/Q is infinite-dimensional. Now let $P_2 := Q$. Inductively we can construct an ascending chain $P_1 \subsetneq P_2 \subsetneq \cdots$ of prime ideals in E such that for each $n \in \mathbb{N}$ the dimension of E/P_n is infinite. However, this contradicts the fact that E is prime Noetherian.   □

The following proposition proves that the finite-dimensional vector lattices are precisely the uniformly complete Archimedean prime Noetherian vector lattices with a strong unit.

Proposition 6.8 *Let E be a uniformly complete Archimedean vector lattice with a strong unit. Then E is prime Noetherian if and only if E is finite-dimensional.*

Proof By Corollary 6.7 every prime ideal in E is a maximal ideal, so that by [10,Theorem 37.6] the quotient vector lattice E/J is Archimedean for every ideal J in E. Since E is uniformly complete, [10,Theorem 61.4] yields that E is lattice isomorphic to the vector lattice $c_{00}(\Omega)$ for some set Ω. Since E has a strong unit, Ω needs to be finite. Hence, the vector lattice E is finite-dimensional.

The converse follows from the fact that finite-dimensional vector lattices have only finitely many ideals.   □

In general, the uniformly complete prime Noetherian vector lattices are characterized as $c_{00}(\Omega)$ for some set Ω.

Theorem 6.9 *A uniformly complete Archimedean vector lattice E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$ for some set Ω.*

Proof Suppose that E is a prime Noetherian vector lattice. Pick a positive vector $x \in E$ and consider the principal ideal I_x which is a uniformly complete vector lattice with a strong unit. Since I_x is prime Noetherian by Proposition 6.1, it follows that I_x is finite-dimensional by Proposition 6.8. Hence, E is lattice isomorphic to $c_{00}(\Omega)$ for some set Ω by [10,Theorem 61.4]. For the converse implication, note that $c_{00}(\Omega)$ is prime Noetherian by Theorem 3.1.   □

If E is equipped with a completely metrizable locally solid topology, then the prime Noetherian property implies that E is finite-dimensional.

Lemma 6.10 *If $c_{00}(\Omega)$ is lattice isomorphic to a completely metrizable locally solid vector lattice E, then Ω is finite.*

Proof Let $\Phi : c_{00}(\Omega) \to (E, \tau)$ be a lattice isomorphism where τ is a completely metrizable locally solid topology on E. Then Φ^{-1} induces a completely metrizable locally solid topology τ' on $c_{00}(\Omega)$. Hence, Φ is an isomorphism between completely metrizable locally solid vector lattices.

Suppose that Ω is infinite and let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence in Ω of distinct points. Denote by x_n the vector $\Phi(\chi_{\omega_n})$ in E. Since τ is metrizable, there exists a local basis $\{U_n : n \in \mathbb{N}\}$ of solid neighborhoods of zero in E with the property that $U_{n+1} + U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Since each set U_n is absorbing, there exists a $\lambda_n > 0$ such that $\lambda_n x_n \in U_n$. Denote the vector $\lambda_1 x_1 + \cdots + \lambda_n x_n$ by s_n.

We claim that the sequence $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (E, τ). Pick any neighborhood U of zero in (E, τ) and find $n_0 \in \mathbb{N}$ such that $U_{n_0} \subseteq U$. By [3,Exercise 2.1.14] for all $m' \geq m > n_0$ we have

$$s_{m'} - s_m = \lambda_{m+1} x_{m+1} + \cdots + \lambda_{m'} x_{m'} \in U_{m+1} + \cdots + U_{m'} \subseteq U_m \subseteq U_{n_0} \subseteq U$$

which proves the claim. Hence, the increasing sequence $(s_n)_{n \in \mathbb{N}}$ converges to some positive vector $s \in E$. Since Φ is an isomorphism, the increasing sequence $(t_n)_{n \in \mathbb{N}}$ where $t_n = \lambda_1 \chi_{\omega_1} + \cdots + \lambda_n \chi_{\omega_n}$ is convergent to the vector $\Phi^{-1}(s)$. However, the inequality $\Phi^{-1}(s) \geq \lambda_n \chi_{\omega_n}$ which holds for each $n \in \mathbb{N}$ yields that $\Phi^{-1}(s)$ has infinite support. This clearly contradicts the definition of the space $c_{00}(\Omega)$. Therefore, Ω is finite and the proof is completed.   □

Proposition 6.11 *A completely metrizable locally solid vector lattice is prime Noetherian if and only if it is finite-dimensional. In particular, a Banach lattice is prime Noetherian if and only if it is finite-dimensional.*

Proof Let E be a completely metrizable locally solid vector lattice with the prime Noetherian property. Since E is uniformly complete, it is lattice isomorphic to $c_{00}(\Omega)$ by Theorem 6.9. By Lemma 6.10 we conclude that E is finite-dimensional.   □

Corollary 6.11 we can construct an example of a vector lattice E that contains a prime Noetherian sublattice as an order dense ideal, however E itself is not prime Noetherian. Indeed, the vector lattice c_0 contains c_{00} as an order dense ideal.

Corollary 6.12 *The following assertions are equivalent for a uniformly complete prime Noetherian Archimedean vector lattice E.*

 (i) E has a strong unit.
 (ii) E is lattice isomorphic to a Banach lattice.
(iii) E is finite-dimensional.

The following example shows that there exists an infinite-dimensional prime Noetherian vector lattice with a strong unit which is not uniformly complete.

Example 6.13 The vector lattice $E := c_{00} + \mathbb{R}\mathbf{1}$ of all eventually constant sequences is not uniformly complete, and every quotient space of E is Archimedean by [10,Exercise 61.5] and therefore, every proper prime ideal in E is a maximal ideal by [10,Theorem 37.6]. Hence, E is an infinite-dimensional atomic prime Noetherian vector lattice with a strong unit.

7 Vector lattices of piecewise polynomials

In this section the prime ideals in vector lattices of piecewise polynomials are studied. It turns out that this class of non-uniformly complete vector lattices is a source of insightful examples when studying principal prime ideals and prime Noetherian properties of vector lattices.

For $n \in \mathbb{N}$ let $\mathrm{PPol}^n([a, b])$ be the vector lattice of piecewise polynomials of degree at most n that are continuous on the interval $[a, b]$, and we shall denote the space of piecewise polynomials that are continuous on the interval $[a, b]$ without any bound on the degree by $\mathrm{PPol}([a, b])$. By the lattice version of the Stone-Weierstrass theorem all these spaces are uniformly dense in $C([a, b])$. For the system of (not necessarily open) neighborhoods $N(t_0)$ of $t_0 \in [a, b]$, we define the ideal

$$I_{N(t_0)} := \left\{ f \in E \colon f^{-1}(\{0\}) \in N(t_0) \right\}.$$

Note that for $t_0 \in (a, b)$ the ideal $I_{N(t_0)}$ is not prime. To see this, consider the functions $f(t) := (t - t_0)_+$ and $g(t) := (t_0 - t)_+$. Then $f \wedge g = 0$, but neither f nor g are in $I_{N(t_0)}$. Furthermore, if we write E for either $\mathrm{PPol}^n([a, b])$ or $\mathrm{PPol}([a, b])$, then it follows from Lemma 4.1 that all maximal ideals in E are of the form $M_{t_0}^E$ for some $t_0 \in [a, b]$. For $t_0 \in (a, b]$ we define

$$L_{t_0} := \left\{ f \in M_{t_0}^E \colon \text{there exists a } \delta > 0 \text{ such that } f(t) = 0 \text{ for } t \in (t_0 - \delta, t_0] \right\}$$

and for $t_0 \in [a, b)$ we define

$$R_{t_0} := \left\{ f \in M_{t_0}^E \colon \text{there exists a } \delta > 0 \text{ such that } f(t) = 0 \text{ for } t \in [t_0, t_0 + \delta) \right\}.$$

In fact, these are exactly the minimal prime ideals in E.

Lemma 7.1 *Let E be either $\mathrm{PPol}^n([a, b])$ or $\mathrm{PPol}([a, b])$. The minimal prime ideals in E are precisely L_{t_0} for $t_0 \in (a, b]$ and R_{t_0} for $t_0 \in [a, b)$.*

Proof It is straightforward to check that L_{t_0} and R_{t_0} are ideals. Suppose $f, g \in E$ are such that $f \wedge g = 0$. Since f and g are piecewise polynomials and have only finitely many zeros when they are not constant, there must be a $\delta > 0$ such that $f(t) = 0$ on either $(t_0 - \delta, t_0]$ or on $[t_0, t_0 + \delta)$, or $g(t) = 0$ on either $(t_0 - \delta, t_0]$ or on $[t_0, t_0 + \delta)$. Hence f is in L_{t_0} or in R_{t_0}, or g is in L_{t_0} or R_{t_0}, and if f is not in L_{t_0}, then g is in L_{t_0}. Similarly, if g is not in L_{t_0}, then f is in L_{t_0}. Thus L_{t_0} is a prime ideal and it follows analogously that R_{t_0} is a prime ideal.

Next we will show that $I_{N(t_0)}$ is contained in every prime ideal in $M_{t_0}^E$. Indeed, if P is a prime ideal in $M_{t_0}^E$ and $f \in I_{N(t_0)}$ is positive, then there is an $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon)$ is in $\{t \in [a, b] : f(t) = 0\}$ and we can construct a piecewise linear continuous function g that is zero outside $(t_0 - \epsilon, t_0 + \epsilon)$ and $g(t_0) = 1$. Since $f \wedge g = 0$, it follows that $f \in P$.

We proceed to show that L_{t_0} and R_{t_0} are the minimal prime ideals in E. Suppose P is a minimal prime ideal in $M_{t_0}^E$, and there is a function $f \in L_{t_0}$ that is not in P and that there is a function $g \in R_{t_0}$ that is not in P. Then $f \wedge g$ is not in P, but $f \wedge g$ is in $I_{N(t_0)}$ contradicting the fact that $I_{N(t_0)} \subseteq P$. Hence, $L_{t_0} \subseteq P$ or $R_{t_0} \subseteq P$. Since P is a minimal prime ideal, it follows that $P = L_{t_0}$ or $P = R_{t_0}$. Moreover, if $t_0 \in (a, b]$, then the prime ideal L_{t_0} contains a minimal prime ideal P. So, there is an $s \in [a, b]$ such that $P = L_s \subseteq L_{t_0}$ or $P = R_s \subseteq L_{t_0}$. Then by Lemma 4.1 it follows that $s = t_0$. If R_{t_0} is contained in L_{t_0}, then we must have that $R_{t_0} = I_{N(t_0)}$, which is impossible. Hence, for all $t_0 \in (a, b]$ the prime ideals L_{t_0} are minimal. Similarly, for all $t_0 \in [a, b)$ the prime ideals R_{t_0} are minimal as well. □

For $t_0 \in (a, b]$ any $f \in \text{PPol}^n([a, b])$ has left derivatives at t_0, which we will denote by $f_-^{(j)}(t_0)$ for $0 \leq j \leq n$ where it is understood that $f_-^{(0)}(t_0) = f(t_0)$. The left j-th derivatives of f yield a map $\varphi_L \colon \text{PPol}^n([a, b]) \to \mathbb{R}^{n+1}$ defined by

$$\phi_L(f) := (f(t_0), -f_-'(t_0), f_-''(t_0), \ldots, (-1)^n f_-^{(n)}(t_0)). \qquad (7.1)$$

Similarly for $t_0 \in [a, b)$ and any $f \in \text{PPol}^n([a, b])$ we have right derivatives $f_+^{(j)}(t_0)$ for $0 \leq j \leq n$ which yield the map $\varphi_R \colon \text{PPol}^n([a, b]) \to \mathbb{R}^{n+1}$ defined by

$$\phi_R(f) := (f(t_0), -f_+'(t_0), f_+''(t_0), \ldots, (-1)^n f_+^{(n)}(t_0)), \qquad (7.2)$$

where we again put $f_+^{(0)}(t_0) = f(t_0)$. It follows that if we equip $\mathbb{R}^{n+1}$ with the lexicographical ordering, the maps φ_L and φ_R are Riesz homomorphisms.

Lemma 7.2 *Equip $\mathbb{R}^{n+1}$ with the lexicographical ordering. Then for $t_0 \in (a, b]$ the map φ_L as in (7.1) is a Riesz homomorphism with kernel L_{t_0} and for $t_0 \in [a, b)$ the map φ_R as in (7.2) is a Riesz homomorphism with kernel R_{t_0}.*

Proof If $f(t_0) > 0$, then $|f| = f$ on a left neighborhood of t_0, and $\phi_L(f) > \phi_L(-f)$, so $\phi_L(|f|) = |\phi_L(f)|$. On the other hand, if $f(t_0) < 0$, then $|f| = -f$ on a left neighborhood of t_0 and $\phi_L(f) < \phi_L(-f)$, so $\phi_L(|f|) = |\phi_L(f)|$. Suppose that

$$f(t_0) = f_-'(t_0) = \ldots = f_-^{(k)}(t_0) = 0$$

for some $0 \leq k < n = \deg(f)$. Then on a left neighborhood of t_0 we have that f is of the form

$$f(t) = a_{k+1}(t - t_0)^{k+1} + \ldots + a_{n-1}(t - t_0)^{n-1} + a_n(t - t_0)^n.$$

If $f_{-}^{(k+1)}(t_0) = (k+1)!a_{k+1} > 0$, then for k odd we have that

$$f(t) = |t - t_0|^{k+1}\left(a_{k+1} + a_{k+2}(t - t_0) + \ldots + a_n(t - t_0)^{n-k-1}\right), \qquad (7.3)$$

so we may choose a sufficiently small left neighborhood of t_0 such that $|f| = f$ on that left neighborhood and as $\phi_L(f) > \phi_L(-f)$, it follows that $\phi_L(|f|) = |\phi_L(f)|$. In case k is even, we have that

$$-f(t) = |t - t_0|^{k+1}\left(a_{k+1} + a_{k+2}(t - t_0) + \ldots + a_n(t - t_0)^{n-k-1}\right), \qquad (7.4)$$

so $|f| = -f$ on a sufficiently small left neighborhood of t_0. Hence $\phi_L(f) < \phi_L(-f)$ and $\phi_L(|f|) = |\phi_L(f)|$. Suppose now that $f_{-}^{(k+1)}(t_0) = (k+1)!a_{k+1} < 0$. If k is odd, then similarly, we find that we may chose a sufficiently small left neighborhood of t_0 such that $|f| = -f$. Since $\phi_L(f) < \phi_L(-f)$, it follows that $\phi_L(|f|) = |\phi_L(f)|$. If k is even, then there is a sufficiently small left neighborhood of t_0 such that $|f| = f$, and as $\phi_L(f) > \phi_L(-f)$, we see that $\phi_L(|f|) = |\phi_L(f)|$. Hence ϕ_L is a Riesz homomorphism with kernel L_{t_0}. A similar argument shows that φ_R is a Riesz homomorphism with kernel R_{t_0}. $\square$

For brevity, write $E := \mathrm{PPol}^n([a, b])$. Let $t_0 \in (a, b]$ and define $L_{t_0}^k$ for $1 \leq k \leq n$ by

$$L_{t_0}^k := \left\{ f \in M_{t_0}^E : f_{-}^{(k)}(t_0) = f_{-}^{(k-1)}(t_0) = \ldots = f_{-}'(t_0) = 0 \right\}, \qquad (7.5)$$

and similarly

$$R_{t_0}^k := \left\{ f \in M_{t_0}^E : f_{+}^{(k)}(t_0) = f_{+}^{(k-1)}(t_0) = \ldots = f_{+}'(t_0) = 0 \right\} \qquad (7.6)$$

for $1 \leq k \leq n$ whenever $t_0 \in [a, b)$. Note that $L_{t_0}^n = L_{t_0}$ and $R_{t_0}^n = R_{t_0}$. It turns out that all non-maximal prime ideals in E are of this form.

Theorem 7.3 *The non-maximal prime ideals in* $\mathrm{PPol}^n([a, b])$ *are of the form* $L_{t_0}^k$ *for some* $1 \leq k \leq n$ *and* $t_0 \in (a, b]$, *or are of the form* $R_{t_0}^k$ *for some* $1 \leq k \leq n$ *and* $t_0 \in [a, b)$.

Proof Write $E := \mathrm{PPol}^n([a, b])$. Let $t_0 \in (a, b]$. Then $L_{t_0}^k$ is the preimage of the prime ideal

$$I_k := \{x \in \mathbb{R}^{n+1} : x_i = 0, \ 1 \leq i \leq k\}$$

under ϕ_L forcing it to be a prime ideal in E as φ_L is a Riesz homomorphism by Lemma 7.2. Indeed, consider the linear map $\sigma_k \colon \mathbb{R}^{n+1} \to \mathbb{R}^k$ by $\sigma_k(x_1, \ldots, x_{n+1}) := (x_1, \ldots, x_k)$. If we equip $\mathbb{R}^{n+1}$ and $\mathbb{R}^k$ with the lexicographical ordering, it follows that σ_k is a Riesz homomorphism. To see this, let $x \in \mathbb{R}^{n+1}$. If $x_1 < -x_1$, then $|x| = -x$ and $\sigma_k(|x|) = |\sigma_k(x)|$, and if $x_1 > -x_1$, then $|x| = x$ and $\sigma_k(|x|) = |\sigma_k(x)|$. Suppose that $x_i = 0$ for $1 \le i \le l < k$ and $x_{l+1} < -x_{l+1}$, then $|x| = -x$ and $\sigma_k(|x|) = |\sigma_k(x)|$, and if $x_{l+1} > -x_{l+1}$, then $|x| = x$ and $\sigma_k(|x|) = |\sigma_k(x)|$. In the case where $x_i = 0$ for all $1 \le i \le k$ it is clear that $\sigma_k(|x|) = |\sigma_k(x)| = 0$. Hence, σ_k is a Riesz homomorphism. Since the kernel of σ_k is I_k, it follows that I_k is an ideal in $\mathbb{R}^{n+1}$, and as the quotient vector lattice $\mathbb{R}^{n+1}/I_k$ is linearly ordered, we conclude that I_k is a prime ideal in $\mathbb{R}^{n+1}$ by [10,Theorem 33.2]. Similarly, we conclude for $t_0 \in [a, b)$ that $R_{t_0}^k$ is a prime ideal in E for all $1 \le k \le n$. Note that the linear maps $\phi_k \colon L_{t_0}^k \to \mathbb{R}$ given by

$$\phi_k(f) := f_-^{(k+1)}(t_0)$$

have kernels $L_{t_0}^{k+1}$ for $1 \le k \le n-1$, and the linear map $\phi_0 \colon M_{t_0}^E \to \mathbb{R}$ given by $\phi_0(f) := f_-'(t_0)$ has kernel $L_{t_0}^1$. Hence

$$L_{t_0}^{n-1}/L_{t_0} \cong \ldots \cong L_{t_0}^k/L_{t_0}^{k+1} \cong \ldots \cong M_{t_0}^E/L_{t_0}^1 \cong \mathbb{R},$$

and similarly, we have

$$R_{t_0}^{n-1}/R_{t_0} \cong \ldots \cong R_{t_0}^k/R_{t_0}^{k+1} \cong \ldots \cong M_{t_0}^E/R_{t_0}^1 \cong \mathbb{R}.$$

It follows that if P is a non-maximal prime ideal in E, then it contains a minimal prime ideal which is of the form L_{t_0} for some $t_0 \in (a, b]$ or R_{t_0} for some $t_0 \in [a, b)$ by Lemma 7.1. Thus, we have $L_{t_0} \subseteq P \subsetneq M_{t_0}^E$ or $R_{t_0} \subseteq P \subsetneq M_{t_0}^E$. If $k \ge 1$ is the smallest number such that $L_{t_0}^k \subseteq P$, then we must have $L_{t_0}^k \subseteq P \subseteq L_{t_0}^{k-1}$, so $P = L_{t_0}^k$ or $P = L_{t_0}^{k-1}$. Similarly, if P is a non-maximal prime ideal such that $R_{t_0} \subseteq P \subseteq M_{t_0}^E$, then $P = R_{t_0}$ or $P = R_{t_0}^k$ for some $k \ge 1$. □

Corollary 7.4 *The vector lattice* $\mathrm{PPol}^n([a, b])$ *is prime Noetherian. Furthermore, any ascending chain of prime ideals is of length at most* n, *and it contains a chain of ascending prime ideals of length* n.

Proof The ascending chains of prime ideals

$$L_{t_0} = L_{t_0}^n \subseteq L_{t_0}^{n-1} \subseteq \ldots \subseteq L_{t_0}^1 \subseteq M_{t_0}^E \quad \text{and} \quad R_{t_0} = R_{t_0}^n \subseteq R_{t_0}^{n-1} \subseteq \ldots \subseteq R_{t_0}^1 \subseteq M_{t_0}^E$$

are of length n. □

For the piecewise polynomials of arbitrary degree $\mathrm{PPol}([a, b])$ and $t_0 \in (a, b]$ we define the linear map $\psi_L \colon \mathrm{PPol}([a, b]) \to c_{00}$ by

$$\psi_L(f) := (f(t_0), -f_-'(t_0), f_-''(t_0), \ldots, (-1)^k f_-^{(k)}(t_0), (-1)^{k+1} f_+^{(k+1)}(t_0), \ldots). \tag{7.7}$$

Similarly for $t_0 \in [a, b)$ we define the linear map $\psi_R \colon \text{PPol}([a, b]) \to c_{00}$ by

$$\psi_R(f) := (f(t_0), -f_+'(t_0), f_+''(t_0), \ldots, (-1)^k f_+^{(k+1)}(t_0), (-1)^{k+1} f_+^{(k+1)}(t_0), \ldots). \tag{7.8}$$

If we equip c_{00} with the lexicographical ordering, then this yields a totally ordered vector lattice which will be denoted by $\text{Lex}(\mathbb{N})$. For a literary reference, see [12, Section 2]. It follows that ψ_L and ψ_R are Riesz homomorphisms.

Lemma 7.5 *For $t_0 \in (a, b]$ the map ψ_L as in (7.7) is a Riesz homomorphism with kernel L_{t_0} and for $t_0 \in [a, b)$ the map ψ_R as in (7.8) is a Riesz homomorphism with kernel R_{t_0}.*

Proof Note that ψ_L is surjective as for any $(x_n)_{n\in\mathbb{N}}$ in c_{00} there is a smallest $N \geq 1$ such that $x_n = 0$ for all $n \geq N + 1$, and the polynomial defined by

$$f(t) := x_1 - x_2(t - t_0) + \ldots + (-1)^{N-1} \frac{1}{(N-1)!} x_N (t - t_0)^{N-1}$$

satisfies $\psi_L(f) = (x_n)_{n\in\mathbb{N}}$. Let $f \in E$. Then it follows that $\psi_L(|f|) = |\psi_L(f)|$ via analogous reasoning for the map ϕ_L in (7.1) where n equals the degree of the polynomial that equals f on a left neighborhood of t_0. Hence, ψ_L is a Riesz homomorphism with kernel L_{t_0}. Analogous to the reasoning in proving that φ_R is a Riesz homomorphism, it follows that ψ_R is a Riesz homomorphism with kernel R_{t_0}. □

If we write $E := \text{PPol}([a, b])$, then as in (7.5) and (7.6) we consider

$$L_{t_0}^k := \left\{ f \in M_{t_0}^E \colon f_-^{(k)}(t_0) = f_-^{(k-1)}(t_0) = \ldots = f_-'(t_0) = 0 \right\}$$

for $t_0 \in (a, b]$ and all $k \in \mathbb{N}$ in this case, and

$$R_{t_0}^k := \left\{ f \in M_{t_0}^E \colon f_+^{(k)}(t_0) = f_+^{(k-1)}(t_0) = \ldots = f_+'(t_0) = 0 \right\}$$

for $t_0 \in [a, b)$ and all $k \in \mathbb{N}$ in this case. Note that $\bigcap_{k=1}^{\infty} L_{t_0}^k = L_{t_0}$ and $\bigcap_{k=1}^{\infty} R_{t_0}^k = R_{t_0}$, and it follows that these are in fact all the non-maximal prime ideals in E.

Theorem 7.6 *The non-maximal and non-minimal prime ideals in $\text{PPol}([a, b])$ are of the form $L_{t_0}^k$ for some $k \in \mathbb{N}$ and $t_0 \in (a, b]$, or are of the form $R_{t_0}^k$ for some $k \in \mathbb{N}$ and $t_0 \in [a, b)$.*

Proof Write $E := \text{PPol}([a, b])$. Let $t_0 \in (a, b]$. Then $L_{t_0}^k$ is the preimage of the prime ideal

$$J_k := \{x \in \text{Lex}(\mathbb{N}) \colon x_i = 0, \ 1 \leq i \leq k\}$$

under the Riesz homomorphism ψ_L. It follows that J_k is the kernel of $\tau_k \colon \text{Lex}(\mathbb{N}) \to \mathbb{R}^k$ defined by $\tau_k((x_n)_{n\in\mathbb{N}}) := (x_1, \ldots, x_k)$, which is proved to be a Riesz homomorphism analogously to showing that σ_k is a Riesz homomorphism in Theorem 7.3.

Similarly, we have prime ideals

$$R_{t_0}^k := \left\{ f \in M_{t_0}^E \colon f_+^{(k)}(t_0) = f_+^{(k-1)}(t_0) = \ldots = f_+'(t_0) = 0 \right\}$$

for all $k \in \mathbb{N}$. Furthermore, as in Theorem 7.3 we have $L_{t_0}^k/L_{t_0}^{k+1} \cong \mathbb{R}$ and $R_{t_0}^k/R_{t_0}^{k+1} \cong \mathbb{R}$ for $k \in \mathbb{N}$, and $M_{t_0}^E/L_{t_0}^1 \cong \mathbb{R}$ and $M_{t_0}^E/R_{t_0}^1 \cong \mathbb{R}$.

Let P be a non-maximal prime ideal in E such that $L_{t_0} \subseteq P \subseteq M_{t_0}^E$ for some $t_0 \in (a, b]$. If P is contained in $L_{t_0}^k$ for all $k \in \mathbb{N}$, then $P \subseteq \bigcap_{k=1}^{\infty} L_{t_0}^k = L_{t_0}$, so $P = L_{t_0}$. On the other hand, if $k \in \mathbb{N}$ is the smallest number such that $L_{t_0}^k \subseteq P$, then $L_{t_0}^k \subseteq P \subseteq L_{t_0}^{k-1}$, so $P = L_{t_0}^k$ or $P = L_{t_0}^{k-1}$ as $L_{t_0}^{k-1}/L_{t_0}^k \cong \mathbb{R}$. Similarly, if P is a non-maximal prime ideal such that $R_{t_0} \subseteq P \subseteq M_{t_0}^E$, then $P = R_{t_0}$ or $P = R_{t_0}^k$ for some $k \in \mathbb{N}$. □

Corollary 7.7 *The vector lattice* $\mathrm{PPol}([a, b])$ *is prime Noetherian and contains ascending chains of prime ideals of arbitrary finite length.*

Proof Write $E := \mathrm{PPol}([a, b])$. Let $P_1 \subseteq P_2 \subseteq \ldots$ be an ascending chain of prime ideals. Then for any $n \in \mathbb{N}$, it follows that $L_{t_0} \subseteq P_n \subseteq M_{t_0}^E$ for some $t_0 \in (a, b]$ or $R_{t_0} \subseteq P_n \subseteq M_{t_0}^E$ for some $t_0 \in [a, b)$. By Theorem 7.6 the only P_n such that $L_{t_0} \subsetneq P_n \subsetneq M_{t_0}^E$ or $R_{t_0} \subsetneq P_n \subsetneq M_{t_0}^E$ must be of the form $P_n = L_{t_0}^k$ or $P_n = R_{t_0}^k$, respectively, for some $k \in \mathbb{N}$. In this case P_n can only be contained in finitely many prime ideals, so the chain must be stationary. Note that for $n \in \mathbb{N}$ we have ascending chains of prime ideals

$$L_{t_0}^n \subseteq L_{t_0}^{n-1} \subseteq \ldots \subseteq L_{t_0}^1 \subseteq M_{t_0}^E \quad \text{and} \quad R_{t_0}^n \subseteq R_{t_0}^{n-1} \subseteq \ldots \subseteq R_{t_0}^1 \subseteq M_{t_0}^E$$

of length n in E. □

In view of Theorem 5.4 there are no non-maximal principal prime ideals in uniformly complete Archimedean vector lattices. However, there is an abundance of principal prime ideals, even ascending chains of arbitrary finite length, in vector lattices of piecewise polynomials.

Theorem 7.8 *Let E be the vector lattice* $\mathrm{PPol}^n([a, b])$ *or* $\mathrm{PPol}([a, b])$. *Then all non-minimal prime ideals in E are principal.*

Proof By Lemma 4.1, every maximal ideal in E is of the form $M_{t_0}^E$ for some $t_0 \in [a, b]$, that is,

$$M_{t_0}^E = \{ f \in E \colon f(t_0) = 0 \}.$$

We will show that $f(t) := |t - t_0|$ generates $M_{t_0}^E$. Indeed, suppose first that $t_0 \in (a, b)$. If $g \in M_{t_0}^E$, then there is a left neighborhood I_{t_0} and a right neighborhood J_{t_0} of t_0 in $[a, b]$ such that g is a polynomial on I_{t_0} and J_{t_0}, and $|t - t_0| \leq 1$ for all $t \in I_{t_0} \cup J_{t_0}$. That is, we have

$$g(t) = a_1(t - t_0) + \ldots + a_n(t - t_0)^n \quad \text{on } I_{t_0} \quad \text{and}$$

$$g(t) = b_1(t - t_0) + \ldots + b_n(t - t_0)^m \quad \text{on } J_{t_0}.$$

Hence, for $\mu := \max\{\sum_{k=1}^n |a_k|, \sum_{k=1}^m |b_k|\}$, we find that $|g(t)| \leq \mu |f(t)|$ for all $t \in I_{t_0} \cup J_{t_0}$. Outside $I_{t_0} \cup J_{t_0}$ we have that f is strictly positive, so gf^{-1} is bounded there and we conclude that $|g| \leq \lambda f$ for some $\lambda > 0$, showing that $M_{t_0}^E$ is generated by f. If t_0 is either a or b, a similar argument using only a left or right neighborhood of t_0 shows that $M_{t_0}^E$ is generated by f as well.

Let P be a non-maximal and non-minimal prime ideal in E. Then by Theorem 7.3 and Theorem 7.6 there is a $t_0 \in [a, b]$ such that $P = L_{t_0}^k$ or $P = R_{t_0}^k$ for some $k \in \mathbb{N}$. Suppose first that $t_0 \in (a, b)$. The functions

$$f_l(t) := \begin{cases} t_0 - t & \text{if } a \leq t \leq t_0 \\ 0 & \text{if } t_0 < t \leq b \end{cases} \quad \text{and} \quad f_r(t) := \begin{cases} 0 & \text{if } a \leq t < t_0 \\ t - t_0 & \text{if } t_0 \leq t \leq b \end{cases}$$

are disjoint, so it follows that either $f_l \in P$ or $f_r \in P$. Note that P cannot contain both f_l and f_r as that would force P to be the maximal ideal $M_{t_0}^E$ as discussed in the paragraph above. Suppose $f_l \in P$. Then $P = R_{t_0}^k$ and since each $f \in E$ can be written as

$$f(t) = a_0 + a_1(t - t_0) + \ldots + a_n(t - t_0)^n$$

locally on a right neighborhood of t_0 for some $n \in \mathbb{N}$, it follows that $f \in P$ if and only if $a_0 = \ldots = a_k = 0$. Define $g_{k+1}(t) := |t - t_0|^{k+1}$ and note that for $f \in P$ there is a $\delta > 0$ such that $|t - t_0| \leq 1$ for all $t \in [t_0, t_0 + \delta)$, and then

$$|f(t)| = |t - t_0|^{k+1} \left| a_{k+1} + a_{k+2}(t - t_0) + \ldots + a_n(t - t_0)^{n-k-1} \right|$$

$$\leq g_{k+1}(t) \sum_{i=k+1}^n |a_i|. \tag{7.9}$$

Since g_{k+1} is strictly positive on $[t_0 + \delta, b]$, it follows that fg_{k+1}^{-1} is bounded there and so there is a $\lambda > 0$ such that $|f| \leq \lambda g_{k+1}$ on $[t_0, b]$. Since $P \subsetneq M_{t_0}^E$, there also is a $\mu > 0$ such that $|f| \leq \mu f_l$ on $[a, t_0]$, which implies that the function

$$g(t) := \begin{cases} f_l(t) & \text{if } a \leq t \leq t_0 \\ g_{k+1}(t) & \text{if } t_0 < t \leq b \end{cases}$$

generates the ideal P. Similarly, in the case where $f_r \in P$ instead of f_l, it follows that $P = L_{t_0}^k$ and the function

$$h(t) := \begin{cases} g_{k+1}(t) & \text{if } a \leq t < t_0 \\ f_r(t) & \text{if } t_0 \leq t \leq b \end{cases}$$

generates the ideal P. Finally, if $t_0 = a$ or $t_0 = b$, then it follows from (7.9) that P is generated by either $(t - a)^{k+1}$ or $(b - t)^{k+1}$, respectively. □

Note that by Theorem 5.2 the vector lattices of piecewise polynomials must contain non-principal prime ideals, which are precisely the minimal prime ideals as shown below.

Corollary 7.9 *Let E be the vector lattice* $\mathrm{PPol}^n([a, b])$ *or* $\mathrm{PPol}([a, b])$. *Then none of the minimal prime ideals in E are principal.*

Proof For any minimal prime ideal P in E there is a $t_0 \in [a, b]$ such that $P = L_{t_0}$ or $P = R_{t_0}$ by Lemma 7.1. For every positive function g in P there is a $\delta > 0$ such that g is zero on either $(t_0 - \delta, t_0]$ or $[t_0, t_0 + \delta)$. Hence, we can construct two disjoint non-zero functions supported in $(t_0 - \delta, t_0]$ or we can construct two disjoint functions supported in $[t_0, t_0 + \delta)$. Hence, neither L_{t_0} nor R_{t_0} can be principal. □

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