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1 Introduction: Indefinites

Indefinites can occur in a nested sequence of quantificational phrases like

1. Every student read every paper that a professor recommended

and also in combination with anaphoric pronouns as in

2. A man smiles. He is happy.

The nesting, on one side, and the anaphoric link, on the other, create an interpretational tension. The nesting favours a quantificational interpretation of the indefinites according to which they behave more like any other quantificational NP, e.g. they enter into scopal (dependency) relations with other quantificational phrases. But the presence of discourse anaphora creates some pressure to interpret the indefinites referentially in a way which makes their semantical behaviour resemble more that of proper names. In a seminal paper Fodor and Sag (1982) argue against the assimilation of indefinites to existential quantifiers, and propose instead an interpretation according to which they are ambiguous. That is, they may have a quantificational reading as well as a referential one. If the latter, they function as a kind of "mental pointer" within the mind of the speaker having, semantically speaking, cross-referential relations with pronominal anaphors which are not in their "local" domain.

Several approaches were developed in the 80's and 90's in order to provide a unitary framework in which the two functions of indefinites could be combined. They may be divided into two groups depending on which one of the two aspects is emphasized. *Dynamic theories* (Discourse Representation Theory (DRT), File Semantics, Dynamic Predicate Logic (DPL)) ended up in defending a so-called *extended existential analysis* of indefinites. According to this analysis an indefinite becomes an existential quantifier and the pronominal anaphor a free variable bound by the existential quantifier accross conjunction. Thus the logical force of (2) is that of

3. $\exists x(M(x) \wedge S(x) \wedge H(x))$.

In the other group, and against this tradition, several theories have emerged which introduce a rich ontology of different types (collective, distributive, group-denoting, arbitrary objects, etc.) to which nominal expressions *refer*. Hilbert's epsilon terms and Fine's arbitrary objects belong to this group. In this paper I will put Fine's work in perspective.

2 Some problems with the dynamic account

Some of the problems with the dynamic account may be introduced though an argument originally formulated by Strawson and extensively discussed by Geach (1962) and Slater (1986):

4. A man has just drunk a pint of sulphuric acid.

5. Nobody who drinks a pint of sulphuric acid lives through the day.

6. Very well then, *he* won't live through the day.

Geach claimed that, despite the appearances, from the two premises, one can derive only the conclusion in which the pronoun is (implicitly) existentially quantified by "A man", and not a referential conclusion, that is, a statement having the form '$\neg L(t)$' (where 't' is a term and $'L$' stands for the predicate 'will live through the day'). In response to Geach, Slater points one major problem with the quantificational account: the conclusion is not truth-conditionally separable from the premises. In order to preserve the structure of the argument, one has to switch to one of the referential approaches.

3 The 'Hilbertian' approach: epsilon terms

The reader interested in the historical developments and the role of the epsilon calculus in Hilbert's program is referred to Avigad and Zach (2016). Here I shall extract from their article few basic things relevant to my concern.

The epsilon calculus is an extension of first-order logic with new terms formed through the clause:

- If A is a formula, ϵxA is a term.

The variable x becomes bound in the term ϵxA. The intended interpretation of '$\epsilon xA'$ is: some x satisfies A, if there is one.

The basic axiom governing the epsilon term is the so-called Hilbert's transfinite axiom:

$$A(x) \to A(\epsilon xA).$$

The epsilon calculus includes a complete set of axioms governing the classical propositional connectives, and the equality symbol. The usual quantifier rules can be defined from the following definitions of the standard quantifiers:

$$\exists xA(x) \leftrightarrow A(\epsilon xA)$$
$$\forall xA(x) \leftrightarrow A(\epsilon x(\neg A)).$$

This shows that the predicate calculus may be embedded into the epsilon calculus. (It is also known that the converse is not true).

Avigad and Zach emphasize two features of epsilon terms:

- An epsilon term is nondeterministic, that is, the calculus leaves it entirely open whether $\epsilon x(x = a \vee x = b)$ refers to either a or b.
- The calculus may be enriched with a schema of extensionality

$$\forall x(A(x) \leftrightarrow B(x)) \to \epsilon x A = \epsilon x B.$$

That is, the epsilon operator assigns the same witness to extensionally equivalent formulae.

Avigad and Zach give two nice examples to illustrate the usefulness of the epsilon terms. We can express

7. The least value satisfying A, if there is one

by

$$A(x) \to A(\epsilon x A(x)) \wedge \epsilon x A(x) \leq x.$$

We can also express

8. If there is a witness satisfying $A(y)$, the epsilon term returns a value whose predecessor does not have this property

by

$$A(y) \to A(\epsilon x A(x)) \wedge \epsilon x A(x) \neq y + 1.$$

I propose an alternative way to read (8), which is:

9. If there is a witness satisfying $A(y)$, then an individual satisfies A and *that* individual is distinct from $y + 1$.

Under this reading the example shows that epsilon terms can be used to express co-referential relations between pronominal anaphors and the indefinites which are their heads. In fact, Slater (1986), p. 29) exploits this property of epsilon terms to represent the Geach-Strawson's argument in a way which yields a "referential" conclusion that preserves:

10. $D(\epsilon x D(x))$

11. $\forall x(D(x) \to \neg L(x))$

12. $D(\epsilon x D(x)) \to \neg L(\epsilon x D(x))$

Hence

13. $\neg L(\epsilon x D(x))$.

It is known that essential to the Hilbert's program was the division between the "real" and the "ideal" part of a mathematical theory. It is only the former which carries ontological commitments, whereas the latter has only a heuristic function. For instance, one could take primite recursive arithmetic to be the real part of a mathematical theory to which Cantorian set theory is added as an ideal part. The lack of ontological import of the ideal part of a theory was supposed to be garanteed through *conservativity* arguments: the ideal part does not prove new real statements. Hilbert's division is of great relevance to the present case. The epsilon terms belong to the "ideal" part of a mathematical theory, i.e. that part which has only a heuristic value. The ontological burden of the theory is carried by the "real" part of the theory, e.g. the quantifier free axioms. There are two conservativity results (see e.g. Avigad and Zach 2016). The first epsilon theorem implies that any detour through first-order predicate logic used to derive a quantifier-free theorem from quantifier-free axioms can ultimately be avoided. The second epsilon theorem shows that any detour through the epsilon calculus used to derive a theorem in the language of the predicate calculus from axioms in the language of the predicate calculus can also be avoided. The two results show in my opinion that the question of the reference of epsilon terms in the context of proof does not arise, or, if it does, it is of a very mild "deflationist" kind.

4 Choice functions

Even if epsilon terms can be dispensed with in proofs, they have been used for the semantic analysis of contingent sentences in natural language, like

14. A man smiles

15. A man smiles. He is happy.

In this context the interpretation of epsilon terms becomes more pressing. Slater (1986) suggests a semantics for epsilon terms in terms of choice functions with restricted domain. Linguists have picked up on this idea. The idea to treat indefinites as choice functions has been explored in Reinhart's work (see e.g. Reinhart 1997) and subsequently developed by the Konstanz school. A choice function f assigns to any non-empty set of individuals a member of this set . That is, every model is enlarged with a choice function f which obeys the condition:

- f is a choice function if and only for any non-empty set P, $P(f(P))$.

When choice functions are used directly in the object language, they become a notational variant of Hilbert's epsilon terms. For instance (14) and (15) may be represented as

16. $ch(f) \wedge S(f(M))$

and

17. $ch(f) \wedge S(f(M)) \wedge H(f(M))$

respectively. (16) is true in a model if the choice function of the model picks up from the set of men an individual who smiles.

Things, however, get more complicated when indefinites occur in "the scope" of other quantified NP's. Winter (1997) discusses the following problematic case:

18. Every professor has invited a colleague from his university.

The Hilbertian analyis renders (18) as

19. $\forall x(P(x) \rightarrow I(x, \epsilon y(C(x, y)))$

(here we use choice functions and epsilon terms interchangably in the syntax) where '$C(x, y)$' stands for 'y is the colleague of x'.

Suppose now that every x has the same colleagues (i.e. the colleagues come from one and the same university). In this case, in the interpretation of (19) the relevant model's choice function f which interprets the epsilon term will have to select for each x, one of $x's$ colleagues. But given that the sets of colleagues are identical, then the principle of extensionality will constrain the choice function to select the same individual. As a result of all this, the truth of (19) will amount to all professors inviting one and the same person. This, however, is not what (18) is intended to say, but rather

20. $\forall x(P(x) \rightarrow \exists y(C(x, y) \wedge I(x, y)))$.

Thus we are forced back to the quantificational account of indefinites, something that we wanted to give up.
[1]

We have reached the following conclusion. Choice functions, like epsilon terms, are adequate to express coreferential mechanisms arising between indefinites and anaphoric pronoun in simple sentences like (15). But when the coreferential and dependence (co-variation) mechanisms are juxtaposed, we run into trouble. One way out is to use Skolem functions (terms) to keep track of the dependence mechanisms. On this proposal (18) has the logical force of

(21) $\forall x(P(x) \rightarrow C(x, f(x)) \wedge I(x, f(x))))$

which is the Skolem form of (20). This is the right moment to introduce the arbitrary objects framework.

[1] Arancha Sanchez pointed out in conversation that if we take

$$\forall x\,[P(x) \rightarrow C(x, \epsilon y(C(x, y) \wedge I(x, y))) \wedge I((x, \epsilon y(C(x, y) \wedge I(x, y)))]$$

as the representation of (18) we avoid the objection raised by Winter. I find this proposal unnatural, for the following reason. I think that if a term has a referential function, then it should refer to the entity it putports to refer at some point of discourse, once and for all, after which the entity in question eventually receive new attributes. On the scheme proposed here one has to wait for the whole discourse to reach an end, and only then introduce a referent which subsumes all the properties mentioned so far.

5 Arbitrary Objects

I will introduce the referential approach to arbitrary objects through an argument in Kleene (1952, pp. 149-150) discussed by Tichý (1988). Kleene's argument shows that "nothing is a P" follows from the assumptions that nothing is a Q and all P's are Q's. It goes like this.

Suppose that

i) Nothing is a Q

and

(ii) All P's are Q's.

Let

(iii) x be a P;

by (ii),

(iv) if x is a P then x is a Q,

which together with (iii) entails

(v) x is a Q;

by (i), on the other hand,

(vi) x is not a Q;

Consequently,

(vii) x is not a P;

Thus,

(viii) Nothing is a P.

In commenting this argument, Tichý (1988) remarks that 'x' is introduced in (iii) as a name for an object which *stands for* "all P". It is thus the representative of a class, which, as Tichy points out, cannot be the name of any particular object, for then the last step of the argument would not be warranted: from the assumption that x is not a P one cannot infer that nothing is a P. Tichy concludes:

> Kleene can only be right if, over and above particular objects, there also are *arbitrary* ones. Kleene shrinks from saying that and resorts instead to a formal mode of speech. The 'x' he suggests, might be called an 'arbitrary' constant. (Tichý 1988, pp. 258-259).

The referential approaches to arbitrary objects do not rest content with Kleene's formal mode of speech. Before discussing Fine's views, let me mention another, more recent aproach to arbitrary objects.

Breckenridge and Magidor (2012) take an "arbitrary" constant to refer *arbitrarily* to an ordinary object, that is, the constant "receives its ordinary kind of semantic-value, though we do not and cannot know which value in particular it receives." (Breckenridge and Magidor 2012, p. 377). This interpretation is very closed to Slater's interpretation of an epsilon term as being an epistemically indeterminate object. In fact Breckenridge and Magidor acknowledge in a footnote that "Our account is in some ways very close to that involved in systems of Hilbert's Epsilon Calculus though as far as we know Hilbert was not particularly concerned with the metaphysically underpinnings of the epsilon operator" (p. 393). There is a difference, however. Slater is concerned with contingent natural language examples whereas Breckenridge and Magidor are concerned with proofs. In this case, as pointed out earlier, Hilbert's conservativity program becomes relevant: the question of the "metaphysical underpinnings" of the epsilon terms does not arise, for they belong to the ideal part of the theory and the two conservativity theorems quoted in section 3 show that they are dispensable.

Fine's approach (Fine and Tennant 1983, Fine 1985a, Fine 1985b) takes "arbitrary" constants to refer to refer to new kind of objects, arbitrary objects. Although his main concern is with proofs and arguments in logic and mathematics, he also thinks that arbitrary objects are useful for the semantic analyis of both mathematical and ordinary language. For the mathematical language he takes the theory of arbitrary objects to explain the role of variables in ordinary mathematical discourse. In a nuttshell his view is like this: universal quantifier phrases introduce unrestricted arbitrary objects; existential quantifiers introduce dependent arbitrary objects, and the scopal relations between them has a counterpart in the relation of dependence between arbitrary objects (Fine and Tennant 1983, pp. 74-75)

Although arbitrary objects belong to a domain disjoint from that of ordinary, individual objects, the two domains are related. An arbitrary object a is associated with a set of individual objects which is the set of values a can take. To take just one example, the set of values of an arbitrary natural number is the set of (individual) natural numbers.

Fine's machinary of arbitrary objects has two components. The first component defines the truth-conditions of formulas containing (names of) arbitrary objects. Roughly:

(G4) If $\varphi(x_1, ..., x_n)$ is a first-order formula containing no name of arbitrary objects, then $\varphi(a_1, ..., a_n)$ is true if and only it is true for all admissible assignments of individuals $i_1, ..., i_n$ to the objects $a_1, ..., a_n$.

(We suppose that the arbitrary objects $a_1, ..., a_n$ name themselves; in addition, we must think of the principle (G4) as applying to the arbitrary objects simultaneously.)

(G4) presupposes a way to determine the class of admissible assignments of individual objects $i_1, ..., i_n$ to the arbitrary objects $a_1, ..., a_n$. This is done by the second component of the framework. Arbitrary objects stand in a relation of *object dependence*, which, we are told, correspond, roughly, to the relation between dependent and independent variables in mathematics. Any relation of dependence at the level of values (individual objects) must be sustained, in one way or another, by a relation of object dependence. That is, when b is an arbitrary object that depends only upon the arbitrary objects $a_1, a_2, ...$, then the values assigned to b must be determined upon the values assigned to $a_1, a_2, ...$ Thus arbitrary objects are divided into *independent* and *dependent* ones. An independent arbitrary object is one which does not depend on any other arbitrary object. Otherwise it is dependent. Identity criteria are provided for both kinds, but this issue will not concern us here.

6 Arbitrary objects and natural language

Although Fine's main concerns is with proofs in predicate logic, he and others inspired by him (e.g. Steedman) have observed that terms that are generated by rules of instantiation in logic proofs have a counterpart in natural language discourse: quantificational NP's and embedded indefinites create a similar juxtaposition of referential and dependence mechanisms that are better handled if one develops a referential view of indefinites and other quantificational NP's. Fine's work on arbitrary objects found an echo among the logically minded linguists and AI people who were looking for alternative frameworks to cope with the problems mentioned in the Introduction of this paper. I will briefly comment on some of Steedman's ideas who draws on the work of Fine (1985b). I prefer, however, to start anachronistically with Steedman (1999) for reasons which will become apparent later on.

Steedman discusses one of the examples due to Geach's 1962 which has populated the philosophical literature *ad nauseaum*:

22. Every farmer who owns a donkey$_i$ beats it$_i$.

One way to account for the possibility of anaphora would be to treat the indefinite "a donkey$_i$" as an existential quantifier that binds the pronominal anaphor "it$_i$". The problem with this kind of solution (the quantificational account), alluded to in the Introduction, is that there is no syntactic theory on the market, which allows the existential quantifier to remain both within the scope of the universal quantifier, and to *c*-command the anaphoric pronoun. As Steedman points out, a different route to go would be to take the anaphoric pronoun to be a non-bound variable, and its relation to the indefinite to be a cross-referential relation, more on the scheme at work in the sentence

23. Everybody who knows Gilbert$_i$ likes him$_i$.

But in order to be able to make this sort of move, one has to construe the indefinite as some sort of referential expression. Steedman's proposal is to take the indefinite to be a dependent arbitrary object whose values are generated by a Skolem term. More exactly, he takes "a donkey" to translate as

$arb'donkey'$, where arb' yields a Skolem term–$Sk_{donkey}(x)$ with the variable x bound by the universal quantifier in whose scope $arb'donkey'$ falls ("Every farmer" in this case). For every individual farmer which is an instantiation of x, Sk_{donkey}produces an individual which is a donkey. The pronominal anaphor is now construed as coreferential with the Skolem term. Of course for this solution to work, the anaphor has to occur in the scope of the universal quantifier, otherwise it will fail to refer. Finally we are told that

> However, by making the pronoun refer instead to a Skolem term or arbitrary object, we free our hands to make the inferences we draw on the basis of such sentences sensitive to world knowledge. For example, if we hear the stan-dard donkey sentence and know that farmers may own more than one donkey, we will probably infer on the basis of knowledge about what makes people beat an arbitrary donkey that she beats all of them. (Idem, p. 304)

The analysis is then extended to other supposed quantifiers such as *some*, *a few*, which may be "better analyzed as referential categories".

Fine's proposal is more radical. I take it that this is partly due to the fact that he considers examples which involves not only cross-referential relations between anaphors and indefinites, but also examples involving pronominal anaphors and universal quantifiers. In addition, Fine voices objections to the use of Skolem functions. I will discuss them in the next section. For the moment let me quickly discuss one of Fine's examples (actually the only one, to the best of my knowledge):

24. Every farmer owns a donkey. He beats it. He feeds it rarely...

Fine points out that on a referential view of the pronominal anaphors there is no individual farmer or individual donkey to which the pronouns can be taken to refer. But, he goes on, once we allow for arbitrary objects, we can take 'He' to refer to the arbitrary farmer introduced by "Every farmer", and 'it' to the arbitrary donkey that he owns introduced by "a donkey". The dependence of the arbitrary donkey on the arbitrary farmer means that for a given individual farmer as the value of the arbitrary farmer, the arbitrary donkey can only take as a value an individual donkey that the farmer owns. Thus the statement 'He beats it' is true if and only if for all values i and j simultaneously assumed by the arbitrary farmer and donkey, it is true that i beats j. (Fine 1983). We get the desired truth-conditions of (24).

7 Skolem functions and multidependencies

Fine (1983) mentions a couple of applications of arbitrary objects to the semantic analysis of mathematical languages. One of them is the analysis of so-called *Henkin quantifiers.* A Henkin quantifier (Henkin, 1961) is a prefix of four quantifiers, two universal, $\forall x$ and $\forall z$, and two existential, $\exists y$ and $\exists z$, such that:

(i) $\exists y$ depends on $\forall x$ but is independent of $\forall z$ and $\exists w$.

(ii) $\exists w$ depends on $\forall z$ but is independent of $\forall x$ and $\exists y$

The partial order of the dependence relation cannot be expressed in the linear notation of first-order logic. Henkin (1961) uses the branching form

$$\left\{ \begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array} \right\}$$

to convey the dependencies in (i) and (ii), and Skolem functions to express the truth-conditions of formulas with Henkin quantifiers:

$$\left\{ \begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array} \right\} R(x, y, z, w) \Leftrightarrow \exists f \exists g \forall x \forall z R(x, f(x), z, g(z)).$$

The functions f and g are generalizations of the Skolem functions we encountered in our earlier examples. It is well known that for certain choices of the formula $R(x, y, z, w)$, the sentence $\left\{ \begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array} \right\} R(x, y, z, w)$ has no equivalent in ordinary first-order logic.

Fine does not enter into details, but it is clear that his arbitrary objects framework provides an alternative analysis of the Henkin prefix. The four quantifiers in the Henkin prefix introduce four arbitrary objects, say a, b, c and d, such that c depends on a and d depends on b. Any set of assignments which satisfy $R(a, c, b, d)$ (we let $a, b, c,$ and d to name themselves) must obey these dependencies making the values of c correlate in the appropriate way with the values of a; and the values of d correlate in the appropriate way with the values of b. If we take the correlations to be functional, we obtain the right side of the equivalence above.

Although Fine does not explicitly compare his interpretation to Henkin's interpretation of the Henkin quantifier, he raises, in a different context, an objection to the use of Skolem functions. The objection is that Skolem functions, unlike arbitrary objects, cannot handle *multi-dependencies*. He illustrates what he has in mind with three arbitrary objects a, b and c such that c depends on b in *a particular way*, say $c = 2b$, and b depends on a in *another way*, say $b = a^2$.

I take Fine's claim of the impossibility of representing multi-dependencies by Skolem functions to be an oversight. Skolem functions provides already a handy way to handle multi-dependencies in connection with the Henkin quantifier. They also do the job in the present example:

25. $\forall x \left[f(g(x)) = 2g(x) \wedge g(x) = x^2 \right]$

Here$f$ and g are unary Skolem functions.

If we try to express the dependencies in Fine's example with quantifiers, we need three of them, $\forall x, \exists y$ and $\exists z$ such that

(i) $\exists y$ depends on $\forall x$ (in a particular way)

(ii) $\exists z$ depends on $\exists y$ but is independent of $\forall x$ (in another way)

We shall call a prefix of quantifiers $\forall x, \exists y$ and $\exists z$ which obey the constraints (i) and (ii) a *signaling* prefix. There is no way to arrange the quantifiers $\forall x, \exists y$ and $\exists z$ in a linear sequence such that (i) and (ii) are satisfied. For no matter how we do it, we end up with one of the two existential quantifiers depending on the other. We need, again, an alternative notation to express the partial order of the dependence relation. We shall tackle this question in the next section.

Let me point out that there is a first-order sentence which expresses the *particular* dependencies Fine has in mind, namely

26. $\forall x \exists y \exists z (y = x^2 \wedge z = 2y)$

But notice that (26) says that$\exists y$ depends on $\forall x$ and $\exists z$ depends on both $\exists y$ and $\forall x$ such that $y = x^2 \wedge z = 2y$ is true. The Skolem form of (26)

27. $\forall x \left[h(x, g(x)) = 2g(x) \wedge g(x) = x^2 \right]$

is distinct from (25). Nevertheless (27) may be shown to be equivalent to (25).

But this is not true in the general case. What we mean by this will become clearer in the next section. For the moment it is enough to conclude, against Fine, that a generalization of the Skolem function approach handles quite nicely multi-dependencies. But Fine deserves credit for pointing out an interesting example of multi-dependencies of arbitrary objects, which, when expressed with quantifiers, leads, like the Henkin prefix, to greater expressive power than ordinary first-order logic. In the fnal section we will also see that this pattern may be found in natural language examples.

8 Independence-friendly languages

Several logical systems, in addition to Henkin quantifiers, have been introduced to deal with arbitrary patterns of dependence and independence between quantifiers. They include:

- Independence-Friendly Logic (Hintikka and Sandu 1989)
- Dependence Logic (Väänänen 2007)
- Independence Logic (Grädel and Väänänen 2013).

Independence-Friendly logic is an extension of ordinary FOL with quantifiers of the form (Qx/W), where $Q \in \{\exists, \forall\}$ and W is a finite set of variables. In this extension, we have formulas like

$$\begin{aligned}
\varphi_{MP}: &\quad \forall x (\exists y / \{x\})\, x = y \\
\varphi_{sig}: &\quad \forall x \exists y (\exists z / \{x\})(z = 2y \wedge y = x^2) \\
\varphi_{H}: &\quad \forall x \forall z (\exists y / \{z\})(\exists w / \{x, y\}) D(x, y, z, w) \\
\varphi_{\infty}: &\quad \forall x \exists y (\exists z / \{x\})(y \neq c \wedge z = x)
\end{aligned}$$

The quantifier (Qx/W) expresses the fact that Qx depends on all the quantifiers in whose syntactical scope it occurs, except for the quantifiers that bind the variables in the slash set W of which Qx is independent. Thus

- φ_{MP} expresses the statement: For all x there is a y which does not depend on (is independent of) x such that x is identical with y
- φ_{sig} expresses Fine's example of multi-dependencies
- φ_H expresses the Henkin quantifier discussed in the previous section
- φ_∞ is identical to φ_{sig} except for the quantifier-free part.

The most intuitive interpretation of IF sentences is through Skolemization. In the Skolemized form of an IF formula (in negation normal form), every existential quantifier $(\exists y/W)$ is replaced with a new function symbol f whose arguments are all the variables quantified by quantifers in whose syntactical scope $(\exists y/W)$ occurs, minus the variables in W. Thus the Skolem form of an IF formula expresses the notion of functional dependence. Here are the Skolem forms of the sentences listed above:

$$\begin{array}{l}
Sk(\varphi_{MP}) = \forall x(x = c) \\
Sk(\varphi_{sig}) = \forall x \left[f(g(x)) = 2g(x) \wedge g(x) = x^2\right] \\
Sk(\varphi_H) = \forall x \forall z D(x, f(x), z, g(z)) \\
Sk(\varphi_\infty) = \forall x \left[f(x) \neq c \wedge g(f(x)) = x\right]
\end{array}$$

(c is a 0-place function symbol, i.e., an individual constant).

Let us denote the Skolem form of an IF formula φ by $Sk(\varphi)$. It is worth noting that $Sk(\varphi)$ is an ordinary first-order formula in a vocabulary which extends that of φ with new function symbols.

Satisfaction of an IF formula φ in a model $\mathbb{M}$ with respect to a partial assignment s whose domain includes the free variables of φ is then defined as:

- $\mathbb{M}, s \models \varphi$ if and only if there are functions $f_1, ..., f_n$ in M to be the interpretations of the new function symbols of $Sk(\varphi)$ such that

$$\mathbb{M}, s, f_1, ..., f_n \models Sk(\varphi).$$

For φ a sentence, we stipulate:

- φ is true in $\mathbb{M}$ if and only if $M, \varnothing \models \varphi$,

where $\varnothing$ is the empty assignment.

Let $\mathbb{M}$ be a set, say $\mathbb{M}$ is the set of natural numbers. It may be checked that φ_{sig} is true in $\mathbb{M}$ if and only if there are unary functions f and g such that such that for every a in the universe of $\mathbb{M}$, $g(a) = a^2$and $f(g(a)) = 2g(a)$. We thus recover the interpretation (25).

The case of $Sk(\varphi_\infty)$ is more subtle. For any model $\mathbb{M}$, φ_∞ is true in $\mathbb{M}$ if and only if there are two unary functions f and g such that the range of

f is distinct from c and for every a in the universe of $\mathbb{M}$: $g(f(a) = a$. The last condition says that f is injective and the conjunction of the two conditions expresses (Dedekind) infinity. Thus φ_∞ is true in $\mathbb{M}$ if and only if $\mathbb{M}$ is infinite. It is well known that infinity is a property which is not definable in ordinary first-order logic.

We are now in a position to substantiate the claim made at the end of the preceding section. The pattern of dependencies expressed by (i) and (ii) in the preceding section leads to properties which are not first-order definable.

A sensible question to ask is which patterns of dependencies and independencies of quantifiers lead to greater expressive power than ordinary FOL. The answer turns out to be: exactly the patterns discussed in the preceding section, the Henkin prefix and the signaling prefix. Let me quickly review the results.

First a bit of terminology. We say that a quantifier (Qx/X) *depends* on another quantifier (Qy/Y) $(Q \in \{\exists, \forall\})$ whenever (Qx/X) in the syntactical scope of(Qy/Y)

$$...(Qy/Y)...(Qx/X)...$$

and, in addition, $y \notin X$. If (Qx/X) does not depend on (Qy/Y), then we say that it is *independent* of (Qy/Y).

Thus(Qx/X) is independent of (Qy/Y) either when (Qx/X) is not in the scope of (Qy/Y), or, alternatively, (Qx/X) is in the scope of (Qy/Y) but $y \in X$.

Sevenster (2014) shows that there are two prefixes of IF quantifiers which lead to greater expressive (and computational) power than FOL: Henkin and signaling prefixes.

A *Henkin prefix* contains at least 4 quantifiers

$$...\forall x...\forall y...(\exists u/U)...(\exists v/V)...$$

in any order such that:

- $(\exists u/U)$ depends on $\forall x$ and is independent of $\forall y$ and $(\exists v/V)$
- $(\exists v/V)$ depends on $\forall y$ and is independent of $\forall x$ and $(\exists u/U)$.

A *signaling prefix* contains at least three quantifiers

$$...(\forall u/U)...(\exists v/V)...(\exists w/W)...$$

such that:

- $(\exists v/V)$ depends on $(\forall u/U)$
- $(\exists w/W)$ depends on $(\exists v/V)$ but is independent of $(\forall u/U)$.

The idea in a signaling prefix is that the second existential quantifier does not "see" the universal quantifier, but it sees another existential quantifier which, in turn, sees the universal quantifier. We also notice that whereas in Henkin prefixes the existential quantifiers are independent of each other, in signaling prefixes, one of the existential quantifier is dependent on the other, which in turn depends on the universal quantifier.

We saw that our earlier sentence φ_∞ expresses a property, (Dedekind) infinity, which is not first-order definable. In a similar way one can show that there is an IF sentenceφ_{Henkin} which contains a Henkin prefix such that φ_{Henkin} expresses properties which are not first-order definable. Sevenster (2014) shows that every sentential IF prefix which is neither Henkin nor signaling is equivalent to a sentential FOL prefix. This is another way of saying that Henkin and signaling prefixes are the only prefixes which lead to greater expressive (and computational) power than ordinary first-order logic.

9 Signaling sequences in natural language

Fine's example of multi-dependencies is intriguing, for we found out it illustrates a pattern of dependencies of arbitrary objects, which, when expressed in terms of quantifiers, is not first-order definable (in the general case). Signaling sequences can also be found in natural language. King (1991) gives the following example to illustrate the application of arbitrary objects to natural language:

28. Every professor at the university of San Clement$_1$ teaches a large lecture class$_2$. The professor$_1$ does all the grading of the class$_2$. The class$_2$ has a final exam$_3$. The final$_3$ is comprehensive. It$_3$ need not be long, however....

The numerical subscripts indicate the anaphoric relations. We notice that:

(i) "a large lecture class" depends on "Every professor at the university of San Clement",

(ii) "a final exam" depends on "a large lecture class" but is independent of "Every professor at the university of San Clement", etc.

In the symbolism of IF logic we render (28) by

29. $\forall x(P(x) \rightarrow \exists y(C(y) \wedge T(x,y) \wedge Gr(x,y) \wedge (\exists z/\{x\})(E(z) \wedge H(y,z) \wedge ...)))$

where the predicate symbols have a self-explanatory meaning. The Skolem form of (28) eliminates the indefinites (existential quantifiers) by Skolem terms

30. $\forall x(P(x) \rightarrow C(f(x)) \wedge T(x, f(x)) \wedge Gr(x, f(x)) \wedge (E(g(f(x))) \wedge H(f(x), g(f(x))) \wedge ...$

We can still go one level up, get rid of the universal quantifiers and the Skolem terms, and reach the stratosphere of arbitrary objects with their dependence relations inhereted from the argument structures of the Skolem terms:

31. $P(a) \rightarrow C(b) \wedge T(a,b) \wedge Gr(a,b) \wedge E(c) \wedge H(b,c) \dots$

where b depends on a, and c depends on b. If we were to represent the dependence relation in the object language (I take Fine to accept this move), we would end up with

32. $P(a) \to C(b) \wedge T(a,b) \wedge Gr(a,b) \wedge E(c) \wedge H(b,c)) \wedge Dep(a,b) \wedge Dep(b,c) \ldots$

We recall the problem that was our starting point at the beginning of the article: The nesting favours a quantificational interpretation of the indefinites according to which they behave more like any other quantificational NP, e.g. they enter into scopal relations with other quantificational phrases. On the other side, pronominal anaphors create a pressure to interpret the indefinites referentially. In trying to solve this predicament, we notice the benefits of the "ontological ascent".

The introduction of the Skolem functions made possible the elimination of the existential quantifiers in favour of Skolem terms. (30) is adequate, as far as I am concerned, as the representation of (28): it contains only universal quantifiers, which means that we do not need dependency relations (*priority scope*) between quantifiers to get the truth-conditions of (30) right. But (30) achieves only a *quasi-referential* interpretation of indefinites: we still need *binding scope* to get the interpretation of the Skolem terms right. I take it that this is what Steedman had in mind when he pointed out that "unless the pronoun is in the scope of the quantifiers that bind any variables in the Skolem term, it will include a variable that is outside the scope of its binder, and fail to refer." (Steedman 1999, p. 303). To get a full-fledged referential interpretation, we still have to get rid of the binding scope. This is what Fine's framework of arbitrary objects accomplishes. I consider them useful fictions: they have a systematic role to play if and to the extend to which the notion of reference is needed in our best linguistic theories. Independently of this, Fine's work provides in my opinion an alternative and exciting framework for the study of Henkin and signaling prefixes in logic.

References

Avigad, J. and Zach, R. (2016). The epsilon calculus. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2016 edition.

Breckenridge, W. and Magidor, O. (2012). Arbitrary reference. *Philosophical Studies*, 158(3):377–400.

Fine, K. (1985a). Natural deduction and arbitrary objects. *Journal of Philosophical Logic*, 14(1):57–107.

Fine, K. (1985b). *Reasoning with Arbitrary Objects*. Blackwell.

Fine, K. and Tennant, N. (1983). A defence of arbitrary objects. *Aristotelian Society Supplementary Volume*, 57(1):55–89.

Fodor, J. D. and Sag, I. (1982). Referential and quantificational indefinites. *Linguistics and philosophy*, 5(3):355–398.

Geach, P. T. (1962). *Reference and Generality*. Ithaca, NY: Cornell University Press.

Grädel, E. and Väänänen, J. (2013). Dependence and independence. *Studia Logica*, 101:399–410.

Hintikka, J. and Sandu, G. (1989). Informational independence as a semantical phenomenon. In Fenstad et al, J. E., editor, *Logic, Methodology and Philosophy of Science VIII*, pages 571–589. Elsevier Science Publishers B.V.

King, J. C. (1991). Instantial terms, anaphora and arbitrary objects. *Philosophical Studies*, 61(3):239–265.

Kleene, S. C. (1952). *Introduction to Metamathematics*. North Holland.

Reinhart, T. (1997). Quantifier scope: How labor is divided between QR and choice functions. *Linguistics and philosophy*, 20(4):335–397.

Sevenster, M. (2014). Dichotomy result for independence-friendly prefixes of generalized quantifiers. *The Journal of Symbolic Logic*, 79(04):1224–1246.

Slater, B. H. (1986). E-type pronouns and ϵ-terms. *Canadian Journal of Philosophy*, 16(1):27–38.

Steedman, M. (1999). Alternating quantifier scope in CCG. In *Proceedings of the 37th annual meeting of the Association for Computational Linguistics on Computational Linguistics*, pages 301–308. Association for Computational Linguistics.

Tichý, P. (1988). *The Foundations of Frege's Logic*. De Gruyter, Berlin-New York.

Väänänen, J. (2007). *Dependence Logic: A New Approach to Independence Friendly Logic*, volume 70 of *London Mathematical Society Student Texts*. Cambridge University Press.

Winter, Y. (1997). Choice functions and the scopal semantics of indefinites. *Linguistics and Philosophy*, 20:399–467.