

Article

# A Note on a Class of Generalized Parabolic Marcinkiewicz Integrals along Surfaces of Revolution

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**Abstract:** In this article, certain sharp  $L^p$  estimates for a specific class of generalized Marcinkiewicz operators with mixed homogeneity associated to surfaces of revolution are established. By virtue of Yano’s extrapolation argument, beside these estimates, the  $L^p$  boundedness of the aforementioned operators under weaker assumptions on the kernels is confirmed. The obtained results in this article are fundamental extensions and improvements of numerous previously known results on parabolic generalized Marcinkiewicz integrals.

**Keywords:** extrapolation; Triebel–Lizorkin space; generalized Marcinkiewicz; rough kernels

**MSC:** 42B05; 42B20



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## 1. Introduction

Let  $s \geq 2$  be an integer, and let  $\mathbb{S}^{s-1}$  denote the unit sphere in the Euclidean space  $\mathbb{R}^s$ , which is equipped with the normalized Lebesgue surface measure  $d\sigma$ .

For  $j \in \{1, 2, \dots, s\}$ , assume that  $\alpha_j \geq 1$  are fixed numbers. Consider the mapping  $\mathcal{V} : \mathbb{R}_+ \times \mathbb{R}^s \rightarrow \mathbb{R}$  by  $\mathcal{V}(\kappa, \omega) = \sum_{j=1}^s \frac{\omega_j^2}{\kappa^{2\alpha_j}}$  with  $\omega = (\omega_1, \omega_2, \dots, \omega_s) \in \mathbb{R}^s$ . Then, for a fixed  $\omega \in \mathbb{R}^s$ , it is easy to see that  $\mathcal{V}(\kappa, \omega)$  is strictly decreasing mapping in  $\kappa > 0$ . So, the equation  $\mathcal{V}(\kappa, \omega) = 1$  has a unique solution, denoted by  $\kappa(\omega) \equiv \kappa$ . The authors of [1] proved that  $(\mathbb{R}^s, \kappa)$  is a metric space and called it the mixed homogeneity space related to  $\{\alpha_j\}_{j=1}^s$ . For  $\kappa > 0$ , let  $D_\kappa$  denote the diagonal  $s \times s$  matrix

$$D_\kappa = \begin{bmatrix} \kappa^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \kappa^{\alpha_s} \end{bmatrix}.$$

The change of variables regarding the space  $(\mathbb{R}^s, \kappa)$  is given as the following:

$$\begin{aligned} \omega_1 &= \kappa^{\alpha_1} \cos \vartheta_1 \dots \cos \vartheta_{n-2} \cos \vartheta_{s-1}, \\ \omega_2 &= \kappa^{\alpha_2} \cos \vartheta_1 \dots \cos \vartheta_{s-2} \sin \vartheta_{s-1}, \\ &\vdots \\ \omega_{s-1} &= \kappa^{\alpha_{s-1}} \cos \vartheta_1 \sin \vartheta_2, \\ \omega_s &= \kappa^{\alpha_s} \sin \vartheta_1. \end{aligned}$$

This leads to  $d\omega = \kappa^{\alpha-1} J(v) dk d\sigma(v)$ , where  $\kappa^{\alpha-1} J(v)$  is the Jacobian of the above transformations,

$$v = D_{\kappa^{-1}} \omega \in \mathbb{S}^{s-1}, \quad \alpha = \sum_{j=1}^s \alpha_j, \quad \text{and} \quad J(v) = \sum_{j=1}^s \alpha_j (v_j)^2.$$

It is worth mentioning that the authors of [1] showed that  $J(v)$  belongs to the space  $C^\infty(\mathbb{S}^{s-1})$  and also showed that there is a real number  $M$  with

$$M \geq J(v) \geq 1, \quad \forall v \in \mathbb{S}^{s-1}.$$

For  $\lambda = \lambda_1 + i\lambda_2$  ( $\lambda_1 \in \mathbb{R}$  and  $\lambda_2 \in \mathbb{R}_+$ ), we define the kernel  $K_{\mathcal{U},h}$  on  $\mathbb{R}^s$  by

$$K_{\mathcal{U},h}(\omega) = \frac{\mathcal{U}(\omega)h(\kappa(\omega))}{\kappa(\omega)^{\alpha-\lambda}},$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{C}$  is a measurable function and  $\mathcal{U}$  is a function belonging to  $L^1(\mathbb{S}^{s-1})$  that satisfies the conditions

$$\mathcal{U}(D_\kappa\omega) = \mathcal{U}(\omega), \quad \forall \kappa > 0, \tag{1}$$

and

$$\int_{\mathbb{S}^{s-1}} \mathcal{U}(v)J(v)d\sigma(v) = 0. \tag{2}$$

For a suitable function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we consider the generalized parabolic Marcinkiewicz operator

$$\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g)(\omega, \omega_{s+1}) = \left( \int_{\mathbb{R}_+} \left| t^{-\lambda} \int_{\kappa(u) \leq t} g(\omega - u, \omega_{s+1} - \Phi(\kappa(u)))K_{\mathcal{U},h}(u)du \right|^{\tau} \frac{dt}{t} \right)^{1/\tau}, \tag{3}$$

where  $g \in \mathcal{S}(\mathbb{R}^{s+1})$  and  $\tau > 1$ .

It is obvious that when  $\alpha_1 = \alpha_2 = \dots = \alpha_s = 1$ , then we have  $\alpha = s$  and  $\kappa(\omega) = |\omega|$ . In this instance, we denote  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  by  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$ . Further, when  $\tau = 2$ ,  $\Phi(\kappa) = \kappa$  and  $h \equiv 1$ , then  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$ , denoted by  $\mathfrak{M}_{\mathcal{U}}$ , reduces to the classical parametric Marcinkiewicz integral operator. Historically, the integral operator  $\mathfrak{M}_{\mathcal{U}}$  was introduced by Stein in [2] where he established the  $L^p$  boundedness of  $\mathfrak{M}_{\mathcal{U}}$  for all  $p \in (1, 2]$  whenever  $\lambda = 1$  and  $\mathcal{U}$  belongs to the space  $Lip_\gamma(\mathbb{S}^{s-1})$  for some  $0 < \gamma \leq 1$ . Afterwards, the above result was improved by Hörmander in [3]. Indeed, he confirmed the  $L^p$  boundedness of  $\mathfrak{M}_{\mathcal{U}}$  for all  $p \in (1, \infty)$  under the conditions  $\lambda > 0$  and  $\mathcal{U} \in Lip_\gamma(\mathbb{S}^{s-1})$  for some  $0 < \gamma \leq 1$ . Later on, the authors of [4] extended and improved the result of Stein. In fact, they proved that  $\mathfrak{M}_{\mathcal{U}}$  is bounded on  $L^p(\mathbb{R}^s)$  for all  $p \in (1, \infty)$  if  $\lambda = 1$  and  $\mathcal{U} \in C^1(\mathbb{S}^{s-1})$ . Subsequently, the  $L^p$  boundedness of the  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  under various assumptions on the kernels has attracted a considerable amount of attention from many mathematicians. For instance, Walsh in [5] obtained the  $L^2$  boundedness of the operator  $\mathfrak{M}_{\mathcal{U}}$  provided that  $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{s-1})$  and  $\lambda = 1$ . Furthermore, he found that  $\mathfrak{M}_{\mathcal{U}}$  will lose the  $L^2$  boundedness when the assumption  $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{s-1})$  is replaced by  $\mathcal{U} \in L(\log L)^v(\mathbb{S}^{s-1})$  for any  $v \in (0, 1/2)$ . On the other hand, the  $L^p$  ( $1 < p < \infty$ ) boundedness for  $\mathfrak{M}_{\mathcal{U}}$  was proved in [6] whenever  $\lambda = 1$  and  $\mathcal{U}$  belongs to the block space  $B_q^{(0,-1/2)}(\mathbb{S}^{s-1})$  for some  $q > 1$ . In the same article, the authors verified that  $-1/2$  in  $B_q^{(0,-1/2)}(\mathbb{S}^{s-1})$  cannot be substituted by any number in  $(-1, -1/2)$ , meaning that  $\mathfrak{M}_{\mathcal{U}}$  is still bounded on  $L^2(\mathbb{R}^s)$ .

Recently, Ali in [7] used Yano’s argument to improve and extend all the above-cited results. In fact, he proved that if  $\mathcal{U} \in L(\log L)^{1/2}(\mathbb{S}^{s-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{s-1})$ ,  $h \in Y_\mu(\mathbb{R}_+)$  for some  $\mu > 1$ , and  $\Phi$  is in  $C^2([0, \infty))$ , an increasing and convex function with  $\Phi(0) = 0$ , then

the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(2),c}$  is bounded on  $L^p(\mathbb{R}^{s+1})$  for all  $|1/p - 1/2| < \min\{1/2, 1/\mu'\}$ , where  $Y_\mu(\mathbb{R}_+)$  denotes the class of all measurable functions  $h : \mathbb{R}_+ \rightarrow \mathbb{C}$  that satisfy

$$\|h\|_{Y_\mu(\mathbb{R}_+)} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(\kappa)|^\mu \frac{d\kappa}{\kappa} \right)^{1/\mu} < \infty.$$

For further information about the significance, development, and recent advances of the discussion about the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(2),c}$ , the readers are referred to [8–16] and the references therein.

Although several problems regarding the boundedness of  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(2),c}$  remain open, the study of the boundedness of the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  has been investigated by many authors. For example, the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  was introduced in [17], where the authors showed that when  $\Phi(\kappa) = \kappa, h \equiv 1, \mathcal{U} \in L^q(\mathbb{S}^{s-1})$  with  $q > 1$ , and  $1 < \tau < \infty$ , then for all  $1 < p < \infty$ ,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}. \tag{4}$$

Later on, this result was improved by Le in [18]. Precisely, he proved that the inequality (4) holds for all  $p \in (1, \infty)$  if  $\Phi(\kappa) = \kappa, \mathcal{U} \in L(\log L)(\mathbb{S}^{s-1})$  and  $h \in \Gamma_{\max\{\tau', 2\}}(\mathbb{R}_+)$ , where  $\tau'$  denotes the conjugate index of  $\tau$ .

Recently, the authors of [19] confirmed that if  $\mathcal{U} \in B_q^{(0, -1/\tau')}(\mathbb{S}^{s-1}) \cup L(\log L)^{1/\tau}(\mathbb{S}^{s-1}), \Phi(\kappa) = \kappa$  and  $h \in Y_\mu(\mathbb{R}_+)$  with  $\mu \in (1, 2]$ , then the boundedness of  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  is satisfied for all  $p \in [\tau, \infty)$ . Further, they proved that if the condition  $\mu \in (1, 2]$  is replaced by  $\mu > 2$ , then the boundedness of  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  is satisfied for all  $p \in (1, \tau)$  if  $2 < \mu < \infty$  and  $\mu \leq \tau'$ , and also for all  $p \in (\mu', \infty)$  if  $2 < \mu \leq \infty$  and  $\mu > \tau'$ . Very recently, the authors of [20] extended the results in [19]. In fact, they confirmed that the above results are true not only for the case  $\Phi(\kappa) = \kappa$ , but also when  $\Phi$  belongs to the class **I** or the class **D**, which were introduced in [21]. Precisely, the class **I** is the collection of all  $C^1$  functions  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  that are non-negative and satisfy the following:

- (i)  $\Phi'$  is monotone and  $\Phi(\kappa) > 0$  for all  $\kappa \in \mathbb{R}_+$ ;
- (ii)  $\Phi(2\kappa) \geq M_1\Phi(\kappa)$  for some fixed  $M_1 > 1$  and  $\Phi(2\kappa) \leq M_2\Phi(\kappa)$  with  $M_2 \geq M_1$ ;
- (iii)  $\kappa\Phi'(\kappa) \geq M_3\Phi(\kappa)$  on  $\mathbb{R}_+$  for some  $0 < M_3 < \log(M_2)$ .

In addition, **D** is the class of all  $C^1$  functions  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  that are non-negative and satisfy the following:

- (i)  $\Phi'$  is monotone and  $\Phi(\kappa) < 0$  for all  $\kappa \in \mathbb{R}_+$ ;
- (ii)  $\Phi(\kappa) \geq M_1\Phi(2\kappa)$  for some fixed  $M_1 > 1$  and  $\Phi(\kappa) \leq M_2\Phi(2\kappa)$  with  $M_2 \geq M_1$ ;
- (iii)  $|\kappa\Phi'(\kappa)| \geq M_3\Phi(\kappa)$  on  $\mathbb{R}_+$  for some  $0 < M_3 < \log(M_2)$ .

For recent advances in the study of the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$ , we refer the readers to consult [19,20,22–25] and the references therein.

Let us recall the definitions of some spaces that are related to this work. For  $m > 0$ , we let  $\Delta_m(\mathbb{R}_+)$  denote the collection of all functions  $h$  that are measurable on  $\mathbb{R}_+$  such that

$$\Lambda_m(h) = \sum_{n=1}^{\infty} 2^n n^m d_n(h) < \infty,$$

where  $d_n(h) = \sup_{j \in \mathbb{Z}} 2^{-j} |U(j, m)|$  with  $U(j, m) = \{\kappa \in (2^j, 2^{j+1}] : 2^{m-1} < |h(\kappa)| \leq 2^m\}$  for  $m \geq 2$  and  $U(j, 1) = \{\kappa \in (2^j, 2^{j+1}] : |h(\kappa)| \leq 2\}$ .

It is clear that  $Y_m(\mathbb{R}_+) \subset \Delta_\nu(\mathbb{R}_+)$  for any  $m \geq 1$  and  $\nu > 0$ .

In this work, let  $L(\log L)^m(\mathbb{S}^{s-1})$  (for  $m > 0$ ) denote the class of all functions  $\mathcal{U}$  that are measurable on  $\mathbb{S}^{s-1}$  such that

$$\|\mathcal{U}\|_{L(\log L)^m(\mathbb{S}^{s-1})} = \int_{\mathbb{S}^{s-1}} |\mathcal{U}(v)| \log^m(|\mathcal{U}(v)| + 2) d\sigma(v) < \infty.$$

In addition, let  $B_q^{(0,v)}(\mathbb{S}^{s-1})$  (with  $v > -1$  and  $q > 1$ ) denote the block space that was introduced in [26]. Furthermore,  $H_e^{\tau,p}$  denotes the homogeneous Triebel–Lizorkin space, which is defined as follows: assume that  $\theta \in \mathbb{R}^s$  and  $\mathcal{W} \in C_0^\infty(\mathbb{R}^s)$  is a radial function satisfying the following:

- (a)  $0 \leq \mathcal{W} \leq 1$ ;
- (b)  $\text{supp } \mathcal{W} \subset \left\{ \theta : \frac{1}{2} \leq |\theta| \leq 2 \right\}$ ;
- (c)  $\mathcal{W}(\theta) \geq M > 0$  if  $\frac{3}{5} \leq |\theta| \leq \frac{5}{3}$ ;
- (d)  $\sum_{n \in \mathbb{Z}} \mathcal{W}(2^{-n}\theta) = 1$  ( $\theta \neq 0$ ).

For  $\varepsilon \in \mathbb{R}$  and  $1 < p, \tau \leq \infty$  with  $(p \neq \infty)$ ,

$$H_e^{\tau,p}(\mathbb{R}^s) = \left\{ g \in \mathcal{S}'(\mathbb{R}^s) : \|g\|_{H_e^{\tau,p}(\mathbb{R}^s)} = \left\| \left( \sum_{n \in \mathbb{Z}} 2^{n\varepsilon\tau} |\mathbf{F}_n * g|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^s)} < \infty \right\},$$

where  $\mathcal{S}'$  denotes the tempered distribution class on  $\mathbb{R}^s$  and  $\widehat{\mathbf{F}}_n(\theta) = \mathcal{W}(2^{-n}\theta)$  for  $n \in \mathbb{Z}$ .

The generalized parabolic Marcinkiewicz operator  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  was recently introduced in [27] under some weak conditions on the kernels. This result was studied in [28] under some weaker conditions on the kernels only for the case  $\tau = 2$ . In view of the result in [19] on the boundedness of classical generalized parametric Marcinkiewicz  $\mathfrak{M}_{\mathcal{U},h,\Phi}^{(\tau),c}$  and of the result in [28] on the parabolic Marcinkiewicz  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(2)}$ , a natural question arises: is the boundedness of the parabolic operator  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  satisfied under the same conditions assumed in [19] while replacing the condition  $\tau = 2$  by a weaker condition  $\tau > 1$  and when  $\Phi$  belongs to **I** or **D**?

In the next section, we shall give an affirmative answer to this question.

## 2. Statement of results

We devote this section to presenting the main results of this article. Indeed, they are formulated as follows.

**Theorem 1.** *Let  $\mathcal{U}$  belong to the space  $L^q(\mathbb{S}^{s-1})$  and  $h$  belong to the space  $Y_\mu(\mathbb{R}_+)$  with  $q, \mu \in (1, 2]$ . Assume that  $\Phi$  lies in **I** or **D**. Then, there is a constant  $C_p > 0$  such that*

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \frac{1}{(q-1)(\mu-1)} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})} \text{ if } 1 < p < \tau, \tag{5}$$

and

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \frac{1}{((q-1)(\mu-1))^{1/\tau}} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})} \text{ if } \tau \leq p < \infty. \tag{6}$$

The constant  $C_p$  is independent of  $\mathcal{U}, h, \Phi, q$ , and  $\mu$ .

**Theorem 2.** *Suppose that  $\mathcal{U}$  and  $\Phi$  are given as in Theorem 1. Let  $h$  belong to the space  $Y_\mu(\mathbb{R}_+)$  for some  $\mu > 2$ . Then, the inequality*

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \frac{1}{(q-1)^{1/\tau}} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}$$

holds for all  $p \in (1, \tau)$  if  $2 < \mu < \infty$  and  $\tau \leq \mu'$  and also holds for all  $p \in (\mu', \infty)$  if  $2 < \mu \leq \infty$  and  $\tau > \mu'$ .

By utilizing the conclusions of Theorems 1 and 2, Yano’s extrapolation argument, and the same method used in [12,29,30], we obtain the following results.

**Theorem 3.** Let  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  be given by (3) and  $\Phi$  be given as in Theorem 1.

(i) If  $\mathcal{U} \in B_q^{(0,-1/\tau')}(\mathbb{S}^{s-1})$  for some  $q > 1$  and  $h \in \Delta_{1/\tau}(\mathbb{R}_+)$ , then for any  $p \in [\tau, \infty)$ ,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{B_q^{(0,-1/\tau')}(\mathbb{S}^{s-1})} + 1 \right) (\Lambda_{1/\tau}(h) + 1) \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}.$$

(ii) If  $\mathcal{U} \in B_q^{(0,0)}(\mathbb{S}^{s-1})$  with  $q > 1$  and  $h \in \Delta_1(\mathbb{R}_+)$ , then for any  $p \in (1, \tau)$ ,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{B_q^{(0,0)}(\mathbb{S}^{s-1})} + 1 \right) (\Lambda_1(h) + 1) \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}.$$

(iii) If  $\mathcal{U} \in L(\log L)^{1/\tau}(\mathbb{S}^{s-1})$  and  $h \in \Delta_{1/\tau}(\mathbb{R}_+)$ , then for any  $p \in [\tau, \infty)$ ,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{L(\log L)^{1/\tau}(\mathbb{S}^{s-1})} + 1 \right) (\Lambda_{1/\tau}(h) + 1) \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}.$$

(iv) If  $\mathcal{U} \in L(\log L)(\mathbb{S}^{s-1})$  and  $h \in \Delta_1(\mathbb{R}_+)$ , then for any  $p \in (1, \tau)$ ,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{L(\log L)(\mathbb{S}^{s-1})} + 1 \right) (\Lambda_1(h) + 1) \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}.$$

**Theorem 4.** Assume that  $\Phi$  is given as in Theorem 1 and  $h \in Y_\mu(\mathbb{R}_+)$  for some  $\mu > 2$ .

(i) If  $\mathcal{U} \in B_q^{(0,-1/\tau')}(\mathbb{S}^{s-1})$  for some  $q > 1$ ; then,

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{B_q^{(0,-1/\tau')}(\mathbb{S}^{s-1})} + 1 \right) \|h\|_{Y_\mu(\mathbb{R}_+)} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}$$

holds for all  $p \in (1, \tau)$  if  $2 < \mu < \infty$  and  $\mu \leq \tau'$  and also for all  $p \in (\mu', \infty)$  if  $2 < \mu \leq \infty$  and  $\mu > \tau'$ .

(ii) If  $\mathcal{U} \in L(\log L)^{1/\tau}(\mathbb{S}^{s-1})$ , then

$$\left\| \mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \left( \|\mathcal{U}\|_{L(\log L)^{1/\tau}(\mathbb{S}^{s-1})} + 1 \right) \|h\|_{Y_\mu(\mathbb{R}_+)} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}$$

holds for all  $p \in (1, \tau)$  if  $2 < \mu < \infty$  and  $\mu \leq \tau'$ , and also for all  $p \in (\mu', \infty)$  if  $2 < \mu \leq \infty$  and  $\mu > \tau'$ .

Here and henceforward, the letter C refers to a positive number whose value does not depend on the primary variables and also that is not necessary the same at each occurrence.

### 3. Some Auxiliary Lemmas

We devote this section to establishing some preliminary lemmas that are needed to prove our main results. Let us begin by introducing some notations. Let  $b \geq 2$ . Define the family of measures  $\{\sigma_{K_{\mathcal{U},h,\Phi,t}} := \sigma_t : t \in \mathbb{R}_+\}$  and the corresponding maximal operators  $\sigma_h^*$  and  $\mathcal{M}_{h,b}$  on  $\mathbb{R}^{s+1}$  by

$$\int_{\mathbb{R}^{s+1}} g d\sigma_t = t^{-\lambda} \int_{t/2 \leq \kappa(\omega) \leq t} g(\omega, \Phi(\kappa(\omega))) K_{\mathcal{U},h}(\omega) d\omega,$$

$$\sigma^*(g) = \sup_{t \in \mathbb{R}_+} |\sigma_t * g|,$$

and

$$\mathcal{M}_{h,b}(g) = \sup_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * g| \frac{dt}{t},$$

where  $|\sigma_t|$  is defined in the same way as  $\sigma_t$ , but replacing  $\mathcal{U}h$  by  $|\mathcal{U}h|$ .

The following two lemmas play a great role in the proofs of Theorems 1 and 2. They can be established by following the exact procedure utilized in [31] (with a very simple minor modification).

For simplicity, we let  $\mathcal{G}(h, \mathcal{U}) = \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})}$ .

**Lemma 1.** *Let  $b \geq 2$ ,  $\mathcal{U} \in L^q(\mathbb{S}^{s-1})$  and  $h \in Y_\mu(\mathbb{R}_+)$  for some  $q, \mu > 1$ . Suppose that  $\Phi$  belongs to **I** or **D**. Then, there are constants  $\delta$  and  $C$  with  $0 < \delta \leq \min\{\frac{1}{2}, \frac{m}{2q'}, \frac{m}{2\alpha}\}$  such that for all  $j \in \mathbb{Z}$ ,*

$$\|\sigma_t\| \leq C\mathcal{G}(h, \mathcal{U})$$

and

$$\int_{b^j}^{b^{j+1}} |\hat{\sigma}_t(\xi, \xi_{s+1})|^2 \frac{dt}{t} \leq C(\ln b)\mathcal{G}^2(h, \mathcal{U}) \min\{|D_{b^j}\xi|^{\frac{2\delta}{m(\ln b)}}, |D_{b^j}\xi|^{-\frac{2\delta}{m(\ln b)}}\},$$

where  $\|\sigma_t\|$  is the total variation of  $\sigma_t$  and  $m$  is denoted to be the distinct numbers of  $\alpha_j$ .

**Lemma 2.** *Suppose that  $b, \mathcal{U}, h$ , and  $\Phi$  are given as in Lemma 1. Then, for  $1 < p < \infty$ , there exists  $C_p > 0$  such that inequalities*

$$\|\sigma^*(g)\|_{L^p(\mathbb{R}^{s+1})} \leq C_p(\ln b)\mathcal{G}(h, \mathcal{U})\|g\|_{L^p(\mathbb{R}^{s+1})}, \tag{7}$$

and

$$\|\mathcal{M}_{h,b}(g)\|_{L^p(\mathbb{R}^{s+1})} \leq C_p(\ln b)\mathcal{G}(h, \mathcal{U})\|g\|_{L^p(\mathbb{R}^{s+1})} \tag{8}$$

hold for all  $g \in L^p(\mathbb{R}^{s+1})$ .

One of the key tools in proving our main results is Plancherel’s Theorem, which states that  $\|g\|_{L^2(\mathbb{R}^{s+1})} = C\|\hat{g}\|_{L^2(\mathbb{R}^{s+1})}$ . A significant step towards handling our main results is to prove the following:

**Lemma 3.** *Let  $b \geq 2$  and  $\Phi$  be in **I** or **D**. Assume that  $\mathcal{U} \in L^q(\mathbb{S}^{s-1})$  for some  $1 < q \leq 2$  and  $h \in Y_\mu(\mathbb{R}_+)$  for some  $1 < \mu \leq 2$ . Then, there is  $C > 0$  such that for arbitrary functions  $\{\mathcal{A}_j(\cdot), j \in \mathbb{Z}\}$  on  $\mathbb{R}^{s+1}$ , nd we have*

$$\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \leq C(\ln b)^{1/\tau} \mathcal{G}(h, \mathcal{U}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \tag{9}$$

for  $1 < \tau \leq p < \infty$ , and

$$\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \leq C(\ln b)\mathcal{G}(h, \mathcal{U}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \tag{10}$$

for  $1 < p < \tau$ .

**Proof.** Firstly, we prove (9). For a fixed  $p \in [\tau, \infty)$ , we find, by duality, that there exists a non-negative function  $\phi \in L^{(p/\tau)'}(\mathbb{R}^{s+1})$  that satisfies  $\|\phi\|_{L^{(p/\tau)' }(\mathbb{R}^{s+1})} \leq 1$  and

$$\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}^\tau = \int_{\mathbb{R}^{s+1}} \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j(\omega, \omega_{s+1})|^\tau \frac{dt}{t} \times \phi(\omega, \omega_{s+1}) d\omega d\omega_{s+1}. \tag{11}$$

Thanks to Hölder’s inequality, we deduce that

$$\begin{aligned} |\sigma_t * \mathcal{A}_j(\omega, \omega_{s+1})|^\tau &\leq C \|h\|_{Y_1(\mathbb{R}_+)}^{(\tau/\tau')} \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau/\tau')} \\ &\times \int_{t/2}^t \int_{\mathbb{S}^{s-1}} J(v) |\mathcal{A}_j(\omega - D_\kappa v, \omega_{s+1} - \Phi(\kappa))|^\tau |\mathcal{U}(v)| |h(\kappa)| \frac{d\kappa}{\kappa}. \end{aligned} \tag{12}$$

Hence, the inequalities (11) and (12) together with Hölder’s inequality lead to

$$\begin{aligned} &\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}^\tau \\ &\leq C \|h\|_{Y_1(\mathbb{R}_+)}^{(\tau/\tau')} \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau/\tau')} \int_{\mathbb{R}^{s+1}} \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j(\omega, \omega_{s+1})|^\tau \right) \mathcal{M}_{|h|, b} \bar{\phi}(-\omega, \omega_{s+1}) d\omega d\omega_{s+1} \\ &\leq C \|h\|_{Y_1(\mathbb{R}_+)}^{(\tau/\tau')} \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau/\tau')} \left\| \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right\|_{L^{(p/\tau)' }(\mathbb{R}^{s+1})} \left\| \mathcal{M}_{|h|, b}(\bar{\phi}) \right\|_{L^{(p/\tau)' }(\mathbb{R}^{s+1})}, \end{aligned}$$

where  $\bar{\phi}(-\omega, \omega_{s+1}) = \phi(\omega, \omega_{s+1})$ . Therefore, by Lemma 2, we find that

$$\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \leq C (\ln b)^{1/\tau} \mathcal{G}(h, \mathcal{U}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \tag{13}$$

for  $\tau < p < \infty$ . Now, let us prove (9) for the case  $p = \tau$ . It is clear that, by using (12) and Hölder’s inequality,

$$\begin{aligned} &\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}^\tau \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau/\tau')} \|h\|_{Y_1(\mathbb{R}_+)}^{(\tau/\tau')} \\ &\times \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{s+1}} \int_{b^j}^{b^{j+1}} \int_{t/2}^t \int_{\mathbb{S}^{s-1}} J(v) |\mathcal{A}_j(\omega - D_\kappa v, \omega_{s+1} - \Phi(\kappa))|^\tau |\mathcal{U}(v)| |h(\kappa)| d\sigma(v) \frac{d\kappa}{\kappa} \frac{dt}{t} d\omega d\omega_{s+1} \\ &\leq C (\ln b) \|h\|_{Y_1(\mathbb{R}_+)}^{(\tau/\tau') + 1} \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau/\tau') + 1} \int_{\mathbb{R}^{s+1}} \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j(\omega, \omega_{s+1})|^\tau \right) d\omega d\omega_{s+1}. \end{aligned}$$

Therefore, the inequality (9) holds for the case  $p = \tau$ . Let us prove (10) for the case  $1 < p < \tau$ . Thanks to the duality, we deduce that a set of functions  $\{\mathcal{F}_j(\omega, \omega_{s+1}, t)\}$  defined on  $\mathbb{R}^{s+1} \times \mathbb{R}_+$  exists such that  $\left\| \left\| \mathcal{F}_j \right\|_{L^{\tau'}([b^j, b^{j+1}], \frac{dt}{t})} \right\|_{\tau'} \left\| \right\|_{L^{p'}(\mathbb{R}^{s+1})} \leq 1$  and

$$\begin{aligned} & \left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} = \int_{\mathbb{R}^{s+1}} \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} (\sigma_t * \mathcal{A}_j(\omega, \omega_{s+1})) \mathcal{F}_j(\omega, \omega_{s+1}, t) \frac{dt}{t} d\omega d\omega_{s+1} \\ & \leq C((\ln b)^{1/\tau} \|(\Psi(\mathcal{F}))^{1/\tau'}\|_{L^{p'}(\mathbb{R}^{s+1})}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}, \end{aligned} \tag{14}$$

where

$$\Psi(\mathcal{F})(\omega, \omega_{s+1}) = \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \overline{\mathcal{F}}_j(\omega, \omega_{s+1}, t)|^{\tau'} \frac{dt}{t} \text{ and } \overline{\mathcal{F}}_j(\omega, \omega_{s+1}, t) = \mathcal{F}_j(-\omega, \omega_{s+1}, t).$$

As  $p' > \tau'$ , there exists a function  $r \in L^{(p'/\tau)'}(\mathbb{R}^{s+1})$  satisfies

$$\|\Psi(\mathcal{F})\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{s+1}} \int_{b^j}^{b^{j+1}} |\sigma_t * \overline{\mathcal{F}}_j(\omega, \omega_{s+1}, t)|^{\tau'} \frac{dt}{t} r(\omega, \omega_{s+1}) d\omega d\omega_{s+1}. \tag{15}$$

Thus, by employing the same procedure as above, we obtain

$$\begin{aligned} \|\Psi(\mathcal{F})\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} & \leq C \|h\|_{Y_\mu(\mathbb{R}_+)}^{(\tau'/\tau)} \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)} \\ & \quad \times \|\sigma^*(\bar{r})\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} \left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\mathcal{F}_j(\cdot, \cdot, t)|^{\tau'} \frac{dt}{t} \right) \right\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} \\ & \leq C(\ln b) \|h\|_{Y_\mu(\mathbb{R}_+)}^{(\tau'/\tau)+1} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})}^{(\tau'/\tau)+1} \|\bar{r}\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})}' \end{aligned} \tag{16}$$

where  $\bar{r}(\omega, \omega_{s+1}) = r(-\omega, \omega_{s+1})$ . Consequently, by using (14) and (16), the inequality (10) is satisfied. So, the proof of Lemma 3 is complete.  $\square$

In the same manner, we get the following lemma.

**Lemma 4.** Suppose that  $b, \Phi$ , and  $\mathcal{U}$  are given as in Lemma 3, and suppose that  $h \in Y_\mu(\mathbb{R}_+)$  for some  $2 \leq \mu < \infty$ . Then, for arbitrary functions  $\{\mathcal{A}_j(\cdot), j \in \mathbb{Z}\}$  on  $\mathbb{R}^{s+1}$ , the inequality

$$\left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \leq C(\ln b)^{1/\tau} \mathcal{G}(h, \mathcal{U}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \tag{17}$$

holds for any  $p \in (1, \tau)$  if  $\tau \leq \mu'$ , and for any  $p \in (\mu', \infty)$  if  $\tau > \mu'$ .

**Proof.** Let us prove (17) for the case  $p \in (1, \tau)$  with  $\tau \leq \mu'$ . By duality, there are functions  $\{\mathcal{H}_j(\omega, \omega_{s+1}, t)\}$  that are defined on  $\mathbb{R}^{s+1} \times \mathbb{R}_+$  such that

$$\left\| \|\mathcal{H}_j\|_{L^{\tau'}([b^j, b^{j+1}], \frac{dt}{t})} \right\|_{L^{\tau'}(\mathbb{R}^{s+1})} \leq 1 \text{ and}$$

$$\begin{aligned} & \left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^\tau \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} = \int_{\mathbb{R}^{s+1}} \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} (\sigma_t * \mathcal{A}_j(\omega, \omega_{s+1})) \mathcal{H}_j(\omega, \omega_{s+1}, t) \frac{dt}{t} d\omega d\omega_{s+1} \\ & \leq C((\ln b)^{1/\tau} \|(\Omega(\mathcal{H}))^{1/\tau'}\|_{L^{p'}(\mathbb{R}^{s+1})}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}, \end{aligned} \tag{18}$$



where

$$\Omega(\mathcal{H})(\omega, \omega_{s+1}) = \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{H}_j(\omega, \omega_{s+1}, t)|^{\tau'} \frac{dt}{t}.$$

As  $1 < \tau \leq \mu' \leq \mu$ , then Hölder’s inequality leads to

$$\begin{aligned} |\sigma_t * \mathcal{H}_j(\omega, \omega_{s+1}, t)|^{\tau'} &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)} \|h\|_{Y_\mu(\mathbf{R}_+)}^{\tau'} \int_{b^j}^{b^{j+1}} \int_{\mathbb{S}^{s-1}} |\mathcal{U}(\omega)| \\ &\quad \times |\mathcal{H}_j(\omega - D_\kappa v, \omega_{s+1} - \Phi(\kappa), t)|^{\tau'} d\sigma(v) \frac{dt}{t}. \end{aligned} \tag{19}$$

Again, as  $p' > \tau'$ , we deduce that there exists a function  $u$  belonging to  $L^{(p'/\tau)'}(\mathbb{R}^{s+1})$  such that

$$\left\| (\Omega(\mathcal{H}))^{1/\tau'} \right\|_{L^{p'}(\mathbb{R}^{s+1})}^{\tau'} = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{s+1}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{H}_j(\omega, \omega_{s+1}, t)|^{\tau'} \frac{dt}{t} u(\omega, \omega_{s+1}) d\omega d\omega_{s+1}.$$

So, by using Lemma 2, Hölder’s inequality, and the inequality (19), we obtain

$$\begin{aligned} \left\| (\Omega(\mathcal{H}))^{1/\tau'} \right\|_{L^{p'}(\mathbb{R}^{s+1})}^{\tau'} &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)} \|h\|_{Y_\mu(\mathbf{R}_+)}^{\tau'} \|\sigma^*(u)\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} \\ &\quad \times \left\| \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\mathcal{H}_j(\cdot, \cdot, t)|^{\tau'} \frac{dt}{t} \right\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})} \\ &\leq C_p \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)+1} \|h\|_{Y_\mu(\mathbf{R}_+)}^{\tau'} \|u\|_{L^{(p'/\tau)'}(\mathbb{R}^{s+1})}. \end{aligned} \tag{20}$$

Consequently, by the last inequality and (18), we get (17) for any  $p \in (1, \tau)$  with  $\mu \leq \tau'$ .

On the other hand, to satisfy (17) for the case  $p \in (\mu', \infty)$  with  $\mu > \tau'$ , we employ the arguments employed in [19]. By invoking Lemma 2, we find that

$$\left\| \sup_{j \in \mathbb{Z}} \sup_{t \in [1, b]} |\sigma_{b^j t} * \mathcal{A}_j| \right\|_{L^p(\mathbb{R}^{s+1})} \leq \left\| \sigma^* \left( \sup_{j \in \mathbb{Z}} |\mathcal{A}_j| \right) \right\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \mathcal{G}(h, \mathcal{U}) \left\| \sup_{j \in \mathbb{Z}} |\mathcal{A}_j| \right\|_{L^p(\mathbb{R}^{s+1})} \tag{21}$$

for all  $\mu' < p < \infty$  with  $\mu \geq 2$ . This leads to

$$\left\| \left\| \sigma_{b^j t} * \mathcal{A}_j \right\|_{L^\infty([1, b], \frac{dt}{t})} \right\|_{L^\infty(\mathbb{Z})} \Big\|_{L^p(\mathbb{R}^{s+1})} \leq C_p \mathcal{G}(h, \mathcal{U}) \left\| \left\| \mathcal{A}_j \right\|_{L^\infty(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^{s+1})}. \tag{22}$$

Thanks again to duality, we deduce that there exists  $\rho \in L^{(p/\mu)'}(\mathbb{R}^{s+1})$  such that  $\|\rho\|_{L^{(p/\mu)'}(\mathbb{R}^{s+1})} \leq 1$  and

$$\begin{aligned}
 & \left\| \left( \sum_{j \in \mathbb{Z}} \int_1^b |\sigma_{b^j t} * \mathcal{A}_j|^{\tau} \frac{dt}{t} \right)^{1/\tau'} \right\|_{L^p(\mathbb{R}^{s+1})}^{\tau'} = \int_{\mathbb{R}^{s+1}} \sum_{j \in \mathbb{Z}} \int_1^b |\sigma_t * \mathcal{A}_j(\omega, \omega_{s+1})|^{\tau} \frac{dt}{t} \rho(\omega, \omega_{s+1}) d\omega d\omega_{s+1} \\
 & \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)} \|h\|_{Y_{\mu}(\mathbb{R}_+)}^{\tau'} \int_{\mathbb{R}^{s+1}} \sum_{j \in \mathbb{Z}} |\mathcal{A}_j(\omega, \omega_{s+1})|^{\tau} \sigma^*(\bar{\rho})(\omega, \omega_{s+1}) d\omega d\omega_{s+1} \\
 & \leq C \ln(b) \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)} \|h\|_{Y_{\mu}(\mathbb{R}_+)}^{\tau'} \left\| \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^{\tau'} \right\|_{L^{(p/\tau')'}(\mathbb{R}^{s+1})} \|\sigma^*(\bar{\rho})\|_{L^{(p/\tau')'}(\mathbb{R}^{s+1})} \\
 & \leq C \ln(b) \|\mathcal{U}\|_{L^1(\mathbb{S}^{s-1})}^{(\tau'/\tau)+1} \|h\|_{Y_{\mu}(\mathbb{R}_+)}^{\tau'} \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^{\tau'} \right)^{\frac{1}{\tau'}} \right\|_{L^p(\mathbb{R}^{s+1})}^{\tau'} , \tag{23}
 \end{aligned}$$

where  $\bar{\rho}(\omega, \omega_{s+1}) = \rho(-\omega, \omega_{s+1})$ . Define the linear operator  $\Theta$  on any function  $\mathcal{A}_j(\omega, \omega_{s+1})$  by  $\Theta(\mathcal{A}_j(\omega, \omega_{s+1})) = \sigma_{b^j t} * \mathcal{A}_j(\omega, \omega_{s+1})$  and then interpolate (22) with (23); thus, we end with

$$\begin{aligned}
 & \left\| \left( \sum_{j \in \mathbb{Z}} \int_{b^j}^{b^{j+1}} |\sigma_t * \mathcal{A}_j|^{\tau} \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \leq \left\| \left( \sum_{j \in \mathbb{Z}} \int_1^b |\sigma_{b^j t} * \mathcal{A}_j|^{\tau} \frac{dt}{t} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})} \\
 & \leq C_p (\ln b)^{1/\tau} \mathcal{G}(h, \mathcal{U}) \left\| \left( \sum_{j \in \mathbb{Z}} |\mathcal{A}_j|^{\tau} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^{s+1})}
 \end{aligned}$$

for all  $p \in (\mu', \infty)$  with  $\mu > \tau'$ . The proof of Lemma 4 is complete.  $\square$

**4. Proof of Theorem 1**

Assume that  $\mathcal{U} \in L^q(\mathbb{S}^{s-1})$  and  $h \in Y_{\mu}(\mathbb{R}_+)$  for some  $q, \mu \in (1, 2]$ . By using Minkowski’s inequality, we obtain that that

$$\begin{aligned}
 \mathfrak{M}_{\mathcal{U}, h, \Phi, \kappa}^{(\tau)}(g)(\omega, \omega_{s+1}) & \leq \sum_{j=0}^{\infty} \left( \int_{\mathbb{R}_+} |t^{-\lambda} \int_{2^{-j-1}t < \kappa(u) \leq 2^{-j}t} g(\omega - u, \omega_{s+1} - \Phi(\kappa(u))) K_{\mathcal{U}, g}(u) du \Big|^{\tau} \frac{dt}{t} \right)^{1/\tau} \\
 & = \frac{2^{\lambda_1}}{2^{\lambda_1} - 1} \left( \int_{\mathbb{R}_+} |\sigma_t * g(\omega, \omega_{s+1})|^{\tau} \frac{dt}{t} \right)^{1/\tau} . \tag{24}
 \end{aligned}$$

Let  $b = 2^{q'\mu'}$ . Then,  $\ln(b) \leq \frac{1}{(q-1)(\mu-1)}$ . Let  $\{\zeta_j\}_{j \in \mathbb{Z}}$  be a collection of smooth functions defined on  $(0, \infty)$  and satisfying the following properties:

$$\begin{aligned}
 \text{supp } \zeta_j & \subseteq [b^{-1-j}, b^{1-j}]; \quad 0 \leq \zeta_j \leq 1; \\
 \sum_{j \in \mathbb{Z}} \zeta_j(\kappa) & = 1; \quad \text{and} \quad \left| \frac{d^n \zeta_j(\kappa)}{d\kappa^n} \right| \leq \frac{C_n}{\kappa^n} .
 \end{aligned}$$

Let  $\widehat{\zeta}_j$  be the multiplier operators defined on  $\mathbb{R}^{s+1}$  by

$$\widehat{\zeta}_j(\widehat{g})(\eta, \eta_{s+1}) = \zeta_j(\kappa(\eta)) \widehat{g}(\eta, \eta_{s+1}) \quad \text{for } (\eta, \eta_{s+1}) \in \mathbb{R}^s \times \mathbb{R} .$$

Thus, for  $g \in \mathcal{S}(\mathbb{R}^{s+1})$  we have that

$$\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}(g)(\omega, \omega_{s+1}) \leq \frac{2^{\lambda_1}}{2^{\lambda_1}-1} \sum_{n \in \mathbb{Z}} \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(\tau)}(g)(\omega, \omega_{s+1}), \tag{25}$$

where

$$\begin{aligned} \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(\tau)}(g)(\omega, \omega_{s+1}) &= \left( \int_{\mathbb{R}_+} |\mathcal{B}_{\mathcal{U},h,\Phi,n,b}(\omega, \omega_{s+1}, t)|^\tau \frac{dt}{t} \right)^{1/\tau}, \\ \mathcal{B}_{\mathcal{U},h,\Phi,n,b}(\omega, \omega_{s+1}, t) &= \sum_{j \in \mathbb{Z}} (\zeta_{j+n} * \sigma_t * g)(\omega, \omega_{s+1}) \chi_{[b^j, b^{j+1})}(t). \end{aligned}$$

The  $L^p$ -norm of  $\mathcal{M}_{\mathcal{U},h,\Phi,n}^{(\tau)}(g)$  is estimated as follows: If  $p = \tau = 2$ , then  $\|g\|_{H_0^{2,2}(\mathbb{R}^{s+1})} = \|g\|_{L^2(\mathbb{R}^{s+1})}$ . So, by Lemma 1 and Plancherel’s Theorem, we get

$$\begin{aligned} &\left\| \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(2)}(g) \right\|_{L^2(\mathbb{R}^{s+1})}^2 \leq \sum_{j \in \mathbb{Z}} \int_{\mathcal{I}_{j+n,b}} \left( \int_{b^j}^{b^{j+1}} |\hat{\sigma}_t(\eta, \eta_{s+1})|^2 \frac{dt}{t} \right) |\hat{g}(\eta, \eta_{s+1})|^2 d\eta d\eta_{s+1} \\ &\leq C_p (\ln b) \mathcal{G}^2(h, \mathcal{U}) \sum_{j \in \mathbb{Z}} \int_{\mathcal{I}_{j+n,b}} \left( \min \left\{ |D_{2^{q'\mu'j}}|^{\frac{2\delta}{mq'\mu'}}, |D_{2^{q'\mu'j}}|^{-\frac{2\delta}{mq'\mu'}} \right\} \right) |\hat{g}(\eta, \eta_{s+1})|^2 d\eta d\eta_{s+1} \\ &\leq C_p 2^{-\varepsilon|n|} (\mu - 1)^{-1} (q - 1)^{-1} \mathcal{G}^2(h, \mathcal{U}) \sum_{j \in \mathbb{Z}} \int_{\mathcal{I}_{j+n,b}} |\hat{g}(\eta, \eta_{s+1})|^2 d\eta d\eta_{s+1} \\ &\leq C_p 2^{-\varepsilon|n|} (\mu - 1)^{-1} (q - 1)^{-1} \|h\|_{Y_\mu(\mathbb{R}_+)}^2 \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})}^2 \|g\|_{L^2(\mathbb{R}^{s+1})}^2, \end{aligned}$$

where  $\mathcal{I}_{j,b} = \{(\eta, \eta_{s+1}) \in \mathbb{R}^{s+1} : \kappa(\eta) \in [b^{-1-j}, b^{1-j}]\}$  and  $0 < \varepsilon < 1$ . Thus,

$$\begin{aligned} &\left\| \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(2)}(g) \right\|_{L^2(\mathbb{R}^{s+1})} \\ &\leq C 2^{-\frac{\varepsilon}{2}|n|} (\mu - 1)^{-1/2} (q - 1)^{-1/2} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{2,2}(\mathbb{R}^{s+1})}. \end{aligned} \tag{26}$$

On the other hand, if  $p \in [\tau, \infty)$ , then by Lemma 3, we obtain that

$$\begin{aligned} &\left\| \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \\ &\leq C (q - 1)^{-1/\tau} (\mu - 1)^{-1/\tau} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}. \end{aligned} \tag{27}$$

Moreover, if  $p \in (1, \tau)$ , we conclude that

$$\begin{aligned} &\left\| \mathcal{M}_{\mathcal{U},h,\Phi,n}^{(\tau)}(g) \right\|_{L^p(\mathbb{R}^{s+1})} \\ &\leq C (q - 1)^{-1} (\mu - 1)^{-1} \|h\|_{Y_\mu(\mathbb{R}_+)} \|\mathcal{U}\|_{L^q(\mathbb{S}^{s-1})} \|g\|_{H_0^{\tau,p}(\mathbb{R}^{s+1})}. \end{aligned} \tag{28}$$

Consequently, when we interpolate (26) with (27) and (28) and then use (25), we directly get (5) and (6). This finishes the proof of Theorem 1.

In the same manner, except invoking Lemma 4 with  $b = 2^{q'}$  instead of Lemma 3, we can prove Theorem 2.

### 5. Conclusions

In this work, we found appropriate  $L^p$  bounds for the generalized parabolic Marcinkiewicz operators  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  whenever  $\mathcal{U}$  belongs to the space  $L^q(\mathbb{S}^{s-1})$ . By employing these bounds

along with Yano's extrapolation argument, we establish the  $L^p$  boundedness of  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  under very weak conditions assumed on the integral kernels. The results in this article represent substantial extensions and improvements to previously known results. In fact, our results improve and extend the results in [2–9,19,22,25]. We notice that the range of  $p$ ,  $|1/p - 1/2| < \{1/\mu - 1/2\}$  becomes a tiny open interval when  $\mu \rightarrow 1^+$ . In future work, we aim to prove the  $L^p$  boundedness of the operator  $\mathfrak{M}_{\mathcal{U},h,\Phi,\kappa}^{(\tau)}$  for the full range of  $p \in (1, \infty)$  and also whenever  $\mathcal{U} \in B_q^{(0,-1/2)}(\mathbb{S}^{s-1}) \cup L(\log L)^{1/2}(\mathbb{S}^{s-1})$ .

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## References

1. Fabes, E.; Rivi re, N. Singular integrals with mixed homogeneity. *Studia Math.* **1966**, *27*, 19–38.
2. Stein, E. On the functions of Littlewood-Paley. Lusin and Marcinkiewicz, *Trans. Amer. Math. Soc.* **1958**, *88*, 430–466.
3. H rmander, L. Estimates for translation invariant operators in  $L^p$  space. *Acta Math.* **1960**, *104*, 93–139.
4. Benedek, A.; Calderon, A.; Panzone, R. Convolution operators on Banach space valued functions. *Proc. Nat. Acad. Sci. USA* **1962**, *48*, 356–365.
5. Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, *44*, 203–217.
6. Al-Qassem, H.; Al-Salman, A. A note on Marcinkiewicz integral operators. *J. Math. Anal. Appl.* **2003**, *282*, 698–710.
7. Ali, M.  $L^p$  estimates for Marcinkiewicz integral operators and extrapolation. *J. Ineq. Appl.* **2014**, 269. <https://doi.org/10.1186/1029-242X-2014-269>.
8. Ding, Y. On Marcinkiewicz integral. In Proceedings of the conference “Singular Integrals and Related Topics, III”, Osaka, Japan, 27–29 January 2001.
9. Ding, Y.; Fan, D.; Pan, Y. On the  $L^p$  boundedness of Marcinkiewicz integrals. *Mich. Math. J.* **2002**, *50*, 17–26.
10. Ding, Y.; Lu, S.; Yabuta, K. A problem on rough parametric Marcinkiewicz functions. *J. Aust. Math. Soc.* **2002**, *72*, 13–21.
11. Al-Qassem, H.; Pan, Y.  $L^p$  estimates for singular integrals with kernels belonging to certain block spaces. *Rev. Mat. Iberoam.* **2002**, *18*, 701–730.
12. Al-Qassem, H.; Pan, Y. On certain estimates for Marcinkiewicz integrals and extrapolation. *Collec. Math.* **2009**, *60*, 123–145.
13. Ali, M.; Al-Senjlawi, A. Boundedness of Marcinkiewicz integrals on product spaces and extrapolation. *Inter. J. Pure Appl. Math.* **2014**, *97*, 49–66.
14. Al-Qassem, H.; Ali, M.  $L^p$  boundedness for singular integral operators with  $L(\log^+ L)^2$  kernels on product spaces. *Kyungpook Math. J.* **2008**, *46*, 377–387.
15. Liu, F. A note on Marcinkiewicz integrals associated to revolution. *J. Aust. Math. Soc.* **2018**, *104*, 380–402.
16. Fan, D.; Pan, Y. A singular integral operators with rough kernel. *Proc. Amer. Math. Soc.* **1979**, *125*, 3695–3703.
17. Chen, J.; Fan, D.; Ying, Y. Singular integral operators on function spaces. *J. Math. Anal. Appl.* **2002**, *276*, 691–708.
18. Le, H. Singular integrals with mixed homogeneity in Triebel-Lizorkin spaces. *J. Math. Anal. Appl.* **2008**, *345*, 903–916.
19. Al-Qassem, H.; Cheng, L.; Pan, Y. On rough generalized parametric Marcinkiewicz integrals. *J. Math. Ineq.* **2017**, *11*, 763–780.
20. Ali, M.; Hamadneh, M. On the functions of generalized Marcinkiewicz integral operators along subvarieties via Extrapolation. *Symmetry* **2022**, *14*, 1607. <https://doi.org/10.3390/sym14081607>.
21. Fan, D.; Pan, Y.; Yang, D. A weighted norm inequality for rough singular integrals. *Tohoku Math. J.* **1999**, *51*, 141–161.
22. Sakamoto, M.; Yabuta, K. Boundedness of Marcinkiewicz functions. *Studia Math.* **1999**, *135*, 103–142.
23. Al-Qassem, H.; Cheng, L.; Pan, Y. On generalized Littlewood-Paley functions. *Collec. Math.* **2018**, *69*, 297–314.
24. Ali, M.; Al-Refai, O. Boundedness of Generalized Parametric Marcinkiewicz Integrals Associated to Surfaces. *Mathematics* **2019**, *7*, 886. <https://doi.org/10.3390/math7100886>.
25. Fan, D.; Wu, H. On the generalized Marcinkiewicz integral operators with rough kernels. *Canad. Math. Bull.* **2011**, *54*, 100–112.
26. Jiang, Y.; Lu, S. A class of singular integral operators with rough kernel on product domains. *Hokk. Math. J.* **1995**, *24*, 1–7.

27. Ali, M.; Katatbeh, Q. Generalized parabolic Marcinkiewicz integrals associated with polynomial compound curves with rough kernels. *Demon. Math.* **2022**, *53*, 44–57.
28. Liu, F.; Zhang, D. Parabolic Marcinkiewicz integrals associated to polynomials compound curves and extrapolation. *Bull. Korean Math. Soc.* **2015**, *52*, 771–788.
29. Ali, M.; Al-Mohammed, O. Boundedness of a class of rough maximal functions. *J. Ineq. Appl.* **2018**, *2018*, 305. <https://doi.org/10.1186/s13660-018-1900-y>.
30. Sato, S. Estimates for singular integrals and extrapolation. *arXiv* **2007**, arXiv:0704.1537.
31. Ali, M.; Al-Qassem, H. A note on parabolic maximal operators along surfaces of revolution via extrapolation. *Symmetry* **2022**, *14*, 1147. <https://doi.org/10.3390/sym14061147>.