On finite groups whose power graph is a cograph

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Abstract

A P_4-free graph is called a cograph. In this paper we partially characterize finite groups whose power graph is a cograph. As we will see, this problem is a generalization of the determination of groups in which every element has prime power order, first raised by Graham Higman in 1957 and fully solved very recently.

First we determine all groups G and H for which the power power graph of $G \times H$ is a cograph. We show that groups whose power graph is a cograph can be characterised by a condition only involving elements whose orders are prime or the product of two (possibly equal) primes. Some important graph classes are also taken under consideration. For finite simple groups we show that in most of the cases their

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power graphs are not cographs: the only ones for which the power graphs are cographs are certain groups PSL$(2, q)$ and Sz(q) and the group PSL$(3, 4)$. However, a complete determination of these groups involves some hard number-theoretic problems.

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1 Introduction

There are various graphs we can define for a group using different group properties [4]. These graphs include the commuting graph, the generating graph, the power graph, the enhanced power graph, deep commuting graph, etc. The power graphs were first seen in early 2000's as the undirected power graphs of semigroups [19]. For a semigroup S, the *directed power graph* of S, denoted by $\vec{P}(S)$, is a directed graph with vertex set $V(\vec{P}(S)) = S$; and two distinct vertices x and y are having an arc $x \to y$ if y is a power of x.

The corresponding undirected graph is called the *undirected power graph* of S, denoted by $P(S)$. The undirected power graph of a semigroup was introduced by Chakrabarty et al. [11] in 2009. So the undirected power graph of S is the graph with vertex set $V(P(S)) = S$, with an edge between two vertices u and v if $u \neq v$ and either v is a power of u or u is a power of v. These concepts are defined for groups as a special case of semigroups. In the sequel, we only consider groups;"power graph" will mean "undirected power graph", and all the groups in this paper are finite.

The power graphs are well studied in the literature [1, 2, 5, 6, 7, 8, 9, 11, 12, 22]. We find several research papers in which researchers give complete or partial characterization of different graph parameters for the power graphs. We mention few notable works in this context:

- $P(G)$ is a complete graph if and only if either G is trivial or a cyclic group of prime power order. (Chakrabarty *et al.* [11])
- $P(G)$ is always connected and we can compute the number of edges in $P(G)$ by the formula $|E(P(G))| = \frac{1}{2}\Big[\sum_{a\in G}(2o(a) - \phi(o(a)) - 1)\Big]$.
- The power graph of a finite group G is Eulerian if and only if G has odd order.

- Curtin *et al.* [16] introduced the concept of proper power graphs. They determine the diameter of the proper power graph of S_n.
- Chattopadhyay *et al.* [12] have provided bounds for the vertex connectivity $P(G)$ where G is a cyclic group.
- Cameron [5] proved that, for any two finite groups G_1 and G_2, if power graphs of G_1 and G_2 are isomorphic then $\vec{P}(G_1)$ and $\vec{P}(G_2)$ are also isomorphic.

In our previous paper [9], we partially characterized finite groups whose power graphs forbid certain induced subgraphs. These subgraphs include P_4 (the path on 4 vertices); C_4 (the cycle on 4 vertices); $2K_2$ (the complement of C_4); etc. A graph forbidding P_4 is called a cograph. In other words, a graph Γ is a *cograph* if it does not contain the 4-vertex path as an induced subgraph. Cographs have various important properties. For example, they form the smallest class of graphs containing the 1-vertex graph and closed under complementation and disjoint union. (The complement of a connected cograph is disconnected.) See [3, 4] for more about these concepts.

We will use the term *power-cograph group*, sometimes abbreviated to *PCG-group*, for a finite group whose power graph is a cograph.

In [9], we completely characterized finite nilpotent power-cograph groups. We proved the following theorem:

Theorem 1.1 ([9], Theorem 3.2)**.** *Let G be a finite nilpotent group. Then $P(G)$ is a cograph if and only if either $|G|$ is a prime power, or G is cyclic of order pq for distinct primes p and q.*

For a given group G, the power graph of any subgroup of G is an induced subgraph of $P(G)$. Thus, if the power graph of a group is a cograph then the power graph of any of its subgroups is also a cograph. In other words, the class of finite power-cograph groups is subgroup-closed.

This gives a necessary condition for a group to be a power-cograph group: any nilpotent subgroup of such a group is either a p-group or isomorphic to C_{pq}, where p and q are distinct primes. So if G has a nilpotent subgroup which is neither a p-group nor isomorphic to C_{pq}, then $P(G)$ is not a cograph. In our previous paper, we have asked the following question: Classify the finite groups G for which $P(G)$ is a cograph. In this paper we provide further results towards the answer to this question.

We now give several equivalent conditions on a finite group which are known to imply that the power graph is a cograph. First we require a few definitions.

- For a finite group G, Let $\pi(G)$ denote the set of all prime divisors of $|G|$. The *prime graph* or *Gruenberg–Kegel graph* of G is a graph with $V = \pi(G)$ and two distinct elements p and q of $\pi(G)$ are connected if and only if G contains an element of order pq.
- The *enhanced power graph* of G is the graph with vertex set G, in which vertices x and y are joined if there exists $z \in G$ such that both x and y are powers of z. Clearly the power graph is a spanning subgraph of the enhanced power graph.
- The group G is an *EPPO group* if every element of G has prime power order.

Theorem 1.2. *For a finite group G, the following conditions are equivalent:*

(a) *G is an EPPO group;*

(b) *the Gruenberg–Kegel graph of G has no edges;*

(c) *the power graph of G is equal to the enhanced power graph.*

If these conditions hold, then the power graph of G is a cograph.

For the equivalence of (a)–(c), see Aalipour *et al.* [1]. The class of EPPO groups was first investigated (though not under that name) by Graham Higman in 1957 [18], and the simple EPPO groups were determined by Suzuki [25, 26]but the complete determination of these groups only appeared in a paper not yet published [10].

Suppose that G is an EPPO group. The *reduced power graph* of G (obtained by removing the identity vertex from $P(G)$) is the disjoint union of reduced power graphs of subgroups of prime power order. Each of this is a cograph, by Theorem 1.1; thus $P(G)$ is a cograph.

Hence our problem is a generalization of the classification of EPPO groups.

We note that the condition that G is a power-cograph group does not imply (a)–(c). Moreover two groups may have the same prime graph, yet one and not the other is a power-cograph group. For example, consider $G_1 = C_{12}$ and $G_2 = D_6$.

In this paper we explore various graph classes and try to identify whether their power graphs are cographs or not. First we discuss direct product two groups. We are able to identify certain solvable groups whose power graph is a cograph. Finally we consider finite simple groups. Our result is as follows:

Theorem 1.3. *Let G be a non-abelian finite simple group. Then G is a power-cograph group if and only if one of the following holds:*

(a) $G = \mathrm{PSL}(2, q)$*, where q is an odd prime power with $q \geq 5$, and each of $(q-1)/2$ and $(q+1)/2$ is either a prime power or the product of two distinct primes;*

(b) $G = \mathrm{PSL}(2, q)$*, where q is a power of 2 with $q \geq 4$, and each of $q - 1$ and $q + 1$ is either a prime power or the product of two distinct primes;*

(c) $G = \mathrm{Sz}(q)$*, where $q = 2^{2e+1}$ for $e \geq 2$, and each of $q - 1$, $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$ is either a prime power or the product of two distinct primes;*

(d) $G = \mathrm{PSL}(3, 4)$*.*

We end the introduction with a remark about this theorem. In the first three cases, determining precisely which groups occur is a purely number-theoretic problem which is likely to be quite difficult. For example, the values of d (at least 2) for which the power graph of $\mathrm{PSL}(2, 2^d)$ is a cograph for $d \leq 200$ are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, and 199, and the values of e (at least 1) for which the power graph of $\mathrm{Sz}(2^{2e+1})$ is a cograph for $e \leq 100$ are 1, 2, 3, 4, 5, 6, 8, 44.

Problem Are there infinitely many non-abelian finite simple groups G which are power-cograph groups?

2 Direct products

Recall that a finite nilpotent group can be written as a direct product of its Sylow subgroups. Thus if $P(G \times H)$ is a cograph then, by using Theorem 1.1, we have only a few choices for the orders of G and H. The following theorem gives a complete characterization for all direct products $G \times H$ such that $P(G \times H)$ is a cograph.

Theorem 2.1. *Let G and H be non-trivial groups. Then $G \times H$ is a power-cograph group if and only if one of the following holds:*

(a) *the orders of G and H are powers of the same prime;*

(b) *G and H are cyclic groups of distinct prime orders;*

(c) *there are primes p and q and an integer $m \geq 1$ such that $q^m \mid (p-1)$; one of G and H is a cyclic group of order q, and the other is the non-abelian group*

$$\langle a, b : a^p = 1, b^{q^m} = 1, b^{-1}ab = a^k \rangle,$$

where k is an integer with multiplicative order $q^m \pmod{p}$.

Proof. Let $G \times H$ be a cograph. If the orders of G and H are each divisible by exactly one prime then $G \times H$ is nilpotent. Therefore by Theorem 1.1, we obtain first two cases. So we can suppose that at least one one of G and H has order divisible by two primes.

Let p and q be distinct primes such that p divides $|G|$ and q divides $|H|$. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. Then $P \times Q$ is a nilpotent subgroup of $G \times H$; so $|P| = p$ and $|Q| = q$.

First we consider that neither G nor H has prime power order. Then it follows that both $|G|$ and $|H|$ are squarefree. For, if $p^2 \mid |G|$, then there is a prime $q \neq p$ dividing $|H|$, a contradiction. Hence both G and H are metacyclic [24, pp.246–247]. So each has a normal subgroup of prime order. Moreover, each of G and H has the property that any abelian subgroup is cyclic of prime order; since otherwise its direct product with a cyclic group of prime order in the other factor would be nilpotent, contradicting Theorem 1.1. Hence each of G and H is non-abelian with order the product of two primes.

Suppose that G is a squarefree group of order $p_1 p_2 \cdots p_r$, where $p_1, p_2, \ldots, p_r$ are primes and $p_1 < p_2 < \cdots < p_r$. Let P_1 be a Sylow p_1-subgroup of G. Then the order of the automorphism group of P_1 is only divisible by primes smaller than p_1, so $C_G(P_1) = N_G(P_1)$. By Burnside's Transfer Theorem, G has a normal p_1-complement Q_1, a normal subgroup of order $p_2 \cdots p_r$. By continuing this procedure, we find eventually a normal subgroup of order p_r, say N.

Now suppose that the power graph of $G \times H$ is a cograph. Then G has no abelian subgroup with order divisible by two primes, or else the direct product of this subgroup with a subgroup of prime order in H is a nilpotent subgroup of $G \times H$ whose power graph contains P_4. So $C_G(N) = N$, and G/N acts faithfully as a group of automorphisms of N. Thus G/N is abelian, and is isomorphic to a complement to N in G. By the remark just made, N is cyclic of prime order, so G is a non-abelian group with order divisible by just two primes, say p and q, with q dividing $p-1$. By symmetry the same applies to H, and indeed the two prime divisors of $|H|$ must also be p and q. For, if $r \mid |H|$ and a, b, c are elements of orders p, q, r respectively, then (a, ac, c, bc) is a copy of P_4 in the power graph.

Now let a be an element of order p in H, and let b and c be two elements of order q in G which are not adjacent in the power graph. Then (b, ab, a, ac, c) is an induced path in $P(G \times H)$, contradicting the assumption that this graph is a cograph.

We have now ruled out the possibility that neither G nor H has prime power order. So one of G and H (say G, without loss of generality) is a group of prime power order q^m. By assumption, H is not a q-group, and so contains a subgroup P of prime order $p \neq q$. Then $G \times P$ is a nilpotent subgroup of $G \times H$ with forbidden structure, unless $|G| = q$. Now p must divide $|H|$ to the first power only, and $|H| = pq^m$ for some $m \geq 1$. We claim next that H has a normal Sylow p-subgroup P. For suppose not, and let b and c be elements of order p not adjacent in the power graph, and a a non-identity element of G. Then (b, ab, a, ac, c) is an induced path in the power graph of $G \times H$, a contradiction. As before, we conclude that $C_H(P) = P$, and so the Sylow q-subgroup of H (which is a complement to P) is cyclic of order dividing $p - 1$. This finally yields the claimed structure for G and H.

Now we prove the converse.

The first two cases are easy to verify. In both these cases $G \times H$ is nilpotent and the result follows from Theorem 1.1.

Now suppose G is the non-abelian group $\langle a, b : a^p = 1, b^{q^m} = 1, b^{-1}ab = a^k \rangle$, where k is an integer with multiplicative order $q^m \pmod{p}$ and $H = C_q$. First we observe the structure of G. Since $p > q$ and G is non-abelian, therefore G has a unique Sylow p-subgroup and p Sylow q-subgroups. So each non-identity element of G has order p or power of q. On the other hand, $P(H)$ is the complete graph K_q.

Now we show that $P(G \times H)$ is a cograph. On the contrary let us assume that $P(G \times H)$ contains an induced P_4, say $(a_1, b_1) \sim (a_2, b_2) \sim (a_3, b_3) \sim (a_4, b_4)$. Since $P(H)$ is complete none of a_1, a_2, a_3, a_4 can be identity. Now

$(a_1, b_1) \sim (a_2, b_2)$ implies either both a_1, a_2 has order p or power of q.
If both of them has order p then as $(a_2, b_2) \sim (a_3, b_3)$, so a_3 is of order p and then all these 3 vertices lies in the same cyclic subgroup of $G \times H$ with same order and so they forms a cycle.
Again, if orders of both a_1, a_2 are powers of q then as G contains elements of order only p or power of q and $(a_3, b_3) \sim (a_2, b_2)$ implies either $o(a_3)|o(a_2)$ or $o(a_2)|o(a_3)$ so a_3 must be order power of q. Similarly, a_4 must be of order power of q. Thus the whole path contained in a subgroup of $G \times H$ whose order is power of q and by Theorem 1.1, power graph of this subgroup of $G \times H$ is a cograph. So our taken path is impossible in any aspect. Hence, $P(G \times H)$ is a cograph. □

Observation 2.2. Let G and H are two groups such that $P(G \times H)$ is a cograph then both $P(G)$ and $P(H)$ are cographs. (For $G \times H$ has subgroups isomorphic to G and H.)

Remark 2.3. The converse of the above is not true in general. Consider $G = C_4$ and $H = C_6$. Then by Theorem 1.1, both $P(G)$ and $P(H)$ are cographs where as $P(G \times H)$ is not a cograph.

3 Minimal non-power-cograph groups

Let $\mathcal{C}$ be the class of finite groups G for which $P(G)$ is a cograph. As noted earlier, $\mathcal{C}$ is subgroup-closed; so it can be characterised by finding all minimal non-$\mathcal{C}$ groups.

Theorem 3.1. *Let G be a finite group. Then $P(G)$ is not a cograph if and only if G contains elements g and h with orders pr and pq respectively, where p, q, r are prime numbers and $p \neq q$, such that*

(a) $g^r = h^q$;

(b) *if* $q = r$, *then* $g^p \notin \langle h^p \rangle$.

Proof. Let G be a minimal non-$\mathcal{C}$ group. Suppose first that G is abelian. By Theorem 1.1, it has order the product of three primes which are not all equal. We distinguish three cases.

- Suppose that $|G| = pqr$ where p, q, r are all distinct. Then G is cyclic; say $G = \langle x \rangle$. Now if we put $g = x^q$ and $h = x^r$, we see that the conditions of the theorem are satisfied.

- Suppose that $|G| = p^2q$, and that the Sylow p-subgroup of G is cyclic, generated by g. Let z be an element of order q, and $h = g^pz$. Take $r = p$ in the conditions of the theorem.
- Finally, suppose that $|G| = p^2q$ and the Sylow p-subgroup is elementary abelian, generated by x and y. Let z be an element of order q. Now take $g = xz$ and $h = yz$. Then g and h have order pq; $g^p = z^p = h^p$, but $g^q = x^q \notin \langle x^p$. So these elements satisfy the conditions of the theorem, if we take $r = q$ and reverse the roles of p and q.

So we can suppose that G is nonabelian.

Since $P(G)$ is not a cograph, there is an induced path (a, b, c, d) in $P(G)$. Now we cannot have $a \to b \to c$ or $c \to b \to a$ in $\vec{P}(G)$, since $\to$ is transitive but a and c are nonadjacent in $P(G)$. So either $a \to b \leftarrow c$ or $a \leftarrow b \to c$. Applying the same reasoning to b, c and d, we see that (up to reversal of the path) we have

$$a \to b \leftarrow c \to d.$$

Now $\langle c\rangle$ is a cyclic group and contains b and c. Since G is nonabelian, it is a proper subgroup, and hence its power graph is a cograph. So the order of c is either a prime power or of the form pq where p and q are distinct primes. The former case is impossible. For the power graph of a cyclic group of prime power order is complete, but b is not joined to d. So the order of c is pq, with $p \neq q$. We may suppose without loss that $b = c^q$ has order p while $d = c^p$ has order q.

Now consider the element a. We know that the order of a is divisible by p (the order of b). By replacing a by a power of itself, we can assume that the order of a is pr, where r is a prime which may or may not be equal to p. (This power is still joined to b, but it cannot be joined to d. For if a and d are joined, then $d \in \langle a\rangle \cap \langle c\rangle = \langle b\rangle$, contradicting the fact that d has order q whereas b has order p. Also a cannot be joined to c, for this would imply that $a \to c$ and hence $a \to d$.)

We have now verified all the conditions of the theorem.

Conversely, if these conditions hold, then $(g, g^r = h^q, h, h^p)$ is an induced path of length 3, so $P(G)$ is not a cograph. $\square$

Remark 3.2. A minimal non-PCG group has nontrivial centre. For such a group is generated by elements g and h as in the theorem, and $g^r = h^q$ is in the centre.

Corollary 3.3. *Let G be a finite group. Let $P_2(G)$ be the set of non-identity elements of G whose orders are either prime or the product of two (not necessarily distinct) prime numbers. Then $P(G)$ is a cograph if and only if the induced subgraph on $P_2(G)$ is a cograph.*

Here is an application, which we will require later. Suppose that G is a finite group containing elements a of order 4 and b of order 6 such that a^2 and b^3 are conjugate. Replacing b by a conjugate, we may assume that $a^2 = b^3$. Now the theorem above implies that G is not a power-cograph group. These conditions can be verified for the simple groups M_{11} and $\mathrm{PSU}(3, 8)$ using the $\mathbb{ATLAS}$ of finite groups [14]. We will use this argument several times, so we refer to it as the 4-6 *test*.

4 Examples

Below we let $P^*(G)$ be the *reduced power graph* of G, the induced subgraph on the set $G^{\#} = G \setminus \{1\}$. Note that $P(G)$ is a cograph if and only if $P^*(G)$ is a cograph.

We also make the following observation.

Theorem 4.1. *Let G be a finite group in which any two distinct maximal cyclic subgroups intersect in the identity. Then $P(G)$ is a cograph if and only if the orders of the maximal cyclic subgroups are either prime powers or products of two distinct primes.*

Proof. Every edge of $P(G)$ is contained in a maximal cyclic subgroup of G. The hypothesis implies that $P^*(G)$ is the union of $P^*(C)$ as C runs over the maximal cyclic subgroups of G. □

Theorem 4.2. *The symmetric group S_n on n symbols is a power-cograph group if and only if $n \leq 5$.*

Proof. for $n \geq 6$, $P(S_n)$ contain a path $(pqr)(xy) \sim (xy) \sim (qrz)(xy) \sim (qzr)$ and thus $P(S_6)$ is not a cograph.

For $n \leq 5$, the maximal cyclic subgroups intersect in the identity, and their orders are in the sets $\{2\}$ (for $n = 2$), $\{2, 3\}$ (for $n = 3$), $\{2, 3, 4\}$ (for $n = 4$), or $\{4, 5, 6\}$ (for $n = 5$), so these symmetric groups are all power-cograph groups, by Theorem 4.1. □

Theorem 4.3. *Let p and q $(< p)$ be primes and G be the semidirect product of C_p by C_{q^m} acting faithfully on C_p. Then $P(G)$ is a cograph.*

Proof. By assumption, there are no elements of order pq, so the orders of the maximal cyclic subgroups are p and q^m. □

Theorem 4.4. *If G is a dihedral group of order $2m$, then G is a power-cograph group if and only if m is either a prime power or the product of two distinct primes.*

Proof. The orders of maximal cyclic subgroups are 2 and m, and any two intersect in the identity. □

4.1 Remarks on solvable groups

Let G be a solvable group and $G \in \mathcal{C}$. Let $F(G)$ be the Fitting subgroup (the maximal normal nilpotent subgroup of G) of G. Then by Theorem 1.1, the order of $F(G)$ is either a prime power or the product of two primes.

Let $F(G) = C_{pq}$ for distinct primes p and q. Then $F(G)$ contains its centraliser, and so is equal to it; so $G/F(G)$ acts as a group of automorphisms of $F(G)$. Thus G is contained in the group $(C_p.C_{p-1}) \times (C_q.C_{q-1})$. But we can't have a direct product larger than $C_p \times C_q$. So the structure of G is $(C_p \times C_q).C_r$ where r divides both $p-1$ and $q-1$, and r is either a prime power or the product of two primes.

Next suppose that $F(G)$ be a p-group. If all the elements of G are of prime power order then the prime graph of G is a null graph, and hence $G \in \mathcal{C}$. Higman [18] gave nice characterization of such groups. And in that $|G|$ has at most two prime divisors and $G/F(G)$ is one of the following:

(a) a cyclic group whose order is a power of a prime other than p.

(b) a generalized quaternion group, p being odd; or

(c) a group of order $p^a q^b$ with cyclic Sylow subgroups, q being a prime of the form $kp^a + 1$.

But the problem arises when G contains elements whose order is not a prime power. One thing is for sure for in that case: If the order of a element is not a prime power then it must be product of two distinct primes. We observe that such group exists and we give two different examples for that. The

Frobenius group F_7 of order 42 whose power graph is a cograph; Here $|F_7|$ is divisible by 3 primes, F_7 contains an element of order 6, and it's fitting subgroup C_7. Let G be the semidirect product of the Heisenberg group H_3 of order 27 by C_2, then G is solvable and $G \in \mathcal{C}$. In that case the fitting subgroup $F(G) = H_3$, and G contains elements of order 6.

Problem 4.5. Classify all solvable $\mathcal{C}$-groups whose Fitting subgroup is a p-group.

5 Finite simple groups

In this section we discuss simple groups whose power graphs are cographs. For each prime p, the simple group C_p has complete power graph, therefore it is a power-cograph group. In the next theorem we classify alternating groups which are power-cograph groups.

Theorem 5.1. *The alternating group A_n is a power-cograph group if and only if $n \leq 6$.*

Proof. For $n \geq 7$, the 4-6 test applies, with $a = (1,2,3,4)$ and $b = (1,3)(2,4)(5,6,7)$.

Now we consider $n \leq 6$.

If $n = 3$ then A_3 is nothing but the cyclic group C_3 and hence its power graph is the complete graph K_3 and hence a cograph.

For $n = 4, 5, 6$ then prime graph of A_n is a null graph and by Theorem 1.2 the power graphs is a cograph. □

In the next few sections we discuss simple groups of Lie type of low rank or over small fields and sporadic simple groups. Information about specific groups is found in the $\mathbb{ATLAS}$ [14], and further information about the simple groups and their subgroups is in Rob Wilson's book [27].

We also use the fact that $\mathcal{C}$ is subgroup-closed; so, if a group G contains a subgroup not in $\mathcal{C}$, then $G \notin \mathcal{C}$.

5.1 Simple groups of Lie type of rank 1

The simple groups of Lie type of rank 1 are $A_1(q) = \mathrm{PSL}(2,q)$, ${}^2A_2(q) = \mathrm{PSU}(3,q)$, ${}^2B_2(q) = \mathrm{Sz}(q)$ where $q = 2^{2e+1}$, and ${}^2G_2(q) = R_1(q)$ where $q = 3^{2e+1}$.

In [5], Cameron proved that, if q is an odd prime power then the power graph of $\mathrm{PSL}(2,p)$ is a cograph if a only if $(q-1)/2$ and $(q+1)/2$ are either prime powers or product of two primes. And if $q \geq 4$ is a power of 2 then power graph of $\mathrm{PSL}(2,p)$ is a cograph if and only if $q-1$ and $q+1$ are either prime powers or products of two distinct primes.

Next we show that power graph of $\mathrm{PSU}(3,q)$ is not a cograph for $q \neq 2$. Since $\mathrm{PSL}(3,2)$ is not simple, there are no simple power-cograph groups of this type.

Theorem 5.2. *Let q be a power of a odd prime p. Then power graph of $PSU(3,q)$ is not a cograph.*

Proof. We use the fact that $\mathrm{PSU}(3,q)$, q odd, has cyclic subgroups of orders $n_1 = (q^2-1)/\gcd(q+1,3)$, $n_2 = (q^2-q+1)/\gcd(q+1,3)$ and $n_3 = p(q+1)/\gcd(q+1,3)$.

So, if the power graph is a cograph, then both $(q-1)/\gcd(q+1,3)$ and $(q+1)/\gcd(q+1,3)$ are primes, or else both are powers of the same prime.

In the second case, the prime must be 2, and since one of $q-1$ and $q+1$ is not divisible by 4, we must have $(q-1,q+1) = (2,4)$ or $(4,6)$, so $q=3$ or 5.

Now for $q=3$ we have $n_3 = 12$, so power graph of $\mathrm{PSU}(3,3)$ is not a cograph. On the other hand $\mathrm{PSU}(3,5)$ contains A_7. Therefore power graph of $PSU(3,q)$ is not a cograph. □

Theorem 5.3. *Let $q \geq 4$ be a power of 2. Then the power graph of $PSU(3,q)$ is not a cograph.*

Proof. Let β be a generator of the multiplicative group of $\mathrm{GF}(q^2)$. Then β^{q-1} has order $q+1$. Let p be a prime factor of $q+1$ greater than 3, and let $d = (q+1)/p$. Then $\alpha = \beta^{d(q-1)}$ has order p. Then $\overline{\alpha} = \beta^{d(q^2-q)}$, so $\alpha\overline{\alpha} = \beta^{d(q^2-1)} = 1$ in $\mathrm{GF}(q^2)$. Consider the elements

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$$

Then g is a element of order 2 and it commutes with h. So $o(gh) = o(hk) = 2p$. On the other hand $k^2 = h$.

Therefore the elements the elements g, gh, h, k induce a path of length 3 in $\mathrm{SU}(2, q)$.

Now observe that $g, h, k \in \mathrm{SU}(3, q) \setminus Z$. Take $\gamma = gZ$ and $\eta = hZ$, $\kappa = kZ$. Then the elements $\gamma, \gamma\eta, \eta, \kappa$ induce a path of length 3 in the power graph of $\mathrm{PSU}(3, q)$.

The argument fails for $q = 8$. But we saw earlier that $\mathrm{PSL}(3, 8)$ is not a power-cograph group, using the 4-6 test. □

Theorem 5.4. *Let $G = {}^2B_2(q) = \mathrm{Sz}(q)$, $q = 2^{2e+1}$. Then $G \in \mathcal{C}$ if and only if each of $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$ is either a prime power or the product of two distinct primes.*

Proof. Any edge of the power graph is contained in a maximal cyclic subgroup. The maximal cyclic subgroups of $\mathrm{Sz}(q)$ have orders 4, $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$. These four numbers are pairwise coprime. (The last three are odd. The difference between the third and fourth is a power of 2, but 2 does not divide either. Suppose that p is a prime dividing both $q-1 = 2^{2e+1}-1$ and $q+\sqrt{2q}+1 = 2^{2e+1}+2^{e+1}+1$. Then p divides their difference, $2^{e+1}+2$; since it is odd, it divides 2^e+1, and hence it divides $2^{2e}-1$, and also $2^{2e+1}-2$. This p divides 1. The argument for $q-1$ and $q-\sqrt{2q}+1$ is similar.) Thus no element can lie in maximal cyclic subgroups of different orders. So, if the power graph contains P_4, then this P_4 must be contained in a maximal cyclic subgroup, so this subgroup must have three prime divisors, not all equal. The converse is clear. □

Now let $G = {}^2G_2(q) = R_1(q)$, $q = 3^{2e+1}$. The centraliser of an involution in G is $C_2 \times \mathrm{PSL}(2, q)$, which contains subgroups $C_2 \times C_{(q\pm1)/2}$. So, if $G \in \mathcal{C}$, then $(q \pm 1)/2$ is either prime or a power of 2. If it is a power of 2, then we have a solution to Catalan's equation, contradicting the result of Mihăilescu's Theorem: see [13, Section 6.11]. The numbers $(q \pm 1)/2$ have opposite parity, so cannot both be prime. So G is not a power-cograph group.

5.2 Simple groups of Lie type of rank 2

The rank 2 simple groups of Lie type are $A_2(q) = \mathrm{PSL}(3,q)$, $C_2(q) = \mathrm{PSp}(4,q)$, ${}^2A_3(q) = \mathrm{PSU}(4,q)$, ${}^2A_4(q) = \mathrm{PSU}(5,q)$, $G_2(q)$, ${}^2F_4(q)$ and ${}^3D_4(q)$. We examine each of the above cases. In the case of $A_2(q)$, we prove a slightly stronger result, for later use.

Theorem 5.5. *Let G be a quotient of $\mathrm{SL}(3,q)$ by a subgroup of the group of scalars. If G is a power-cograph group, then $q = 2$ or $q = 4$.*

Proof. We work in $\mathrm{SL}(3,q)$. Suppose that q is odd. Consider the elements

$$g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is easily checked that

$$g^2 = h^3 = \begin{pmatrix} -I_2 & O \\ O & 1 \end{pmatrix}$$

So $(h^2, h, h^3 = g^2, g)$ is an induced path of length 3 in the power graph.

Now observe that neither g nor h contains any non-identity scalar matrix. So these elements project onto elements with the same property in the quotient when a group of scalars is factored out.

Now we consider q to be a power of 2, with $q > 4$. If q is an odd power of 2, then $q-1$ is not divisible by 3, while if q is an even power of 2, then $q-1$ cannot be a power of 3 (according to the solution of Catalan's equation) and so must have a larger prime divisor.

Let α be an element of the multiplicative group of $\mathrm{GF}(q)$ of prime order p greater than 3. Consider the elements

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad k = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-2} & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

It is routine to check that

$$g^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

an element of order 2; and that g^2 commutes with k, so that g^2k has order $2p$, and $(g^2k)^p = g^2$.

Putting $h = g^2k$, we have $h^p = g^2$, so the elements $g, g^2 = h^p, h, h^2$ induce a path of length 3.

No power of any of these elements except the identity is a scalar. (For this we need $p > 3$, since if $p = 3$ then $\alpha^{-2} = \alpha$.) So factoring out a group of scalars we get elements with the same properties.

Finally we note that $\mathrm{PSL}(3,2)$ and $\mathrm{PSL}(3,4)$ are power-cographs (their Gruenberg–Kegel graphs are null). However, $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$, so this group does not need to be included in the statement of the theorem. □

Theorem 5.6. *Let $G = \mathrm{PSp}(4,q)$. Then $P(G)$ is not a cograph.*

Proof. A 4-dimensional symplectic space is the direct sum of two 2-dimensional symplectic spaces; and the 2-dimensional symplectic group is the special linear group. So $G = \mathrm{PSp}(4,q)$ contains a subgroup which is the direct product of two copies of $\mathrm{PSL}(2,q)$ if q is even, or the central product of two copies of $\mathrm{SL}(2,q)$ if q is odd.

Thus G contains the direct product of cyclic groups of orders $q \pm 1$ if q is even, and a quotient of this by a subgroup of order 2 if q is odd.

For q even, $q-1$ and $q+1$ are coprime, so $P(G)$ is a cograph only if both are primes; since one is divisible by 3, this requires $q = 2$ or $q = 4$.

For q odd, one of $(q-1)/2$ and $(q+1)/2$ is even, so the order of the cyclic subgroup is divisible by 4 and (if $q > 3$) by at least one further prime. So $P(G)$ is a cograph only if $q = 3$.

Now $\mathrm{PSp}(4,2) \cong S_6$ is not simple; $\mathrm{PSp}(4,3)$ contains elements of order 12; and $\mathrm{PSp}(4,4)$ is ruled out by the 4-6 test. □

Theorem 5.7. *The power graph of $G_2(q)$ is not a cograph.*

Proof. The group $G_2(q)$ contains both $\mathrm{SL}(3,q)$ and $\mathrm{SU}(3,q)$ [15, 21]. Now $\mathrm{SL}(3,q) = \mathrm{PSL}(3,q)$ if $q \not\equiv 1 \pmod 3$, while $\mathrm{SU}(3,q) = \mathrm{PSL}(3,q)$ if $q \not\equiv -1 \pmod 3$. So, for any q, $G_2(q)$ contains either $\mathrm{PSL}(3,q)$ or $\mathrm{PSU}(3,q)$. Now the former is in $\mathcal{C}$ only for $q = 2$ or $q = 4$, and the latter is never in $\mathcal{C}$ except for $q = 2$ (this group is not simple). So the only case needing further consideration is $q = 2$; but $G_2(2)$ is not simple, and is not in $\mathcal{C}$ (it contains $\mathrm{PSU}(3,3)$ as a subgroup of index 2). □

Below we give arguments for rest of the simple groups of Lie type of rank 2. We find that in each of the following cases the power graph is not a cograph.

- Let $G = {}^2A_3(q) = \mathrm{PSU}(4, q)$. This group contains $\mathrm{PSp}(4, q)$, so we only need consider $q = 2$. But $\mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$.
- The group $G = {}^2A_4(q) = \mathrm{PSU}(5, q)$ contains $\mathrm{PSU}(4, q)$. So $G \notin \mathcal{C}$.
- The group ${}^2F_4(2^d)$ contains ${}^2F_4(2)$ for all odd d (Malle [23]), and ${}^2F_4(2)$ is ruled out by the 4-6 test.
- The group $G = {}^3D_4(q)$ contains $G_2(q)$ (see Kleidman [20]).

5.3 Higher rank

Let G be a simple group of Lie type of higher rank. We show that $P(G)$ is not a cograph.

Since the Dynkin diagram of G contains a single bond in all cases, G has a subgroup of a Levi factor which is a quotient of $\mathrm{SL}(3, q)$ by a group of scalars. The results of the preceding section give the desired conclusion if $q \notin \{2, 4\}$.

It remains to deal with groups over the fields of 2 or 4 elements.

Now $\mathrm{PSL}(4, 2) \cong A_8$, so its power graph is not a cograph, while $\mathrm{PSp}(6, 2)$ is excluded by the 4-6 test. Moreover, $\mathrm{PSL}(4, 4)$ contains $\mathrm{PSL}(4, 2)$, and $\mathrm{PSp}(6, 4)$ contains $\mathrm{PSp}(6, 2)$ (by restricting scalars). The orthogonal and unitary groups of Lie rank 3 all contain $\mathrm{PSp}(4, q)$ for $q = 2$ or $q = 4$. So $P(G)$ is not a cograph.

5.4 Sporadic simple groups

Now we prove that there exist no sporadic simple group whose power graph is cograph. Recall that there are 26 sporadic simple groups [14], namely, the five Mathieu groups (M_{11}, M_{12}, M_{22}, M_{23} and M_{24}), four Janko groups (J_1, J_2, J_3 and J_4), three Conway groups (Co_1, Co_2 and Co_3), three Fischer groups (Fi_{22}, Fi_{23} and Fi_{24}), Higman–Sims group (HS), the McLaughlin group (M^cL), the Held group He, the Rudvalis group (Ru), the Suzuki group (Suz), the O'Nan group ($O'N$), the Harada–Norton group HN, the Lyons group (Ly), the Thompson group (Th) the Baby Monster group (B)and the Monster group (M). Amongst these 26 groups the the Mathieu group M_{11} is of smallest order ($|M_{11}| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$).

Observation 5.8. We observed, using information in the 𝔸𝕋𝕃𝔸𝕊 of Finite Groups [14], that M_{11} is not a power-cograph group, by the 4-6 test; it contains elements a, b of orders 4 and 6 respectively with $a^2 = b^3$.

Theorem 5.9. *Let G be a sporadic simple group. Then $P(G)$ is not a cograph.*

Proof. Observation 5.8 shows that the power graph of the Mathieu group M_{11} is not a cograph. Now the Mathieu group M_{11} is a subgroup of all the other sporadic simple groups except J_1, M_{22}, J_2, J_3, He, Ru and Th. So the power graphs of these groups are also not cographs.

For the other seven groups we look for subgroups which are not power-cograph groups. We observe that J_1 contains $D_3 \times D_5$, M_{22} contains A_7, J_2 contains $A_4 \times A_5$, J_3 contains $C_3 \times A_6$, He contains S_7, Ru contains A_8 and Th contains $\mathrm{PSL}(2, 19) : C_2$. By Theorems 2.1, 4.2 and 5.1, the power graphs of these subgroups are not cographs. Hence the power graphs of the original groups are not cographs. □

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