Farey Maps, Spectra and Integer Continued Fractions

Thesis

Farey maps, spectra and integer continued fractions

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This thesis is dedicated to my husband David. It would not have been possible without his unfailing understanding, encouragement and support.

Abstract

We examine the structure of Farey maps, a class of graph embeddings on surfaces that have received significant attention recently. When the Farey graph is embedded in the hyperbolic plane it induces a tessellation by ideal triangles. Farey maps are the quotients of this tessellation by the principal congruence subgroups of the modular group. We describe how the Farey maps of different levels are related to each other through regular coverings and parallel products, and use this to find their complete spectra. We then generalise Farey maps to include those defined by non–principal congruence subgroups of the modular group, finding their spectra and diameter. We also examine a similar class of maps defined by Hecke groups, again obtaining results for their spectra and diameter. Most of this work is the subject of [63], which has been published in Acta Mathematica Universitatis Comenianae.

Fundamental to the theory of continued fractions is the fact that every infinite continued fraction with positive integer coefficients converges; however, this is not so if the coefficients are integers which are not necessarily positive. We show that integer continued fractions can be represented as paths on the Farey graph, and use this to develop a simple test that determines whether an integer continued fraction converges or diverges. In addition, for convergent continued fractions, the test specifies whether the limit is rational or irrational. This work, carried out jointly with Ian Short, is the subject of [57], which has been published in the Proceedings of the American Mathematical Society.

Finally further work is described, including practical applications of our spectral results, and a search for interesting expansions of real numbers as generalised continued fractions.

Declaration

I confirm that this thesis is my own work, apart from Chapter 5 and Section 6.4 of Chapter 6, which are the result of joint work with Ian Short. It has not been submitted for another qualification to this or any other university or institution.

Margaret Stanier

September 25, 2022

Publications

Much of the work in Chapters 3 and 4 has been published in:

M. Stanier, *Regular coverings and parallel products of Farey maps*, Acta Math. Univ. Comenian. (N.S.) **91** (2022), no. 1, 1–18. (ArXiv:2104.03905)

Much of the work in Chapter 5 has been published in:

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Contents

Chapter 1

Introduction

This thesis is an account of the class of graph embeddings on surfaces known as *Farey maps*. A feature of these maps is that paths on them can represent integer continued fractions; we use this to find conditions for the convergence of these fractions. We also study Farey map spectra, proving conjectures developed from experimental results by considering how different Farey maps are connected through coverings and parallel products.

In this introduction we give a selection of our most significant results, then explain in detail the motivation and context for each one. The formal definitions of the terms we use will be given in Chapter 2, together with the background results from classical theory which we use.

1.1 Main results

Our first main results concern the way in which Farey maps are connected through *coverings*. This work is explained in detail in Chapter 3, and is the subject of [63]. We will use coverings to find the spectra of Farey and related maps, obtaining by more straightforward means results given recently in [50] using the theory of finite valuation fields. We also use coverings to give an alternative proof of a recent result in [38] concerning certain map diameters.

Maps are graphs embedded in surfaces in such a way that the components of their complements are homeomorphic to the unit disc. As we explain in more detail later, the universal triangular map is shown in [59] to be the regular triangulation of the hyperbolic plane known as the *Farey tessellation.* Its automorphism group is the modular group. Farey maps are regular maps on oriented surfaces which are constructed by taking quotients of the Farey tessellation by congruence subgroups of the modular group of different *levels.* A formal definition is given in Section 3.3.

Our first two theorems concern the way in which Farey maps of different levels are connected to each other through coverings and *parallel products.*

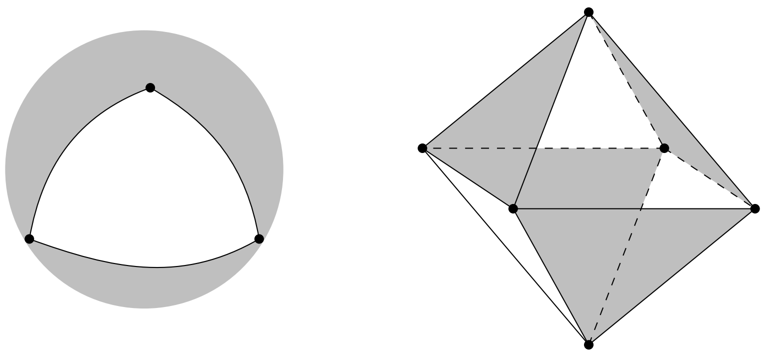

Figure 1.1.1: The Farey maps of level 2, a triangle, and level 4, an octahedron. The octahedron is a 4-sheeted covering of the triangle, ramified at the vertices, with ramification index 2.

The formal definitions of the terms used are given in Section 2.7. We give an informal example in Figure 1.1.1, which shows the Farey map of level 2, a triangle embedded in a sphere, and the Farey map of level 4, an octahedron. A *net* of the octahedron consists of 4 rhombuses, pairs of white and grey triangles, which could be cut and folded in such a way that each rhombus covers the 2 faces of the Farey map of level 2. We say that the octahedron is a 4-*sheeted covering* of the Farey map of level 2. It has 4 times as many faces and 4 times as many edges, but only twice as many vertices as each vertex is in 2 sheets. The covering is *ramified* at the vertices, with *ramification index* 2. The covering would be *unramified* if each vertex were in exactly one sheet - a map consisting of 4 separated triangles embedded in a surface topologically equivalent to 4 spheres would be an unramified 4-sheeted covering of the Farey map of level 2.

The parallel product of two maps is the smallest map which is a covering of both maps.

Theorem 1.1.1. *For a prime p and a positive integer k, the Farey map of level p^k is a regular* *covering of the Farey map of level* p^{k-1}*, ramified at the vertices with ramification index* p*. The covering has* 4 *sheets if* $p^k = 4$*, and otherwise has* p^3 *sheets.*

Theorem 1.1.2. *Let* m *be a positive integer.*

(i) *If* m *is odd, the Farey map of level* $2m$ *is the parallel product of the the Farey map of level* 2 *and the Farey map of level* m*.*

(ii) *If* l *and* m *are coprime integers, and neither* l *nor* m *is twice an odd integer, then the Farey map of level* lm *is a regular unramified double covering of the parallel product of the Farey map of level* l *and the Farey map of level* m*.*

Work has also been developed in [30, 38] on similar quotients of other universal tessellations, the *Hecke maps.* In [30] it is shown that the universal map whose faces are q sided polygons is a tessellation whose automorphism group is the Hecke group of level q; in fact the modular group is the Hecke group of level 3. Hecke maps are constructed from the Hecke group in a way analogous to the construction of Farey maps from the modular group. We will prove the following theorem in Chapter 3, using graph coverings rather than map coverings.

Theorem 1.1.3. *For odd* n*, the underlying graph of the Hecke map of type* $(4, n)$ *is a double graph covering of the underlying graph of the Farey map of type* $(3, n)$*. If* n *is not a multiple of* 3*, the underlying graph of the Hecke map of type* $(6, n)$ *is a double graph covering of the underlying graph of the Farey map of type* $(3, n)$*.*

These theorems then enable us to find the complete spectra of all Farey maps and of some Hecke maps, as is explained in detail in Chapter 4.

The second part of this thesis presents new necessary and sufficient conditions which depend solely on the coefficients for the convergence of a continued fraction with integer coefficients. This result is explained in detail in Chapter 6.4, and is the subject of [57], which has been published in the Proceedings of the American Mathematical Society. A *regular continued fraction* is a continued fraction

$$(b_0, b_1, \dots) = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}. \tag{1.1.1}$$

such that the *coefficients* b_i are strictly positive integers. Conditions for the convergence of these fractions have been well known since the work of Euler and Lagrange in the eighteenth century, but we do not know of any such results for *integer continued fractions*, whose coefficients can be any integer, positive, negative or zero. However integer continued fractions can be interpreted as paths on the Farey graph or tessellation, and we have been able to use this idea to obtain, for the first time, criteria for their convergence.

We give further technical details on continued fractions in Section 2.8. It is usual to write an infinite integer continued fraction as in (1.1.1). However we prefer to use the *negative continued fractions* defined by

$$[b_0, b_1, \dots] = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots}}}, \tag{1.1.2}$$

where the coefficients b_i can be positive or negative integers, or zero. We can easily switch between this expression and the more usual one by using the formula

$$(b_0, b_1, b_2, b_3, \dots) = [b_0, -b_1, b_2, -b_3, \dots]. \tag{1.1.3}$$

We will define an iterating function mapping the collection of negative continued fractions to itself. Then $p^{(n)}$, the *key position*, is the position of the first coefficient equal to 0,1, or -1 after n iterations of this function, and $q^{(n)}$ is the modulus of the coefficient in the position preceding $\liminf p^{(n)}$ if that limit is finite. We will show that, if $p^{(n)} \to \infty$, the continued fraction stabilises as a limit continued fraction $[b_0^*, b_1^*, \dots]$ (in a sense which we will make more precise later). We will prove the following result in Chapter 5.

Theorem 1.1.4. *Let* $[b_0, b_1, \dots]$ *be a negative continued fraction.*

1. *Suppose that* $p^{(n)} \to \infty$.
   (a) *If* $[b_0^*, b_1^*, \dots]$ *has the same tail as* $[2, 2, \dots]$ *or* $[-2, -2, \dots]$, *then* $[b_0, b_1, \dots]$ *converges to a rational.*
   (b) *Otherwise,* $[b_0, b_1, \dots]$ *converges to an irrational.*
2. *Suppose that* $p^{(n)} \not\to \infty$.
   (a) *If* $q^{(n)} \to \infty$, *then* $[b_0, b_1, \dots]$ *converges to a rational or to infinity.*
   (b) *If* $q^{(n)} \not\to \infty$, *then* $[b_0, b_1, \dots]$ *diverges.*

We now give details of the context for our results.

1.2 Background to the study of Farey maps

Cartographic maps, which in this thesis we simply refer to as *maps*, are graphs embedded in surfaces in such a way that the surface is divided into regions with interiors homeomorphic to the unit disc. We use the theory of maps on surfaces given in [59] and [33]. In these papers universal maps on Riemann surfaces are introduced, and shown to be regular tessellations by regular polygons. All finite maps on oriented surfaces are quotients of these universal maps by various groups. Specifically, in [59], it is shown that the universal triangular map is the Farey tessellation, which was first introduced by Hurwitz in 1894 in the context of quadratic forms. It is embedded in the hyperbolic plane, and its quotients are embedded in Riemann surfaces.

Background to the study of Riemann surfaces can be found in [17] and [32]. In developing their study of these surfaces and their representation as algebraic curves in 1880, Klein and his student Dyck were particularly interested in two objects which describe the symmetries of algebraic curves, and which we now call maps. These are the *Klein map*, a regular map of degree 7, which has 24 vertices and is embedded in the surface of genus 3 known as the

Klein quartic; and the *Dyck map*, a regular map of degree 8 with 12 vertices, which is also embedded in a surface of genus 3. They were able to describe these maps as graph embeddings in certain algebraic curves (representing Riemann sufaces) with equations, in homogeneous complex variables, $x^3y + y^3z + z^3x = 0$ and $x^4 + y^4 + z^4 = 0$ respectively. In [43] there are studies of the Klein quartic and of the Klein map, which turns out to be an example of the family of maps which is the main focus of our study, the Farey maps. In fact, in [30], the Klein map is constructed as the Farey map of level 7. In Chapter 3 we also study a related family of maps which includes Dyck's map.

Much current work defines maps in terms of their *flags*, a term which describes a triple consisting of a vertex, an incident edge, and a face incident to that edge. This is necessary for the study of maps embedded in surfaces which are not oriented. As the hyperbolic plane and its quotients are oriented, we will instead define maps on oriented surfaces in terms of their directed edges, or *darts*, as do [26, 27, 33, 59].

1.3 Spectra of graphs and maps

One of the most important practical applications of graph theory is to the design of communication networks. As is explained in [40], it is important that these networks be as fast as possible, which requires a small diameter, but also as reliable as possible, so that they are not put completely out of action by the failure of one or two circuits. It follows that it is not desirable for the number of cuts needed to separate a set of vertices from the rest of the network to be too small. Considerable emphasis has been given to the study of *expanders*, which are families of graphs each of which, for a given diameter, maximises the number of cuts needed to separate a set of vertices from its complement. Expanders can be used to construct large reliable networks. A useful measure of reliability is the *isoperimetric constant*, which is the normalised minimum of the number of edges joining a set of vertices to its complement. As is shown, for instance in [40] and [68], the isoperimetric constant is related to certain eigenvalues of the adjacency matrix of the network.

It is also desirable that the *mixing time* of a network, that is the time needed for any input to reach a stationary distribution, be as short as possible. A well–known classical result ([45, Theorem 3.1]) relates mixing time to the modulus of a certain graph eigenvalue; this needs to be as small as possible. In [20] an expected value for this modulus is found for a random graph. Graphs that are 'better than random' are known as *Ramanujan graphs* and are the subject of considerable research (see for instance [16, 23, 50]).

Both these important practical considerations motivate the study of the spectra, or sets of eigenvalues, of graphs and maps, the basic theory for which is found in [7, 22]. We have found results for the spectra of Farey maps and of some other families of maps by considering the way they are connected to each other through *regular coverings* and through the *parallel products* developed in [66, 67].

1.4 Continued fractions and paths on Farey maps

Our results on integer continued fractions are motivated in part by the work of Beardon, Hockman and Short, [3], who prove that all integer continued fractions can be represented by paths on the Farey tessellation. Caroline Series in [56], uses a different geometric representation of regular continued fractions on the Farey tessellation.

It is explained in [39, 64] how the search for good approximations to π led Lord Brouckner to propose to Wallis in 1655 the *continued fraction* expansion

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}.$$

In the eighteenth century Euler found a wealth of further expansions, as shown for instance in [18, 19].

The theory of regular continued fractions, both finite and infinite, is now a central part of number theory and is explored in many texts, such as [24]. During the early twentieth century the theory of continued fractions was extended to those with rational, real and complex coefficients.

There is an extensive literature on the convergence of continued fractions – from Perron's classic treatise [51, 52] to more recent texts such as [44] – and there are a wealth of algorithms for generating different types of integer continued fractions (see, for example, [28, Chapter 4] for algorithms in the metric theory of continued fractions, and [37] for algorithms in the geometric theory of continued fractions). A wide variety of tests for convergence of continued fractions with real and complex coefficients are known. For instance, if $|b_n| \geqslant 2$ for $n = 0, 1, \dots$, then the integer continued fraction $(b_0, b_1, \dots)$ given by equation (1.1.1) converges – even if the coefficients b_n are complex numbers – and some slightly stronger tests are known when the coefficients are integers (such as [36, Lemma 1.1]). There has, however, been relatively little work on the general theory of integer continued fractions. The second part of this thesis, which gives a necessary and sufficient condition for their convergence, extends results concerning the convergence of paths on the Farey tessellation in [58] and applies them to integer continued fractions. The detail of this work is set out in Chapter 5.

Chapter 2

Preliminaries

In this chapter we give the theoretical background for our research in detail, collecting the definitions and results which we need.

We will use the following notation: $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ the set of natural numbers, and we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the real number line, $\mathbb{C}$ the complex plane, and $\mathbb{Q}$ the set of rationals. If $z \in \mathbb{C}$, we will write $z = x + iy$, where $i^2 = -1$, $x = \mathrm{Re}(z)$ is the real part of z, and $y = \mathrm{Im}(z)$ is the imaginary part of z.

Adjoining the point at infinity, we define the *extended set of integers* by $\mathbb{Z}_\infty = \mathbb{Z} \cup \infty$, and, similarly, the extended sets of real numbers, complex numbers and rationals by $\mathbb{R}_\infty = \mathbb{R} \cup \infty$, $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and $\mathbb{Q}_\infty = \mathbb{Q} \cup \infty$. We will represent a member of $\mathbb{Q}$ by the *reduced rational* a/b, with $a, b \in \mathbb{Z}, b > 0$, and $\gcd(a, b) = 1$, and use the convention that $1/0$ represents ∞ when considered a member of $\mathbb{Q}_\infty$; in this case we also take $1/0$ to be a reduced rational.

2.1 Some linear and other groups

We will use the *general linear group* $\mathrm{GL}_n(K)$, whose members are the invertible symmetrical $n \times n$ matrices with elements in K, where K can be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We also use the *special linear group*, which is the subgroup $\mathrm{SL}_n(K)$ of $\mathrm{GL}_n(K)$ whose elements have determinants equal to 1. The *projective special linear group* $\mathrm{PSL}_2(K)$, is the quotient of $\mathrm{SL}_2(K)$ by the subgroup $\{\pm I\}$, where I is the identity of $\mathrm{SL}_2(K)$. In order to avoid the frequent use of the $\pm$ symbol, we will often write the members of $\mathrm{PSL}_2(K)$ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K).$$

We will also frequently use the group of *Möbius transformations* acting on $K_\infty = K \cup \infty$, given, for all $a, b, c, d \in K$, with $ad - bc \neq 0$, by

$$\begin{array}{rcl} K_\infty & \longrightarrow & K_\infty \\ z & \longmapsto & \dfrac{az+b}{cz+d}, \end{array}$$

with the usual conventions for the point at infinity.

We can define a left action of $\mathrm{GL}_2(K)$ on K by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

and noting that, for all $\lambda \in K, \quad \lambda \neq 0$

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az+b}{cz+d},$$

we see that the action of the group of Möbius transformation coincides with that of the left action of the *projective special linear group* $\mathrm{PSL}_2(K)$.

If s and t are Möbius transformations, and the equivalent members of $\mathrm{PSL}_2(K)$ are S and T respectively, then we write the action of s followed by the action of t as $t \circ s$. Its equivalent matrix form is the matrix product TS.

The set of all possible permutations of the elements of a collection Ω of objects form the group $\mathrm{Sym}(\Omega)$. In group theory the action of a permutation x on the elements of Ω is a right action. If $h_1, h_2, h_3 \in \Omega$, x takes h_1 to h_2, and y takes h_2 to h_3, we write $h_1 x = h_2$, and $h_1 xy = h_3$. Note that if Ω is a set of objects which can be mapped to each other by Möbius transformations, and the actions of the Möbius transformations s and t coincide with that of the permutations x and

y respectively, we write $t \circ s(h_1) = h_3$, but $h_1 xy = h_3$.

Let G and H be two groups with group operations $+$ and $\times$ respectively. Then a transformation $\sigma : G \longrightarrow H$ is a *group homomorphism* if, for all $u, v \in G, \sigma(u+v) = \sigma(u) \times \sigma(v)$. The *kernel* of σ in G, $\mathrm{Ker}(\sigma)$, is the set of members of G which σ maps onto the identity of H. If H is a normal subgroup of G, and σ is a surjective homomorphism, $\mathrm{Ker}(\sigma)$ is a normal subgroup of G.(See for instance [54, Theorems 3.9 and 3.22].) We will use the following result

$$G/\mathrm{Ker}(\sigma) \quad \cong \quad H. \tag{2.1.1}$$

2.2 The hyperbolic plane

The *upper half plane* is the set $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The Riemannian or hyperbolic metric is

$$|ds|^2 = \frac{dx^2 + dy^2}{y} = \frac{|dz|^2}{\mathrm{Im}(z)}.$$

The upper half plane model of the *hyperbolic plane* is the upper half plane endowed with the Riemannian metric. We define the unit disc to be $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Using the Möbius transformation

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathbb{D} \\ z & \longmapsto & \dfrac{i(1+z)}{1-z}, \end{array}$$

we can transfer the hyperbolic metric from $\mathbb{H}$ to $\mathbb{D}$ to give the *unit disc model* of the hyperbolic plane. We will mainly use the upper half plane model.

The paths of shortest length on a surface are its *geodesics* (also known as *hyperbolic lines*). The geodesics between points on $\mathbb{H}$ are either vertical lines or segments of semicircles centered on the real axis. Then the action on $\mathbb{H}$ of any member of the subgroup of the group of Möbius transformations such that $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ is an isometry, and is an orientation preserving automorphism. In particular, geodesics are preserved.

Note that, as $\mathbb{H}$ does not include its limit points, which are ∞ and the points on the real axis, it is not compact.

Further details and proofs on this topic can be found in texts such as [17, 21, 32].

2.3 Graphs

A *graph* consists of a set of vertices and a set of unordered pairs of vertices called edges. Two vertices u and v in the same edge $\{u, v\}$ are said to be *adjacent*. We write $v \sim u$. Alternatively, we say that v is a *neighbour* of u, and vice-versa. The edge $\{u, u\}$ is a *loop*. A *simple* graph has no vertices which are not members of an edge, and no loops. For a simple graph, the number of neighbours of a vertex is its *valency*. A simple graph in which every vertex has the same valency is *regular*. The vertex valency is then termed the *degree* of the graph. A directed edge, or *dart*, of a graph is, if the unordered pair $\{u, v\}$ is an edge, one of the ordered pairs of vertices (u, v) or (v, u). We say that u is the *initial vertex* and v the *final vertex* of the dart (u, v). The graphs considered in this thesis are all simple graphs, which we shall just call graphs. A simple graph can be completely specified by its set of darts. Unless we state otherwise, we consider *undirected* graphs, that is graphs for which, if (u, v) is a dart, (v, u) is also a dart.

A *path* in a graph $\mathcal{G}$ is is a sequence $v_1, v_2, \dots, v_N$ of vertices of $\mathcal{G}$ such that $v_i \sim v_{i+1}$ for $i = 1, 2, \dots, N-1$. The *length* of this path, or the *distance* between the two vertices v_1 and v_N along this path, is $N-1$. We denote the shortest distance between 2 vertices u and v of a graph by $d(u, v)$. The diameter of a graph is the largest value of $d(u, v)$ between any two of its vertices.

A graph *isomorpism* is a bijection from one graph to another that preserves adjacency between the vertex sets of the two graphs. An isomorphism from a graph to itself is a *graph automorphism.* The set of all automorphisms of a graph $\mathcal{G}$ form its *automorphism group* $\mathrm{Aut}(\mathcal{G})$. From [22, Lemma 1.3.1], if a graph is connected, its graph automorphisms permute the vertices of equal valency amongst themselves. A *graph homomorphism* is a transformation from one graph to another that preserves adjacency between the vertex sets of the two graphs. A surjective graph homomorphism $\pi : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a *graph covering transformation* if π induces a bijection between the set of vertices adjacent to any vertex v of $\mathcal{G}_2$ and the set of vertices adjacent in $\mathcal{G}_1$ to any vertex in $\pi^{-1}(v)$. $\mathcal{G}_1$ is then a *graph covering* of $\mathcal{G}_2$, as defined in [22].

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are graphs with dart sets Ω_1 and Ω_2, respectively, the *graph direct product* or *graph tensor product* $\mathcal{G}_1 \times \mathcal{G}_2$ is the graph whose dart set is the cartesian product of Ω_1 and Ω_2.

The *adjacency matrix* A of an N vertex graph $\mathcal{G}$ is the $N \times N$ square matrix whose rows and columns are indexed by the vertices $v_0, v_1, \ldots, v_{N-1}$ of $\mathcal{G}$. The entry of A corresponding to the vertices v_i and v_j is 1 if $v_i \sim v_j$, and 0 otherwise. The *eigenvalues* of A are the solutions of the characteristic equation $|A - \lambda I| = 0$; their multiplicity as solutions of this equation is their *algebraic multiplicity.* The total number of eigenvalues counting multiplicities is N.

If the eigenvalues of A are $\lambda_1 \geq \cdots \geq \lambda_N$, the *spectrum* of $\mathcal{G}$ is the multiset of these eigenvalues with their algebraic multiplicities, which we write $\mathrm{sp}(\mathcal{G}) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_i^{(m_i)}\}$ if its i distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_i$ with algebraic multiplicities $m_1, m_2, \ldots, m_i$ respectively.

The *complete graph* on n vertices, K_n is a graph all of whose vertices are adjacent to every other vertex. K_n is regular of degree $n - 1$. It is well known (see for instance [7, Section 1.4.1]) that

$$\mathrm{sp}(K_n) = \{-1^{(n-1)}, n - 1\}. \tag{2.3.1}$$

2.4 The Farey graph

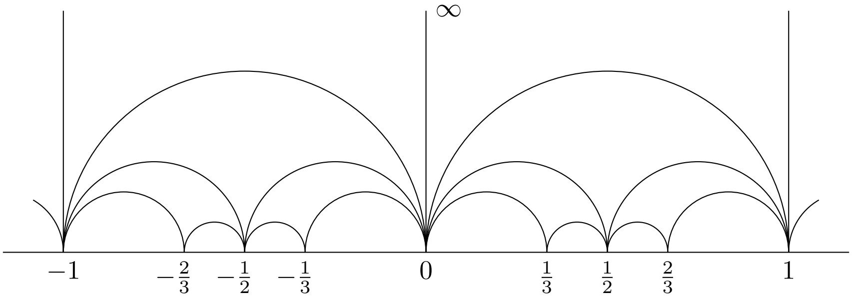

Figure 2.4.1: Part of the Farey graph drawn on the hyperbolic plane.

The extended rationals in $\mathbb{Q}_\infty$ lie on the boundary $\mathbb{R}_\infty$ of $\mathbb{H}$. The *Farey graph* is the graph whose vertices are the members of $\mathbb{Q}_\infty$ represented by reduced rationals (including $\infty = 1/0$), and edges comprising those pairs a/b and c/d of reduced rationals for which $ad - bc = \pm 1$.

We represent the edge incident to a/b and c/d by the unique geodesic in $\mathbb{H}$ between those two boundary points. The collection of all edges creates an embedding of the Farey graph in the hyperbolic plane which is a tessellation of $\mathbb{H}$ by triangles, the *Farey tessellation* $\mathscr{F}$, part of which is shown in Figure 2.4.1.

The *modular group* Γ is the group of unimodular Möbius transformations with coefficents in $\mathbb{Z}$, given, for all $a, b, c, d \in \mathbb{Z}$, with $ad - bc = 1$, by

$$\begin{aligned} K_\infty &\longrightarrow K_\infty \\ z &\longmapsto \frac{az+b}{cz+d}, \end{aligned}$$

The modular group is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$. It acts on $\mathbb{C}_\infty$, on $\mathbb{R}_\infty$, and on $\mathbb{Q}_\infty$, with the standard conventions regarding the point ∞. It acts on $\mathbb{H}$ as a group of orientation preserving hyperbolic isometries. It is straightforward to check that the action of any member of Γ preserves adjacency between the vertices of $\mathscr{F}$, so Γ also acts on $\mathscr{F}$; therefore, as well as being an orientation preserving automorphism of $\mathbb{H}$, each element of Γ induces a graph automorphism of $\mathscr{F}$.

2.5 Surfaces

A *topological space* is a set X of elements which we call points, together with a collection of its subsets, which are called its *open sets*. Both X and the empty set are open sets, and the collection of open sets is such that any union of open sets is open, and the intersection of a finite number of open sets is open.

A *Hausdorff space* is a topological space in which any two distinct points are in open sets whose intersection is empty.

A transformation between two topological spaces is *continuous* if the inverse image of every open set is open. A *homeomorphism* between two topological spaces is a transformation which is bijective and bicontinous.

We define a *topological surface*, as in [21, 32, 49], as a Hausdorff space $\mathcal{S}$ with an *atlas*. This is a collection of pairs (U_i, ϕ_i), where U_i is an open set of $\mathcal{S}$ and $\bigcup_i U_i$ covers $\mathcal{S}$; ϕ_i is a homeomorphism, called the *chart* of U_i, from U_i to an open set of the complex plane $\mathbb{C}$, and, if U_1 and U_2 with charts $\phi_1 : U_1 \to \mathbb{C}$ and $\phi_2 : U_2 \to \mathbb{C}$ respectively have a non-empty intersection, the *transition function* $\phi_2 \circ \phi_1^{-1}$, which is a mapping between open subsets of $\mathbb{C}$, is also a homeomorphism. If all the transition functions of the atlas are holomorphic (that is differentiable everywhere), the surface is a *Riemann surface*.

It is shown in [49, chapter 3] that every surface is homeomorphic to a space obtained from the unit sphere by adding either *g handles*, in which case the surface is an *orientable surface of genus g*, or *h crosscaps*, in which case it is a *non-orientable surface of genus h*.

A *path* on a surface is the image of the unit interval $[0, 1]$ in $\mathbb{C}$ under a homeomorphism. A path which starts and finishes at the same point is a *closed* path. If, given any two points on a surface, there is a path between them, then that surface is said to be *connected*. If the closed paths on a surface can be continuously deformed onto each other, the surface is *simply connected*.

2.6 Maps as graph embeddings in surfaces

We define a *topological map* $\mathcal{M}_T$ as the embedding of a finite graph $\mathcal{G}$ (the *underlying graph*) in a compact surface $\mathcal{S}$ (the *supporting surface*) such that each component of the complement of $\mathcal{G}$ in $\mathcal{S}$ is homeomorphic to an open disc.

The components of the complement of $\mathcal{G}$ in $\mathcal{S}$ are the *faces* of the map. We write the numbers of vertices, edges and faces of the map as, respectively, $V(\mathcal{M}_T), E(\mathcal{M}_T)$ and $F(\mathcal{M}_T)$, or simply as V, E, F if there is no ambiguity. It is well known (see for instance [49, Section 3.1]) that if a graph is embedded in an oriented surface its genus, that is the genus g of its supporting surface, is given by

$$2 - 2g = V - E + F. \tag{2.6.1}$$

Much work on topological maps deals with them algebraically. This was put on a firm theoretical basis in [33]. In that paper it is shown, as we briefly sketch below, that any map defined topologically is equivalent to an object defined purely algebraically called an algebraic map. This enables the powerful tools of group algebra to be used to study maps and their underlying graphs.

As in [33], we define an *algebraic map* as a quadruple $\mathcal{M}_A = (G, \Omega, x, y)$, where Ω is a set of objects, G is a subgroup of the group $\mathrm{Sym}(\Omega)$ of all permutations of Ω, x and y generate G, $x^2 = e$, where e is the identity element of G, and G is transitive on Ω.

If a topological map $\mathcal{M}_T$ is the embedding of a finite connected graph $\mathcal{G}$ in an orientable compact surface $\mathcal{S}$ which has no boundary, we construct an algebraic map $\mathrm{Alg}(\mathcal{M}_T) = (G, \Omega, x, y)$ by making Ω the set of darts or directed edges of $\mathcal{G}$. Then we define x as the permutation of Ω which sends each dart to the other dart on the same edge, and y as the permutation of Ω which sends each dart to the next dart incident to its initial vertex in the direction of the positive orientation of the surface $\mathcal{S}$. We define G as the group generated by x and y. We then define the vertices of $\mathrm{Alg}(\mathcal{M}_T)$ algebraically as the left cosets of $\langle y \rangle$ in G, and the edges as the left cosets of $\langle x \rangle$ in G.

The action of the permutation xy^{-1} on a dart h consists of mapping h onto the other dart on the same edge, which is hx, and then sending that dart to the next dart around the final vertex of h in a direction opposite to that of the positive orientation of $\mathcal{S}$, which is hxy^{-1}. As shown in Figure 2.6.1, this is the dart following h in a path around a face. If m is the order of xy^{-1}, the repeated action of xy^{-1} takes h to m succesive darts in a closed path in the direction of the positive orientation of $\mathcal{S}$ ending at the inital vertex of h. This path encloses a face of the topological map, so we define the faces of the algebraic map as the left cosets of xy^{-1} in G, with incidence defined by non-empty intersection.

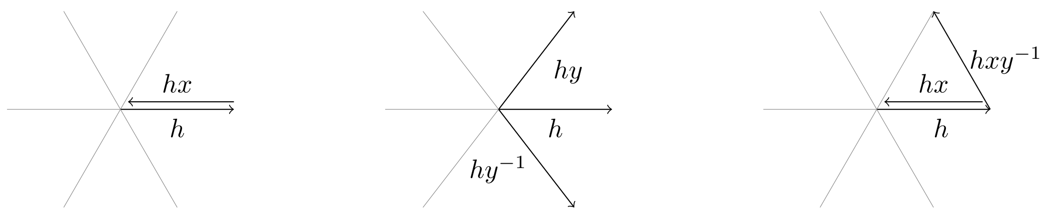

Figure 2.6.1: The actions of the permutations x, y, y^{-1} and xy^{-1} on the dart h of a topological map embedded in a surface whose positive orientation is anticlockwise.

If m is the order of xy^{-1} in G, and n the order of y in G, we say that both $\mathcal{M}_T$ and $\mathrm{Alg}(\mathcal{M}_T)$ are of *type* (m, n). It is shown in [33, Corollary 5.2, Proposition 5.3] that, from $\mathrm{Alg}(\mathcal{M}_T) = (G, \Omega, x, y)$, a topological map $\mathcal{M}_R$ of the same type can be constructed in some

Riemann surface, and that $\mathcal{M}_T$ is isomorphic to $\mathcal{M}_R$. Unless the context is not clear, we will in what follows simply refer to a map $\mathcal{M}$, using the algebraic definition but also referring to its topological model where this is helpful. If $\mathcal{G}$ is the underlying graph of a map $\mathcal{M}$, the spectrum of $\mathcal{M}$ is defined as $\mathrm{sp}(\mathcal{M}) = \mathrm{sp}(\mathcal{G})$.

A *topological map automorphism* is an angle preserving homeomorphism between the supporting surface and itself which induces a graph automorphism of the underlying graph. An *algebraic map automorphism* of an algebraic map (G, Ω, x, y) is the subgroup of $\mathrm{Sym}(\Omega)$ which preserves the incidence of edges and vertices. From [33, Corollary 3.2], we know that there is an epimorphism θ between the topological map automorphism group $\mathrm{Aut}(\mathcal{M}_T)$ and the algebraic map automorphism group $\mathrm{Aut}(\mathrm{Alg}(\mathcal{M}_T))$. Its kernel $\mathrm{Ker}(\theta)$ is the group of those angle preserving homeomorphisms of the supporting surface which leave each dart of the underlying graph invariant. So $\mathrm{Aut}(\mathcal{M}_T)/\mathrm{Ker}(\theta)$ is isomorphic to $\mathrm{Aut}(\mathrm{Alg}(\mathcal{M}_T))$. We will refer to this algebraic automorphism group simply as the *map automorphism group*, and denote it $\mathrm{Aut}(\mathcal{M})$.

We define a *regular map* as a connected map whose automorphism group is transitive on its darts. Then the automorphism group of its underlying graph is also transitive on its darts, and, by [22, Lemma 1.3.1], takes any vertex to another vertex with the same valency, so the graph is regular. If $\mathcal{M} = (G, \Omega, x, y)$ is a regular map of type (m, n), each vertex is incident to exactly n darts, so n is the *vertex valency*, and, with our definiton, each face is incident to m darts, so m is the *face valency*. Let $h, h' \in \Omega$. Then there is a $g \in G$ with a right action on h such that $hg = h'$, and, since the map is regular, an $a \in \mathrm{Aut}(\mathcal{M})$ with a right action on h such that $ha = h' = hg$. So there is an isomorphism between the actions of $\mathrm{Aut}(\mathcal{M})$ and of G on the set Ω. We choose a base dart $h_B \in \Omega$. Then any $h \in \Omega$ can be written $h = h_B g$ for a unique $g \in G$, and conversely for any $g' \in G$ there is a unique $h' \in \Omega$ such that $h' = h_B g'$. So we can identify the darts of a regular map $\mathcal{M} = (G, \Omega, x, y)$ with the elements of G, identifying h_B with e. In this way we will define a regular map algebraically simply by the triple (G, x, y), where $x^2 = e$ and $G = \langle x, y \rangle$.

The 5 *platonic solids* can all be considered regular maps embedded in a topological sphere. The tetrahedron is of type $(3, 3)$, the cube of type $(4, 3)$, the octahedron of type $(3, 4)$, the icosahedron of type $(3, 5)$ and the dodecahedron of type $(5, 3)$.

2.7 Map coverings

We will use coverings to compare maps. We use the term covering as it is used in [26,27]. In [33] the term morphism describes the same concept, and in [47] the term homomorphism is used.

As in [21, Definition 1.64], we define a *surface covering transformation* ϕ of *order* r from a topological surface $\mathcal{S}$ to a topological surface $\mathcal{S}'$ as a continuous transformation such that every point $x' \in \mathcal{S}'$ is in an open set B' of S' such that $\phi^{-1}(B') = \bigcup B_i$, where the r open sets $B_1, B_2, \ldots, B_r$ are pairwise disjoint and the restriction $\phi|_{B_i}$ from B_i to B' is a homeomorphism. We say that $\mathcal{S}$ is a *covering* of $\mathcal{S}'$ of *order* r, or an r*-sheeted covering*, or alternatively a covering with r *sheets*. Then a *topological map covering transformation* is a surface covering transformation ϕ from the supporting surface of a map $\mathcal{M}$ with underlying graph $\mathcal{G}$ to that of a map $\mathcal{M}'$ with underlying graph $\mathcal{G}'$ whose restriction $\phi|_{\mathcal{G}}$ from $\mathcal{G}$ to $\mathcal{G}'$ is a surjective graph homomorphism. We then say that $\mathcal{M}$ is a *topological covering* of $\mathcal{M}'$. If h' is a dart and v' a vertex of $\mathcal{M}'$, the sets $\phi^{-1}(h)$ and $\phi^{-1}(v)$ are, respectively, the *fibres* of h and of v. The *covering transformation group* $\mathrm{CT}(\phi)$ is the subgroup of $\mathrm{Aut}(\mathcal{M})$ such that, if $g \in \mathrm{CT}(\phi)$ and h is a dart of $\mathcal{M}$, $\phi(hg) = \phi(h)$. The covering is *regular* if $\mathrm{CT}(\phi)$ is transitive on all fibres, in which case the order of the group $\mathrm{CT}(\phi)$ is r, the order of the covering.

The above definition is that of an *unramified* covering. We encounter cases where for a transformation ϕ, $|\phi^{-1}(x')| = r$ for most points x' of a surface $\mathcal{S}'$, but, for a finite number of points y' of $\mathcal{S}'$, $|\phi^{-1}(y')| = s < r$. We then say that the covering of surfaces is *ramified* at the points y'. We define a ramified map covering in the same way. Many interesting map coverings are ramified at the map vertices or at the map face centres, or at both.

As in [33], we define an *algebraic covering transformation* from a map $\mathcal{M} = (G, \Omega, x, y)$ to a map $\mathcal{M}' = (G', \Omega', x', y')$ as a pair of surjective functions (σ, τ), $\sigma : G \longrightarrow G'$, $\tau : \Omega \longrightarrow \Omega'$, where σ is a group homomorphism, $\sigma(x) = x'$, $\sigma(y) = y'$ and, for any $g \in G$ and $h \in \Omega$, $\tau(hg) = \tau(h)\sigma(g)$. In [33, Section 3] it is shown that, for any algebraic covering transformation (σ, τ) between two algebraic maps, a topological covering transformation ϕ can be determined between the corresponding topological maps; conversely, if ϕ is a covering transformation between two topological maps, an algebraic covering transformation (σ, τ) can be determined between the corresponding algebraic maps.

If $\mathcal{M} = (G, x, y)$ and $\mathcal{M}'(G', x', y')$ are two regular maps of type (m, n) and (m', n') respectively, and σ is a group homomorphism from the group G to the group G', with $\sigma(x) = x'$ and $\sigma(y) = y'$, then the pair (σ, σ) is an algebraic map covering transformation as we identify the group elements with the darts of the maps. We will refer to both this transformation and the corresponding topological map covering transformation as the map covering transformation σ.

The automorphism group of $\mathcal{M}$ is G. Let $h \in G$. Then $g \in G$ is in $\mathrm{CT}(\sigma)$ if and only if $\sigma(hg) = \sigma(h)\sigma(g) = \sigma(h)$. Therefore $g \in G$ is in $\mathrm{CT}(\sigma)$ if and only if $g \in \mathrm{Ker}(\sigma)$, the kernel of σ in G. So $\mathrm{Ker}(\sigma)$ is the covering transformation group.

Let $h' \in G'$ be a dart of $\mathcal{M}'$, and let $h_1, h_2 \in \sigma^{-1}(h')$. Then, for some $g_2 \in \mathrm{Ker}(\sigma)$,

$$\sigma(h_2 g_2) = \sigma(h_2) = \sigma(h_1) = h'.$$

Now there is a $g \in G$ such that $h_1 g = h_2$, so

$$\sigma(h_1 g g_2) = \sigma(h_2) = h' = \sigma(h_1).$$

Therefore $gg_2 \in \mathrm{Ker}(\sigma)$, and so $g \in \mathrm{Ker}(\sigma)$. This shows that $\mathrm{Ker}(\sigma)$ is transitive on the fibres of the covering, which is therefore regular. So we recover a result given in [47, Theorem 3.2]: 'Any covering between regular maps is regular'.

If $n \neq n'$, since $\sigma(x^n) = \sigma(x)^n = (x')^n$, and $x^n = e$, then $\sigma(x^n) = (x')^n = \sigma e = e'$, where e and e' are, respectively, the identities of G and G'. But n' is the smallest integer such that $(x')^{n'} = e'$, so n' is an divisor of n. The covering is ramified at the vertices, with vertex *ramification index* n/n'. In the same way, we can show that if $m \neq m'$, m' is an divisor of m, and the covering is ramified at the face centres, with face ramification index m/m'.

2.8 Continued fractions

Continued fractions are defined as finite or infinite expressions of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\cdots + \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n + \cdots}}}}. \tag{2.8.1}$$

The coefficients a_i, b_i can be members of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or of rings such as $\mathbb{Z}/n\mathbb{Z}$. If the expression is finite it can be evaluated. The result of evaluating the expression having truncated it after the n^{th} coefficient is known as the n^{th} *convergent* of the continued fraction.

For a continued fraction with coefficients in the ring K, we will define the transformation $s_n(z) = b_n + a_n/z$ for $z \in K_\infty$. We will also sometimes write s_n as the left action on K_∞ of either of the matrices $\pm \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix}$. To avoid frequent use of the $\pm$ sign, where it causes no confusion, we will write

$$s_n = \left\{ \pm \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix} \right\} = \begin{bmatrix} b_n & a_n \\ 1 & 0 \end{bmatrix}.$$

We also define $S_n = s_0 \circ s_1 \circ \cdots \circ s_n$, for $n = 0, 1, \ldots$. Then it can be shown that the n^{th} convergent of the continued fraction is

$$S_n(\infty) = b_0 + \cfrac{a_0}{b_1 + \cfrac{a_1}{\cdots + \cfrac{a_{n-2}}{b_{n-1} + \cfrac{a_{n-1}}{b_n}}}}.$$

If the coefficents are integers, this convergent is a member of $\mathbb{Q}_\infty$ which we write p_n/q_n. We define $p_k, q_k, r_k, s_k \in \mathbb{Z}$ by

$$S_k(z) = \frac{p_k z + r_k}{q_k z + s_k}.$$

Then $p_0 = b_0, r_0 = a_0, q_0 = 1, s_0 = 0,\ p_1 = b_0 b_1 + a_0, r_1 = a_1 b_0, q_1 = b_1$, and $s_1 = a_1$. We see that, for $k = 1, r_k = a_k p_{k-1}$ and $s_k = a_k q_{k-1}$. Assume this is true for all k such that $0 < k \leq n$. Then, as $S_{k+1} = S_k \circ s_{k+1}$, writing s_n and S_n in matrix form,

$$S_{k+1}(z) = \begin{bmatrix} p_{k+1} & r_{k+1} \\ q_{k+1} & s_{k+1} \end{bmatrix} z = \begin{bmatrix} p_k & a_k p_{k-1} \\ q_k & a_k q_{k-1} \end{bmatrix} \begin{bmatrix} b_{k+1} & a_{k+1} \\ 1 & 0 \end{bmatrix} z = \begin{bmatrix} p_k b_{k+1} + a_k p_{k-1} & a_{k+1} p_k \\ q_k b_{k+1} + a_k q_{k-1} & a_{k+1} q_k \end{bmatrix} z.$$

So

$$S_k(z) = \frac{p_k z + a_k p_{k-1}}{q_k z + a_k q_{k-1}} \quad \text{for all} \quad k > 0,$$

and we have the recurrences

$$p_{k+1} = p_k b_{k+1} + a_k p_{k-1} \tag{2.8.2}$$

$$q_{k+1} = q_k b_{k+1} + a_k q_{k-1}. \tag{2.8.3}$$

If $a_i = 1$ for all i, and $b_i \in \mathbb{N}$, the expression is a *regular continued fraction.* If $a_i = 1$ for all i, and $b_i \in \mathbb{Z}$, the expression is an *integer continued fraction.*

It is usual to write an infinite integer continued fraction as

$$(b_0, b_1, \dots) = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \tag{2.8.4}$$

Finite integer continued fractions are written $(b_0, b_1, \dots, b_n)$, with the obvious use of notation. A continued fraction is considered to converge if its sequence of convergents converges. It diverges if it does not converge. The limit, when it exists, is called the value of the continued fraction. If the coefficients are integers, the convergents belong to the set of extended rationals $\mathbb{Q}_\infty$, but the value of the continued fraction could be irrational or infinity, and lies in $\mathbb{R}_\infty$.

We say that two continued fractions $(b_0, b_1, \ldots)$ and $(c_0, c_1, \ldots)$ have the same *tail* if there are positive integers r and s such that the two sequences $b_r, b_{r+1}, \ldots$ and $c_s, c_{s+1}, \ldots$ coincide.

We will often, as in Theorem 1.1.4 and in Chapter 6.4, find it more convenient to use the following negative expression:

$$[b_0, b_1, \ldots] \quad = \quad b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots}}} \tag{2.8.5}$$

We can easily switch between this expression and the more usual one by using the formula

$$(b_0, b_1, b_2, b_3, \ldots) \quad = \quad [b_0, -b_1, b_2, -b_3, \ldots]. \tag{2.8.6}$$

These two continued fractions have the same sequence of convergents, so one converges if and only if the other does, and if they do converge then they converge to the same value. An advantage of using the negative expression is that we have

$$s_n = b_n - 1/z, \quad \text{or} \quad s_n(z) = \begin{bmatrix} b_n & -1 \\ 1 & 0 \end{bmatrix} z,$$

and so s_n and S_n are members of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. A consequence of this, as we will see in Chapter 5, is that it is straightforward to represent them as paths on the Farey tessellation.

Chapter 3

Farey maps and related maps

3.1 Introduction and main results

The *Farey map of level* n is the quotient of the Farey tessellation by the principal congruence subgroup of level n of the modular group. We define related maps by taking the quotients of the Farey tessellation by other congruence subgroups. We also study maps defined by congruence subgroups of Hecke groups. In this chapter, we study how the different Farey and Hecke maps are related to each other by regular coverings and parallel products. We prove Theorems 1.1.1, 1.1.2 and 1.1.3 given in the introduction; we restate them here.

Theorem 1.1.1 *For a prime* p *and a positive integer* k*, the Farey map of level* p^k *is a regular covering of the Farey map of level* p^{k-1}*, ramified at the vertices, with ramification index* p*. The covering has* 4 *sheets if* $p^k = 4$*, and otherwise has* p^3 *sheets.*

Theorem 1.1.2 *Let* m *be a positive integer.*

(i) *If* m *is odd, the Farey map of level* $2m$ *is the parallel product of the the Farey map of level* 2 *and the Farey map of level* m*.*

(ii) *If* l *and* m *are coprime integers, and neither* l *nor* m *is twice an odd integer, then the Farey map of level* lm *is a regular unramified double covering of the parallel product of the Farey map of level* l *and the Farey map of level* m*.*

Theorem 1.1.3 *For odd n, the underlying graph of the Hecke map of type $(4, n)$ is a double graph covering of the underlying graph of the Farey map of type $(3, n)$. If n is not a multiple of 3, the underlying graph of the Hecke map of type $(6, n)$ is a double graph covering of the underlying graph of the Farey map of type $(3, n)$.*

We also give an alternative proof of the following theorem from [38].

Theorem 3.1.1. *Both the Hecke map of type $(4, n)$ (for odd n) and the Hecke map of type $(6, n)$ (for $3 \nmid n$) have diameter 4.*

3.2 Background theory

Recently, there has been significant research on Farey maps, see [29, 30, 38, 50, 60, 61]. In this section we define them formally and describe them, mainly following the approach in [30, 38, 60], and introduce the notation we will use.

In [59], the definition of an algebraic map is extended to include a map corresponding to the infinite modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Recall from Chapter 2 that Γ is the quotient by $\{\pm I\}$, where I is the 2×2 identity matrix, of the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

and that, in order to avoid the frequent use of the $\pm$ symbol, we write the members of Γ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The elements $X = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, of order 3, and $Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, of infinite order, generate Γ.

$$\text{Note also that} \qquad XY = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and that } XY \text{ is of order 2.}$$

We define an infinite regular algebraic map $\mathscr{F} = (\Gamma, XY, Y)$, of type $(3, \infty)$. In the same way as we established in Section 2.6 that the darts of the topological representation of a regular finite map (G, x, y) can be identified with elements of G, we identify any $\gamma \in \Gamma$ with a dart of the topological representation of $\mathscr{F}$. Its vertices are the cosets

$$\gamma\langle Y\rangle = \left\{ \begin{bmatrix} a & ar+b \\ c & cr+d \end{bmatrix} : r \in \mathbb{Z} \right\}.$$

Given $a, c \in \mathbb{Z}$ such that $\gcd(a, c) = 1$, we can determine a unique vertex of $\mathscr{F}$: we find $b, d \in \mathbb{Z}$ such that $ad - bc = 1$. Then, if $\gamma \in \Gamma$ is the matrix with entries a, b, c and d, the vertex corresponding to the ordered pair (a, c) is $\gamma\langle Y\rangle$. Hence we can identify the vertices with the reduced rationals a/c, with the usual convention that $1/0 = \infty$.

The coset $\gamma\langle XY\rangle$ is an edge consisting of the two darts γ and γXY. We say that a/c is the initial vertex of γ and that, as b/d is the initial vertex of γXY, it is the final vertex of γ. The reduced rationals a/c and b/d are vertices incident to the same edge, or *adjacent* vertices, if and only if $ad - bc = \pm 1$. We recognise $\mathscr{F}$ as the Farey tessellation introduced in section 2.4, and shown in Figure 2.4.1.

It is shown in [59, Theorem 1] that any map with triangular faces is the quotient of $\mathscr{F}$ by a subgroup of the modular group. In this sense, $\mathscr{F}$ is the *universal triangular map.* This subgroup is the stabiliser of the darts of the map in Γ, and is called the *map subgroup.* By [33, Theorem 6.3], the map is regular if and only if the map subgroup is a normal subgroup of the modular group.

The *principal congruence subgroup* $\Gamma(n)$ *of level* n is the normal subgroup of Γ given by:

$$\Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{n} \right\}.$$

The *Farey map of level* n is the regular map $\mathcal{M}_3(n) = (\Gamma/\Gamma(n), XY\Gamma(n), Y\Gamma(n))$. It is of type $(3, n)$. The group $\Gamma/\Gamma(n)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. The members of $\Gamma/\Gamma(n)$, which we identify with the darts of $\mathcal{M}_3(n)$, are, for $\gamma \in \Gamma$, the cosets $\gamma\Gamma(n)$. It is well known (see for instance [17, 30]) that the order of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is

$$n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right),$$

and so the order of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ is

$$\frac{n^3}{2} \prod_{p|n} \left(1 - \frac{1}{p^2}\right). \tag{3.2.1}$$

As we wish to compare Farey maps of different levels, we will use the following notation, where $\gamma \in \Gamma$, $\quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\gamma\Gamma(n) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n = \left\{ \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \pmod{n} \right\}.$$

Recall that the subgroup $\Gamma_1(n)$ of Γ is given by

$$\Gamma_1(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n \in \Gamma/\Gamma(n) : \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}_n \quad : \quad r = 0, 1, \dots, n-1. \right\}$$

Then we note that $\Gamma_1(n)$ is the group generated by $Y\Gamma(n)$, and so the vertices of $\mathcal{M}_3(n)$ are, for $\gamma \in \Gamma$, $\quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the cosets $\gamma\Gamma(n)\langle Y\Gamma(n)\rangle = \gamma\langle Y\Gamma(n)\rangle = \gamma\Gamma_1(n)$, or

$$\gamma\Gamma_1(n) = \left\{ \begin{bmatrix} a & ar+b \\ c & cr+d \end{bmatrix}_n \quad : \quad r = 0, 1, \dots, n-1. \right\}.$$

Given $a, c \in \mathbb{Z}$ such that $\gcd(a, c, n) = 1$, we can determine a unique vertex of $\mathcal{M}_3(n)$: we find $b, d \in \mathbb{Z}$ such that $ad - bc \equiv 1 \pmod{n}$. Then, if $\gamma \in \Gamma$ is the matrix with entries a, b, c and d, the vertex corresponding to the ordered pair (a, c) is $\gamma\Gamma_1(n)$, which we denote by $[a/c]_n$. There is a bijection between these vertices and the *Farey fractions* defined in [60], which are the equivalence classes $\{(a', c') \in \mathbb{Z} \times \mathbb{Z} : \gcd(a', c', n) = 1,\ (a', c') \equiv \pm(a, c) \,(\text{mod } n)\}$. Note that, whereas 2/2 is not a reduced rational, and so not a vertex of $\mathscr{F}$, $[2/2]_5$ is a vertex of $\mathcal{M}_3(5)$, and $[2/0]_5$ is not the same vertex as $[1/0]_5$. The n darts incident to $[a/c]_n$ all have $[a/c]_n$ as initial vertex, and one of $[b + ar/d + cr]_n$ for $r = 0, 1, 2, 3, 4$ as final vertex.

An edge of $\mathcal{M}_3(n)$ is a coset $\gamma\Gamma(n)\langle XY\rangle$, which consists of the two darts $\gamma\Gamma(n)$ and $\gamma\Gamma(n)XY$. So if $\gamma \in \Gamma$ is the matrix with entries a, b, c and d, the coset

$$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}_n, \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_n\right\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}_n, \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}_n\right\}$$

is an edge of $\mathcal{M}_3(n)$. The vertices $[a/c]_n$ and $[b/d]_n$ are adjacent in $\mathcal{M}_3(n)$ if they are incident to the same edge, that is if and only if $ad - bc \equiv \pm 1(\text{mod } n)$.

A face of $\mathcal{M}_3(n)$ is, for some $\gamma \in \Gamma$, a matrix with entries a, b, c, and d, the coset

$$\gamma\Gamma(n)\langle X\Gamma(n)\rangle = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}_n \right\rangle = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}_n, \begin{bmatrix} -b & a+b \\ -d & c+d \end{bmatrix}_n, \begin{bmatrix} -a-b & a \\ -c-d & c \end{bmatrix}_n\right\}.$$

So the dart with initial vertex $[a/c]_n$ and final vertex $[b/d]_n$ is incident to the triangular face with vertices $[a/c]_n$, $[b/d]_n$, and $[(a + b)/(c + d)]_n$.

The number of darts of $\mathcal{M}_3(n)$ is the order of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$, which is given by equation (3.2.1). As n darts are incident to each vertex, the number of vertices is

$$V(\mathcal{M}_3(n)) = \frac{n^2}{2} \prod_{p|n} \left(1 - \frac{1}{p^2}\right). \tag{3.2.2}$$

As 2 darts are incident to each edge, the number of edges is

$$E(\mathcal{M}_3(n)) = \frac{n^3}{4} \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

Then, as 3 darts are incident to each face, the number of faces is

$$F(\mathcal{M}_3(n)) = \frac{n^3}{6}\prod_{p|n}\left(1-\frac{1}{p^2}\right).$$

The map is embedded in a surface whose genus g is given by equation (2.6.1), so

$$\begin{aligned} g &= 1-\frac{1}{2}(E(\mathcal{M}_3(n)-V(\mathcal{M}_3(n))-F(\mathcal{M}_3(n)) \\ &= 1+\frac{n^2}{24}(n-6)\prod_{p|n}\left(1-\frac{1}{p^2}\right). \end{aligned}$$

Note that g is always a non-negative integer. If $n<6$, $g=0$ and the supporting surface of the map $\mathcal{M}_3(n)$ is the sphere. If $n=6$, $g=1$, so the supporting surface of $\mathcal{M}_3(6)$ is a torus.

Example 3.2.1. If $n=5$, as $\gcd(2,2,5)=1$, we can find the vertex of $\mathcal{M}_3(5)$ corresponding to the pair $(2,2)$. We note that $3\times 2-0\times 2=1 \pmod{n}$, so it is the coset

$$[2/2]_5 = \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}\Gamma_1 = \left\{ \begin{bmatrix} 2 & 2r+3 \\ 2 & 2r \end{bmatrix}_5 \quad : \quad r=0,1,2,3,4. \right\}.$$

Note that $[a/c]_5 = [2/2]_5$ if and only if $(a,c)\equiv \pm(2,2) \pmod{n}$, and so we have, for instance, $[3/3]_5 = [7/2]_5 = [2/2]_5$. The 5 darts incident to $[2/2]_5$ all have $[2/2]_5$ as initial vertex, and one of $[3+2r/2r]_5$ for $r=0,1,2,3,4$ as final vertex. The edge of the map incident to the dart

$$\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}_5 \quad \text{is the coset} \quad \left\{ \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}_5, \begin{bmatrix} -3 & 2 \\ 0 & 2 \end{bmatrix}_5 \right\},$$

which is determined by the unordered pair of vertices $[2/2]_5, [3/0]_5$. The face incident to the dart

$$\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}_5 \quad \text{is the coset} \quad \left\{ \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}_5, \begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix}_5, \begin{bmatrix} -5 & 2 \\ -2 & 2 \end{bmatrix}_5 \right\},$$

So the dart with initial vertex $[2/2]_5$ and final vertex $[3/0]_5$ is incident to the triangular face with vertices $[2/2]_5$, $[3/0]_5$, and $[5/2]_5$.

The number of vertices, edges and faces of $\mathcal{M}_3(5)$ is given by

$$\begin{aligned} V(\mathcal{M}_3(5)) &= \frac{25}{2}(1-1/25) = 12 \\ E(\mathcal{M}_3(5)) &= 5 \times 12/2 = 30, \quad \text{and} \\ F(\mathcal{M}_3(5)) &= 5 \times 12/3 = 20. \end{aligned}$$

A representation of $\mathcal{M}_3(5)$ is the well-known icosahedron embedded in the sphere, as shown in Figure 3.2.1

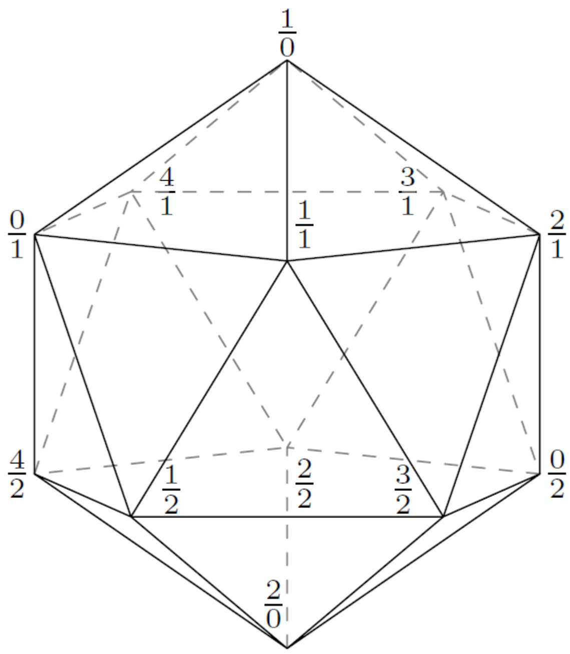

Figure 3.2.1: The map $\mathcal{M}_3(5)$, an icosahedron embedded in a sphere, from [31]. The suffix 5 has been omitted from the vertex labels.

Example 3.2.2. We can describe $\mathcal{M}_3(7)$, the Klein map, in the same way. The number of its vertices, and its genus, are given by

$$\begin{aligned} V(\mathcal{M}_3(7)) &= \frac{49}{2}(1-1/49) = 24, \\ g &= 1 + \frac{49}{24}(1-1/49) = 3. \end{aligned}$$

The underlying surface of this map is the well-known Klein quartic, a surface of genus 3. It has many interesting properties, which can be found for instance in [43], along with attractive illustrations. A representation of the map, with the vertices labelled in a way similar to that which we have used, is in [30, Figure 3].

3.3 Regular coverings

We prove Theorem 1.1.1 by giving a more general result, which we will also use later.

Lemma 3.3.1. *If $n = dm$, the Farey map $\mathcal{M}_3(n)$ is a regular map covering of $\mathcal{M}_3(m)$ of order d^3 if $d \neq 4$, or 4 if $d = 2$ and $m = 2$, which is ramified at the vertices with ramification index d. The covering transformation takes the dart $\gamma\Gamma(n)$ to the dart $\gamma\Gamma(m)$, and the vertex $[a/c]_n$ to the vertex $[a/c]_m$.*

Proof. We have $\mathcal{M}_3(n) = (\Gamma/\Gamma(n), XY\Gamma(n), Y\Gamma(n))$ and $\mathcal{M}_3(m) = (\Gamma/\Gamma(m), XY\Gamma(m), Y\Gamma(m))$. Since, as m divides n, $\Gamma(n)$ is a subgroup of $\Gamma(m)$, the mapping

$$\begin{aligned} \sigma : \Gamma/\Gamma(n) &\longrightarrow \Gamma/\Gamma(m) \\ X\Gamma(n) &\longmapsto X\Gamma(m); \\ Y\Gamma(n) &\longmapsto Y\Gamma(m), \end{aligned}$$

is an epimorphism and so a covering transformation. The kernel of σ has order d^3 if $n \neq 4$, or 4 if $d = 2$ and $m = 2$, since

$$\sigma\left(\begin{bmatrix} 1+km & rm \\ sm & 1-km \end{bmatrix}_n\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_m \quad \text{for } k, r, s = 0, 1, \ldots, d-1.$$

As $\Gamma = \langle X, Y \rangle$, we have, for all $\gamma \in \Gamma$, $\quad \sigma(\gamma\Gamma(n)) = \gamma\Gamma(m)$. The vertices of $\mathcal{M}_3(n)$ are the cosets $\gamma\Gamma_1(n) = [a/c]_n$. So $\sigma(\gamma\Gamma_1(n)) = \gamma\Gamma_1(m) = [a/c]_m$. □

Putting $n = p^{k-1}, d = p$ in Lemma 3.3.1 then proves Theorem 1.1.1.

Example 3.3.2. Figure 3.3.1 shows $\mathcal{M}_3(2)$ and $\mathcal{M}_3(4)$. The underlying surface of each of these maps is a sphere. $\mathcal{M}_3(4)$ is a four sheeted covering of $\mathcal{M}_3(2)$, ramified at all the vertices with ramification index 2. The covering transformation takes $X\Gamma(4)$ to $X\Gamma(2)$, and $Y\Gamma(4)$ to $Y\Gamma(2)$. So we have $\gamma\Gamma(4) \longrightarrow \gamma\Gamma(2)$ and $[a/c]_4 \longrightarrow [a/c]_2$. For instance

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_4 \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_2, \qquad [1/0]_4 \longrightarrow [1/0]_2, \quad \text{and} \quad [2/1]_4 \longrightarrow [0/1]_2.$$

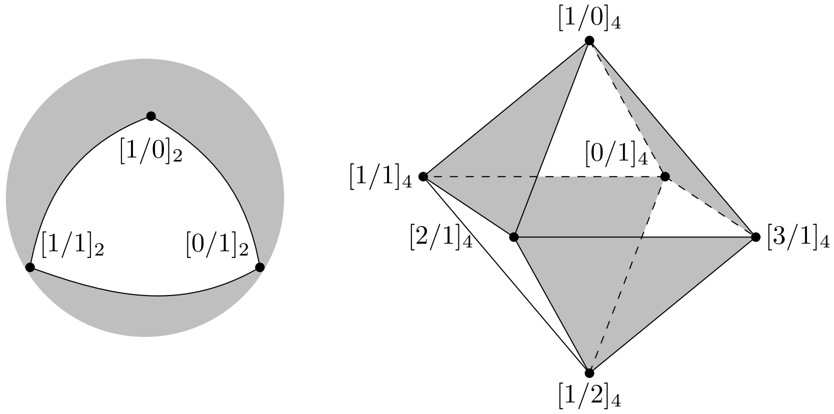

Figure 3.3.1: The Farey maps $\mathcal{M}_3(2)$ and $\mathcal{M}_3(4)$.

3.4 Parallel products

In order to decompose $\mathcal{M}_3(n)$ for a composite n, we use a product of maps which is consistent with the tensor product of their underlying graphs. This is the *parallel product of maps*, the minimal common covering of a set of maps, introduced in [67], and used recently in [26]. It is analogous to the join of hypermaps defined in [5], and to the blend of polytopes used in [48].

The definition of the parallel product of groups, and then of maps, is given in [67] as follows.

Definition 3.4.1. Let G_1 be a finite group generated by x_1 and y_1, and G_2 a finite group generated by x_2 and y_2. The *parallel product* of (G_1, x_1, y_1) and (G_2, x_2, y_2) is the group $(E, (x_1, x_2), (y_1, y_2))$, where E is the subgroup of $G_1 \times G_2$ generated by (x_1, x_2) and (y_1, y_2).

If x_1 and x_2 are both of order 2, so is (x_1, x_2). Then the triple $(E, (x_1, x_2), (y_1, y_2))$ defines a regular map, the parallel product of the maps $\mathcal{M}_1 = (G_1, x_1, y_1)$ and $\mathcal{M}_2 = (G_2, x_2, y_2)$. If Γ is the modular group, and H and K are the map subgroups of $\mathcal{M}_1$ and $\mathcal{M}_2$, so that $G_1 = \Gamma/H$ and $G_2 = \Gamma/K$, from [26, Lemma 3(vii)] and the discussion following [5, Proposition 3.1], the parallel product of $\mathcal{M}_1$ and $\mathcal{M}_2$ is a map with map subgroup $H \cap K$.

If l and m are coprime integers, [66, Theorem 2] shows that, if $\mathcal{G}_3(l)$ and $\mathcal{G}_3(m)$ are the underlying graphs of $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$, the graph $\mathcal{G}_3(l) \times \mathcal{G}_3(m)$ can be embedded as a map on a surface. This map is the parallel product of the maps $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$.

We now prove Theorem 1.1.2, which gives an iterative method for decomposing the Farey map $\mathcal{M}_3(n)$ into a series of regular coverings and parallel products given the prime decomposition of the integer n.

Proof of Theorem 1.1.2. Let l and m be positive coprime integers. If $a \equiv b \pmod{lm}$, then we have $a \equiv b \pmod{l}$ and $a \equiv b \pmod{m}$, so $\Gamma(lm) \subset \Gamma(l) \cap \Gamma(m)$. Therefore we can define the group epimorphism

$$\begin{aligned} \sigma : \Gamma/\Gamma(lm) &\longrightarrow \Gamma/(\Gamma(l) \cap \Gamma(m)) \\ \gamma\Gamma(lm) &\longmapsto \gamma(\Gamma(l) \cap \Gamma(m)). \end{aligned}$$

Then σ is a regular map covering transformation. It maps each member of $\Gamma/\Gamma(lm)$, a coset $\gamma\Gamma(lm)$, onto the coset $\gamma\Gamma(l)\cap\Gamma(m)$ in which it is contained, which is a member of $\Gamma/\Gamma(l))\cap\Gamma(m)$. In particular the kernel of σ consists of those members of $\Gamma/\Gamma(lm)$ which are mapped onto the identity of $\Gamma/\Gamma(l) \cap \Gamma(m)$. This is the coset of $\Gamma(l) \cap \Gamma(m)$ in Γ containing the identity of Γ, which is $\Gamma(l) \cap \Gamma(m)$.

If a matrix with entries a, b, c and d is a member of $(\Gamma/\Gamma(lm)) \cap (\Gamma(l) \cap \Gamma(m))$, then

$$\begin{aligned} a &\equiv d \equiv \pm 1 \pmod{l}, b \equiv c \equiv 0 \pmod{l} \text{ and} \\ a &\equiv d \equiv \pm 1 \pmod{m}, b \equiv c \equiv 0 \pmod{m}. \end{aligned}$$

Then $b \equiv c \equiv 0 \pmod{lm}$ and either $a \equiv d \equiv \pm 1 \pmod{lm}$ or $a \equiv d \equiv \pm u \pmod{lm}$ for

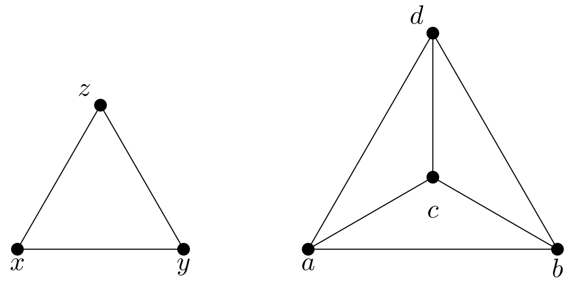

Figure 3.4.1: The maps $\mathcal{M}_3(2)$, with vertices x, y, z, and $\mathcal{M}_3(3)$, with vertices a, b, c, d.

any $u \in U$, where $U = \{u \in \mathbb{Z} : u \equiv 1 \pmod{l}, u \equiv -1 \pmod{m}$ and $u \not\equiv \pm 1 \pmod{lm}\}$.

If either $l = 2$ or $m = 2$, then $U = \emptyset$, so that $\Gamma(2) \cap \Gamma(m) = \Gamma(2m)$, and $\mathcal{M}_3(2m)$ is the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(m)$.

If neither l nor m is equal to 2, as $\gcd(l, m) = 1$, there is exactly one u, modulo lm, such that $u \equiv 1 \pmod{l}, u \equiv -1 \pmod{m}$ and $u \not\equiv \pm 1 \pmod{lm}$. So, as $\sigma^{-1}(\Gamma(l) \cap \Gamma(m)) = \{\Gamma(lm), u\Gamma(lm)\}$ is of order 2, $\mathcal{M}_3(lm)$ is a double covering of the parallel product of $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$, unramified at the vertices as both maps have vertex valency lm. $\square$

Example 3.4.2. As 6 is twice an odd number, $\mathcal{M}_3(6)$ is the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$. We designate the 3 vertices of $\mathcal{M}_3(2)$ by $x = [1/0]_2, y = [0/1]_2$, and $z = [1/1]_2$, and the 4 vertices of $\mathcal{M}_3(3)$ by $a = [1/0]_3, b = [0/1]_3, c = [1, 1]_3$ and $d = [2/1]_3$.

Then the 12 vertices of the parallel product are xa, xb, xc, xd; ya, yb, yc, yd; and za, zb, zc, zd. The vertex xa is adjacent to the vertices yb, yc, yd, zb, zc and zd, and similarly the other vertices all have 6 neighbours, so the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$ is of degree 6.

This parallel product is shown in Figures 3.4.2, and we can see that is is isomorphic to the map $\mathcal{M}_3(6)$ shown in Figure 3.4.3. For instance the vertex $[1/0]_6$ of $\mathcal{M}_3(6)$ corresponds to the vertex xa of the parallel product, and its neighbours $[0/1]_6$,$[1/1]_6$,$[2/1]_6$,$[3/1]_6$,$[4/1]_6$ and $[5/1]_6$ correspond, in order, to yb, zc, yd, zb, yc and zd.

Example 3.4.3. As 15 is not twice an odd number, the second part of Theorem 1.1.2 tells us that $\mathcal{M}_3(15)$ is a double cover of the parallel product of $\mathcal{M}_3(3)$ and $\mathcal{M}_3(5)$.

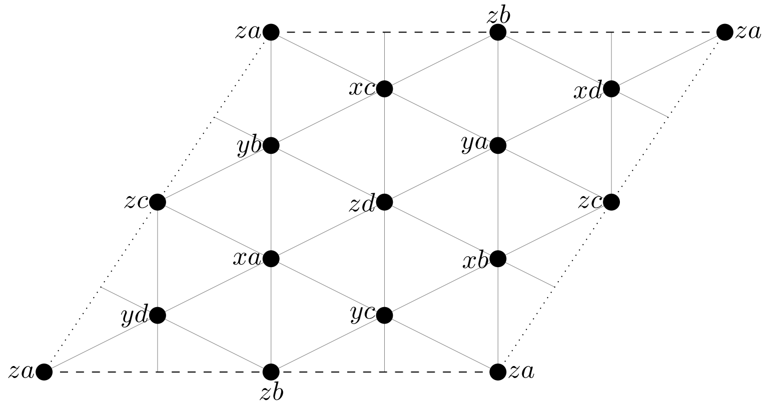

Figure 3.4.2: The map of the parallel product of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$, drawn on a torus.

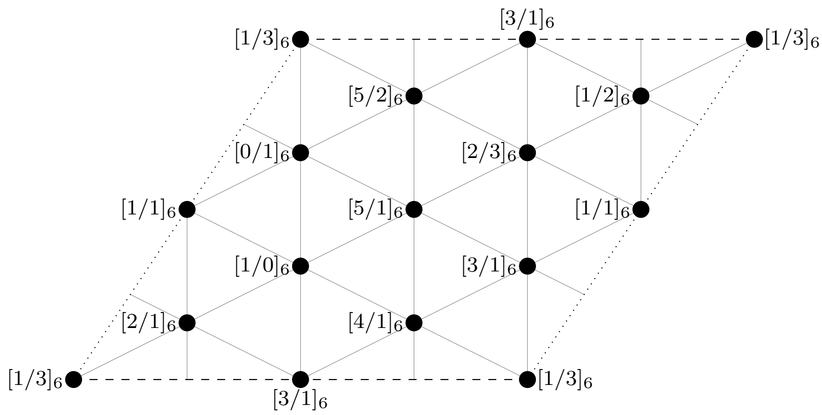

Figure 3.4.3: The map $\mathcal{M}_3(6)$, drawn on a torus.

3.5 Farey maps and complete graphs

We show that the underlying graphs of Farey maps of prime level are graph coverings of complete graphs. The spectra of complete graphs are known, so this will enable us to find the spectra of the Farey maps. We first collect some necessary information about the vertices of $\mathcal{M}_3(n)$.

The *poles* of the map $\mathcal{M}_3(n)$ are defined in [60] as the vertices $[a/0]_n$, where a is coprime to n. If d is such that $ad \equiv \pm 1 \pmod{n}$ and $1 \leq d \leq n/2$, then the vertices $[b/d]_n$ for $b = 0, \ldots, n-1$ are adjacent to $[a/0]_n$.

The *star* of a vertex consists of that vertex and all vertices adjacent to it. In [60, Theorem 7] it is shown that the stars of the $h = \frac{1}{2}(p-1)$ poles of $\mathcal{M}_3(p)$ for an odd prime p are disjoint, each contain $p+1$ vertices, and, together, include all of the $\frac{1}{2}(p^2-1) = h(p+1)$ vertices of $\mathcal{M}_3(p)$. The map $\mathcal{M}_3(2)$ has 3 vertices, $[1/0]_2$,$[0/1]_2$ and $[1/1]_2$; only one vertex, $[1/0]_2$, is a pole, its star consists of all three vertices of the map.

Let v be any vertex of $\mathcal{M}_3(n)$ and let M be an automorphism of $\mathcal{M}_3(n)$ which is such that $M([1,0]_n) = v$. Then we define the *copoles* of v as $M([a,0]_n)$ for $a = 1, \ldots, h$.

Theorem 3.5.1. *The underlying graph of the Farey map $\mathcal{M}_3(p)$, for an odd prime p, is a graph covering of order $\frac{1}{2}(p-1)$ of the complete graph K_{p+1} on $p+1$ vertices. The underlying graph of the Farey map $\mathcal{M}_3(2)$ is a graph covering of order 1 of the complete graph K_3.*

Proof. We label the $p+1$ vertices of K_{p+1} as $0, 1, \ldots, p$. We define a transformation ϕ, which takes each vertex of $\mathcal{M}_3(p)$ together with all its copoles to the same vertex of K_{p+1}.

$$\begin{aligned}
\phi: \text{ vertices of } \mathcal{M}_3(p) \quad &\longrightarrow \quad \text{vertices of } K_{p+1} \\
[a/0]_p \quad &\longmapsto \quad p \\
\text{for } b \neq 0, \quad [a/b]_p \quad &\longmapsto \quad ab^{-1}.
\end{aligned}$$

There is a bijection τ between $[1/0]_p$ and its adjacent vertices, and the vertices of K_{p+1}:

$$\begin{aligned} \tau: \text{ vertices of the star of } [1/0]_p &\longrightarrow \text{ vertices of } K_{p+1} \\ [1/0]_p &\longmapsto p \\ \text{for } b = 0, \ldots, p-1, \quad [b/1]_p &\longmapsto b. \end{aligned}$$

Let v be any vertex of $\mathcal{M}_3(p)$, and let A be an automorphism of $\mathcal{M}_3(p)$ which takes the vertex v to $[1/0]_p$. Then the transformation $\tau \circ A$ is a bijection between the star of v and the set of vertices of K_{p+1}, so ϕ is a graph covering. If p is an odd prime the covering has $h = \frac{1}{2}(p-1)$ sheets as p has h pre-images. If $p = 2$, the covering has one sheet. □

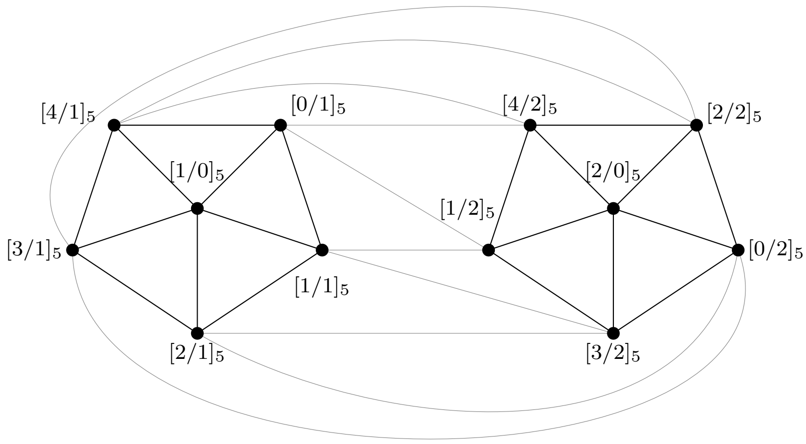

Figure 3.5.1: The map $\mathcal{M}_3(5)$, showing the stars of the two poles $[1/0]_5$ and $[2/0]_5$. The edges joining vertices in the same star are shown in black, the other edges in grey.

Example 3.5.2. The underlying graph of $\mathcal{M}_3(5)$ is a double graph cover of K_6. The poles of $\mathcal{M}_3(5)$ are $[1/0]_5$ and $[2/0]_5$, and $h = 2$, so the map can be drawn showing two stars, as in

Figure 3.5.1. Then, as $2^{-1} = 3$ in $\mathbb{Z}/5\mathbb{Z}$,

$$\begin{aligned} \phi : \text{ vertices of } \mathcal{M}_3(5) &\longrightarrow \text{ vertices of } K_6 \\ [a/0]_5 &\longmapsto 6 \text{ for } a = 1, 2 \\ [a/1]_5 &\longmapsto a \text{ for } a = 0, 1, 2, 3, 4 \\ [a/2]_5 &\longmapsto 3a \pmod 5 \text{ for } a = 0, 1, 2, 3, 4 \end{aligned}$$

$\mathcal{M}_3(5)$ has 12 vertices and 25 edges, whereas K_6 has 6 vertices and 15 edges. The graph covering is a double covering of vertices, and maps vertices to adjacent vertices, but there is not a mapping between the edges.

3.6 The Cayley graph of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$

Before describing some families of maps related to the Farey maps, we give an interesting representation of the Farey maps, showing their connexion with the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$.

Definition 3.6.1. If G is a group and S a subset of G, the *Cayley graph* of G with respect to the defining set S, $Cay(G, S)$, is a graph with the elements of G as vertices. If v and w are members of G, vw is an edge of $Cay(G, S)$ if and only if $sv = w$ for some $s \in S$.

Before generalising, we consider the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$, using as defining set

$$S = \{h, g, g^{-1}\} \text{ where } h = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_4 \text{ and } g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_4 .$$

The 24 elements of $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ are:

$$
\begin{aligned}
I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_4, \; g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_4, \; g^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_4, g^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}_4, \\
h &= \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}_4, \; gh = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}_4, \; g^2h = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}_4, g^3h = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}_4, \\
a &= hg \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}_4, \; ghg = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_4, \; g^2hg = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}_4, g^3hg = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}_4, \\
b &= hg^2 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}_4, \; ghg^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}_4, \; g^2hg^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}_4, \; g^3hg^2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}_4, \\
c &= hg^3 = \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}_4, \; ghg^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}_4, \; g^2hg^3 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}_4, \; g^3hg^3 = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}_4, \\
d &= hg^2h = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}_4, \; ghg^2h = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}_4, \; g^2hg^2h = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}_4, \text{ and } g^3hg^2h = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}_4.
\end{aligned}
$$

The Cayley graph is shown in Figure 3.6.1. Note that h is self-inverse, and that, although for clarity only the directed edges corresponding to g are shown, since the defining set also contains g^{-1} the graph is not directed. It is clear that by shrinking the squares (and identifying $[1/3]_4$ with $[3/1]_4$) we obtain the Farey map $\mathcal{M}_3(4)$ shown in Figure 3.3.1.

This is an example of a general result concerning regular maps. We first give a definition of the *truncation* of a map, as in [62].

Definition 3.6.2. Let $\mathcal{M}$ be a map, and v_i and v_j two adjacent vertices. The edges incident to v_i are labelled anticlockwise around v_i as $e_{ij}, e_{i,j+1}, \ldots, e_{i,j+d_i-1}$, and similarly the edges incident to v_j are labelled as $e_{ji}, e_{j,i+1}, \ldots, e_{j,i+d_j-1}$, where d_i and d_j are the vertex valencies of v_i and v_j respectively.

Identify, on the edge e_{ij} joining v_i to v_j, two points t_{ij} and t_{ji} such that the path along the edge from t_{ji} to v_i is through t_{ij} and the path along the edge from t_{ij} to v_{ji} is through t_{ji}. Repeat this for each edge.

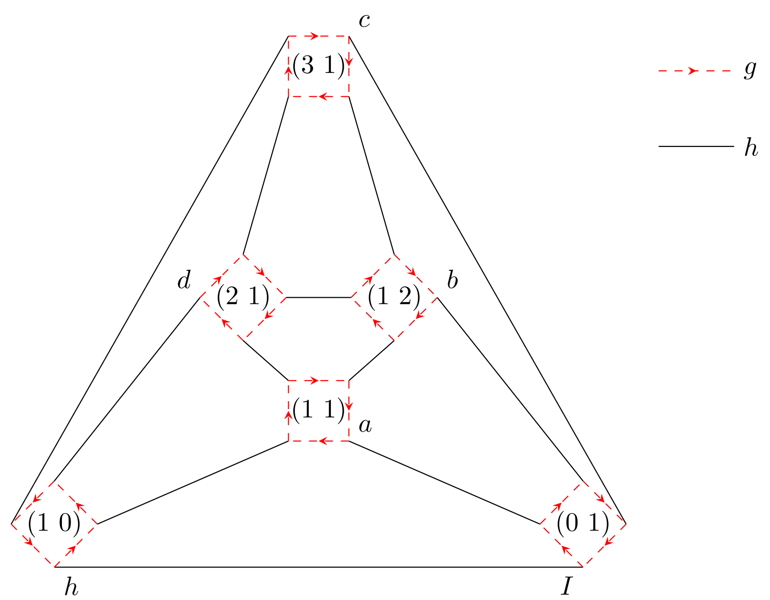

Figure 3.6.1: The Cayley graph of $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ with generating set $\{h, g, g^{-1}\}$. I is the identity matrix, $a = hg, b = hg^2, c = hg^3$ and $d = hg^2h$. The ordered pairs in each square are the bottom rows of the matrices corresponding to its vertices.

The *truncation* of $\mathcal{M}$, $T(\mathcal{M})$ is the map whose vertices are the points t_{ij}. Any vertex t_{ij} is adjacent to t_{ji}, and also to $t_{i,j+1}$ and $t_{i,j+d_i-1}$, as shown in Figure 3.6.2.

Theorem 3.6.3. *The truncation of a regular map $\mathcal{M} = (G, h, g)$ represents a Cayley graph whose vertices are the members of the group $G = \langle h, g\rangle$, and whose defining set is $S = \{h, g, g^{-1}\}$.*

Proof. From [34, Theorem 2.1], if $\mathcal{M}$ is regular then $C(G) \cong G$, where $C(G)$ is the centraliser of G. Each vertex t_{ij} of $T(\mathcal{M})$ corresponds to a dart of $\mathcal{M}$. Since G acts transitively on these darts there is a member of G which sends e, the dart with initial vertex v_0 and final vertex v_1, to any other directed edge. This member is unique as, from [33, Proposition 3.3], $C(G)$ and therefore G are semiregular. Let the permutation h send e to the dart corresponding to t_{10}, and g send e to the dart corresponding to t_{02}. If $a \in G$ sends e to t_{ij}, then $ha(e) = h(a(e))$ sends e to t_{ji}, $ga(e) = g(a(e))$ sends e to $t_{i,j+1}$, and $g^{-1}a(e) = g^{-1}(a(e))$ sends e to $t_{i,j-1}$. So the three neighbours of a on the Cayley graph of G with respect to $\{h, g, g^{-1}\}$ correspond to the three neighbours of t_{ij} on the truncation of $\mathcal{M}$. This shows that there is an incidence preserving bijection between the vertices and edges of the Cayley graph and the truncation, so we say that

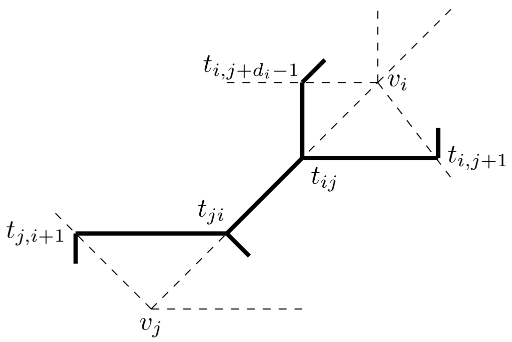

Figure 3.6.2: The truncation $T(\mathcal{M})$ of a map $\mathcal{M}$, where v_i and v_j are vertices of $\mathcal{M}$. The vertices of the truncation are t_{kl}. The edges of $\mathcal{M}$ are shown as dotted lines.

the truncation represents the Cayley graph. $\square$

We have the following straightforward corollary.

Corollary 3.6.4. *The Cayley graph of* $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ *for* $n > 2$ *with respect to the defining set* $\{g, h, h^{-1}\}$ *with*

$$g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_n, \textit{ and } h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_n$$

is the underlying graph of a truncation of $\mathcal{M}_3(n)$*.*

3.7 Generalised Farey maps

In [48], McMullen, Monson and Weiss construct polyhedra with triangular faces from rotation groups which are quotients of certain linear groups. They show that they can be considered as regular maps, including Dycke's map, which are are given realisations by describing their vertices as ordered pairs in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

We combine this approach with that which we took to define Farey maps in the previous section. Whereas there the ordered pair (a, b) is identified with $(-a, -b)$ in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, we identify it with ordered pairs (ua, ub), where u is a unit of $\mathbb{Z}/n\mathbb{Z}$. This gives us a new family of regular maps on Riemann surfaces. We will show how Dyck's map can be modelled by the representation shown in Figure 3.7.1.

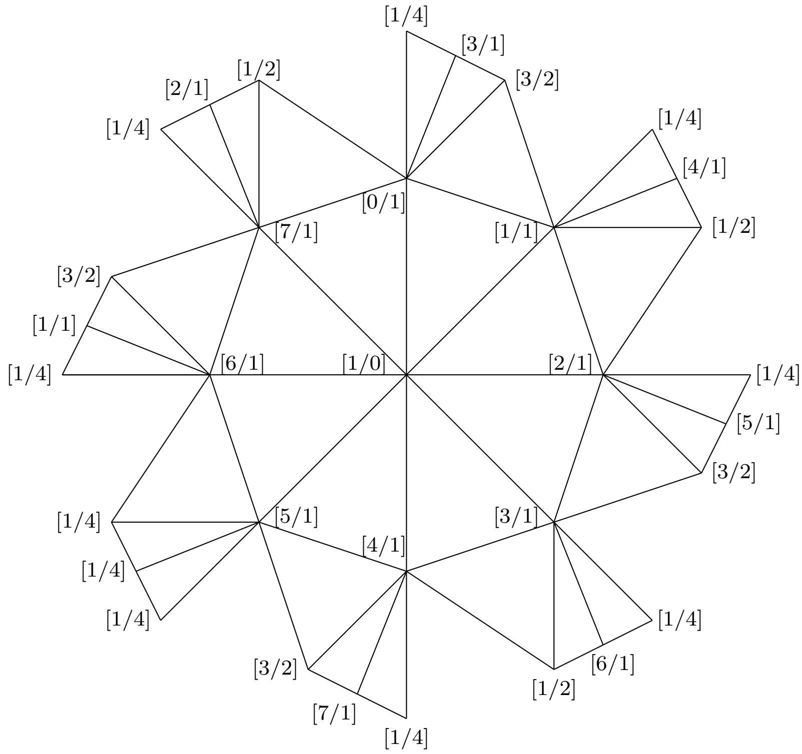

Figure 3.7.1:
A realisation of Dyck's map on a Riemann surface of genus 3, adapted from Figure 1 of [55]. Vertices with the same label are to be identified. The suffix $8, \{\pm 1, \pm 3\}$ has been omitted from all vertices for clarity.

A *congruence subgroup* of the modular group Γ is a subgroup which contains one of the principal congruence subgroups $\Gamma(n)$. We consider the congruence subgroups

$$\Gamma(n, U) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \pmod{n},\ u \in U \right\}.$$

where U is a subgroup of the group of units of $\mathbb{Z}/n\mathbb{Z}$ containing $\{-1, 1\}$ and whose elements u satisfy $u^2 \equiv 1 \pmod{n}$.

Lemma 3.7.1. *$\Gamma(n, U)$ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, $\Gamma(n)$ is a normal subgroup of $\Gamma(n, U)$, and $|\Gamma(n, U) : \Gamma(n)| = \frac{1}{2}|U|$.*

Proof. As the matrices in $\Gamma(n, U)$ are diagonal, they commute with those in $\mathrm{PSL}_2(\mathbb{Z})$.

Let $U' = U/\{\pm 1\}$ be the group in which each element of U is identified with its negative. Since, if $u \in U, -u \in U$, and, if n is even, $n/2$ is not a unit, we have $|U'| = \frac{1}{2}|U|$. Let u' be the member of U' corresponding to $\pm u$. Then the group homomorphism

$$\Phi_U : \Gamma(n, U)) \longrightarrow U; \;\; \Phi_U\left(\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}\right) = u'$$

is a surjection with kernel $\Gamma(n)$, which proves the result. $\square$

So, from [59, Theorem 1], the quotient of the Farey tessellation $\mathscr{F}$ by $\Gamma(n, U)$ is a regular map $\mathcal{M}_3(n, U) = (\Gamma/\Gamma(n, U), XY\Gamma(n, U), Y\Gamma(n, U))$, which we call a *generalised Farey map of level n*.

A member of $\Gamma/\Gamma(n, U)$ is the coset

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Gamma(n, U), \qquad \text{where} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

which can be written

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Gamma(nU) &= \left\{\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \Gamma : \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \equiv \begin{bmatrix} ua & ub \\ uc & ud \end{bmatrix} \pmod{n}, u \in U\right\} \\
&= \left\{\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}_n \in \Gamma/\Gamma(n) : \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}_n = \begin{bmatrix} ua & ub \\ uc & ud \end{bmatrix}_n : u \in U\right\}
\end{aligned}$$

We will denote a member of $\Gamma/\Gamma(n, U)$, which is a dart of $\mathcal{M}_3(n, U)$ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Gamma(n, U) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U} \qquad \text{where} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

We define the subgroup $\Gamma_1(n,U)$ of Γ as

$$\begin{aligned}\Gamma_1(n,U) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n \in \Gamma/\Gamma(n) : \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}_n = \begin{bmatrix} u & ur \\ 0 & u \end{bmatrix}_n \quad : \quad r = 0,1,\ldots,n-1;\ u \in U \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U} \in \Gamma/\Gamma(n,U) : \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U} = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}_{n,U} \quad : \quad r = 0,1,\ldots,n-1 \right\}.\end{aligned}$$

Then we note that $\Gamma_1(n,U)$ is the group generated by $Y\Gamma(n,U)$, and so the vertices of $\mathcal{M}_3(n,U)$ are, for $\gamma \in \Gamma$, the cosets $\gamma\Gamma(n,U)\,\langle Y\Gamma(n,U)\rangle = \gamma\langle Y\Gamma(n,U)\rangle = \gamma\Gamma_1(n,U)$, or

$$\gamma\Gamma_1(n,U) = \left\{ \begin{bmatrix} a & ar+b \\ c & cr+d \end{bmatrix}_{n,U} \quad : \quad r = 0,1,\ldots,n-1. \right\}.$$

Given $a, c \in \mathbb{Z}$ such that $\gcd(a,c,n) = 1$, we can determine a unique vertex of $\mathcal{M}_3(n,U)$: we find $b, d \in \mathbb{Z}$ such that $ad - bc \equiv 1 \pmod{n}$. Then, if $\gamma \in \Gamma$ is the matrix with entries a, b, c and d, the vertex corresponding to the ordered pair (a,c) is $\gamma\Gamma_1(n,U)$, which we denote by $[a/c]_{n,U}$, omitting the suffix where this does not cause confusion.

There is a bijection between these vertices and the equivalence classes

$$\{(a',c') \in \mathbb{Z}\times\mathbb{Z} : \gcd(a',c',n) = 1,\ (a',c') \equiv (ua,uc) \pmod{n};\ u \in U\}.$$

The n darts incident to $[a/c]_{n,U}$ all have $[a/c]_{n,U}$ as initial vertex, and one of $[b+ar/d+cr]_{n,U}$ for $r = 0,1,2,\ldots,n-1$ as final vertex.

We can now define the edges and faces of $\mathcal{M}_3(n,U)$ in the same way as we did those of $\mathcal{M}_3(n)$, as follows.

An edge of $\mathcal{M}_3(n)$ is a coset $\gamma\Gamma(n,U)\langle XY\rangle$, which consists of the two darts $\gamma\Gamma(n,U)$ and $\gamma\Gamma(n,U)XY$ that is, if $\gamma \in \Gamma$ is the matrix with entries a, b, c and d,

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{n,U} \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U}, \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}_{n,U} \right\}.$$

The vertices $[a/c]_{n,U}$ and $[b/d]_{n,U}$ are adjacent in $\mathcal{M}_3(n,U)$ if they are incident to the same edge, that is if and only if $ad - bc \equiv u \pmod{n}$ for some $u \in U$.

A face of $\mathcal{M}_3(n,N)$ is the coset

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U} \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}_{n,U} \right\rangle = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n,U}, \begin{bmatrix} -b & a+b \\ -d & c+d \end{bmatrix}_{n,U}, \begin{bmatrix} -a-b & a \\ -c-d & c \end{bmatrix}_{n,U} \right\}.$$

So the dart with initial vertex $[a/c]_{n,U}$ and final vertex $[b/d]_{n,U}$ is incident to the triangular face with vertices $[a/c]_{n,U}$, $[b/d]_{n,U}$, and $[(a+b)/(c+d)]_{n,U}$.

We have

$$|\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(n,U)||\Gamma(n,U) : \Gamma(n)| = |\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(n)|,$$

so the number of darts of $\mathcal{M}_3(n,U)$ is

$$|\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(n,U)| = \frac{n^3}{|U|} \prod_{p|n} \left(1 - \frac{1}{p^2}\right),$$

where the product is over all prime divisors of n. Therefore the number of vertices, edges and faces of $\mathcal{M}_3(n,U)$, and its genus, g, are given by:

$$\begin{aligned} V(\mathcal{M}_3(n,U)) &= \frac{n^2}{|U|} \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \\ E(\mathcal{M}_3(n,U)) &= \frac{n^3}{2|U|} \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \\ F(\mathcal{M}_3(n,U)) &= \frac{n^3}{3|U|} \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \\ g(\mathcal{M}_3(n,U)) &= 1 + \frac{n^2(n-6)}{12|U|} \prod_{p|n} \left(1 - \frac{1}{p^2}\right). \end{aligned}$$

If $U = \{\pm 1\}$, the maps $\mathcal{M}_3(n,U)$ are the Farey maps. As suggested in [48], introducing larger sets U enables us to study additional maps, as the following examples show.

Example 3.7.2. The smallest map $\mathcal{M}_3(n,U)$ where $U \neq \{\pm \mathbf{1}\}$ is $\mathcal{M}_3(8, \{\pm 1, \pm 3\})$, Dyck's map. It was first studied in 1880 along with Klein's quartic by Dyck, a student of Klein's, in connection with work on Riemann surfaces. Details, together with illustrations of how the map can be realised as a polyhedron in three dimensional Euclidean space can be found in [6] and in [55]. Its 12 vertices are (omitting the suffix $8, \{\pm 1, \pm 3\}$):

$$[1/0], [0/1], [1/1], [2/1], [3/1], [4/1], [5/1], [6/1], [7/1], [1/2], [3/2], \text{ and } [1/4].$$

The map has 48 edges and 32 faces. Its genus is 3. Its vertex valency is 8, and its faces are triangles, so it is a map of type $(3, 8)$. Its group of orientation preserving automorphisms is the quotient of $\mathrm{PSL}_2(\mathbb{Z}/8\mathbb{Z})$ by the subgroup $\{I, 3I\}$, where I is the identity of $\mathrm{PSL}_2(\mathbb{Z}/8\mathbb{Z})$, and is of order 96. As stated in [55], the whole group of automorphisms, including those which do not preserve orientation, is of order 192. A realisation of this map is shown in Figure 3.7.1.

To find other maps in this family, we determine the values of n for which there are units u of $\mathbb{Z}/n\mathbb{Z}$, other than ± 1 , with $u^2 \equiv 1 \pmod{n}$.

From equation (2.2) of [48], if $n = {p_0}^{\alpha_0} {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_t}^{\alpha_t}$, there are 2^t units of $\mathbb{Z}/n\mathbb{Z}$ with $u^2 = 1 \pmod{n}$ if $\alpha_0 < 2$, 2^{t+1} if $\alpha_0 = 2$, and 2^{t+2} if $\alpha_0 > 2$.
So if p is an odd prime we have $|U| = 2$ and $\mathcal{M}_3(p^k, U) = \mathcal{M}_3(p^k)$ for any k. If $n = 2^k$, $|U| = 4$ for $k > 2$. For many composite values of n we can have $|U| > 2$, and $|U|$ can be large if n has many different prime factors. We give some examples of these maps with $n > 8$.

Example 3.7.3. The map $\mathcal{M}_3(12, \{1, 5, 7, 11\})$.

The 24 vertices of this map are, omitting the suffix $12, \{1, 5, 7, 11\}$:
[1/0], [0/1],[1/1],. . . ,[11/1],
[1/2],[3/2],[1/3],[2/3],[4/3],[7/3],[8/3],[11/3],
[1/4],[3/4], [1/6].
It has 144 edges and 96 faces. Its genus is 13.

Example 3.7.4. The map $\mathcal{M}_3(15, \{1, 4, 11, 14\})$.

The 48 vertices of this map are, omitting the suffix $15, \{1, 4, 11, 14\}$:
[1/0],[2/0],
[0/1],[1/1],. . . ,[14/1],

[0/2],[1/2],...,[14/2], [1/3],[2/3],[4/3],[5/3],[7/3],

[1/5],[2/5],[3/5],[4/5],[6/5],[7/5],

[1/6],[2/6],[4,6],[5/6],[7/6].

It has 360 edges and 240 faces. Its genus is 37.

Example 3.7.5. The map $M_3(60, \{1, 11, 19, 29, 31, 41, 49, 59\})$.

This map has 120 vertices, 1080 edges and 720 faces. Its genus is 145.

The following theorem extends some of the results in [48, Section 4]:

Theorem 3.7.6. *If $n = dm$, the generalised Farey map $\mathcal{M}_3(n, U)$ is a regular map covering of $\mathcal{M}_3(m, U)$ of order d^3 if $d \neq 2$, or 4 if $d = 2$, which is ramified at the vertices with ramification index d. The covering transformation takes the dart $\gamma\Gamma(n, U)$ to the dart $\gamma\Gamma(m, U)$, and the vertex $[a/c]_{n,U}$ to the vertex $[a/c]_{m,U}$.*

Proof. We note that if U is a set of units in $\mathbb{Z}/mU$ including $\{\pm 1\}$, with $u^2 = 1$ if $u \in U$, then U is also a set of units in $\mathbb{Z}/nU$ including $\{\pm 1\}$, with $u^2 = 1$ if $u \in U$. The proof of Lemma 3.3.1 can then be repeated, simply replacing $\Gamma/\Gamma(n)$ by $\Gamma/\Gamma(n, U)$. $\square$

We also note the following result:

Theorem 3.7.7. *For any U and any n, $\mathcal{M}_3(n)$ is a $|U|/2$ covering of $\mathcal{M}_3(n, U)$.*

Proof. As $\Gamma(n)$ is a subgroup of $\Gamma(n, U)$, the mapping

$$\begin{aligned} \sigma : \Gamma/\Gamma(n) &\longrightarrow \Gamma/\Gamma(n, U) \\ X\Gamma(n) &\longmapsto X\Gamma(n, U); \\ Y\Gamma(n) &\longmapsto Y\Gamma(n, U), \end{aligned}$$

is an epimorphism and so a covering transformation. So, as the identity of $\Gamma/\Gamma(n, U)$ is

$$\left\{ \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}_n : u \in U \right\},$$

the kernel of σ is
$$\left\{ \pm u \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_n : \right\} : u \in U \right\},$$
which is of order $|U|/2$. $\square$

So, for instance, $\mathcal{M}_3(8)$ is a regular double cover of Dycke's map. Note that $\mathcal{M}_3(8)$ is of genus 5, whereas Dycke's map is of genus 3.

We can also extend a recent result from [60]:

Theorem 3.7.8. *The diameter of $\mathcal{M}_3(n, U)$ is at most 3.*

Proof. The proof in [60] can adapted in a straightforward way. Alternatively, from [60, Theorem 11] we know that between any 2 vertices v_1 and v_2 of $\mathcal{M}_3(n)$ there is a path $\langle v_1, v, v', v_2 \rangle$ of length 3. Let $w_1 = [a/c]_{n,U}$ and $w_2 = [b/d]_{n,U}$ be any two vertices of $\mathcal{M}_3(n, U)$. Then since $\{-1, 1\} \subset U$, $v_1 = [a/c]_n$ and $v_2 = [b/d]_n$ are vertices of $\mathcal{M}_3(n)$, and from [60, Theorem 11] there is a path in $\mathcal{M}_3(n)$ of length 3 joining them. Let this path be $\langle v_1, v, v', v_2 \rangle$ and let σ be the mapping defined in the previous theorem. Then, since σ preserves adjacency, there is a path $\langle \sigma(v_1), \sigma(v), \sigma(v'), \sigma(v_2) \rangle$ of length 3 in $\mathcal{M}_3(n, U)$ joining $w_1 = \sigma(v_1)$ and $w_2 = \sigma(v_2)$. $\square$

The maps developed in [48] also include those where U includes units u such that $u^2 = -1$. The corresponding matrices are not members of $\mathrm{PSL}_2(\mathbb{Z})$, and the surfaces on which the maps are embedded are not orientated. It could also be possible to study the case of U being any group of units, including the whole group of units.

3.8 Maps defined by Hecke groups

We first summarise the theoretical background developed in [30]. The Hecke group H^q is a discrete subgroup of infinite index in $\mathrm{PSL}_2(\mathbb{Z}[\lambda_q])$ generated by the matrices
$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}, \quad \text{where } \lambda_q = 2\cos\pi/q.$$

The universal Hecke map $\widehat{\mathcal{M}_q}$ is the tessellation of the upper hyperbolic plane whose darts are the dart from infinity to zero and its images under H^q. If $q = 3$, then $\lambda_q = 1$, so H^3 is the modular group and $\widehat{\mathcal{M}_3}$ is the Farey tessellation.

Hecke maps are quotients of $\widehat{\mathcal{M}_q}$ by the congruence subgroups of H^q. We are particularly interested in the map $\mathcal{M}_q(n)$ of type (q, n), which is the quotient of $\widehat{\mathcal{M}_q}$ by the subgroup $H^q(n) = H^q/(n)$ defined by the ideal $(n) = \{n(a + b\lambda_q) : a, b \in \mathbb{Z}\}$ of $\mathbb{Z}[\lambda_q]$.

As in [30, 38], we will consider the maps $\mathcal{M}_4(n)$ and $\mathcal{M}_6(n)$ corresponding to the Hecke groups H^4 and H^6. These are relatively straightforward to deal with as $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$. The faces of $\mathcal{M}_4(n)$ and $\mathcal{M}_6(n)$ are, respectively, quadrilaterals and hexagons.

As is shown in [30], $\mathcal{M}_4(n)$ has two types of vertices. If (a, c) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ such that $\gcd(a, c, n) = 1$ and $\gcd(a, 2, n) = 1$, an *even vertex*, which we will write $[a/c\sqrt{2}]_n$, is the equivalence class of ordered pairs

$$\{(a', c') \in \mathbb{Z} \times \mathbb{Z} : \gcd(a', c', n) = 1,\ (a', c') \equiv \pm(a, c) \pmod{n}\}.$$

If (a, c) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ such that $\gcd(a, c, n) = 1$ and $\gcd(c, 2, n) = 1$, an *odd vertex*, which we will write $[a\sqrt{2}/c]_n$, is the equivalence class of ordered pairs

$$\{(a', c') \in \mathbb{Z} \times \mathbb{Z} : \gcd(a', c', n) = 1,\ (a', c') \equiv \pm(a, c) \pmod{n}\}.$$

If $[a/c\sqrt{2}]_n$ and $[b\sqrt{2}/d]_n$ with $a, b, c, d \in \mathbb{Z}/n\mathbb{Z}$ are two vertices of $\mathcal{M}_4(n)$, then those vertices are adjacent if and only if $ad - 2bc \equiv \pm 1 \pmod{n}$. Odd vertices are adjacent to even vertices, and vice-versa. The vertex valency of $\mathcal{M}_4(n)$ is n. Replacing 2 by 3 and 4 by 6 gives analogous results for $\mathcal{M}_6(n)$. $\mathcal{G}_4(n)$ is the underlying graph of $\mathcal{M}_4(n)$, and $\mathcal{G}_6(n)$ that of $\mathcal{M}_6(n)$.

We use the ideas of coverings as in the previous sections to obtain a new result linking some Hecke maps to Farey maps. This is Theorem 1.1.3, which states that, for odd n, $\mathcal{G}_4(n)$ is a double graph covering of $\mathcal{G}_3(n)$, and if n is not a multiple of 3, $\mathcal{G}_6(n)$ is a double graph covering of $\mathcal{G}_3(n)$. These are graph coverings, not, in general, map coverings, but this result will be sufficient to enable us to find the spectra of these maps in the next chapter.

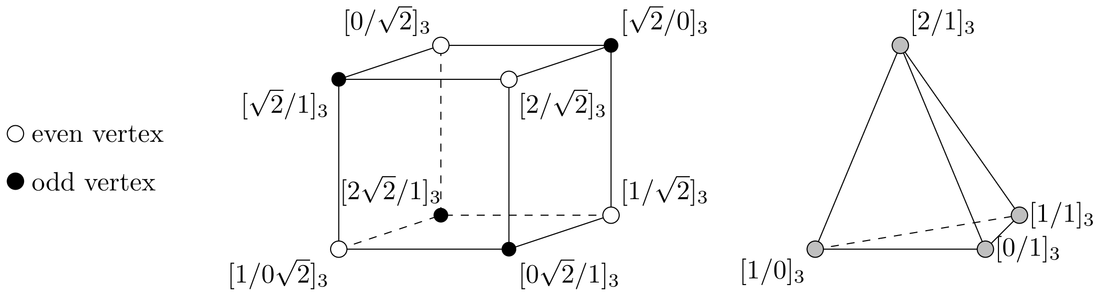

Figure 3.8.1: The Hecke map $\mathcal{M}_4(3)$ and the Farey map $\mathcal{M}_3(3)$.

Figure 3.8.1 shows the cube with skeleton $\mathcal{G}_4(3)$ as a double graph covering of the tetrahedron with skeleton $\mathcal{G}_3(3)$. Both these graphs can be embedded as maps on the sphere. This is not a map covering as there is no mapping from the 6 faces of $\mathcal{M}_4(3)$ to the 4 faces of $\mathcal{M}_3(3)$.

Proof of Theorem 1.1.3. Consider the mapping

$$\begin{array}{rcl} \sigma : \text{ vertices of } \mathcal{M}_4(n) & \longrightarrow & \text{vertices of } \mathcal{M}_3(n) \\ [a/c\sqrt{2}]_n & \longmapsto & [a/c]_n \\ [a\sqrt{2}/c]_n & \longmapsto & [a/c]_n \end{array}$$

Then σ is a bijection between the odd vertices of $\mathcal{M}_4(n)$ and the vertices of $\mathcal{M}_3(n)$, and also between the even vertices of $\mathcal{M}_4(n)$ and the vertices of $\mathcal{M}_3(n)$. So it is a graph covering transformation between the underlying graphs, of order 2 as each vertex of $\mathcal{M}_3(n)$ has two pre-images. Replacing 2 by 3 gives the corresponding result for $\mathcal{M}_6(n)$ if 3 is not a factor of n. □

We also obtain the diameter of certain Hecke maps, partially recovering a result in [38] by a different method.

Theorem 3.1.1 states that both $\mathcal{M}_4(n)$ (for odd n) and $\mathcal{M}_6(n)$ (for $3 \nmid n$) have diameter 4.

Proof of Theorem 3.1.1. Define σ as in Theorem 1.1.3, and let $P = \langle w_1, w_2, \ldots, w_k \rangle$ be a path in $\mathcal{M}_3(n)$. Then $\sigma^{-1}(w_i)$ consists of an even vertex of $\mathcal{M}_4(n)$, v_i, and an odd vertex u_i. As even vertices are adjacent to odd vertices and vice-versa, P is lifted by σ^{-1} to two paths in $\mathcal{M}_4(n)$: $P_1 = \langle v_1, u_2, v_3, \ldots \rangle$, and $P_2 = \langle u_1, v_2, u_3 \ldots \rangle$.

A lift of a path of even length in $\mathcal{M}_3(n)$ will join two vertices of the same parity in $\mathcal{M}_4(n)$, and a lift of a path of odd length two vertices of opposite parity.

Now let v_a and v_b be any two distinct vertices of $\mathcal{M}_4(n)$, and let $w_a = \sigma(v_a)$ and $w_b = \sigma(v_b)$. If $w_a = w_b$, v_a and v_b are of opposite parities. There is a path of length 3 round the edges of a triangle incident to w_a; this lifts to a path of length 3 between v_a and v_b, so $d(v_a, v_b) \leq 3$.

From [60], the diameter of $\mathcal{M}_3(n)$ for any n is 3, so if $w_a \neq w_b$, $d(w_a, w_b) \leq 3$. If P is a path of shortest length between w_1 and w_i, the vertices preceding w_{i-1} cannot be adjacent to w_i. The edge $w_{i-1}w_i$ is incident to two triangles. Let w be the third vertex of one of these triangles. Then there is a path $\langle w_1, \ldots, w_{i-1}, w, w_i \rangle$ from w_1 to w_i of length $d(w_1, w_i) + 1$.

So since $d(w_a, w_b) \leq 3$, there is always both a path of odd length less than 4 and a path of even length less than or equal to 4 between w_a and w_b, which lifts to a path of length less than or equal to 4 between v_a and v_b whether or not they are of the same parity. The result for $\mathcal{M}_6(n)$ if $3 \nmid n$ follows similarly.

The further result that this is true for all n is given by [38, Theorem 14].

Chapter 4

Spectra of graphs and maps

4.1 Introduction and main results

In [10–12] the spectra of graphs are found using coverings ramified at the face centres of maps. In this chapter we use the regular coverings ramified at map vertices which are specified by Theorems 1.1.1 and 1.1.2 to find, for each positive integer n, the spectrum of the Farey map $\mathcal{M}_3(n)$ from the prime decomposition of n.

To express our results, we define a product of multisets as follows. Suppose that

$$\mathrm{sp}(\mathcal{M}_1) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_i^{(m_i)}\} \quad \text{and} \quad \mathrm{sp}(\mathcal{M}_2) = \{\mu_1^{(l_1)}, \mu_2^{(l_2)}, \ldots, \mu_j^{(l_j)}\}.$$

$$\text{Then we define} \qquad \mathrm{sp}(\mathcal{M}_1)\mathrm{sp}(\mathcal{M}_2) = \{\lambda_r \mu_s^{(m_r l_s)} : r = 1, \ldots, i;\ s = 1, \ldots, j\}.$$

$$\text{Also, for } k \in \mathbb{Z},\ \text{we define} \qquad k\mathrm{sp}(\mathcal{M}_1) = \{k\lambda_1^{(m_1)}, k\lambda_2^{(m_2)}, \ldots, k\lambda_i^{(m_i)}\}.$$

We simplify notation by writing $\mathrm{sp}(\mathcal{M}_3(n)) = \mathrm{sp}_3(n)$. Then we have, from equation (2.3.1), $\mathrm{sp}_3(2) = \{-1^{(2)}, 2\}$ and $\mathrm{sp}_3(3) = \{-1^{(3)}, 3\}$ and we will show that $\mathrm{sp}_3(4) = \{-2^{(2)}, 0^{(3)}, 4\}$. For higher values of n, we prove the following two theorems.

Theorem 4.1.1. *Let p be a prime number, and k a positive integer.*

(i) *If $p > 3$, then $\mathrm{sp}_3(p) = \{-\sqrt{p}^{(m)}, -1^{(p)}, \sqrt{p}^{(m)}, p\}$, where $m = \frac{1}{4}(p-3)(p+1)$.*

(ii) *If* $p^k > 4$, *then* $\mathrm{sp}_3(p^k) = p\,\mathrm{sp}_3(p^{k-1}) \cup \{-\sqrt{p^k}^{(\frac{1}{2}pc)}, 0^{(c)}, \sqrt{p^k}^{(\frac{1}{2}pc)}\}$,
where $c = (p-1)V(\mathcal{M}_3(p^{k-1}))$.

Theorem 4.1.2. *Let* m *be a positive integer.*

(i) *If* m *is odd, then* $\mathrm{sp}_3(2m) = \mathrm{sp}_3(2)\mathrm{sp}_3(m)$.

(ii) *If* l *and* m *are coprime integers, and neither* l *nor* m *is twice an odd integer, then*

$$\mathrm{sp}_3(lm) = \mathrm{sp}_3(l)\mathrm{sp}_3(m) \cup \{-\sqrt{lm}^{(N/4)}, \sqrt{lm}^{(N/4)}\}, \text{ where } N = V(\mathcal{M}_3(lm)).$$

For example we have the following, first using (3.2.2) to find

$$V(\mathcal{M}_3(7)) = \frac{7^2}{2}\left(1 - \frac{1}{7^2}\right) = 24, \quad \text{and} \quad V(\mathcal{M}_3(28)) = \frac{28^2}{2}\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{7^2}\right) = 72,$$

$$\begin{aligned}
\mathrm{sp}_3(7) &= \{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\}, \\
\mathrm{sp}_3(49) &= 7\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \{-7^{(7\times3\times24)}, 0^{(6\times24)}, 7^{(7\times3\times24)}\} \\
&= \{-7\sqrt{7}^{(8)}, -7^{(511)}, 0^{(144)}, 7^{(504)}, 7\sqrt{7}^{(8)}, 49\}, \\
\mathrm{sp}_3(14) &= \mathrm{sp}_3(2)\mathrm{sp}_3(7) \\
&= \{-1^{(2)}, 2\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \\
&= \{-1^{(2)}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \{2\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \\
&= \{-7^{(2)}, -2\sqrt{7}^{(8)} - \sqrt{7}^{(16)}, -2^{(7)}, 1^{(14)}, \sqrt{7}^{(16)}, 2\sqrt{7}^{(8)}, 14\}, \\
\mathrm{sp}_3(28) &= \mathrm{sp}_3(4)\mathrm{sp}_3(7) \cup \{-\sqrt{28}^{(72)}, \sqrt{28}^{(72)}\} \\
&= \{-2^{(2)}, 0^{(3)}, 4\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \{-2\sqrt{7}^{(72)}, 2\sqrt{7}^{(72)}\} \\
&= \{-2^{(2)}\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \{0^{(3)}\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \\
&\quad \{4\}\{-\sqrt{7}^{(8)}, -1^{(7)}, \sqrt{7}^{(8)}, 7\} \cup \{-2\sqrt{7}^{(72)}, 2\sqrt{7}^{(72)}\} \\
&= \{-14^{(2)}, -4\sqrt{7}^{(8)}, -2\sqrt{7}^{(88)}, -4^{(7)}, 0^{(72)}, 2^{(14)}, 2\sqrt{7}^{(88)}, 4\sqrt{7}^{(8)}, 28\}.
\end{aligned}$$

4.2 Evaluation of spectra using regular coverings.

In this section we give some general results on coverings for regular graphs and maps. Recall that, if $\mathcal{M}_1$ is a regular covering of $\mathcal{M}_2$ with covering transformation σ, and h is a dart of $\mathcal{M}_2$, then $\sigma^{-1}(h)$ is the fibre of h. Also, we define the fibre of a vertex v of $\mathcal{M}_2$ to be $\sigma^{-1}(v)$.

Lemma 4.2.1. *Let $\mathcal{M}_1 = (G_1, x_1, y_1)$ and $\mathcal{M}_2 = (G_2, x_2, y_2)$ be regular maps such that $\mathcal{M}_1$ is a regular covering of $\mathcal{M}_2$, and let their vertex valencies be n and m respectively. If v and v' are adjacent vertices of $\mathcal{M}_2$, then each of the vertices in the fibre of v is adjacent to exactly $d = n/m$ of the vertices in the fibre of v'.*

Proof. Recall from Section 2.7 that m divides n. Let σ be the covering transformation, and let the number of sheets of the covering be $r = |\mathrm{Ker}(\sigma)|$. Let h be a dart of $\mathcal{M}_2$ with initial vertex v and final vertex v'. Let $s = |\sigma^{-1}(v)|$, and let w be a vertex of $\mathcal{M}_1$ in $\sigma^{-1}(v)$. Then the orbit of w under the action of $\mathrm{Ker}(\sigma)$ is $\sigma^{-1}(v)$. Let d be the number of vertices in $\sigma^{-1}(v')$ adjacent to w. Then the stabiliser of w in $\mathrm{Ker}(\sigma)$ is the set of darts in $\sigma^{-1}(h)$ incident to w. By the orbit stabiliser theorem $r = sd$. Now let D be the set of all darts in the fibres of all of the m darts incident to v. We have $|D| = rm = sn$, so $dm = n$. □

Lemma 4.2.2. *Let a graph $\mathcal{G}_1$ be a covering of a graph $\mathcal{G}_2$ with graph covering transformation ϕ. Let v and v' be two adjacent vertices of $\mathcal{G}_2$. Then any vertex in $\phi^{-1}(v)$ is adjacent to exactly one of the vertices in $\phi^{-1}(v')$.*

Proof. Let w be a member of $\phi^{-1}(v)$. The transformation ϕ maps the vertices adjacent to w bijectively onto the vertices adjacent to v, so ϕ^{-1} maps v' to just one vertex adjacent to w. □

Recall that the N eigenvalues of a graph or map with N vertices are those of its adjacency matrix, which is an $N \times N$ square matrix whose rows and columns are indexed by the graph vertices. For a graph $\mathcal{G}$ with vertices $v_1, v_2, \ldots, v_N$, the entry in the i^{th} row and the j^{th} column is e_{ij}, where $e_{ij} = 1$ if v_i and v_j are adjacent, and otherwise $e_{ij} = 0$. An adjacency matrix has real entries; for the simple, undirected graphs we are considering, it is symmetrical and so has real eigenvalues; all its diagonal entries are 0.

If A is the adjacency matrix of a graph with N vertices, an eigenvector x has N components $x_1, \dots x_N$, which are indexed in the same way as the rows and columns of the adjacency matrix, that is by the graph vertices. We will denote by x_v the component of the eigenvector corresponding to the vertex v, so that, if v is the vertex v_i, we have $x_v = x_i$.

Lemma 4.2.3. *Let $\mathcal{G}$ be a graph with adjacency matrix A. Then λ is an eigenvalue of A with eigenvector x if and only if, for all vertices v of $\mathcal{G}$,*

$$\lambda x_v = \sum_{u \sim v} x_u,$$

where the sum is over all the vertices of $\mathcal{G}$ adjacent to v.

Proof. By definition, x is an eigenvector for the eigenvalue λ of A if and only if $Ax = \lambda x$. That is equivalent to $\sum_{j=1}^{N} e_{ij} x_j = \lambda x_i$ for all $i = 1, \dots, N$ or, as $e_{ij} = 0$ unless v_i is adjacent to v_j, $\sum_{u \sim v} x_u = \lambda x_v$. □

Lemma 4.2.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two regular graphs with vertex valencies n and m respectively, where $n = dm$, and suppose either that they are the underlying graphs of two regular maps $\mathcal{M}_1$ and $\mathcal{M}_2$, and that $\mathcal{M}_1$ is a regular map covering of $\mathcal{M}_2$, or, if $d = 1$, that $\mathcal{G}_1$ is a graph covering of $\mathcal{G}_2$. Let N be the number of vertices of $\mathcal{G}_2$. Then*

(i) *if $d = 1$, $\mathrm{sp}(\mathcal{G}_2) \subset \mathrm{sp}(\mathcal{G}_1)$, and*

(ii) *if $d > 1$, $d\,\mathrm{sp}(\mathcal{G}_2) \cup \{0^{(\gamma)}\} \subset \mathrm{sp}(\mathcal{G}_1)$, where $\gamma \geq N(d-1)$.*

Proof. Let π be the regular map covering transformation or graph covering transformation of $\mathcal{G}_2$ by $\mathcal{G}_1$. Let w be a vertex of $\mathcal{G}_1$ and v a vertex of $\mathcal{G}_2$ with $\pi(w) = v$. Then, from Lemmas 4.2.1 and 4.2.2, if v' is adjacent to v in $\mathcal{G}_2$, w is adjacent to d of the vertices in $\pi^{-1}(v')$. Let λ be an eigenvalue of $\mathcal{G}_2$ with eigenvector x.

(i) Define the vector y by $y_w = x_{\pi(w)}$ for all vertices w of $\mathcal{G}_1$. Then

$$\sum_{w' \sim w} y_{w'} = d \sum_{v' \sim v} x_{v'} = d\lambda x_v = d\lambda y_w.$$

So $d\lambda$ is an eigenvalue of $\mathcal{G}_1$ with eigenvector y. Since a different vector y corresponds to each vector x in the eigenspace of λ, the geometric multiplicity of $d\lambda$ in $\mathcal{G}_1$ is at least that of λ in $\mathcal{G}_2$.

(ii) Let v be a vertex of $\mathcal{G}_2$, and let w be a vertex in the fibre of v. Let the m neighbours of v be v'_i for $i = 1, \ldots, m$. For each $w \in \pi^{-1}(v)$, label the d vertices in the fibre of each v'_i adjacent to w as $w'_{i,1}, \ldots, w'_{i,d}$. Let K be an integer such that $1 \leq K < d$. Then, for any pair (v, K), we define a vector z by its components z_u (here the component z_u corresponds to the vertex u of $\mathcal{G}_1$):

$$z_u = \begin{cases} 1 \text{ if } u = w'_{i,K} & \text{for some } w \in \pi^{-1}(v) \text{ and some } v_i \text{ adjacent to } v, \\ -1 \text{ if } u = w'_{i,(K+1)} & \text{for some } w \in \pi^{-1}(v) \text{ and some } v_i \text{ adjacent to } v, \\ 0 \text{ otherwise.} \end{cases}$$

Let A be the adjacency matrix of $\mathcal{G}_1$. Then, for each pair (v, K), the r'th component of the vector Az is, putting $F = \pi^{-1}(v)$,

$$\sum_u e_{ur} z_u = \sum_{w \in F} \left(\sum_{i=1}^{m} e_{w'_{i,K},w} - \sum_{i=1}^{m} e_{w'_{i,(K+1)},w} \right) = \sum_{w \in F} \sum_{i=1}^{m} (1-1) = 0 z_r.$$

So 0 is an eigenvalue of $\mathcal{G}_1$ with eigenvector z. The vectors z together with y form a linearly independent set. So, as $\mathcal{G}_2$ has N vertices v, and K takes $d-1$ values, 0 is an eigenvalue of $\mathcal{G}_1$ with (possibly additional) geometric multiplicity at least $N(d-1)$.

□

We can immediately use this lemma to find the spectrum of $\mathcal{M}_3(4)$. That map is a covering of $\mathcal{M}_3(2)$, so since the spectrum of $\mathcal{M}_3(2)$ is $\{(-1)^{(2)}, 2\}$, $d = 2$, and $\mathcal{M}_3(2)$ has 3 vertices, the lemma gives us 6 eigenvalues counting multiplicities, so as $\mathcal{M}_3(4)$ has just 6 vertices and therefore just 6 eigenvalues we have the following result.

Corollary 4.2.5. *The spectrum of* $\mathcal{M}_3(4)$ *is* $\{(-2)^{(2)}, 0^{(3)}, 4\}$.

We will also use the following result.

Lemma 4.2.6. *Let A be an $N \times N$ symmetric matrix, and suppose that, for some integer n,*

$$A^2 - nI \quad = \quad \begin{pmatrix} C & C & \cdots & C \\ C & C & \cdots & C \\ \vdots & \vdots & \ddots & \vdots \\ C & C & \cdots & C \end{pmatrix},$$

where C is an $r \times r$ symmetric matrix. Then either $\sqrt{n}$ or $-\sqrt{n}$ or both are eigenvalues of A with total algebraic multiplicity greater than or equal to $N - \mathrm{rank}(C)$.

Proof. The rows and columns of $A^2 - nI$ are not linearly independent, so its determinant is 0 and n is an eigenvalue of A^2. Since $|A^2 - nI| = |A - \sqrt{n}I||A + \sqrt{n}I|$, $\sqrt{n}$ or $-\sqrt{n}$ or both are eigenvalues of A with total algebraic multiplicity equal to the algebraic multiplicity of n as an eigenvalue of A^2.

We will denote the geometric multiplicity of an eigenvalue λ of a matrix M as $\gamma(\lambda)$. It is the dimension of the space generated by its eigenvectors, and equal to the nullity of $M - \lambda I$. The geometric multiplicity of an eignenvalue can never exceed its algebraic multiplicity. If X is the dimension of the space on which M acts,

$$\gamma(\lambda) \quad = \quad X - \mathrm{rank}(M - \lambda I).$$

Then as the rank of $A^2 - nI$ is equal to the rank of C, n is an eigenvalue of A^2 with geometric multiplicity equal to $N - \mathrm{rank}(C)$, and therefore the total of the algebraic multiplicities of $\sqrt{n}$ and of $-\sqrt{n}$ as eigenvalues of A is at least $N - \mathrm{rank}(C)$. □

4.3 Farey maps of prime level

We denote the adjacency matrix of $\mathcal{M}_3(n)$ by $A(n)$, and index its rows and columns according to the labelling we choose for the vertices of $\mathcal{M}_3(n)$. In particular we will label the vertex $[1/0]_n$ as v_0, so that it corresponds to the first row and column of $A(n)$.

The entry of $A(n)^2$ corresponding to the vertices v_i and v_j is equal to the number of walks of length 2 connecting them (see, for instance, [22, Lemma 8.1.2]). We find this entry by extending the results given in [60, Theorems 11-15] for the distance between two Farey map vertices.

Lemma 4.3.1. *Let the integers β and Δ be such that* $\gcd(\beta, \Delta, n) = 1$*, so $v_k = [\beta/\Delta]_n$ is a vertex of $\mathcal{M}_3(n)$, and let* $\gcd(\Delta, n) = r$*. Then the number of walks of length 2 between v_0 and v_k is*

$$\begin{cases} 2 \text{ if } r = 1, \\ 0 \text{ if } r \text{ is not a divisor of either of } \beta \pm 1, \\ r \text{ if } r \text{ is a divisor of one of } \beta \pm 1, \text{and} \\ 4 \text{ if } r = 2 \text{ and } \beta \text{ is odd }. \end{cases}$$

Proof. Since all vertices adjacent to $[1/0]_n$ are of the form $[x/1]_n$ for $x \in \mathbb{Z}$, the vertex v_k has a walk of length 2 to $[1/0]_n$ if and only if there is a vertex $[x/1]_n$ adjacent to $[\beta/\Delta]_n$, that is if and only if there is an integer x such that

$$x\Delta \equiv \beta \pm 1 \pmod{n}.$$

Then the number of walks of length 2 from v_k to $[1,0]_n$ is equal to the total number of solutions modulo n to these 2 congruences. The result follows, recalling, for instance from [2, Theorem 5.12-5.14], that each congruence has one solution modulo n if and only if $\gcd(\Delta, n) = 1$, no solutions if $\gcd(\Delta, n) = r$ and r does not divide either of $\Delta \pm 1$, and r solutions if $r \neq 2$ and r divides either $\beta + 1$ or $\beta - 1$. If $r = 2$ and β is odd, 2 divides both $\beta + 1$ and $\beta - 1$, so there are 4 walks of length 2. □

Given any two vertices $v_i = [a/c]_n$ and $v_j = [b/d]_n$ of $\mathcal{M}_3(n)$ we will use this lemma to find the ij^{th} entry of $A(n)^2$. Define $\Delta_n(ij) = ad - bc$, and let $\lambda_n(i), \mu_n(i) \in \mathbb{Z}$ be any two integers such that $a\lambda_n(i) + c\mu_n(i) + \nu n = 1$ for some $\nu \in \mathbb{Z}$. Then the automorphism

$$M_n(i) = \begin{bmatrix} \lambda_n(i) & \mu_n(i) \\ -c & a \end{bmatrix}_n$$

takes v_i to v_0, and v_j to $v_l = [\beta_n(ij)/\Delta_n(ij)]_n$, where $\beta_n(ij) = \lambda_n(i)b + \mu_n(i)d$. The ij^{th} entry of $A(n)^2$ is then given by Lemma 4.3.1.

Example 4.3.2. Consider the vertices $v_i = [1/3]_{10}$ and $v_j = [3/2]_{10}$ of $\mathcal{M}_3(10)$. Put $\lambda_{10}(i) = 1$ and $\mu_{10}(i) = 0$. Then

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}_{10} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}_{10} = \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}_{10},$$

so $M_{10}(i)$ takes $[1/3]_{10}$ to $[1/0]_{10}$, and $[3/2]_{10}$ to $[3/-7]_{10} = [3/3]_{10}$. Then, as $\gcd(3, 10) = 1$, there are 2 walks of length 2 from $[1/0]_{10}$ to $[3/3]_{10}$, and therefore there are 2 walks of length 2 from $[1/3]_{10}$ to $[3/2]_{10}$, and the ij^{th} entry in $A(10)^2$ is 2.

Example 4.3.3. Consider the vertices $v_i = [3/2]_{10}$ and $v_j = [1/4]_{10}$ of $\mathcal{M}_3(10)$. Put $\lambda_{10}(i) = 1$ and $\mu_n(i) = -1$. Then

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}_{10} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}_{10} = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}_{10},$$

so $M_{10}(i)$ takes $[3/2]_{10}$ to $[1/0]_{10}$, and $[1/4]_{10}$ to $[-3/0]_{10} = [3/0]_{10}$. Then as $\gcd(0, 10) = 10$ does not divide 3 ± 1, there are no walks of length 2 from $[1/0]_{10}$ to $[3/0]_{10}$, and therefore there are no walks of length 2 from $[3/2]_{10}$ to $[1/4]_{10}$, and the ij^{th} entry in $A(10)^2$ is 0.

Lemma 4.3.4. *If $\Delta_n(ij) \equiv 0 \pmod{n}$, the ij^{th} entry of the matrix $A(n)^2 - nI$ is 0.*

Proof. If $\Delta_n(ij) = \nu n$ for some $\nu \in \mathbb{Z}$, the automorphism $M_n(i)$ takes v_i and v_j to $[1/0]_n$ and $[\beta_n(ij)/\nu n]_n$. Then $r = n$ divides $\beta_n(ij) \pm 1$ if and only if $\beta_n(ij) = \pm 1 + \kappa n$ for some $\kappa \in \mathbb{Z}$, in which case $[\beta_n(ij)/\nu n]_n = v_0$; so $v_i = v_j$, and the ii^{th} entry of $A(n)^2$ is n, as expected, since there are n paths to and from v_i along each of the n edges incident to v_i. If $\beta_n(ij) \not\equiv \pm 1 \pmod{n}$, $v_i \neq v_j$. In this case the ij^{th} entry of $A(n)^2$ is 0, as [60, Theorem 14] shows that the shortest path between these vertices is of length 3. $\square$

The underlying graphs of $\mathcal{M}_3(2)$ and $\mathcal{M}_3(3)$ are the complete graphs on 3 and 4 vertices respectively. The spectrum of the complete graph on $k+1$ vertices is $\{-1^{(k)}, k\}$, therefore $\mathrm{sp}_3(2) = \{-1^{(2)}, 2\}$ and $\mathrm{sp}_3(3) = \{-1^{(3)}, 3\}$. We can now prove Theorem 4.1.1(i).

Proof of Theorem 4.1.1(i). All congruences in this proof are modulo p. If $a \in \mathbb{Z}$, we define the integers $a_C : a_C \in \mathbb{Z}, 0 \leq a_C < p, a_C \equiv a$, and $a_H : a_H \in \mathbb{Z}, 0 \leq a_H < p/2, a_H \equiv a$ or $p - a_H \equiv a$.

As the spectrum of K_{p+1} is $\{-1^{(p)}, p\}$, from Lemmas 4.2.4 and 3.5.1, $\{-1^{(p)}, p\} \subset \mathrm{sp}(p)$.

If $\Delta_p(ij) \not\equiv 0$, $\gcd(\Delta_p(ij), p) = 1$ and the ij^{th} entry of $A(p)^2$ is 2.

We order the vertices of $\mathcal{M}_3(p)$ so that, if $v_i = [a/c]_p$,

$$i = \begin{cases} (a_H - 1)(p+1) & \text{if} \quad c \equiv 0 \\ ((c^{-1})_H - 1)(p+1) + (ac^{-1})_C + 1 & \text{if} \quad c \not\equiv 0. \end{cases}$$

We show that this ensures the $p+1$ vertices of each of the $h = \frac{1}{2}(p-1)$ stars are together, with the poles $p+1$ positions apart. We note that $\Delta_p(ij) \equiv 0$ if, putting $v_j = [b/d]_n$, $ad - bc \equiv 0$.
If $c \equiv 0$, as $a \not\equiv 0, d \equiv 0$, so $i = (a_H - 1)(p+1)$ and $j = (b_H - 1)(p+1)$, and therefore $j - i$ is a multiple of $p+1$.
If $c \not\equiv 0, i = ((c^{-1})_H - 1)(p+1) + (ac^{-1})_C + 1, j = ((d^{-1})_H - 1)(p+1) + (bd^{-1})_C + 1$, and $ac^{-1} - bd^{-1} \equiv 0$, so that $j - i$ is again a multiple of $p+1$.

So the ij^{th} entry of $A(p)^2$ is 0 if and only if i and j are a multiple of $p+1$ positions apart. Therefore

$$A(p)^2 - pI = \begin{pmatrix} C & C & \cdots & C & C \\ C & C & \cdots & C & C \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C & C & \cdots & C & C \\ C & C & \cdots & C & C \end{pmatrix} \quad \text{where} \quad C = \begin{pmatrix} 0 & 2 & \cdots & 2 & 2 \\ 2 & 0 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 0 & 2 \\ 2 & 2 & \dots & 2 & 0 \end{pmatrix}.$$

The matrix $A(p)^2 - pI$ is arranged as $h \times h$ copies of the $(p+1) \times (p+1)$ matrix C, whose ij elements are equal to 2 if $i \neq j$, or to 0 if $i = j$. Then, from Lemma 4.2.6, $\sqrt{p}$ or $-\sqrt{p}$ or both are eigenvalues of $\mathcal{M}_3(p)$ with total algebraic multiplicity greater than or equal to $\frac{1}{2}(p+1)(p-3)$. The result follows as the total algebraic multiplicity of the eigenvalues of $\mathcal{M}_3(p)$ is the number of its vertices, which is $\frac{1}{2}(p+1)(p-1)$. $\square$

Example 4.3.5. We use the example of $\mathcal{M}_3(7)$ to show how the rows and columns of $A(p)$, the adjacency matrix of $\mathcal{M}_3(p)$, and those of its square $A(p)^2$ are indexed. We have $p = 7$ and $h = 3$. The graph has $3(7+1) = 24$ vertices, so the matrices are of size 24×24. The three poles are $[1/0]_7, [2/0]_7$ and $[3/0]_7$. (We will omit the suffix 7 for the rest of this example).

We will divide the vertices into three blocks corresponding to the 3 stars of these poles, with 8 vertices each. We let $[1/0]$ and its neighbours $[a/1]$ for $a = 0, \ldots, 6$ correspond, in order, to the first 8 rows and columns of the matrices, that is $[1/0] = v_0$, and $[a/1] = v_{a+1}$ for $a = 0, \ldots, 6$. Also $[2/0] = v_8$ and the vertices v_i for $i = 9, \ldots, 15$ are the 7 vertices in the star of $[2, 0]$, with for instance $[1/3] = v_{11}$ as $1 \times 8 + 2 \times 1 + 1 = 11$. $[3/0] = v_{16}$, and, for instance, $[2/2] = v_{23}$.

Table 4.1 shows part of the matrix $A(7)^2 - 7I$, together with the indexing of its rows and columns. The way in which the indexing has been done ensures that the matrix can be subdivided into 9 submatrices, the diagonal entries of which are 0.

4.4 Farey maps of prime power level

We begin this section by showing an example of the matrices which we are investigating.

Example 4.4.1. From Theorem 1.1.1, the map $\mathcal{M}_3(25)$ is a covering of $\mathcal{M}_3(5)$ with 125 sheets, ramified at the vertices with ramification index 5. Since $\mathcal{M}_3(5)$ has 12 vertices, $\mathcal{M}_3(25)$ has $12 \times 25 = 300$ vertices. We index these vertices in the following way: we designate as w_0 to w_{11} the 12 vertices $[a/c]_{25}$, where $[a/c]_5$ are the 12 vertices of $\mathcal{M}_3(5)$, ordered in the same way as those in example 4.3.5; then the vertices w_{12} to w_{23} are $[a/(c+5)]_{25}$, the vertices w_{24} to w_{35} are $[a/(c+10)]_{25}$, the vertices w_{36} to w_{47} are $[a/(c+15)]_{25}$, and the vertices w_{48} to w_{59} $[a/(c+20)]_{25}$; a second set of 60 vertices w_{60} to w_{119} is found by replacing a by $a+5$, a third set by replacing a by $a+10$, and so on. We have 5 sets of 60 vertices, each comprising 5 smaller sets of 12 vertices each. Part of the square of the adjacency matrix of $\mathcal{M}_3(25)$ is shown in Table 4.2, its entries are found using Lemma 4.3.1. They are $e_{ii} = 25$, $e_{ij} = 0$ if $i \equiv j \equiv 0 \pmod{60}$ and for vertices on the corresponding diagonal, $e_{ij} = 5$ if $i \neq j$, and $i \equiv j \equiv 0 \pmod 6$, but either $i \not\equiv 0 \pmod{60}$ or $j \not\equiv 0 \pmod{60}$, and for vertices on the corresponding diagonals, and 2 otherwise.

To prove Theorem 4.1.1(ii) we need the following lemmas. We put $q = p^{k-1}$.

		v_0 [1/0]	v_1 [0/1]	v_2 [1/1]	v_3 [2/1]	v_4 [3/1]	v_5 [4/1]	v_6 [5/1]	v_7 [6/1]	v_8 [2/0]	v_9 [0/1]	$\dots$ $\dots$
v_0	[1/0]	0	2	2	2	2	2	2	2	0	2	
v_1	[0/1]	2	0	2	2	2	2	2	2	2	0	$\dots$
v_2	[1/1]	2	2	0	2	2	2	2	2	2	2	$\dots$
v_3	[2/1]	2	2	2	0	2	2	2	2	2	2	$\dots$
v_4	[3/1]	2	2	2	2	0	2	2	2	2	2	$\dots$
v_5	[4/1]	2	2	2	2	2	0	2	2	2	2	$\dots$
v_6	[5/1]	2	2	2	2	2	2	0	2	2	2	$\dots$
v_7	[6/1]	2	2	2	2	2	2	2	0	2	2	$\dots$
v_8	[2/0]	0	2	2	2	2	2	2	2	0	2	$\dots$
v_9	[0/3]	2	0	2	2	2	2	2	2	2	0	$\dots$
v_{10}	[4/3]	2	2	0	2	2	2	2	2	2	2	$\dots$
v_{11}	[1/3]	2	2	2	0	2	2	2	2	2	2	$\dots$
v_{12}	[5/3]	2	2	2	2	0	2	2	2	2	2	$\dots$
v_{13}	[2/3]	2	2	2	2	2	0	2	2	2	2	$\dots$
v_{14}	[6/3]	2	2	2	2	2	2	0	2	2	2	$\dots$
v_{15}	[3/3]	2	2	2	2	2	2	2	0	2	2	$\dots$
v_{16}	[3/0]	0	2	2	2	2	2	2	2	0	2	$\dots$
v_{17}	[0/2]	2	0	2	2	2	2	2	2	2	0	$\dots$
v_{18}	[5/2]	2	2	0	2	2	2	2	2	2	2	$\dots$
v_{19}	[3/2]	2	2	2	0	2	2	2	2	2	2	$\dots$
v_{20}	[1/2]	2	2	2	2	0	2	2	2	2	2	$\dots$
v_{21}	[6/2]	2	2	2	2	2	0	2	2	2	2	$\dots$
v_{22}	[4/2]	2	2	2	2	2	2	0	2	2	2	$\dots$
v_{23}	[2/2]	2	2	2	2	2	2	2	0	2	2	$\dots$

Table 4.1: Part of the matrix $A(7)^2 - 7I$, showing the indexing of its rows and columns.

Lemma 4.4.2. *Let Δ and a be integers such that $\Delta + aq \not\equiv 0 \,(\mathrm{mod}\ pq)$. Then we have*

$$\gcd(\Delta, q) = \gcd(\Delta + aq, pq).$$

Proof. Let $r = \gcd(\Delta, q)$ and $s = \gcd(\Delta + aq, pq)$. It is straightforward to see that $r \leqslant s$. Then, as $\Delta + aq \not\equiv 0 \,(\mathrm{mod}\ pq)$, $s = p^l$, where $0 \leq l \leqslant k-1$. Hence s divides q, and so, since s divides $\Delta + aq$, s divides Δ. Consequently $s \leqslant r$. Therefore $s = r$, as required. $\square$

Lemma 4.4.3. *If $p^k > 4$, and the number of vertices of $\mathcal{M}_3(p^{k-1})$ is N, then*

$$\{(-\sqrt{p^k})^{(m)}, (\sqrt{p^k})^{(m)}\} \subset \mathrm{sp}(p^k), \quad \textit{where } m = \frac{1}{2}p(p-1)N.$$

		w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\ldots$
		[1/0]	[0/1]	[1/1]	[2/1]	[3/1]	[4/1]	[2/0]	[0/3]	[2/3]	[4/3]	$\ldots$
w_0	[1/0]	25	2	2	2	2	2	5	2	2	2	$\ldots$
w_1	[0/1]	2	25	2	2	2	2	2	5	2	2	$\ldots$
w_2	[1/1]	2	2	25	2	2	2	2	2	5	2	$\ldots$
w_3	[2/1]	2	2	2	25	2	2	2	2	2	5	$\ldots$
w_4	[3/1]	2	2	2	2	25	2	2	2	2	2	$\ldots$
w_5	[4/1]	2	2	2	2	2	25	2	2	2	2	$\ldots$
w_6	[2/0]	5	2	2	2	2	2	25	2	2	2	$\ldots$
w_7	[0/3]	2	5	2	2	2	2	2	25	2	2	$\ldots$
w_8	[2/3]	2	2	5	2	2	2	2	2	25	2	$\ldots$
w_9	[4/3]	2	2	2	5	2	2	2	2	2	25	$\ldots$
w_{10}	[1/3]	2	2	2	2	5	2	2	2	2	2	$\ldots$
w_{11}	[3/3]	2	2	2	2	2	5	2	2	2	2	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
w_{60}	[6/0]	0	2	2	2	2	2	5	2	2	2	$\ldots$
w_{61}	[5/1]	2	0	2	2	2	2	2	5	2	2	$\ldots$
w_{62}	[6/1]	2	2	0	2	2	2	2	2	5	2	$\ldots$
w_{63}	[7/1]	2	2	2	0	2	2	2	2	2	5	$\ldots$
w_{64}	[8/1]	2	2	2	2	0	2	2	2	2	2	$\ldots$
w_{65}	[9/1]	2	2	2	2	2	0	2	2	2	2	$\ldots$
w_{66}	[7/0]	5	2	2	2	2	2	0	2	2	2	$\ldots$
w_{67}	[5/3]	2	5	2	2	2	2	2	0	2	2	$\ldots$
w_{68}	[7/3]	2	2	5	2	2	2	2	2	0	2	$\ldots$
w_{69}	[9/3]	2	2	2	5	2	2	2	2	2	0	$\ldots$
w_{70}	[6/3]	2	2	2	2	5	2	2	2	2	2	$\ldots$
w_{71}	[8/3]	2	2	2	2	2	5	2	2	2	2	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 4.2: Part of the matrix $A(25)^2$, showing the indexing of its rows and columns.

Proof. From Lemma 3.3.1, there is a regular covering of $\mathcal{M}_3(q)$ by $\mathcal{M}_3(pq)$. Let $v_i = [a/c]_q$ and $v_j = [b/d]_q$ be vertices of $\mathcal{M}_3(q)$. Let w_f and w_g be vertices of $\mathcal{M}(pq)$ such that w_f is in the fibre of v_i, so that $w_f = [a + sp/c + tp]_{pq}$ for some $s, t \in \mathbb{Z}$, and w_g is in the fibre of v_j.

We compare the entries of $A(pq)^2$ and $A(q)^2$. We can check that $\beta_{pq}(fg) = \beta_q(ij) + \rho q$, and $\Delta_{pq}(fg) = \Delta_q(ij) + \tau q$ for some $\rho, \tau \in \mathbb{Z}$. If $\Delta_{pq}(fg) \equiv 0 \pmod{pq}$, $\tau = 0$, and $\Delta_{pq}(fg) = \Delta_q(ij)$. So $\gcd(\Delta_{pq}(fg), pq) = \gcd(\Delta_q(ij), pq) = q = r$.

Now assume that $\Delta_{pq}(fg) \not\equiv 0 \pmod{pq}$. We order the rows and columns of the $p^2N \times p^2N$ matrix $A(pq)^2$ so that $w_f = [a + sp/c + tp]_{pq}$ is in position $f = sqN + tN + i$ if $v_i = [a/c]_q$ is in position i in $\mathcal{M}_3(q)$. This ensures that, for any pair of integers $s, t = 0, \ldots, p-1$, the set of vertices $w_l : l = sqN + tN + i, i = 0, \ldots, N-1$ contains exactly one vertex in each of the fibres of the vertices v_i of $\mathcal{M}_3(q)$. Then, apart from the entries corresponding to $\Delta_{pq} \equiv 0 \pmod{pq}$, the matrix $A(pq)^2$ consists of $p^2 \times p^2$ copies of $A(q)^2$.

If $\Delta_{pq} \equiv 0 \pmod{pq}$, the fg entry of $A(pq)^2 - pqI$ is 0. The first row of $A(pq)^2 - pqI$ consists of p^2 copies of the first row of $A(q)^2$, apart from entries equal to 0 corresponding to the vertices $[1 + lq/0]_{pq}$, which are in positions lpN for $l = 0, \ldots, p-1$. We can check that an automorphism takes $[1/0]_{pq}$ to w_f, and $[1 + lq/0]_{pq}$ to a vertex in a position a multiple of pN from w_f.

Define the $N \times N$ matrices $B = A^2(q)$, $T = A^2(q) - qI$, and let D be the $pN \times pN$ matrix consisting of p blocks of rows each comprising $p-1$ copies of B, with one copy of T in the diagonal position, that is

$$D = \begin{pmatrix} T & B & \cdots & B & B \\ B & T & \cdots & B & B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B & B & \cdots & T & B \\ B & B & \cdots & B & T \end{pmatrix}.$$

Then

$$A^2(pq) - pqI = \begin{pmatrix} D & D & \cdots & D & D \\ D & D & \cdots & D & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D & D & \cdots & D & D \\ D & D & \cdots & D & D \end{pmatrix}.$$

The rank of D is less than or equal to pN. So, from Lemma 4.2.6, $\sqrt{pq}$ or $-\sqrt{pq}$ or both are eigenvalues of $\mathcal{M}(pq)$ with total algebraic multiplicity greater than or equal to $p(p-1)N$. $\square$

We can now prove Theorem 4.1.1(ii), which states that if $p^k > 4$, then

$$\mathrm{sp}_3(p^k) = p\,\mathrm{sp}_3(p^{k-1}) \cup \{-\sqrt{p^k}^{(\frac{1}{2}pc)}, 0^{(c)}, \sqrt{p^k}^{(\frac{1}{2}pc)}\}$$

where $c = (p-1)V(\mathcal{M}_3(p^{k-1})) = N$.

Proof of Theorem 4.1.1(ii). We put $p^{k-1} = q$. The number of vertices of $\mathcal{M}(pq)$, and so the number of its eigenvalues, is p^2N. Either $\sqrt{pq}$ or $-\sqrt{pq}$, or both, are eigenvalues of $\mathcal{M}_3(pq)$ with total algebraic multiplicity greater than or equal to $p(p-1)N$, and 0 is an eigenvalue of $\mathcal{M}_3(pq)$ with algebraic multiplicity greater than or equal to $(p-1)N$. Also, from Lemma 4.2.4, $p\,\mathrm{sp}_3(q) \subset \mathrm{sp}_3(pq)$. The sum of the lower bound of the multiplicities of all the eigenvalues of $\mathcal{M}_3(pq)$ we have found is $p(p-1)N + (p-1)N + N = Np^2$, which is the total number of eigenvalues of $\mathcal{M}_3(pq)$, so we take the lower bound for all multiplicities and there are no more eigenvalues. Since the entries on the main diagonal of $A(pq)$ are all zero, its trace is zero, so the sum of its eigenvalues is 0. Therefore, as the eigenvalues of $\mathcal{M}_3(q)$ also sum to 0, $\sqrt{pq}$ and $-\sqrt{pq}$ have the same multiplicity as eigenvalues of $\mathcal{M}(pq)$. $\square$

Example 4.4.4. We find $\mathrm{sp}_3(27)$, the spectrum of $\mathcal{M}_3(27)$. We first find the spectrum of $\mathcal{M}_3(9)$, using Theorem 4.1.1(ii), putting $p = 3$ and $k = 2$. Then the number of vertices of $\mathcal{M}_3(3)$ is 4, so $c = 2 \times 4 = 8$ and $\frac{1}{2}pc = 12$. We have $\mathrm{sp}_3(3) = \{-1^{(3)}, 3\}$, so

$$\begin{aligned} 3\{-1^{(3)}, 3\} = \{-3^{(3)}, 9\} &\subset \mathrm{sp}_3(9). \\ \{-3^{(12)}, 0^{(8)}, 3^{(12)}\} &\subset \mathrm{sp}_3(9), \end{aligned}$$

and therefore

$$\mathrm{sp}_3(9) = \{-3^{(15)}, 0^{(8)}, 3^{(12)}, 9\}.$$

Next we obtain $\mathrm{sp}_3(27)$ by again using Theorem 4.1.1(ii), putting $p = 2$ and $k = 3$. As the number of vertices of $\mathcal{M}_3(9)$ is 36, $c = 2 \times 36 = 72$, and $\frac{1}{2}pc = 108$. Then

$$\begin{aligned} 3\{-3^{(15)}, 0^{(8)}, 3^{(12)}, 9\} &\subset \mathrm{sp}_3(27), \\ \{-3\sqrt{3}^{(108)}, 0^{(72)}, 3\sqrt{3}^{(108)}\} &\subset \mathrm{sp}_3(27), \end{aligned}$$

and therefore

$$\mathrm{sp}_3(27) = \{-9^{(15)}, -3\sqrt{3}^{(108)}, 0^{(80)}, 3\sqrt{3}^{(108)}, 9^{(12)}, 27\}.$$

4.5 Farey maps of composite level

To find the spectrum of $\mathcal{M}_3(n)$ for a composite n we need the parallel product of maps introduced in Section 3.4. From [66, Theorem 2], if the vertex valencies of two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are coprime, the underlying graph of their parallel product $\mathcal{M}$ is the tensor product $\mathcal{G}_1 \times \mathcal{G}_2$ of their underlying graphs. Then, from [25, Theorem 4.2.12], $\mathrm{sp}(\mathcal{M}) = \mathrm{sp}(\mathcal{M}_1)\mathrm{sp}(\mathcal{M}_2)$.

Lemma 4.5.1. *If l and m are coprime integers, neither of which is twice an odd integer, then $\sqrt{lm}$ or $-\sqrt{lm}$ or both are eigenvalues of $\mathcal{M}_3(lm)$ with total algebraic multiplicity greater than or equal to half the number of its vertices.*

Proof. From Theorem 1.1.2, $\mathcal{M}_3(lm)$ is a double covering of the parallel product of $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$. Let $u \in \mathbb{Z}, 1 < u < lm$ be such that $u \equiv 1 \pmod{l}$ and $u \equiv -1 \pmod{m}$. The covering transformation takes both the vertices $w_f = [a/c]_{lm}$ and $w_{f'} = [ua/uc]_{lm}$ of $\mathcal{M}_3(lm)$ to the vertex $v_i = ([a/c]_l, [a/c]_m)$ of the parallel product of $\mathcal{M}_3(l)$ and $\mathcal{M}_3(m)$. Let the number of vertices of $\mathcal{M}_3(lm)$ be $2V$. Then the parallel product has V vertices. We order the vertices of $\mathcal{M}_3(lm)$ so that $f' = f + V$.

Let $w_g = [b/d]_{lm}$ be a vertex of $\mathcal{M}_3(lm)$, with $w_g \neq w_f$ and $w_g \neq w'_f$. We compare the fg and $f'g$ entries of the matrix $(A(lm))^2 - lmI$. We have $\Delta(fg) = ad - bc$ and $\Delta(f'g) = uad - buc$. Then $\gcd(\Delta(fg), lm) = \gcd(\Delta(f'g), lm) = r$. We note that $u\lambda(fg)ua + u\mu(fg)uc + \nu lm = 1$, so we put $\lambda(f'g) = u\lambda(fg)$ and $\mu(f'g) = u\mu(fg)$, and obtain $\beta(f'g) = \beta(fg)$. So the $(fg)^{\text{th}}$ and $(f+V, g)^{\text{th}}$ entries of $A(lm)^2 - lmI$ are equal.

Therefore we can write $A^2(lm) - lmI$ as 2×2 blocks of an $V \times V$ matrix and apply Lemma 4.2.6. It follows that $\pm\sqrt{lm}$ are eigenvalues of $(A(lm))^2$ with total algebraic multiplicity greater than or equal to V. □

It is now straightforward to prove Theorem 4.1.2, which states that, if m is a positive integer,

(i) if m is odd, then $\mathrm{sp}_3(2m) = \mathrm{sp}_3(2)\mathrm{sp}_3(m)$.

(ii) if l and m are coprime integers, and neither l nor m is twice an odd integer, then

$$\mathrm{sp}_3(lm) = \mathrm{sp}_3(l)\mathrm{sp}_3(m) \cup \{-\sqrt{lm}^{(N/4)}, \sqrt{lm}^{(N/4)}\}, \text{ where } N = |V(\mathcal{M}_3(lm))|.$$

Proof of Theorem 4.1.2. Both parts of this theorem now follow from Theorem 1.1.2. For part (ii), we also need Lemma 4.5.1, and we note that, as the spectrum of the parallel product is a subset of the spectrum of $\mathcal{M}_3(lm)$, and it has V eigenvalues counting algebraic multiplicities, the total algebraic multiplicity of $\pm\sqrt{lm}$ as eigenvalues of $(A(lm))^2$ cannot exceed V as the total number of eigenvalues counting multiplicities is $2V$. Also, as the trace of the adjacency matrix of the parallel product is zero, its eigenvalues sum to zero; therefore the eigenvalues $\sqrt{lm}$ and $-\sqrt{lm}$ have the same multiplicity since the eigenvalues of $\mathcal{M}_3(lm)$ also sum to zero. $\square$

Example 4.5.2. We find $\mathrm{sp}_3(10)$, the spectrum of $\mathcal{M}_3(10)$.

We have

$$\mathrm{sp}_3(2) = \{-1^{(2)}, 2\}, \quad \text{and} \quad \mathrm{sp}_3(5)\{-\sqrt{5}^{(3)}, -1^{(5)}, \sqrt{5}^{(3)}, 5\},$$

so by the first part of Theorem 4.1.2

$$\begin{aligned} \mathrm{sp}_3(10) &= \{-1^{(2)}, 2\}\{-\sqrt{5}^{(3)}, -1^{(5)}, \sqrt{5}^{(3)}, 5\} \\ &= \{(-1^{(2)})(-\sqrt{5}^{(3)}), (-1^{(2)})(-1^{(5)}), (-1^{(2)})(\sqrt{5}^{(3)}), 5(-1^{(2)}), -2\sqrt{5}^{(3)}, -2^{(5)}, 2\sqrt{5}^{(3)}, 10\} \\ &= \{-5^{(2)}, -2\sqrt{5}^{(3)}, -\sqrt{5}^{(6)}, -2^{(5)}, 1^{(10)}, \sqrt{5}^{(6)}, 2\sqrt{5}^{(3)}, 10\}. \end{aligned}$$

Example 4.5.3. We find $\mathrm{sp}_3(20)$, the spectrum of $\mathcal{M}_3(20)$. We have

$$\mathrm{sp}_3(4) = \{-2^{(2)}, 0^{(3)}, 4\}, \quad \text{and} \quad \mathrm{sp}_3(5) = \{-\sqrt{5}^{(3)}, -1^{(5)}, \sqrt{5}^{(3)}, 5\}.$$

Since 20 is not twice a odd number, we use the second part of Theorem 4.1.2, so

$$\begin{aligned} \mathrm{sp}_3(20) &= \left(\{-2^{(2)}, 0^{(3)}, 4\}\{-\sqrt{5}^{(3)}, -1^{(5)}, \sqrt{5}^{(3)}, 5\}\right) \cup \{-2\sqrt{5}^{(36)}, 2\sqrt{5}^{(36)}\} \\ &= \{-10^{(2)}, -4\sqrt{5}^{(3)}, -2\sqrt{5}^{(6)}, -4^{(5)}, 0^{(36)}, 2^{(10)}, 2\sqrt{5}^{(6)}, 4\sqrt{5}^{(3)}, 20\} \cup \{-2\sqrt{5}^{(36)}, 2\sqrt{5}^{(36)}\} \\ &= \{-10^{(2)}, -4\sqrt{5}^{(3)}, -2\sqrt{5}^{(42)}, -4^{(5)}, 0^{(36)}, 2^{(10)}, 2\sqrt{5}^{(42)}, 4\sqrt{5}^{(3)}, 20\}. \end{aligned}$$

The spectra of all the Farey maps $\mathcal{M}_3(n)$ for $n = 1, \ldots, 36$, and for $n = 49,\ 64,\ 81$ and 125 are given in Appendix A.

4.6 Hecke maps

Theorem 4.6.1. *The spectrum of $\mathcal{M}_4(n)$ for odd n, and that of $\mathcal{M}_6(n)$ for $3 \nmid n$, is the multiset $-\mathrm{sp}_3(n) \cup \mathrm{sp}_3(n)$, where $\mathrm{sp}_3(n)$ is the spectrum of $\mathcal{M}_3(n)$.*

Proof. From Theorem 1.1.3 and Lemma 4.2.4, if λ is an eigenvalue of $\mathcal{M}_3(n)$, then it is also an eigenvalue of $\mathcal{M}_4(n)$. In $\mathcal{M}_4(n)$ the vertices adjacent to an even vertex are all odd, and the vertices adjacent to an odd vertex are all even, so that if we partition the vertices into the set of even vertices and the set of odd vertices every edge of the map has one vertex in each set, and therefore $\mathcal{M}_4(n)$ is *bipartite*. Then, by Theorem 8.8.2 of [22], if λ is an eigenvalue of a bipartite graph, $-\lambda_i$ also an eigenvalue with the same algebraic multiplicity. Since $\mathcal{M}_4(n)$ has twice as many vertices as $\mathcal{M}_3(n)$ it has twice as many eigenvalues, so there are no more eigenvalues and the result follows. Replacing 4 by 6, we have the corresponding result for $\mathcal{M}_6(n)$ if 3 is not a factor of n. □

4.7 Ramanujan graphs

A Ramanujan graph is a regular graph for which $\lambda < 2\sqrt{n-1}$, where n is the degree of the graph and λ is the graph eigenvalue with the second largest modulus. (See for instance [40, Definition 3.7] or [46, Definition 1.1].) Ramanujan graphs are the subject of much research as they can be used to make particularly good communication networks, as we will explain in Section 6.5.

Lemma 4.7.1. *Suppose that $n > 4$, and let p_1 be the smallest prime divisor of n. Then $\lambda(n)$, the eigenvalue of $\mathcal{M}_3(n)$ with the second largest modulus, is given by*

$$\lambda(n) = \begin{cases} \frac{1}{2}n & \textit{if} \quad p_1 = 2, \\ \frac{1}{3}n & \textit{if} \quad p_1 = 3, \\ \frac{1}{\sqrt{p_1}}n & \textit{otherwise.} \end{cases}$$

Proof. In the spectrum of $\mathcal{M}_3(p^k)$, the largest eigenvalue is p^k, and, from Theorem 4.1.1, the eigenvalue with the second largest modulus is $p^k/\sqrt{p_1}$ for $p_1 > 3$, $p^k/2$ for $p_1 = 2$, and $p^k/3$

for $p_1 = 3$. We give the full proof of this lemma for the case $p_1 > 3$; it is straightforward to prove the corresponding results for $p_1 = 2$ and $p_1 = 3$ in the same way.

We proceed by induction on the number r of prime divisors of n. If $r = 1$, then $n = p_1^k$ and $\lambda(n) = n/\sqrt{p_1}$. Now assume that the lemma is true for any integer greater than 4 with $r - 1$ prime divisors. Let n be any integer with r prime divisors the smallest of which is p_1. Write $n = p_1^k m$ where m is coprime to p_1. We have $m > 4$, as otherwise $p_1 = 2$ or $p_1 = 3$, so that m is an integer greater that 4 with $r-1$ prime divisors the smallest of which is $p_2 > p_1$. Then, by the inductive hypothesis, the eigenvalue with the second largest modulus of $\mathcal{M}_3(n)$ is the largest of $p_1^k \dfrac{m}{\sqrt{p_2}}$ and $\dfrac{p_1^k}{\sqrt{p_1}} m$, which is equal to $\dfrac{n}{\sqrt{p_1}}$. □

From this lemma we recover a result proved in a very different way in [16, Theorem 1], [23, Theorem 4.2], and [50, Theorem 1.12,(i)].

Corollary 4.7.2. *The underlying graph of the Farey map $\mathcal{M}_3(n)$ is a Ramanujan graph if and only if n is prime or equal to one of* $4, 6, 8, 9, 10, 12, 14, 15, 21, 27$ *or* 33*.*

Proof. The graph is a Ramanujan graph if and only if $\lambda < 2\sqrt{n-1}$.

Suppose first that $p_1 = 2$. Then the graph is a Ramanujan graph if and only if $n/2 < 2\sqrt{n-1}$. For $n \in \mathbb{N}$, this is equivalent to $n^2 < 16(n-1)$, or $n < 15$. So, if $n = 2, 4, 6, 8, 10, 12$ or 14, the graph is a Ramanujan graph.

Suppose next that $p_1 = 3$. Then the graph is a Ramanujan graph if and only if $n/3 < 2\sqrt{n-1}$. For $n \in \mathbb{N}$, this is equivalent to $n^2 < 36(n-1)$, or $n < 35$. So, if $n = 3, 9, 15, 21, 27$ or 33, the graph is a Ramanujan graph.

Then if $p_1 > 3$, the condition $\lambda < 2\sqrt{n-1}$ is equivalent to $n/\sqrt{p_1} < 2\sqrt{n-1}$, or

$$\frac{n^2}{n-1} = n + 1 + \frac{1}{n-1} < 4p_1, \quad \text{or} \quad n < 4p_1 - 2.$$

Let $n = mp_1$. Then the graph is a Ramanujan graph if and only if $mp_1 < 4p_1 - 2$. Now neither 2 nor 3 can be divisors of m so the inequality is true if and only if $m = 1$, that is if n is prime.

□

Chapter 5

The convergence of integer continued fractions

5.1 Introduction and main results

This chapter describes joint work carried out with Ian Short. It is the subject of [57], which has been published in the Proceedings of the American Mathematical Society. Building on results concerning the convergence of paths on the Farey tessellation in [58], we have found a necessary and sufficient condition for the convergence of integer continued fractions. It is well known that every infinite continued fraction $(b_0, b_1, \dots)$ with positive integer coefficients converges (to an irrational number). However, if we stipulate that the coefficients are integers, but not necessarily positive integers, then the continued fraction need not converge; for example, the periodic continued fraction $(1, -1, 1, -1, \dots)$ diverges as its sequence of convergents oscillates between the three values 1, 0 and ∞.

We will prove Theorem 1.1.4 given in the introduction, which we restate here. We express our results in terms of the negative continued fractions defined by equation (2.8.5) in Section 2.8, and in this chapter we use the phrase 'continued fraction' to refer to a negative continued fraction with integer coefficients. We will define a function Φ on a negative continued fraction $[b_0, b_1, \cdots]$. Then $p^{(n)}$, the *key position*, is the position of the first coefficient equal to 0,1, or -1 after n iterations of Φ, and $q^{(n)}$ is the modulus of the coefficient in the position preceding $\liminf p(n)$ if that limit is finite. Then our main result is:

Theorem 1.1.4 *Let $[b_0, b_1, \dots]$ be a continued fraction.*

1. *Suppose that $p^{(n)} \to \infty$.*
   - *(a) If $[b_0^*, b_1^*, \dots]$ has the same tail as $[2, 2, \dots]$ or $[-2, -2, \dots]$, then $[b_0, b_1, \dots]$ converges to a rational.*
   - *(b) Otherwise, $[b_0, b_1, \dots]$ converges to an irrational.*

2. *Suppose that $p^{(n)} \not\to \infty$.*
   - *(a) If $q^{(n)} \to \infty$, then $[b_0, b_1, \dots]$ converges to an extended rational.*
   - *(b) If $q^{(n)} \not\to \infty$, then $[b_0, b_1, \dots]$ diverges.*

5.2 The algorithm for continued fraction convergence

We first define the notation used in the statement of Theorem 1.1.4.

Let $\mathscr{C}$ denote the collection of all continued fractions $[b_0, b_1, \dots]$. We define a function $\Phi \colon \mathscr{C} \longrightarrow \mathscr{C}$ as follows. If $b_n \neq 0, 1, -1$ for every *positive* integer n (ignore b_0), then Φ fixes $[b_0, b_1, \dots]$. Otherwise, let m be the least positive integer for which b_m is 0, 1 or -1. Then

$$\Phi([b_0, b_1, \dots]) = \begin{cases} [b_0, b_1, \dots, b_{m-2}, b_{m-1} + b_{m+1}, b_{m+2}, \dots], & \text{if } b_m = 0, \\ [b_0, b_1, \dots, b_{m-2}, b_{m-1} - 1, b_{m+1} - 1, b_{m+2}, \dots], & \text{if } b_m = 1, \\ [b_0, b_1, \dots, b_{m-2}, b_{m-1} + 1, b_{m+1} + 1, b_{m+2}, \dots], & \text{if } b_m = -1. \end{cases}$$

In each case, Φ removes the coefficient b_m, and adjusts the two coefficients on either side, merging them when $b_m = 0$. The operations induced by Φ are familiar in the theory of continued fractions, where they are sometimes referred to as 'singularization' operations (see, for example

[28, Section 4.2]). We later give a geometric description of the function Φ, which will help to explain its definition.

Given $[b_0, b_1, \dots]$ in $\mathscr{C}$, we define $p^{(n)}$ to be the first position (not zero) at which a 0, 1 or -1 appears in the continued fraction $\Phi^n([b_0, b_1, \dots])$, and we define $p^{(n)}$ to be ∞ if there is no such position. Let $p = \liminf p^{(n)}$. Suppose for the moment that $p = \infty$. In this case, for each non-negative integer k, the sequence of integers obtained by taking the k^{th} coefficient of $\Phi^n([b_0, b_1, \dots])$, for $n = 1, 2, \dots$, eventually fixes on some value b_k^*. Thus we obtain a limit continued fraction $[b_0^*, b_1^*, \dots]$, where $|b_n^*| \geqslant 2$, for $n = 1, 2, \dots$. Suppose now that $p < \infty$. In this case, we define $q^{(n)}$ to be the modulus of the coefficient of $\Phi^n([b_0, b_1, \dots])$ in positon $p-1$, for $n = 1, 2, \dots$, to give a sequence of nonnegative integers $(q^{(n)})$.

We highlight a corollary of part (i) of Theorem 1.1.4 for continued fractions with no coefficients (other than perhaps b_0) equal to 0, 1 or -1, which generalises the known result of the same type that has $b_n \geqslant 2$, for $n = 1, 2, \dots$ (see, for example, [35, Section 1]).

Theorem 5.2.1. *Suppose that $|b_n| \geqslant 2$, for $n = 1, 2, \dots$. Then the continued fraction $[b_0, b_1, \dots]$ converges to a rational if it has the same tail as $[2, 2, \dots]$ or $[-2, -2, \dots]$, and otherwise it converges to an irrational.*

The following examples illustrate how Theorem 1.1.4 can be applied.

Example 5.2.2. Consider the continued fraction

$$[b_0, b_1, \dots] = [3, 1, 3, 4, 1, 2, 3, 5, 1, 2, 2, 3, 6, 1, 2, 2, 2, 3, \dots].$$

Some numbers are shaded to indicate how the pattern of coefficients continues. If these were the coefficients of a regular continued fraction, then that continued fraction would converge, because all the coefficients are positive. However,$[b_0, b_1, \dots]$ is a negative continued fraction, and to determine whether it converges we can use Theorem 1.1.4. To do this, we apply Φ repeatedly to give the sequence $\Phi^n([b_0, b_1, \dots])$, for $n = 1, 2, \dots$, which is

$$[2, 2, 4, 1, 2, 3, 5, 1, \dots] \to [2, 2, 3, 1, 3, 5, 1, \dots] \to [2, 2, 2, 2, 5, 1, \dots] \to \cdots.$$

Continuing in this way we see that $p^{(n)} \to \infty$, and the limit continued fraction is

$$[b_0^*, b_1^*, \dots] = [2, 2, \dots].$$

Hence $[b_0, b_1, \dots]$ converges to a rational number (namely 1, the value of $[2, 2, \dots]$).

Example 5.2.3. Consider the continued fraction

$$[b_0, b_1, \dots] = [1, 2, 1, 3, 1, 4, 1, 5, 1, \dots].$$

Applying Φ repeatedly, we obtain

$$\begin{aligned}&[1, 1, 2, 1, 4, 1, 5, 1, 6, 1, 7, \dots] \to [0, 1, 1, 4, 1, 5, 1, 6, 1, 7, \dots] \to [-1, 0, 4, 1, 5, 1, 6, 1, 7, \dots]\\ &\to [3, 1, 5, 1, 6, 1, 7, \dots] \to [2, 4, 1, 6, 1, 7, \dots] \to [2, 3, 5, 1, 7, \dots] \to \cdots .\end{aligned}$$

Again we see that $p^{(n)} \to \infty$, and this time the limit continued fraction is

$$[b_0^*, b_1^*, \dots] = [2, 3, 4, 5, \dots].$$

Hence $[b_0, b_1, \dots]$ converges to an irrational number.

Example 5.2.4. Consider the continued fraction

$$[b_0, b_1, \dots] = [1, 0, 2, 0, 3, 0, 4, 0, \dots].$$

Applying Φ repeatedly gives

$$[3, 0, 3, 0, 4, 0, 5, 0 \dots] \to [6, 0, 4, 0, 5, 0 \dots] \to [10, 0, 5, 0 \dots] \to \cdots .$$

By continuing with this process it becomes clear that, for each positive integer n, the coefficient of $\Phi^n([b_0, b_1, \dots])$ in position 1 is 0, and the coefficient in position 0 is $\frac{1}{2}(n+1)(n+2)$. Hence $p = \liminf p^{(n)} = 1$ and $q^{(n)} \to \infty$. Therefore $[b_0, b_1, \dots]$ converges to an extended rational – to infinity, in fact (this could easily be ascertained by other means).

Example 5.2.5. Consider the continued fraction

$$[b_0, b_1, \dots] = [3, 0, -3, 3, 3, 0, -3, -3, 3, 3, 3, 0, -3, -3, -3, \dots].$$

Once more, applying Φ repeatedly we obtain

$$\begin{aligned}&[0, 3, 3, 0, -3, -3, 3, 3, 3, 0, -3, -3, -3, \dots] \to [0, 3, 0, -3, 3, 3, 3, 0, -3, -3, -3, \dots]\\ &\to [0, 0, 3, 3, 3, 0, -3, -3, -3, \dots] \to [3, 3, 3, 0, -3, -3, -3, \dots] \to \cdots .\end{aligned}$$

In this case the sequence of coefficients of $\Phi^n([b_0, b_1, \dots])$ in position 0, for $n = 1, 2, \dots,$ takes the value 0 infinitely often, as does the sequence of coefficients in position 1. Hence
$p = \liminf p^{(n)} = 1$ and $q^{(n)} \not\to \infty$, so $[b_0, b_1, \dots]$ diverges.

To obtain these results, as we explain in the next section, we use the model of continued fractions as paths on the Farey tessellation $\mathscr{F}$. We also use the geometry of $\mathscr{F}$ to establish the following theorem about convergence of paths in $\mathscr{F}$. It is a generalisation of [58, Theorem 1.4], which states (in a more general context) that if an infinite path in $\mathscr{F}$ does not return to any vertex infinitely often, then it converges.

Theorem 5.2.6. *An infinite path in the Farey tessellation converges if and only if it does not return to any two distinct vertices infinitely often.*

In the language of continued fractions, this theorem says that a continued fraction $[b_0, b_1, \dots]$ converges if and only if there are not two distinct extended rationals that each appear infinitely many times in the sequence of convergents of $[b_0, b_1, \dots]$.

5.3 Continued fractions as paths on the Farey graph

Given two vertices u and v of $\mathscr{F}$, we write $u \sim v$ if u and v are adjacent. An *infinite path* in $\mathscr{F}$ is a sequence $v_0, v_1, \dots$ of vertices of $\mathcal{F}$ such that $v_i \sim v_{i+1}$, for $i = 0, 1, \dots$. We denote this path by $\langle v_0, v_1, \dots \rangle$, and represent it by directed edges along the hyperbolic geodesics joining v_i to v_{i+1} for $i = 0, 1, \dots$. Occasionally we consider finite paths in $\mathscr{F}$, such as $\langle v_0, v_1, \dots, v_k \rangle$, for

$i = 0, 1, \ldots, k$. An infinite path $\langle v_0, v_1, \ldots \rangle$ is said to *converge* in $\mathscr{F}$ if the sequence $v_0, v_1, \ldots$ converges in $\mathbb{R}_\infty$. Note that we allow paths to pass through the same vertex more than once; in graph theory the term 'walk' would be used in place of 'path'.

In [3, Theorem 3.1] it is shown that an integer continued fraction can be represented by a path on the Farey tessellation. We adapt the proof, giving it in full for completeness, to show that this is also true if the continued fraction is represented in the negative form given by equation (2.8.5).

Theorem 5.3.1. *The extended rationals $v_0, v_1, \ldots$ are the consecutive convergents of a negative continued fraction if and only if $\langle \infty, v_0, v_1, \ldots \rangle$ is an infinite path on $\mathscr{F}$.*

Proof. For a continued fraction $[b_0, b_1, \ldots]$, we define the Mobius transformations $s_n(z) = b_n - 1/z$ and $S_n = s_0 \circ s_1 \circ \cdots \circ s_n$, for $n = 0, 1, \ldots$. Both s_n and S_n belong to the modular group Γ. Let us define $v_n = S_n(\infty)$, so that, as shown in Section 2.8, v_n is the the n^{th} convergent $[b_0, b_1, \ldots, b_n]$ of $[b_0, b_1, \ldots]$. Since 0 and ∞ are adjacent vertices of $\mathscr{F}$, and $S_n \in \Gamma$, it follows that $S_n(0) \sim S_n(\infty) = v_n$. But $S_n(0) = S_{n-1} \circ s_n(0) = S_{n-1}(\infty) = v_{n-1}$, so $v_{n-1} \sim v_n$, and $\langle \infty, v_0, v_1, \ldots \rangle$ is an infinite path in $\mathscr{F}$.

Conversely, given an infinite path $\langle \infty, v_0, v_1, \ldots \rangle$ in $\mathscr{F}$, we find a unique infinite continued fraction $[b_0, b_1, \ldots]$ whose sequence of convergents is $v_0, v_1, \ldots$. First put $b_0 = v_0$. Then assume that, for all $i \leq n$, there are integers b_i such that $S_i(\infty) = S_{i-1}(b_i) = v_i$. Put $b_{n+1} = S_n^{-1}(v_{n+1})$. Then, putting $s_{n+1}(z) = b_{n+1} - 1/z$, we have $v_{n+1} = S_n(b_{n+1}) = S_{n+1}(\infty)$. Now v_{n+1} is adjacent to v_n in the Farey tessellation, that is $S_n(b_{n+1})$ is adjacent to $S_n(\infty)$. As S_n^{-1} is in the modular group it preserves adjacency, so b_{n+1} is adjacent to ∞, and therefore it is an integer. It follows that the $(n+1)^{\text{th}}$ convergent in the continued fraction $[b_0, b_1, \ldots b_{n+1} \ldots]$ is v_{n+1}, and an induction argument shows that any path on the Farey tessellation is a sequence of convergents of a continued fraction. $\square$

Because of this correspondence between convergents and paths, we denote the sequence of convergents $v_0, v_1, \ldots$ of a continued fraction by $\langle v_0, v_1, \ldots \rangle$ (omitting ∞ as an initial vertex).

Example 5.3.2. Consider the continued fraction $[0, -3, -2, 0, 3, -2, \ldots]$. We calculate the convergents in the following table using equations (2.8.2) and (2.8.3), then show the corresponding path on the Farey tessellation.

n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	$1/3$
2	-2	2	5	$2/5$
3	0	1	3	$1/3$
4	3	1	4	$1/4$
5	-2	-3	-11	$3/11$

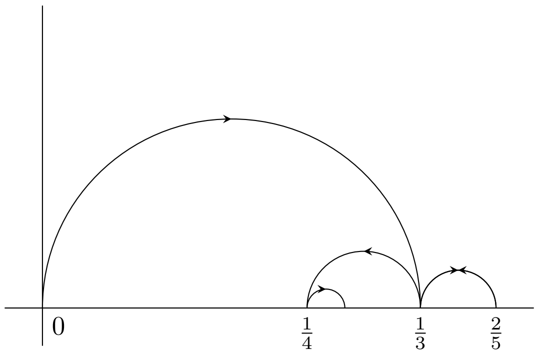

Figure 5.3.1: The path on the Farey tessellation from Example 5.3.2.

The connection between continued fractions and paths in the Farey tessellation has been explored before, in, for example, [3,58]. Using this perspective we can describe the function Φ in geometric terms, by looking at how it modifies the sequence of convergents. To see this, suppose first that $b_m = 0$. Then $s_m(z) = -1/z$, so

$$S_m(\infty) = S_{m-1} \circ s_m(\infty) = S_{m-1}(0),$$

and hence

$$v_m = S_m(\infty) = S_{m-1}(0) = S_{m-2}(\infty) = v_{m-2}.$$

In this case the path of convergents $\langle v_0, v_1, \dots \rangle$ travels from v_{m-2} to v_{m-1} and then back to $v_{m-2} = v_m$. The effect of applying Φ is to remove the vertices v_{m-1} and v_m from the path of convergents. This will be proved formally in Lemma 5.5.2.

We show how this applies to the continued fraction in example 5.3.2.

Example 5.3.3. We denote both the continued fraction and its path by γ.
Let $\gamma = [0, -3, -2, 0, 3, -2, \dots]$, so that $\Phi(\gamma) = [0, -3, 1, -2, \dots]$. As $b_3 = 0$, both v_2 and v_3, that is 2/5 and the second occurrence of 1/3, are removed from the path.

γ : n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	1/3
2	-2	2	5	2/5
3	0	1	3	1/3
4	3	1	4	1/4

$\Phi(\gamma)$: n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	1/3
2	1	-1	-4	1/4
3	-2	3	11	3/11

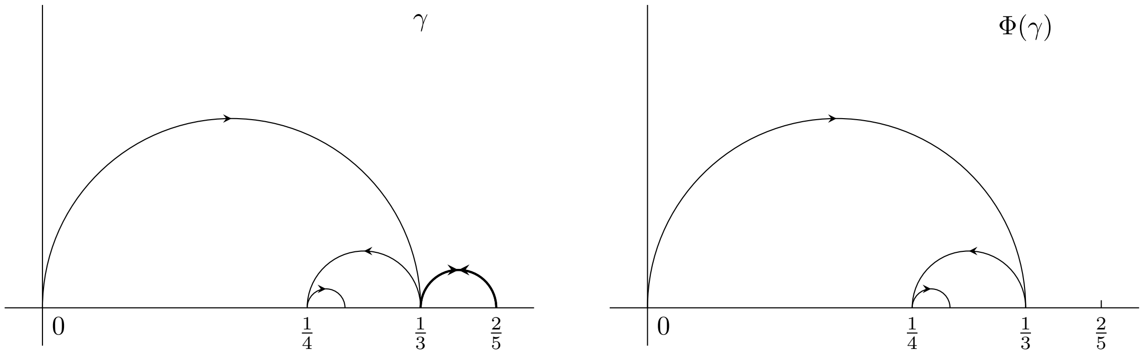

Figure 5.3.2: The paths γ and $\Phi(\gamma)$ from Example 5.3.3.

Now suppose that $b_m = 1$, in which case $s_m(1) = 0$, so $S_m(1) = S_{m-1} \circ s_m(1) = S_{m-1}(0)$. Then $v_m = S_m(\infty)$, $v_{m-1} = S_{m-1}(\infty) = S_m(0)$ and $v_{m-2} = S_{m-2}(\infty) = S_{m-1}(0) = S_m(1)$. Since 1, 0 and ∞ are the vertices of a triangle in $\mathscr{F}$, so are v_{m-2}, v_{m-1} and v_m. So v_{m-2} is adjacent to v_m in $\mathscr{F}$, and the effect of applying Φ is to remove the vertex v_{m-1} and proceed directly from v_{m-2} to v_m. The vertices v_k for $k < m - 1$ do not change, and for $k \geq m$, the vertex in position k moves to position $k - 1$.

For $b_m = -1$, we note that $s_m(-1) = 0$, so $S_m(1) = S_{m-1} \circ s_m(-1) = S_{m-1}(0)$. The proof is then identical to that for $b_m = 1$, as $\infty, 0$ and -1 also form a triangle in $\mathscr{F}$. The next two examples demonstrate the cases $b_m = 1$ and $b_m = -1$.

Example 5.3.4. Let $\gamma = [0, -3, 2, 1, -3, 2\ldots]$. So $\Phi(\gamma) = [0, -3, 1, -4, 2\ldots]$.

γ : n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	$1/3$
2	2	-2	-7	$2/7$
3	1	-1	-4	$1/4$
4	-3	5	19	$5/19$

$\Phi(\gamma)$: n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	$1/3$
2	1	-1	-4	$1/4$
3	-4	5	19	$5/19$
4	2	11	42	

The path goes straight from 1/3 to 1/4, removing convergent 2/7 as shown in Figure 5.3.

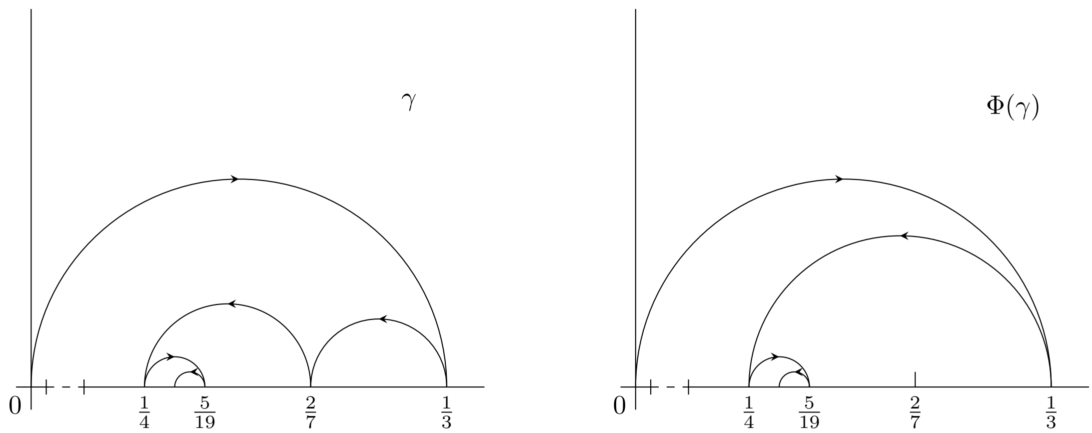

Figure 5.3.3: The paths γ and $\Phi(\gamma)$ from Example 5.3.3.

Example 5.3.5. Let $\gamma = [0, -3, 2, -1, -3, 2\ldots]$. So $\Phi(\gamma) = [0, -3, 3, -2, 2\ldots]$.

γ : n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	$1/3$
2	2	-2	-7	$2/7$
3	-1	3	10	$3/10$
4	-3	-7	-23	$7/23$

$\Phi(\gamma)$: n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-3	-1	-3	$1/3$
2	3	-3	-10	$3/10$
3	-2	7	23	$7/23$
4	2	17	56	

The path goes straight from 1/3 to 3/10, removing the convergent 2/7 as shown in Figure 5.3.4.

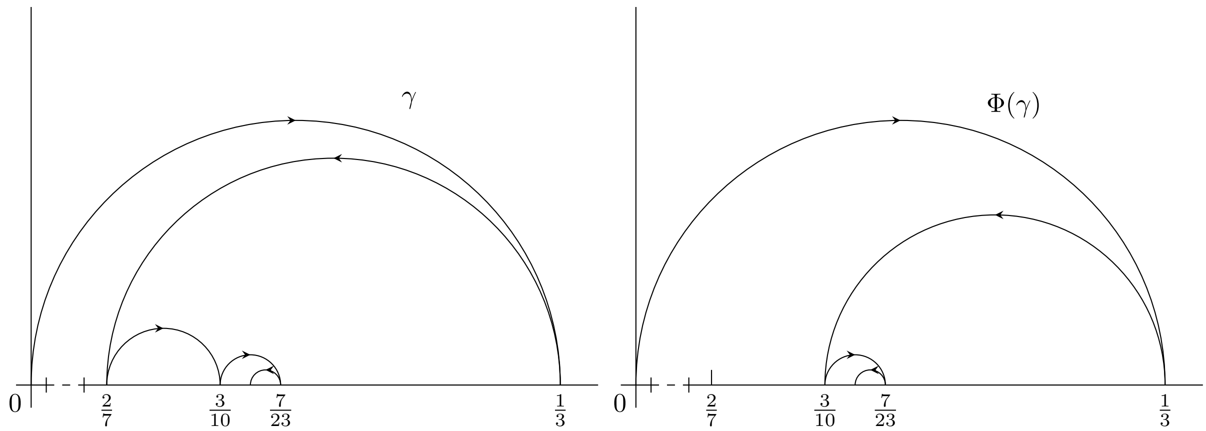

Figure 5.3.4: The paths γ and $\Phi(\gamma)$ from Example 5.3.5.

We also give, in Figure 5.3.5, a schematic representaion of the effect of Φ for $b_m = 0$ and $b_m = 1$.

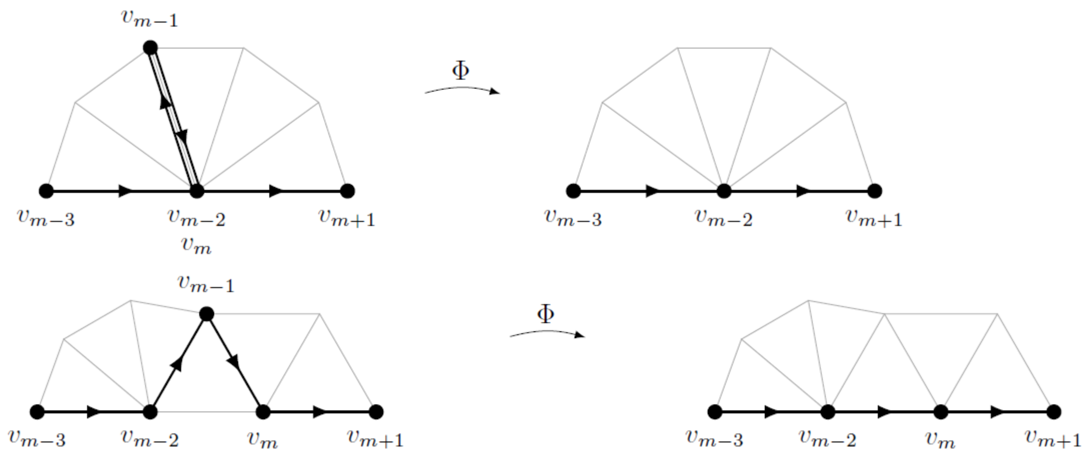

Figure 5.3.5: Schematic representations of the action of Φ on a Farey graph path in the cases $b_m = 0$ (upper) and $b_m = 1$ (lower). The case $b_m = -1$ is similar to that of $b_m = 1$.

We now prove Theorem 5.2.6. We need the following lemma, which will be used several times.

Lemma 5.3.6. *Let u and v be two adjacent vertices of $\mathscr{F}$, and let γ be a finite path in $\mathscr{F}$ with initial and final vertices in different components of $\mathbb{R}_\infty \setminus \{u, v\}$. Then γ passes through one or both of u and v.*

Proof. Observe that the two vertices of an edge of γ cannot lie in different components of

$\mathbb{R}_\infty \setminus \{u, v\}$, for if they did then this edge would intersect the edge of $\mathscr{F}$ between u and v. It follows, then, that γ must pass through u or v. $\square$

It is helpful to highlight another elementary lemma.

Lemma 5.3.7. *Let α and β be distinct elements of $\mathbb{R}_\infty$ that are not adjacent vertices of $\mathscr{F}$. Then the hyperbolic line between α and β intersects some edge of $\mathscr{F}$.*

Proof. Let ℓ be the hyperbolic line between α and β. If α and β are irrational, then ℓ must intersect some edge of $\mathscr{F}$, for otherwise the vertices of $\mathscr{F}$ in one of the components of $\mathbb{R}_\infty \setminus \{\alpha, \beta\}$ are disconnected from the vertices in the other component. On the other hand, if one of the two vertices (α, say) is rational, then for some automorphism $M \in \Gamma$, $M(\alpha) = \infty$, in which case, as $M(\beta)$ is not an integer since α and β are not adjacent, the vertical line joining ∞ to $M(\beta)$ intersects the edge between n and $n+1$, where n is the integer part of β. $\square$

We are now able to prove Theorem 5.2.6.

Proof of Theorem 5.2.6. We will prove the contrapositive statement of Theorem 5.2.6, namely that an infinite path in the Farey graph diverges if and only if it returns to two distinct vertices infinitely often.

Let $\gamma = \langle v_0, v_1, \dots \rangle$ be a path in $\mathscr{F}$ that diverges. Then γ must have two convergent subsequences with distinct limit points α and β in $\mathbb{R}_\infty$.

Suppose for the moment that α and β are not adjacent vertices of $\mathscr{F}$. By Lemma 5.3.7, there is an edge of $\mathscr{F}$ that intersects the hyperbolic line between α and β. Let u and v be the vertices of this edge. Then α and β lie in different components of $\mathbb{R}_\infty \setminus \{u, v\}$. Since γ approaches each of α and β infinitely often, we see from Lemma 5.3.6 that γ passes through one of u or v infinitely many times.

Suppose now that α and β are adjacent vertices of $\mathscr{F}$. After applying an automorphism, that is an element of the modular group Γ, we can assume that they are 0 and ∞. Then γ must pass in and out of one of the intervals $[-1, 0]$ or $[0, 1]$ infinitely often. Applying Lemma 5.3.6 once more, we see again that γ passes through a vertex of $\mathscr{F}$ infinitely many times.

Thus, in both cases, and after applying another automorphism, we can assume that γ passes through the vertex ∞ infinitely often. However, γ diverges, so it must enter some interval $[n, n+1]$ infinitely often, where n is an integer. Since n and $n+1$ are adjacent vertices of $\mathscr{F}$, we can apply Lemma 5.3.6 yet again, to see that γ passes through one of n or $n+1$ infinitely often.

Therefore γ returns to two distinct vertices of $\mathscr{F}$ infinitely often, as required. The converse implication is immediate. □

5.4 Convergence to a rational

This section proves Theorem 5.2.1. Although this theorem is a corollary of Theorem 1.1.4, we prove it independently, and then later use it to prove the stronger theorem.

Lemma 5.4.1. *The continued fraction* $[b_0, b_1, \dots]$*, where* $|b_n| \geqslant 2$*, for* $n = 1, 2, \dots,$ *converges to a value in* $[b_0 - 1, b_0 + 1]$*. Furthermore, it converges to* $b_0 - 1$ *if and only if* $b_1 = b_2 = \cdots = 2$*, and it converges to* $b_0 + 1$ *if and only if* $b_1 = b_2 = \cdots = -2$*.*

Proof. We prove the lemma when $b_0 = 0$; the more general case follows immediately by applying a translation.

Let $t_n(z) = -1/(b_n + z)$, for $n = 1, 2, \dots$, and let $T_n = t_1 \circ t_2 \circ \cdots \circ t_n$. (It is marginally more convenient to use these mappings in place of the mappings $s_n = b_n - 1/z$ and $S_n = s_1 \circ s_2 \circ \cdots \circ s_n$ defined earlier.) Then the nth convergent of $[0, b_1, b_2 \dots]$ is $T_n(0)$, for $n = 1, 2, \dots$. Observe that t_n takes any point $z \in [-1, 1]$ to a point $t_n(z) \in [-1, 1]$, and preserves the order of points in that interval. It follows that T_n also takes points in $[-1, 1]$ to points in $[-1, 1]$, and preserves order in that interval. It is straightforward to check that $t_{n+1}(-1) \geq -1$, so

$$T_{n+1}(-1) = T_n(t_{n+1}(-1)) \geq T_n(-1),$$

and similarly

$$T_{n+1}(1) = T_n(t_{n+1}(1)) \leq T_n(1).$$

Furthermore, $t_n(-1) = -1$ if and only if $b_n = 2$, and $t_n(1) = 1$ if and only if $b_n = -2$. Thus

$$-1 \leqslant T_1(-1) \leqslant T_2(-1) \leqslant \cdots \leqslant T_2(1) \leqslant T_1(1) \leqslant 1,$$

where equality holds in all of the left set of inequalities if and only if all coefficients b_n equal 2, and equality holds in all of the right set of inequalities if and only if all coefficients b_n equal -2.

Now, $T_n(0) \in (T_n(-1), T_n(1))$, but $T_n(0) = T_{n+1}(\infty)$, so $T_n(0) \notin [T_{n+1}(-1), T_{n+1}(1)]$. Therefore either $T_n(-1) < T_n(0) < T_{n+1}(-1)$ or $T_{n+1}(1) < T_n(0) < T_n(1)$. From this we see that all the points $T_1(0), T_2(0), \dots$ are distinct. For any $\varepsilon > 0$ there are only finitely many edges of $\mathscr{F}$ with vertices in $[-1, 1]$ and with Euclidean diameter greater than ε. Since $T_n(0), T_n(1) \in [-1, 1]$, and they are the vertices of an edge of $\mathscr{F}$ (because 0 and 1 are adjacent in $\mathscr{F}$ and $T_n \in \Gamma$), we see that $|T_n(0) - T_n(1)| \to 0$, and similarly $|T_n(0) - T_n(-1)| \to 0$.

Reasoning in this way we deduce that the sequences $(T_n(-1))$ and $(T_n(1))$ converge to the same limit between -1 and 1, which is the value of the continued fraction. Furthermore, this limit is -1 if and only if all coefficients b_n equal 2, and it is 1 if and only if all coefficients b_n equal -2. $\square$

We can now prove Theorem 5.2.1, which says that if $|b_n| \geqslant 2$, for $n = 1, 2, \dots$, then the continued fraction $[b_0, b_1, \dots]$ converges to a rational if it has the same tail as $\pm[2, 2, \dots]$, and otherwise it converges to an irrational.

Proof of Theorem 5.2.1. We use the notation $s_n(z) = b_n - 1/z$, $S_n = s_1 \circ s_2 \circ \cdots \circ s_n$ and $v_n = S_n(\infty)$, for $n = 0, 1, \dots$, which was presented earlier. The convergents $\langle v_0, v_1, \dots \rangle$ of the continued fraction form a path in $\mathscr{F}$, which converges to a limit α, a real number, by Lemma 5.4.1.

Suppose that α is rational. We claim that there is a nonnegative integer m for which $v_m \sim \alpha$ and $v_{m+1} \neq \alpha$. (Recall that $\sim$ denotes adjacency in $\mathscr{F}$.) To prove the claim, first suppose that $v_k = \alpha$ for some position k. Then $v_{k+1} \sim \alpha$. Now $s_{k+2}(\infty) = b_{k+2}$ and $s_{k+1}^{-1}(z) = 1/(b_{k+1} - z)$, so $s_{k+1}^{-1}(\infty) = 0$; therefore since $b_{k+2} \neq 0$ we see that $s_{k+2}(\infty) \neq s_{k+1}^{-1}(\infty)$, and so

$$v_{k+2} = S_{k+2}(\infty) = S_{k+1} \circ s_{k+2}(\infty) \neq S_{k+1} \circ s_{k+1}^{-1}(\infty) = S_k(\infty) = v_k = \alpha.$$

Hence in this case we can choose $m = k + 1$.

We can now suppose that no vertex v_k is equal to α. Choose vertices u and v of $\mathscr{F}$ each adjacent to α with $u < \alpha < v$ such that the vertex v_0 of the path does not lie in $[u, v]$. By Lemma 5.3.6, the path γ must pass through one of u or v, so there is a postion m for which $v_m \sim \alpha$ (and $v_{m+1} \neq \alpha$). This proves the claim. Observe that $v_m = S_m(\infty)$, so $\infty = S_m^{-1}(v_m)$. Since $v_m \sim \alpha$, it follows that $\infty \sim S_m^{-1}(\alpha)$, so $S_m^{-1}(\alpha)$ is an integer. We have

$$s_0 \circ s_1 \circ \cdots \circ s_m \circ s_{m+1} \circ \cdots = S_m \circ s_{m+1} \circ \cdots ,$$

and therefore $S_m^{-1}(\alpha) = [b_{m+1}, b_{m+2}, \cdots]$. Now $v_{m+1} = S_{m+1}(\infty) = S_m(b_{m+1})$, so we have $b_{m+1} = S_m^{-1}(v_{m+1})$, which is distinct from $S_m^{-1}(\alpha)$. So we can apply Lemma 5.4.1 to the continued fraction $[b_{m+1}, b_{m+2}, \cdots]$ to see that $S_m^{-1}(\alpha) = b_{m+1} - 1$ and $b_{m+2} = b_{m+3} = \cdots = 2$, or else $S_m^{-1}(\alpha) = b_{m+1} + 1$ and $b_{m+2} = b_{m+3} = \cdots = -2$.

It remains to prove that if $[b_0, b_1, \ldots]$ has the same tail as $[2, 2, \ldots]$ or $[-2, -2, \ldots]$, then α is rational. Suppose then that $b_{m+2} = b_{m+3} = \cdots = 2$, for some position m. Then

$$\alpha = [b_0, b_1, \ldots] = S_{m+1}([2, 2, \ldots]) = S_{m+1}(1),$$

so α is rational, and similarly we can see that α is rational if $b_{m+2} = b_{m+3} = \cdots = -2$. □

The following example shows the representation on the Farey tessellation of a continued fraction which converges to zero.

Example 5.4.2. Consider the continued fraction $[0, -1, 1, 2, 2, 2, 2, 2, \ldots]$. We calculate the first 6 convergents in the following table, then in Figure 5.4.1 show the corresponding path on the Farey tessellation, which we see converges very slowly to zero.

n	b_n	p_n	q_n	v_n
0	0	0	1	0
1	-1	-1	-1	1
2	1	-1	-2	1/2
3	2	-1	-3	1/3
4	2	-1	-4	1/4
5	2	-1	-5	1/5

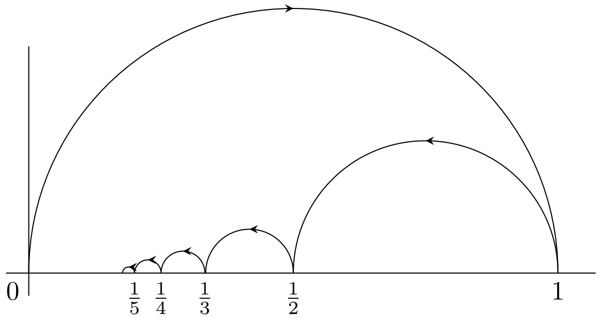

Figure 5.4.1: The path on the Farey tessellation of $[0, -1, 1, 2, 2, 2, 2, 2, \ldots]$.

5.5 Convergence if the key position increases to infinity

In this section we prove part (i) of Theorem 1.1.4. First we gather several more elementary results. In these results we use the usual notation $s_n(z) = b_n - 1/z$ and $S_n = s_0 \circ s_1 \circ \cdots \circ s_n$, for $n = 0, 1, \ldots$, associated to the continued fraction $[b_0, b_1, \ldots]$.

Lemma 5.5.1. *Suppose that the finite continued fraction $[b_0, b_1, \ldots, b_m]$, where $|b_i| \geqslant 2$ for $i = 1, 2, \ldots, m$, has as value the reduced rational c/d. Then $m < d$.*

Proof. From equation (2.8.3), if $S_n(z) = (c_n z - c_{n-1})/(d_n z - d_{n-1})$, $d_n = b_n d_{n-1} - d_{n-2}$ for $n = 1, 2, \ldots, m$, where $d_0 = 1$ and $d_{-1} = 0$. So

$$|d_n| = |b_n d_{n-1} - d_{n-2}| \geqslant 2|d_{n-1}| - |d_{n-2}|,$$

and therefore $|d_n| - |d_{n-1}| \geqslant |d_{n-1}| - |d_{n-2}|$. Since $|d_0| - |d_{-1}| = 1$, it follows that

$$|d_m| = (|d_m| - |d_{m-1}|) + (|d_{m-1}| - |d_{m-2}|) + \cdots + (|d_0| - |d_{-1}|) > m.$$

But $c_m/d_m = c/d$, so $|d_m| = d$, and the result follows. □

The following lemma describes how the convergents of a continued fraction are modified under an application of the function Φ.

Lemma 5.5.2. *Let $[b_0, b_1, \dots]$ and $[b'_0, b'_1, \dots]$ be continued fractions with convergents $\langle v_0, v_1, \dots\rangle$ and $\langle v'_0, v'_1, \dots\rangle$, respectively, and suppose that $[b'_0, b'_1, \dots] = \Phi([b_0, b_1, \dots])$. Let m be the first position at which a coefficient equal to 0, 1 or -1 appears in the sequence $b_1, b_2, \dots$ (assuming there is one). Then*

$$\langle v'_0, v'_1, \dots\rangle = \begin{cases} \langle v_0, v_1, \dots, v_{m-2}, v_{m+1}, \dots\rangle, & \textit{if } b_m = 0, \\ \langle v_0, v_1, \dots, v_{m-2}, v_m, \dots\rangle, & \textit{if } b_m = \pm 1. \end{cases}$$

In the case $m = 1$ this formula should be interpreted to say that $\langle v'_0, v'_1, \dots\rangle$ is equal to $\langle v_2, v_3, \dots\rangle$ when $b_m = 0$ and $\langle v'_0, v'_1, \dots\rangle$ is equal to $\langle v_1, v_2, \dots\rangle$ when $b_m = \pm 1$.

Proof. We use the usual notation $s_n(z) = b_n - 1/z$, $S_n = s_0 \circ s_1 \circ \cdots \circ s_n$ and $v_n = S_n(\infty)$, for $n = 0, 1, \dots$; also $s'_n(z) = b'_n - 1/z$, $S'_n = s'_0 \circ s'_1 \circ \cdots \circ s'_n$ and $v'_n = S'_n(\infty)$, for $n = 0, 1, \dots$.

First consider the case when $b_m = 0$. Then $s'_n(z) = s_n(z)$, for $n = 0, 1, \dots, m-2$, so $v'_n = S'_n(\infty) = S_n(\infty) = v_n$. Now, $s_m(z) = -1/z$, so $s_{m-1} \circ s_m(z) = b_{m-1} + z$. Hence

$$s_{m-1} \circ s_m \circ s_{m+1}(z) = b_{m-1} + b_{m+1} - 1/z = s'_{m-1}(z).$$

Also, $s'_n(z) = s_{n+2}(z)$, for $n \geqslant m$. Hence, for $n \geqslant m-1$, we have

$$v'_n = S'_n(\infty) = S_{n+2}(\infty) = v_{n+2}.$$

Now suppose that $b_m = 1$ (the case $b_m = -1$ is similar). As in the previous case, $s'_n(z) = s_n(z)$, for $n = 0, 1, \ldots, m-2$, so $v'_n = S'_n(\infty) = S_n(\infty) = v_n$. This time $s_m(z) = 1 - 1/z$, so

$$s_{m-1} \circ s_m(z) = b_{m-1} - 1 - \frac{1}{z-1} \quad \text{and} \quad s_{m-1} \circ s_m \circ s_{m+1}(z) = b_{m-1} - 1 - \frac{1}{b_{m+1} - 1/z - 1}$$

Now $b'_{m-1} = b_{m-1} - 1$, and $b'_m = b_{m+1} - 1$, so

$$s'_{m-1} \circ s'_m(z) = b_{m-1} - 1 - \frac{1}{b_{m+1} - 1 - 1/z} = s_{m-1} \circ s_m \circ s_{m+1}(z)$$

Also, $s'_n(z) = s_{n+1}(z)$, for $n \geqslant m+1$. Hence, for $n \geqslant m-1$, we have

$$v'_n = S'_n(\infty) = S_{n+1}(\infty) = v_{n+1}. \qquad \square$$

For the remainder of this section we will use the following notation associated to a continued fraction $[b_0, b_1, \ldots]$. As usual, the sequence of convergents of $[b_0, b_1, \ldots]$ is denoted by $\langle v_0, v_1, \ldots \rangle$. We define $[b_0^{(n)}, b_1^{(n)}, \ldots] = \Phi^n([b_0, b_1, \ldots])$, and we let $\langle v_0^{(n)}, v_1^{(n)}, \ldots \rangle$ be the associated sequence of convergents. Then $p^{(n)}$ is the first position at which a 0, 1 or -1 appears in $[b_0^{(n)}, b_1^{(n)}, \ldots]$, and $p = \liminf p^{(n)}$. By the definition of Φ we can see that, for $0 \leqslant i < p-1$ (where possibly $p = \infty$), each of the sequences $b_i^{(0)}, b_i^{(1)}, \ldots$ is eventually constant, with value b_i^*, say. We denote the sequence of convergents of $[b_0^*, b_1^*, \ldots, b_{p-2}^*]$ by $\langle v_0^*, v_1^*, \ldots, v_{p-2}^* \rangle$. Of course, if $p = \infty$, then $[b_0^*, b_1^*, \ldots]$ is an infinite continued fraction with an infinite sequence of convergents $\langle v_0^*, v_1^*, \ldots \rangle$.

Whereas the position of various vertices on the path after several iterations of Φ seems obvious intuitively, the more rigorous treatment which follows is necessary for a proof of Theorem 1.1.4.

Consider a particular convergent v_k of $[b_0, b_1, \ldots]$. We define inductively a sequence $E(k)$ of non-negative integers $e^{(0)}(k), e^{(1)}(k), \ldots$, chosen such that the vertex at position $e^{(n)}(k)$ of $\langle v_0^{(n)}, v_1^{(n)}, \ldots \rangle$ is v_k. The sequence $E(k)$ may be finite or infinite.

Firstly, as v_k is at position k in $\langle v_0, v_1, \ldots \rangle$, $e^{(0)}(k) = k$.

Now suppose that $e^{(0)}(k), e^{(1)}(k), \ldots, e^{(n)}(k)$ have all been defined, for some non-negative integer n. Let $m = p^{(n)}$, the position of the first coefficient (ignoring $b_0^{(n)}$) equal to 0, 1 or -1 in $[b_0^{(n)}, b_1^{(n)}, \ldots]$.

If $b_m^{(n)} = 0$, and $e^{(n)}(k)$ equals $m-1$ or m, then the sequence $E(k)$ terminates at $e^{(n)}(k)$.

If $b_m^{(n)} = \pm 1$, and $e^{(n)}(k) = m-1$, then, again, the sequence $E(k)$ terminates at $e^{(n)}(k)$. Otherwise, we define

$$e^{(n+1)}(k) = \begin{cases} e^{(n)}(k), & \text{if } e^{(n)}(k) \leqslant m-2, \\ e^{(n)}(k) - 2, & \text{if } e^{(n)}(k) > m \text{ and } b_m^{(n)} = 0, \\ e^{(n)}(k) - 1, & \text{if } e^{(n)}(k) > m-1 \text{ and } b_m^{(n)} = \pm 1. \end{cases}$$

This ensures that the $e^{(n+1)}(k)^{\text{th}}$ vertex of $\langle v_0^{(n+1)}, v_1^{(n+1)}, \dots \rangle$ is equal to the $e^{(n)}(k)^{\text{th}}$ vertex of the path $\langle v_0^{(n)}, v_1^{(n)}, \dots \rangle$.

The resulting sequence $E = e^{(0)}(k), e^{(1)}(k), \dots$ is decreasing, though not necessarily strictly decreasing; if it is infinite, then it must therefore eventually be constant. We record two further properties in the following lemmas. We write $\mathbb{N}_0$ for the set $\{0, 1, 2, \dots\}$.

Lemma 5.5.3. *If $k < l$, then $e^{(n)}(k) < e^{(n)}(l)$, for all $n \in \mathbb{N}_0$ for which both expressions exist.*

Proof. We proceed by induction. As $e^{(0)}(k) = k$ and $e^{(0)}(l) = l$, $e^{(0)}(k) < e^{(0)}(l)$. Assume that $e^{(i)}(k) < e^{(i)}(l)$ for all non-negative integers $i \leq n$. Then if $e^{(n)}(k) < m-1$ and $e^{(n)}(l) < m-1$, $e^{(n+1)}(k) < e^{(n+1)}(l)$. Suppose that $b_m = 0$. If $e^{(n)}(l) = m-1$ or $e^{(n)}(l) = m$, the sequence $E(l)$ terminates at $e^{(n)}(l)$. If $e^{(n)}(l) > m-1$, then $e^{(n)}(l) > e^{(n)}(k) + 2$, so $e^{(n+1)}(k) < e^{(n+1)}(l)$. If $e^{(k)}(m) = m-1$ or $e^{(n)}(k) = m$, the sequence $E(k)$ terminates at $e^{(n)}(k)$. If $e^{(n)}(k) > m$, we also have $e^{(n)}(l) > m$, and $e^{(n+1)}(k) < e^{(n+1)}(l)$. A similar proof applies to the case $b_m = \pm 1$. $\square$

Lemma 5.5.4. *Let $n, r \in \mathbb{N}_0$. Then there exists a unique integer $k \in \mathbb{N}_0$ with $e^{(n)}(k) = r$.*

Proof. We prove by induction on n that for any $r \in \mathbb{N}_0$ there exists $k \in \mathbb{N}_0$ with $e^{(n)}(k) = r$. For $n = 0$, we can choose $k = r$, because $e^{(0)}(r) = r$. Suppose next that the induction statement is true for all non-negative integers up to and including n. Let $m = p^{(n)}$. Given $r \in \mathbb{N}_0$, we define

$$s = \begin{cases} r, & \text{if } r \leqslant m-2, \\ r+2, & \text{if } r \geqslant m-1 \text{ and } b_m^{(n)} = 0, \\ r+1, & \text{if } r \geqslant m-1 \text{ and } b_m^{(n)} = \pm 1. \end{cases}$$

We can choose $k \in \mathbb{N}_0$ such that $e^{(n)}(k) = s$. Then $e^{(n+1)}(k) = r$ for each of the three cases used to define s. This completes the inductive proof. For uniqueness, we observe that if $k < l$ then $e^{(n)}(k) < e^{(n)}(l)$, so $e^{(n)}(k)$ and $e^{(n)}(l)$ cannot both equal r. □

Lemma 5.5.5. *Suppose that $p^{(n)} \to \infty$. Then for any extended rational v there are only finitely many convergents v_k equal to v.*

Proof. We write v as a reduced rational c/d. Choose a positive integer R such that if $n \geqslant R$, then $p^{(n)} \geqslant d+2$. It follows that

$$[b_0^{(n)}, b_1^{(n)}, \dots, b_d^{(n)}] = [b_0^*, b_1^*, \dots, b_d^*],$$

for $n \geqslant R$. From Lemma 5.5.4 there is a non-negative integer S such that $e^{(R)}(S) = d$. Then $e^{(R)}(S) \leqslant p^{(n)} - 2$, so $e^{(n)}(S) \leqslant p^{(n)} - 2$, for $n \geqslant R$, and it follows from the definition of the sequence $e^{(0)}(S), e^{(1)}(S), \dots$ that $e^{(n)}(S) = d$, for all $n \geqslant R$.

Let us assume that $k > S$ and $v_k = v$. We will establish a contradiction, thereby proving that only finitely many convergents are equal to v.

Suppose for the moment that the sequence $e^{(0)}(k), e^{(1)}(k), \dots$ is infinite, so it is eventually constant, with value m, say. Choose any sufficiently large integer $n \geqslant R$ for which $e^{(n)}(k) = m$ and $p^{(n)} > m$. Since $k > S$, it follows that $e^{(n)}(k) > e^{(n)}(S)$, so $m > d$. Now, from the definition of $e^{(0)}(k), e^{(1)}(k), \dots,$ we know that the vertex at position $m = e^{(n)}(k)$ of $\langle v_0^{(n)}, v_1^{(n)}, \dots \rangle$ is equal to v_k; that is, $v_m^{(n)} = v_k = v = c/d$. However, this contradicts Lemma 5.5.1, because $|b_i^{(n)}| \geqslant 2$, for $i = 1, 2, \dots, m$ (since $p^{(n)} > m$), and $m > d$.

Suppose instead that $e^{(0)}(k), e^{(1)}(k), \dots$ is of finite length, and that the final term is $e^{(n)}(k)$.

Since $k > S$, it follows that $e^{(n)}(k) > e^{(n)}(S) = d$.

Let $m = p^{(n)}$. Because the final term of the sequence $e^{(0)}(k), e^{(1)}(k), \dots$ is $e^{(n)}(k)$, we see from the definition of that sequence that either $b_m^{(n)} = 0$ and $e^{(n)}(k)$ equals $m-1$ or m, or $b_m^{(n)} = \pm 1$ and $e^{(n)}(k) = m-1$. So, in all cases $m > d$.

Suppose that $b_m^{(n)} = 0$. If $e^{(n)}(k) = m-1$, $[b_0^{(n)}, b_1^{(n)}, \dots, b_{m-1}^{(n)}]$ has value $v_{m-1}^{(n)} = v_k = v$, which contradicts Lemma 5.5.1, because $|b_i^{(n)}| \geqslant 2$, for $i = 1, 2, \dots, m-1$ (since $p^{(n)} = m$), and $m > d$. And if $e^{(n)}(k) = m$, $[b_0^{(n)}, b_1^{(n)}, \dots, b_{m-2}^{(n)}]$ has value $v_{m-2}^{(n)} = v_m^{(n)} = v_k = v$, which again contradicts Lemma 5.5.1, because $|b_i^{(n)}| \geqslant 2$, for $i = 1, 2, \dots, m-2$ (since $p^{(n)} = m$), and $m > d$. We obtain a similar contradiction when $b_m^{(n)} = \pm 1$.

It follows that there are no integers $k > S$ with $v_k = v$. Hence there are only finitely many convergents equal to v. □

Lemma 5.5.6. *Suppose that $p^{(n)} \to \infty$. Then the sequence of convergents of the limit continued fraction $[b_0^*, b_1^*, \dots]$ is a subsequence of the sequence of convergents of $[b_0, b_1, \dots, b_n]$.*

Proof. With the usual notation, the lemma says that $\langle v_0^*, v_1^*, \dots \rangle$ is a subsequence of $\langle v_0, v_1, \dots \rangle$. To see why this is so, choose any non-negative integer r, and let N be a positive integer for which $p^{(n)} \geqslant r+2$, for $n \geqslant N$. Then $v_r^{(n)} = v_r^*$, for $n \geqslant N$, because $r \leqslant p^{(n)} - 2$. By Lemma 5.5.4 there is a nonnegative integer k_r with $e^{(N)}(k_r) = r$; and in fact $e^{(n)}(k_r) = r$, for $n \geqslant N$, because $r \leqslant p^{(n)} - 2$. Hence $v_r^* = v_r^{(n)} = v_{k_r}$, by definition of $e^{(0)}(k_r), e^{(1)}(k_r), \dots$.

Now, we know that if $k_r < k_s$, then $e^{(n)}(k_r) < e^{(n)}(k_s)$. Hence $k_0 < k_1 < \cdots$. It follows that $\langle v_0^*, v_1^*, \dots \rangle$ is a subsequence of $\langle v_0, v_1, \dots \rangle$. □

We can now prove the first part of Theorem 1.1.4.

Proof of Theorem 1.1.4(i). Part (i) of Theorem 1.1.4 assumes that $p^{(n)} \to \infty$. In this case we see from Lemma 5.5.5 that the sequence of convergents $\langle v_0, v_1, \dots \rangle$ does not return to any vertex infinitely often. Hence, by Theorem 5.2.6, the continued fraction converges to some value α.

Now, by Lemma 5.5.6, the sequence of convergents of $[b_0^*, b_1^*, \dots]$ is a subsequence of that of $[b_0, b_1, \dots]$. The former continued fraction certainly converges – by Theorem 5.2.1 – so it must converge to α. Moreover, Theorem 5.2.1 tells us that α is rational if $[b_0^*, b_1^*, \dots]$ has the same tail as $[2, 2, \dots]$ or $[-2, -2, \dots]$, and otherwise it is irrational, as required. $\square$

The following example shows a continued fraction for which $p^{(n)} \to \infty$ as $n \to \infty$, with a tail different to $[2, 2, 2, 2, 2, \dots]$. It converges to an irrational number.

Example 5.5.7. Consider the continued fraction

$$[0, 2, -2, -1, -2, 2, 1, -1, 1, 2, -2, -1, 1, -1, 1, 2, -2, -1, 1, -1, 1, -1, -2, 2, \dots]$$

The following table shows the result of applying Φ 10 times:

n	$p^{(n)}$	${b_1}^{(n)}$	${b_2}^{(n)}$	${b_3}^{(n)}$	${b_4}^{(n)}$	${b_5}^{(n)}$	${b_6}^{(n)}$	${b_7}^{(n)}$	${b_8}^{(n)}$	${b_9}^{(n)}$	$\dots$
0	3	2	−2	−1	−2	2	1	−1	1	2	$\cdots$
1	2	2	−1	−1	2	1	−1	1	2	−2	$\cdots$
2	2	3	0	2	1	−1	1	2	−2	−1	$\cdots$
3	2	5	1	−1	1	2	−2	−1	1	−1	$\cdots$
4	3	4	−2	1	2	−2	−1	1	−1	1	$\cdots$
5	3	4	−3	1	−2	−1	1	−1	1	2	$\cdots$
6	4	4	−4	−3	−1	1	−1	1	2	−2	$\cdots$.
7	5	4	−4	−2	2	−1	1	2	−2	−1	$\cdots$
8	8	4	−4	−2	3	2	2	−2	−1	1	$\cdots$
9	7	4	−4	−2	3	2	2	−1	2	−1	$\cdots$
10	8	4	−4	−2	3	2	3	3	−1	1	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

It is clear from this table (and straightforward to prove assuming that the pattern for the coefficients continues) that $p^{(n)} \to \infty$ as $n \to \infty$. The first 7 convergents are given in the following table:

m	b_m	p_m	q_m	v_m
		1	0	
0	0	0	1	0
1	2	-1	2	-0.5
2	-2	2	-5	-0.4
3	-1	-1	3	-0.3
4	-2	0	-1	0
5	2	1	-5	-0.2
6	1	1	-4	-0.25
7	-1	-2	9	

We show the original path and that after the first 3 applications of the algorithm in Figure 5.5.1.

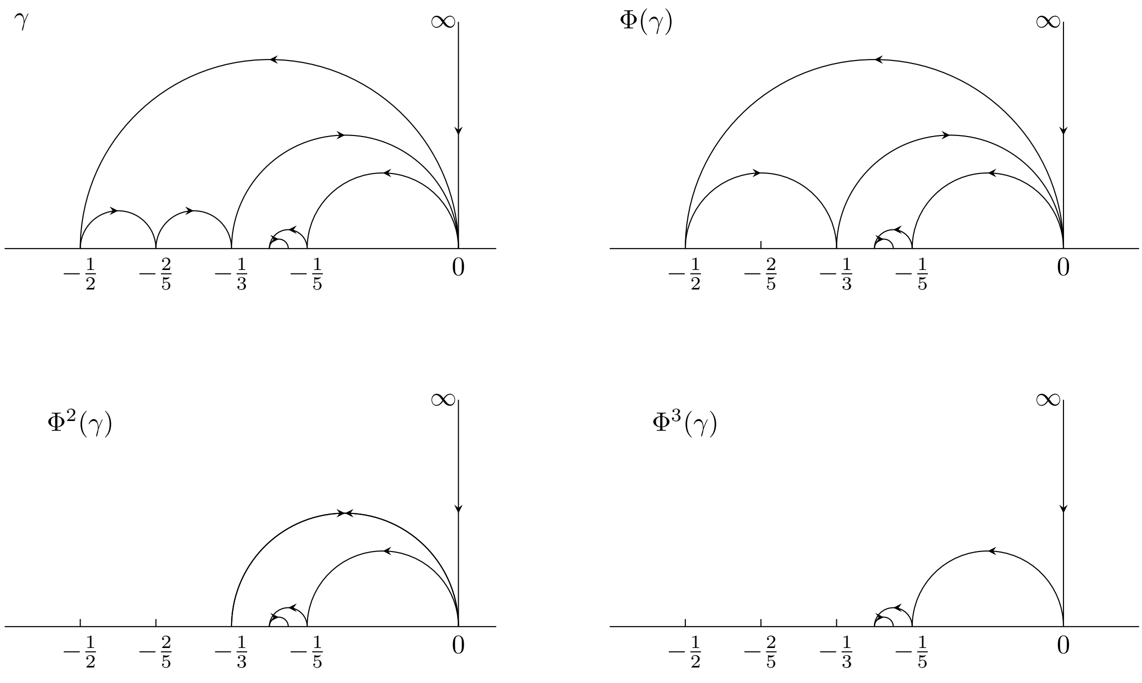

Figure 5.5.1: The paths γ, $\Phi(\gamma), \Phi^2(\gamma)$ and $\Phi^3(\gamma)$ from Example 5.5.7.

5.6 Convergence if the key position is bounded

The second part of Theorem 1.1.4 will be proved after two preliminary lemmas.

Lemma 5.6.1. *Consider a continued fraction* $[b_0, b_1, \dots]$ *with* $|b_i| \geqslant 2$*, for* $i = 1, 2, \dots, n$*. Let* $\langle v_0, v_1, \dots \rangle$ *be the corresponding sequence of convergents. If* $1 \leqslant k \leqslant n$*, then*

$$|v_n - v_{k-1}| \leqslant \frac{1}{|b_k| - 1}.$$

This inequality remains true if $1 \leqslant k < n$ *and* $|b_i| \geqslant 2$*, for* $i = 1, 2, \dots, n-1$*, but* $b_n = 0$*.*

Proof. By applying a translation we can assume that $b_0 = 0$ (so $v_0 = 0$). Now define $t_m(z) = -1/(b_m + z)$ and $T_m = t_1 \circ t_2 \circ \cdots \circ t_m$, for $m = 1, 2, \dots$, as we did in proving Lemma 5.4.1. Let T_0 denote the identity transformation. Recall that $(T_m(0))$ is the sequence of convergents of $[b_0, b_1, \dots]$, and $t_m([-1, 1]) \subset [-1, 1]$ for $m = 1, 2, \dots, n$. Observe also that $|t'_m(z)| = 1/|b_m + z|^2 \leqslant 1$, for $z \in [-1, 1]$ and $1 \leqslant m \leqslant n$. Applying the chain rule, we see that

$$|T'_m(z)| = |t'_1(z_1)||t'_2(z_2)| \cdots |t'_m(z_m)| \leqslant 1,$$

for $z \in [-1, 1]$ and $1 \leqslant m \leqslant n$, where $z_i = t_{i+1} \circ t_{i+2} \circ \cdots \circ t_m(z)$ (and $z_m = z$).

Next, we have $v_k = T_k(0)$, for $k = 1, 2, \dots, n$. Observe that $t_{k+1} \circ t_{k+2} \circ \cdots \circ t_n(0) \in [-1, 1]$. Hence

$$v_n \in T_k([-1, 1]) = T_{k-1}([t_k(-1), t_k(1)]).$$

Suppose that $b_k > 0$ (the case $b_k < 0$ is similar). Then $t_k(1) = -1/(b_k + 1) < 0$, so

$$v_n \in T_{k-1}([t_k(-1), 0]).$$

Therefore $v_n = T_{k-1}(u_n)$, for some point u_n in $[t_k(-1), 0]$. Now,

$$|t_k(-1) - 0| = \left| \frac{-1}{b_k - 1} \right| = \frac{1}{|b_k| - 1}.$$

Hence

$$|v_n - v_{k-1}| = |T_{k-1}(u_n) - T_{k-1}(0)| = |T'_{k-1}(c_k)||u_n - 0|,$$

for some real number c_k between u_n and 0. The required inequality follows, since $|T'_{k-1}(c_k)| \leqslant 1$ and $|u_n - 0| \leqslant 1/(|b_k| - 1)$.

It remains to prove the final part of the lemma, in which $b_n = 0$. In this case, $v_n = v_{n-2}$, so the lemma continues to hold if $k \leqslant n - 2$. And clearly it also holds if $k = n - 1$. □

Lemma 5.6.2. *Suppose that $p^{(n)} \not\to \infty$. Then it follows that only a finite number of the sequences $e^{(0)}(k), e^{(1)}(k), \ldots$, for $k \in \mathbb{N}_0$, are of infinite length.*

Proof. Let $p = \liminf p^{(n)}$. Suppose that the sequence $e^{(0)}(k), e^{(1)}(k), \ldots$ is of infinite length for some positive integer k. Choose a positive integer N such that, for $n \geqslant N$, all terms $e^{(n)}(k)$ of the sequence are equal to some value e.

Now suppose, in order to reach a contradiction, that $e > p$. Choose an integer $n \geqslant N$ for which $p^{(n)} = p$. Then $b_p^{(n)}$ is 0, 1 or -1, and we see from the definition of the sequence $e^{(0)}(k), e^{(1)}(k), \ldots$ that $e^{(n+1)}(k) < e^{(n)}(k)$. This is the contradiction we need, since $e^{(n+1)}(k) = e^{(n)}(k) = e$.

It follows, then, that $e \leqslant p$. But for each nonnegative integer $e \leqslant p$ there is a *unique* integer k such that $e^{(n)}(k) \to e$ as $n \to \infty$. Hence there are only finitely many infinite sequences $e^{(0)}(k), e^{(1)}(k), \ldots$, for $k \in \mathbb{N}_0$. □

We now prove the second part of Theorem 1.1.4.

Proof of Theorem 1.1.4(ii). We assume that $p^{(n)} \not\to \infty$. Let $p = \liminf p^{(n)}$.

Suppose that $q^{(n)} \not\to \infty$. Since $p = \liminf p^{(n)}$, we can find a positive integer N such that

$$[b_0^{(n)}, b_1^{(n)}, \ldots, b_{p-2}^{(n)}] = [b_0^*, b_1^*, \ldots, b_{p-2}^*],$$

for $n \geqslant N$. Observe that the sequence $b_{p-1}^{(1)}, b_{p-1}^{(2)}, \ldots$ does not stabilise on a fixed value. And recall that $q^{(n)} = |b_{p-1}^{(n)}|$, by definition.

Now let $m_1, m_2, \dots$ be the complete list of positive integers n greater than N, written in increasing order, for which $p^{(n)} = p$. That is, $m_1, m_2, \dots$ are the positive integers $n > N$, in order, for which $b_p^{(n)}$ is 0, 1 or -1.

The sequence $b_{p-1}^{(n)}$, for $n = N+1, N+2, \dots$, can only change value when n is equal to one of $m_1, m_2, \dots$. Since $q^{(n)} \not\to \infty$, we can find a subsequence $n_1, n_2, \dots$ of $m_1, m_2, \dots$ for which every term $b_{p-1}^{(n_i)}$ is equal to some fixed integer b, where $|b| \geqslant 2$. And by restricting to a further subsequence, we can assume that all the terms $b_p^{(n_i)}$ are equal to precisely one of 0, 1 or -1.

Observe that $v_{p-1}^{(n_i)} = u$ and $v_p^{(n_i)} = v$, for $i = 1, 2, \dots$, where u and v are two fixed adjacent vertices of $\mathscr{F}$. Let r_i and s_i be non-negative integers for which $e^{(n_i)}(r_i) = p-1$ and $e^{(n_i)}(s_i) = p$, in which case $v_{r_i} = u$ and $v_{s_i} = v$. Now, the sequence $e^{(0)}(r_i), e^{(1)}(r_i), \dots$ has length exactly n_i+1; the final term is $e^{(n_i)}(r_i) = p-1$ because $b_p^{(n_i)}$ is 0, 1 or -1. It follows that all the integers r_i are distinct from one another, so there are infinitely many of them. With similar reasoning we can see that the collection of integers s_i is infinite too.

We deduce that the sequence (v_n) is equal to u for infinitely many indices n, and likewise it is equal to v for infinitely many indices n. It follows that (v_n) diverges, so $[b_1, b_2, \dots]$ diverges.

Suppose now that $q^{(n)} \to \infty$. As before, we choose a positive integer N for which $p^{(n)} \geqslant p$, for $n \geqslant N$, so

$$[b_0^{(n)}, b_1^{(n)}, \dots, b_{p-2}^{(n)}] = [b_0^*, b_1^*, \dots, b_{p-2}^*].$$

Next, by Lemma 5.6.2, we can choose a positive integer M such that, for $k \geqslant M$, the sequence $e^{(0)}(k), e^{(1)}(k), \dots$ is of finite length. Let us also choose $M > 2N + (p-1)$.

Let $k \geqslant M$. By definition of the sequence $e^{(0)}(k), e^{(1)}(k), \dots$, we can see that

$$k - e^{(n)}(k) = (k - e^{(1)}(k)) + (e^{(1)}(k) - e^{(2)}(k)) + \cdots + (e^{(n-1)}(k) - e^{(n)}(k)) \leqslant 2n.$$

Suppose, in order to reach a contradiction, that $e^{(n)}(k) < p-1$, for some positive integer n. Then $k - (p-1) < 2n$, and since $M > 2N + (p-1)$ and $k \geqslant M$, we have that $n > N$. However, if $e^{(n)}(k) \leqslant p-2 \leqslant p^{(n)} - 2$, then $e^{(n+1)}(k) = e^{(n)}(k)$, so the sequence $e^{(0)}(k), e^{(1)}(k), \dots$ is infinite. This is the required contradiction. Therefore $e^{(n)}(k) \geqslant p-1$ for $k \geqslant M$ and $n \in \mathbb{N}_0$.

Next, for each integer $k > N$, define r_k to be the positive integer such that $e^{(r_k)}(k)$ is the last term of the sequence $e^{(0)}(k), e^{(1)}(k), \dots$. Let $n \in \mathbb{N}$ and $m = p^{(n)}$. Then $r_k = n$ if and only if either $b_m = 0$ and $e^{(n)}(k)$ equals $m-1$ or m, or $b_m = \pm 1$ and $e^{(n)}(k) = m-1$. Hence there are at most two values of k for which $r_k = n$. Consequently, we deduce that $r_k \to \infty$ as $k \to \infty$.

The vertex at position $e^{(r_k)}(k)$ of $\langle v_0^{(r_k)}, v_1^{(r_k)}, \dots\rangle$ is equal to v_k, so we can apply Lemma 5.6.1 to the continued fraction $[b_0^{(r_k)}, b_1^{(r_k)}, \dots]$ to see that

$$|v_k - v_{p-2}^{(r_k)}| \leqslant \frac{1}{|b_{p-1}^{(r_k)}| - 1} = \frac{1}{q^{(r_k)} - 1}.$$

Observe that $v_{p-2}^{(r_k)} = v_{p-2}^*$, for sufficiently large values of k. Since $r_k \to \infty$ as $k \to \infty$, and hence $q^{(r_k)} \to \infty$ as $k \to \infty$, we see that $v_k \to v_{p-2}^*$ as $k \to \infty$. Therefore $[b_0, b_1, \dots]$ converges to the extended rational v_{p-2}^*. $\square$

The following two examples give two continued fractions for which $p^{(n)} \not\to \infty$. In the first, we have $q^{(n)} \to \infty$ and the continued fraction converges, in the second, $q^{(n)} \not\to \infty$, and the continued fraction does not converge.

Example 5.6.3. Consider the continued fraction $[0, -1, 0, -1, 0, -1, 0, \dots]$. Applying Φ gives:

n	$p^{(n)}$	${b_1}^{(n)}$	${b_2}^{(n)}$	${b_3}^{(n)}$	${b_4}^{(n)}$	${b_5}^{(n)}$	${b_6}^{(n)}$	${b_7}^{(n)}$	${b_8}^{(n)}$	${b_9}^{(n)}$	$\cdots$
0	2	-1	0	-1	0	-1	0	-1	0	-1	$\cdots$
1	2	-2	0	-1	0	-1	0	-1	0	-1	$\cdots$
2	2	-3	0	-1	0	-1	0	-1	0	-1	$\cdots$
3	2	-4	0	-1	0	-1	0	-1	0	-1	$\cdots$
4	2	-5	0	-1	0	-1	0	-1	0	-1	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

We see that $q^{(n)} = b_1^{(n)} \to -\infty$. The first 8 convergents are given in the following table.

m	b_m	p_m	q_m	v_m
		1	0	
0	0	0	1	0
1	-1	-1	-1	1
2	0	0	-1	0
3	-1	1	2	$1/2$
4	0	0	1	0
5	-1	-1	-3	$1/3$
6	0	0	-1	0
7	-1	1	4	$1/4$

We show the original path and that after the first 3 applications of the algorithm in Figure 5.6.1.

The path converges to 0.

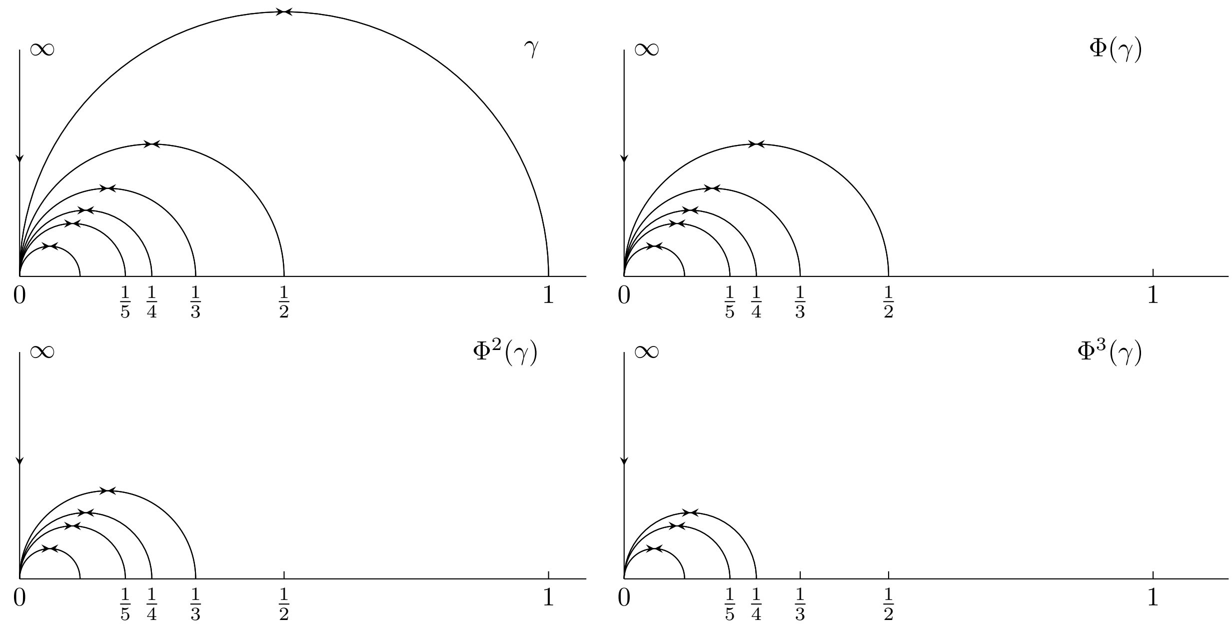

Figure 5.6.1: The paths γ, $\Phi(\gamma), \Phi^2(\gamma)$ and $\Phi^3(\gamma)$ from Example 5.6.3.

Example 5.6.4. Consider the continued fraction $[0, 1, 2, -1, 1, -1, 1, \ldots]$. The results of applying Φ four times are shown below.

n	$p^{(n)}$	$b_1{}^{(n)}$	$b_2{}^{(n)}$	$b_3{}^{(n)}$	$b_4{}^{(n)}$	$b_5{}^{(n)}$	$b_6{}^{(n)}$	$\ldots$
0	3	-1	2	1	1	1	1	$\ldots$
1	2	-1	1	0	1	1	1	$\ldots$
2	2	-2	-1	1	1	1	1	$\ldots$
3	3	-1	2	1	1	1	1	$\ldots$
4	2	-1	1	0	1	1	1	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

The convergents are shown in the following table:

m	b_m	p_m	q_m	v_m
		1	0	
0	0	0	1	0
1	-1	-1	-1	1
2	2	-2	-3	2/3
3	1	-1	-2	1/2
4	1	1	1	1
5	1	2	3	2/3
6	1	1	2	1/2
7	1	-1	-1	1

with the paths in Figure 5.6.2.

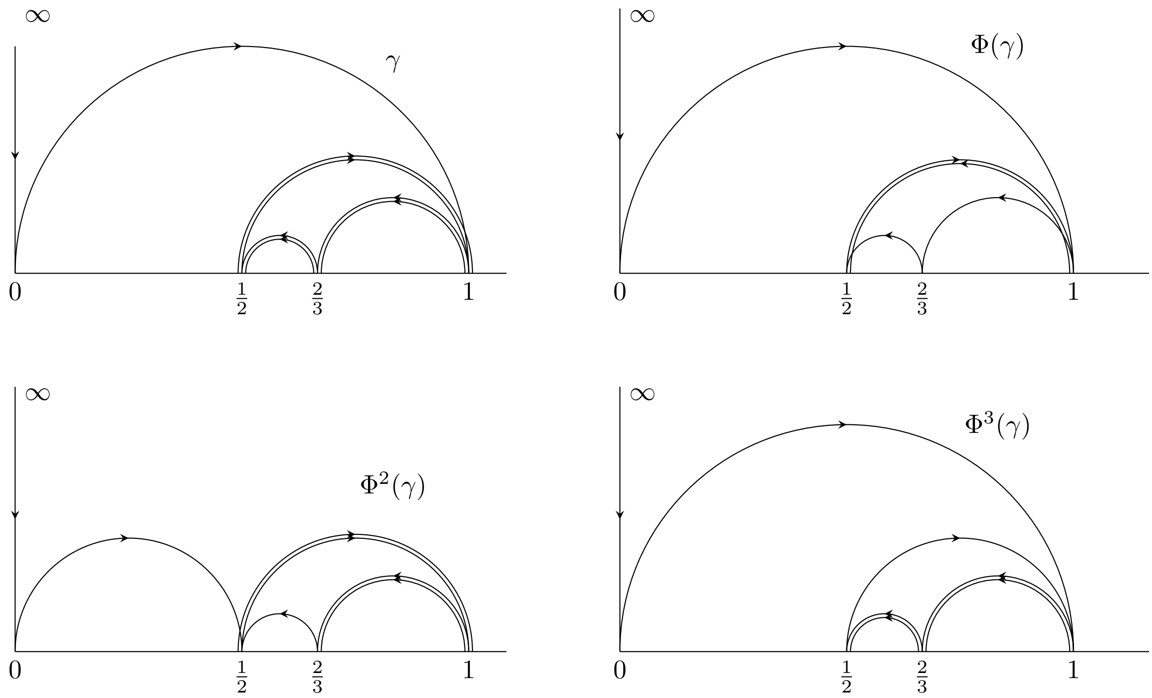

Figure 5.6.2: The paths γ, $\Phi(\gamma), \Phi^2(\gamma)$ and $\Phi^3(\gamma)$ from Example 5.6.4.

We see that the paths $\Phi^3(\gamma)$ and $\Phi^4(\gamma)$ repeat the paths γ and $\Phi(\gamma)$, and that of $\Phi^5(\gamma)$ will repeat that of $\Phi^2(\gamma)$. All these paths have the same two initial convergents, and the paths are periodic paths through $1/2$, 1 and $2/3$. So γ goes infinitely often through these three convergents, and does not converge.

Chapter 6

Further work

Most of the work detailed in previous chapters has been set out in [57] and [63]. In this chapter we describe other recent work which could be developed further.

6.1 Random walks on Farey maps

Our work on spectra detailed in Chapter 4 was motivated by the theory of random walks on graphs, and that of families of expander graphs. Here we explore this connection further.

A *random walk* on a graph is a path which consists of a succession of random steps between adjacent vertices. At each vertex, there is an equal probability of the next step being to each of its neighbours.

Let the N vertices of a graph be $v_1, \ldots, v_N$, and let $p_i(t)$ be the probability that the walker is at vertex v_i after t steps. Then we define the *probability distribution* after t steps as

$$\mathbf{p}(t) = (p_1(t), \ldots, p_N(t))^T.$$

Note that $t \in \mathbb{N}$, and that the distribution is an N-dimensional vector. Since the walker must be at one of the vertices, we have $\Sigma_{i=1}^{N} p_i(t) = 1$, justifying the use of the word distribution. The initial distribution of a walk starting at v_1 is $\mathbf{p}(0) = (1, 0, \ldots, 0)^T$. After one step on a regular graph with vertex valency d the walker is at one of the d neighbours of v_1 with probability $1/d$, so that $\mathbf{p}(1) = \frac{1}{d}A\mathbf{p}(0)$, where A is the adjacency matrix of the graph. We define the *probability matrix* of the graph as $P = \frac{1}{d}A$. Then

$$\mathbf{p}(t) = P^t\mathbf{p}(0).$$

The distribution in which the probability of the walker being at any vertex is the same,which is called the *uniform distribution*, is $\boldsymbol{\pi} = \frac{1}{N}(1, 1, \ldots, 1)^T$. Suppose that a graph is regular with vertex valency d, and that the distribution is uniform after t steps. Then, as each row of A contains exactly d entries equal to 1, after $t + 1$ steps the distribution is

$$\mathbf{p}(t+1) = \frac{1}{N}P(1, 1, \ldots, 1)^T \quad = \quad \frac{1}{dN}A(1, 1, \ldots, 1)^T = \boldsymbol{\pi} \tag{6.1.1}$$

If, for instance, the graph represents an electrical circuit, random walks model the journey of electrons, after a switch has been turned on, to a light bulb elsewhere in the circuit. For the light bulb not to flicker, the probability of an electron reaching the bulb should not change with time (and practical experience tells us that this seems to happen either immediately or after a short time). A distribution which does not change with time is a *stationary distribution*. We say that the probability distribution *converges* to a stationary distribution $\boldsymbol{\sigma}$ if, using the l_2 norm, $||\mathbf{u}|| = (\mathbf{u}^T\mathbf{u})^{1/2}$,

$$\lim_{t\to\infty} ||\mathbf{p}(t) - \boldsymbol{\sigma}|| = 0.$$

From equation (6.1.1), for a regular graph, the uniform distribution $\boldsymbol{\pi}$ does not change with time, and so $\boldsymbol{\sigma} = \boldsymbol{\pi}$, and the stationary distribution is the uniform distribution.

Given a small value δ, which we will call the *tolerance*, the *mixing time* for a circuit is the number of steps t needed so that $||\mathbf{p}(t) - \boldsymbol{\sigma}|| < \delta$. For instance, in case of a light bulb which has been switched on, if the tolerance δ is the shortest time interval in which the human eye can detect that a light is flickering, the mixing time, measured in a unit equal to the time taken by an electron to travel between the vertices or nodes of the circuit, is the time before the light stops flickering. It is desirable that the mixing time be as small as possible for a given δ. For a circuit represented by a regular graph, a well–known classical result relates the mixing time to the eigenvalue of the graph with the second highest modulus, λ. From [45, Theorem 3.1], for a random walk on a non–bipartite d-regular graph, if θ is the eigenvalue with the second largest modulus of the probability matrix of the graph, then after t steps

$$||\mathbf{p}(t) - \boldsymbol{\pi}|| \quad \leq \quad |\theta|^t, \tag{6.1.2}$$

For a regular graph $\theta = \dfrac{\lambda}{d}$. From the Perron-Frobenius Theorem (see [22, Section 8.8]) the highest eigenvalue of the graph is the single eigenvalue $\lambda_0 = d$, so $|\theta| < 1$, and therefore the probability distribution converges to the uniform distribution. For the underlying graph of a Farey map $\mathcal{M}_3(n)$, using Lemma 4.7.1, we have, in addition, the following theorem, showing that the mixing time is relatively short.

Theorem 6.1.1. *For a given a tolerance δ, the mixing time t for a random walk on the underlying graph of a Farey map $\mathcal{M}_3(n)$ can be found, if p_1 is the smallest prime dividing n, from*

$$\delta \leq C^t, \quad \text{with } C \quad = \quad \begin{cases} \frac{1}{2} & \text{if} \quad p_1 = 2, \\ \frac{1}{3} & \text{if} \quad p_1 = 3 \\ \frac{1}{\sqrt{p_1}} & \text{otherwise.} \end{cases} \tag{6.1.3}$$

We are not aware of this result elsewhere in the literature. It shows that an even value of n will give the best result for the mixing time.

Example 6.1.2. If n is even, we find the number of steps t after which the probability of a random walk on the underlying graph of $\mathcal{M}_3(n)$ not reaching its stationary distribution is 10^{-k}.

Here $\delta = 10^{-k}$, so, from (6.1.3), $-k \leq -t \log_{10} 2$, so $t \geq \dfrac{1}{\log_{10} 2} k$, which gives $t \geq 3.33k$.

So, to find the number of steps after which a random walk on the underlying graph of $\mathcal{M}_3(50)$ will be at any vertex with equal probability with 99% confidence, we have $k = 2$, which gives, since $t \in \mathbb{N}$, $t \geq 7$.

6.2 Continued fractions over $\mathbb{Z}/n\mathbb{Z}$

In this section we consider continued fractions with coefficients in $\mathbb{Z}/n\mathbb{Z}$ and their convergents. Recall that the set of Farey fractions modulo n is the set of equivalence classes

$$F = \{(a', c') \in \mathbb{Z} \times \mathbb{Z} : \gcd(a', c', n) = 1,\ (a', c') \equiv \pm(a, c) \pmod{n}\}.$$

We write these fractions as $[a/c]_n$, or as the column vector $[a\ c]_n^T$. The convergents of continued fractions over $\mathbb{Z}/n\mathbb{Z}$ can take any value in this set of Farey fractions. We show that they quickly reach a uniform distribution in that set. We obtain this result by again considering integer continued fractions as walks on the Farey graph.

Using the notation for members of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ given in Section 3.2, for any $b_i \in \mathbb{Z}/n\mathbb{Z}$ we define $s_i \in \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ as $s_i = \begin{bmatrix} b_i & -1 \\ 1 & 0 \end{bmatrix}_n$, and, for any ordered list $b_0, b_1, \ldots, b_k$ of members of $\mathbb{Z}/n\mathbb{Z}$, we define $S_k = s_0 \circ s_1 \circ \cdots \circ s_k$.

Definition 6.2.1. A finite *negative continued fraction expression over* $\mathbb{Z}/n\mathbb{Z}$ consists of a finite ordered set of coefficients in $\mathbb{Z}/nZ$, $[b_o, b_1 \ldots, b_m]$, and an ordered pair $[p/q]_n \in F$ such that

$$\begin{bmatrix} b_0 & -1 \\ 1 & 0 \end{bmatrix}_n \begin{bmatrix} b_1 & -1 \\ 1 & 0 \end{bmatrix}_n \cdots \begin{bmatrix} b_m & -1 \\ 1 & 0 \end{bmatrix}_n \begin{bmatrix} 1 \\ 0 \end{bmatrix}_n = \begin{bmatrix} p \\ q \end{bmatrix}_n .$$

For $k < m$, let

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix}_n = \begin{bmatrix} b_1 & -1 \\ 1 & 0 \end{bmatrix}_n \cdots \begin{bmatrix} b_k & -1 \\ 1 & 0 \end{bmatrix}_n \begin{bmatrix} 1 \\ 0 \end{bmatrix}_n .$$

We say that $[p_k/q_k]_n$ is the k^{th} *convergent* of the continued fraction $[b_1, \ldots, b_m]$.

Theorem 6.2.2. *Let $v_0, v_1, \ldots, v_k$ be Farey fractions modulo n. Then they are the consecutive convergents of a negative continued fraction over $\mathbb{Z}/n\mathbb{Z}$ if and only if $\langle \infty, v_1, \ldots, v_k\rangle$ is a path on the Farey map $\mathcal{M}_3(n)$.*

Proof. With the notation we have defined, the proof of Theorem 5.3.1 can be replicated, replacing ∞ by the Farey fraction $[1/0]_n$, 0 by the Farey fraction $[0/1]_n$, and using the definitions we have stated above for s_n and S_n. $\square$

So the members of F are both Farey fractions and vertices of $\mathcal{M}_3(n)$. We showed in Chapter 3 that they can be written $[a/c]_n$, with $a, c \in \mathbb{Z}$, and $\gcd(a, c, n) = 1$.

The definition of a finite continued fraction over $\mathbb{Z}/n\mathbb{Z}$ can be extended in the obvious way to define an infinite continued fraction. From Theorem 6.1.1, using the notation given in the previous section, we have the following corollary.

Corollary 6.2.3. *The set of convergents of a continued fraction over $\mathbb{Z}/n\mathbb{Z}$ is a finite set V of vertices of $\mathcal{M}_3(n)$. Let $|V| = N$. Then the convergents will reach a stationary distribution in which the probability of any convergent being a specific member v of V is $1/N$. The probability of the k^{th} convergent being at v differs from $1/N$ by C^k, where, if p_1 is the smallest prime divisor of n,*

$$C = \begin{cases} \frac{1}{2} & \text{if} \quad p_1 = 2, \\ \frac{1}{3} & \text{if} \quad p_1 = 3 \\ \frac{1}{\sqrt{p_1}} & \text{otherwise.} \end{cases}$$

6.3 An expander–type result for Farey maps

The study of *expanders* deals with families of regular graphs of increasing size with the same degree d. If the graph represents a communication circuit, a useful measure of its reliability is the *isoperimetric constant*, which is a measure of the relative number of cuts to edges of a connected graph which result in it becoming disconnected. It is formally defined, for a graph X with vertex set V, as

$$h(X) = \min \left\{ \frac{|E(A, \overline{A})|}{|A|} : A \subset V, |A| \leq \frac{|V|}{2} \right\},$$

where $\overline{A}$ is the complement of A in X, and $E(A, \overline{A})$ is the set of edges of X connecting vertices in A to vertices in $\overline{A}$. If the isoperimetric constant of a sequence of graphs does not tend to zero as the size of the grapsh increases it is possible to use it to construct large reliable networks. A sequence of d-regular graphs$\{X_n\}$ is an *expander* family if there is a positive number ε such that the isoperimetric constant $h(X_k)$ is greater than ε for all k.

It is well known that, if $\lambda_1(X)$ is the second largest eigenvalue of X, then $d - \lambda_1(X) \leq 2h(X)$. (See for instance [40, Proposition 1.84].) We call $d - \lambda_1(X)$ the *spectral gap* of the graph, and the sequence of graphs is an expander family if the spectral gap is bounded below. From [40, Proposition 3.1] we have

$$\liminf_{n\to\infty} \lambda_1(X_n) \geq 2\sqrt{d-1}.$$

If the graph represents a circuit with a good mixing time, we need the eigenvalue with the second largest modulus, λ, to be as small as possible, and the Alon-Boppana theorem gives the further result

$$\liminf_{n\to\infty} \lambda(X_n) \geq 2\sqrt{d-1}.$$

(See for instance [40, Proposition 3.6] or[46, Proposition 4.2]). So a graph with good communication properties will have $\lambda \leq 2\sqrt{d-1}$. We call such graphs Ramanujan graphs.

In [4] it is shown that if $S \subset \mathrm{SL}_2(\mathbb{Z})$, and S is Zariski dense in $\mathrm{SL}_2(\mathbb{Z})$, then the Cayley graphs of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ with respect to the projection of S in $\mathbb{Z}/n\mathbb{Z}$ form a family of expanders. It follows that if $S \subset \mathrm{PSL}_2(\mathbb{Z})$ is Zariski dense in $\mathrm{PSL}_2(\mathbb{Z})$, then the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ with respect to the projection of S in $\mathbb{Z}/n\mathbb{Z}$ form a family of expanders. Now $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the set $S = \{x', y'\}$, where $x' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. S is Zariski dense in $\mathrm{PSL}_2(\mathbb{Z})$. The natural projection of $\{x', y'\}$ onto $\mathbb{Z}/n\mathbb{Z}$ is $\{x, y\}$, where $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_n$ and $y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_n$. This proves that the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ with generating set $\{x, x^{-1}, y\}$ form a family of expanders.

We use this to obtain a result for Farey maps of different levels. As they are not of equal degree they do not form an expander family, but they do have similar properties. The following theorem shows that a circuit which is represented by the underlying graph of a Farey map is very strongly connected.

Theorem 6.3.1. *If the isoperimetric constant of the Farey map $\mathcal{M}_3(n)$, is $h(\mathcal{M}_3(n))$, then there is a constant ε independent of n such that $h(\mathcal{M}_3(n)) > \varepsilon n$.*

Proof. The supporting graphs of the family of truncations $T(\mathcal{M}_3(n))$ of the maps $\mathcal{M}_3(n)$ are, from Theorem 3.6.4, the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. These form a family of expanders, so that, if X is a subset of vertices of $T(\mathcal{M}_3(n))$, we have, for all n, $\dfrac{|E(X,\overline{X})|}{|X|} > \varepsilon$ for some positive number ε. Assume that X consists of the union of the sets of n truncation vertices around each map vertex in some set V of map vertices. The number of map edges incident to a map vertex is n, and the only edges leaving X are those from some of the n truncation points around each map vertex. So, $|E(X,\overline{X})| \leq |E(V,\overline{V})|$ and therefore we have, since $|X| = n|V|$,

$$\frac{|E(V,\overline{V})|}{|V|} \geq \frac{n|E(X,\overline{X})|}{|X|}, \text{ so } \frac{|E(V,\overline{V})|}{|V|} > \varepsilon n, \text{ and } h(\mathcal{M}_3(n)) > \varepsilon n.$$

□

We recall that the underlying graphs of Farey maps also have diameter 3, so if they could be realised as communication networks they would be both fast and secure. This does not, however, seem to be practical because of the relatively large number of connections needed.

6.4 Generalised regular continued fraction expansions

There have been several studies of generalisations of the regular continued fraction expansions first introduced by Wallis, Lord Brouncker, and Euler (see for instance [18]). Here we consider continued fractions of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{6.4.1}$$

with $a_i, b_i \in \mathbb{N}$, and b_0 can be zero.

We summarise recent results in the literature, set out recent work we have undertaken with Ian Short, and consider possibilities for further investigation.

We first list some known results.

(1) In [42], it is shown that for any sequence of positive integers (a_n) and any positive $x \in \mathbb{R}$, there exists a finite or infinite sequence of integers (b_n) for $n \geq 0$ with $b_i \geq a_i$ for $i \geq 1$, and $b_0 = \lfloor x \rfloor$, such that x can be written as (6.4.1).

(2) In [8] it is pointed out that, for any positive $x \in \mathbb{R}$, given its regular continued fraction expansion $[b_0, b_1, \ldots]$, for any $N \in \mathbb{N}$, we also have

$$x = b_0 + \cfrac{N}{Nb_1 + \cfrac{N}{b_2 + \cfrac{N}{Nb_3 + \cdots}}} \tag{6.4.2}$$

thus showing that for any positive $x \in \mathbb{R}$ and any $N \in \mathbb{N}$ there is a generalised continued fraction expansion with $a_n = N$ for all n.

(3) In [1] the case $a_n = N$ is explored further. A form of the Euclidian continued fraction algorithm is used to allow at each i^{th} iteration the choice of an integer b_i such that

$$x_i - N \leq b_i \leq \lfloor x_i \rfloor,$$

and so to show that, for $N \geq 2$, every positive irrational has infinitely many generalised continued fraction expansions with $b_n = N$. [1] also defines a *best expansion*, obtained by putting $b_i = \lfloor x_i \rfloor$, and shows that for a best expansion $b_i \geq N$.

(4) In [9] it is pointed out that the best expansion can be obtained by using the transformation

$$T_N : [0, N] \to [0, N] : T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, x \neq 0, T_N(0) = 0,$$

which then gives

$$b_1(x) = \left\lfloor \frac{N}{x} \right\rfloor; \quad b_n(x) = \left\lfloor \frac{N}{T_N^{n-1}(x)} \right\rfloor.$$

(5) In [65] the Seidel-Stern theorem is recalled. It states that the generalised continued fraction

given by (6.4.1) converges if and only if

$$b_1 + b_2\frac{1}{a_1} + b_3\frac{a_1}{a_2} + b_4\frac{a_2}{a_1a_3} + b_5\frac{a_1a_3}{a_2a_4} + b_6\frac{a_2a_4}{a_1a_3a_5} + \cdots = +\infty.$$

which is true if (a_n) is a bounded sequence. Another version of the Euclidean continued fraction algorithm is then used to find expansions for $b_n = N$, showing that any positive $x \in \mathbb{R}$ has infinitely many such expansions.

(6) In [53] a way to find interesting expansions using a computer algorthim is suggested, which is curently being implemented by the website "Ramanujan's Machine".

In joint work with Ian Short, we develop the idea of choice functions for continued fractions introduced by Dani and Nogueira [14] and Dani [13]. They worked with Gaussian integer continued fractions. Here we focus on integer continued fractions.

Definition 6.4.1. Let (a_n) be a sequence of positive integers for which

$$\frac{1}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_1a_3} + \frac{a_1a_3}{a_2a_4} + \frac{a_2a_4}{a_1a_3a_5} + \cdots = +\infty.$$

A *choice function* for (a_n) is a function $\beta\colon \mathbb{N} \times [1, +\infty) \longrightarrow \mathbb{N}$ for which $0 \leqslant x - \beta_n(x) < a_n$ for all $n \in \mathbb{N}$ and $x \in [1, +\infty)$, with equality in the left-hand inequality when x is an integer.

Here we have written $\beta_n(x)$ for $\beta(n, x)$. A consequence of the inequality $x - \beta_n(x) \geqslant 0$ is that $\beta_n(x) \leqslant x$. Since β is integer valued $\beta_n(x) \leqslant \lfloor x \rfloor$. So, for example, $\beta_n(x) = 1$ for $x \in [1, 2)$.

We now describe our choice continued fraction algorithm, which is similar to Euclid's algorithm. For convenience we use the language of Möbius transformations.

Definition 6.4.2. Let β be a choice function for a sequence (a_n). The *β-algorithm* is the following algorithm, which from a positive number $w \geqslant 1$ generates a sequence of Möbius transformations $t_n(x) = b_n + a_n/x$, for $n = 1, 2, \ldots$, where $b_n \in \mathbb{N}$. The sequence may be finite or infinite. We use the notation $T_n = t_1 \circ t_2 \circ \cdots \circ t_n$.

(1) Define $b_1 = \beta_1(w)$ and $t_1(x) = b_1 + a_1/x$. Stop the algorithm if $T_1^{-1}(w) = \infty$; otherwise proceed to (2).

(2) Suppose that $t_1, t_2, \ldots, t_{n-1}$ have been constructed, where $n \geqslant 2$. Define $b_n = \beta_n(T_{n-1}^{-1}(w))$ and $t_n(x) = b_n + a_n/x$. Stop the algorithm if $T_n^{-1}(w) = \infty$; otherwise repeat (2).

Observe that $T_n^{-1}(w) \neq \infty$ unless t_n is the last transformation in the sequence.

By applying Euclid's algorithm to a positive number $w \geqslant 1$, we obtain the usual simple or regular continued fraction expansion of w. In a similar way, when we apply the β-algorithm to w, we obtain a β-dependent continued fraction expansion of w.

Definition 6.4.3. Given a choice function β for a sequence (a_n), and $w \geqslant 1$, we define the *β-continued fraction expansion* of w to be the continued fraction

$$b_1 + \cfrac{a_1}{b_2 + \cfrac{a_2}{b_3 + \cfrac{a_3}{b_4 + \cdots}}},$$

where b_n are the positive integers that result from applying the β-algorithm to w. The continued fraction may be finite or infinite.

From the Seidel–Stern theorem we know that if the β-continued fraction expansion of w is infinite, then it converges.

We define integers $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ by the equations

$$\begin{pmatrix} A_n & a_n A_{n-1} \\ B_n & a_n B_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 & a_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & a_2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix},$$

for $n = 1, 2, \ldots$. In particular, $A_0 = 1$, $B_0 = 0$, $A_1 = b_1$, $B_1 = 1$, and otherwise $A_n, B_n \in \mathbb{N}$. The composition of matrices on the right corresponds to the composition $t_1 \circ t_2 \circ \cdots \circ t_n$. Thus

$$T_n(x) = \frac{A_n x + a_n A_{n-1}}{B_n x + a_n B_{n-1}}, \quad \text{so} \quad T_n^{-1}(x) = -a_n \left(\frac{B_{n-1}x - A_{n-1}}{B_n x - A_n} \right).$$

Suppose that the β-expansion of $w \geqslant 1$ is finite. Then $T_n^{-1}(w) = \infty$, for some n, so $w = T_n(\infty)$

and hence w has a finite β-continued fraction expansion. In this case w is rational. However, if w is rational, then it need not have a finite β-expansion as can be seen from example 6.4.6.

Theorem 6.4.4. *Suppose that the β-continued fraction expansion of $w \geqslant 1$ is infinite. Then that continued fraction converges to w.*

Proof. We have $T_n(\infty) = A_n/B_n$, the nth convergent of the continued fraction. By Theorem 1.1.4, the sequence (A_n/B_n) converges.

Now $b_n = \beta_n(T_{n-1}^{-1}(w))$, so, by definition, $0 < T_{n-1}^{-1}(w) - b_n < a_n$. Hence

$$T_n^{-1}(w) = t_n^{-1}(T_{n-1}^{-1}(w)) = \frac{a_n}{T_{n-1}^{-1}(w) - b_n} > 1.$$

Observe that

$$T_n^{-1}(w) = -a_n \left(\frac{B_{n-1}w - A_{n-1}}{B_n w - A_n} \right) = -\frac{a_n B_n}{B_{n-1}} \left(\frac{w - A_{n-1}/B_{n-1}}{w - A_n/B_n} \right).$$

Since $-a_n B_n/B_{n-1}$ is always negative, we see that $w - A_{n-1}/B_{n-1}$ and $w - A_n/B_n$ differ in sign. Consequently, w must lie between A_{n-1}/B_{n-1} and A_n/B_n. So $A_n/B_n \to w$ as $n \to \infty$. $\square$

Example 6.4.5. Choose $a_n = 1$ for all n and $\beta_n(x) = \lfloor x \rfloor$. Then β is a choice function for (a_n), and the algorithm results in the usual regular continued fraction. In fact, if $a_n = 1$ for all n, then we have $0 \leqslant x - \beta_n(x) < 1$ for all x, so $\beta_n(x) = \lfloor x \rfloor$ is the only possible choice function for the sequence (a_n).

Example 6.4.6. Choose $a_n = 2$ for all n and $\beta_n(x) = \lfloor x \rfloor$. Then β is a choice function for (a_n). For example, as can be seen from Appendix B, putting $w = 3.1$ gives

$$3.1 = 3 + \cfrac{2}{19 + \cfrac{2}{2 + \cdots}},$$

and, putting $w = 5.684$,

$$5.68 = 5 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{16 + \cdots}}}.$$

Example 6.4.7. Suppose that (a_n) is the positive integer sequence $2, 2, 3, 4, 5, \ldots$. Choose $\beta_n(x) = \lfloor x \rfloor$. Then, applying the β-algorithm to the real number $w = e$, we obtain the following β-continued fraction expansion, as shown in Appendix B.

$$e = 2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{3}{5 + \cdots}}}.$$

Example 6.4.8. As shown in Appendix B, putting $a_n = 2$ for all n and $w = \sqrt{2}$ we obtain

$$\sqrt{2} = 1 + \cfrac{2}{4 + \cfrac{2}{2 + \cfrac{2}{4 + \cfrac{2}{2 + \cfrac{2}{4 + \cdots}}}}}$$

In summary, for any positive real number, we can find generalised continued fraction expansions with numerators which are any sequence of positive integers, and also generalised continued fraction expansions for which the denominator is any constant positive integer. Using the choice continued fraction algorithm allows a choice of denominators if the sequence of numerators is bounded. An outstanding problem is that, as far as we are aware, there is no way of determining whether any given sequences of numerators and denominators give a convergent continued fraction expansion.

Attractive continued fraction expansions such as Lord Brouncker's for π and the expansion for e given in Example 6.4.7 have specific sequences for both numerator and denominator. Whereas a number of these have been found over the past 300 years by various mathematicians, it would be interesting to have a general method for generating them, and this could result from further work building on the above results.

Bibliography

[1] M. Anselm and S.H. Weintrab, A generalization of continued fraction, J. Number Theory **131** (2011), no 12, 2442–2460.

[2] T.M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[3] A.F. Beardon, M. Hockman and I. Short, Geodesic continued fractions, Michigan Math. J. **61** (2012), no. 1, 133–150.

[4] J. Bourgain and P. P. Varjú, Expansion in $SL_d(\mathbf{Z}/q\mathbf{Z})$, q arbitrary, Invent. Math. **188** (2012), no. 1, 151–173.

[5] A. Breda D'Azevedo and R. Nedela, Join and intersection of hypermaps, Acta Univ. M. Belii Ser. Math. No. 9 (2001), 13–28.

[6] U. Brehm, Maximally symmetric polyhedral realizations of Dyck's regular map. Mathematika 34 (1987), no. 2, 229–236

[7] A.E. Brouwer and W.H. Haemers, *Spectra of graphs*, Universitext. Springer, New York, 2012

[8] E.B. Burger, J Gell-Redman, R. Kravitz, D. Walton, and N. Yates, Shrinking the period length of continued fractions while still capturing convergents, J. Number Theory **128** (2008), no 1, 144–153.

[9] K. Dajani, C. Kraaikamp and N.D.S. Langeveld, Continued fraction expansions with variable numerators, Ramanujan J.**37** (2015) no. 3, 617–639.

[10] C. Dalfó, M. A. Fiol, M. Miller, J. Ryan and J. Širáň, An algebraic approach to lifts of digraphs, Discrete Appl. Math. **269** (2019), 68–76.

[11] C. Dalfó, M. A. Fiol,, S. Pavlíková and J. Širáň, Spectra and eigenspaces of arbitrary lifts of graphs, Acta Math. Univ. Comenian. (N.S.) **88** (2019), no. 3, 593–600.

[12] C. Dalfó, M. A. Fiol and J. Širáň, The spectra of lifted digraphs, J. Algebraic Combin. **50** (2019), no. 4, 419–426.

[13] S.G. Dani, continued fraction expressions for complex numbers – a general approach, Acta Arith, **171** (2015), no. 4, 355–369.

[14] S.G. Dani and A. Nogueira, Continued fractions for complex numbers and values of binary quadratic forms, Trans. Amer. Mat. Soc. **366**, no 7, 3553–3583.

[15] G. Davidoff, P. Sarnak and A. Valette, *Elementary number theory, group theory, and Ramanujan graphs*, London Mathematical Society Student Texts, 55, Cambridge University Press, Cambridge, 2003.

[16] M. DeDeo, D. Lanphier and M. Minei, The spectrum of Platonic graphs over finite fields, Discrete Math. **307** (2007), no. 9-10, 1074–1081.

[17] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, 228, Springer-Verlag, New York, 2005.

[18] L. Euler, De fractionibus continuis dissertatio, Comm. Acaf. Sci. Petropol **9** (1744), 98-137

[19] L. Euler, De fractiones continuis observationes (1750). Euler Archive–All Works.123.

[20] J. Friedman, On the second eigenvalue and random walks in random d-regular graphs, Combinatorica **11** (1991), no. 4, 331–362.

[21] E. Girondo and G. Gonzalez-Diez, *Introduction to compact Riemann surfaces and dessins d'enfants*, London Mathematical Society Student Texts, 79, Cambridge University Press, Cambridge, 2012.

[22] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, 207, Springer-Verlag, New York, 2001.

[23] P. E. Gunnells, Some elementary Ramanujan graphs, Geom. Dedicata **112** (2005), 51–63.

[24] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, sixth edition, Oxford University Press, Oxford, 2008.

[25] R.A. Horn and C.R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1991.

[26] K. Hu, R. Nedela and N.-E. Wang, Branched cyclic regular coverings over platonic maps, European J. Combin. **36** (2014), 531–549.

[27] K. Hu, G.A. Jones, R. Nedela and N.-E. Wang, Non-abelian almost totally branched coverings over the platonic maps, European J. Combin. **51** (2016), 1–11.

[28] M. Iosifescu and C. Kraaikamp, *Metrical theory of continued fractions*, Mathematics and its Applications, 547, Kluwer Academic Publishers, Dordrecht, 2002.

[29] I. Ivrissimtzis, N. Peyerimhoff and A. Vdovina, Trivalent expanders, $(\Delta - Y)$-transformation, and hyperbolic surfaces, Groups Geom. Dyn. **13** (2019), no. 3, 1103–1131.

[30] I. Ivrissimtzis and D. Singerman, Regular maps and principal congruence subgroups of Hecke groups, European J. Combin. **26** (2005), no. 3-4, 437–456.

[31] I. Ivrissimtzis, D. Singerman and J. Strudwick, From Farey fractions to the Klein quartic and beyond, Ars Math. Contemp. **20** (2021), no. 1, 37–50.

[32] G. A. Jones and D. Singerman, *Complex functions*, Cambridge University Press, Cambridge, 1987.

[33] G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. (3) **37** (1978), no. 2, 273–307.

[34] G. A. Jones and J. Wolfart, *Dessins d'enfants on Riemann surfaces*, Springer Monographs in Mathematics, Springer, Cham, 2016.

[35] S. Katok, Coding of closed geodesics after Gauss and Morse, Geom. Dedicata **63** (1996), no. 2, 123–145.

[36] S. Katok and I. Ugarcovici, Geometrically Markov geodesics on the modular surface, Mosc. Math. J. **5** (2005), no. 1, 135–155.

[37] S. Katok and I. Ugarcovici, Symbolic dynamics for the modular surface and beyond, Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 1, 87–132.

[38] D. Kattan and D. Singerman, The diameter of some Hecke–Farey maps, Albanian Journal of Mathematics **15**(2021), no. 1, 39-60.

[39] S. Khrushchev, Two great theorems of Lord Brouncker and his formula, Math. Intelligencer **32** (2010), no. 4, 19–31.

[40] M. Krebs and A. Shaheen, *Expander families and Cayley graphs*, Oxford University Press, Oxford, 2011.

[41] L. J. Lange, An elegant continued fraction for π, Amer. Math. Monthly **106** (1999), no. 5, 456–458.

[42] W. Leighton, Proper continued fractions, Amer. Math,. Monthly **47**(1940) 274–280.

[43] S. Levy, *The eightfold way*, Mathematical Sciences Research Institute Publications, 35, Cambridge University Press, Cambridge, 1999.

[44] L. Lorentzen and H. Waadeland, *Continued fractions. Vol. 1*, second edition, Atlantis Studies in Mathematics for Engineering and Science, 1, Atlantis Press, Paris, 2008.

[45] L. Lovász and P. Winkler, Mixing of random walks and other diffusions on a graph, in *Surveys in combinatorics, 1995 (Stirling)*, 119–154, London Math. Soc. Lecture Note Ser., 218, Cambridge Univ. Press, Cambridge.

[46] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, Combinatorica **8** (1988), no. 3, 261–277.

[47] A. Malnič, R. Nedela and M. Škoviera, Regular homomorphisms and regular maps, European J. Combin. **23** (2002), no. 4, 449–461.

[48] P. McMullen, B. Monson and A. I. Weiss, Regular maps constructed from linear groups, European J. Combin. **14** (1993), no. 6, 541–552.

[49] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.

[50] B. Nica, Unimodular graphs and Eisenstein sums. J. Algebraic Combin. 45 (2017), no. 2, 423–454.

[51] O. Perron, *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche*, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954.

[52] O. Perron, *Die Lehre von den Kettenbrüchen. Dritte, verbesserte und erweiterte Aufl. Bd. II*, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1957.

[53] G. Raayoni, S. Gottlieb, Y. Manor et al., Generating conjectures on fundamental constants with the Ramanujan Machine. Nature 590 (2021), 67–73.

[54] J.S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[55] E. Schulte and J. M. Wills, Geometric realizations for Dyck's regular map on a surface of genus 3, Discrete Comput. Geom. **1** (1986), no. 2, 141–153.

[56] C. Series, The modular surface and continued fractions, J. London Math. Soc. (2) **31** (1985), no. 1, 69–80.

[57] I. Short and M. Stanier, Necessary and sufficient conditions for convergence of integer continued fractions. Proc. Amer. Math. Soc. 150 (2022), pp. 617–631.

[58] I. Short and M. Walker, Geodesic Rosen continued fractions, Q. J. Math. **67** (2016), no. 4, 519–549

[59] D. Singerman, Universal tessellations, Rev. Mat. Univ. Complut. Madrid **1** (1988), no. 1-3, 111–123.

[60] D. Singerman and J. Strudwick, The Farey maps modulo n, Acta Math. Univ. Comenian. (N.S.) **89** (2020), no. 1, 39–52.

[61] D. Singerman and J. Strudwick, Petrie polygons, Fibonacci sequences and Farey maps, Ars Math. Contemp. **10** (2016), no. 2, 349–357.

[62] D. Singerman and R.I. Syddall, The Riemann surface of a uniform dessin. Beiträge Algebra Geom. **44** (2003), no. 2, 413–430.

[63] M. Stanier, Regular coverings and parallel products of Farey maps, Acta Math. Univ. Comenian. (N.S.) **91** (2022), no. 1, 1–18.

[64] J.A. Stedall, Catching Proteus: The collaborations of Wallis and Brouncker. II. Number problems, Notes Rec. R. Soc. Lond. **54** (3), 317–331 (2000).

[65] T. Törmä, Generalised continued fraction expansions with constant partial denominators, J. Aust. Math Soc. **107**(2019).

[66] S. E. Wilson, Families of regular graphs in regular maps, J. Combin. Theory Ser. B **85** (2002), no. 2, 269–289.

[67] S. E. Wilson, Parallel products in groups and maps, J. Algebra **167** (1994), no. 3, 539–546.

[68] A. Yehudayoff, Proving expansion in 3 steps *ACM SIGACT News* (2012) Vol 43, no 3, 67–84.

Appendix A

Table of spectra of Farey maps

n	Spectrum of $\mathcal{M}_3(n)$(eigenvalues with multiplicities as indices)
1	$\{1\}$
2	$\{(-1)^{(2)}, 2\}$
3	$\{(-1)^{(3)}, 3\}$
4	$\{(-2)^{(2)}, 0^{(3)}, 4\}$
5	$\{(-\sqrt{5})^{(3)}, (-1)^{(5)}, (\sqrt{5})^{(3)}, 5\}$
6	$\{(-3)^{(2)}, (-2)^{(3)}, 1^{(6)}, 6\}$
7	$\{(-\sqrt{7})^{(8)}, (-1)^{(7)}, (\sqrt{7})^{(8)}, 7\}$
8	$\{(-4)^{(2)}, (-\sqrt{8})^{(6)}, 0^{(9)}, (\sqrt{8})^{(6)}, 8\}$
9	$\{(-3)^{(15)}, 0^{(8)}, 3^{(12)}, 9\}$
10	$\{(-5)^{(2)}, (-2\sqrt{5})^{(3)}, (-\sqrt{5})^{(6)}, (-2)^{(5)}, 1^{(10)}, (\sqrt{5})^{(6)}, (2\sqrt{5})^{(3)}, 10\}$
11	$\{(-\sqrt{11})^{(24)}, (-1)^{(11)}, (\sqrt{11})^{(24)}, 11\}$
12	$\{(-6)^{(2)}, (-4)^{(3)}, (-2\sqrt{3})^{(12)}, 0^{(12)}, (2\sqrt{3})^{(12)}, 12\}$
13	$\{(-\sqrt{13})^{(35)}, (-1)^{(13)}, (\sqrt{13})^{(35)}, 13\}$
14	$\{(-7)^{(2)}, (-2\sqrt{7})^{(8)}, (-\sqrt{7})^{(16)}, (-2)^{(7)}, 1^{(14)}, (\sqrt{7})^{(16)}, (2\sqrt{7})^{(8)}, 14\}$
15	$\{(-3\sqrt{5})^{(3)}, (-5)^{(3)}, (-\sqrt{15})^{(24)}, (-3)^{(5)}, (-\sqrt{5})^{(9)}, 1^{(15)}, (\sqrt{5})^{(9)}, (\sqrt{15})^{(24)}, (3\sqrt{5})^{(3)}, 15\}$
16	$\{(-8)^{(2)}, (-4\sqrt{2})^{(6)}, (-4)^{(24)}, 0^{(33)}, 4^{(24)}(4\sqrt{2})^{(6)}, 16\}$
17	$\{(-\sqrt{17})^{(63)}, (-1)^{(17)}, (\sqrt{17})^{(63)}, 17\}$
18	$\{(-9)^{(2)}, (-6)^{(15)}, (-3)^{(24)}, 0^{(24)}, 3^{(30)}, (6)^{(12)}, 18\}$
19	$\{(-\sqrt{19})^{(80)}, (-1)^{(19)}, (\sqrt{19})^{(80)}, 19\}$
20	$\{(-10)^{(2)}, (-4\sqrt{5})^{(3)}, (-2\sqrt{5})^{(42)}, (-4)^{(5)}, 0^{(36)}, 2^{(10)}), (2\sqrt{5})^{(42)}, (4\sqrt{5})^{(3)}, 20\}$
21	$\{(-3\sqrt{7})^{(8)}, (-7)^{(3)}, (-\sqrt{21})^{(48)}, (-3)^{(7)}, (-\sqrt{7})^{(24)}, 1^{(21)}, (\sqrt{7})^{(24)}, (\sqrt{21})^{(48)}, (3\sqrt{7})^{(8)}, 21\}$
22	$\{(-11)^{(2)}, (-2\sqrt{11})^{(24)}, (-\sqrt{11})^{(48)}, (-2)^{(11)}, 1^{(22)}, (\sqrt{11})^{(48)}, (2\sqrt{11})^{(24)}, 22\}$
23	$\{(-\sqrt{23})^{(120)}, (-1)^{(23)}, (\sqrt{23})^{(120)}, 23\}$

n	Spectrum of $\mathcal{M}_3(n)$(eigenvalues with multiplicities as indices)
24	$\{(-12)^{(2)},(-6\sqrt{2})^{(6)},(-8)^{(3)},(-4\sqrt{3})^{(12)},(-2\sqrt{6})^{(24)},(-2\sqrt{2})^{(18)},0^{(60)},(2\sqrt{2})^{(18)},4^{(24)}(4\sqrt{2})^{(6)},(2\sqrt{6})^{(24)},(4\sqrt{3})^{(12)},(6\sqrt{2})^{(6)},24\}$
25	$\{(-5\sqrt{5})^{(3)}(-5)^{(125)},0^{(48)},5^{(120)},(5\sqrt{5})^{(3)},25\}$
26	$\{(-13)^{(2)},(-2\sqrt{13})^{(35)},(-\sqrt{13})^{(70)},(-2)^{(13)},1^{(26)},(\sqrt{13})^{(70)},(2\sqrt{13})^{(35)},26\}$
27	$\{(-9)^{(15)},(-3\sqrt{3})^{(108)},0^{(80)},(3\sqrt{3})^{(108)},9^{(12)},27\}$
28	$\{(-14)^{(2)},(-4\sqrt{7})^{(8)},(-2\sqrt{7})^{(88)},(-4)^{(7)},0^{(72)},2^{(14)}),(2\sqrt{7})^{(88)},(4\sqrt{7})^{(8)},28\}$
29	$\{(-\sqrt{29})^{(195)},(-1)^{(29)},(\sqrt{29})^{(195)},29\}$
30	$\{(-15)^{(2)},(-6\sqrt{5})^{(3)},(-10)^{(3)},(-2\sqrt{15})^{(24)},(-3\sqrt{5})^{(6)},(-2\sqrt{2})^{(18)},(-6)^{(5)},(-2\sqrt{5})^{(9)},(-\sqrt{15})^{(48)},(-\sqrt{5})^{(18)},(-1)^{(30)},(2)^{(15)},(\sqrt{5})^{(18)},(3)^{(10)},(\sqrt{15})^{(48)},(2\sqrt{5})^{(9)},(5)^{(6)},(3\sqrt{5})^{(6)},(6\sqrt{5})^{(3)},30\}$
31	$\{(-\sqrt{31})^{(224)},(-1)^{(31)},(\sqrt{13})^{(224)},31\}$
32	$\{(-16)^{(2)},(-8\sqrt{2})^{(6)},(-8)^{(24)},(-4\sqrt{2})^{(96)},0^{(129)},(4\sqrt{2})^{(96)},8^{(24)}(4\sqrt{2})^{(6)},(8\sqrt{2})^{(6)},32\}$
33	$\{(-11)^{(3)}(-3\sqrt{11})^{(24)},(-\sqrt{33})^{(120)},(-\sqrt{11})^{(72)}(-3)^{(11)},1^{(33)},(\sqrt{11})^{(72)},(\sqrt{33})^{(120)},(3\sqrt{11})^{(24)},33\}$
34	$\{(-17)^{(2)},(-2\sqrt{17})^{(63)},(-\sqrt{17})^{(126)},(-2)^{(17)},1^{(34)},(\sqrt{17})^{(126)},(2\sqrt{17})^{(63)},34\}$
35	$\{(-7\sqrt{5})^{(3)},(-5\sqrt{7})^{(8)},(-7)^{(5)},(-\sqrt{35})^{(192)},(-5)^{(7)},(-\sqrt{7})^{(40)},(-\sqrt{5})^{(21)},1^{(35)},(\sqrt{5})^{(21)}(\sqrt{7})^{(40)},(\sqrt{35}),^{(192)}(3\sqrt{7})^{(8)},(5\sqrt{7})^{(8)},(7\sqrt{5})^{(3)},35\}$
36	$\{(-18)^{(2)},(-12)^{(15)},(-6\sqrt{3})^{(12)},(-6)^{(96)},(0)^{(180)},(6)^{(102)},(6\sqrt{3}),^{(12)}(12)^{(12)},36)\}$
49	$\{(-7\sqrt{7})^{(8)}(-7)^{(511)},0^{(144)},7^{(504)},(7\sqrt{7})^{(8)},49\}$
64	$\{(-32)^{(2)},(-16\sqrt{2})^{(6)},(-16)^{(24)},(-8\sqrt{2})^{(96)},(-8)^{(384)},0^{(513)},(8)^{(384)},(8\sqrt{2})^{(96)},(16)^{(24)},\ (16\sqrt{2})^{(6)},64\}$
81	$\{(-27)^{(15)},(-9\sqrt{3})^{(108)},(-9)^{(972)},0^{(728)},(9)^{(972)},(9\sqrt{3})^{(108)},(27)^{(12)},81\}$
125	$\{(-25\sqrt{5})^{(3)},(-25)^{(125)},(-5\sqrt{5})^{(3000)},0^{(1248)},(5\sqrt{5})^{(3000)},9^{(12)},25^{(120)},(25\sqrt{5})^{(3)},125\}$

Appendix B

Continued fraction expansions

We give print-outs of the results found for the examples given in Section 6.4 using the choice algorithm. The notation is that given in Section 6.4.

Example 6.4.6

Putting $a_n = 2$ for all n, $\beta_n(x) = \lfloor x \rfloor$ and $w = 3.1$ gives:

n	$T_n^{-1}(w)$	a_n	b_n	$T_n(w)$	A_n/B_n
1	20.0	2	3.0	3.64516129032	3.0
2	2.0	2	19.0	3.10471204188	3.10526315789
3	5.62949953421e+13	2	2.0	3.10169491525	3.1
4	9.84615384615	2	5.62949953421e+13	3.1	3.1

The expansion terminates as $T_3^{-1}(w) = \infty$.

Putting $a_n = 2$ for all n, $\beta_n(x) = \lfloor x \rfloor$ and $w = 5.68$ gives:

n	$T_n^{-1}(w)$	a_n	b_n	$T_n(w)$	A_n/B_n
1	2.94117647059	2	5.0	5.35211267606	5.0
2	2.125	2	2.0	5.74626865672	6.0
3	16.0	2	2.0	5.74626865672	5.66666666667
4	5.86406201481e+12	2	16.0	5.67990074442	5.68
5	6.00586510264	2	5.86406201480e+12	5.68	5.68

The expansion terminates as $T_4^{-1}(w) = \infty$.

Example 6.4.7

Putting $(a_n) = (2, 2, 3, 4, 5, \ldots)$, $\beta_n(x) = \lfloor x \rfloor$ and $w = e$ gives:

n	$T_n^{-1}(w)$	a_n	b_n	$T_n(w)$	A_n/B_n
1	2.78442238235	2	2.0	2.73575888234	2.0
2	2.5496467783	2	2.0	2.73575888234	3.0
3	5.45805073079	3	2.0	2.76057272385	2.66666666667
4	8.73265717329	4	5.0	2.71617508002	2.72222222222
5	6.82447423201	5	8.0	2.71821355086	2.71794871795
6	7.27736509768	6	6.0	2.71827938403	2.71832358674
7	25.2374940414	7	7.0	2.71828504954	2.71828036462
8	33.6850556433	8	25.0	2.71828182402	2.71828184189
9	13.1376189488	9	33.0	2.71828182829	2.71828182818
10	72.6644120472	10	13.0	2.71828182845	2.71828182846
11	16.5559911905	11	72.0	2.71828182846	2.71828182846
12	21.5830757837	12	16.0	2.71828182846	2.71828182846
13	22.2955580786	13	21.0	2.71828182846	2.71828182846
14	47.3680166907	14	22.0	2.71828182846	2.71828182846
15	40.7590209279	15	47.0	2.71828182846	2.71828182846
16	21.0797876749	16	40.0	2.71828182846	2.71828182846
17	213.065489296	17	21.0	2.71828182846	2.71828182846
18	274.854074861	18	213.0	2.71828182846	2.71828182846
19	22.2462934733	19	274.0	2.71828182846	2.71828182846
20	81.2039382556	20	22.0	2.71828182846	2.71828182846

Example 6.4.8

Putting $a_n = 2$, $\beta_n(x) = \lfloor x \rfloor, w = \sqrt{2}$ gives

n	$T_n^{-1}(w)$	a_n	b_n	$T_n(w)$	A_n/B_n
1	4.828427124746189	2	1.0	2.414213562373095	1.0
2	2.4142135623730985	2	4.0	1.4530818393219729	1.5
3	4.82842712474615	2	2.0	1.4248894475888325	1.4
4	2.414213562373212	2	4.0	1.4153426780554785	1.4166666666666667
5	4.828427124744825	2	2.0	1.414526684006436	1.4137931034482758
6	2.414213562377074	2	4.0	1.4142467875622948	1.4142857142857144
7	4.828427124699808	2	2.0	1.414222778823099	1.4142011834319526
8	2.4142135625082615	2	4.0	1.4142145404201956	1.4142156862745099
9	4.828427123170575	2	2.0	1.414213833679194	1.4142131979695431
10	2.414213566964775	2	4.0	1.4142135911641038	1.4142136248948696
11	4.828427071221649	2	2.0	1.4142135703596002	1.4142135516460548
12	2.41421371835505	2	4.0	1.4142135632206234	1.4142135642135643
13	4.828425306487962	2	2.0	1.4142135626081958	1.4142135620573204
14	2.4142188611775124	2	4.0	1.414213562398044	1.4142135624272734

The outputs were obtained using the following Python script:

```
import numpy as np
def alpha(n,x):
      return (2*n-1)**2
def beta(n,x):
      if n==1:
             return 3
      if 1<n<5:
             return 6
      else:
             return (x//1)-2*n+5
def poly(n):
      if n<5:
             return 1
      else:
             return 2*n-5
def result(i,w):
      n=1
      x=w
      A=1
      C=0
      B=0
      D=1
      P=1
      a=1
      while n<i+1 :
             E=C
             C=A
             F=D
             D=B
             c=a
             a=alpha(n,x)
             b=beta(n,x)
             p=poly(n)
             A=(b*C)+(c*E)
             B=(b*D)+(c*F)
             T=((x*A)+(a*C))
             S=((x*B)+(a*D))
             P=p*P/1.0
             x=a/(x-b)
             print n,"&",x,"&",a,"&",b,"&",T/S,"&",A/B,"\\"
             n=n+1
```