

VIRTUAL EQUIVARIANT GROTHENDIECK-RIEMANN-ROCH FORMULA

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Received: November 8, 2020

Revised: December 26, 2021

Communicated by Thomas Geisser

ABSTRACT. For a G -scheme X with a given equivariant perfect obstruction theory, we prove a virtual equivariant Grothendieck-Riemann-Roch formula, this is an extension of a result of Fantechi-Göttsche ([FG10]) to the equivariant context. We also prove a virtual non-abelian localization theorem for schemes over \mathbb{C} with proper actions.

2020 Mathematics Subject Classification: Primary 14C15, 14C40; Secondary 14L30, 19L47

Keywords and Phrases: Riemann-Roch theorems, equivariant Chow groups, equivariant K -theory

1 INTRODUCTION

As several moduli spaces that one encounters in algebraic geometry have a well-defined expected dimension, one is interested in constructing a fundamental class of the expected dimension in its Chow group. Interesting numerical invariants like Gromov-Witten invariants and Donaldson-Thomas invariants are obtained by integrating certain cohomology classes over such a fundamental class. This motivated the construction of virtual fundamental classes. Given a moduli space with a perfect obstruction theory, such classes were defined for complex analytic spaces by Li and Tian in [LT98] and in the algebraic sense for Deligne-Mumford stacks by Behrend and Fantechi in [BF97].

Given a Deligne-Mumford stack \mathcal{X} , Behrend and Fantechi construct an algebraic stack $\mathfrak{C}_{\mathcal{X}}$ over \mathcal{X} , called the intrinsic normal cone of \mathcal{X} . The choice of a perfect obstruction theory E^{\bullet} over \mathcal{X} is then used to define a vector bundle stack \mathfrak{E} over \mathcal{X} , which contains $\mathfrak{C}_{\mathcal{X}}$ as a closed substack. The virtual fundamental class associated to E^{\bullet} is a class in the Chow group of \mathcal{X} obtained by intersecting $\mathfrak{C}_{\mathcal{X}}$ with the zero section of \mathfrak{E} . Following a similar construction, Lee in [Le04] defined the virtual structure sheaf in the Grothendieck group of

coherent sheaves of \mathcal{X} . For schemes, the Riemann-Roch transformation provides a natural isomorphism between K -theory of coherent sheaves and Chow groups with rational coefficients and it is natural to ask if there is a relation between the virtual fundamental class and the image of the virtual structure sheaf under the Riemann-Roch transformation. This question goes back to the definition of the virtual fundamental class for quasi-manifolds by Kontsevich (see [Ko95, § 1.42]), where the class is defined via the Riemann-Roch map using the definition of the virtual structure sheaf. In [FG10], Fantechi and Göttsche prove that for a scheme with a perfect obstruction theory, the image of the virtual structure sheaf is indeed the virtual fundamental class twisted by the Todd class of the virtual tangent bundle, thus verifying the formula proposed by Kontsevich. A similar result in the context of dg -manifolds was obtained previously by Ciocan-Fontanine and Kapranov in [CK09].

A theory of Chow groups for algebraic stacks has been developed by various authors. For Deligne-Mumford stacks this was introduced by Vistoli in [Vi89], for quotient stacks by Edidin-Graham in [EG98] and for algebraic stacks which admit stratifications by quotient stacks, this was developed by Kresch in [Kr99]. Quotient stacks which are obtained as the quotient of a scheme (or an algebraic space) with the action of linear algebraic group turn out to be some of the most well understood algebraic stacks as one can approach questions related to them as equivalent problems in equivariant geometry. In this text, we study the virtual classes associated to equivariant perfect (relative) obstruction theories, that is perfect obstruction theory associated to a morphism of stacks $[X/G] \rightarrow [Y/G]$, where the map is induced by a G -equivariant morphism between G -schemes $X \rightarrow Y$. In [EG00], Edidin-Graham define an equivariant Riemann-Roch transformation τ_-^G from equivariant K -theory $G_0(G, -)_{\mathbb{Q}}$ to the equivariant Chow groups $\mathrm{CH}_*^G(-)_{\mathbb{Q}}$. We show that for a scheme with group action and a given equivariant perfect obstruction theory, the virtual structure sheaf and virtual fundamental class satisfy Kontsevich's formula. More precisely, we prove the following:

THEOREM 1.1. *(see Theorem 3.9) Let $X, Y \in \mathbf{Sch}_k^G$ and $\tilde{f} : X \rightarrow Y$ be a G -equivariant morphism such that Y is smooth, G -equivariantly connected and pure dimensional. Suppose that there exists a G -equivariant closed immersion $X \hookrightarrow M$ for some $M \in \mathbf{Sm}_k^G$ such that the assumptions in 3.8 are satisfied. Then for an equivariant perfect relative obstruction theory $E^\bullet \rightarrow L_{X/Y}$ on X with respect to Y :*

$$\tau_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\mathrm{vir}}) = Td^G(T_{[X/Y, E^\bullet]}^{\mathrm{vir}}) \cap [G, X^{\mathrm{vir}}, E^\bullet] \quad (1.1)$$

in $\mathrm{CH}_*^G(X)_{\mathbb{Q}}$.

Here $\mathcal{O}_{[G, X, E^\bullet]}^{\mathrm{vir}}$, $[G, X^{\mathrm{vir}}, E^\bullet]$ and $T_{[X/Y, E^\bullet]}^{\mathrm{vir}}$ denote the virtual structure sheaf, the virtual fundamental class and the virtual tangent bundle associated to the G -equivariant perfect relative obstruction theory E^\bullet . This is an extension of [FG10, Proposition 3.4] to the case of quotient stacks. As an immediate

consequence of the above result, we prove the following virtual equivariant Grothendieck Riemann-Roch theorem which generalizes [FG10, Theorem 3.5].

COROLLARY 1.2. (see Corollary 3.10) *Let X be a G -scheme satisfying the assumptions in 3.8 and let $E^\bullet \rightarrow L_X$ be an equivariant perfect obstruction theory on X and let $\alpha \in K_0(G, X)$.*

1. (Virtual Equivariant Grothendieck-Riemann-Roch). *Let $f : X \rightarrow Y \in \mathbf{Sch}_k^G$ where f is proper and Y is smooth and G -quasi-projective. Then:*

$$\begin{aligned} ch^G(f_*(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})) \cdot Td^G(T_Y) \cap [Y] \\ = f_*(ch^G(\alpha) \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}) \cap [G, X^{\text{vir}}, E^\bullet]) \end{aligned} \tag{1.2}$$

in $CH_*^G(X)_{\mathbb{Q}}$.

2. (Virtual Equivariant Hirzebruch-Riemann-Roch). *If X is a complete G -scheme, then:*

$$\chi_{\text{vir}}^G(X, E^\bullet, \alpha) = \int_{[G, X^{\text{vir}}, E^\bullet]} ch^G(\alpha) \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}). \tag{1.3}$$

There are several interesting examples of perfect obstruction theories on schemes which are equivariant with respect to an underlying group action. For a smooth complex projective toric threefold Y with its natural split torus action, it has been recently shown in the work of Ricolfi that the natural perfect obstruction theories on the Hilbert scheme $\text{Hilb}^n(Y)$ of n -points on Y , and the Quot scheme of length n quotients of a torus equivariant exceptional locally free sheaf on Y are all in fact equivariant with respect to the torus action (see [Ri20, Theorem B, Example 4.6]). On a smooth projective complex curve X , the moduli stack of G -local systems on X for a complex reductive group G is a quotient stack with an equivariant perfect obstruction theory (see e.g. [BZN18]).

The next goal of this paper is to study the relation between the virtual classes of a quotient stack and its inertia stack. The equivariant Riemann-Roch map fails to be an isomorphism even in simple cases of trivial actions. For example, for $X = \text{Spec } \mathbb{C}$ with a trivial $G = \mathbb{Z}/n\mathbb{Z}$ action, while $G_0(G, X)_{\mathbb{Q}}$ is an n -dimensional \mathbb{Q} -vector space $CH_*^G(X)_{\mathbb{Q}}$ is a 1-dimensional \mathbb{Q} -vector space. However this anomaly was rectified by Edidin-Graham in [EG05] for separated quotient Deligne-Mumford stacks over \mathbb{C} by taking complex coefficients and by constructing a Riemann-Roch transformation which maps to the Chow group of the inertia. For a G -scheme X with finite stabilizers, let

$$I\tau_X^G : G_0(G, X)_{\mathbb{C}} \rightarrow CH_*^G(I_X)_{\mathbb{C}},$$

denote this map, where I_X denotes the inertia scheme of X . The map $I\tau_X^G$ factors through the geometric part of the K -theory of the inertia scheme, i.e.,

$$I\tau_X^G : G_0(G, X)_{\mathbb{C}} \xrightarrow{\vartheta_X^G} (G_0(G, I_X)_{\mathbb{C}})_{\text{m}_1} \xrightarrow{\tau_{I_X}^G} CH_*^G(I_X)_{\mathbb{C}},$$

where the map ϑ_X^G is the Atiyah-Segal map and \mathfrak{m}_1 is the augmentation ideal. For finite group actions the Atiyah-Segal map was introduced by Vistoli in [Vi91] and studied by Edidin-Graham in [EG05] and [EG08] in the construction of their Riemann-Roch transformation to the Chow group of the inertia. In [KS20], whose terminology is followed in this paper to describe this map, the second author and Krishna extend the Atiyah-Segal map to higher K -theory for quotient Deligne-Mumford stacks over \mathbb{C} and prove that it is an isomorphism. In Section 4, we describe the image of the virtual structure sheaf under the Atiyah-Segal map. And as a corollary, we give an explicit formula for the virtual structure sheaf under the Riemann-Roch transformation to inertia in terms of Chern characters and Todd classes. This can be seen as a virtually smooth analog of [EG08, Theorem 6.5(c)]. Note that in Theorem 1.3 and Corollary 1.4, all K -groups and Chow groups are taken with complex coefficients.

THEOREM 1.3. (see Theorem 4.9) *Let $X \in \mathbf{Sch}_{\mathbb{C}}^G$ such that G acts properly on X . Suppose that there exists a G -equivariant closed immersion $X \hookrightarrow M$ for some $M \in \mathbf{Sm}_{\mathbb{C}}^G$. Let $E^\bullet \rightarrow L_{X/k}$ be a G -equivariant perfect obstruction theory and let $\tilde{E}^\bullet \rightarrow L_{I_X/k}$ be the induced G -equivariant perfect obstruction theory on I_X (see Lemma 4.5). Then under the Atiyah-Segal isomorphism $\vartheta_X^G : G_0(G, X) \rightarrow G_0(G, I_X)_{\mathfrak{m}_1}$, we have the following:*

$$\vartheta_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = \sum_{\psi \in \Sigma_X^G} \frac{\mathcal{O}_{[G, I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}}{\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}])}$$

Here $N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}$ denotes the virtual conormal bundle of the inertia (see Section 4.3). The above theorem is closely related to the virtual localization theorems for torus actions for Chow groups due to Graber-Pandharipande in [GP99] and for K -theory due to Qu in [Qu18]. The perfect obstruction theory induced on the inertia scheme is obtained using the proof of [GP99, Lemma 1]. Combining the above theorem with Theorem 1.1 we have the following corollary.

COROLLARY 1.4. (see Corollary 4.10) *Under the notations of Theorem 1.3 further suppose assumptions in 3.8 are satisfied. Then we have the following equality in $\text{CH}_*^G(I_X)$:*

$$I\tau_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = \sum_{\psi \in \Sigma_X^G} \text{ch}^G(\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}])^{-1}) Td^G(T_{[I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}) \cap [G, I_X^{\psi, \text{vir}}, \tilde{E}^\bullet].$$

We briefly describe some easy consequences of these results. Let X be a scheme with a perfect obstruction theory $\phi : E^\bullet \rightarrow L_{X/k}$. In [Si04], Siebert shows that the virtual fundamental class in the Chow group is independent of the map ϕ and only depends on X and the class of E^\bullet in $K_0(X)$. As noted in [FMR20], for a scheme with G -action and an equivariant perfect obstruction theory, Siebert’s proof goes through and one has a similar statement for the equivariant virtual

fundamental class. On the other hand in [Th18], Thomas defines a K -theoretic analogue of the Fulton class and shows that the virtual structure sheaf is independent of ϕ and only depends on the class of E^\bullet by showing that the virtual structure sheaf can be calculated in terms of the K -theoretic Fulton class. Note that in Corollary 1.4 the terms $ch^G(\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}]))$ and $Td^G(T_{[I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}})$ are independent of the map ϕ and only depend on $[E^\bullet]$ in $K_0(G, X)$ (see Section 4 for the definitions). Now as $I\tau_X$ is an isomorphism, it follows from the equivariant form of Siebert’s result for equivariant Chow groups and Corollary 1.4 that the virtual structure sheaf $[\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}]$ in $G_0(G, X)_{\mathbb{C}}$ is also independent of the map ϕ .

For a complete G -scheme X with proper action satisfying the assumptions of Corollary 1.4, we can combine the results of Theorem 1.3 and Corollary 1.2 (2) applied to the Riemann-Roch transformation for the inertia stack to obtain the following form of the virtual Kawasaki-Riemann-Roch formula:

$$\chi_{\text{vir}}^G(X, E^\bullet, \alpha) = \int_{[G, I_X^{\text{vir}}, \tilde{E}^\bullet]} ch^G \left(\frac{\alpha_{I_X}}{\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}])} \right) \cdot Td^G(T_{[I_X, \tilde{E}^\bullet]}^{\text{vir}}),$$

for a choice of an equivariant perfect obstruction theory $E^\bullet \rightarrow L_X$ on X and for any $\alpha \in K_0(G, X)$. Here $\tilde{E}^\bullet \rightarrow L_{I_X/k}$ is the induced equivariant perfect obstruction theory on I_X and α_{I_X} denotes the twisted image of α in $K_0(G, I_X)$ and is defined as $\mu(t^{-1}(i^!(\alpha)))$, where $i^!$, t and μ denote the pullback on K -theory (see [EG08, § 5.2]), the twist and the Morita maps, respectively. A virtual Kawasaki-Riemann-Roch formula in terms of the Euler characteristic was obtained earlier by Tonita in [To14], which has applications in studying the quantum K -theory of complex manifolds.

We briefly describe other works in literature that are closely related to the results obtained in this paper. The relationship between the virtual structure sheaf and the virtual fundamental class has been studied by several authors. For torus actions a version of this relationship was established by Thomas in [Th20, Proposition 2.13]. More recently, this was proved by Khan [Kh19, § 3.5] for all quasi-smooth derived algebraic stacks. In the setting of Bredon style homology theories this has been established by Joshua in [Jo07]. A virtual Atiyah-Segal type isomorphism has been established using the theory of θ -stratifications for quasi-smooth derived stacks by Halpern-Leistner (see [Ha18]). A K -theoretic virtual localization theorem for finite group actions has also been parallelly proved in [Gu20].

OUTLINE OF THE PAPER

In Section 2 we first setup the notations and conventions that would be used in the rest of this text. To keep the paper reasonably self contained we recall the definitions of perfect relative obstruction theories and virtual classes associated to them. We also recall the construction of the equivariant Riemann-Roch map from [EG98], as the precise construction plays an important role in the proof

of our main theorems. Section 3 is devoted to the proof of Theorem 3.9. In Section 4 we study the image of the virtual structure sheaf under the Atiyah-Segal map. We recall the definition of the Atiyah-Segal map from [EG08] and [KS20] in Section 4.1. In Section 4.2 we discuss the construction of the induced perfect obstruction theory on the inertia scheme and prove that the Morita isomorphism respects virtual classes. In Section 4.3 we study the image of the virtual structure sheaf under various maps in the definition of the Atiyah-Segal map and finally prove the main theorem of this section which is Theorem 4.9.

ACKNOWLEDGEMENTS

We thank Richard Thomas for his inputs and suggestions. The first author would like to thank Adeel Khan for several useful conversations on the subject. The second author thanks Amalendu Krishna for suggesting him to think about an equivariant analogue of [FG10] and Bumsig Kim for discussions. C.R. was supported by SFB 1085 Higher Invariants, Universität Regensburg. B.S. was supported by KIAS Individual Grant MG062803 at Korea Institute for Advanced Study. We thank the referee for a careful reading of the text and for several comments which helped in improving the exposition.

2 PRELIMINARIES

In this section we recall some preliminaries related to virtual classes and equivariant Chow groups. We first begin by fixing the notations that would be used in the rest of this text.

2.1 NOTATIONS AND CONVENTIONS

In this note we assume that our base is a field k . Let \mathbf{Sch}_k denote the category of separated schemes of finite type over k . \mathbf{Sm}_k shall denote the full subcategory of smooth schemes in \mathbf{Sch}_k . For $X \in \mathbf{Sch}_k$ we let $\mathrm{CH}_j(X)$ denote the j th Chow group of X .

A stack shall mean an algebraic stack of finite type over k . A Deligne-Mumford stack over k will be abbreviated to a \mathcal{DM} -stack. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of stacks, f is called a \mathcal{DM} -type morphism if for any morphism $S \rightarrow \mathcal{Y}$ from an algebraic space S , the fiber product $S \times_{\mathcal{Y}} \mathcal{X}$ is a \mathcal{DM} -stack.

A group scheme G will refer to a linear algebraic group over k , that is a smooth affine group scheme over k . For a group scheme G , let \mathfrak{g} denote the lie algebra of G , which is viewed as a G -representation via the adjoint representation of G . Let \mathbf{Sch}_k^G denote the category of schemes over k with G -action. For any $X \in \mathbf{Sch}_k^G$ there is an associated quotient algebraic stack $\mathcal{X} = [X/G]$, we let BG denote the classifying stack $[\mathrm{Spec} k/G]$ of G . For a (G -equivariant) coherent sheaf V over $\mathrm{Spec} k$, a finite dimensional k -vector space (representation of G over k), by abuse of notation we also use V to denote the corresponding (G -equivariant) vector bundle over $\mathrm{Spec} k$.

For a stack \mathcal{X} , let $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ denote the categories of quasicoherent $\mathcal{O}_{\mathcal{X}}$ -modules and coherent $\mathcal{O}_{\mathcal{X}}$ -modules on \mathcal{X} respectively and let $\mathrm{G}(\mathcal{X})$ denote the Quillen K -theory spectrum of the exact category $\mathrm{Coh}(\mathcal{X})$. For $X \in \mathbf{Sch}_X^G$, we let $\mathrm{Coh}^G(X)$ denote the abelian category of G -equivariant coherent \mathcal{O}_X -modules on X and the K -theory spectrum of $\mathrm{Coh}^G(X)$ is denoted by $\mathrm{G}(G, X)$. $\mathrm{G}_i(\mathcal{X})$ and $\mathrm{G}_i(G, X)$ are defined to be the i th homotopy groups of $\mathrm{G}(\mathcal{X})$ and $\mathrm{G}(G, X)$ respectively, for $i = 0$, $\mathrm{G}_0(\mathcal{X})$ and $\mathrm{G}_0(G, X)$ are therefore the Grothendieck groups of the exact categories $\mathrm{Coh}(\mathcal{X})$ and $\mathrm{Coh}^G(X)$ respectively. Any coherent $\mathcal{O}_{\mathcal{X}}$ -module $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$ (respectively $\mathrm{Coh}^G(X)$) defines an element of $\mathrm{G}_0(\mathcal{X})$ (respectively $\mathrm{G}_0(G, X)$) which is denoted by $[\mathcal{F}]$. For a G -scheme X , descent along the G -torsor $X \rightarrow \mathcal{X} := [X/G]$ induces an equivalence of categories $\mathrm{Coh}^G(X) \xrightarrow{\cong} \mathrm{Coh}(\mathcal{X})$, which induces a homotopy equivalence of spectra $\mathrm{G}(G, X) \xrightarrow{\cong} \mathrm{G}(\mathcal{X})$. $\mathrm{K}(\mathcal{X})$ and $\mathrm{K}(G, X)$ are defined to be the K -theory spectrum of the exact categories of locally free $\mathcal{O}_{\mathcal{X}}$ -modules and G -equivariant locally free \mathcal{O}_X -modules respectively which are again homotopy equivalent. And similar to the case of coherent modules the 0th homotopy groups are the Grothendieck groups of locally free sheaves. There is a natural map of spectra $\mathrm{K}(\mathcal{X}) \rightarrow \mathrm{G}(\mathcal{X})$ which is not in general an equivalence.

For a G -scheme X , let X/G denote the geometric quotient scheme of X by G [GIT, Definition 0.6], provided it exists. Let $m : G \times X \rightarrow X$ denote the map defining the action of G on X . Recall that G is said to act freely on X if the map $(m, pr_X) : G \times X \rightarrow X \times X$ is a closed immersion, where pr_X denotes the projection to X . Suppose that G acts freely on X and the geometric quotient X/G exists as a scheme, then $X \rightarrow X/G$ is a G -torsor (see [GIT, Proposition 0.9]) and therefore we have a canonical equivalence of categories between $\mathrm{Coh}(X/G)$ and $\mathrm{Coh}^G(X)$ which further induces an isomorphism between the Grothendieck groups and we denote by Inv_X^G the natural isomorphism $\mathrm{G}_0(G, X) \rightarrow \mathrm{G}_0(X/G)$.

We note that we follow the cohomological notation for complexes, that is for a complex (E^\bullet, d^\bullet) , we have the differential $d^i : E^i \rightarrow E^{i+1}$. Let $h^i(E^\bullet)$ denote the i^{th} -cohomology sheaf of the complex E^\bullet . For a morphism of complexes $\phi : E^\bullet \rightarrow F^\bullet$, $h^i(\phi)$ will denote the induced map on cohomologies $h^i(\phi) : h^i(E^\bullet) \rightarrow h^i(F^\bullet)$. For any additive category \mathcal{A} , $D(\mathcal{A})$ denotes the derived category of \mathcal{A} and $D^b(\mathcal{A})$ denotes the derived category of bounded complexes on \mathcal{A} . For $E^\bullet \in D^b(\mathrm{Coh}(\mathcal{X}))$ one can naturally define $[E^\bullet] = \sum_i (-1)^i [E^i]$ as an element of $\mathrm{G}_0(\mathcal{X})$ and an element of $\mathrm{K}_0(\mathcal{X})$ if E^\bullet is a complex of locally free sheaves.

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a \mathcal{DM} -type morphism of algebraic stacks then $\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$ will denote the relative cotangent complex of \mathcal{X} over \mathcal{Y} . For notational convenience this would also be denoted by L_f . When $\tilde{f} : X \rightarrow Y$ is a G -equivariant map of G -schemes, the relative cotangent complex is in fact a complex of G -equivariant sheaves and $\mathrm{L}_{X/Y}$ or $\mathrm{L}_{\tilde{f}}$ in this case would refer to this complex as an object of $D(\mathrm{QCoh}^G(X))$. Let $f : \mathcal{X} := [X/G] \rightarrow \mathcal{Y} := [Y/G]$ be the map induced by \tilde{f} , then under the equivalence $D(\mathrm{QCoh}^G(X)) \simeq D(\mathrm{QCoh}(\mathcal{X}))$, the complex $\mathrm{L}_{X/Y}$ corresponds to $\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$.

2.2 EQUIVARIANT CHOW GROUPS

Equivariant Chow groups were defined by Edidin-Graham in [EG98] based on the construction of the Chow groups of classifying spaces by Totaro [To99]. In this subsection we briefly recall the definition of equivariant Chow groups and the construction of the equivariant Riemann-Roch transformation, for further details see [EG98],[EG00]. Some of the definitions are standard and we recall them here primarily to fix notations which would be repeatedly used in what follows.

2.2.1 EQUIVARIANT CHOW GROUPS

Let G be a linear algebraic group of dimension g and let $X \in \mathbf{Sch}_k^G$ be a scheme with $\dim(X) = n$. Let V be an l -dimensional representation of G over k and let U be a G -invariant open subset of V such that G acts freely on U . The diagonal action on $X \times U$ is also free, so there is a quotient in the category of algebraic spaces $X \times U \rightarrow X \times^G U$, which in general may not be a scheme. The pair (V, U) is called an l -dimensional *good pair* for an integer j if $\text{codim}(U, V) > j$ and the geometric quotient U/G exists as a scheme. It follows from [EG98, Lemma 9] a good pair exists for every integer j .

DEFINITION 2.1. [EG98, Definition-Proposition 1] The i th *equivariant Chow group* of X is defined as $\text{CH}_i^G(X) := \text{CH}_{i+l-g}(X \times^G U)$, where (V, U) is an l -dimensional good pair for the integer $n - i$ and CH_p is the usual Chow group. We write $\text{CH}_*^G(X) = \prod_{-\infty < i \leq n} \text{CH}_i^G(X)$.

The definition of equivariant Chow groups in Definition 2.1 is independent of the choice of a good pair (V, U) up to unique isomorphisms as shown in [EG98, Definition-Proposition 1]. Equivariant Chow groups have the same functoriality as ordinary Chow groups for equivariant proper, flat, smooth, regular embedding and l.c.i morphisms.

Remark 2.2. It is shown in [EG98, Proposition 16] that for a G -scheme X the groups $\text{CH}_i^G(X)$ depend only on the stack $[X/G]$ and not on its presentation as a quotient. More precisely, suppose \mathcal{Z} is an algebraic stack which has two presentations $[X/G]$ and $[Y/H]$ as a quotient stack, where $X \in \mathbf{Sch}_k^G$ and $Y \in \mathbf{Sch}_k^H$, for group schemes G and H . Then there is a natural isomorphism $\text{CH}_{i+g}^G(X) \simeq \text{CH}_{i+h}^H(Y)$, where $g = \dim(G)$ and $h = \dim(H)$.

Remark 2.3. Chow groups for more general algebraic stacks have been defined by Kresch in [Kr99]. For \mathcal{X} an algebraic stack, let $A_*(\mathcal{X})$ denote the Chow groups defined in *loc.cit.* In particular, if $\mathcal{X} = [X/G]$ is in fact a quotient stack it follows from [Kr99, Theorem 2.1.12], that there is a natural map $\text{CH}_*^G(X) \rightarrow A_*(\mathcal{X})$ which is a graded isomorphism. The groups $\text{CH}_*^G(X)$ are denoted by $\widehat{A}_*([X/G])$ and are defined as a limit over the naive Chow groups of vector bundles over the stack (see [Kr99, Definition 2.1.4]). The reason the two definitions coincide is due the fact that the limit stabilizes and further

vector bundles which are pullbacks from BG form a cofinal system (see [Kr99, Remark 2.1.7, Remark 2.1.17]).

2.3 EQUIVARIANT RIEMANN-ROCH THEOREM

We let $\tau_{(-)} : G_0(-)_{\mathbb{Q}} \rightarrow CH_*(-)_{\mathbb{Q}}$ denote the Riemann-Roch transformation constructed by Baum-Fulton-Macpherson in [BFM75]. An equivariant analogue of the Riemann-Roch transformation, denoted τ^G , was constructed in [EG00] and for $X \in \mathbf{Sch}_k^G$ it defines a map

$$\tau_X^G : G_0(G, X)_{\mathbb{Q}} \rightarrow CH_*^G(X)_{\mathbb{Q}}, \tag{2.1}$$

from equivariant G -theory to equivariant Chow groups. In this section we recall briefly the construction of this map in terms of the Riemann-Roch map of [BFM75] as this would be key to our proofs of the equivariant virtual Riemann-Roch theorems.

If G acts freely on a scheme X , then there are natural isomorphisms $G_0(G, X) \xrightarrow{\cong} G_0(X/G)$ and $CH_*^G(X) \xrightarrow{\cong} CH_*(X/G)$ and these isomorphisms identify τ_X^G and $\tau_{X/G}$. Although unlike $\tau_{(-)}$, τ_X^G need not be an isomorphism in general.

2.3.1 EQUIVARIANT RIEMANN-ROCH MAP

Let $X \in \mathbf{Sch}_k^G$ be of dimension n , let i be an integer and let (V, U) be an l -dimensional good pair for $n - i$. Let $j : U \hookrightarrow V$ denote the open immersion and let $j_X : X \times U \hookrightarrow X \times V$ denote the open immersion given by base change. Let $\bar{p}_X : X \times V \rightarrow X$ and $p_U : X \times U \rightarrow U$ denote the natural projection maps. As $X \times U \times V$ is a G -equivariant vector bundle over $X \times U$, it descends to define a vector bundle $X \times^G (U \times V)$ over $X \times^G U$. The Todd class $Td(X \times^G (U \times V))$ of the vector bundle $X \times^G (U \times V)$ over $X \times^G U$, defines an invertible map from $CH_*(X \times^G U)_{\mathbb{Q}} \rightarrow CH_*(X \times^G U)_{\mathbb{Q}}$. The equivariant Riemann-Roch map $\tau_X^G : G_0(G, X)_{\mathbb{Q}} \rightarrow CH_*^G(X)_{\mathbb{Q}}$ is determined by defining its projection to $CH_i^G(X)_{\mathbb{Q}}$ for each i . For a given i , the i th component of τ_X^G is defined by the following commutative diagram:

$$\begin{array}{ccccc}
 G_0(G, X \times V)_{\mathbb{Q}} & \xrightarrow{j_X^*} & G_0(G, X \times U)_{\mathbb{Q}} & \xrightarrow[\text{Inv}_{X \times U}^G]{\cong} & G_0(X \times^G U)_{\mathbb{Q}} \\
 \uparrow \bar{p}_X^* & & \nearrow s_U & & \downarrow \left(\frac{\tau_{X \times^G U}}{Td(X \times^G (U \times V))} \right)_{i+l-g} \\
 G_0(G, X)_{\mathbb{Q}} & \xrightarrow{(\tau_X^G)_i} & & & CH_{i+l-g}(X \times^G U)_{\mathbb{Q}} =: CH_i^G(X)_{\mathbb{Q}}.
 \end{array} \tag{2.2}$$

The left vertical map \bar{p}_X^* is an isomorphism by homotopy invariance of G -theory, j_X^* is the flat pullback, and the right vertical map is the $(i + l - g)$ th

projection of the map given by the Riemann-Roch map $\tau_{X \times^G U}$ multiplied by the inverse of $Td(X \times^G (U \times V))$. Let $s_U : G_0(G, X)_{\mathbb{Q}} \rightarrow G_0(X \times^G U)_{\mathbb{Q}}$ denote the composition of maps $\text{Inv}_{X \times U}^G \circ p_X^*$, where $p_X := \bar{p}_X \circ j_X : X \times U \rightarrow X$ is the projection map. Therefore $(\tau_X^G)_i$ is defined as $\left(\frac{\tau_{X \times^G U}}{Td(X \times^G (U \times V))} \right)_{i+l-g} \circ s_U$, and we can define τ_X^G by making an appropriate choice of good pair (V, U) for each i . And it follows from [EG00, Proposition 3.1] that this definition of τ^G is independent of the choices of good pairs.

2.4 PERFECT OBSTRUCTION THEORIES AND VIRTUAL CLASSES

We recall the notion of relative perfect obstruction theory for \mathcal{DM} -type morphisms and the virtual classes with respect to them. The main references are [BF97] and [Ma12].

DEFINITION 2.4. [BF97, Definition 4.4][Ma12, Definition 2.5] Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a \mathcal{DM} -type morphism of algebraic stacks. An *obstruction theory* on the stack \mathcal{X} relative to \mathcal{Y} would consist of the following data:

1. A complex $E^\bullet \in D(\text{QCoh}(\mathcal{X}))$ such that $h^i(E^\bullet) = 0$ for all $i > 0$ and $h^i(E^\bullet)$ is coherent for $i = -1, 0$.
2. A map $\phi : E^\bullet \rightarrow L_f$ in $D(\text{QCoh}(\mathcal{X}))$, where L_f denotes the relative cotangent complex of the stack \mathcal{X} over \mathcal{Y} .
3. $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

DEFINITION 2.5. [BF97, Definition 5.1] An obstruction theory E^\bullet is called a *perfect relative obstruction theory* on \mathcal{X} with respect to \mathcal{Y} if the complex E^\bullet is a perfect complex of amplitude contained in $[-1, 0]$. The *relative virtual dimension* of \mathcal{X} relative to \mathcal{Y} with respect to the perfect relative obstruction theory E^\bullet is defined to be the virtual rank of the perfect complex E^\bullet (which is in general a \mathbb{Z} -valued locally constant function and an integer when \mathcal{X} is connected).

DEFINITION 2.6. Let $\tilde{f} : X \rightarrow Y$ be a G -equivariant map between G -schemes. An *equivariant perfect relative obstruction theory* on X relative to Y is a map $\phi : E^\bullet \rightarrow L_{\tilde{f}}$ in $D(\text{QCoh}^G(X))$, the derived category of G -equivariant quasicoherent sheaves on X , such that forgetting the G -action, the underlying non-equivariant map $\phi : E^\bullet \rightarrow L_{\tilde{f}}$ in $D(\text{QCoh}(X))$ is a perfect relative obstruction theory in the sense of Definition 2.5. In the case when $Y = \text{Spec } k$ is the base scheme, $\phi : E^\bullet \rightarrow L_{\tilde{f}}$ is called an *equivariant perfect obstruction theory* on the G -scheme X . An equivariant perfect relative obstruction theory on X relative to Y is equivalent to a perfect relative obstruction theory on the quotient stack $\mathcal{X} := [X/G]$ relative to the quotient stack $\mathcal{Y} := [Y/G]$ for the representable and hence \mathcal{DM} -type morphism from \mathcal{X} to \mathcal{Y} induced by \tilde{f} and an equivariant perfect obstruction theory on X is the same as a perfect relative obstruction theory on the quotient stack $\mathcal{X} := [X/G]$ relative to BG . Note that

this is therefore different from a perfect obstruction theory on the stack \mathcal{X} even when \mathcal{X} is a \mathcal{DM} -stack.

Remark 2.7. A perfect complex $E^\bullet \in D(\mathrm{QCoh}(\mathcal{X}))$ of perfect amplitude contained in $[a, b]$ is said to admit a *global resolution* if it is (globally on \mathcal{X}) quasi-isomorphic to a complex of the form $F^\bullet = F^a \rightarrow \cdots \rightarrow F^b$, where each F^i is a locally free sheaf. If \mathcal{X} has the resolution property, i.e every quasicohherent $\mathcal{O}_{\mathcal{X}}$ -module is a quotient of a direct sum of locally free sheaves, then every perfect complex on \mathcal{X} has a global resolution. Moreover, if F^\bullet as above is a global resolution of E^\bullet , then the virtual rank of E^\bullet is equal to $\sum_{i=a}^b (-1)^i \mathrm{rank}(F^i)$.

DEFINITION 2.8. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a \mathcal{DM} -type morphism of algebraic stacks with \mathcal{Y} smooth. Let $E^\bullet \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ be a perfect relative obstruction theory on \mathcal{X} relative to \mathcal{Y} . Suppose E^\bullet and $L_{\mathcal{Y}/\mathrm{Spec} k}$ admit global resolutions. We define the virtual relative tangent bundle of \mathcal{X} over \mathcal{Y} corresponding to E^\bullet to be

$$T_{[\mathcal{X}/\mathcal{Y}, E^\bullet]}^{\mathrm{vir}} := [E^{\bullet \vee}] + f^*([L_{\mathcal{Y}/\mathrm{Spec} k}^\vee]) \in K_0(\mathcal{X}).$$

Example 2.9. Under the notations of Definition 2.6 further assume Y is smooth. If E^\bullet admits a global resolution by a bounded complex of equivariant locally free sheaves, then the equivariant virtual tangent bundle corresponding to E^\bullet is given by

$$T_{[X/Y, E^\bullet]}^{\mathrm{vir}} = [E^{\bullet \vee}] + f^*[T_Y] - [\mathfrak{g}_X],$$

where \mathfrak{g}_X denotes the pullback of the Lie algebra of G to X .

For the rest of this section let us fix the following notations. See [BF97, Definitions 3.6, 3.10] for detailed constructions of the intrinsic normal sheaf and the intrinsic normal cone in the absolute case and see [Ma12, Definitions 2.12, 2.14] for the relative case.

SETUP 2.10. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a \mathcal{DM} -type morphism of algebraic stacks, where \mathcal{Y} is a stack of pure dimension. For a perfect complex E^\bullet , let $h^1/h^0(E^{\bullet \vee})$ denote the stack quotient $h^1/h^0(\tau_{[0,1]}(E^{\bullet \vee}))$. Let $\mathfrak{N}_{\mathcal{X}/\mathcal{Y}} := h^1/h^0(L_{\mathcal{X}/\mathcal{Y}}^\vee)$ denote the intrinsic normal sheaf of \mathcal{X} over \mathcal{Y} and let the closed subcone stack $\mathfrak{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \mathfrak{N}_{\mathcal{X}/\mathcal{Y}}$ denote the intrinsic normal cone of \mathcal{X} over \mathcal{Y} . Given a perfect relative obstruction theory $E^\bullet \rightarrow L_{\mathcal{X}/\mathcal{Y}}$, there is an associated vector bundle stack $h^1/h^0(E^{\bullet \vee})$ over \mathcal{X} such that $\mathfrak{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow h^1/h^0(E^{\bullet \vee})$ is a closed immersion (see [Ma12, Proposition 3.11]).

2.4.1 VIRTUAL STRUCTURE SHEAF

Before we get to the definition of the virtual structure sheaf, we recall some facts related to the G-theory of vector bundle stacks.

LEMMA 2.11. *If $\psi : \mathcal{M} \rightarrow \mathcal{X}$ is an affine bundle, then $\psi^* : G(\mathcal{X}) \rightarrow G(\mathcal{M})$ is a homotopy equivalence, and in particular when \mathcal{M} is a vector bundle this establishes the homotopy invariance of G-theory.*

Proof. Associated to the affine bundle $\psi : \mathcal{M} \rightarrow \mathcal{X}$, there exists a short exact sequence of locally free sheaves

$$0 \rightarrow F \rightarrow E \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0$$

on \mathcal{X} such that we have a closed immersion $\mathbb{P}(F) \hookrightarrow \mathbb{P}(E)$ with complement \mathcal{M} . By Quillen's localization theorem and dévissage theorem, we have the following localization sequence for G-theory:

$$G(\mathcal{M}) \rightarrow G(\mathbb{P}(E)) \rightarrow G(\mathbb{P}(F)),$$

and now using the projective bundle formula for G-theory [KR18, Section 3C] we have the required equivalence. \square

PROPOSITION 2.12. *Let $\pi : \mathfrak{F} \rightarrow \mathcal{X}$ be a vector bundle stack on an algebraic stack \mathcal{X} such that \mathfrak{F} has a global presentation as $[E/F]$ for some morphism of vector bundles $F \rightarrow E$ on \mathcal{X} , then the map $\pi^* : G(\mathcal{X}) \rightarrow G(\mathfrak{F})$ is a homotopy equivalence.*

Proof. Since \mathfrak{F} is of the form $[E/F]$ for a morphism of vector bundles $d : F \rightarrow E$ on \mathcal{X} for the (additive) action of F on E induced by d (see [BF97, Section 1]), the map π factors as $\mathfrak{F} \xrightarrow{p} [\mathcal{X}/F] \rightarrow \mathcal{X}$, where \mathcal{X} has trivial F -action and $p : \mathfrak{F} \rightarrow [\mathcal{X}/F]$ is a vector bundle. By homotopy invariance of G-theory for vector bundles Lemma 2.11, it follows that p^* is an equivalence, and therefore it suffices to consider the case when \mathfrak{F} is the classifying stack of a vector bundle. In this case, the structure map $s : \mathcal{X} \rightarrow \mathfrak{F}$ is an affine bundle and $\pi \circ s = id_{\mathcal{X}}$ and so the equivalence again follows from Lemma 2.11. \square

Remark 2.13. If \mathcal{X} is an algebraic stack with resolution property, then every coherent $\mathcal{O}_{\mathcal{X}}$ -module on \mathcal{X} is the quotient of a locally free sheaf. Therefore, by [De73, Proposition 1.4.15], every vector bundle stack on \mathcal{X} has a global presentation as $[E/F]$, for some morphism of vector bundles $F \rightarrow E$ on \mathcal{X} .

DEFINITION 2.14. [Le04, § 2.3, Definition]) Under the notations of Setup 2.10, let $\pi : h^1/h^0(E^{\bullet\vee}) \rightarrow \mathcal{X}$ be the vector bundle stack. The closed embedding $\mathfrak{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow h^1/h^0(E^{\bullet\vee})$ defines a class $[\mathcal{O}_{\mathfrak{C}_{\mathcal{X}/\mathcal{Y}}}]$ in $G_0(h^1/h^0(E^{\bullet\vee}))$. The *virtual structure sheaf* $\mathcal{O}_{[\mathcal{X}, E^{\bullet}]}^{\text{vir}}$ for the perfect (relative) obstruction theory E^{\bullet} is defined to be the unique element in $G_0(\mathcal{X})$ such that $\pi^*(\mathcal{O}_{[\mathcal{X}, E^{\bullet}]}^{\text{vir}}) = [\mathcal{O}_{\mathfrak{C}_{\mathcal{X}/\mathcal{Y}}}]$, which is well defined by Proposition 2.12. If we assume $\mathcal{X} := [X/G]$ is a quotient stack and E^{\bullet} is an equivariant perfect (relative) obstruction theory on X (relative to a G -scheme Y) (see Definition 2.6), the virtual structure sheaf can be naturally realized as an element of $G_0(G, X)$ and we denote it by $\mathcal{O}_{[G, X, E^{\bullet}]}^{\text{vir}}$.

2.4.2 VIRTUAL FUNDAMENTAL CLASS

The virtual fundamental class associated to a (relative) perfect obstruction theory on a stack \mathcal{X} (\mathcal{DM} over a pure dimensional stack \mathcal{Y}) is constructed

in [BF97] by intersecting the intrinsic normal cone with the zero section of the vector bundle stack associated to the obstruction theory to produce a class in $\mathrm{CH}_*(\mathcal{X})_{\mathbb{Q}}$. This was done in [BF97] in the case when the perfect obstruction theory has a global resolution and \mathcal{X} is of \mathcal{DM} -type, and can be done more generally for any perfect obstruction theory [Kr99, Section 5.2] and for any \mathcal{X} which admits a stratification by global quotient stacks [Ma12, Corollary 3.12].

DEFINITION 2.15. Let notations be as in Setup 2.10 and let $\pi : h^1/h^0(E^{\bullet\vee}) \rightarrow \mathcal{X}$ be the vector bundle stack. The intrinsic normal cone defines a closed subscheme and therefore a cycle $[\mathfrak{C}_{\mathcal{X}/\mathcal{Y}}]$ on $h^1/h^0(E^{\bullet\vee})$. Recall that $\pi^* : A_*(\mathcal{X}) \rightarrow A_*(h^1/h^0(E^{\bullet\vee}))$ is an isomorphism (see [Kr99, Theorem 4.3.2]), therefore one can define $[\mathcal{X}^{\mathrm{vir}}, E^{\bullet}] \in A_*(\mathcal{X})$ as the unique element such that $\pi^*([\mathcal{X}^{\mathrm{vir}}, E^{\bullet}]) = [\mathfrak{C}_{\mathcal{X}/\mathcal{Y}}]$.

From [Kr99, Theorem 2.1.12], it follows for a G -scheme X and $\mathcal{X} = [X/G]$, the associated quotient stack, there is a natural map $\psi : \mathrm{CH}_*^G(X) \rightarrow A_*(\mathcal{X})$ which is an isomorphism. We define the virtual fundamental class in $\mathrm{CH}_*^G(X)_{\mathbb{Q}}$ as $\psi^{-1}([\mathcal{X}^{\mathrm{vir}}, E^{\bullet}])$ and denote it by $[X^{\mathrm{vir}}, E^{\bullet}]$. If $\mathcal{Y} = [Y/G]$ is also a quotient stack for a G -scheme Y and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is induced by a G -equivariant map from $X \rightarrow Y$, then we denote $[X^{\mathrm{vir}}, E^{\bullet}]$ by $[G, X^{\mathrm{vir}}, E^{\bullet}]$, where now E^{\bullet} is also used to notate its corresponding G -equivariant perfect obstruction theory.

3 VIRTUAL CLASSES AND THE EQUIVARIANT RIEMANN-ROCH MAP

In this section we derive the equivariant virtual Riemann-Roch formula, which for a G -scheme X and an equivariant perfect (relative) obstruction theory E^{\bullet} on X gives a relation between the image of the virtual structure sheaf under the Edidin-Graham Riemann-Roch map $\tau_X^G(\mathcal{O}_{[G, X, E^{\bullet}]}^{\mathrm{vir}})$ and the virtual fundamental class $[G, X^{\mathrm{vir}}, E^{\bullet}]$ in the Totaro-Edidin-Graham equivariant Chow group $\mathrm{CH}_*^G(X)_{\mathbb{Q}}$ of X .

3.1 INDUCED PERFECT OBSTRUCTION THEORY

The following lemma shows that given an equivariant perfect (relative) obstruction theory on a G -scheme W with finite étale stabilizers (relative to a smooth G -scheme), it descends to a perfect obstruction theory on the associated quotient stack $[W/G]$. This is well known to the experts in the area (see [KP19, Example 4.14]), but we spell out the proof here as the explicit construction of the induced perfect obstruction theory on $[W/G]$ is used in the proof of our main theorem. See Definition 2.6 for the definition of equivariant perfect obstruction theories.

LEMMA 3.1. *Let $\mathcal{W} \xrightarrow{f_1} \mathcal{Y} \xrightarrow{f_2} \mathcal{Z}$ be a morphism of stacks, where \mathcal{Z} is smooth. Further assume that f_2 is smooth and $f_1, f_2 \circ f_1$ are of \mathcal{DM} -type. Then any relative perfect obstruction theory $\phi : F^{\bullet} \rightarrow L_{\mathcal{W}/\mathcal{Y}}$ defines a (non-unique) perfect obstruction theory $\phi' : \tilde{E}^{\bullet} \rightarrow L_{\mathcal{W}/\mathcal{Z}}$ on \mathcal{W} .*

Proof. The composition of maps $\mathcal{W} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ gives the following distinguished triangle of cotangent complexes in $D(\text{QCoh}(\mathcal{W}))$:

$$f_1^*L_{\mathcal{Y}/\mathcal{Z}} \rightarrow L_{\mathcal{W}/\mathcal{Z}} \rightarrow L_{\mathcal{W}/\mathcal{Y}} \xrightarrow{+1} . \tag{3.1}$$

Note that $L_{\mathcal{Y}/\mathcal{Z}}$ is concentrated in degree $[0, 1]$ as f_2 is assumed to be a smooth morphism. Consider the following commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc} \tilde{E}^\bullet & \longrightarrow & F^\bullet & \xrightarrow{a \circ \phi} & f_1^*L_{\mathcal{Y}/\mathcal{Z}}[1] & \xrightarrow{+1} & \\ \downarrow \phi' & & \downarrow \phi & & \downarrow Id & & \\ L_{\mathcal{W}/\mathcal{Z}} & \longrightarrow & L_{\mathcal{W}/\mathcal{Y}} & \xrightarrow{a} & f_1^*L_{\mathcal{Y}/\mathcal{Z}}[1] & \xrightarrow{+1} & , \end{array} \tag{3.2}$$

where $a : L_{\mathcal{W}/\mathcal{Y}} \rightarrow f_1^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ is the map in the distinguished triangle (3.1), \tilde{E}^\bullet is defined as the object in $D(\text{QCoh}(\mathcal{W}))$ that makes the top horizontal diagram a distinguished triangle and the (non-unique) map ϕ' is defined by the axioms for triangulated categories. To prove the lemma we need to show that $\phi' : \tilde{E}^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Z}}$ defines a perfect obstruction theory, that is we need to show that $h^{-1}(\phi')$ is surjective, $h^0(\phi')$ is an isomorphism and \tilde{E}^\bullet is of perfect amplitude contained in $[-1, 0]$.

From (3.2) we obtain the following long exact sequence of cohomology sheaves:

$$\begin{array}{ccccccc} h^{-1}(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) & \longrightarrow & h^{-1}(\tilde{E}^\bullet) & \longrightarrow & h^{-1}(F^\bullet) & \longrightarrow & \\ \downarrow \simeq & & \downarrow h^{-1}(\phi') & & \downarrow h^{-1}(\phi) & & \\ h^{-1}(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) & \longrightarrow & h^{-1}(L_{\mathcal{W}/\mathcal{Z}}) & \longrightarrow & h^{-1}(L_{\mathcal{W}/\mathcal{Y}}) & \longrightarrow & \\ \\ h^0(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) & \longrightarrow & h^0(\tilde{E}^\bullet) & \longrightarrow & h^0(F^\bullet) & \longrightarrow & h^1(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) \\ \downarrow \simeq & & \downarrow h^0(\phi') & & \downarrow \simeq & & \downarrow \simeq \\ h^0(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) & \longrightarrow & h^0(L_{\mathcal{W}/\mathcal{Z}}) & \longrightarrow & h^0(L_{\mathcal{W}/\mathcal{Y}}) & \longrightarrow & h^1(f_1^*L_{\mathcal{Y}/\mathcal{Z}}) \end{array} \tag{3.3}$$

It now follows from the five lemma applied to the above diagram that $h^0(\phi')$ is an isomorphism and the four lemma establishes the surjectivity of $h^{-1}(\phi')$. To prove that \tilde{E}^\bullet is of perfect amplitude contained in $[-1, 0]$ consider the top distinguished triangle of (3.2):

$$\tilde{E}^\bullet \rightarrow F^\bullet \xrightarrow{a \circ \phi'} f_1^*(L_{\mathcal{Y}/\mathcal{Z}})[1] \xrightarrow{+1} . \tag{3.4}$$

Since $f_1^*(L_{\mathcal{Y}/\mathcal{Z}})$ is of perfect amplitude contained in $[0, 1]$ and F^\bullet is of perfect amplitude contained in $[-1, 0]$, it suffices to check that $h^1(\tilde{E}^\bullet)$ vanishes. From

the long exact sequence of cohomology sheaves associated to (3.4), it follows that

$$h^1(\tilde{E}^\bullet) \simeq h^1(L_{\mathcal{Y}/\mathcal{Z}})/\text{Im}(h^0(F^\bullet) \xrightarrow{h^0(a \circ \phi')} h^1(L_{\mathcal{Y}/\mathcal{Z}})),$$

since $h^1(F^\bullet)$ vanishes. Now by considering the bottom distinguished triangle of (3.2) and since $\mathcal{W} \rightarrow \mathcal{Z}$ is a \mathcal{DM} -type morphism (which implies $h^1(L_{\mathcal{W}/\mathcal{Z}})$ vanishes), we conclude that $h^0(a)$ is surjective. Further since $h^0(\phi')$ is an isomorphism as proved above it follows that $h^1(\tilde{E}^\bullet)$ vanishes. \square

Remark 3.2. In Lemma 3.1 we need to assume that the morphisms are of \mathcal{DM} -type only to conclude that the induced obstruction theory is of tor amplitude contained in $[-1, 0]$.

Remark 3.3. As in [GP99, Appendix B] one can make a canonical choice of the perfect obstruction theory $\phi' : \tilde{E}^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Z}}$ in Lemma 3.1. We thank the referee for this observation. However the choice in the lemma does not affect the construction of the virtual classes (see Remark 3.7).

The below construction shows how an equivariant perfect (relative) obstruction theory on a G -scheme X (relative to a smooth G -scheme) can be lifted to a perfect obstruction theory on the scheme $X \times^G U$ for a good pair (V, U) . Refer to Section 2.2.1 for a definition of good pairs.

Construction 3.4. Let $\tilde{f} : X \rightarrow Y \in \mathbf{Sch}_k^G$ where $Y \in \mathbf{Sm}_k^G$ with $\dim(X) = n$. Let $\mathcal{X} := [X/G]$ and $\mathcal{Y} := [Y/G]$ and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be the map induced on the quotient stacks. Fix an integer i and let (V, U) be a be an l -dimensional good pair for $n - i$, and let $\mathcal{V} := [V/G]$ with structure map $q_{\mathcal{V}} : \mathcal{V} \rightarrow \text{BG}$ and let $\mathcal{U} := [U/G]$. We have the following distinguished triangle in $D^b(\text{Coh}(\mathcal{V}))$ given by the composition of maps $\mathcal{V} \rightarrow \text{BG} \rightarrow \text{Spec}(k)$:

$$q_{\mathcal{V}}^* L_{\text{BG}/\text{Spec } k} \rightarrow L_{\mathcal{V}/\text{Spec } k} \rightarrow L_{\mathcal{V}/\text{BG}} \xrightarrow{+1} . \tag{3.5}$$

Note that the cotangent complex of \mathcal{V} , $L_{\mathcal{V}/\text{Spec } k} \simeq [\Omega_{\mathcal{V}/\text{Spec } k} \rightarrow \mathfrak{g}^{\vee}]$ is concentrated in degrees 0 and 1. It follows that $L_{\mathcal{V}/\text{BG}}$ is precisely $\Omega_{\mathcal{V}/\text{Spec } k} \simeq V^{\vee}$ (with the underlying G -action) where we abuse notations and denote the global sections of V also by V .

Consider the following diagram of cartesian squares:

$$\begin{array}{ccccc} \mathcal{X} \times_{\text{BG}} \mathcal{U} & \xrightarrow{f_{\mathcal{U}}} & \mathcal{Y} \times_{\text{BG}} \mathcal{U} & \xrightarrow{\bar{p}_{\mathcal{U}}} & \mathcal{U} \\ \downarrow p_{\mathcal{X}} & & \downarrow p_{\mathcal{Y}} & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} & \longrightarrow & \text{BG}. \end{array} \tag{3.6}$$

We have natural isomorphisms $\mathcal{X} \times_{\text{BG}} \mathcal{U} \simeq [X \times U/G] \simeq X \times^G U$, where the last two terms are isomorphic as the G -action on U is free. As $p_{\mathcal{Y}}$ is flat, this implies that the left cartesian square of (3.6) is Tor-independent. We therefore have a natural isomorphism:

$$L_{\mathcal{X} \times_{\text{BG}} \mathcal{U}/\mathcal{Y}} \xrightarrow{\simeq} p_{\mathcal{X}}^* L_{\mathcal{X}/\mathcal{Y}} \oplus f_{\mathcal{U}}^* L_{\mathcal{Y} \times_{\text{BG}} \mathcal{U}/\mathcal{Y}}. \tag{3.7}$$

As the right square of (3.6) is cartesian and all the maps are flat, it follows that $L_{\mathcal{Y} \times_{\text{BG}} \mathcal{U}/\mathcal{Y}} \simeq \bar{p}_{\mathcal{U}}^* L_{\mathcal{U}/\text{BG}} \simeq \bar{p}_{\mathcal{U}}^* \Omega_{\mathcal{U}/\text{BG}}$. Therefore, we have

$$L_{\mathcal{X} \times_{\text{BG}} \mathcal{U}/\mathcal{Y}} \simeq p_{\mathcal{X}}^* L_{\mathcal{X}/\mathcal{Y}} \oplus p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\text{BG}},$$

where $p_{\mathcal{U}} : \mathcal{X} \times_{\text{BG}} \mathcal{U} \rightarrow \mathcal{U}$ denotes the projection to \mathcal{U} .

Let $E^\bullet \rightarrow L_{\mathcal{X}/\mathcal{Y}}$ be a perfect relative obstruction theory for the map $f : \mathcal{X} \rightarrow \mathcal{Y}$. Then

$$F^\bullet := p_{\mathcal{X}}^* E^\bullet \oplus p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\text{BG}} \rightarrow p_{\mathcal{X}}^* L_{\mathcal{X}/\mathcal{Y}} \oplus p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\text{BG}} \simeq L_{(\mathcal{X} \times_{\text{BG}} \mathcal{U})/\mathcal{Y}} \tag{3.8}$$

defines a perfect relative obstruction theory on $\mathcal{X} \times_{\text{BG}} \mathcal{U}$ over \mathcal{Y} .

Recall that $\mathcal{X} \times_{\text{BG}} \mathcal{U} \simeq X \times^G U$ is an algebraic space and therefore a \mathcal{DM} -stack. Now it follows from Lemma 3.1 applied to the composition of morphisms $\mathcal{X} \times_{\text{BG}} \mathcal{U} \rightarrow \mathcal{Y} \rightarrow \text{BG}$ that $F^\bullet \rightarrow L_{(\mathcal{X} \times_{\text{BG}} \mathcal{U})/\mathcal{Y}}$ induces a perfect obstruction theory:

$$\tilde{E}^\bullet \rightarrow L_{(X \times^G U)/\text{Spec } k}, \tag{3.9}$$

such that \tilde{E}^\bullet fits into a distinguished triangle in $D(\text{QCoh}(X \times^G U))$:

$$p_{\mathcal{X}}^* \circ f^*(L_{\mathcal{Y}/\text{Spec } k}) \rightarrow \tilde{E}^\bullet \rightarrow F^\bullet \xrightarrow{+1} . \tag{3.10}$$

In the rest of this section we study the relation between the virtual classes corresponding to a given equivariant perfect obstruction theory and the induced perfect obstruction theories constructed above. The following proposition shows that the pullback from \mathcal{X} to $\mathcal{X} \times_{\text{BG}} \mathcal{U}$ of the virtual class associated to a perfect relative obstruction theory on \mathcal{X} with respect to \mathcal{Y} is the virtual class associated to the pullback of the perfect relative obstruction theory (as constructed above).

PROPOSITION 3.5. *With notations as in Construction 3.4, let us further assume that \mathcal{Y} is a pure dimensional stack. Then we have the following:*

1. $p_{\mathcal{X}}^*(\mathcal{O}_{[\mathcal{X}, E^\bullet]}^{\text{vir}}) = \mathcal{O}_{[\mathcal{X} \times_{\text{BG}} \mathcal{U}, F^\bullet]}^{\text{vir}}$ in $G_0(\mathcal{X} \times_{\text{BG}} \mathcal{U})$,
2. $p_{\mathcal{X}*}[\mathcal{X}^{\text{vir}}, E^\bullet] = [\mathcal{X} \times_{\text{BG}} \mathcal{U}^{\text{vir}}, F^\bullet]$ in $A_*(\mathcal{X} \times_{\text{BG}} \mathcal{U})_{\mathbb{Q}}$.

Proof. Consider the composition of maps $\mathcal{X} \times_{\text{BG}} \mathcal{U} \xrightarrow{p_{\mathcal{X}}} \mathcal{X} \xrightarrow{f} \mathcal{Y}$ and let us denote the composite map by g . Note that all the morphisms are representable, hence in particular they are of \mathcal{DM} -type. Let $f_{E^\bullet}^! : G_0(\mathcal{Y}) \rightarrow G_0(\mathcal{X})$ and $g_{F^\bullet}^! : G_0(\mathcal{Y}) \rightarrow G_0(\mathcal{X} \times_{\text{BG}} \mathcal{U})$ be the virtual pull-back maps defined in [Qu18, Definition 2.2]. By Definition 2.14 (and identifying the Gysin pullback in *loc. cit.* with the inverse of π^*) we have

$$\mathcal{O}_{[\mathcal{X}, E^\bullet]}^{\text{vir}} = f_{E^\bullet}^!(\mathcal{O}_{\mathcal{Y}}) \quad \text{and} \quad \mathcal{O}_{[\mathcal{X} \times_{\text{BG}} \mathcal{U}, F^\bullet]}^{\text{vir}} = g_{F^\bullet}^!(\mathcal{O}_{\mathcal{Y}}). \tag{3.11}$$

Clearly by construction we have a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
 p_{\mathcal{X}}^* E^\bullet & \longrightarrow & F^\bullet & \longrightarrow & p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\mathrm{BG}} & \xrightarrow{+1} & \\
 \downarrow & & \downarrow & & \downarrow \simeq & & \\
 p_{\mathcal{X}}^* L_f & \longrightarrow & L_g & \longrightarrow & L_{p_{\mathcal{X}}} & \xrightarrow{+1} & ,
 \end{array} \tag{3.12}$$

which shows that the perfect obstruction theories are compatible and therefore satisfy the hypothesis of [Qu18, Proposition 2.11]. Hence we have

$$\begin{aligned}
 \mathcal{O}_{[\mathcal{X} \times_{\mathrm{BG}} \mathcal{U}, F^\bullet]}^{\mathrm{vir}} &= g_{F^\bullet}^! (\mathcal{O}_{\mathcal{Y}}) \\
 &= (p_{\mathcal{X}})_{(p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\mathrm{BG}})}^! (f_{E^\bullet}^! (\mathcal{O}_{\mathcal{Y}})) \\
 &= p_{\mathcal{X}}^* [\mathcal{O}_{[\mathcal{X}, E^\bullet]}^{\mathrm{vir}}],
 \end{aligned} \tag{3.13}$$

where the second equality follows from [Qu18, Proposition 2.11] and the last equality follows from [Qu18, Remark 2.3] as $p_{\mathcal{X}}$ is smooth. This completes the proof of (1).

To prove (2) we argue similarly, this time applying the virtual pullback defined for Chow groups. Let $f_{E^\bullet}^! : A_*(\mathcal{Y})_{\mathbb{Q}} \rightarrow A_*(\mathcal{X})_{\mathbb{Q}}$ and $g_{F^\bullet}^! : A_*(\mathcal{Y})_{\mathbb{Q}} \rightarrow A_*(\mathcal{X} \times_{\mathrm{BG}} \mathcal{U})_{\mathbb{Q}}$ be the virtual pullback maps defined in [Ma12, Construction 3.6]. From [Ma12, Corollary 3.12] it follows that

$$[\mathcal{X}^{\mathrm{vir}}, E^\bullet] = f_{E^\bullet}^! ([\mathcal{Y}]) \quad \text{and} \quad [\mathcal{X} \times_{\mathrm{BG}} \mathcal{U}^{\mathrm{vir}}, F^\bullet] = g_{F^\bullet}^! ([\mathcal{Y}]). \tag{3.14}$$

Now we have the following:

$$\begin{aligned}
 [\mathcal{X} \times_{\mathrm{BG}} \mathcal{U}^{\mathrm{vir}}, F^\bullet] &= g_{F^\bullet}^! ([\mathcal{Y}]) \\
 &= (p_{\mathcal{X}})_{(p_{\mathcal{U}}^* \Omega_{\mathcal{U}/\mathrm{BG}})}^! (f_{E^\bullet}^! ([\mathcal{Y}])) \\
 &= p_{\mathcal{X}}^* ([\mathcal{X}^{\mathrm{vir}}, E^\bullet]),
 \end{aligned} \tag{3.15}$$

where the second equality follows from [Ma12, Theorem 4.8] and the last equality follows as $p_{\mathcal{X}}$ is smooth and the perfect obstruction theory is trivial (see e.g. [Kr99, Section 5.1] or [Ma12, Remark 3.10]). \square

Next we show that the virtual classes associated to the given perfect relative obstruction theory on \mathcal{W}/\mathcal{Y} and the descended perfect relative obstruction theory on \mathcal{W}/\mathcal{Z} of Lemma 3.1 are the same.

LEMMA 3.6. *Under the notations of Lemma 3.1, further assume that \mathcal{Y} and \mathcal{Z} are pure dimensional. Then we have the following:*

1. $\mathcal{O}_{[\mathcal{W}, F^\bullet]}^{\mathrm{vir}} = \mathcal{O}_{[\mathcal{W}, \tilde{E}^\bullet]}^{\mathrm{vir}}$ in $G_0(\mathcal{W})$;
2. $[\mathcal{W}^{\mathrm{vir}}, F^\bullet] = [\mathcal{W}^{\mathrm{vir}}, \tilde{E}^\bullet]$ in $A_*(\mathcal{W})_{\mathbb{Q}}$.

Proof. The proof of both the claims are similar and (2) is a straight forward generalization of [KKP03, Proposition 3] to the relative case, and the proof follows directly using their techniques. For the convenience of the reader we recall the proof in the case of (1). Arguing as in the proof of [BF97, Proposition 3.14] and using (3.2) we have the following cartesian square of cone stacks:

$$\begin{array}{ccc}
 \mathfrak{C}_{\mathcal{W}/\mathcal{Y}} & \longrightarrow & \mathfrak{C}_{\mathcal{W}/\mathcal{Z}} \\
 \downarrow & & \downarrow \\
 h^1/h^0(F^{\bullet\vee}) & \longrightarrow & h^1/h^0(\tilde{E}^{\bullet\vee}),
 \end{array} \tag{3.16}$$

where the bottom horizontal map is smooth. Since $h^1/h^0(F^{\bullet\vee})$ and $h^1/h^0(\tilde{E}^{\bullet\vee})$ are vector bundle stacks over \mathcal{W} , (1) follows from Definition 2.14 and the functoriality of flat pull-back for G -theory. \square

Remark 3.7. Let $F^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Y}}$ and $\tilde{E}^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Z}}$ be as in Lemma 3.1. Although $\tilde{E}^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Z}}$ is not uniquely determined by $F^\bullet \rightarrow L_{\mathcal{W}/\mathcal{Y}}$, Lemma 3.6 allows us to conclude that the virtual classes $\mathcal{O}_{[\mathcal{W}, \tilde{E}^\bullet]}^{\text{vir}}$ and $[\mathcal{W}^{\text{vir}}, \tilde{E}^\bullet]$ are indeed uniquely determined by the construction in Lemma 3.1 as they coincide with $\mathcal{O}_{[\mathcal{W}, F^\bullet]}^{\text{vir}}$ and $[\mathcal{W}^{\text{vir}}, F^\bullet]$ respectively.

3.2 VIRTUAL EQUIVARIANT RIEMANN-ROCH FORMULA

In this section we prove equivariant versions of virtual Riemann-Roch theorems. Throughout this section we make the following assumptions regarding G and $X \in \mathbf{Sch}_k^G$ which ensure that the algebraic space $X \times^G U$ associated to good pairs (V, U) are in fact schemes.

ASSUMPTIONS 3.8. Let $X \in \mathbf{Sch}_k^G$ such that there is a G -equivariant closed immersion $X \hookrightarrow M$ where $M \in \mathbf{Sm}_k^G$. Further assume that one of the following conditions holds:

1. M is quasi-projective (and hence M and X are G -quasi-projective by [Br17, Corollary 2.14]), or
2. G is connected.

The fact that for any good pair (V, U) , the principal bundle quotient $X \times U \rightarrow X \times^G U$ exists in the category of schemes under assumptions 3.8 follows from [EG98, Proposition 23]. Also note that under the given assumptions the stack $[X/G]$ has the resolution property (see for example [Th87-1, Lemma 2.11], [To04, Theorem 2.1]) and therefore every perfect complex on $[X/G]$ admits a global resolution (see Remark 2.7).

Recall that a G -equivariant map $f : X \rightarrow Y$ is said to be G -quasi-projective if there exists a G -equivariant locally free \mathcal{O}_Y -module E on Y and a quasi-compact G -equivariant immersion $X \hookrightarrow \mathbb{P}(E)$ over Y . And $X \in \mathbf{Sch}_k^G$ is said to be G -quasi-projective if $X \rightarrow \text{Spec } k$ is a G -quasi-projective map.

THEOREM 3.9. *Let $X, Y \in \mathbf{Sch}_k^G$ and $\tilde{f} : X \rightarrow Y$ be a G -equivariant morphism such that Y is smooth, G -equivariantly connected and pure dimensional. Suppose that there exists a G -equivariant closed immersion $X \hookrightarrow M$ for some $M \in \mathbf{Sm}_k^G$ such that the assumptions in 3.8 are satisfied. Then for an equivariant perfect relative obstruction theory $E^\bullet \rightarrow L_{X/Y}$ on X with respect to Y :*

$$\tau_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = Td^G(T_{[X/Y, E^\bullet]}^{\text{vir}}) \cap [G, X^{\text{vir}}, E^\bullet] \tag{3.17}$$

in $\text{CH}_*^G(X)_{\mathbb{Q}}$, where Td^G denotes the equivariant Todd class (cf. [EG00, Definition 3.1]).

Proof. Equivariant Chow groups of X are by definition given by the Chow groups of the quotient scheme $X \times^G U$. Recall from Section 2.3.1 that the equivariant Riemann-Roch map is defined by precomposing the Riemann-Roch map for schemes with a flat pullback map on equivariant G -theory. To prove the theorem we first show that the virtual structure sheaf corresponding to the equivariant perfect relative obstruction theory pulls back to the virtual structure sheaf corresponding to an absolute perfect obstruction theory on $X \times^G U$. We then apply [FG10, Proposition 3.4 (2)] to write the image of the virtual structure sheaf on $X \times^G U$ under the Riemann-Roch map in terms of the virtual fundamental class and the Todd class of the virtual tangent bundle. Finally we reinterpret these classes in terms of the equivariant classes corresponding to the given equivariant perfect relative obstruction theory.

Let $\mathcal{X} := [X/G], \mathcal{Y} := [Y/G]$ and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be the map induced by \tilde{f} . Let d be the virtual dimension of X with respect to the equivariant perfect relative obstruction theory E^\bullet , which implies that $[G, X^{\text{vir}}, E^\bullet] \in \text{CH}_{\dim(Y)+d}^G(X)$. Let (V, U) be an l -dimensional good pair for $\dim(X) - j$, where $j := \dim(Y) + d$. Step 1: Let $s_U : G_0(G, X)_{\mathbb{Q}} \rightarrow G_0(X \times^G U)_{\mathbb{Q}}$ be the map defined in Section 2.3.1. For $F^\bullet \rightarrow L_{X \times^G U/\mathcal{Y}}$ the perfect relative obstruction theory defined in (3.8), it follows from Proposition 3.5 that

$$s_U(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = \mathcal{O}_{[X \times^G U, F^\bullet]}^{\text{vir}} \tag{3.18}$$

in $G_0(X \times^G U)$.

Step 2: Next we compute the Todd class of $X \times^G U$ corresponding to the perfect obstruction theory \tilde{E}^\bullet of (3.9) in terms of the equivariant Todd class $Td^G(T_{[X/Y, E^\bullet]}^{\text{vir}})$. By abuse of notation we let $Td^G(V)$ denote the Todd class $Td(X \times^G (U \times V)) = r^*Td^G(V)$, where $r : X \times U \rightarrow \text{Spec}(k)$ is the structure map in \mathbf{Sch}_k^G .

Consider the following:

$$\begin{aligned} T_{[X \times^G U, \tilde{E}^\bullet]}^{\text{vir}} &= {}^1 Td(\tilde{E}^\bullet) \\ &= {}^2 Td(F^{\bullet \vee}) \cdot Td(p_X^* \circ f^*(L_{\mathcal{Y}/\text{Spec } k}^\vee)) \\ &= {}^3 p_X^*(Td^G(E^{\bullet \vee})) \cdot p_U^*(Td^G(\Omega_U^\vee)) \cdot p_X^*(Td^G(\tilde{f}^*T_Y)) \cdot Td^G(\mathfrak{g}_X)^{-1} \\ &= {}^4 p_X^*(Td^G(T_{[X, E^\bullet]}^{\text{vir}})) \cdot Td^G(V), \end{aligned}$$

where $=^1$ follows from Definition 2.8 and $=^2$ follows from the multiplicative property of Td and the distinguished triangle (3.10) in $D^b(\text{Coh}(X \times^G U))$. $=^3$ follows from the definition of F^\bullet (see (3.8)), the definition of the equivariant Todd class Td^G and since the perfect complex $L_{\mathcal{Y}/\text{Spec } k}^\vee$ on \mathcal{Y} corresponds to the equivariant perfect complex $[\mathfrak{g}_Y \rightarrow T_Y]$ on Y . The last equality, $=^4$ follows from Definition 2.8 (see Example 2.9).

Step 3: Note that by definition $\text{CH}_j^G(X) := \text{CH}_{j+l-g}(X \times^G U)$. We next claim that:

$$[G, X^{\text{vir}}, E^\bullet] = [X \times^G U^{\text{vir}}, F^\bullet] \tag{3.19}$$

in $\text{CH}_{(j+l-g)}(X \times^G U)$. First note that if the virtual dimension of X with respect to E^\bullet is d then the virtual dimension of $X \times^G U$ with respect to F^\bullet is $(l + d)$ (by construction (3.8)), thus both the fundamental classes lie in $\text{CH}_{(j+l-g)}(X \times^G U)$ as $\dim(\mathcal{Y}) = \dim(Y) - g$. From the naturality of the isomorphism $\widehat{A}_*(\) \rightarrow A_*(\)$ for quotient stacks (see [Kr99, Theorem 2.1.12]) it follows that the following diagram commutes,

$$\begin{CD} \text{CH}_j^G(X) @>{p_X^*}>> \text{CH}_{j+l}^G(X \times U) = \text{CH}_{(j+l-g)}(X \times^G U) \\ @VV \simeq V @VV \simeq V \\ A_j(\mathcal{X}) @>{p_X^*}>> A_{j+d}([(X \times U)/G]), \end{CD} \tag{3.20}$$

where the top right equality follows from [EG98, Proposition 8(a)]. Now (3.19) follows by combining Definition 2.15 and Lemma 3.6.

Step 4: By assumption 3.8 the scheme X has a G -equivariant embedding in a smooth scheme M . This induces an embedding of schemes $X \times^G U \hookrightarrow M \times^G U$ such that $M \times^G U \in \mathbf{Sm}_k$. Thus one can apply [FG10, Proposition 3.4 (2)] to the perfect obstruction theory \tilde{E}^\bullet on the scheme $X \times^G U$ to conclude:

$$\tau_{X \times^G U}([\mathcal{O}_{[X \times^G U, \tilde{E}^\bullet]}^{\text{vir}}]) = Td(T_{X \times^G U, \tilde{E}^\bullet}^{\text{vir}}) \cap [X \times^G U^{\text{vir}}, \tilde{E}^\bullet]. \tag{3.21}$$

Now we have the following:

$$\begin{aligned} \tau_X^G([\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}]) &=^1 Td^G(V)^{-1} \cap \tau_{X \times^G U}([\mathcal{O}_{[X \times^G U, F^\bullet]}^{\text{vir}}]) \\ &=^2 Td^G(V)^{-1} \cap \tau_{X \times^G U}([\mathcal{O}_{[X \times^G U, \tilde{E}^\bullet]}^{\text{vir}}]) \\ &=^3 Td^G(V)^{-1} \cdot Td(T_{[X \times^G U, \tilde{E}^\bullet]}^{\text{vir}}) \cap [X \times^G U^{\text{vir}}, \tilde{E}^\bullet] \\ &=^4 Td^G(V)^{-1} \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}) \cdot Td^G(V) \cap [X \times^G U^{\text{vir}}, F^\bullet] \\ &=^5 Td^G(T_{[X, E^\bullet]}^{\text{vir}}) \cap [G, X^{\text{vir}}, E^\bullet], \end{aligned} \tag{3.22}$$

where $=^1$ follows from Step 1 and the definition of the equivariant Riemann-Roch map, $=^2$ follows from Lemma 3.6 (2), $=^3$ follows from (3.21), $=^4$ follows from Step 2 and finally $=^5$ follows from Step 3. This completes the proof of the theorem. \square

Let ch^G denote the equivariant chern character map of [EG00, Definition 3.1]). Let $\pi : X \rightarrow \text{Spec } k \in \mathbf{Sch}_k^G$ be a proper map, i.e. X is a complete G -scheme. Then for an equivariant perfect obstruction theory $E^\bullet \rightarrow L_X$, the virtual equivariant Euler characteristic of $\alpha \in K_0(G, X)$ is defined as

$$\chi_{\text{vir}}^G(X, E^\bullet, \alpha) := \chi^G(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}),$$

where $\chi^G(\beta)$ is defined as $ch^G(\pi_*(\beta))$ for $\beta \in G_0(G, X)$ (see [EG00]). Theorem 3.9 allows us to prove equivariant analogues of the virtual Riemann-Roch Theorems (see [BF97, Theorem 3.5]).

COROLLARY 3.10. *Let X be a G -scheme satisfying the assumptions in 3.8 and let $E^\bullet \rightarrow L_X$ be an equivariant perfect obstruction theory on X and let $\alpha \in K_0(G, X)$.*

1. (Virtual Equivariant Grothendieck-Riemann-Roch). *Let $f : X \rightarrow Y \in \mathbf{Sch}_k^G$ where f is proper and Y is smooth and G -quasi-projective. Then:*

$$\begin{aligned} ch^G(f_*(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})) \cdot Td^G(T_Y) \cap [Y] & \quad (3.23) \\ & = f_*(ch^G(\alpha) \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}) \cap [G, X^{\text{vir}}, E^\bullet]) \end{aligned}$$

in $\text{CH}_*^G(X)_{\mathbb{Q}}$.

2. (Virtual Equivariant Hirzebruch-Riemann-Roch). *If X is a complete G -scheme, then:*

$$\chi_{\text{vir}}^G(X, E^\bullet, \alpha) = \int_{[G, X^{\text{vir}}, E^\bullet]} ch^G(\alpha) \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}). \quad (3.24)$$

Proof. The proof of (3.23) is identical to [FG10, Theorem 3.5] using the properties of the equivariant Riemann-Roch map given in [EG00, Theorem 3.1]. By the module property of τ^G given by Theorem 3.1(c) and the covariance with respect to equivariant proper maps of [EG00, Theorem 3.1(b)]:

$$f_*(ch^G(\alpha) \cdot Td^G(T_{[X, E^\bullet]}^{\text{vir}}) \cap [G, X^{\text{vir}}, E^\bullet]) = \tau_Y^G(f_*(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})).$$

Again by the module property of τ^G and since $K_0(G, Y) \rightarrow G_0(G, Y)$ is an isomorphism for $Y \in \mathbf{Sm}_k^G$ (cf. [Th87, Theorem 5.7]):

$$\tau_Y^G(f_*(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})) = ch^G(f_*(\alpha \otimes \mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})) \cdot \tau_Y^G([\mathcal{O}_Y]).$$

Now for $g : Y \rightarrow \text{Spec } k$ smooth and G -quasi-projective, using [EG00, Theorem 3.1(d)(ii)] we have

$$\begin{aligned} \tau_Y^G([\mathcal{O}_Y]) & = Td^G(T_Y) \cap g^* \tau_{\text{Spec } k}^G([\mathcal{O}_{\text{Spec } k}]) \\ & = Td^G(T_Y) \cap [Y]. \end{aligned}$$

This proves (1). (2) is a special case of (1) for the morphism $X \rightarrow \text{Spec } k$. \square

4 VIRTUAL STRUCTURE SHEAF AND THE ATIYAH-SEGAL MAP

In this section we work over the base field \mathbb{C} and we consider algebraic K -theory and Chow groups with complex coefficients; but for simplicity of notation we shorten $G_0(G, X) \otimes_{\mathbb{Z}} \mathbb{C}$ as $G_0(G, X)$ and $\mathrm{CH}_*^G(X) \otimes_{\mathbb{Z}} \mathbb{C}$ as $\mathrm{CH}_*^G(X)$. We recall the construction of the Atiyah-Segal map in Section 4.1 and study the image of the virtual structure sheaf under this map for a given equivariant perfect obstruction theory in Section 4.3.

4.1 ATIYAH-SEGAL ISOMORPHISM

Let $X \in \mathbf{Sch}_{\mathbb{C}}^G$ such that G acts properly on X . For X smooth the Atiyah-Segal map was defined by Edidin-Graham as a part of [EG05, Theorem 6.7] and for general X in [EG08, Theorem 6.5]. This was extended to higher K -theory in [KS20]. We recall here briefly the construction of the map for G_0 from the works of Edidin-Graham [EG05] and [EG08] without getting into the technical details. We refer to their papers for proof of the non-trivial statements which are stated as facts below, but are in fact quite involved. The purpose of this section is to set notations and give an outline of the definitions which would aid in the proof of Theorem 4.9.

For a linear algebraic group G , let $R(G)$ denote the representation ring of G , which in our notation is $K_0(G, \mathrm{Spec} \mathbb{C})$. There is a bijection between the semi-simple conjugacy classes in G and the maximal ideals in $R(G)$ (see [EG05, Proposition 2.5]). Given a semi-simple conjugacy class ψ_i we denote by \mathfrak{m}_{ψ_i} the corresponding maximal ideal in $R(G)$ under this bijection. The maximal ideal $\mathfrak{m}_1 \subset R(G)$, defined as the kernel of the rank map $R(G) \rightarrow \mathbb{C}$, is called the augmentation ideal of G .

Recall that G is said to act properly on X , if the map $(m, pr_X) : G \times X \rightarrow X \times X$ is proper, where $m : G \times X \rightarrow X$ defines the group action on X . Let X be a \mathbb{C} -scheme with proper G -action. The algebraic analogue of the Atiyah-Segal theorem from topology provides an isomorphism between the localization of the equivariant algebraic K -theory of X at a given maximal ideal with the localization of a corresponding component of the inertia scheme I_X at the augmentation ideal. We remark that in the case of proper actions, the equivariant K -theory is supported at a finite number of maximal ideals of $R(G)$, therefore the localization and completion of $G_0(G, X)$ at maximal ideals coincide, which justifies the comparison of the map at the level localizations of the K -theory to the classical topological Atiyah-Segal map. Recall that the inertia scheme I_X is the G -scheme defined as the pullback in the following square of G -schemes:

$$\begin{array}{ccc} I_X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ G \times X & \xrightarrow{(m, pr_X)} & X \times X. \end{array} \quad (4.1)$$

The closed points of I_X correspond to pairs (g, x) such that $g \cdot x = x$. Since the G -action on X is proper, there is a finite set of semi-simple conjugacy classes $\Sigma_X^G = \{\psi_1, \dots, \psi_n\}$ in G such that the fixed point locus $X^g \subset X$ is non-empty if and only if $g \in \psi_i$ for some $\psi_i \in \Sigma_X^G$. Moreover, the natural map given by localizations at maximal ideals

$$G_0(G, X) \xrightarrow{\cong} \bigoplus_{\psi \in \Sigma_X^G} G_0(G, X)_{\mathfrak{m}_\psi}$$

is an isomorphism of $R(G)$ -modules (see [EG05, Proposition 3.6]). Also, in this case the inertia scheme has a G -equivariant decomposition as $I_X = \coprod_{\psi \in \Sigma_X^G} I_X^\psi$, where I_X^ψ corresponds to the set of closed points (g, x) of I_X such that $g \in \psi$.

4.1.1 THE ATIYAH-SEGAL MAP

In this section we recall the definition of the Atiyah-Segal map from [EG08, Theorem 6.5] closely following the presentation of [KS20, § 6.3]. Let $g \in G$ be such that $X^g \neq \emptyset$ and let $\psi \in \Sigma_X^G$ denote the corresponding semi-simple conjugacy class. Let Z_g denote centralizer of g in G . Let $\mathfrak{m}_g \subset R(Z_g)$ denote the maximal ideal of Z_g corresponding to the conjugacy class $\psi \cap Z_g = \{g\}$ of Z_g . Let $\vartheta_X^\psi : G_0(G, X)_{\mathfrak{m}_\psi} \rightarrow G_0(G, I_X^\psi)_{\mathfrak{m}_1}$ be defined as the composite map:

$$G_0(G, X)_{\mathfrak{m}_\psi} \xrightarrow{(i_1^g)^{-1}} G_0(Z_g, X^g)_{\mathfrak{m}_g} \xrightarrow{t_g^{-1}} G_0(Z_g, X^g)_{\mathfrak{m}_1} \xrightarrow{\mu_*^g} G_0(G, I_X^\psi)_{\mathfrak{m}_1}, \tag{4.2}$$

where the maps are as defined below.

The map i_1^g . As $i^g : X^g \hookrightarrow X$ is a Z_g -equivariant closed immersion, there is a natural pushforward map $i_*^g : G_0(Z_g, X^g) \rightarrow G_0(Z_g, X)$ and it follows from [EG08, Theorem 5.1] that i_*^g is an isomorphism after localization at \mathfrak{m}_g . The restriction of groups defines an exact functor from $\text{Coh}^G(X) \rightarrow \text{Coh}^{Z_g}(X)$, which induces a map $\text{res}_{Z_g}^G : G_0(G, X) \rightarrow G_0(Z_g, X)$ and it follows from [EG08, Corollary 5.5(b)] that on localizations, the restriction map $\text{res}_{Z_g}^G : G_0(G, X)_{\mathfrak{m}_\psi} \rightarrow G_0(Z_g, X)_{\mathfrak{m}_g}$ is again an isomorphism. One defines i_1^g as the composition $(\text{res}_{Z_g}^G)^{-1} \circ i_*^g$.

The twisting map t_g . Let $P = \langle g \rangle$ denote the central cyclic subgroup of Z_g generated by g . As G acts properly on X , g is an element of finite order. Then P is a finite abelian group and is therefore semi-simple, let \hat{P} denote the group of characters of P . Now as g acts trivially on X^g it follows that for any Z_g -equivariant coherent sheaf \mathcal{F} on X^g , we have an eigensheaf decomposition $\mathcal{F} = \bigoplus_{\chi \in \hat{P}} \mathcal{F}_\chi$, where each $\mathcal{F}_\chi \in \text{Coh}^{Z_g}(X^g)$. Now following [EG05, § 6.2], one defines an action of P on $G_0(Z_g, X^g)$ as follows. For any $\mathcal{F} \in \text{Coh}^{Z_g}(X^g)$, let $[\mathcal{F}] \in G_0(Z_g, X^g)$ denote its corresponding class. For each $a \in P$ one defines $t_a([\mathcal{F}]) = \bigoplus_{\chi \in \hat{P}} \chi(a^{-1}) \mathcal{F}_\chi$, which extends linearly to define an automorphism of $G_0(Z_g, X^g)$. On $R(Z_g)$, $t_g^{-1}(\mathfrak{m}_g) = \mathfrak{m}_1$, hence it follows that on restricting to localizations we have an isomorphism $t_g^{-1} : G_0(Z_g, X^g)_{\mathfrak{m}_g} \rightarrow G_0(Z_g, X^g)_{\mathfrak{m}_1}$.

Morita isomorphism μ_*^g . The map $\mu_*^g : G_0(Z_g, X^g)_{\mathfrak{m}_1} \rightarrow G_0(G, I_X^\psi)_{\mathfrak{m}_1}$ is the Morita isomorphism discussed below in Example 4.4 and since μ_*^g is $R(G)$ -linear (see [EG05, Remark 3.2]) where the $R(G)$ -module structure on $G_0(Z_g, X^g)$ is given by the restriction $R(G) \rightarrow R(Z_g)$, it follows that it induces an isomorphism $G_0(Z_g, X^g)_{\mathfrak{m}_1} \rightarrow G_0(G, I_X^\psi)_{\mathfrak{m}_1}$ (where the localization on the left is with respect to the augmentation ideal in $R(Z_g)$ and the localization on the right is with respect to the augmentation ideal in $R(G)$).

Note that all the maps in (4.2) are isomorphisms and therefore the map ϑ_X^ψ is an isomorphism for each $\psi \in \Sigma_X^G$.

Remark 4.1. Although a priori the definition of the map ϑ_X^ψ depends on the choice of a $g \in \psi$, it follows from [EG05, Lemma 6.5] (or [KS20, Lemma 6.7]) that the map is indeed independent of this choice, i.e., the composite maps $\mu_*^g \circ t_g^{-1} \circ (i_!^g)^{-1}$ and $\mu_*^h \circ t_h^{-1} \circ (i_!^h)^{-1}$ coincide if $g, h \in \psi$. This is proved by showing that the following diagram commutes:

$$\begin{array}{ccccccc}
 G_0(G, X)_{\mathfrak{m}_\psi} & \xrightarrow{(i_!^g)^{-1}} & G_0(Z_g, X^g)_{\mathfrak{m}_g} & \xrightarrow{t_g^{-1}} & G_0(Z_g, X^g)_{\mathfrak{m}_1} & \xrightarrow{\mu_*^g} & G_0(G, I_X^\psi)_{\mathfrak{m}_1} \\
 \downarrow Id & & \downarrow \theta_{X^*} & & \downarrow \theta_{X^*} & & \downarrow Id \\
 G_0(G, X)_{\mathfrak{m}_\psi} & \xrightarrow{(i_!^h)^{-1}} & G_0(Z_h, X^h)_{\mathfrak{m}_h} & \xrightarrow{t_h^{-1}} & G_0(Z_h, X^h)_{\mathfrak{m}_1} & \xrightarrow{\mu_*^h} & G_0(G, I_X^\psi)_{\mathfrak{m}_1},
 \end{array} \tag{4.3}$$

where θ_X is an isomorphism between the stacks $[X^g/Z_g]$ and $[X^h/Z_h]$ which is constructed as follows. Since g, h are in the same conjugacy class, there exists a $l \in G$ such that $h = lgl^{-1}$. We have isomorphisms between the centralizers Z_g and Z_h given by $\phi : Z_g \xrightarrow{\cong} Z_h$; $\phi(z) = lzl^{-1}$ and between the fixed point loci $u : X^g \rightarrow X^h$ given by $x \mapsto lx$. If we let Z_g act on X^h via the canonical action of Z_h composed with ϕ , then u is Z_g -equivariant. Therefore we have an isomorphism of stacks $\theta_X : [X^g/Z_g] \rightarrow [X^h/Z_h]$ defined as the composite

$$[X^g/Z_g] \xrightarrow[u]{\cong} [X^h/Z_g] \xrightarrow[\phi]{\cong} [X^h/Z_h]. \tag{4.4}$$

DEFINITION 4.2. Let G act properly on X . The Atiyah-Segal map $\vartheta_X^G : G_0(G, X) \rightarrow G_0(G, I_X)_{\mathfrak{m}_1}$ is defined to be the composite isomorphism

$$G_0(G, X) \xrightarrow{\cong} \bigoplus_{\psi \in \Sigma_X^G} G_0(G, X)_{\mathfrak{m}_\psi} \xrightarrow{\bigoplus \vartheta_X^\psi} \bigoplus_{\psi \in \Sigma_X^G} G_0(G, I_X^\psi)_{\mathfrak{m}_1} \xrightarrow{\cong} G_0(G, I_X)_{\mathfrak{m}_1}. \tag{4.5}$$

4.2 MORITA EQUIVALENCE

Let us begin by briefly recalling the Morita isomorphism in our context. Note that this works in general over any base. Let G be a linear algebraic group and let $H \subset G$ be a closed subgroup. Let W be an H -scheme, one can consider $G \times W$ as a $G \times H$ -scheme, where the action is given by $((g, h), (g', w)) \mapsto (gg'h^{-1}, hw)$ where $(g, h) \in G \times H$ and $(g', w) \in G \times W$. Both the G -action

and the H -action on $G \times W$ are free. Let $Y := G \times^H W$ denote the geometric quotient under the H -action (which exists as an algebraic space). We briefly recall the Morita isomorphism which provides an isomorphism of stacks $\mu : [W/H] \xrightarrow{\simeq} [Y/G]$. Consider the following,

$$W \xleftarrow{\tilde{\pi}_G} G \times W \xrightarrow{\tilde{\pi}_H} Y, \tag{4.6}$$

where $\tilde{\pi}_G$ and $\tilde{\pi}_H$ denote the quotient by the actions of $G \times 1$ and $1 \times H$, respectively. As the actions are free it follows that $\tilde{\pi}_G$ is a principal G -bundle and $\tilde{\pi}_H$ is a principal H -bundle. It is easy to see that this induces an isomorphism of stacks (see for e.g. [KS20, Lemma 2.3]):

$$[W/H] \xleftarrow[\pi_G]{\simeq} [G \times W/G \times H] \xrightarrow[\pi_H]{\simeq} [Y/G]. \tag{4.7}$$

We let $\mu = \pi_H \circ \pi_G^{-1} : [W/H] \rightarrow [Y/G]$ denote the Morita isomorphism.

PROPOSITION 4.3. *Let notations be as above and let $\phi : F^\bullet \rightarrow L_{W/k}$ be an H -equivariant perfect obstruction theory on W . Then we can naturally define a G -equivariant perfect obstruction theory, $\tilde{E}^\bullet \rightarrow L_{Y/k}$ on Y . Further, under the Morita isomorphism $\mu_* : G_0(H, W) \rightarrow G_0(G, Y)$, we have $\mu_*(\mathcal{O}_{[H, W, F^\bullet]}^{\text{vir}}) = \mathcal{O}_{[G, Y, \tilde{E}^\bullet]}^{\text{vir}}$.*

Proof. We first construct a G -equivariant perfect obstruction theory on Y . Consider the following commutative diagram of Tor-independent, Cartesian squares:

$$\begin{array}{ccccc}
 & & \xrightarrow{\pi_G} & & \\
 [G \times W/G \times H] & \xrightarrow{p_W} & [W/G \times H] & \xrightarrow{j_1} & [W/H] \\
 \downarrow p_G & \searrow t_1 & \downarrow s & & \downarrow t_2 \\
 [G/G \times H] & \xrightarrow{i_1} & B(G \times H) & \xrightarrow{q_H} & BH,
 \end{array} \tag{4.8}$$

where p_W , p_G and q_H are induced by the projection maps from $G \times W$ to W and G and $G \times H$ to H , respectively. The map i_1 is induced by the structure map $G \rightarrow \text{Spec } k$, where $G \times H$ acts on G by $(g, h).g' = gg'h^{-1}$. The maps s , t_1 and t_2 are also induced by the structure maps to $\text{Spec } k$ and j_1 is induced by the quotient map as W has a trivial G -action.

We have a $G \times H$ -equivariant perfect obstruction theory on $G \times W$ defined by

$$\psi : F_1^\bullet := p_G^* L_{i_1} \oplus \pi_G^* F^\bullet \xrightarrow{(id, \pi_G^*(\phi))} p_G^* L_{i_1} \oplus p_W^* L_s \simeq L_{t_1}.$$

Now consider the following commutative diagram:

$$\begin{array}{ccc}
 [G \times W/G \times H] & \xrightarrow{\pi_H} & [Y/G] \\
 t_1 \downarrow & & \downarrow t_3 \\
 B(G \times H) & \xrightarrow{q_G} & BG,
 \end{array} \tag{4.9}$$

where t_3 is induced by the structure map to $\text{Spec } k$ and q_G denotes the map from $B(G \times H) \rightarrow BG$ induced by the group homomorphism $G \times H \rightarrow G$ given by projection (which is not in general representable). Since π_H is an isomorphism, the composite map $t_3 \circ \pi_H$ is representable. Applying Lemma 3.1 to the composition of maps $[G \times W/G \times H] \rightarrow B(G \times H) \rightarrow BG$, we obtain a perfect obstruction theory $G_1^\bullet \rightarrow L_{(t_3 \circ \pi_H)}$. Since π_H^* is an isomorphism, we get a perfect obstruction theory $\tilde{E}^\bullet \rightarrow L_{t_3}$ which under π_H^* gives $G_1^\bullet \rightarrow L_{(t_3 \circ \pi_H)}$. Now $\tilde{E}^\bullet \rightarrow L_{t_3}$ is the required G -equivariant perfect obstruction theory on Y . To complete the proof we need to show that μ_* respects the virtual sheaves, which reduces to showing $\pi_H^*(\mathcal{O}_{[[W/H], F^\bullet]}^{\text{vir}}) = \pi_G^*(\mathcal{O}_{[[Y/G], \tilde{E}^\bullet]}^{\text{vir}})$ in $G_0(G \times H, G \times W)$. Consider the following commutative diagram obtained from (4.8):

$$\begin{array}{ccc}
 [G \times W/G \times H] & \xrightarrow{\pi_G} & [W/H] \\
 t_1 \downarrow & & \downarrow t_2 \\
 B(G \times H) & \xrightarrow{q_H} & BH, .
 \end{array} \tag{4.10}$$

Since π_G is an isomorphism, as before $F^\bullet \rightarrow L_{t_2}$ defines a relative perfect obstruction theory $\pi_G^*(\phi) : \pi_G^*F^\bullet \rightarrow L_{t_2 \circ \pi_G}$. The composite map $[G \times W/G \times H] \xrightarrow{t_1} B(G \times H) \xrightarrow{q_H} BH$ induces a distinguished triangle of cotangent complexes in $D(\text{QCoh}([G \times W/G \times H]))$:

$$L_{q_H \circ t_1} \rightarrow L_{t_1} \xrightarrow{+1} t_1^*L_{q_H}[1] \rightarrow,$$

and the map $L_{t_1} \xrightarrow{+1} t_1^*L_{q_H}[1]$ is split by the natural map of cotangent complexes $t_1^*L_{q_H}[1] = p_G^*i_1^*L_{q_H}[1] \xrightarrow{u^{-1}} p_G^*L_{i_1} \rightarrow L_{i_1 \circ p_G} = L_{t_1}$, where $p_G^*L_{i_1} \xrightarrow{+1} p_G^*i_1^*L_{q_H}[1]$ is an isomorphism since $q_H \circ i_1$ is an isomorphism. Therefore we have a commutative diagram of distinguished triangles on $[G \times W/G \times H]$:

$$\begin{array}{ccccccc}
 \pi_G^*F^\bullet & \longrightarrow & F_1^\bullet & \longrightarrow & p_G^*L_{i_1} & \longrightarrow & \\
 \downarrow \pi_G^*(\phi) & & \downarrow \psi & & \downarrow u & & \\
 L_{q_H \circ t_1} & \longrightarrow & L_{t_1} & \longrightarrow & p_G^*i_1^*L_{q_H}[1] & \longrightarrow & .
 \end{array} \tag{4.11}$$

It follows from Lemma 3.6 that

$$\mathcal{O}_{[[G \times W/G \times H], F^\bullet]}^{\text{vir}} = \mathcal{O}_{[[G \times W/G \times H], \pi_G^*F^\bullet]}^{\text{vir}} \tag{4.12}$$

in $G_0([G \times W/G \times H])$. Next consider the composite map $[G \times W/G \times H] \xrightarrow{\pi_G} [W/H] \xrightarrow{t_3} \text{BH}$. As π_G is an isomorphism, it follows from definitions that $\mathcal{O}_{[[G \times W/G \times H], \pi_G^* F^\bullet]}^{\text{vir}} = \pi_G^*(\mathcal{O}_{[[W/H], F^\bullet]}^{\text{vir}})$ in $G_0([G \times W/G \times H])$. Therefore from (4.12) we can conclude that:

$$\pi_G^*(\mathcal{O}_{[[W/H], F^\bullet]}^{\text{vir}}) = \mathcal{O}_{[[G \times W/G \times H], F_1^\bullet]}^{\text{vir}}. \tag{4.13}$$

As $G_1^\bullet \rightarrow L_{t_2 \circ \pi_H}$ is induced from Lemma 3.1 applied to the perfect obstruction theory $F_1^\bullet \rightarrow L_{t_1}$ it follows from Lemma 3.6 that:

$$\mathcal{O}_{[[Y/G], G_1^\bullet]}^{\text{vir}} = \mathcal{O}_{[[G \times W/G \times H], F_1^\bullet]}^{\text{vir}}. \tag{4.14}$$

It then follows immediately that

$$\begin{aligned} \pi_H^*(\mathcal{O}_{[[Y/G], \tilde{E}^\bullet]}^{\text{vir}}) &= \mathcal{O}_{[[G \times W/G \times H], \pi_H^* \tilde{E}^\bullet]}^{\text{vir}} \\ &= \mathcal{O}_{[[G \times W/G \times H], G_1^\bullet]}^{\text{vir}}. \end{aligned}$$

The proposition now follows by comparing the above equality with (4.14) and (4.13). □

Example 4.4. [EG05, Section 4.1], [EG08, Section 5.4] Let $g \in G$ be a semi-simple element of G and let us denote its conjugacy class by ψ and let the centralizer of g in G be denoted by Z_g . Let $X \in \mathbf{Sch}_{\mathbb{C}}^G$ be a scheme with proper action. Then the natural map $G \times X^g \rightarrow I_X^\psi$ is a principal Z_g -bundle which induces an isomorphism of stacks $[I_X^\psi/G] \cong [G \times^{Z_g} X^g/G]$. By (4.7) we have an induced Morita isomorphism $\mu^g : [X^g/Z_g] \rightarrow [I_X^\psi/G]$

4.3 VIRTUAL STRUCTURE SHEAF ON THE INERTIA SCHEME

Given an equivariant perfect obstruction theory on a G -scheme X with a proper G -action, the fixed point locus X^g of an element $g \in G$ carries an associated g -fixed perfect obstruction theory, which is equivariant with respect to the action of the centralizer subgroup Z_g of g , and this is compatible with the given perfect obstruction theory on X . For the case of \mathbb{C}^* -actions, this was observed by Graber-Pandharipande in [GP99] and we note in the following lemma that their proof also works for other groups. And we use this to induce a compatible G -equivariant perfect obstruction theory on the inertia scheme I_X under the Morita equivalence.

LEMMA 4.5. [GP99, Proposition 1] *Let $X \in \mathbf{Sch}_{\mathbb{C}}^G$ such that the G -action is proper. Suppose that there exists a G -equivariant closed immersion $X \hookrightarrow M$ for some $M \in \mathbf{Sm}_{\mathbb{C}}^G$. Let $E^\bullet \rightarrow L_{X/\mathbb{C}}$ be a G -equivariant perfect obstruction theory. Then we have a canonical induced G -equivariant perfect obstruction theory on I_X .*

Proof. Let notations be as in Section 4.1. Let $g \in \psi$ such that $X^g \neq \emptyset$. The first step is to construct a canonical Z_g -equivariant perfect obstruction theory on X^g . Let $P = \langle g \rangle$ denote the finite cyclic subgroup of G generated by g . Now as $E^\bullet \rightarrow L_{X/\mathbb{C}}$ is a G -equivariant perfect obstruction theory on X , it can also be viewed, by restriction of groups, as Z_g -equivariant and P -equivariant perfect obstruction theories on X . Let $i^g : X^g \hookrightarrow X$ denote the Z_g -equivariant closed immersion. It follows from the proof of [GP99, Proposition 1], that $F_g^\bullet := (i^{g,*} E^\bullet)^{\text{fix}} \rightarrow L_{X^g/\mathbb{C}}$ defines a Z_g -equivariant perfect obstruction theory on X^g , where $(-)^{\text{fix}}$ denotes the invariant (or fixed) part under the eigensheaf decomposition on X^g with respect to the P -action. Recall from Example 4.4 that $I_X^\psi \simeq G \times^{Z_g} X^g$ as G -schemes, therefore it follows from Proposition 4.3 that $F_g^\bullet \rightarrow L_{X^g/\mathbb{C}}$ induces a G -equivariant perfect obstruction theory on I_X^ψ , which we denote by \tilde{E}_ψ^\bullet . As I_X is a finite disjoint union of I_X^ψ it follows that $\tilde{E}^\bullet := \bigoplus_{\psi \in \Sigma_X^G} \tilde{E}_\psi^\bullet \rightarrow L_{I_X/\mathbb{C}}$ is a G -equivariant perfect obstruction theory on I_X . \square

For a scheme with a \mathbb{C}^* -action, the K -theoretic virtual localization formula was proved by Qu in [Qu18, Theorem 3.3]. For a scheme X with G -action and $g \in G$ such that $X^g \neq \emptyset$, one can consider the analogous virtual localization statement with respect to the g -fixed locus X^g if we restrict to the Z_g -action. Let us briefly recall the setup as in *loc. cit.* Given a G -equivariant perfect obstruction theory E^\bullet on X , the virtual conormal bundle to the Z_g -equivariant closed immersion $i^g : X^g \hookrightarrow X$ is defined as $N_{[i^g, E^\bullet]}^{\text{vir}\vee} := (i^{g,*} E^\bullet)^{\text{mov}}$, where as discussed before under the eigensheaf decomposition on X^g with respect to the P -action, we have a decomposition

$$i^{g,*} E^\bullet = (i^{g,*} E^\bullet)^{\text{fix}} \oplus (i^{g,*} E^\bullet)^{\text{mov}}$$

into fixed and moving parts. As E^\bullet admits a global resolution, in particular this implies that $N_{[i^g, E^\bullet]}^{\text{vir}\vee}$ has a global resolution. Also note that the map of $R(Z_g)$ -modules given by

$$\cap \Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}\vee}]) : G_0(Z_g, X^g) \rightarrow G_0(Z_g, X^g)$$

is invertible after localizing at \mathfrak{m}_g . This would essentially follow from the proof of [EG05, Theorem 3.3(b)] by using the fact that $N_{[i^g, E^\bullet]}^{\text{vir}\vee}$ is a complex of sheaves, such that the characters in their eigensheaf decompositions with respect to the P -action are all non-trivial.

LEMMA 4.6. *With notations as above, under the proper pushforward $i_*^g : G_0(Z_g, X^g)_{\mathfrak{m}_g} \rightarrow G_0(Z_g, X)_{\mathfrak{m}_g}$:*

$$i_*^g \left(\frac{\mathcal{O}_{[Z_g, X^g, F_g^\bullet]}^{\text{vir}}}{\Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}\vee}])} \right) = \mathcal{O}_{[Z_g, X, E^\bullet]}^{\text{vir}} \tag{4.15}$$

in $G_0(Z_g, X)_{\mathfrak{m}_g}$.

Proof. The proof is identical to the proof of [Qu18, Theorem 3.3] once we note that $(i^g)_* : G_0(Z_g, X^g)_{m_g} \rightarrow G_0(Z_g, X)_{m_g}$ is an isomorphism by [EG08, Theorem 5.1]. \square

The following lemma shows that the equivariant perfect obstruction theory induced by restriction of groups produces compatible virtual structure sheaves under the restriction functor on K -theories.

LEMMA 4.7. *Let $H \hookrightarrow G$ be a closed subgroup and let $W \in \mathbf{Sch}_k^G$. Let $E^\bullet \rightarrow L_{W/k}$ be a G -equivariant perfect obstruction theory on W . Then $\text{res}_H^G(\mathcal{O}_{[G,W,E^\bullet]}^{\text{vir}}) = \mathcal{O}_{[H,W,E^\bullet]}^{\text{vir}}$ in $G_0(H, W)$.*

Proof. As $H \hookrightarrow G$ is a closed subgroup of a smooth affine group scheme G , the induced morphism $f : BH \rightarrow BG$ is smooth (since fppf locally on BG , $BH \times_{BG} \text{Spec } k = G/H$ is smooth). Consider the following Cartesian square:

$$\begin{CD} [W/H] @>g>> [W/G] \\ @VqVV @VVpV \\ BH @>>f>> BG \end{CD}$$

Let us denote by $E^\bullet \rightarrow L_p$ the relative perfect obstruction theory (corresponding to the G -equivariant perfect obstruction theory on W), it induces a relative perfect obstruction $g^*E^\bullet \rightarrow L_q$ (the H -equivariant perfect obstruction theory on X obtained by restricting to the H -action). The restriction functor $\text{QCoh}^G(W) \rightarrow \text{QCoh}^H(W)$ is given by $g^* : \text{QCoh}([W/G]) \rightarrow \text{QCoh}([W/H])$ under the identification of equivariant quasi-coherent sheaves and quasi-coherent sheaves on quotient stacks (see [AOV08, Section 2.1]). Under this identification,

$$\begin{aligned} \text{res}_H^G(\mathcal{O}_{[G,W,E^\bullet]}^{\text{vir}}) &= g^*(\mathcal{O}_{[G,W,E^\bullet]}^{\text{vir}}) \\ &\stackrel{1}{=} g^*p_{E^\bullet}^!(\mathcal{O}_{BG}) \\ &\stackrel{2}{=} q_{E^\bullet}^!f^*(\mathcal{O}_{BG}) \\ &= q_{E^\bullet}^!(\mathcal{O}_{BH}) \\ &\stackrel{3}{=} \mathcal{O}_{[H,W,E^\bullet]}^{\text{vir}}, \end{aligned}$$

where $\stackrel{1}{=}$ and $\stackrel{3}{=}$ follow from Definition 2.14 and [Qu18, Definition 2.2] and $\stackrel{2}{=}$ follows from the basic properties of virtual pullbacks [Qu18, Remark 2.3, Proposition 2.5]. \square

We now prove a virtually smooth analogue of the Edidin-Graham non-abelian localization theorem [EG05, Theorem 5.1]. We show that under the Atiyah-Segal morphism the virtual structure sheaf on X maps to the virtual structure sheaf of the inertia scheme I_X up to an automorphism induced by the virtual

conormal bundle appearing in Lemma 4.6. As we mentioned before the Atiyah-Segal map is independent of the choice of $g \in \psi$ (see Remark 4.1), we prove in the following lemma that the image of the class of the virtual conormal bundle in I_ψ is also independent of the choice of $g \in \psi$.

LEMMA 4.8. *Given a G -equivariant perfect obstruction theory E^\bullet on X , and $g, h \in \psi$, $\theta_X^*(\Lambda_{-1}([N_{[i^h, E^\bullet]}^{\text{vir}}]))) = \Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}}]))$ in $K_0(Z_g, X^g)$, where θ_X is defined in (4.4).*

Proof. Let P_g and P_h denote the cyclic subgroups of G generated by g and h , respectively. The proof follows from the fact that the map $u : X^g \rightarrow X^h$ (see Remark 4.1) takes the fixed and moving parts under the P_h -action to the fixed and moving parts under the P_g -action, respectively. \square

In view of the above lemma and (4.3), we denote $\mu_*^g(t_{g^{-1}}(\Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}}])))$, the image of the class of the virtual conormal bundle in $K_0(G, I_X^\psi)$, by $\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}}]))$. Moreover, since $t_{g^{-1}}$ and μ_*^g are isomorphisms, $\cap \Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}}])) : G_0(G, I_X^\psi) \rightarrow G_0(G, I_X^\psi)$ is again an invertible map after localizing at \mathfrak{m}_1 .

THEOREM 4.9. *Let $X \in \text{Sch}_{\mathbb{C}}^G$ such that G acts properly on X . Suppose that there exists a G -equivariant closed immersion $X \hookrightarrow M$ for some $M \in \text{Sm}_{\mathbb{C}}^G$. Let $E^\bullet \rightarrow L_{X/k}$ be a G -equivariant perfect obstruction theory and let $\tilde{E}^\bullet \rightarrow L_{I_X/k}$ be the induced G -equivariant perfect obstruction theory on I_X (see Lemma 4.5). Then under the Atiyah-Segal isomorphism $\vartheta_X^G : G_0(G, X) \rightarrow G_0(G, I_X)_{\mathfrak{m}_1}$, we have the following:*

$$\vartheta_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = \sum_{\psi \in \Sigma_X^G} \frac{\mathcal{O}_{[G, I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}}{\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}}]))}.$$

Proof. Fix an element $g \in \psi$ for each conjugacy class $\psi \in \Sigma_X^G$, under the notations of perfect obstruction theories given in Lemma 4.5, we have:

$$\begin{aligned} \vartheta_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) &=^1 \bigoplus_{\psi \in \Sigma_X^G} \vartheta_X^\psi(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) \\ &=^2 \sum_{\psi} \mu_*^g(t_g^{-1}(i_*^{g^{-1}}(\text{res}_{Z_g}^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}})))) \\ &=^3 \sum_{\psi} \mu_*^g(t_g^{-1}(i_*^{g^{-1}}(\mathcal{O}_{[Z_g, X, E^\bullet]}^{\text{vir}}))) \\ &=^4 \sum_{\psi} \mu_*^g \left(t_g^{-1} \left(\frac{\mathcal{O}_{[Z_g, X^g, F_g^\bullet]}^{\text{vir}}}{\Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}}]))} \right) \right) \\ &=^5 \sum_{\psi} \frac{\mu_*^g(\mathcal{O}_{[Z_g, X^g, F_g^\bullet]}^{\text{vir}})}{\mu_*^g(t_{g^{-1}}(\Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}}])))} \end{aligned}$$

$$\begin{aligned}
 &=^6 \sum_{\psi} \frac{\mathcal{O}_{[G, I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}}{\mu_*^g(t_{g^{-1}}(\Lambda_{-1}([N_{[i^g, E^\bullet]}^{\text{vir}^\vee}])))} \\
 &= \sum_{\psi} \frac{\mathcal{O}_{[G, I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}}{\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}])}
 \end{aligned}$$

where $=^1, =^2$ follow from Definitions 4.2, (4.2) and $=^3, =^4$ follow from Lemmas 4.7, 4.6, respectively. $=^5$ is a consequence of the fact that F_g^\bullet is $<g>$ -invariant and $=^6$ is given by Proposition 4.3. \square

Recall from [KS20, Definition 10.8] that the Riemann-Roch isomorphism to the inertia scheme is defined as the composition of the Atiyah-Segal map and the Riemann-Roch transformation of the inertia scheme:

$$I\tau_X := \tau_{I_X}^G \circ \vartheta_X^G : G_0(G, X) \rightarrow \bigoplus_{\psi \in \Sigma_G^X} \text{CH}_*^G(I_X).$$

We have the following analogue of Theorem 3.9 relating the virtual structure sheaf of a G -scheme with a given G -equivariant perfect obstruction theory and the virtual fundamental class of the inertia scheme corresponding to the naturally induced G -equivariant perfect obstruction theory, under the Riemann-Roch transformation to the inertia.

COROLLARY 4.10. *Under the notations of Theorem 4.9 further suppose assumptions in 3.8 are satisfied. Then we have the following equality in $\text{CH}_*^G(I_X)$:*

$$I\tau_X^G(\mathcal{O}_{[G, X, E^\bullet]}^{\text{vir}}) = \sum_{\psi \in \Sigma_G^X} ch^G(\Lambda_{-1}([N_{[i^\psi, E^\bullet]}^{\text{vir}^\vee}])^{-1}Td^G(T_{[I_X^\psi, \tilde{E}^\bullet]}^{\text{vir}}) \cap [G, I_X^{\psi^{\text{vir}}}, \tilde{E}^\bullet]).$$

Proof. The proof follows directly from Theorems 4.9 and 3.9, by noting that for any $\epsilon \in K_0(G, I_X^\psi)$ and $\alpha \in G_0(G, I_X^\psi)$, $\tau_{I_X^\psi}^G(\epsilon\alpha) = ch^G(\epsilon)\tau_{I_X^\psi}^G(\alpha)$ (cf. [EG00, Theorem 3.1(c)]). \square

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