On correlations and fluctuations of time-averaged densities and currents with general time-dependence

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Abstract. We present technical results required for the description and understanding of correlations and fluctuations of the empirical density and current as well as diverse time-integrated and time-averaged thermodynamic currents of diffusion processes with a general time dependence on all time scales. In particular, we generalize the results from arXiv:2105.10483 (*Phys. Rev. Lett.* , article in press), arXiv:2204.06553 (*Phys. Rev. Research*, article in press), and arXiv:2206.04034 to additive functionals with explicit time dependence and transient or non-ergodic overdamped diffusion. As an illustration we apply the results to two-dimensional harmonically confined overdamped diffusion in a rotational flow evolving from a non-stationary initial distribution.

1. Introduction

"Time-average statistical mechanics" focuses on the study of additive functionals of stochastic paths and is important in the analysis of single-particle tracking [1–3], large deviation theory [4–8], and stochastic thermodynamics [9–16], to name but a few. The most important functionals from a physical point of view include the "empirical density" (also known as local or occupation time) [17–26], time-integrated and time-averaged currents [4–7, 15, 16, 27–35], and the time-averaged mean squared displacement (see e.g. [1, 2, 36–40]).

Fluctuations of time-averaged observables have a noise floor—they are bounded from below by the dissipation in a system, which is embodied within the "thermodynamic uncertainty relation" (TUR) [10, 41–49]. One may fruitfully exploit this universal lower bound on current fluctuations, e.g. to gauge the thermodynamic cost of precision [41, 50, 51], infer dissipation from fluctuations [34, 35, 43, 44], or to derive thermodynamic limits on the temporal extent of anomalous diffusion [52].

Recent works addressed fluctuations of additive functionals in transient non-equilibrium systems [45–47], as well as in periodically [10, 48, 49] and generally driven systems [10].

Our aim here is to generalize the direct, stochastic-calculus approach we developed for steady-state systems in [34, 35] to transients and systems as well as functionals with explicit time dependence. Note that this includes non-ergodic systems (see e.g. [20, 25]).

The paper is structured as follows. We first set up the formal background and define the additive functionals in Sec. 2. In Section 3 we evaluate the first moments. In Sec. 4 we present our main result—a Lemma that allows a direct evaluation of fluctuations and correlations of general additive functionals in systems with explicit time dependence—and derive general results for current fluctuations and current-density correlations. In Sec. 5 we illustrate how to apply the newly developed results by evaluating current-density correlations in overdamped diffusion in a rotational flow evolving from a non-stationary initial distribution. We conclude with a brief outlook.

2. Set-up

Consider overdamped Langevin dynamics with possibly multiplicative noise and explicit time dependence, described by the anti-Itô (or Hänggi-Klimontovich [53, 54]) stochastic differential equation

$$\mathrm{d}\mathbf{x}_\tau = \mathbf{F}(\mathbf{x}_\tau,\tau)\mathrm{d}\tau + \boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\circledast\mathrm{d}\mathbf{W}_\tau, \tag{1}$$

with positive definite diffusion matrix $\mathbf{D}(\mathbf{x}_\tau,\tau)=\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\boldsymbol{\sigma}^T(\mathbf{x}_\tau,\tau)/2$. Assume that the drift $\mathbf{F}(\mathbf{x}_\tau,\tau)$ and noise amplitude $\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)$ are sufficiently well-behaved for Eq. (1) to be well-defined with a unique strong solution (e.g. assume that a weak solution exists and $\mathbf{F}$ and $\boldsymbol{\sigma}$ are locally Lipschitz continuous [55]). The anti-Itô convention $\circledast\mathrm{d}\mathbf{W}_\tau=\mathbf{W}_\tau-\mathbf{W}_{\tau-\mathrm{d}\tau}$ is the thermodynamically consistent choice [13, 33, 35], in particular it ensures Boltzmann statistics if the drift $\mathbf{F}(\mathbf{x}_\tau,\tau)$ is such that the system settles into thermodynamic equilibrium [35]. The time-evolution of the probability density $P(\mathbf{x},\tau)$ for any initial density $P(\mathbf{x},\tau=0)$ obeys a Fokker-Planck equation [56, 57]

$$\begin{aligned}\partial_\tau P(\mathbf{x},\tau) &= [-\nabla_\mathbf{x}\cdot\mathbf{F}(\mathbf{x},\tau)+\nabla_\mathbf{x}^T\mathbf{D}(\mathbf{x},\tau)\nabla_\mathbf{x}]P(\mathbf{x},\tau)\\ &\equiv L(\mathbf{x},\tau)P(\mathbf{x},\tau),\end{aligned} \tag{2}$$

which is equivalent to a continuity equation $[\partial_\tau+\nabla_\mathbf{x}\cdot\hat{\mathbf{j}}(\mathbf{x},\tau)]P(\mathbf{x},\tau)=0$, with the current operator

$$\hat{\mathbf{j}}(\mathbf{x},\tau)\equiv\mathbf{F}(\mathbf{x},\tau)-\mathbf{D}(\mathbf{x},\tau)\nabla_\mathbf{x} \tag{3}$$

that gives the instantaneous current as $\mathbf{j}(\mathbf{x},\tau)=\hat{\mathbf{j}}(\mathbf{x},\tau)P(\mathbf{x},\tau)$.

As a special case of Eq. (1) we will also study time-homogeneous non-equilibrium steady-state systems, where the stochastic equation of motion reads (curly brackets throughout denote that derivatives only act inside brackets)

$$\mathrm{d}\mathbf{x}_\tau = \left[\mathbf{D}(\mathbf{x}_\tau)\{\nabla\log p_\mathrm{s}\}(\mathbf{x}_\tau)+\frac{\mathbf{j}_\mathrm{s}(\mathbf{x}_\tau)}{p_\mathrm{s}(\mathbf{x}_\tau)}\right]\mathrm{d}\tau+\boldsymbol{\sigma}(\mathbf{x}_\tau)\circledast\mathrm{d}\mathbf{W}_\tau, \tag{4}$$

where $p_{\rm s}$ and $\mathbf{j}_{\rm s}$ denote the steady-state density and current [35]. Note that (as opposed to [34, 35]) we do *not* assume that the initial distribution is sampled from $p_{\rm s}$.

Based on the dynamics defined in Eqs. (1) or (4), we consider time-averaged density and current functionals of the trajectories $[\mathbf{x}_\tau]_{0\le\tau\le t}$ defined as

$$\begin{aligned}\rho_t^V &= \frac{1}{t}\int_0^t V(\mathbf{x}_\tau,\tau)\mathrm{d}\tau \\ \mathbf{J}_t^U &= \frac{1}{t}\int_{\tau=0}^{\tau=t} U(\mathbf{x}_\tau,\tau)\circ \mathrm{d}\mathbf{x}_\tau,\end{aligned} \tag{5}$$

with U, V differentiable and square integrable functions and $\circ$ denotes the Stratonovich convention of the stochastic integral. The density functional ρ_t^V measures the time spent in the region $V(\mathbf{x}) \neq 0$, weighted by $V(\mathbf{x})$, while the current $\mathbf{J}_t^U$ functional measures weighted displacements accumulated in U. In particular, for positive V, U that are centered around some point $\mathbf{x}$ and decay on a finite length scale, one can interpret ρ_t^V and $\mathbf{J}_t^U$ as the coarse-grained empirical density and current at $\mathbf{x}$ [34, 35].

In the following, we will derive expressions for the mean values, correlations and fluctuations of these stochastic quantities and illustrate them with an example, thereby generalizing the results in [34,35] to non-steady-state initial conditions and even systems with explicit time-dependence, and thus in particular also without the existence of a steady state.

3. First moments

Consider overdamped Langevin dynamics as defined in Eq. (1) starting from an arbitrary initial density $P(\mathbf{x},\tau=0)$. Let $P(\mathbf{x},\tau)$ be the probability density to find the particle at position $\mathbf{x}$ after time τ, i.e. the solution of the Fokker-Planck equation in Eq. (2). Then the mean value of the density functional in Eq. (5) is given by

$$\begin{aligned}\left\langle \rho_t^V \right\rangle &= \frac{1}{t}\int_0^t \left\langle V(\mathbf{x}_\tau,\tau)\right\rangle \mathrm{d}\tau \\ &= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}^d x V(\mathbf{x},\tau)P(\mathbf{x},\tau).\end{aligned} \tag{6}$$

The mean value of the current is in turn given accordingly by (following closely the approach [35] using that the Itô-d$\mathbf{W}_\tau$-term vanishes on average, integrating by parts, and using $\mathbf{D}=\mathbf{D}^T$)

$$
\begin{aligned}
\left\langle \mathbf{J}_t^U \right\rangle &= \frac{1}{t}\int_0^t \langle U(\mathbf{x}_\tau,\tau)\circ \mathrm{d}\mathbf{x}_\tau\rangle \\
&= \frac{1}{t}\int_{\tau=0}^{\tau=t} \langle U(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{x}_\tau\rangle + \frac{1}{t}\int_{\tau=0}^{\tau=t} \frac{1}{2}\langle \mathrm{d}U(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{x}_\tau\rangle_{\mathrm{s}} \\
&= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} P(\mathbf{x},\tau)\left[U(\mathbf{x},\tau)\mathbf{F}(\mathbf{x},\tau) + \left\{\nabla_{\mathbf{x}}^T \mathbf{D}(\mathbf{x},\tau)\right\} U(\mathbf{x},\tau) + \mathbf{D}(\mathbf{x},\tau)\left\{\nabla_{\mathbf{x}} U(\mathbf{x},\tau)\right\}\right] \\
&= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} P(\mathbf{x},\tau)\left[U(\mathbf{x},\tau)\mathbf{F}(\mathbf{x},\tau) + \nabla_{\mathbf{x}}^T \mathbf{D}(\mathbf{x},\tau) U(\mathbf{x},\tau)\right] \\
&= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} U(\mathbf{x},\tau)\left[\mathbf{F}(\mathbf{x},\tau) - \mathbf{D}(\mathbf{x},\tau)\nabla_{\mathbf{x}}\right] P(\mathbf{x},\tau) \\
&= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} U(\mathbf{x},\tau)\hat{\mathbf{j}}(\mathbf{x},\tau) P(\mathbf{x},\tau) \\
&= \frac{1}{t}\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} U(\mathbf{x},\tau)\mathbf{j}(\mathbf{x},\tau).
\end{aligned} \tag{7}
$$

The expressions Eq. (6) and (7) average the probability density and current over the function $U(\mathbf{x},\tau)$ and over time $\tau\in[0,t]$, i.e. one can interpret ρ_t^V and $\mathbf{J}_t^U$ as estimators of space and time averages of $P(\mathbf{x},\tau)$ and $\mathbf{j}(\mathbf{x},\tau)$. Note that for time-homogeneous steady-state dynamics (see Eq. (4)) these results are unchanged. They only further simplify for dynamics in Eq. (4) if also the initial condition is sampled from the steady state $P(\mathbf{x},\tau=0)=p_{\mathrm{s}}(\mathbf{x})$, in which case $P(\mathbf{x},\tau)=p_{\mathrm{s}}(\mathbf{x})$ and $\mathbf{j}(\mathbf{x},\tau)=\mathbf{j}_{\mathrm{s}}(\mathbf{x})$ implies that $\left\langle \rho_t^V \right\rangle$ and $\left\langle \mathbf{J}_t^U \right\rangle$ become independent of t.

4. Correlations and fluctuations

We now derive second moments and linear correlations of the time-averaged density and current in Eq. (5). The derivations for higher moments of currents are more involved than the first moments but as in [35] we solve the complications in the derivation by means of a single Lemma derived in the following subsection. Note that one could alternatively derive the following results using a Feynman-Kac approach (and optionally functional calculus) by appropriately generalizing the approach in [58].

4.1. Lemma

In the derivation of expressions for fluctuations and correlations of the time-averaged quantities we must evaluate correlations of noise increments $\mathrm{d}\mathbf{W}_\tau$ and functions of $\mathbf{x}_{\tau'}$. Correlations for $\tau'\leq\tau$ vanish by the properties of the Wiener process. Conversely, correlations for $\tau'>\tau$ are non-trivial. This problem was solved for steady-state dynamics in [35] and via Doob conditioning [5, 13, 59] for general time-homogeneous Langevin systems in the Supplemental Material of [16]. We now generalize the direct approach from [35] to overdamped Langevin systems with explicit time-dependence.

Consider the k-th component $[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k$ of a noise increment in an expectation value $\langle f(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\,[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k\rangle$ with some (differentiable, square integrable) function f. For $\tau'\leq\tau$ this term vanishes due to vanishing correlations and zero mean of $\mathrm{d}\mathbf{W}_\tau$. Now consider $\tau'>\tau$.

Given a point $\mathbf{x}_\tau=\mathbf{x}$ and writing $\boldsymbol{\varepsilon}\equiv\boldsymbol{\sigma}(\mathbf{x},\tau)\mathrm{d}\mathbf{W}_\tau$, the equation of motion (1) rewritten in Itô form (writing out the anti-Itô correction term) implies a displacement $\mathrm{d}\mathbf{x}_\tau(\mathbf{x},\tau,\boldsymbol{\varepsilon})=[\mathbf{F}(\mathbf{x},\tau)+\nabla_{\mathbf{x}}^T\mathbf{D}(\mathbf{x},\tau)]\mathrm{d}\tau+\boldsymbol{\varepsilon}$. With this we can write the expectation $\langle f(\cdots)\,[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k\rangle$ as ε_k integrated over the probability to be at points $\mathbf{x},\mathbf{x}+\mathrm{d}\mathbf{x}_\tau(\mathbf{x},\tau,\boldsymbol{\varepsilon}),\mathbf{y}$ at times $\tau<\tau+\mathrm{d}\tau<\tau'$, i.e. (with joint density $P(\mathbf{y},\tau';\mathbf{x},\tau)$ and conditional density $P(\mathbf{y},\tau'|\mathbf{x},\tau)\equiv P(\mathbf{y},\tau';\mathbf{x},\tau)/P(\mathbf{x},\tau)$; we write $\mathbb{1}_{\tau<\tau'}$ for 1 if $\tau<\tau'$ and 0 else)

$$\begin{aligned}&\langle f(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\,[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k\rangle\\&=\mathbb{1}_{\tau<\tau'}\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}f(\mathbf{x},\mathbf{y},\tau,\tau')\int\mathrm{d}\boldsymbol{\varepsilon}\,\mathbb{P}(\boldsymbol{\varepsilon})\varepsilon_kP(\mathbf{y},\tau'|\mathbf{x}+\mathrm{d}\mathbf{x}_\tau(\mathbf{x},\tau,\boldsymbol{\varepsilon}),\tau+\mathrm{d}\tau)P(\mathbf{x},\tau)\,,\end{aligned}\tag{8}$$

where the probability $\mathbb{P}(\boldsymbol{\varepsilon})$ is given by a Gaussian distribution with zero mean and covariance matrix $2\mathbf{D}(\mathbf{x},\tau)\mathrm{d}\tau$. Since this distribution is symmetric around $\mathbf{0}$, only terms with even powers of the components of $\boldsymbol{\varepsilon}$ survive the $\mathrm{d}\boldsymbol{\varepsilon}\mathbb{P}(\boldsymbol{\varepsilon})$-integration. Note that

$$P(\mathbf{y},\tau'|\mathbf{x}+\mathrm{d}\mathbf{x}_\tau(\mathbf{x},\tau,\boldsymbol{\varepsilon}),\tau+\mathrm{d}\tau)\overset{\mathrm{d}\tau\to0}{\longrightarrow}[1+\mathrm{d}\mathbf{x}_\tau(\mathbf{x},\tau,\boldsymbol{\varepsilon})\cdot\nabla_{\mathbf{x}}]P(\mathbf{y},\tau'|\mathbf{x},\tau)+\mathcal{O}(\mathrm{d}\tau),\tag{9}$$

and we can neglect the higher orders $\mathcal{O}(\mathrm{d}\tau)$ since $\varepsilon_k\mathcal{O}(\mathrm{d}\tau)=\mathcal{O}(\mathrm{d}\tau^{3/2})$ which (unlike $\varepsilon_k\mathcal{O}(\mathrm{d}\tau^{1/2})$) will still give zero after integration in τ. From the zeroth and first order contribution, we see that the only even power of the components of $\boldsymbol{\varepsilon}$ in the above integration gives

$$\begin{aligned}&\langle f(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\,[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k\rangle\\&=\mathbb{1}_{\tau<\tau'}\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}f(\mathbf{x},\mathbf{y},\tau,\tau')P(\mathbf{x},\tau)\int\mathrm{d}\boldsymbol{\varepsilon}\,\mathbb{P}(\boldsymbol{\varepsilon})\varepsilon_k\boldsymbol{\varepsilon}\cdot\nabla_{\mathbf{x}}P(\mathbf{y},\tau'|\mathbf{x},\tau),\end{aligned}\tag{10}$$

which, using $\int\mathrm{d}\boldsymbol{\varepsilon}\mathbb{P}(\boldsymbol{\varepsilon})\varepsilon_k\varepsilon_j=2D_{kj}(\mathbf{x},\tau)\mathrm{d}\tau$, yields the result for $\tau<\tau'$

$$\begin{aligned}&\langle f(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\,[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau]_k\rangle\\&=\mathbb{1}_{\tau<\tau'}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}P(\mathbf{x},\tau)f(\mathbf{x},\mathbf{y},\tau,\tau')\,[2\mathbf{D}(\mathbf{x},\tau)\nabla_{\mathbf{x}}P(\mathbf{y},\tau'|\mathbf{x},\tau)]_k\,.\end{aligned}\tag{11}$$

For scalar products with vector valued functions $\mathbf{f}$ the result (11) can be summed over components f_k to obtain

$$\begin{aligned}&\langle\mathbf{f}(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\cdot\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau\rangle\\&=\mathbb{1}_{\tau<\tau'}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}P(\mathbf{x},\tau)\mathbf{f}(\mathbf{x},\mathbf{y},\tau,\tau')\cdot2\mathbf{D}(\mathbf{x},\tau)\nabla_{\mathbf{x}}P(\mathbf{y},\tau'|\mathbf{x},\tau).\end{aligned}\tag{12}$$

Eq. (12) is the central result of this work that allows us to directly deduce expressions for fluctuations and correlations of densities and currents. Upon integrating by parts and using symmetry $\mathbf{D}^T(\mathbf{x},\tau)=\mathbf{D}(\mathbf{x},\tau)$ Eq. (12) could also be rewritten as

$$\begin{aligned}&\langle \mathbf{f}(\mathbf{x}_\tau,\mathbf{x}_{\tau'},\tau,\tau')\cdot\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau\rangle\\&=-\mathbb{1}_{\tau<\tau'}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}P(\mathbf{y},\tau'|\mathbf{x},\tau)\nabla_\mathbf{x}\cdot[P(\mathbf{x},\tau)2\mathbf{D}(\mathbf{x},\tau)\mathbf{f}(\mathbf{x},\mathbf{y},\tau,\tau')].\end{aligned}\tag{13}$$

4.2. Fluctuations and correlations of densities and currents

Following the developed approach and generalizing the results obtained in [35] we now derive expressions for fluctuations and correlations of densities and currents for arbitrary initial conditions.

For two time-averaged densities ρ_t^U,ρ_t^V, the covariance (variance for $U=V$) is given by

$$\begin{aligned}\left\langle\rho_t^U\rho_t^V\right\rangle-\left\langle\rho_t^U\right\rangle\left\langle\rho_t^V\right\rangle&=t^{-2}\int_0^t\mathrm{d}\tau\int_0^t\mathrm{d}\tau'\left[\langle U(\mathbf{x}_\tau,\tau)V(\mathbf{x}_{\tau'},\tau')-\langle U(\mathbf{x}_\tau,\tau)\rangle\left\langle V(\mathbf{x}_{\tau'},\tau')\right\rangle\rangle\right]\\&=t^{-2}\int_0^t\mathrm{d}\tau\int_0^t\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}U(\mathbf{x},\tau)V(\mathbf{y},\tau')\left[P(\mathbf{x},\tau;\mathbf{y},\tau')-P(\mathbf{x},\tau)P(\mathbf{y},\tau')\right].\end{aligned}\tag{14}$$

Note that this result can be interpreted as correlations caused by differences of $P(\mathbf{x},\tau;\mathbf{y},\tau')$ and $P(\mathbf{x},\tau)P(\mathbf{y},\tau')$, averaged over time and over functions U,V. More precisely, the two-point function $P(\mathbf{x},\tau;\mathbf{y},\tau')$ can be understood to be characterized by all paths with $\mathbf{x}_\tau=\mathbf{x}$ and $\mathbf{x}_{\tau'}=\mathbf{y}$. For further interpretation, in particular for the case of steady-state dynamics, see [34,35].

For the correlation of $\mathbf{J}_t^U$ and ρ_t^V we first consider the expectation of the product and carry out the same steps as in Eq. (7),

$$\begin{aligned}t^2\left\langle\mathbf{J}_t^U\rho_t^V\right\rangle&=\int_0^t\mathrm{d}\tau'\int_{\tau=0}^{\tau=t}\langle U(\mathbf{x}_\tau,\tau)\circ\mathrm{d}\mathbf{x}_\tau V(\mathbf{x}_{\tau'},\tau')\rangle\\&=\int_0^t\mathrm{d}\tau\int_0^t\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}U(\mathbf{x},\tau)V(\mathbf{y},\tau')\hat{\mathbf{j}}(\mathbf{x},\tau)P(\mathbf{y},\tau';\mathbf{x},\tau)\\&\quad+\int_0^t\mathrm{d}\tau'\int_{\tau=0}^{\tau=t}\langle U(\mathbf{x}_\tau,\tau)\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau V(\mathbf{x}_{\tau'},\tau')\rangle\,.\end{aligned}\tag{15}$$

Comparing with the calculation in Eq. (7), the noise term no longer vanishes since terms with $\tau<\tau'$ give non-trivial correlations according to Eq. (11), which in turn gives

$$\begin{aligned}&\int_0^t\mathrm{d}\tau'\int_{\tau=0}^{\tau=t}\langle U(\mathbf{x}_\tau,\tau)\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau V(\mathbf{x}_{\tau'},\tau')\rangle=\int_0^t\mathrm{d}\tau'\int_0^t\mathrm{d}\tau\mathbb{1}_{\tau<\tau'}\times\\&\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}U(\mathbf{x},\tau)V(\mathbf{y},\tau')\left[2P(\mathbf{x},\tau)\mathbf{D}(\mathbf{x},\tau)\nabla_\mathbf{x}P(\mathbf{x},\tau)^{-1}\right]P(\mathbf{y},\tau';\mathbf{x},\tau),\end{aligned}\tag{16}$$

where we rewrote $P(\mathbf{y},\tau'|\mathbf{x},\tau) = P(\mathbf{x},\tau)^{-1}P(\mathbf{y},\tau';\mathbf{x},\tau)$. Introducing the adapted current operator

$$\hat{\mathbf{j}}^{\ddagger}(\mathbf{x},\tau) \equiv \hat{\mathbf{j}}(\mathbf{x},\tau) + 2P(\mathbf{x},\tau)\mathbf{D}(\mathbf{x},\tau)\nabla_{\mathbf{x}}P(\mathbf{x},\tau)^{-1}, \tag{17}$$

we thus obtain from Eq. (15) an expression for the current-density correlation that reads

$$\begin{aligned}\left\langle \mathbf{J}_t^U \rho_t^V \right\rangle - \left\langle \mathbf{J}_t^U \right\rangle \left\langle \rho_t^V \right\rangle = t^{-2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{y} U(\mathbf{x},\tau)V(\mathbf{y},\tau')\times \\ \left[\mathbb{1}_{\tau>\tau'}\hat{\mathbf{j}}(\mathbf{x},\tau) + \mathbb{1}_{\tau<\tau'}\hat{\mathbf{j}}^{\ddagger}(\mathbf{x},\tau)\right]\left[P(\mathbf{y},\tau';\mathbf{x},\tau) - P(\mathbf{x},\tau)P(\mathbf{y},\tau')\right].\end{aligned} \tag{18}$$

Note that to write the expression more compactly, we used that $\hat{\mathbf{j}}(\mathbf{x},\tau)P(\mathbf{x},\tau) = \hat{\mathbf{j}}^{\ddagger}(\mathbf{x},\tau)P(\mathbf{x},\tau) = \mathbf{j}(\mathbf{x},\tau)$. For symmetry reasons and since the difference vanishes, we wrote $\mathbb{1}_{\tau>\tau'}$ instead of $\mathbb{1}_{\tau\geq\tau'}$.

The expression (18) is a natural generalization of Eq. (14) with the current operators $\hat{\mathbf{j}}, \hat{\mathbf{j}}^{\ddagger}$ appearing. Recall that $\hat{\mathbf{j}}$ is the current operator entering the Fokker-Planck equation, see Eqs. (2)-(3). The adapted operator $\hat{\mathbf{j}}^{\ddagger}$ defined in Eq. (17) accounts for the fact that trajectories contributing to $P(\mathbf{y},\tau';\mathbf{x},\tau)$ that first visit $\mathbf{x}$ and later $\mathbf{y}$ (i.e. $\tau<\tau'$) have, compared to the Fokker-Planck evolution, altered statistics, since displacements at $\mathbf{x}$ correlate with probabilities of reaching $\mathbf{y}$ later. For the particular case of steady-state systems, the special case of the correlation result (18) and the adapted current operator were discussed in detail, and explained using a generalized time-reversal symmetry, in [34, 35]. Note that for the case of time-homogeneous dynamics (in particular steady-state dynamics defined in Eq. (4)), the Fokker-Planck current operator Eq. (3) does not have an explicit time dependence such that $\hat{\mathbf{j}}(\mathbf{x},\tau)$ in Eq. (18) simplifies to $\hat{\mathbf{j}}(\mathbf{x})$. However, the adapted current operator $\hat{\mathbf{j}}^{\ddagger}(\mathbf{x},\tau)$ defined in Eq. (17) retains explicit time-dependence even for time-homogeneous dynamics. Only in the case of steady-state systems with steady-state initial conditions (where $P(\mathbf{x},\tau) = p_{\mathrm{s}}(\mathbf{x})$ for all τ) $\hat{\mathbf{j}}^{\ddagger}$ has no explicit time dependence, and simplifies to the negative $\hat{\mathbf{j}}$ with inverted steady-state current $\mathbf{j}_{\mathrm{s}} \to -\mathbf{j}_{\mathrm{s}}$ [34, 35].

Covariances of components m, n of time-integrated currents $J_{t,m}^U$ and $J_{t,n}^V$ can be obtained analogously by considering

$$t^2\left\langle J_{t,m}^U J_{t,n}^V \right\rangle = \int_{\tau'=0}^{\tau'=t}\int_{\tau=0}^{\tau=t} \left\langle U(\mathbf{x}_\tau,\tau)\circ \mathrm{d}x_\tau^m V(\mathbf{x}_{\tau'},\tau')\circ \mathrm{d}x_{\tau'}^n\right\rangle, \tag{19}$$

where both $\circ\mathrm{d}\mathbf{x}_t$ increments split into $\mathrm{d}t$ and $\mathrm{d}\mathbf{W}_t$ terms. The $\mathrm{d}\tau\mathrm{d}\tau'$ terms give rise to the current operator $\hat{\mathbf{j}}$ as in Eqs. (7),(15), but now its components $\hat{\mathrm{j}}_m(\mathbf{x},\tau)$ and $\hat{\mathrm{j}}_n(\mathbf{y},\tau')$ appear. The $\mathrm{d}W_\tau \mathrm{d}W_{\tau'}$ term yields (by Itô's isometry, i.e. "delta-correlated white noise")

$$\begin{aligned}&\int_{\tau'=0}^{\tau'=t}\int_{\tau=0}^{\tau=t}\left\langle U(\mathbf{x}_\tau,\tau)\left[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau\right]_m V(\mathbf{x}_{\tau'},\tau')\left[\boldsymbol{\sigma}(\mathbf{x}_{\tau'},\tau')\mathrm{d}\mathbf{W}_{\tau'}\right]_n\right\rangle \\ &= \int_0^t \left\langle U(\mathbf{x}_\tau,\tau)V(\mathbf{x}_\tau,\tau)2D_{mn}(\mathbf{x}_\tau)\right\rangle \mathrm{d}\tau \\ &= 2\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} U(\mathbf{x},\tau)V(\mathbf{x},\tau)D_{mn}(\mathbf{x})P(\mathbf{x},\tau).\end{aligned} \tag{20}$$

The mixed term $\mathrm{d}\tau'\mathrm{d}\mathbf{W}_\tau$ (and equivalently $\mathrm{d}\tau\mathrm{d}\mathbf{W}_{\tau'}$) in Eq. (19) according to calculations as in Eq. (7) and using Eq. (11) gives

$$\int_0^t \mathrm{d}\tau' \int_{\tau=0}^{\tau=t} \langle U(\mathbf{x}_\tau,\tau)\left[\boldsymbol{\sigma}(\mathbf{x}_\tau,\tau)\mathrm{d}\mathbf{W}_\tau\right]_m \left[V(\mathbf{x}_{\tau'},\tau')\mathbf{F}(\mathbf{x}_{\tau'},\tau') + \{\nabla \mathbf{D} V\}(\mathbf{x}_{\tau'},\tau')\right]_n\rangle = \mathbb{1}_{\tau<\tau'}\times$$
$$\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{y} U(\mathbf{x},\tau)V(\mathbf{y},\tau')\hat{\jmath}_n(\mathbf{y},\tau')\left[2P(\mathbf{x},\tau)\nabla_\mathbf{x} P(\mathbf{x},\tau)^{-1}\right]_m P(\mathbf{y},\tau';\mathbf{x},\tau). \tag{21}$$

Collecting all terms and using the and notation $\hat{\jmath}^\ddagger_m$ for the components of $\hat{\mathbf{j}}^\ddagger$ in Eq. (17), we obtain for Eq. (19)

$$t^2\left\langle J^U_{t,m} J^V_{t,n}\right\rangle = 2\int_0^t \mathrm{d}\tau \int \mathrm{d}\mathbf{x} U(\mathbf{x},\tau)V(\mathbf{x},\tau)D_{mn}(\mathbf{x},\tau)P(\mathbf{x},\tau) + \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{y}\times$$
$$U(\mathbf{x},\tau)V(\mathbf{y},\tau')\left[\mathbb{1}_{\tau<\tau'}\hat{\jmath}^\ddagger_m(\mathbf{x},\tau)\hat{\jmath}_n(\mathbf{y},\tau') + \mathbb{1}_{\tau>\tau'}\hat{\jmath}_m(\mathbf{x},\tau)\hat{\jmath}^\ddagger_n(\mathbf{y},\tau')\right] P(\mathbf{y},\tau';\mathbf{x},\tau). \tag{22}$$

From the derivation one sees that the first term is the $\tau = \tau'$-contribution (see also [35]). This is the natural generalization of the results in Eqs. (14) and (18), with the interpretation of non-trivial displacements (and thus $\hat{\mathbf{j}}^\ddagger$ instead of $\hat{\mathbf{j}}$) for currents evaluated at earlier times (see above and [34, 35]). As before, for time-homogeneous dynamics $\hat{\mathbf{j}}(\mathbf{x},\tau)$ simplifies to $\hat{\mathbf{j}}(\mathbf{x})$ and in the special case of steady-state dynamics (see Eq. (4)) with steady-state initial conditions, $\hat{\mathbf{j}}^\ddagger(\mathbf{x},\tau)$ simplifies to $\hat{\mathbf{j}}^\ddagger(\mathbf{x})$. This special case was discussed and explained using generalized time-reversal symmetry in [34, 35].

5. Example

To present a concrete minimal example, we consider a two-dimensional harmonically confined overdamped diffusion in a rotational flow (i.e. an irreversible Ornstein-Uhlenbeck process)

$$\mathrm{d}\mathbf{x}_t = -\begin{bmatrix} 1 & -\Omega \\ \Omega & 1 \end{bmatrix}\mathbf{x}\mathrm{d}t + \sqrt{2}\mathrm{d}\mathbf{W}_t. \tag{23}$$

Assuming that the initial density $P(\mathbf{x},\tau=0)$ is Gaussian, the solution $P(\mathbf{x},\tau)$ of the Fokker-Planck equation corresponding to Eq. (23) is well known to be a Gaussian density for all $\tau \geq 0$ (see e.g. [57]). We choose U to be a two-dimensional Gaussian centered at $\mathbf{z}$ with width h, i.e.

$$U_\mathbf{z}(\mathbf{x}) = \frac{1}{2\pi h^2}\exp\left[-\frac{(\mathbf{x}-\mathbf{z})^2}{2h^2}\right]. \tag{24}$$

Due to the Gaussianity of $P(\mathbf{x},\tau)$ and $U_\mathbf{z}(\mathbf{x})$, all spatial integrals entering the results Eqs. (14), (18) and (22) can be performed analytically, e.g. using the computer algebra system SymPy [60] (as outlined in the Supplemental Material of [34]). The two remaining time-integrals are computed numerically. For simplicity we only consider the (non-steady-state) initial condition in a point, i.e. $P(\mathbf{x},\tau=0)=\delta(\mathbf{x}-\mathbf{x}_0)$. For this initial condition, via a left-right decomposition for the process Eq. (23) (see e.g. [56]) or by solving the Lyapunov equation, we have the time-dependent density

$$P(\mathbf{x},\tau)=\frac{1}{2\pi(1-\mathrm{e}^{-2\tau})}\exp\left[\frac{-\left(\mathbf{x}-\mathrm{e}^{-\tau}\begin{bmatrix}\cos(\Omega\tau) & \sin(\Omega\tau)\\ -\sin(\Omega\tau) & \cos(\Omega\tau)\end{bmatrix}\mathbf{x}_0\right)^2}{2(1-\mathrm{e}^{-2\tau})}\right], \tag{25}$$

i.e. the mean value $\langle\mathbf{x}_\tau\rangle=\mathrm{e}^{-\tau}\begin{bmatrix}\cos(\Omega\tau) & \sin(\Omega\tau)\\ -\sin(\Omega\tau) & \cos(\Omega\tau)\end{bmatrix}\mathbf{x}_0$ moves on a spiral shape towards the center. The case $\Omega=0$ corresponds to the equilibrium process, i.e. harmonically confined overdamped diffusion without rotational flow.

For this example, we compute the density-current correlation vector as in Eq. (18),

$$\begin{aligned}\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)\equiv\left\langle\mathbf{J}_t^{U_\mathbf{z}}\rho_t^{U_\mathbf{z}}\right\rangle_{\mathbf{x}_0}-\left\langle\mathbf{J}_t^{U_\mathbf{z}}\right\rangle_{\mathbf{x}_0}\left\langle\rho_t^{U_\mathbf{z}}\right\rangle_{\mathbf{x}_0}=t^{-2}\int_0^t\mathrm{d}\tau\int_0^t\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{y}\times\\ U_\mathbf{z}(\mathbf{x})U_\mathbf{z}(\mathbf{y})\left[\mathbb{1}_{\tau>\tau'}\hat{\mathbf{j}}(\mathbf{x})+\mathbb{1}_{\tau<\tau'}\hat{\mathbf{j}}^\ddagger(\mathbf{x},\tau)\right]\left[P(\mathbf{y},\tau';\mathbf{x},\tau)-P(\mathbf{x},\tau)P(\mathbf{y},\tau')\right].\end{aligned} \tag{26}$$

with Gaussian $U_\mathbf{z}$ as in Eq. (24). In Fig. 1 we show the time evolution and spatial dependence of this correlation vector. For long times without driving $\Omega=0$, we see that $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)\to 0$. This corresponds to the limit when the initial condition is forgotten, i.e. for long times $\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$ approaches the result of $\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t)$ for steady-state initial conditions where in equilibrium ($\Omega=0$) we have $P(\mathbf{y},\tau';\mathbf{x},\tau)=P(\mathbf{x},\tau';\mathbf{y},\tau)$ (time-reversal symmetry) and $\hat{\mathbf{j}}^\ddagger(\mathbf{x})=-\hat{\mathbf{j}}(\mathbf{x})$ [35], implying $\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t)=\mathbf{0}$ at all $\mathbf{z}$. In the case $\Omega\neq 0$, the correlation $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$ becomes constant for long times, where $\mathbf{C}_{\mathbf{j}\rho}\propto t^{-1}$ represents the large-deviation limit of the correlation result, which agrees with the large-deviation limit for the process starting in steady-state initial conditions [35]. This has a spatial dependence similar to the steady-state current $\mathbf{j}_\mathrm{s}(\mathbf{z})$ but averaged over the Gaussian $U_\mathbf{z}$. By comparison with the color gradient we see in all panels in Fig. 1, as expected, that large values of the correlation can only occur at points that are visited for a significant amount of time, i.e. with not too small $\left\langle\rho_t^{U_\mathbf{z}}\right\rangle$.

In addition to the qualitative behavior shown in Fig. 1, we present a quantitative evaluation of the correlation result multiplied by time, $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$, for a single $\mathbf{z}$ in Fig. 2. Simulations shown in Fig. 2 confirm the theoretical result in Eq. (18) (re-stated in Eq. (26)). As mentioned above this result approaches the large-deviation limit for long times. Moreover, for long times the initial condition will become irrelevant, i.e. $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$ approaches the result for $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t)$ for steady-state initial conditions [35]. First note that, due to the time-integration, deviations for short times are only slowly 'forgotten' with order t^{-1} (instead of exponentially fast with some Poincaré time scale). Interestingly, we see in Fig. 2a that for substantial coarse-graining (i.e. rather large

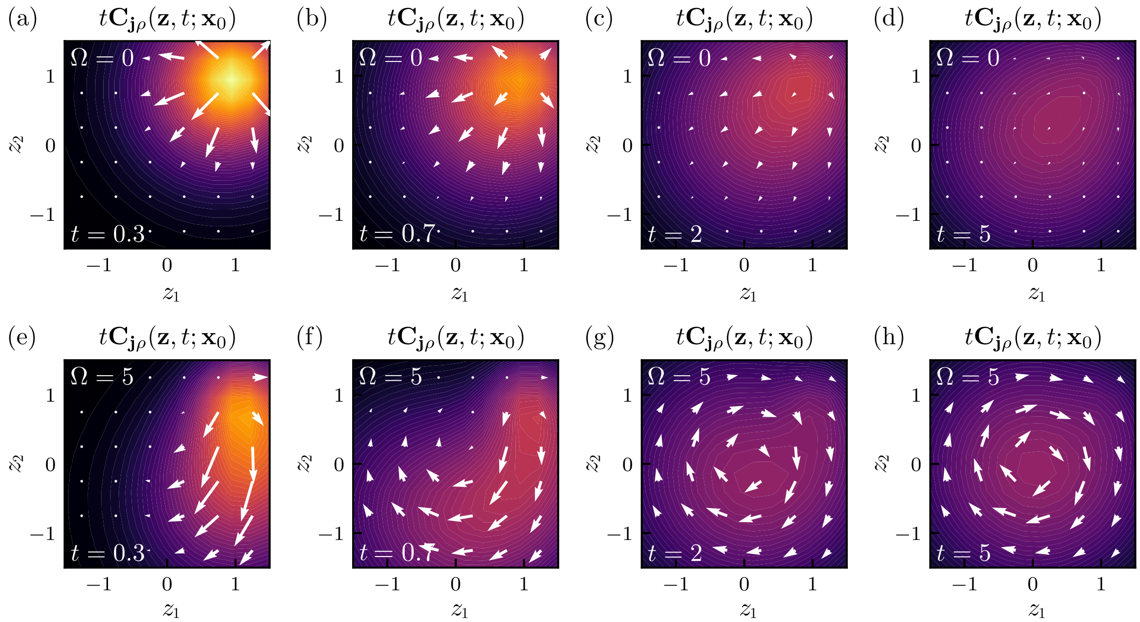

Figure 1. White arrows depict the correlation result multiplied by time, $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$ as in Eq. (18) with $\mathbf{x}_0 = (1,1)^T$ for the process in Eq. (23) with $\Omega = 0$ in (a-d), $\Omega = 5$ in (e-h), and $U_{\mathbf{z}}$ as in Eq. (24). The position $\mathbf{z} = (z_1, z_2)^T$ around which the correlation is evaluated varies along the respective axes. The color gradient depicts the mean time-averaged density $\left\langle \rho_t^{U_{\mathbf{z}}} \right\rangle$, i.e. the time spent around $\mathbf{z}$ weighted by $U_{\mathbf{z}}$. Time increases from left (a,e) to right (d,h), $t = 0.3$, 0.7, 2, 5

$h = 0.5$ in $U_{\mathbf{z}}$ in Eq. (24)), the result for $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z},t;\mathbf{x}_0)$ starting in a point only approaches the corresponding value for steady-state initial condition (green curve) in the large deviation regime (black line), but *not* before. Going to smaller coarse graining $h = 0.15$ in Fig. 2b, we see that the process starting in the center $\mathbf{x}_0 = (0,0)^T$ (blue line) features the arguably more intuitive behavior, by first approaching the steady-state (green line) and later the large deviation result (black line). However, for a different initial condition (see orange line) the steady state curve is again only approached in the large deviation limit. This highlights the long-lasting and non-trivial effects of the time-integration and underscores why interpreting time-average observables, in particular those involving currents, remains challenging.

6. Conclusion

To summarize, we presented a new Lemma (12) that enabled us to derive results Eqs. (14), (18) and (22) for correlations and fluctuations of the time-averaged density and current Eq. (5) for general Langevin dynamics defined in Eq. (1) with general initial conditions. This generalization of the recent results derived for non-equilibrium steady states [34, 35] may improve the understanding of inference of densities and currents with the estimators ρ_t^V and $\mathbf{J}_t^U$ (in particular in connection with the notion of coarse graining [34, 35]) in cases where the dynamics does *not* evolve from the steady-state, or is not

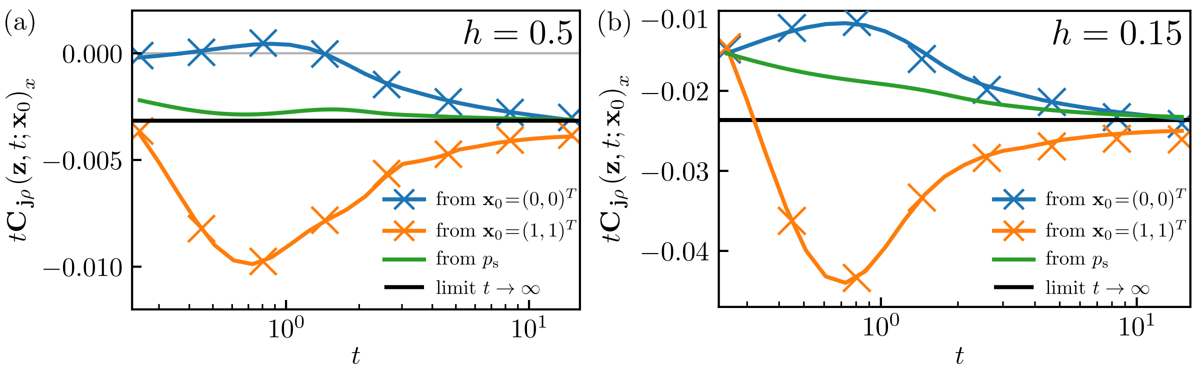

Figure 2. Quantitative depiction of the time-dependence of the x-component of the current-density correlation $t\mathbf{C}_{\mathbf{j}\rho}(\mathbf{z}, t; \mathbf{x}_0)_x$ with $\mathbf{z} = (0, -0.2)^T$ for the process in Eq. (23) with $\Omega = 3$ and $U_{\mathbf{z}}$ as in Eq. (24) with (a) $h = 0.5$ and (b) $h = 0.15$ for different initial conditions (colors). The new analytical result (blue and orange lines; Eq. (18)) is confirmed by simulations (crosses; for each t, (a) 10^5 and (b) 10^6 trajectories with 10^3 time-steps each were simulated according to the stochastic Euler algorithm). For $t \to \infty$, irrespective of the initial condition, all result approach the same large-deviation limit.

time-homogeneous. Importantly, the strategy of inferring dissipation from the current variance (see Eq. (22)) via the thermodynamic uncertainty relation (TUR) [41–44] remains valid. Generalized versions of the TUR, e.g. for general initial conditions [45] or time-dependent dynamics [10], already exist. A recently improved version of the TUR that includes current-density correlations (see Eq. (18)) is, however, so far only available for steady-state systems with steady-state initial conditions [15]. Notably, as we will show in a forthcoming publication, Lemma (12) allows the correlation-TUR to also be proved for transient dynamics.

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References

[1] A. Rebenshtok and E. Barkai, “Weakly non-ergodic statistical physics,” *J. Stat. Phys.* **133** (2008) 565.

[2] S. Burov, J.-H. Jeon, R. Metzler, and E. Barkai, “Single particle tracking in systems showing anomalous diffusion: the role of weak ergodicity breaking,” *Phys. Chem. Chem. Phys.* **13** (2011) 1800.

[3] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, “Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking,” *Phys. Chem. Chem. Phys.* **16** (2014) 24128.

[4] H. Touchette, “The large deviation approach to statistical mechanics,” *Phys. Rep.* **478** (2009) 1.

[5] R. Chetrite and H. Touchette, “Nonequilibrium Markov processes conditioned on large deviations,” *Annales Henri Poincaré* **16** (2014) 2005.

[6] H. Touchette, “Introduction to dynamical large deviations of Markov processes,” *Phys. A* **504** (2018) 5.

[7] E. Mallmin, J. du Buisson, and H. Touchette, "Large deviations of currents in diffusions with reflective boundaries," *J. Phys. A: Math. Theor.* **54** (2021) 295001.

[8] F. Coghi, R. Chetrite, and H. Touchette, "Role of current fluctuations in nonreversible samplers," *Phys. Rev. E* **103** (2021) 062142.

[9] U. Seifert, "Stochastic thermodynamics: From principles to the cost of precision," *Phys. A* **504** (2018) 176.

[10] T. Koyuk and U. Seifert, "Thermodynamic uncertainty relation for time-dependent driving," *Phys. Rev. Lett.* **125** (2020) 260604.

[11] P. Pietzonka, A. C. Barato, and U. Seifert, "Universal bounds on current fluctuations," *Phys. Rev. E* **93** (2016) 052145.

[12] U. Seifert, "Entropy production along a stochastic trajectory and an integral fluctuation theorem," *Phys. Rev. Lett.* **95** (2005) 040602.

[13] S. Pigolotti, I. Neri, É. Roldán, and F. Jülicher, "Generic properties of stochastic entropy production," *Phys. Rev. Lett.* **119** (2017) 140604.

[14] U. Seifert, "Stochastic thermodynamics, fluctuation theorems and molecular machines," *Rep. Prog. Phys.* **75** (2012) 126001.

[15] A. Dechant and S.-i. Sasa, "Improving thermodynamic bounds using correlations," *Phys. Rev. X* **11** (2021) 041061.

[16] A. Dechant and S.-i. Sasa, "Continuous time reversal and equality in the thermodynamic uncertainty relation," *Phys. Rev. Research* **3** (2021) 042012.

[17] M. Kac, "On distributions of certain Wiener functionals," *Trans. Am. Math. Soc.* **65** (1949) 1.

[18] D. A. Darling and M. Kac, "On occupation times for Markoff processes," *Trans. Am. Math. Soc.* **84** (1957) 444.

[19] E. Aghion, D. A. Kessler, and E. Barkai, "From non-normalizable Boltzmann-Gibbs statistics to infinite-ergodic theory," *Phys. Rev. Lett.* **122** (2019) 010601.

[20] S. Carmi and E. Barkai, "Fractional Feynman-Kac equation for weak ergodicity breaking," *Phys. Rev. E* **84** (2011) 061104.

[21] S. N. Majumdar and A. Comtet, "Local and occupation time of a particle diffusing in a random medium," *Phys. Rev. Lett.* **89** (2002) 060601.

[22] S. N. Majumdar and D. S. Dean, "Exact occupation time distribution in a non-Markovian sequence and its relation to spin glass models," *Phys. Rev. E* **66** (2002) 041102.

[23] S. N. Majumdar, "Brownian functionals in physics and computer science," *Curr. Sci.* **89** (2005) 2075.

[24] A. J. Bray, S. N. Majumdar, and G. Schehr, "Persistence and first-passage properties in nonequilibrium systems," *Adv. Phys.* **62** (2013) 225.

[25] G. Bel and E. Barkai, "Weak ergodicity breaking in the continuous-time random walk," *Phys. Rev. Lett.* **94** (2005) 240602.

[26] A. Lapolla, D. Hartich, and A. Godec, "Spectral theory of fluctuations in time-average statistical mechanics of reversible and driven systems," *Phys. Rev. Research* **2** (2020) 043084.

[27] C. Maes, K. Netočný, and B. Wynants, "Steady state statistics of driven diffusions," *Phys. A* **387** (2008) 2675.

[28] S. Kusuoka, K. Kuwada, and Y. Tamura, "Large deviation for stochastic line integrals as L^p -currents," *Probab. Theory Relat. Fields* **147** (2009) 649.

[29] R. Chetrite and H. Touchette, "Nonequilibrium microcanonical and canonical ensembles and their equivalence," *Phys. Rev. Lett.* **111** (2013) 120601.

[30] A. C. Barato and R. Chetrite, "A formal view on level 2.5 large deviations and fluctuation relations," *J. Stat. Phys.* **160** (2015) 1154.

[31] J. Hoppenau, D. Nickelsen, and A. Engel, "Level 2 and level 2.5 large deviation functionals for systems with and without detailed balance," *New J. Phys.* **18** (2016) 083010.

[32] C. Monthus, "Inference of Markov models from trajectories via large deviations at level 2.5 with applications to random walks in disordered media," *J. Stat. Mech: Theory Exp.* **2021** (2021)

063211.

[33] D. Hartich and A. Godec, "Emergent memory and kinetic hysteresis in strongly driven networks," *Phys. Rev. X* **11** (2021) 041047.

[34] C. Dieball and A. Godec, "Mathematical, thermodynamical, and experimental necessity for coarse graining empirical densities and currents in continuous space," `arXiv:2105.10483 [cond-mat.stat-mech]`.

[35] C. Dieball and A. Godec, "Coarse graining empirical densities and currents in continuous-space steady states," `arXiv:2204.06553 [cond-mat.stat-mech]`.

[36] D. S. Grebenkov, "Probability distribution of the time-averaged mean-square displacement of a gaussian process," *Phys. Rev. E* **84** (2011) 031124.

[37] D. S. Grebenkov, "Time-averaged quadratic functionals of a gaussian process," *Phys. Rev. E* **83** (2011) 061117.

[38] D. Boyer, D. S. Dean, C. Mejía-Monasterio, and G. Oshanin, "Optimal fits of diffusion constants from single-time data points of brownian trajectories," *Phys. Rev. E* **86** (2012) 060101.

[39] D. Boyer, D. S. Dean, C. Mejía-Monasterio, and G. Oshanin, "Optimal estimates of the diffusion coefficient of a single brownian trajectory," *Phys. Rev. E* **85** (2012) 031136.

[40] D. Boyer, D. S. Dean, C. Mejía-Monasterio, and G. Oshanin, "Distribution of the least-squares estimators of a single brownian trajectory diffusion coefficient," *Journal of Statistical Mechanics: Theory and Experiment* **2013** (2013) P04017.

[41] A. C. Barato and U. Seifert, "Thermodynamic uncertainty relation for biomolecular processes," *Phys. Rev. Lett.* **114** (2015) 158101.

[42] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, "Dissipation bounds all steady-state current fluctuations," *Phys. Rev. Lett.* **116** (2016) 120601.

[43] T. R. Gingrich, G. M. Rotskoff, and J. M. Horowitz, "Inferring dissipation from current fluctuations," *J. Phys. A: Math. Theor.* **50** (2017) 184004.

[44] J. M. Horowitz and T. R. Gingrich, "Thermodynamic uncertainty relations constrain non-equilibrium fluctuations," *Nat. Phys.* **16** (2019) 15–20.

[45] A. Dechant and S.-i. Sasa, "Current fluctuations and transport efficiency for general Langevin systems," *J. Stat. Mech: Theory Exp.* **2018** (2018) 063209.

[46] P. Pietzonka, F. Ritort, and U. Seifert, "Finite-time generalization of the thermodynamic uncertainty relation," *Phys. Rev. E* **96** (2017) 012101.

[47] K. Liu, Z. Gong, and M. Ueda, "Thermodynamic uncertainty relation for arbitrary initial states," *Phys. Rev. Lett.* **125** (2020) 140602.

[48] T. Koyuk and U. Seifert, "Operationally accessible bounds on fluctuations and entropy production in periodically driven systems," *Phys. Rev. Lett.* **122** (2019) 230601.

[49] T. Koyuk, U. Seifert, and P. Pietzonka, "A generalization of the thermodynamic uncertainty relation to periodically driven systems," *J. Phys. A* **52** (2018) 02LT02.

[50] A. C. Barato and U. Seifert, "Cost and precision of brownian clocks," *Phys. Rev. X* **6** (2016) 041053.

[51] U. Seifert, "Stochastic thermodynamics: From principles to the cost of precision," *Physica A* **504** (2018) 176. Lecture Notes of the 14th International Summer School on Fundamental Problems in Statistical Physics.

[52] D. Hartich and A. Godec, "Thermodynamic uncertainty relation bounds the extent of anomalous diffusion," *Phys. Rev. Lett.* **127** (2021) 080601.

[53] P. Hänggi and H. Thomas, "Stochastic processes: Time evolution, symmetries and linear response," *Phys. Rep.* **88** (1982) 207–319.

[54] Y. Klimontovich, "Ito, Stratonovich and kinetic forms of stochastic equations," *Physica A* **163** (1990) 515–532.

[55] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North Holland, 1st edition ed., 1981. eBook ISBN: 9780080960128.

[56] H. Risken, *The Fokker-Planck Equation*. Springer Berlin Heidelberg, 1989.

[57] C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry, and the natural sciences*. Springer-Verlag, Berlin New York, 1985.

[58] C. Dieball and A. Godec, "Feynman-Kac theory of time-integrated functionals: Itô versus functional calculus," `arXiv:2206.04034 [cond-mat.stat-mech]`.

[59] J. L. Doob, "Conditional Brownian motion and the boundary limits of harmonic functions," *Bull. Soc. Math. Fra.* **85** (1957) 431.

[60] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. R. Terrel, v. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, and A. Scopatz, "SymPy: symbolic computing in Python," *PeerJ Comput. Sci.* **3** (2017) e103.