On the Hilbert scheme of linearly normal curves in $\mathbb{P}^r$ with small index of speciality

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Abstract. We study the Hilbert scheme $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ parametrizing smooth, irreducible, non-degenerate and linearly normal curves of degree d and genus g in $\mathbb{P}^r$ whose complete and very ample hyperplane linear series $\mathcal{D}$ have relatively small index of speciality $i(\mathcal{D}) = g-d+r$. In particular we show the existence (and non-existence as well in some sporadic cases) of every Hilbert scheme of linearly normal curves with $i(\mathcal{D}) = 4$. We also determine the irreducibility of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $3 \leq r \leq 8$, which are rather peculiar families in a certain sense.

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1. An overview, preliminaries and basic set-up

Let $\mathcal{H}_{d,g,r}$ be the Hilbert scheme of smooth, irreducible and non-degenerate curves of degree d and genus g in $\mathbb{P}^r$. We denote by $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ the union of those components of $\mathcal{H}_{d,g,r}$ whose general element is linearly normal. By abuse of terminology, we say that a component of the Hilbert scheme $\mathcal{H}_{d,g,r}$ has index of speciality α, if the hyperplane series $\mathcal{D} = g^r_d$ of a general element of the component has index of speciality α, i.e.

$$h^1(C, \mathcal{D}) = g - d + \dim |\mathcal{D}| = \alpha \geq g - d + r.$$

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In case $\alpha \gneq g-d+r$, the linear series $\mathcal{D}$ is incomplete and it is possible that there may exist a component of the Hilbert scheme which has index of speciality strictly greater than $g-d+r$; cf. [12] and references therein, especially in the paragraph before the statement of Main Theorem in the Introduction, in which the authors also provide useful remarks on connections between Hilbert scheme business and several other areas in algebraic geometry. However the index of speciality of any component of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ is $g-d+r$, which one may define as the **index of speciality of a Hilbert scheme of linearly normal curves**.

We recall the following modified assertion of Severi, which has been given attention by some authors recently; cf. [5, 19].

Modified Assertion of Severi. A nonempty $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ is irreducible for any triple (d,g,r) in the Brill-Noether range

$$\rho(d,g,r)=g-(r+1)(g-d+r)\geq 0,$$

where $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ is the union of those components of $\mathcal{H}_{d,g,r}$ whose general element is linearly normal.

We note that the modified assertion of Severi makes sense only if the index of speciality of Hilbert scheme of linearly normal curves $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ is non-negative. Through a preliminary attempt to settle down the Modified Assertion of Severi, which is still in its infancy, one has quite extensive knowledge about the Hilbert scheme of linearly normal curves of small index of speciality $\alpha \leq 3$ which can be summarized as follows; cf. [16, 5, 19].

- $\alpha=0$: $\mathcal{H}^{\mathcal{L}}_{g+r,g,r}\neq\emptyset$ and irreducible.
- $\alpha=1$: $\mathcal{H}^{\mathcal{L}}_{g+r-1,g,r}\neq\emptyset$ and irreducible for $g\geq r+1$ and is empty for $g\leq r$.
- $\alpha=2$: $\mathcal{H}^{\mathcal{L}}_{g+r-2,g,r}\neq\emptyset$ and irreducible for every $g\geq r+3$ and is empty if $g\leq r+2$.
- $\alpha=3$: $\mathcal{H}^{\mathcal{L}}_{g+r-3,g,r}=\emptyset$ for $g\leq r+4$. For $r\geq 5$ and $g\geq r+5$, $\mathcal{H}^{\mathcal{L}}_{g+r-3,g,r}\neq\emptyset$ unless $g=r+6$ and $r\geq 10$. Moreover $\mathcal{H}^{\mathcal{L}}_{g+r-3,g,r}$ is irreducible for every $g\geq 2r+3$ and is reducible for almost all g in the range $r+5\leq g\leq 2r+2$.

One obvious advantage in considering the Hilbert scheme of linearly normal curves according to its index of speciality α is that the residual series of the very ample hyperplane series corresponding to a general element of any component of the Hilbert scheme has fixed dimension $\alpha-1$ regardless of values of genus g and degree d. Therefore one may work more effectively in exploring out several properties of Hilbert schemes under consideration by looking at the family of curves in a fixed projective space $\mathbb{P}^{\alpha-1}$ induced by the residual series of hyperplane series. Especially when α is low, approaching this way becomes somewhat useful.

In case $\alpha=3$, the residual series of hyperplane series are nets and it is possible to derive the basic properties such as irreducibility, existence as well

as the number of moduli of the Hilbert scheme of linearly normal curves by considering the corresponding property of the Severi variety of plane curves, which has been proven to be effective to a certain extent in some previous works on the subject; [16, 5, 18, 19]. In this paper we consider Hilbert scheme of linear normal curves with index of speciality $\alpha = 4$. Equivalently we study the Hilbert schemes of non-degenerate, smooth and linearly normal curves in $\mathbb{P}^r$ $(r \geq 3)$ of degree $d = g + r - 4$. We show the existence as well as non-existence of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ for every triple (d, g, r) with $g - d + r = 4$. We also determine the irreducibility of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ for specific triples $(d, g, r) = (2r+4, r+8, r)$ with $3 \leq r \leq 8$ which is of some particular interest. We also study $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ for the triples $(d, g, r) = (2r + 5, r + 9, r)$ $(3 \leq r \leq 11)$ and give a partial irreducibility result.

For notation and conventions, we usually follow those in [3, 4, 13]; e.g. $\pi(d, r)$ is the maximal possible arithmetic genus of an irreducible and non-degenerate curve of degree d in $\mathbb{P}^r$. $\pi_1(d, r)$ is the so-called the second Castelnuovo genus bound of an irreducible and non-degenerate curve of degree d in $\mathbb{P}^r$ not lying on a surface of degree $r-1$; cf. [13, Theorem 3.13, 3.15]. In particular if $r = 3$, $\pi_1(d, 3)$ is the maximal possible arithmetic genus of an irreducible and non-degenerate curve of degree d in $\mathbb{P}^3$ not lying on a quadric surface. Following classical terminology, a linear series of degree d and dimension r on a smooth irreducible curve C is usually denoted by g^r_d. A base-point-free linear series g^r_d $(r \geq 2)$ on a smooth curve C is called birationally very ample when the morphism $C \to \mathbb{P}^r$ induced by g^r_d is generically one-to-one (or birational) onto its image. A base-point-free linear series g^r_d on C is said to be compounded of a covering (compounded for short) if the morphism induced by g^r_d gives rise to a non-trivial covering map $C \to C'$ of degree $k \geq 2$. For a complete linear series $\mathcal{E}$ on a smooth curve C, the residual series $|K_C - \mathcal{E}|$ is denoted by $\mathcal{E}^\vee$. We work over the field of complex numbers.

The organization of this paper is as follows. We briefly recall some terminologies and basic preliminaries in the remainder of this section. In the second section we determine all the triples (d, g, r), $r \geq 3$ for which $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} \neq \emptyset$.

In subsequent sections, we determine the irreducibility of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for $g = r + 8$ by using the irreducibility of Severi varieties of nodal curves on a non-singular quadric surface in $\mathbb{P}^3$. In the final section – with an open end – we make a discussion on a couple of related aspects of our study, such as the next case $d = g + r - 5$ for further investigation, in addition to the existence and partial irreducibility result of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for $g = r + 9$ $(3 \leq r \leq 11)$.

We briefly recall several fundamental facts for our study of Hilbert schemes which are well-known; cf. [4] or [2, §1 and §2]. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. Given an isomorphism class $[C] \in \mathcal{M}_g$ corresponding to a smooth irreducible curve C, there exist a neighborhood $U \subset \mathcal{M}_g$ of the class $[C]$ and a smooth connected variety $\mathcal{M}$ which is a finite ramified covering $h : \mathcal{M} \to U$, as well as varieties $\mathcal{C}$, $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ proper over $\mathcal{M}$ with the following properties:

(1) $\xi : \mathcal{C} \to \mathcal{M}$ is a universal curve, i.e. for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$,
(2) $\mathcal{W}^r_d$ parametrizes the pairs (p, L) where L is a line bundle of degree d and $h^0(L) \geq r+1$ on $\xi^{-1}(p)$,
(3) $\mathcal{G}^r_d$ parametrizes the couples $(p, \mathcal{D})$, where $\mathcal{D}$ is possibly an incomplete linear series of degree d and dimension r on $\xi^{-1}(p)$.

Given an irreducible family $\mathcal{F} \subset \mathcal{G}^r_d$ with some geometric meaning, if a general element of $\mathcal{F}$ is complete, the closure of the family $\{\mathcal{E}^\vee | \mathcal{E} \in \mathcal{F},\ \mathcal{E} \text{ is complete}\} \subset \mathcal{G}^{g-d+r-1}_{2g-2-d}$ is sometimes denoted by $\mathcal{F}^\vee$.

Let $\widetilde{\mathcal{G}}$ ($\widetilde{\mathcal{G}}_{\mathcal{L}}$, respectively) be the union of components of $\mathcal{G}^r_d$ whose general element $(p, \mathcal{D})$ corresponds to a very ample (very ample and complete, respectively) linear series $\mathcal{D}$ on the curve $C = \xi^{-1}(p)$. By recalling that an open subset of $\mathcal{H}_{d,g,r}$ consisting of elements corresponding to smooth irreducible and non-degenerate curves is a $\mathbb{P}\mathrm{GL}(r+1)$-bundle over an open subset of $\widetilde{\mathcal{G}}$, the irreducibility of $\widetilde{\mathcal{G}}$ guarantees the irreducibility of $\mathcal{H}_{d,g,r}$. Likewise, the irreducibility of $\widetilde{\mathcal{G}}_{\mathcal{L}}$ ensures the irreducibility of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$. We recall the following regarding the schemes $\mathcal{G}^r_d$ which is also well-known; cf. [13, 2.a] and [4, Ch. 21, §3, 5, 6, 11, 12].

Proposition 1.1. *For non-negative integers d, g and $r \geq 1$, let*

$$\rho(d, g, r) := g - (r+1)(g - d + r)$$

be the Brill-Noether number. The dimension of any component of $\mathcal{G}^r_d$ is at least

$$\lambda(d, g, r) := 3g - 3 + \rho(d, g, r),$$

hence the minimal possible dimension of any component of $\mathcal{H}_{d,g,r}$ is

$$\mathcal{X}(d, g, r) := \lambda(d, g, r) + \dim \mathbb{P}GL(r+1).$$

Moreover, if $\rho(d, g, r) \geq 0$, there exists a unique component (called the principal component) $\mathcal{G}_0$ of $\widetilde{\mathcal{G}}$ which dominates $\mathcal{M}$ (or $\mathcal{M}_g$).

2. Existence and non-existence of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$

In this section we prove the existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for every $g \geq r+7$ ($r \geq 5$) unless $g = r+8$ ($r \geq 9$) or $g = r+9$ ($r \geq 12$), in which cases we have $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} = \emptyset$. For $r = 3,\ 4$, we quote the existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ which are known to some people and to the author (or as folklores) as follows.

Remark 2.1. (i) For $r = 3$, it is easy to see that

$$\mathcal{H}^{\mathcal{L}}_{g+r-4,g,3} = \mathcal{H}^{\mathcal{L}}_{g-1,g,3} \neq \emptyset \text{ if and only if } g \geq r+6 = 9.$$

Note that by Castelnuovo genus bound, $\mathcal{H}^{\mathcal{L}}_{g-1,g,3} = \emptyset$ for $g \leq 8$. For $g = 9$, (extremal) curves of type $(4, 4)$ on quadrics form the only irreducible component of degree $d = 8$ and genus $g = 9$ of dimension

$$\dim \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)) + \dim \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 33 \gneq 4 \cdot 8 = \mathcal{X}(8, g, 3).$$

For $g = 10$, there exist curves of type $(3, 6)$ on smooth quadrics which form an irreducible family of dimension

$$\dim \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3, 6)) + \dim \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 36.$$

Moreover, complete intersections of two irreducible cubics form another family of the same expected dimension

$$\dim \mathbb{G}(1, \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)))) = \dim \mathbb{G}(1, 19) = 36.$$

Hence $\mathcal{H}_{9,10,3} = \mathcal{H}^{\mathcal{L}}_{9,10,3} \neq \emptyset$ and is reducible. We further note that the inequality

$$g \leq \frac{(g-2)(g-3)}{6} = \frac{(d-1)(d-2)}{6} \leq \pi_1(d, 3)$$

is valid for every pair $(d, g) = (g - 1, g)$ with $g \geq 11$, which guarantees the existence of a smooth curve of degree $g-1$ and genus g in $\mathbb{P}^3$; cf. [11, Theorem 2.4].

(ii) For the existence of smooth curves of genus g and degree $d = g+r-4 = g$ in $\mathbb{P}^4$, one may deduce that

$$\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} = \mathcal{H}^{\mathcal{L}}_{g,g,4} \neq \emptyset \text{ if and only if } g \geq 11 = r + 7$$

as follows. In general, in the range $d \leq g + r$ inside the Brill-Noether range $\rho(d, g, r) \geq 0$, the principal component $\mathcal{G}_0$ which has the expected dimension $\lambda(d, g, r)$ is one of the components of $\widetilde{\mathcal{G}}_{\mathcal{L}}$ (cf. [13, 2.1 page 70]), and hence $\widetilde{\mathcal{G}}_{\mathcal{L}}$ or $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ is non-empty. In particular for $g \geq 20$, $\mathcal{H}^{\mathcal{L}}_{g,g,4} \neq \emptyset$. For $(d, g, 4) = (g, g, 4)$ outside the Brill-Noether range $11 \leq g \leq 19$, we refer [20, Theorem 1.0.2]. We also remark that the proof of our existence theorem (Theorem 2.4) works for several cases $(d, g, r) = (g, g, 4)$ with negative Brill-Noether number. For $g = 16$ or $g = 19$, there exist smooth curves of degree $d = g$ on Bordiga surface in $\mathbb{P}^4$. The Castelnuovo genus bound implies the the non-existence of $\mathcal{H}^{\mathcal{L}}_{g,g,4}$ for $g \leq 10$; cf. Table 1 at the end of this section.

The following lemma - regarding multiples of the unique pencil g^1_k on a general k-gonal curve - is useful to show the existence of a very ample linear series having index of speciality $\alpha \leq 4$.

Lemma 2.2. [7, Proposition 1.1] *Assume $2k - g - 2 < 0$. Let C be a general k-gonal curve of genus g, $k \geq 2$, $0 \leq m$, $n \in \mathbb{Z}$ such that*

$$g \geq 2m + n(k - 1) \tag{1}$$

and let $D \in C_m$ be an effective divisor of degree m on C. Assume that there is no $F \in g^1_k$ with $F \leq D$. Then $\dim |ng^1_k + D| = n$.

The following easy lemma – which is adapted to our specific situation – will be used to show the non-existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for $g = r + 8$ $(r \geq 9)$ and $g = r + 9$ $(r \geq 12)$.

Lemma 2.3. *Let $\mathcal{E} = g^3_e$ (with $e = 10, 11$) be a complete and special linear series (possibly with non-empty base locus Δ) on a smooth curve C of genus g. Suppose $\mathcal{E}$ is compounded. Then $\mathcal{E}^\vee$ – the residual series of $\mathcal{E}$ – is not very ample.*

Proof. Let k be the degree of the morphism $C \stackrel{\phi}{\to} E \subset \mathbb{P}^3$ induced by the base-point-free part of $\mathcal{E}$. Since $e \geq k \cdot \deg \phi(C)$ and $\deg \phi(C) \geq 3$, the following three cases are possible;

(a) $k = 3$ and C is trigonal with $\mathcal{E} = |3g^1_3 + \Delta|$, $\Delta \neq \emptyset$.

(b) $k = 2$ and C is bielliptic with $\mathcal{E} = |\phi^*(g^3_4) + \Delta|$, where $C \stackrel{\phi}{\to} E$ is a double cover of an elliptic curve E.

(c) $k = 2$ and $C \stackrel{\phi}{\to} E$ is a double cover of a curve E of genus $h = 2$, $\mathcal{E} = |\phi^*(g^3_5) + \Delta|$.

(a) If C is trigonal, let $q \in \Delta$ and let $q + t + s \in g^1_3$ be the unique trigonal divisor containing q. We then have

$$\begin{aligned} \dim |\mathcal{E} + t + s| &= \dim |3g^1_3 + \Delta + t + s| = \dim |3g^1_3 + (q + t + s) + (\Delta - q)| \\ &= \dim |4g^1_3 + (\Delta - q)| \geq \dim \mathcal{E} + 1. \end{aligned}$$

(b) If $k = 2$ and C is bielliptic, then for any $u \in E$,

$$\dim |\mathcal{E} + \phi^*(u)| = \dim |\phi^*(g^3_4 + u) + \Delta| = \dim |\phi^*(g^4_5) + \Delta| \geq \dim \mathcal{E} + 1$$

(c) For any $u \in E$, we also have

$$\dim |\mathcal{E} + \phi^*(u)| = \dim |\phi^*(g^3_5 + u) + \Delta| = \dim |\phi^*(g^4_6) + \Delta| \geq \dim \mathcal{E} + 1.$$

By Riemann-Roch, it follows that

$$\dim |\mathcal{E}^\vee - (r_1 + r_2)| \geq \dim \mathcal{E}^\vee - 1$$

for some $r_1 + r_2 \in C_2$ and hence $\mathcal{E}^\vee$ is not very ample. □

We assume $r \geq 5$ in the next theorem, just because the existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for $r = 3, 4$ is completely known as we discussed in Remark 2.1.

Theorem 2.4. (a) $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} = \emptyset$ *for* $g \leq r + 6$, $r \geq 5$.

(b) $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} \neq \emptyset$ *for* $g = r + 7$ *or for any* $g \geq r + 10$, $r \geq 5$.

(c) $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} \neq \emptyset$ *if and only if* $5 \leq r \leq 8$ *for* $g = r + 8$, $r \geq 5$.

(d) $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} \neq \emptyset$ *if and only if* $5 \leq r \leq 11$ *for* $g = r + 9$, $r \geq 5$.

Proof. (a) For $r \geq 5$, we have $g \leq \pi(g + r - 4, r) = 2g - r - 7$ if $m := [\frac{g+r-5}{r-1}] = 2$ (or $g \leq \pi(g + r - 4, r) = g - 4$ if $m = 1$) by Castelnuovo genus bound, which is not compatible with the assumption $g \leq r + 6$.

(b) For $g = r + 7$ and $r \geq 5$, we have $\pi(g + r - 4, r) = r + 7$. Hence the curve corresponding to a general element of a component of $\mathcal{H}^{\mathcal{L}}_{2r+3,r+7,r}$ is an extremal curve, a curve whose arithmetic genus attains its maximal value $\pi(d, r)$. Therefore $\mathcal{H}^{\mathcal{L}}_{2r+3,r+7,r} \neq \emptyset$ which is irreducible except for $r = 5$; cf. [13, Corollary 3.12, page 92]. We now show the existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for every $g \geq r + 10$ and $r \geq 5$. We will establish the existence of corresponding curves in $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ on a suitable family of k-gonal curves.

Let $e := g - r + 2 = \deg \mathcal{E} = \deg |K_C - \mathcal{D}|$; $(p, \mathcal{D}) \in \mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}} \subset \mathcal{G}^r_{g+r-4}$, $C = \xi^{-1}(p)$. We distinguish the following three cases.

(i) $e \equiv 0 \pmod 3$: By the assumption $g \geq r+10$, we may set $e = g-r+2 = 3k \geq 12$ for some $k \geq 4$. Note that

$$\rho(k, g, 1) = 2k - g - 2 = -k - r \lneqq 0$$

and we consider a general k-gonal curve C. Taking $n = 3, m = 2$ in Lemma 2.2, the numerical condition (1) is satisfied; note that $g = 3k + r - 2 \geq 3k + 1$ for $r \geq 3$. Furthermore there is no $F \in g^1_k$ with $F \leq D$ (where $D \in C_2$) since $k \geq 4$. Hence we have

$$\dim |3g^1_k + D| = 3$$

for any $D \in C_2$ by Lemma 2.2 and therefore

$$|K_C - 3g^1_k| = g^r_{g+r-4}$$

is very ample.

(ii) $e \equiv 1 \pmod 3$: Again, by the assumption $g \geq r + 10$, we may set $e = g - r + 2 = 3k + 1 \geq 12$ for some $k \geq 4$. Note that

$$\rho(k, g, 1) = 2k - g - 2 = -k - r - 1 \lneqq 0$$

and we again consider a general k-gonal curve C. For a general choice $q \in C$ and for any $t + s \in C_2$, we take $D = q + t + s$, $m = 3$ and $n = 3$ in Lemma 2.2. The numerical condition (1) holds ($g = 3k + r - 1 \geq 3k + 3$ if $r \geq 4$) and there is no $F \in g^1_k$ with $F \leq D$ just because $k \geq 4$. Hence $\dim |3g^1_k + D| = 3$ which implies that

$$|K_C - 3g^1_k - q| = g^r_{2g-2-3k-1} = g^r_{g+r-4}$$

is very ample.

(iii) $e \equiv 2 \pmod 3$: Again, by the assumption $g \geq r + 10$, we may set $e = g - r + 2 = 3k + 2 \geq 12$ for some $k \geq 4$. Note that

$$\rho(k, g, 1) = 2k - g - 2 = -k - r - 2 \lneqq 0$$

and we consider a general k-gonal curve C. We take $q + q' \in C$ such that $q + q'$ is not in the same fiber of the k-sheeted map onto $\mathbb{P}^1$ defined by the unique g^1_k. For any $t + s \in C_2$, we take $D = q + q' + t + s$, $m = 4$ and $n = 3$ in Lemma 2.2. Again the numerical condition (1) holds (by $r \geq 5$) and there is no $F \in g^1_k$ with $F \leq D$ by our choice of $q + q' \in C_2$. Hence $\dim |3g^1_k + D| = 3$ which implies that

$$|K_C - 3g^1_k - q - q'| = g^r_{2g-2-3k-2} = g^r_{g+r-4}$$

is very ample.

We remark that if $r = 3$ and $g - 1$ is a multiple of 3, the proof (i) in the above is still valid and

$$|K_C - 3g^1_k| = g^3_{g-1}$$

is very ample. Hence $\mathcal{H}^{\mathcal{L}}_{g-1,g,3} \neq \emptyset$ for g such that $g-1 \equiv 0 \pmod 3$. Likewise, for $r=4$ and if $g-2$ is a multiple of 3, then

$$|K_C - 3g^1_k| = g^4_g$$

is very ample. Moreover, if $g-2 = 3k+1$ for some $k \geq 4$ and $r=4$, the proof (ii) in the above remains valid and one may deduce that

$$|K_C - 3g^1_k - q| = g^4_g$$

is very ample for a general $q \in C$. From this observation we have the existence of a smooth linearly normal curves of degree d and genus g in $\mathbb{P}^4$ for $(d,g) = (11,11), (14,14), (15,15), (17,17), (18,18)$; cf. Table 1.

(c) We show the non-existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ in case $g = r+8$ and $r \geq 9$ ($g = r+9$ and $r \geq 12$, respectively) as follows. We set $\mathcal{E} := g^3_{g-r+2} = g^3_e = \mathcal{D}^\vee$ for a general very ample $(p, \mathcal{D}) \in \mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}} \subset \mathcal{G}^r_{g+r-4}$ which has degree $e = 10$ ($e = 11$, respectively) for $g = r+8$ (for $g = r+9$, respectively). If $g = r+8 \gneqq \pi(10,3) = 16$ (if $g = r+9 \gneqq \pi(11,3) = 20$, respectively), $\mathcal{E}$ is compounded inducing a k-sheeted map onto a curve of degree $f \geq 3$ in $\mathbb{P}^3$. Since $k \cdot f \leq e$, a simple numerical calculation leads to the following possibilities;

$(k,f) = (2,5)$ and C is a double cover of a curve of genus $h = 2$
$(k,f) = (2,4)$ and C is bielliptic
$(k,f) = (2,3)$ and C is hyperelliptic
$(k,f) = (3,3)$ and C is trigonal and $\mathcal{E}$ has non-empty base locus.

However $\mathcal{D} = \mathcal{E}^\vee$ is not very ample by Lemma 2.3, leading to a contradiction.

For the existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r} = \mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ with $5 \leq r \leq 8$ and $g = r+8$, we make the following usual convention. Let $S_{t+1} \xrightarrow{\pi} \mathbb{P}^2$ be the surface blown up $\mathbb{P}^2$ at $t+1$ general points $\{p_1, \cdots, p_{t+1}\} \subset \mathbb{P}^2$ with exceptional divisors $E_1, \cdots, E_{t+1}$ whose linear equivalence classes are denoted by $\{e_i\}_{i=1}^{t+1}$. Let $l \in \mathrm{Pic}(S_{t+1})$ be the class of $\pi^*(L)$, where $L \subset \mathbb{P}^2$ is a line. We use the abbreviation

$$(a; b_1, \cdots, b_{t+1})$$

for the linear system

$$|\pi^*(aL) - \sum_{i=1}^{t+1} b_i E_i|$$

on S_{t+1}.

Smooth curves in the very ample linear system $(8; 3^2, 2^2) := (8; 3, 3, 2, 2)$ on S_4 embedded by the anti-canonical linear system $(3; 1^4) := (3; 1, 1, 1, 1)$ has degree

$$(8; 3^2, 2^2) \cdot (3; 1^4) = 14 = 2 \cdot 5 + 4 = 2r + 4$$

in $\mathbb{P}^5$ and genus

$$g = p_a(8l) - 2 \cdot 3 - 2 \cdot 1 = 13 = 5 + 8 = r + 8.$$

Likewise, a smooth curve in $(8; 3, 3, 2)$ ($(8; 3, 3)$ resp.) is embedded into $\mathbb{P}^6$ ($\mathbb{P}^7$ resp.) as a curve of degree $d = 16$ (18 resp.) and genus $g = 14$ ($g = 15$

resp.). For $r = 8$, a curve of degree $e = 10$ and genus $g = 16$ in $\mathbb{P}^3$ - which is a (smooth) extremal curve of type $(5, 5)$ on a (smooth) quadric surface X - is embedded into $\mathbb{P}^8$ by the 2-tuple embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \overset{(2,2)}{\hookrightarrow} \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$$

induced by the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of degree

$$d = (5, 5) \cdot (2, 2) = 2 \cdot e = 20 = 2 \cdot 8 + 4.$$

We also see that $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r} = \mathcal{H}^{\mathcal{L}}_{20,16,8}$ is irreducible, which can be argued as follows. We recall that a general $\mathcal{E} \in \mathcal{G}^{\vee} \subset \mathcal{G}^3_e = \mathcal{G}^3_{10}$ ($\mathcal{G}$ is a component of $\widetilde{\mathcal{G}}_{\mathcal{L}}$) induces a morphism onto a smooth curve of type $(5, 5)$ on a smooth quadric in $\mathbb{P}^3$. On the other hand, curves of type $(5, 5)$ on smooth quadrics in $\mathbb{P}^3$ form the **only** irreducible component of $\mathcal{H}^{\mathcal{L}}_{10,16,3}$. Therefore the family of curves induced by the residual series of the (very ample) hyperplane series of general members of the irreducible Hilbert scheme $\mathcal{H}^{\mathcal{L}}_{10,16,3}$ is also irreducible, which is our $\mathcal{H}^{\mathcal{L}}_{20,16,8}$. $\mathcal{H}^{\mathcal{L}}_{20,16,8}$ has dimension more than expected;

$$\begin{aligned} \dim \mathcal{G} &= \dim \mathcal{G}^{\vee} = \dim \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(5, 5)) \\ &+ \dim \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) - \dim \mathbb{P}GL(4) > \lambda(d, g, r). \end{aligned}$$

We will see in the next section that curves of this type – curves embedded into $\mathbb{P}^r$ by the residual series of the hyperplane series of appropriate (nodal) curves on a smooth quadric in $\mathbb{P}^3$ – form the only irreducible component of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $5 \leq r \leq 8$ except $r = 7$. For $r = 7$, there exists another irreducible component consisting of images of smooth curves in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 6)|$ on smooth quadrics via the embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \overset{(1,3)}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$$

which have degree $(4, 6) \cdot (1, 3) = 18 = 2 \cdot 7 + 4$. The very ampleness of linear systems $(a; b_1, \cdots, b_{t+1})$ on S_{t+1} appearing in this proof follows from [9, Theorem].

(d) This will be treated in the final section together with some partial irreducibility result; cf. Table 3. □

The following table exhibits the existence of smooth curves of degree $d = g$ in $\mathbb{P}^4$, which is a partial application of the proof of Theorem 2.4 (b). We stress that curves described in the list may not form an open dense subset of a component. Furthermore the irreducibility of $\mathcal{H}^{\mathcal{L}}_{g,g,4}$ is not known (to the author) in general except for some particular cases. A smooth curve of degree $d = 11$ and genus $g = 11$ in $\mathbb{P}^4$ is not extremal but nearly extremal ($\pi_1(d, 4) < g < \pi(11, 4)$), which lies on a rational normal scroll in $\mathbb{P}^4$ and such curves form one irreducible family; cf. [13, Corollary 3.16, p. 100]. The irreducibility of $\mathcal{H}^{\mathcal{L}}_{d,g,4}$ for $(d, g) = (12, 12)$ needs more argument, which will be shown in the section following the next section. In the table, X denotes a smooth quadric surface in $\mathbb{P}^3$.

TABLE 1. Smooth curves in $\mathcal{H}^{\mathcal{L}}_{g,g,4}$ for $11 \leq g \leq 19$

(d, g)	Description	Remark
$(11, 11)$	A curve in $\lvert 3H + 2L\rvert$ on a rational normal scroll in $\mathbb{P}^4$ with a very ample $\lvert K_C - 3g^1_3\rvert$	Trigonal
$(12, 12)$	A smooth member in $(8; 3^2, 2^3) := (8; 3, 3, 2, 2, 2)$ on a Del Pezzo, and is birational to a curve in $\lvert(5, 5)\rvert$ on $X \subset \mathbb{P}^3$ with 4 nodes	Pentagonal
$(13, 13)$	A smooth member in $(8; 3, 3, 2, 2, 1)$ on a Del Pezzo, birational to a curve in $\lvert(5, 5)\rvert$ on $X \subset \mathbb{P}^3$ with 3 nodes	Pentagonal
$(14, 14)$	A curve with a very ample $\lvert K_C - 3g^1_4\rvert$	Tetragonal
$(15, 15)$	A curve with a very ample $\lvert K_C - 3g^1_4 - q\rvert$	Tetragonal
$(16, 16)$	On a Bordiga surface	[20, Theorem 1.0.2]
$(17, 17)$	A curve with a very ample $\lvert K_C - 3g^1_5\rvert = g^4_{17}$	Pentagonal
$(18, 18)$	A curve with a very ample $\lvert K_C - 3g^1_5 - q\rvert = g^4_{18}$	Pentagonal
$(19, 19)$	On a Bordiga surface	[20, Theorem 1.0.2]

3. Irreducibility of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ for $g = r + 8$

For $g = r + 8$, $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ is non-empty only for $3 \leq r \leq 8$ by Remark 2.1 and Theorem 2.4. In the next two sections, we determine the irreducibility of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$. For this, we need to recall a couple of generalities regarding the Severi variety of nodal curves on a smooth quadric surface $X \subset \mathbb{P}^3$.

Definition 3.1. (i) Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth quadric surface and we fix $\mathcal{M} = \lvert\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b)\rvert$ a very ample linear system on X. Let

$$p_a(\mathcal{M}) = (a - 1)(b - 1)$$

be the arithmetic genus of curves belonging to $\mathcal{M}$.

(ii) Given an integer g such that $0 \leq g \leq p_a(\mathcal{M})$, we set $\delta = p_a(\mathcal{M}) - g$. We denote by

$$\Sigma_{\mathcal{M},\delta} \subset \lvert\mathcal{M}\rvert = \mathbb{P}(H^0(X, \mathcal{M}))$$

the (equi-singular) Severi variety which is the closure of the locus of integral curves in the linear system $\mathcal{M}$ whose singular locus consists of exactly δ nodes.

(iii) Let $\Sigma_{\mathcal{M},g}$ be the (equi-generic) Severi variety which is the closure of the locus of integral curves of geometric genus g in the linear system $\mathcal{M}$.

We shall make use the following well-known results adopted for our current specific situation. The results quoted below are known to be true in a more general context. Readers are advised to refer [8], which provides an excellent account on Severi varieties on surfaces in general.

Remark 3.2. (i) A general member of every irreducible component of the equi-generic Severi variety $\Sigma_{\mathcal{M},g}$ is a nodal curve; cf. [14, Proposition 2.1].

(ii) The equi-singular Severi variety $\Sigma_{\mathcal{M},\delta}$ is irreducible of dimension

$$\dim |\mathcal{M}| - \delta,$$

if non-empty; cf. [21, Proposition 2.11, Theorem 3.1].

Notations and Remark 3.3. (i) Let $\mathcal{G}_{\Sigma_{\mathcal{M},g}} \subset \mathcal{G}_e^3$ be the locus which the irreducible equi-generic Severi variety $\Sigma_{\mathcal{M},g}$ sits over. To be precise, let $\mathcal{G}_{\Sigma_{\mathcal{M},g}} \subset \mathcal{G}_e^3$ be the locus consisting of (complete) webs cut out by hyperplanes on space curves corresponding to general elements of the Severi variety $\Sigma_{\mathcal{M},g}$. We note that an open subset of $\Sigma_{\mathcal{M},g}$ consisting of nodal curves is an $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle over an open subset of $\mathcal{G}_{\Sigma_{\mathcal{M},g}} = \mathcal{G}_{\Sigma_{\mathcal{M},\delta}}$. Since $\Sigma_{\mathcal{M},g}$ is irreducible, $\mathcal{G}_{\Sigma_{\mathcal{M},g}}$ is irreducible.

(ii) For each component $\mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}} \subset \mathcal{G}_d^r$, we set

$$\mathcal{G}^\vee := \{\mathcal{D}^\vee | \mathcal{D} \in \mathcal{G}\} \subset \mathcal{G}_e^3$$

where $\mathcal{D}^\vee$ is the residual series of $\mathcal{D} \in \mathcal{G}$, $e = 2g-2-d$ and $d = g+r-4$. We also set

$$\mathcal{G}' := \bigcup_{\mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}}} \mathcal{G}^\vee \subset \mathcal{G}_e^3$$

(iii) We will determine the irreducibility of $\mathcal{H}^{\mathcal{L}}_{d,g,r}$ ($d = g + r - 4$, $g = r + 8$, $5 \leq r \leq 8$) according to the following intermediate steps.

(a) We first check that for a general $\mathcal{D} \in \mathcal{G}$, $\mathcal{E} = \mathcal{D}^\vee$ is birationally very ample and induces a morphism into a quadric surface in $\mathbb{P}^3$.

(b) We then collect all the possible $\mathcal{M}_j = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c_j, d_j)| \in Pic(X)$ such that $c_j + d_j \leq e$ and $p_a(\mathcal{M}_j) \geq g$. Note that this is just a simple numerical game which is not too complicated, especially when the value $e = \deg \mathcal{E} = \deg \mathcal{D}^\vee$ is low. Hence it trivially follows that

$$\mathcal{G}' = \cup \mathcal{G}^\vee \subset \bigcup_j \mathcal{G}_{\Sigma_{\mathcal{M}_j,g}}.$$

(c) For each $\mathcal{M}_j$, we determine if a general element of $\mathcal{G}_{\Sigma_{\mathcal{M}_j,g}}$ has a very ample residual series, which implies

$$\mathcal{G}_{\Sigma_{\mathcal{M}_j,g}} \subset \mathcal{G}'.$$

(iv) If there is only one $\mathcal{M}_j$ left such that $\mathcal{G}_{\Sigma_{\mathcal{M}_j,g}} \subset \mathcal{G}'$, say $\mathcal{M}_j = \mathcal{M}_0$ we then may deduce that $\mathcal{G}_{\Sigma_{\mathcal{M}_0,g}} = \mathcal{G}'$.

(v) The irreducibility of $\widetilde{\mathcal{G}}_{\mathcal{L}} \overset{bir}{\cong} \mathcal{G}' = \mathcal{G}_{\Sigma_{\mathcal{M}_0,g}}$ follows from the irreducibility of $\Sigma_{\mathcal{M}_0,g}$ or $\mathcal{G}_{\Sigma_{\mathcal{M}_0,g}}$.

We deal with the case $5 \leq r \leq 8$ using the above roadmap in the following theorem. The proofs of the irreducibility of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $r = 3, 4$ are of somewhat different flavor which shall be presented in the next section.

Theorem 3.4. *For $5 \leq r \leq 8$, $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ is irreducible unless $r = 7$.*

Proof. Set $\mathcal{E} = g^3_{10} = \mathcal{D}^\vee$ for a general $\mathcal{D} \in \mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}} \subset \mathcal{G}^r_d$. We recall that $\mathcal{E}$ is birationally very ample; cf. Lemma 2.3 or the proof of the non-existence of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $r \geq 9$ in Theorem 2.4 (c). Furthermore, $\mathcal{E}$ is base-point-free if $g = r+8 \geq 13$; if $\mathcal{E}$ has non-empty base locus, then $g \leq \pi(9,3) = 12$. We also note that the image curve $C_{\mathcal{E}} \subset \mathbb{P}^3$ induced by $\mathcal{E}$ lies on a quadric surface; if $C_{\mathcal{E}}$ does not lie on a quadric, then one has

$$g \leq p_a(C_{\mathcal{E}}) \leq \pi_1(10,3) = \frac{(10-1)(10-2)}{6} = 12.$$

Since curves on a quadric cone is a specialization of curves on a non-singular quadric surface by [15, Chapt. 2] or [6, Theorem, page 74], we may assume that $C_{\mathcal{E}}$ lies on a non-singular quadric surface. We note that for every genera $g = 13, 14, 15, 16$ there exists an integral curve of degree 10 lying on a non-singular quadric surface $X \subset \mathbb{P}^3$ with $\delta = 0, 1, 2, 3$ nodes as its only singularities and hence $\Sigma_{\mathcal{M},\delta} \neq \emptyset$ for some $\mathcal{M} \in Pic(X)$; cf. [1, Corollary 4.6].

The case $r = 8$ was treated already in the proof of Theorem 2.4, and we assume $5 \leq r \leq 7$. In the previous section, we demonstrated the existence of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ by showing that a general member in a certain very ample linear system $(a; b_1, \cdots, b_{t+1})$ on $S_{t+1} \to \mathbb{P}^2$ is embedded into a Del-Pezzo surface in $\mathbb{P}^r = \mathbb{P}^{9-t-1}$ as a curve of degree $d = 2r+4$ and genus $g = r+8$; say $a = 8, b_1 = b_2 = 3$ and $b_i = 2$ $(3 \leq i \leq t+1)$ if $t = 2, 3$. Here we show that these are the only possible family of curves if $r \neq 7$. Indeed, what we shall show is that $C_{\mathcal{E}} \in \Sigma_{\mathcal{M},\delta}$ for $\delta = t = 2, 3$ where $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(5,5)|$.

Recall that a general element of any component of the equi-generic Severi variety $\Sigma_{\mathcal{M},g}$ is a nodal curve by Remark 3.2 (i), and hence we may assume that $C_{\mathcal{E}}$ is a nodal curve with exactly t nodes $q_1, \cdots, q_t$.

Let $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c,d)|$. Since $\mathcal{E}$ is base-point-free,

$$c + d = 10 \text{ and } g \leq p_a(C_{\mathcal{E}}) = (c-1)(d-1),$$

there are just two possibilities for (c,d) if $13 \leq g \leq 15$; $(c,d) = (5,5)$ or $(4,6)$. Hence we have

$$\mathcal{G}' = \cup \mathcal{G}^\vee \subset \mathcal{G}_{\Sigma_{\mathcal{M}.g}} \cup \ \mathcal{G}_{\Sigma_{\mathcal{N}.g}},$$

where $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(5,5)|$ and $\mathcal{N} = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,6)|$.

First assume that $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(5,5)|$ i.e. $C_{\mathcal{E}} \in \mathcal{M}$, $g = p_a(C_{\mathcal{E}}) - t = 16 - t$, where $t = 1, 2, 3, 4$. As a matter of fact, we only need up to $t \leq 3$ in this theorem. However, in order to handle the case $g = 12$ and $r = 4$ in the next section, we assume $t \leq 4$ and the all the computations remain valid.

First we blow up the non-singular quadric X at q_1 to get surface S_2 which is the surface blown up at two points in the projective plane; recall that a non-singular quadric in $\mathbb{P}^3$ is a blowing up of $\mathbb{P}^2$ at two points and then blown down along the proper transformation of the line through the two points.
If $t \geq 2$, we first blow up at q_1 to get S_2 and then blow up at the remaining points $\tilde{q}_2. \cdots, \tilde{q}_t$ – which are the inverse images of $q_2, \cdots q_t$ under the first

blow up $S_2 \to X$ at q_1 – arriving at the surface S_{t+1}. Now the proper transformation $\widetilde{C}_{\mathcal{E}} \subset S_{t+1}$ of $C_{\mathcal{E}}$ under these successive blow ups is smooth since $C_{\mathcal{E}}$ is a nodal curve. We set

$$\widetilde{C}_{\mathcal{E}} \in (a; b_1, \cdots b_{t+1}).$$

Since the exceptional divisor of the first blow up at the node $q_1 \in C_{\mathcal{E}} \subset X$ is $l - e_1 - e_2$,

$$\widetilde{C}_{\mathcal{E}} \cdot (l - e_1 - e_2) = a - b_1 - b_2 = 2. \tag{2}$$

Since the proper transformations of the two rulings of $X \subset \mathbb{P}^3$ – which are of the classes $l - e_i$ $(i = 1, 2)$ on S_{t+1} – cut out two distinct g^1_5's on $\widetilde{C}_{\mathcal{E}}$, we have

$$(l - e_i) \cdot (a; b_1, \cdots, b_{t+1}) = a - b_i = 5, \ i = 1, 2. \tag{3}$$

Let $C^1_{\mathcal{E}} \subset S_2$ be the proper transformation of $C_{\mathcal{E}} \subset \mathbb{P}^3$ under the first blow up $S_2 \to X$ at q_1; hence $C^1_{\mathcal{E}} = \widetilde{C}_{\mathcal{E}}$ if $t = 1$. For $t \geq 2$, note that $\tilde{q}_2, \cdots, \tilde{q}_t$ remain as nodal singular points of $C^1_{\mathcal{E}}$ and hence

$$e_{i+1} \cdot (a; b_1, \cdots, b_{t+1}) = b_{i+1} = 2, \quad i = 2, \cdots, t$$

where e_{i+1} $(i = 2, \cdots, t)$ are the exceptional divisors over $\tilde{q}_i$ under the blow up $S_{t+1} \to S_2$.

Solving equations (2) and (3), we have $(a; b_1, b_2) = (8; 3, 3)$ and hence

$$\widetilde{C}_{\mathcal{E}} \in (a; b_1, b_2, b_3, \cdots, b_{t+1}) = (8; 3, 3, 2, \cdots, 2) = (8; 3^2, 2^{t-1}).$$

We check that $\mathcal{E}^\vee$ is very ample. We have

$$\begin{aligned} K_{S_{t+1}} + \widetilde{C}_{\mathcal{E}} &= (-3; -1, -1, -1, \cdots, -1) + (8; 3, 3, 2, \cdots, 2) \\ &= (-3; -1^{t+1}) + (8; 3^2, 2^{t-1}) = (5; 2^2, 1^{t-1}). \end{aligned}$$

Since $\mathcal{E}$ is cut out by $|2l - e_1 - e_2| = (2; 1^2, 0^{t-1})$ on $\widetilde{C}_{\mathcal{E}}$, $\mathcal{E}^\vee$ is cut out on $\widetilde{C}_{\mathcal{E}}$ by

$$\begin{aligned} |K_{S_{t+1}} + \widetilde{C}_{\mathcal{E}} - (2l - e_1 - e_2)| &= |(5; 2^2, 1^{t-1}) - (2l - e_1 - e_2)| \\ &= |(5; 2^2, 1^{t-1}) - (2; 1^2, 0^{t-1}| = |(3; 1^{t+1})| \end{aligned}$$

which is very ample on S_{t+1}. Therefore, $\mathcal{E}^\vee$ is very ample which gives an embedding

$$\widetilde{C}_{\mathcal{E}} \hookrightarrow \mathbb{P}^{9-(t+1)} = \mathbb{P}^{8-t} = \mathbb{P}^r$$

as a curve of degree

$$\begin{aligned} d &= (K_{S_{t+1}} + \widetilde{C}_{\mathcal{E}} - (2l - e_1 - e_2)) \cdot \widetilde{C}_{\mathcal{E}} = (3, 1^{t+1}) \cdot (8, 3^2, 2^{t-1}) \\ &= 24 - 6 - 2 \cdot (t - 1) = 2 \cdot (8 - t) + 4 = 2r + 4, \end{aligned}$$

and therefore we have

$$\mathcal{G}_{\Sigma_{\mathcal{M},g}} \subset \mathcal{G}' = \cup \mathcal{G}^\vee.$$

We now check if $\mathcal{G}_{\Sigma_{\mathcal{N},g}} \subset \mathcal{G}'$. Let $\mathcal{E} \in \mathcal{G}_{\Sigma_{\mathcal{N},g}}$, i.e. $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 6)|$.
(a) Assume $g = 15$ (and $r = 7$). In this case, $C_{\mathcal{E}}$ is smooth and $\mathcal{E}^\vee$ is cut out by

$$|K_X + C_{\mathcal{E}} - (1, 1)| = |(-2, -2) + (4, 6) - (1, 1)| = |(1, 3)|$$

which is very ample on X inducing an embedding

$$C_{\mathcal{E}} \subset \mathbb{P}^1 \times \mathbb{P}^1 \overset{(1,3)}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$$

as a curve of degree

$$(4,6)\cdot(1,3) = 12+6 = 18 = 2r+4.$$

Therefore we have $\mathcal{G}_{\Sigma_{\mathcal{N},g}} \subset \mathcal{G}'$ and hence

$$\mathcal{G}' = \cup\mathcal{G}^{\vee} = \mathcal{G}_{\Sigma_{\mathcal{N},g}} \cup \mathcal{G}_{\Sigma_{\mathcal{M},g}},$$

showing that

$$\widetilde{\mathcal{G}}_{\mathcal{L}} = (\mathcal{G}_{\Sigma_{\mathcal{N},g}})^{\vee} \cup (\mathcal{G}_{\Sigma_{\mathcal{M},g}})^{\vee}$$

is reducible with two distinct irreducible components of the same dimension;

$$\begin{aligned}\dim \mathcal{G}_{\Sigma_{\mathcal{M},g}} &= \dim \Sigma_{\mathcal{M},g} - \dim \mathrm{Aut}(X) = \dim \Sigma_{\mathcal{M},\delta} - \dim \mathrm{Aut}(X)\\ &= \dim |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)| - 1 - \dim \mathrm{Aut}(X) = 28\\ \dim \mathcal{G}_{\Sigma_{\mathcal{N},g}} &= \dim |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,6)| - \dim \mathrm{Aut}(X) = 28.\end{aligned}$$

(b) For $r = 5, 6$ ($g = 13, 14$) we show the irreducibility of $\mathcal{G}'$ i.e.

$$\mathcal{G}' = \mathcal{G}_{\mathcal{M},g}$$

by showing that

$$\mathcal{G}_{\mathcal{N},g} \nsubseteq \mathcal{G}'$$

which is enough to show that the residual series of a general element of $\mathcal{G}_{\Sigma_{\mathcal{N},g}}$ is not very ample. We carry out the same computation as we did for the case $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|$. Since $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,6)|$, we set $g = p_a(C_{\mathcal{E}}) - s = 15 - s$, where $s = \delta = 1, 2, 3$; here we only need up to $s \leq 2$. However in order to handle the case $r = 4$ in the next section, we make the computation even for $s = 3$. For $s = \delta = 1$ and $g = 14$, we blow up X at the node $q_1 \in C_{\mathcal{E}} \subset X$ to get S_2. For $s \geq 2$, we first blow up X at a node q_1 to get S_2 and then blow up at the remaining nodes $\tilde{q}_2, \cdots, \tilde{q}_s$ – which are the inverse images of remaining nodes $q_2, \cdots q_s$ of $C_{\mathcal{E}}$ under the first blow up $S_2 \to X$ at q_1 – arriving at the surface S_{s+1}. The proper transformation $\widetilde{C}_{\mathcal{E}} \subset S_{s+1}$ of $C_{\mathcal{E}}$ under these successive blow ups is smooth and we set

$$\widetilde{C}_{\mathcal{E}} \in (a; b_1, \cdots b_{s+1}).$$

Since $q_1 \in C_{\mathcal{E}} \subset X$ is a node and the exceptional divisor over q_1 is $l - e_1 - e_2$,

$$\widetilde{C}_{\mathcal{E}} \cdot (l - e_1 - e_2) = a - b_1 - b_2 = 2. \tag{4}$$

Since the proper transformations of the two rulings of $X \subset \mathbb{P}^3$ – which are of the classes $l - e_i$ $(i = 1, 2)$ on S_{s+1} – cut out two base-point-free pencils g^1_4 and g^1_6 on $\widetilde{C}_{\mathcal{E}}$, we have

$$(l - e_1)\cdot\widetilde{C}_{\mathcal{E}} = a - b_1 = 4,\ (l - e_2)\cdot\widetilde{C}_{\mathcal{E}} = a - b_2 = 6. \tag{5}$$

Let $C^1_{\mathcal{E}} \subset S_2$ be the proper transformation of $C_{\mathcal{E}} \subset \mathbb{P}^3$ under the first blow up $S_2 \to X$ at q_1; hence $C^1_{\mathcal{E}} = \widetilde{C}_{\mathcal{E}}$ if $s = 1$. For $s \geq 2$, note that $\tilde{q}_2, \cdots, \tilde{q}_s$ remain as nodal singular points of $C^1_{\mathcal{E}}$ and hence

$$e_{i+1}\cdot(a; b_1, \cdots, b_{s+1}) = b_{i+1} = 2, \quad i = 2, \cdots, s$$

where e_{i+1} $(i = 2, \cdots, s)$ are the exceptional divisors over $\tilde{q}_i$ under the blow up $S_{s+1} \to S_2$.
Solving the equations (4) and (5), we have $(a; b_1, b_2) = (8; 4, 2)$ and hence

$$\widetilde{C}_{\mathcal{E}} \in (a; b_1, b_2, b_3, \cdots, b_{s+1}) = (8; 4, 2, 2^{s-1}).$$

We claim that $\mathcal{E}^\vee$ is not very ample. We have

$$K_{S_{s+1}} + \widetilde{C}_{\mathcal{E}} = (-3; -1^{s+1}) + (8; 4, 2, 2^{s-1}) = (5; 3, 1^s).$$

Since $\mathcal{E} = g^3_{10}$ is cut out by $|2l - e_1 - e_2| = (2; 1^2, 0^{s-1})$ on $\widetilde{C}_{\mathcal{E}}$, $\mathcal{E}^\vee$ is cut out on $\widetilde{C}_{\mathcal{E}}$ by

$$\begin{aligned} |K_{S_{s+1}} + \widetilde{C}_{\mathcal{E}} - (2l - e_1 - e_2)| &= |(5; 3, 1^s) - (2l - e_1 - e_2)| \\ &= |(5; 3, 1^s) - (2; 1^2, 0^{s-1})| = |(3; 2, 0, 1^{s-1})| \end{aligned}$$

which is not very ample on S_{s+1}; cf. [9]. Moreover, $(3; 2, 0, 1^{s-1})_{|\widetilde{C}_{\mathcal{E}}}$ is not very ample; note that

$$\widetilde{C}_{\mathcal{E}} \cdot e_2 = (8; 4, 2^s) \cdot e_2 = 2.$$

We also note that $(3; 2, 0, 1^{s-1}) \cdot e_2 = 0$ and hence e_2 is contracted by the morphism induced by the linear system $(3; 2, 0, 1^{s-1})$ by which $\mathcal{E}^\vee$ on $\widetilde{C}_{\mathcal{E}}$ is cut out. Therefore it follows that

$$|K_{\widetilde{C}_{\mathcal{E}}} - \mathcal{E}| = (3; 2, 0, 1^{s-1})_{|\widetilde{C}_{\mathcal{E}}}$$

is not very ample. This shows that

$$\mathcal{G}_{\mathcal{N},g} \nsubseteq \mathcal{G}'$$

and hence

$$\mathcal{G}' = \mathcal{G}_{\Sigma_{\mathcal{M},g}},$$

from which the irreducibility of $\widetilde{\mathcal{G}}_{\mathcal{L}}$ and $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $r = 5, 6$ follows. □

4. Irreducibility of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $r = 3, 4$.

In this section, we continue to prove the irreducibility of the peculiar Hilbert schemes $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ for $r = 3, 4$, whose proofs are lengthier and utilizing somewhat different methods from the cases $r \geq 5$. In the course of the proof of Theorem 4.1, we also prove the reducibility of $\mathcal{H}^{\mathcal{L}}_{g-2,g,3}$ for $g = 12$.

Theorem 4.1. *$\mathcal{H}^{\mathcal{L}}_{12,12,4}$ is irreducible.*

Proof. Let $\mathcal{E} = \mathcal{D}^\vee$ for a general element $\mathcal{D} \in \mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}}$. Recall that $\mathcal{E}$ is birationally very ample (possibly with non-empty base locus) and $C_{\mathcal{E}} \subset \mathbb{P}^3$ denotes the image curve induced by $\mathcal{E}$. Since $g = 12 = \pi_1(10, 3)$, $C_{\mathcal{E}}$ does not necessarily lie on a quadric surface in $\mathbb{P}^3$. $C_{\mathcal{E}}$ may lie either on a quadric or on an irreducible surface of higher degree. We need to consider these two cases separately.

(i) Assume that $C_{\mathcal{E}} \subset \mathbb{P}^3$ lies on a quadric. Again, by Remark 3.2 (i) and [15] we may assume that $C_{\mathcal{E}}$ lies on a non-singular quadric $X \subset \mathbb{P}^3$ and has δ nodes as its only singularities. The possibilities are

(a) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|$ and $\delta = 4$,

(b) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,6)|$ and $\delta = 3$,

(c) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,7)|$ and $C_{\mathcal{E}}$ is non-singular,

(d) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,5)|$, $\mathcal{E}$ has non-empty base locus and $C_{\mathcal{E}}$ is non-singular. Hence we have

$$\mathcal{G}' \subset \mathcal{G}_{\Sigma_{(5,5),g}} \cup \mathcal{G}_{\Sigma_{(4,6),g}} \cup \mathcal{G}_{\Sigma_{(3,7),g}} \cup \mathcal{G}_{\Sigma_{(4,5),g}}.$$

We check if the reverse inclusion holds.

(a) The case $C_{\mathcal{E}} \in \mathcal{M} = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|, \delta = 4$ was treated in the proof of Theorem 3.4; we concluded that $\widetilde{C}_{\mathcal{E}} \in (8; 3^2, 2^3)$ for $\mathcal{E} \in \mathcal{G}_{\Sigma_{\mathcal{M},\delta}}$. We also showed that $\mathcal{G}_{\Sigma_{\mathcal{M},g}} \subset \mathcal{G}'$ (or equivalently $(\mathcal{G}_{\Sigma_{\mathcal{M},g}})^\vee \subset \widetilde{\mathcal{G}}_{\mathcal{L}}$) by showing that a general $\mathcal{E} \in \mathcal{G}_{\Sigma_{\mathcal{M},g}}$ has very ample residual series.

(b) $C_{\mathcal{E}} \in \mathcal{N} = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,6)|$ and $\delta = 3$; we showed in the proof of Theorem 3.4 that $\mathcal{E}^\vee$ is not very ample for a general element $\mathcal{E} \in \mathcal{G}_{\Sigma_{\mathcal{N},g}}$ and hence $\mathcal{G}_{\Sigma_{\mathcal{N},g}} \not\subseteq \mathcal{G}'$.

(c) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,7)|$ and $C_{\mathcal{E}}$ is non-singular; $\mathcal{E}^\vee$ is cut out by the series

$$|K_X + C_{\mathcal{E}} - (1,1)| = |(-2,-2) + (3,7) + (1,1)| = |(0,4)|$$

on $C_{\mathcal{E}}$, which is not very ample and hence $\mathcal{G}_{\Sigma_{(3,7),g}} \not\subseteq \mathcal{G}'$. However, it is worthwhile to remark that non-singular curves in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,7)|$ on smooth quadrics in $\mathbb{P}^3$ form an irreducible family $\mathcal{I}_2 \subset \mathcal{H}^{\mathcal{L}}_{10,12,3}$ of the expected dimension

$$\dim \mathcal{G}_{\Sigma_{(3,7),g}} + \dim \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 40.$$

(d) $C_{\mathcal{E}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,5)|$, $\mathcal{E}$ has non-empty base locus and $C_{\mathcal{E}}$ is non-singular: Let $\mathcal{E}' = |\mathcal{E} - q|$ be the moving part $\mathcal{E}$, where q is the base point of $\mathcal{E}$. We argue that $\mathcal{E}^\vee$ is not very ample as follows. Note that the residual series of $\mathcal{E}'$ is cut out on the non-singular curve $C_{\mathcal{E}'} = C_{\mathcal{E}}$ by the very ample linear system

$$|K_X + C_{\mathcal{E}'} - (1,1)| = |(-2,-2) + (4,5) - (1,1)| = |(1,2)|$$

which induces an embedding $X \xrightarrow{\iota} \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Under this embedding ι, the non-singular curve $C_{\mathcal{E}'}$ has the image $\iota(C_{\mathcal{E}'}) \subset \mathbb{P}^5$ which is a curve of degree $2g - 2 - \deg \mathcal{E}' = 13$. Note that

$$|K_{C_{\mathcal{E}'}} - \mathcal{E}'| = |K_{C_{\mathcal{E}'}} - \mathcal{E} + q|,$$

and $\mathcal{E}^\vee = |K_{C_{\mathcal{E}'}} - \mathcal{E}|$ gives rise to a projection of $\iota(C_{\mathcal{E}'})$ into $\mathbb{P}^4$ with center at $\iota(q) \in \iota(C_{\mathcal{E}'}) \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Let L be one of the rulings of X such that $q \in L$, say $L = (0,1) \subset X$. We note that the image of the line L under ι is still a line on the image scroll $\iota(X) \subset \mathbb{P}^5$; $(1,2)\cdot(0,1) = 1$. We also note that $\iota(L)$ is a 4-secant line of the non-singular curve $\iota(C_{\mathcal{E}'})$ containing $\iota(q)$ since $(4,5)\cdot(0,1) = 4$. However, a projection from the point $\iota(q)$ on the 4-secant

line $\iota(L)$ of $\iota(C_{\mathcal{E}'})$ produces a singularity and hence $\mathcal{E}^\vee$ is not very ample. Therefore $\mathcal{G}_{\Sigma_{\mathcal{L},g}} \nsubseteq \mathcal{G}'$ for $\mathcal{L} = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,5)|$.

(ii) We assume $C_{\mathcal{E}}$ does not lie on a quadric. Since $g = 12 \leq p_a(C_{\mathcal{E}}) \leq \pi_1(10,3) = 12$ (cf. [13, Theorem 3.13]), we see that $C_{\mathcal{E}}$ is non-singular. By the exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathcal{I}_{C_{\mathcal{E}}}(3)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(C_{\mathcal{E}}, \mathcal{O}_{C_{\mathcal{E}}}(3))$$

one checks that $C_{\mathcal{E}}$ lies on an (unique) irreducible cubic surface.

(ii-a) We assume that $C_{\mathcal{E}}$ lies on a smooth cubic surface $S = S_6 \subset \mathbb{P}^3$. We compute the class of $C_{\mathcal{E}}$ as follows. Setting $C_{\mathcal{E}} \sim al - \sum_{i=1}^6 b_i e_i$, we have

$$\deg C_{\mathcal{E}} = 3a - \sum b_i = 10, C_{\mathcal{E}}^2 = a^2 - \sum b_i^2 = 2g-2-K_S\cdot C_{\mathcal{E}} = 22+\deg C_{\mathcal{E}} = 32.$$

By Schwartz's inequality, one has

$$(\sum b_i)^2 \leq 6(\sum b_i^2)$$

and substituting $\sum b_i = 3a - 10$, $\sum b_i^2 = a^2 - 32$ we obtain

$$3a^2 - 60a + 292 \leq 0,$$

implying $9 \leq a \leq 11$. Hence, we arrive at the following possibilities for the 7-tuple $(a; b_1, \cdots, b_6)$;

$$\text{(i) } (9; 3^5, 2) \qquad \text{(ii) } (10; 4^2, 3^4) \qquad \text{(iii) } (11; 4^5, 3) \tag{6}$$

and by a simple numerical check, in all the three cases the curve $C_{\mathcal{E}}$ is linearly equivalent on S to $D + 3H$ where D is one of the 27 lines on the cubic.
For each of the above three types of 7-tuples $(a; b_1, \cdots, b_6)$, we set $L = \mathcal{O}_S(a; b_1, \cdots, b_6)$. By Riemann-Roch on S we have

$$\begin{aligned} h^0(S,L) &= h^1(S,L) - h^2(S,L) + \chi(S) + \frac{1}{2}(L^2 - L\cdot\omega_S) \\ &= h^1(S,L) - h^2(S,L) + 1 + \frac{1}{2}(C_{\mathcal{E}}^2 + \deg C_{\mathcal{E}}) \\ &= h^1(S,L) - h^2(S,L) + 1 + \frac{1}{2}(32+10) \end{aligned}$$

By Serre duality we have,

$$h^1(S,L) = h^1(S, \omega_S \otimes L^{-1}) = h^1(S, \mathcal{O}_S(-(a+3)l + \sum (b_i+1)e_i).$$

Since $E := (a+3)l - \sum(b_i+1)e_i$ is (very) ample we have

$$h^1(S,L) = h^1(S, \mathcal{O}_S(-(a+3)l + \sum (b_i+1)e_i)) = 0$$

by Kodaira's vanishing theorem. We further note the divisor $-E$ is not linearly equivalent to an effective divisor; if it were, one would have $-E\cdot l = -(a+3) \geq 0$ whereas $l^2 = 1 \geq 0$, a contradiction. Hence it follows that

$$h^2(S,L) = h^0(S, \omega_S \otimes L^{-1}) = h^0(S, \mathcal{O}_S(-E) = 0$$

and we obtain

$$h^0(S,L) = 22.$$

Therefore the sublocus $\mathcal{I}_3$ of $\mathcal{H}_{10,12,3}$ consisting of curves lying on a smooth cubic has dimension equal to the minimal possible dimension of a component of the Hilbert scheme $\mathcal{H}_{10,12,3}$, i.e.

$$\dim \mathcal{I}_3 = \dim H^0(\mathbb{P}^3, \mathcal{O}(3)) - 1 + \dim |\mathcal{O}_S(a; b_1, \cdots, b_6)| = 40 = 4 \cdot 10$$

We also note that $\mathcal{I}_3$ is not in the closure $\overline{\mathcal{I}_2}$ or vice versa since $\dim \mathcal{I}_2 = \dim \mathcal{I}_3$. So far we have shown that $\mathcal{H}^{\mathcal{L}}_{10,12,3}$ has two irreducible families $\overline{\mathcal{I}_2}$ and $\overline{\mathcal{I}_3}$ of the same dimension $4 \cdot 10$.

(ii-b) We assume that $C_{\mathcal{E}}$ lies on a singular cubic surface. We now argue that a family of non-singular curves lying on a singular cubic surface does not constitute a component of $\mathcal{H}_{10,12,3}$. Note that every singular cubic surface $S \subset \mathbb{P}^3$ is one of the following three types.

(1) S is a normal cubic surface with some double points only.
(2) S is a normal cubic cone.
(3) S is not normal, which may possibly be a cone.

For the case (1), let S be a normal cubic surface which is not a cone. By a work due to John Brevik [6, Theorem 5.24], every curve on S is a specialization of curves on a smooth cubic surface. Therefore we are done for the case (1).
For the case (2), we let C be a smooth curve of degree d and genus g on a normal cubic cone S. We recall by [11, Proposition 2.12] that

(2-a) $g = 1 + d(d-3)/6 - 2/3$ if C passes through the vertex of S
(2-b) $g = 1 + d(d-3)/6$, otherwise.

Note that $(d, g) = (10, 12)$ satisfies only (a) and C passes through the vertex q of the normal cone S. Hence C is a triple covering of an elliptic curve through the projection with center at the vertex q. The following naive dimension count shows that there is no component of $\mathcal{H}^{\mathcal{L}}_{10,12,3}$ (or $\mathcal{H}^{\mathcal{L}}_{12,12,4}$) whose general element corresponds to a triple covering of an elliptic curve. Assume the existence of such a component $\mathcal{G}^{\vee} \subset \mathcal{G}^3_{10}$. By Castelnuovo-Severi inequality, a triple covering $C \stackrel{\tau}{\to} E$ of an elliptic curve of genus $g = 12$ is not trigonal, bielliptic or a smooth plane quintic. Hence by a theorem of Mumford [3, Theorem 5.2, page 193], one has

$$\dim W^3_{10}(C) \leq 10 - 2 \cdot 3 - 2$$

and therefore

$$\dim \mathcal{G}^{\vee} \leq \dim W^3_{10}(C) + \dim \mathcal{X}_{3,1} = 2g \lneq \lambda(10, 12, 3) = \lambda(12, 12, 4),$$

leading to a contradiction. Here $\mathcal{X}_{n,h}$ denotes the locus in $\mathcal{M}_g$ corresponding to curves which are n-fold coverings of smooth curves of genus h, which has pure dimension

$$2g + (2n-3)(1-h) - 2;$$

cf. [4, Theorem 8.23, p.828]. Therefore we are done for the case (2).
For the case (3), we let $C = C_{\mathcal{E}}$ be a smooth curve of degree d and genus g on a non-normal cubic surface S. Recall that if S is a cone, then S is a cone over a singular plane cubic, in which case S is a projection of a cone S' over

a twisted cubic in a hyperplane in $\mathbb{P}^4$ from a point not on S'. Furthermore, the minimal desingularisation $\tilde{S}$ of S' is isomorphic to the ruled surface

$$\mathbb{F}_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3)),$$

which is the blow-up of the cone S' at the vertex. If S is not a cone, then S is a projection of a rational normal scroll

$$S'' \cong \tilde{S} \cong \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \subset \mathbb{P}^4$$

from a point not on S''. In both cases, we have $\mathrm{Pic}\ \tilde{S} = \mathbb{Z}h \oplus \mathbb{Z}f \cong \mathbb{Z}^{\oplus 2}$, where f is the class of a fiber of $\tilde{S} \to \mathbb{P}^1$ and $h = \pi^*(\mathcal{O}_S(1))$ with $\tilde{S} \xrightarrow{\pi} S$. Note that $h^2 = 3, f^2 = 0$, $h \cdot f = 1$ and $K_{\tilde{S}} \equiv -2h + f$. Denoting by $\tilde{C} \subset \tilde{S}$ the proper transformation of the curve $C \subset S$, we let $k := (\tilde{C} \cdot f)_{\tilde{S}}$ be the intersection number of $\tilde{C}$ and the fiber f on $\tilde{S}$. We have

$$\tilde{C} \equiv kh + (d - 3k)f = kh + (10 - 3k)f.$$

By adjunction formula, it follows that

$$g = 12 = \frac{(2 \cdot 10 - 3k - 2)(k - 1)}{2},$$

which does have an integer solution and we are done with the case (3).

Since we have exhausted all the possibilities, the reducibility of $\mathcal{H}^{\mathcal{L}}_{10,12,3} = \mathcal{H}_{10,12,3}$ which has two components $\overline{\mathcal{I}_2}$ and $\overline{\mathcal{I}_3}$ of the same minimal dimension readily follows.

To conclude the irreducibility of $\mathcal{H}^{\mathcal{L}}_{12,12,4}$, it remains to show that the residual series of a very ample $\mathcal{E} = g^3_{10}$ – which induces an embedding into a smooth cubic surface $S = S_6 \subset \mathbb{P}^3$ – is not very ample.

Recall that the residual series of a complete hyperplane series of a general element $C_{\mathcal{E}} \in \mathcal{I}_3$ is cut out by the linear system

$$|K_S + C_{\mathcal{E}} - H| = |2K_S + C_{\mathcal{E}}|,$$

i.e. $|2K_S + C_{\mathcal{E}}|_{|C_{\mathcal{E}}} = |K_{C_{\mathcal{E}}} - \mathcal{E}|$. We recall our previous computation (6).
(i) $C_{\mathcal{E}} = (9, 3^5, 2)$: Note that $|2K_S + C_{\mathcal{E}}| = (3, 1^5, 0)$ which is not very ample on S_6. We further note that $C_{\mathcal{E}} \cdot e_6 = 2$ whereas $(2K_S + C_{\mathcal{E}}) \cdot e_6 = 0$. Hence the restriction of the morphism on S_6 induced by $|2K_S + C_{\mathcal{E}}| = (3, 1^5, 0)$ – which is the blowing down of S_6 along e_6 – produces a singularity on the image curve in $S_5 \subset \mathbb{P}^4$. Hence $|K_{C_{\mathcal{E}}} - \mathcal{E}|$ is not very ample.
(ii) $C_{\mathcal{E}} = (10; 4^2, 3^4)$: Note that

$$|2K_S + C_{\mathcal{E}}| = (4; 2^2, 1^4),$$

and $(4; 2^2, 1^4) \cdot (l - e_1 - e_2) = 0$ whereas $C_{\mathcal{E}} \cdot (l - e_1 - e_2) = 2$. Hence the restriction on $C_{\mathcal{E}}$ of the morphism induced by $|2K_S + C_{\mathcal{E}}|$ – which contracts $(l - e_1 - e_2)$ to a point – produces a singularity.
(iii) $C_{\mathcal{E}} \in (11; 4^5, 3)$: In this case,

$$|2K_S + C_{\mathcal{E}}| = (5; 2^5, 1),$$

and $(5;2^5,1)\cdot(2l-e_1-e_2-e_3-e_4-e_5)=0$ whereas $C_{\mathcal{E}}\cdot(2l-e_1-e_2-e_3-e_4-e_5)=2$. Hence $|K_{C_{\mathcal{E}}}-\mathcal{E}|$ is not very ample by the same reason.

Finally, we may conclude that

$$\mathcal{G}'=\mathcal{G}_{\Sigma_{(5,5),\delta=4}},$$

therefore it follows that $\widetilde{\mathcal{G}}_{\mathcal{L}}$ and $\mathcal{H}^{\mathcal{L}}_{12,12,4}$ are irreducible. □

We quote the reducibility of $\mathcal{H}_{10,12,3}=\mathcal{H}^{\mathcal{L}}_{g-2,g,3}$ which we showed in the proof of the above theorem as a corollary.

Corollary 4.2. *$\mathcal{H}_{10,12,3}=\mathcal{H}^{\mathcal{L}}_{g-2,g,3}$ is reducible consisting of two components.*

We go down to the case $r=3$ and show the irreducibility of $\mathcal{H}^{\mathcal{L}}_{10,11,3}$.

Theorem 4.3. *For $r=3$, $d=g+r-4$ and $g=r+8$,*

$$\mathcal{H}_{d,g,3}=\mathcal{H}^{\mathcal{L}}_{d,g,3}=\mathcal{H}^{\mathcal{L}}_{10,11,3}$$

is irreducible of the expected dimension.

Proof. We first note that a smooth curve $C\subset\mathbb{P}^3$ of degree $d=10$ and genus $g=11$ does not lie on a quadric surface just because there is no integer solution to the equations

$$a+b=10=d,\ (a-1)(b-1)=11=g.$$

We remark that the existence of $\mathcal{H}_{10,11,3}$ is clear by the existence of a non-singular curve in the very ample linear system $(8;3^2,2^4)$ on a smooth cubic surface.

Since $\deg C=10$, C lies on at most one cubic surface S. Assume S is a smooth cubic surface. We recall that smooth space curves of degree d and genus g on a smooth cubic surface form a finite union of locally closed irreducible family in $\mathcal{H}_{d,g,3}$ of dimension $d+g+18$ if $d\geq 10$ by [11, Proposition B.1]. Since $d+g+18<4d$ for $(d,g)=(10,11)$, it follows that the family $\mathcal{I}_3$ consisting of curves lying on a smooth cubic does not constitute a component of $\mathcal{H}_{10,11,3}$. If C lies on a singular cubic surface, we may use the same argument as in the proof (ii-b) of Theorem 4.1 to conclude that such case does not occur.

Let C corresponds to a general element of a component of $\mathcal{H}_{10,11,3}$. By the above discussion we may assume that C does not lie on a quadric or a cubic. Note that every smooth curve C of genus $g=11$ of degree $d=10$ in $\mathbb{P}^3$ lies on at least

$$h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(4))-h^0(C,\mathcal{O}_C(4))=35-(40-11+1)=5$$

independent quartics. Hence C is is residual to a curve $D\subset\mathbb{P}^3$ of degree $e=6$ and genus $h=3$ in a complete intersection of two (irreducible) quartics by the following well-known formula relating degrees d,e and genus g,h of two directly linked curves in $\mathbb{P}^3$ in a complete intersection of surfaces of degrees s and t;

$$2(g-h)=(s+t-4)(d-e). \tag{7}$$

Consider the locus

$$\Sigma \subset \mathbb{G}(1, \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)))) = \mathbb{G}(1, 34)$$

of pencils of quartic surfaces whose base locus consists of a curve C of degree $d = 10$ and genus $g = 11$ and a sextic D of genus $h = 3$ which are directly linked via a complete intersection of quartics, together with two obvious maps

$$\begin{array}{ccc} \mathbb{G}(1,34) \supset \Sigma & \overset{\pi_C}{\dashrightarrow} & \mathcal{I}_4 \subset \mathcal{H}_{10,11,3} \\ \\ \downarrow \pi_D & & \\ \\ \mathcal{H}_{6,3,3} & & \end{array}$$

where $\mathcal{I}_4$ is the image of Σ under π_C. A sextic $D \subset \mathbb{P}^3$ of genus $h = 3$ lies on at least

$$h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) - h^0(D, \mathcal{O}_D(4)) = 35 - (24 - 3 + 1) = 13$$

independent quartics. Note that $C \in \mathcal{H}_{10.11.3}$ is directly linked to $D \in \mathcal{H}_{6,3,3}$ which in turn is directly linked to a twisted cubic F via complete intersection of two cubics. From the basic relation

$$\dim H^1(\mathbb{P}^3, \mathcal{I}_C(m)) = \dim H^1(\mathbb{P}^3, \mathcal{I}_D(s + t - 4 - m))$$

where C and D are directly linked via complete intersection of surfaces of degrees s and t, we have

$$h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = h^1(\mathbb{P}^3, \mathcal{I}_D(0)) = h^1(\mathbb{P}^3, \mathcal{I}_F(3 + 3 - 4 - 0)) = 0 \qquad (8)$$

and

$$h^1(\mathbb{P}^3, \mathcal{I}_D(4)) = h^1(\mathbb{P}^3, \mathcal{I}_F(3 + 3 - 4 - 4)) = 0.$$

Therefore π_D is generically surjective with fibers open subsets of $\mathbb{G}(1, 12)$. Since $\mathcal{H}_{6.3.3}$ is known to be irreducible (cf. [10]), it follows that Σ is irreducible and

$$\dim \Sigma = \dim \mathbb{G}(1, 12) + \dim \mathcal{H}_{6.3.3} = 22 + 4 \cdot 6 = 46.$$

On the other hand, since every smooth curve $C \in \mathcal{I}_4$ of degree $d = 10$ and $g = 11$ lies on exactly 5 independent quartics by (8), π_C is generically surjective with fibers open in $\mathbb{G}(1, 4)$. Finally it follows that $\overline{\mathcal{I}_4} = \mathcal{H}_{10.11.3}$ – which is the image of (irreducible) Σ under π_C – is irreducible of dimension

$$\dim \Sigma - \dim \mathbb{G}(1, 4) = 46 - 6 = 4 \cdot 10.$$

□

The following table is a summary of our discussion in sections 3 and 4 regarding $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$. $C_{\mathcal{E}}$ denotes possibly singular curve in $\mathbb{P}^3$ determined by the residual series $\mathcal{E}$ of the hyperplane series of $[C] \in \mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$. Note that the irreducibility of $\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r}$ has been fully worked out in Theorems 3.4, 4.1 4.3 and all curves described in the table form a dense open set in a component of $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$, except $r = 3$ and $C \in (8; 3^2, 2^4)$, which form an irreducible family of dimension $39 < 40$.

TABLE 2. $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$=$\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r} \neq \emptyset$ for $3 \leq r \leq 8$

(d,g,r)	Description of $C \in \mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$	$C_{\mathcal{E}}$	Irreducibility
$(10,11,3)$	General element is directly linked to $\widetilde{C} \in \mathcal{H}^{\mathcal{L}}_{6,3,3}$ in a complete intersection of two quartics.		Yes
	$C \in (8;3^2,2^4)$ on a Del Pezzo in $\mathbb{P}^3$ which does **not** constitute a component.	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,5}$	
$(12,12,4)$	$C \in (8;3^2,2^3)$ on a Del Pezzo in $\mathbb{P}^4$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,4}$	Yes
$(14,13,5)$	$C \in (8;3^2,2^2)$ on a Del Pezzo in $\mathbb{P}^5$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,3}$	Yes
$(16,14,6)$	$C \in (8;3^2,2)$ on Del-Pezzo in $\mathbb{P}^6$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,2}$	Yes
$(18,15,7)$	$C \in (8;3^2)$ on Del-Pezzo in $\mathbb{P}^7$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,1}$	**No**
	or		
	$C \cong C_{\mathcal{E}} \in \lvert(4,6)\rvert$ on $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ embedded into $\mathbb{P}^7$ by $\lvert(1,3)\rvert$ on X	$C_{\mathcal{E}} \in \Sigma_{\lvert(4,6)\rvert,0}$	
$(20,16,8)$	$C \cong C_{\mathcal{E}} \in \lvert(5,5)\rvert$ on $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ embedded into $\mathbb{P}^8$ by $\lvert(2,2)\rvert$ on X	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,0}$	Yes
$(2r+4,r+8,r)$ $r \geq 9$	$\mathcal{H}^{\mathcal{L}}_{2r+4,r+8,r} = \emptyset$		

5. Irreducibility and existence of $\mathcal{H}^{\mathcal{L}}_{g+r-4,r+9,r}$ and beyond

We have seen another peculiar Hilbert scheme of linearly normal curves $\mathcal{H}^{\mathcal{L}}_{g+r-4,r+9,r}$ which is non-empty only for $3 \leq r \leq 11$; Theorem 2.4 (d). The following table indeed provides the existence of the corresponding Hilbert scheme which has been postponed. We stress that the curves described in the following table do not necessarily represent a general element in a component. As in the previous table, $C_{\mathcal{E}}$ denotes possibly singular curve in $\mathbb{P}^3$ determined by the residual series $\mathcal{E}$ of the complete and very ample hyperplane series of $[C] \in \mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$ and $\Delta_{\mathcal{E}}$ is the base locus of $\mathcal{E}$. We also stress that the irreducibility of $\mathcal{H}^{\mathcal{L}}_{g+r-4,r+9,r}$ has not been determined in a couple of cases. In a remark after the table, we provide some brief explanations for those $\mathcal{H}^{\mathcal{L}}_{g+r-4,r+9,r}$ which we know of their irreducibility (or reducibility).

TABLE 3. $\mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}=\mathcal{H}^{\mathcal{L}}_{2r+5,r+9,r} \neq \emptyset$ for $3 \leq r \leq 11$

(d,g,r)	Description of $C \in \mathcal{H}^{\mathcal{L}}_{g+r-4,g,r}$	$C_{\mathcal{E}}$	Irreducibility
$(11,12,3)$	$C \in (8;3^2,2^3,1)$ on $S_6 \hookrightarrow \mathbb{P}^3$ which do **not** form a component	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,4}$ and $\Delta_{\mathcal{E}} \neq \emptyset$	Yes
	or		
	General element is directly linked to $\widetilde{C} \in \mathcal{H}_{5,0,3}$ in a c.i. of two quartics		
$(13,13,4)$	$C \in (8;3^2,2^2,1)$ on $S_5 \hookrightarrow \mathbb{P}^4$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,3}$ and $\Delta_{\mathcal{E}} \neq \emptyset$	Don't know yet
	or		
	$C \in (9;4,3,2^6)$ on $S_8 \to \mathbb{P}^2$ embedded into $\mathbb{P}^4$ by $(4;2,1^7)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,7}$ and $\Delta_{\mathcal{E}} = \emptyset$	
$(15,14,5)$	$C \in (8;3^2,2,1)$ on $S_4 \hookrightarrow \mathbb{P}^5$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,2}$ and $\Delta_{\mathcal{E}} \neq \emptyset$	Don't know yet.
	or		
	$C \in (9;4,3,2^5)$ on $S_7 \to \mathbb{P}^2$ embedded into $\mathbb{P}^5$ by $(4;2,1^6)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,6}$ and $\Delta_{\mathcal{E}} = \emptyset$	
$(17,15,6)$	$C \cong C_{\mathcal{E}} \in (10;4,3^5)$ on $S_6 \hookrightarrow \mathbb{P}^3$ embedded into $\mathbb{P}^6$ by $(4;2,1^5)$	$C_{\mathcal{E}}$ is on cubic. and $\Delta_{\mathcal{E}} = \emptyset$	No
	or		
	$C \in (9;4,3,2^4)$ on $S_6 \to \mathbb{P}^2$ embedded into $\mathbb{P}^6$ by $(4;2,1^5)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,5}$ and $\Delta_{\mathcal{E}} = \emptyset$	
$(19,16,7)$	$C \in (9;4^2)$ on $S_2 \hookrightarrow \mathbb{P}^7$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,5)\rvert,0}$	Don't know yet
	or		
	$C \in (9;4,3,2^3)$ on $S_5 \to \mathbb{P}^2$ embedded into $\mathbb{P}^7$ by $(4;2,1^4)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,4}$ and $\Delta_{\mathcal{E}} = \emptyset$	
$(21,17,8)$	$C \in (9;4,3,2^2)$ on $S_4 \to \mathbb{P}^2$ embedded into $\mathbb{P}^8$ by $(4;2,1^3)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,3}$ and $\Delta_{\mathcal{E}} = \emptyset$	Yes
$(23,18,9)$	$C \cong C_{\mathcal{E}} \in \lvert(4,7)\rvert$ on $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ embedded into $\mathbb{P}^9$ by $\lvert(1,4)\rvert$ on X	$C_{\mathcal{E}} \in \Sigma_{\lvert(4,7)\rvert,0}$ and $\Delta_{\mathcal{E}} = \emptyset$	No
	or		
	$C \in (9;4,3,2)$ on $S_3 \to \mathbb{P}^2$ embedded into $\mathbb{P}^9$ by $(4;2,1^2)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,2}$ and $\Delta_{\mathcal{E}} = \emptyset$	
$(25,19,10)$	$C \in (9;4,3)$ on $S_2 \to \mathbb{P}^2$ embedded into $\mathbb{P}^{10}$ by $(4;2,1)$	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,1}$ and $\Delta_{\mathcal{E}} = \emptyset$	Yes
$(27,20,11)$	$C \cong C_{\mathcal{E}} \in \lvert(5,6)\rvert$ on $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ embedded into $\mathbb{P}^{11}$ by $\lvert(2,3)\rvert$ on X	$C_{\mathcal{E}} \in \Sigma_{\lvert(5,6)\rvert,0}$ and $\Delta_{\mathcal{E}} = \emptyset$	Yes

Remark 5.1. (i) $(d,g,r) = (11,12,3)$: The proof of Theorem 4.3 using elementary linkage theory works verbatim. $\mathcal{H}_{11,12,3} = \mathcal{H}^{\mathcal{L}}_{11,12,3}$ is irreducible and has the expected dimension, its general element corresponds to a curve which is directly linked to a rational quintic.

(ii) $(d,g,r)=(17,15,6)$: By noting that $g=\pi_1(11,3)=15$ and analyzing $C_{\mathcal{E}}\subset\mathbb{P}^3$ in a way similar to part (ii) of the proof of Theorem 4.1, we arrive at the two possibilities; (a) $C_{\mathcal{E}}\in\Sigma_{|(5,6)|,5}$ or (b) $C_{\mathcal{E}}\in(10;4,3^5)$ on $S_6\subset\mathbb{P}^3$. Two families form different components by semi-continuity; note that $h^0(\mathbb{P}^3,\mathcal{I}_{C_2}(2))>h^0(\mathbb{P}^3,\mathcal{I}_{C_3}(2))$, $C_2\in\Sigma_{|(5,6)|,5}$, $C_3\in(10;4,3^5)$ whereas

$$\dim\mathcal{G}_{(10;4,3^5)}=29<\dim\mathcal{G}_{\Sigma_{|(5,6)|,5}}=30.$$

(iii) $(d,g,r)=(19,16,7)$: We note that $g=r+9\gneq\pi_1(11,3)$ if $r\geq 7$, hence $C_{\mathcal{E}}\subset\mathbb{P}^3$ lies on a quadric. Therefore we have $C_{\mathcal{E}}\in\Sigma_{|(5,6)|,4}\cup\Sigma_{|(4,7)|,2}$ if $\mathcal{E}$ is base-point-free. One may also show that $\mathcal{G}_{\Sigma_{|(4,7)|,2}}\nsubseteq\mathcal{G}'$ by arguing that $\mathcal{E}^\vee$ is not very ample for $C_{\mathcal{E}}\in(4,7)$. However, if $\mathcal{E}=|g^3_{10}+q|=|\mathcal{E}'+q|$ has a base point q,

$$|K_{C_{\mathcal{E}}}-\mathcal{E}|=|K_{C_{\mathcal{E}'}}-\mathcal{E}|=|K_{C_{\mathcal{E}'}}-\mathcal{E}'-q|,$$

which is still very ample for a general q. The family of curves consisting of (smooth) curves in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|$ which is embedded into $\mathbb{P}^8$ induced by $|K_{C_{\mathcal{E}'}}-\mathcal{E}'|=(2,2)_{|C_{\mathcal{E}}=C_{\mathcal{E}'}}$ and then projected from a general point on the image curve in $\mathbb{P}^8$

$$C_{\mathcal{E}}=C_{\mathcal{E}'}\overset{(2,2)}{\hookrightarrow}\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8\xrightarrow{\pi_q}\mathbb{P}^7$$

has dimension $30\geq\lambda(d,g,r)=29$. However, it is not totally clear if this family is in the boundary of the family of curves arising from $\mathcal{G}^\vee_{\Sigma_{|(5,6)|,4}}$, which has dimension 31.

(iv) $(d,g,r)=(21,17,8)$: We have $C_{\mathcal{E}}\in\Sigma_{|(5,6)|,3}\cup\Sigma_{|(4,7)|,1}$. However it turns out that $\mathcal{G}_{\Sigma_{|(4,7)|,1}}\nsubseteq\mathcal{G}'$. Hence $\mathcal{G}'=\mathcal{G}_{\Sigma_{|(5,6)|,3}}$ is irreducible. The irreducibility of $\mathcal{H}^{\mathcal{L}}_{21,17,8}$ follows from the irreducibility of $\Sigma_{|(5,6)|,3}$.

(v) $(d,g,r)=(23,18,9)$: One can check $C_{\mathcal{E}}\in\Sigma_{|(5,6)|,2}\cup\Sigma_{|(4,7)|,0}$ and $\mathcal{G}'=\mathcal{G}_{\Sigma_{|(5,6)|,2}}\cup\mathcal{G}_{\Sigma_{|(4,7)|,0}}$. Hence $\mathcal{H}^{\mathcal{L}}_{23,18,9}$ is reducible with two components.

(vi) $(d,g,r)=(25,19,10)$: $C_{\mathcal{E}}\in\Sigma_{|(5,6)|,1}$ is the only possibility. i.e. $\mathcal{G}'=\mathcal{G}_{\Sigma_{|(5,6)|,1}}$ and hence $\mathcal{H}^{\mathcal{L}}_{25,19,10}$ is irreducible.

(vii) $(d,g,r)=(27,20,11)$: $C_{\mathcal{E}}\subset\mathbb{P}^3$ is an extremal curve in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,6)|$ on a smooth quadric and the irreducibility of $\mathcal{H}^{\mathcal{L}}_{27,20,11}$ follows from the irreducibility of $\mathcal{H}^{\mathcal{L}}_{11,20,3}$.

The following statement regarding the Hilbert scheme of linearly normal curves with index of speciality $\alpha=5$ is rather crude compared with Theorem 2.4. We omit the proof which is more involved but is similar to the proof of Theorem 2.4. We assume $r\geq 6$ in the following statement for the sake of brevity, i.e. in order to avoid subtle complexities in lower dimensions $3\leq r\leq 5$, which need to be considered separately. A proof of the following proposition in a more general formulation shall appear in an article under preparation; [17].

Proposition 5.2. (a) $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r}=\emptyset$ *for* $g\leq r+8$, $r\geq 6$.
(b) $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r}\neq\emptyset$ *for* $g=r+9$ *or for any* $g\geq r+13$, $r\geq 6$.

(c) $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r} = \emptyset$ *for* $g = r + 10$, $r \geq 9$.
(d) $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r} = \emptyset$ *for* $g = r + 11$, $r \geq 12$.
(e) $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r} \neq \emptyset$ *for* $g = r + 12$, $r \geq 13$.

Remark 5.3. (i) In three cases (1) $6 \leq r \leq 8$ and $g = r + 10$; (2) $8 \leq r \leq 11$ and $g = r+11$; (3) $g = r+12$ and $10 \leq r \leq 14$, the curve $C_{\mathcal{E}}$ induced by birationally very ample $\mathcal{E} = g^4_e = \mathcal{D}^\vee$ for general $\mathcal{D} \in \mathcal{G} \subset \widetilde{\mathcal{G}}_{\mathcal{L}}$ lies on a rational normal scroll $S \subset \mathbb{P}^4$; $\pi_1(e, 4) \lneq g \leq \pi(e, 4)$. Therefore, as we did in previous sections, we expect that the existence and the irreducibility (or reducibility in certain cases) would follow by looking at (possibly singular) curves on a scroll and studying the Severi variety of nodal curves on the scroll S.

(ii) For $g = r+12$, $\mathcal{E} = \mathcal{D}^\vee = g^4_{15}$ may induce a triple covering of an elliptic curve $C \xrightarrow{\phi} E$, i,e, $\mathcal{E} = \phi^*(g^4_5)$. One may show that $\mathcal{E}^\vee$ is very ample if $r \geq 13$, which implies $\mathcal{H}^{\mathcal{L}}_{g+r-5,g,r} \neq \emptyset$ for $g = r + 12$, $r \geq 13$.

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