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A homological model for $U_q\mathfrak{sl}2$ Verma modules and their braid representations

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We extend Lawrence's representations of the braid groups to relative homology modules and we show that they are free modules over a ring of Laurent polynomials. We define homological operators and we show that they actually provide a representation for an integral version for $U_q\mathfrak{sl}(2)$. We suggest an isomorphism between a given basis of homological modules and the standard basis of tensor products of Verma modules and we show it preserves the integral ring of coefficients, the action of $U_q\mathfrak{sl}(2)$, the braid group representation and its grading. This recovers an integral version for Kohno's theorem relating absolute Lawrence representations with the quantum braid representation on highest-weight vectors. This is an extension of the latter theorem as we get rid of generic conditions on parameters, and as we recover the entire product of Verma modules as a braid group and a $U_q\mathfrak{sl}(2)$–module.

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1 Introduction

We give two definitions for the braid group on n strands.

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Definition 1.1 Let $n \in \mathbb{N}$.

- The *braid group* on n strands $\mathcal{B}_n$ is the group generated by $n-1$ elements satisfying the so-called *"braid relations"*
$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \dots, n-2 \end{matrix} \right\rangle.$$
- The braid group on n strands is the mapping class group of the punctured disk D_n (defined in Section 2),
$$\mathcal{B}_n = \operatorname{Mod}(D_n).$$

These two definitions provide two different directions to build representations: quantum representations and homological representations. In this work we relate the two theories. Quantum representations are built from the category of modules on a given quantum group, which are purely algebraic tools, so their topological meaning is quite mysterious. We study the quantum group arising from the quantized deformation of $\mathfrak{sl}(2)$, namely $U_q\mathfrak{sl}(2)$, and the question of its topological content is then natural. The goal is to relate Verma modules for $U_q\mathfrak{sl}(2)$ to homological modules, in the sense that we want to preserve both the action of $U_q\mathfrak{sl}(2)$ and that of the braid group, and an integral structure of coefficients.

1.1 Homological representations

In [18], R Lawrence builds graded homological representations of braid groups relying on the fact that braids are associated with homeomorphisms of the punctured disk. Indeed, this generalizes to configuration spaces of r points in the punctured disk, denoted by X_r and defined in Definition 2.1. This becomes a linear representation when lifted to homology, namely to modules denoted by $\mathcal{H}_r^{\text{abs}}$ and defined precisely in Definition 2.7. Lawrence builds a family of graded representations for the braid groups over $\mathcal{H}_r^{\text{abs}}$ ($r \in \mathbb{N}$ is the grading) with local coefficients in a ring of Laurent polynomials $\mathcal{R}_{\max}$ over the configuration space of points inside the punctured disk [18]. She developed this idea around 1990 in her thesis, at which time it was already for the purpose of finding topological information in the Jones polynomial, an invariant of knots defined out of quantum representations of braid groups. S Bigelow used her ideas to recover a definition of the Jones polynomial from these homology modules in [3]. This provides a formula for the Jones polynomial as a pairing between homology classes, allowing it to be compared with homological invariants by Droz and Wagner [8] and Manolescu [20], for instance.

The value of Lawrence's representations comes from D Krammer [17] and Bigelow's [2] work, showing their faithfulness at the second level of the grading, the one we refer to as the *BKL representation*. It is the first known finite-dimensional and faithful linear representation of braid groups. In [21], L Paoluzzi and L Paris show that the BKL representation only recovers a subrepresentation of the entire homological representation with coefficients in the ring of Laurent polynomials.

1.2 Quantum representations

On the quantum side, one can build braid representations over tensor products of *Verma modules* for $U_q\mathfrak{sl}(2)$. Namely, let V be the Verma module of $U_q\mathfrak{sl}(2)$ (we suppress the parameter it depends on in notation, for economy); for $n \in \mathbb{N}$, the module $V^{\otimes n}$ is endowed with a quantum action of the braid group $\mathcal{B}_n$. Let $r \in \mathbb{N}$, $W_{n,r}$ be the subspace of $V^{\otimes n}$ generated by vectors of subweight r and $Y_{n,r}$ be the one generated by the highest-weight vectors of $W_{n,r}$. Spaces $W_{n,r}$ and $Y_{n,r}$ are subrepresentations of $\mathcal{B}_n$, and $V^{\otimes n} = \bigoplus_{r\in\mathbb{N}} W_{n,r}$. All these definitions are rigorously given in Definition 5.8.

In [12], C Jackson and T Kerler establish explicitly an isomorphism between the BKL representation $\mathcal{H}_2^{\text{abs}}$ and that on *highest-weight vectors* and subweights 2, denoted by $Y_{n,2}$. In [16], T Kohno shows Lawrence's representations are isomorphic to those from KZ monodromy restricted to highest-weight vectors, themselves previously shown to be isomorphic to the braid representations on highest-weight vectors $Y_{n,r}$ by Drinfeld [7] and Kohno [14]. This establishes a direct and deep relation between Lawrence's representations and the $U_q\mathfrak{sl}(2)$ R–matrix that is summed up by Ito [11, Theorem 4.5]. Homological and quantum representations depend on $n+1$ variables. One can treat them as parameters, or can take as a ground ring of coefficients Laurent polynomials in these variables, denoted by $\mathcal{R}_{\max}$ in this work (considering *integral versions* for quantum modules). Yet Kohno's isomorphism (between $\mathcal{B}_n$ representations $Y_{n,r}$ and $\mathcal{H}_r^{\text{abs}}$) holds for a generic set of parameters (it is not a morphism on the ring of Laurent polynomials, but on $\mathbb{C}$ when quantum parameters are evaluated at "generic" values) and does not recover the whole product of Verma modules, but only the braid group action over the $Y_{n,r}$ for $r \in \mathbb{N}$. In [9], G Felder and C Wieczerkowski build an action of the quantum group $U_q\mathfrak{sl}(2)$ on some module generated by topological objects of the punctured disk — *r–loops* — together with a natural action of the braid groups which commutes with the quantum one. The homological interpretations of this module remain conjectures [9, Conjectures 6.1 and 6.2] as well as its links with Lawrence's theory. Finally, in [23], V Schechtman and A Varchenko obtain representations of quantum groups on some local system homology on configuration spaces of points. We sum up the brief history of Lawrence's representations in three results.

Theorem 1.2 (i) *For all* $r \in \mathbb{N}$, $\mathcal{H}_r^{\mathrm{abs}}$ *is a representation of* $\mathcal{B}_n$ [18].

(ii) *The representation* $\mathcal{H}_2^{\mathrm{abs}}$ *is faithful* [2; 17].

(iii) *There exists an isomorphism of* $\mathcal{B}_n$*–representations between* $\mathcal{H}_r^{\mathrm{abs}}$ *and the quantum module* $Y_{n,r}$ (by [12] for the case $r = 2$ and Kohno [15] for generic values of the parameters q and α_k).

1.3 Results of the paper

The present work extends Lawrence's representations via relative homology; it clarifies and generalizes their links with quantum representations of braid groups obtained on tensor products of $U_q\mathfrak{sl}(2)$ Verma modules. Inspired by [9], we extend Lawrence modules to relative homology modules, denoted by $\mathcal{H}_r^{\mathrm{rel}-}$ and defined in Definition 2.7. We endow these modules with a homological action of the quantum group $U_q^{L/2}\mathfrak{sl}(2)$ (an integral version for $U_q\mathfrak{sl}(2)$ defined in Section 5.1) via homological actions of its generators (defined in Section 6.1), which leads to the following result:

Theorem 1.3 (Theorem 1 in Section 6.1.3) *The module* $\mathcal{H} = \bigoplus_{r\in\mathbb{N}} \mathcal{H}_r^{\mathrm{rel}-}$ *over the ring of Laurent polynomials* $\mathcal{R}_{\max}$ *is a representation of* $U_q^{L/2}\mathfrak{sl}(2)$.

In Proposition 3.6, we show that modules $\mathcal{H}_r^{\mathrm{rel}-}$ are free modules on the ring of Laurent polynomials $\mathcal{R}_{\max}$, and that a basis (said "integral") is given by the family of *multiarcs*; see Corollary 4.13. This helps us to recognize this $U_q^{L/2}\mathfrak{sl}(2)$ representation as a tensor product of Verma modules, which we sum up in the following statement:

Theorem 1.4 (Theorem 2 in Section 6.2.3) *For all* $n \in \mathbb{N}$, *there exists a morphism of* $U_q^{L/2}\mathfrak{sl}(2)$*–modules*

$$V^{\otimes n} \to \mathcal{H}$$

such that the standard integral basis of $V^{\otimes n}$ *is sent to the* ***multiarcs*** *basis. The integer* n *corresponds to the number of punctures of the disk* D_n *used to define the configuration space* X_r.

Finally, we extend the natural Lawrence action of braid groups over these homological modules and we show that it is the R–matrix representation obtained using $U_q^{L/2}\mathfrak{sl}(2)$ Verma modules.

Theorem 1.5 (Theorem 3 in Section 6.3.2) *For all $n \in \mathbb{N}$ and all $r \in \mathbb{N}$, the morphism*

$$W_{n,r} \to \mathcal{H}_r^{\mathrm{rel}-}$$

induced by the previous theorem is an isomorphism of $\mathcal{B}_n$–representations, so the morphism

$$V^{\otimes n} \to \mathcal{H} = \bigoplus_{r \in \mathbb{N}} \mathcal{H}_r^{\mathrm{rel}-}$$

from the previous theorem is a morphism of $U_q^{L/2}\mathfrak{sl}(2)$–modules **and** *of $\mathcal{B}_n$–modules.*

We provide an integral basis for homology (ie basis as a module on an integral ring of Laurent polynomials). The $U_q^{L/2}\mathfrak{sl}(2)$–action and the $\mathcal{B}_n$–action preserve this structure, as does the isomorphism to the tensor product of Verma modules. This is an improvement over previous models, and has potential for topological quantum invariants built from those braid representations that need parameters to be evaluated. For instance, q being a root of unity is required to study several quantum invariants, and was not recovered by the generic conditions of Kohno's theorem (Theorem 1.2(iii)).

We show that the long exact sequence of relative homology becomes, in this model, a short one,

$$0 \to \mathcal{H}_r^{\mathrm{abs}} \to \mathcal{H}_r^{\mathrm{rel}-} \to H_{r-1}(X_r^-) \to 0$$

(where X_r^- is as defined in Definition 2.7), so that $\mathcal{H}_r^{\mathrm{rel}-}$ extend Lawrence's representations. This work thus allows an extension of Kohno's theorem beyond highest-weight vectors, and recovers homologically the entire tensor product of $U_q\mathfrak{sl}(2)$ Verma modules. Lawrence's representations are subrepresentations of it, so Kohno's theorem is a corollary of this work. Generic hypotheses are clarified and become algebraic thanks to the fact that all isomorphisms preserve the integral structure of coefficients, and the links between an integral basis (multiarcs) and the multifork basis from Kohno's theorem are explicit. All of this is summed up in Corollary 7.1 and Proposition 7.2 in Section 7.1.

The obtained homological representations are a generalization of Lawrence's representations, so they are generically faithful. They allow a homological recovering of several properties of the category of $U_q\mathfrak{sl}(2)$–modules.

We illustrate the weight structure of tensor product of Verma modules in the following diagram, at level r of the grading:

$$\begin{array}{ccc}
\vdots & & \vdots \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r} & \longleftrightarrow & \mathcal{H}_r^{\mathrm{rel}-} \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r+1} & \longleftrightarrow & \mathcal{H}_{r+1}^{\mathrm{rel}-} \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
\vdots & & \vdots
\end{array}$$

Horizontal arrows correspond to isomorphisms of braid representations from Theorem 1.5, while vertical arrows correspond to the action of $U_q\mathfrak{sl}(2)$ generators E and F: the quantum ones on the left side and the homological ones (homological definitions are given in this work) on the right side, which rules the weight structure on Verma modules. The direct sum of all spaces aligned vertically on the left gives the tensor product of Verma modules $V^{\otimes n}$, while the one of all spaces aligned on the right corresponds to the homological module $\mathcal{H}$. The homological interpretation of $U_q\mathfrak{sl}(2)$ generators follows, together with the ones of relations they satisfy and the R–matrix built using these generators.

Plan of the paper

In Section 2 we define topological spaces and homology modules used to build homological representations. In Section 3 we give examples of homology classes in $\mathcal{H}_r^{\mathrm{rel}-}$, representing them by multiarcs diagrams, then we study the structure of the homology complexes of interest. Namely, we prove the crucial Proposition 3.6, stating that modules $\mathcal{H}_r^{\mathrm{rel}-}$ are free over the ring of Laurent polynomials $\mathcal{R}_{\max}$, and that these are the only nonvanishing modules of the entire homology complex. In Section 4 we state all the rules we need to do computation in $\mathcal{H}_r^{\mathrm{rel}-}$, and we use them to show that the family of multiarcs is a basis of $\mathcal{H}_r^{\mathrm{rel}-}$ as a module over $\mathcal{R}_{\max}$ in Corollary 4.13. In Section 5 we recall definitions and notation for quantum algebra. Namely, we define an integral version (ie as a free $\mathcal{R}_{\max}$–module) of $U_q\mathfrak{sl}(2)$, denoted by $U_q^{L/2}\mathfrak{sl}(2)$, and its version for Verma modules. We then present the braid representations defined over a tensor product of Verma modules, and how to get a finite-dimensional representation out of them in Remark 5.13. Finally, the main results of this paper can be found in Section 6. In Section 6.1 we define homological operators corresponding to generators of $U_q^{L/2}\mathfrak{sl}(2)$ and we prove Theorem 1, stating that this provides a representation of $U_q^{L/2}\mathfrak{sl}(2)$. In

Section 6.2 we compute the homological action of $U_q^{L/2}\mathfrak{sl}(2)$ in the multiarcs basis and we prove Theorem 2, saying that this homological representation is isomorphic to a tensor product of Verma modules. In Section 6.3 we recall how to build a homological action of braid groups over homological modules. Then we prove Theorem 3, saying that the isomorphism of $U_q^{L/2}\mathfrak{sl}(2)$–modules relating homological modules with Verma modules is also an isomorphism of $\mathcal{B}_n$–representations and that it preserves their grading. In Section 7.1 we show that Theorem 3 recovers Kohno's theorem (Theorem 1.2(iii)) in an integral version and exhibits previous generic conditions on parameters required. In Section 7.2 we give partially positive answers to Conjecture 6.2 of [9]. In the appendix we recall definitions of homology theories we use, namely the locally finite (Borel–Moore) version of singular homology and the local ring of coefficients setup.

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2 Configuration space and homology

Definition 2.1 Let $r \in \mathbb{N}$, $n \in \mathbb{N}$, D be the unit disk and $\{w_1, \dots, w_n\} \in D^n$ be points lying on the real line in the interior of D. Let $D_n = D \setminus \{w_1, \dots, w_n\}$ be the unit disk with n punctures. Let

$$\mathrm{Conf}_r(D_n) := \{(z_1, \dots, z_r) \in (D_n)^r \mid z_i \neq z_j \text{ for all } i \neq j\}$$

be the configuration space of points in the punctured disk D_n. We define

$$X_r(w_1, \dots, w_n) = \mathrm{Conf}_r(D_n)/\mathfrak{S}_r \tag{1}$$

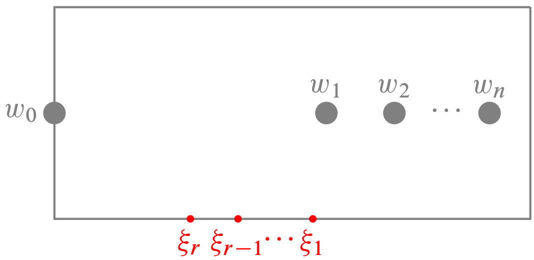
to be the space of *unordered* configurations of r points inside D_n, where the permutation group $\mathfrak{S}_r$ acts by permutation on coordinates.

When no confusion arises in what follows, we omit the dependence on $w_1, \dots, w_n$ to simplify notation. All the following computations rely on a choice of basepoint, which we fix from now on.

Notation (basepoint) Let $\boldsymbol{\xi}^r = \{\xi_1, \dots, \xi_r\}$ be the basepoint of X_r, chosen so that $\xi_i \in \partial D_n$ for all i with negative imaginary parts, and so that

$$\Re(w_0) < \Re(\xi_r) < \Re(\xi_{r-1}) < \cdots < \Re(\xi_1) < \Re(w_1).$$

We illustrate the disk with chosen points in the following figure:



We draw a square boundary for the disk, in order for the reader not to confuse it with arcs we will be drawing inside.

We give a presentation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ as a braid subgroup (the *mixed braid group*), which can be deduced from the one given in the introduction of [24], and will be explained with drawings.

Remark 2.2 The group $\pi_1(X_r, \boldsymbol{\xi}^r)$ is isomorphic to the subgroup of $\mathcal{B}_{r+n}$ generated by

$$\langle \sigma_1, \dots, \sigma_{r-1}, B_{r,1}, \dots, B_{r,n} \rangle,$$

where the σ_i for $i = 1, \dots, r-1$ are standard generators of $\mathcal{B}_{r+n}$, and $B_{r,k}$ (for $k = 1, \dots, n$) is the pure braid

$$B_{r,k} = \sigma_r \cdots \sigma_{r+k-2} \sigma_{r+k-1}^2 \sigma_{r+k-2}^{-1} \cdots \sigma_r^{-1}.$$

To see the correspondence between loops in X_r and generators of the above braid subgroup, we draw two examples.

Example 2.3 Two types of braid generators for $\pi_1(X_r, \boldsymbol{\xi}^r)$ are given in Remark 2.2, which correspond to two types of loops generating $\pi_1(X_r, \boldsymbol{\xi}^r)$. We give examples for both kinds:

- The braid σ_1 corresponds to a loop swapping ξ_r and ξ_{r-1} leaving other basepoint coordinates fixed. This can be seen by drawing the movie of the loop in Figure 1.

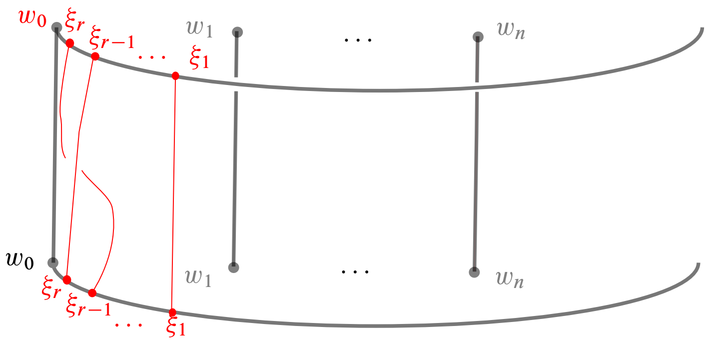


Figure 1: Generator σ_1.

- The braid $B_{r,k}$ for $k \in \{1, \ldots, n\}$ corresponds to ξ_1 running once around w_k before going back, keeping other basepoint coordinates fixed. The correspondence in terms of standard braid generators can be seen by drawing the movie of this loop in Figure 2.

Remark 2.4 In all of what follows and in the above example, braids representing loops in X_r are read from top to bottom.

Using this setup, we define the local system of interest.

Definition 2.5 (local system L_r) Let $L_r(w_1, \ldots, w_n)$ be the local system defined by the algebra morphism

$$\rho_r : \mathbb{Z}[\pi_1(X_r, \boldsymbol{\xi}^r)] \to \mathbb{Z}[s_i^{\pm 1}, t^{\pm 1}]_{i=1,\ldots,n}, \qquad \sigma_i \mapsto t, \quad B_{r,k} \mapsto s_k^2.$$

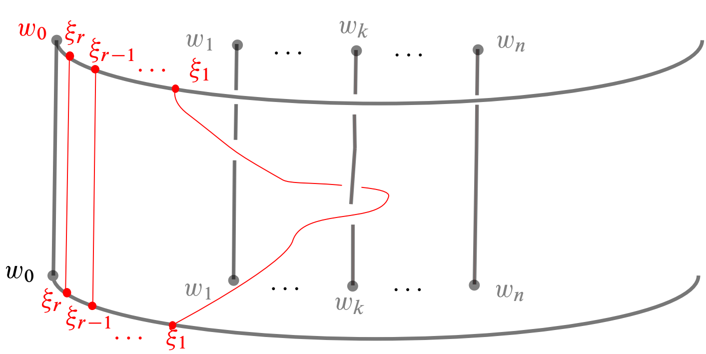


Figure 2: Generator $B_{r,k}$.

In what follows we will use the notation $q^{\alpha_k} := s_k$ for all $k = 1, \dots, n$. Using this notation, the morphism becomes

$$\rho_r : \mathbb{Z}[\pi_1(X_r, \boldsymbol{\xi}^r)] \to \mathbb{Z}[q^{\pm\alpha_i}, t^{\pm 1}]_{i=1,\dots,n}, \qquad \sigma_i \mapsto t, \quad B_{r,k} \mapsto q^{2\alpha_k}.$$

When no confusion is possible we will omit the dependence on $(w_1, \dots, w_n)$ in the notation for economy.

Remark 2.6 As this is a 1–dimensional local system, it is abelian in the sense that

$$\rho_r(s_1 s_1) = \rho_r(s_1)\rho_r(s_2) = \rho_r(s_2)\rho_r(s_1)$$

for $s_1, s_2 \in \pi_1(X_r, \boldsymbol{\xi}^r)$. Moreover, this local system corresponds to the maximal abelian cover of X_r; see [16, Section 2].

We will use homology modules with coefficients in this local system, so we fix notation from now on.

Definition 2.7 Let $w_0 = -1$ be the leftmost point on the boundary of the disk, we define the set

$$X_r^-(w_1, \dots, w_n) = \big\{\{z_1, \dots, z_r\} \in X_r(w_1, \dots, w_n) \mid z_i = w_0 \text{ for some } i\big\}.$$

Let $r \in \mathbb{N}$ and $\mathcal{R}_{\max} = \mathbb{Z}[q^{\pm\alpha_i}, t^{\pm 1}]_{i=1,\dots,n}$. We let H^{lf} designate the homology of locally finite chains, and for homology with local coefficients in the ring $\mathcal{R}_{\max}$ we write

$$\mathcal{H}_r^{\mathrm{abs}} := \mathrm{H}_r^{\mathrm{lf}}(X_r; L_r) \quad \text{and} \quad \mathcal{H}_r^{\mathrm{rel}-} := \mathrm{H}_r^{\mathrm{lf}}(X_r, X_r^-; L_r).$$

The second one is the homology of the pair (X_r, X_r^-). See the appendix for a summary of these homology theories (locally finite/Borel–Moore, with local coefficients).

Remark 2.8 Every local system construction of homology classes (see the appendix) depends on a choice of a lift of basepoint $\boldsymbol{\xi}$, which we make here; namely, let $\widehat{\boldsymbol{\xi}}$ be a lift of $\boldsymbol{\xi}$ in the cover corresponding to the local system L_r. For a different choice $\widehat{\boldsymbol{\xi}'}$ of lift, all the classes are multiplied by the same (invertible) monomial $\rho_r(\widehat{\boldsymbol{\xi}} \to \widehat{\boldsymbol{\xi}'})$ of $\mathcal{R}_{\max}$, namely the local coefficient of a path relating $\widehat{\boldsymbol{\xi}}$ and $\widehat{\boldsymbol{\xi}'}$.

We recall the signature and permutation of braids that will be needed.

Notation We will call *signature* of a braid the signature of the permutation induced by the braid, and we will use the following notation for morphisms:

$$\mathrm{sign} \colon \mathcal{B}_n \xrightarrow{\mathrm{perm}} \mathfrak{S}_n \xrightarrow{\mathrm{sign}} \{\pm 1\}.$$

3 Structure of the homology

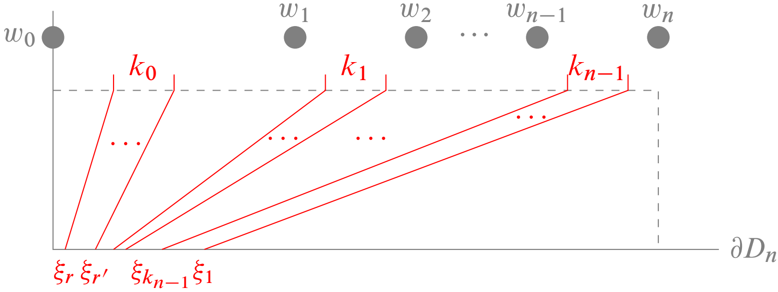
3.1 Examples of classes

Definition 3.1 We define the set of partitions of n in r integers as

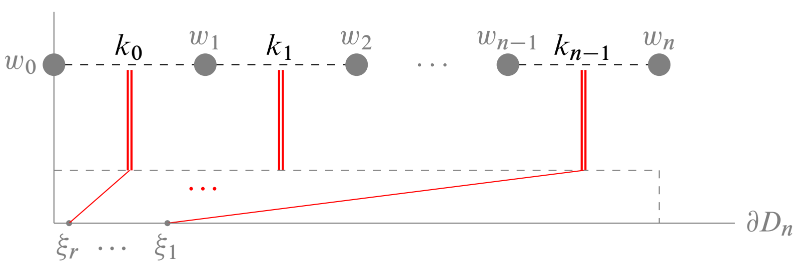
$$E^0_{n,r} = \left\{(k_0, \dots, k_{n-1}) \in \mathbb{N}^n \mid \sum k_i = r\right\}.$$

We now define two families of topological objects indexed by $E^0_{n,r}$, which will correspond to classes in $\mathcal{H}^{\text{rel}-}_r$.

Notation We draw topological objects inside the punctured disk, without drawing the boundary of the disk entirely, for easier reading. The gray color is used to draw the punctured disk. Red arcs are going from a coordinate of the basepoint $\boldsymbol{\xi}$ of X_r lying in its boundary to a dashed black arc. Dashed black arcs are oriented, from left to right if nothing is specified and if no confusion arises. Finally, for all the following objects, the red arcs will end up going like in the following picture inside the dashed box, so that all families of red arcs are attached to the basepoint $\{\xi_1, \dots, \xi_r\}$ of X_r (here $r' = r - k_0$):

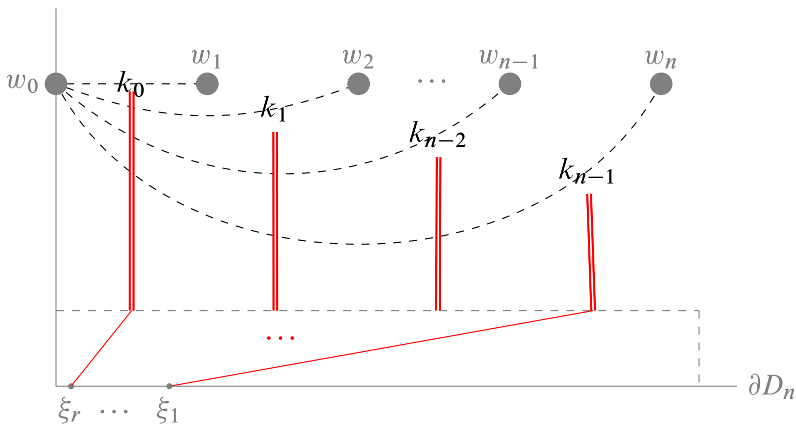


Code sequences Let $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E^0_{n,r}$ we define the *code sequence* $U_{\boldsymbol{k}} = U(k_0, \dots, k_{n-1})$ to be the drawing



The indices k_i serve to illustrate the fact that k_i configuration points are embedded in the corresponding dashed segment, as we explain in what follows. We have attached to an index k_i dashed arc a red arc, called a (k_i)–*handle*. It is represented by a little red tube, which is a simpler representation used to represent k_i parallel red arcs, which are called *handles*. We let $\mathcal{U} = \{U(k_0, \dots, k_{n-1})\}_{\boldsymbol{k} \in E^0_{n,r}}$. The definition of these objects comes from [4].

Multiarcs By analogy, for $\boldsymbol{k} \in E^0_{n,r}$ we define a multiarc $A'_{\boldsymbol{k}} = A'(k_0, \dots, k_{n-1})$ to be the picture



As for code sequences, there is a (k_i)–handle arriving to a dashed arc indexed by k_i, this will be used to define the associated homology class. We write the family of all standard multiarcs as $\mathcal{A}' = \{A'(k_0, \dots, k_{n-1})\}_{\boldsymbol{k} \in E^0_{n,r}}$. This family of objects is new in the literature.

We provide a natural way to assign a class in $\mathcal{H}^{\mathrm{rel}-}_{\boldsymbol{r}}$ to these drawings. Let X be the letter U or A', to treat both cases at the same time. Let $\boldsymbol{k} \in E^0_{n,r}$ and, for all $i = 1, \dots, n$, let

$$\phi_i \colon I_i \to D_n$$

be the embedding of the dashed black arc number i of $X(k_0, \dots, k_{n-1})$ indexed by k_{i-1}, where I_i is a unit interval. Let Δ^k be the standard (open) k simplex

$$\Delta^k = \{0 < t_1 < \cdots < t_k < 1\}$$

for $k \in \mathbb{N}$. For all i, we consider the map

$$\phi^{k_{i-1}} \colon \Delta^{k_{i-1}} \to X_{k_{i-1}}, \quad (t_1, \dots, t_{k_{i-1}}) \mapsto \{\phi_i(t_1), \dots, \phi_i(t_{k_{i-1}})\},$$

which is a singular locally finite (k_{i-1})–chain and moreover a cycle in $X_{k_{i-1}}$. One can think of the image of the simplex $\Delta^{k_{i-1}}$ as the space of configurations of k_{i-1}

points inside the dashed arc. It provides a locally finite cycle, as going to a face of the simplex corresponds to going to a collision between either two configuration points, or a configuration point with a puncture. Namely, points in the boundary of the simplex are removed points of the configuration space X_r; these simplices are closed submanifolds going to infinity, so that they are locally finite cycles; see the appendix. There is a cycle associated with each dashed arc, so that, by considering the product of maps $(\phi^{k_0}, \dots, \phi^{k_{n-1}}) \in \mathrm{Conf}_r(D_n)$, which is naturally sent to X_r, one generalizes this fact by associating an r–cycle of X_r with each object $X(k_0, \dots, k_{n-1})$; see Remark 3.3. This shows how the union of dashed arcs defines a class in the homology with coefficients in $\mathbb{Z}$.

To get a class in the local system homology, one has to choose a lift of the chain to the maximal abelian cover L_r associated with the morphism ρ_r. The way to do so is using the red handles of $X(k_0, \dots, k_{n-1})$, with which is naturally associated a path

$$\boldsymbol{h} = \{h_1, \dots, h_r\}\colon I \to X_r$$

joining the basepoint $\boldsymbol{\xi}$ and the r–chain assigned to the union of the dashed arcs. At the cover level $\widehat{X}_r$, there is a unique lift $\hat{\boldsymbol{h}}$ of $\boldsymbol{h}$ that starts at $\widehat{\boldsymbol{\xi}}$. The lift of $X(k_0, \dots, k_{n-1})$ containing $\widehat{\boldsymbol{\xi}}(1)$ defines a cycle in $\mathrm{C}_r^{\mathrm{rel}-}$, and we still call $X(k_0, \dots, k_{n-1})$ the associated class in $\mathcal{H}_r^{\mathrm{rel}-}$ as we will only use this class out of the original object.

Remark 3.2 If ϕ_i and ϕ_i' are two parametrizations of the dashed arc $D^{k_{i-1}}$, then ϕ_i and ϕ_i' are homotopic, as are the associated maps $\phi^{k_{i-1}}$ and ${\phi'}^{k_{i-1}}$. Then the homology classes associated with $\phi^{k_{i-1}}$ and ${\phi'}^{k_{i-1}}$ are equal and this guarantees that objects are well defined.

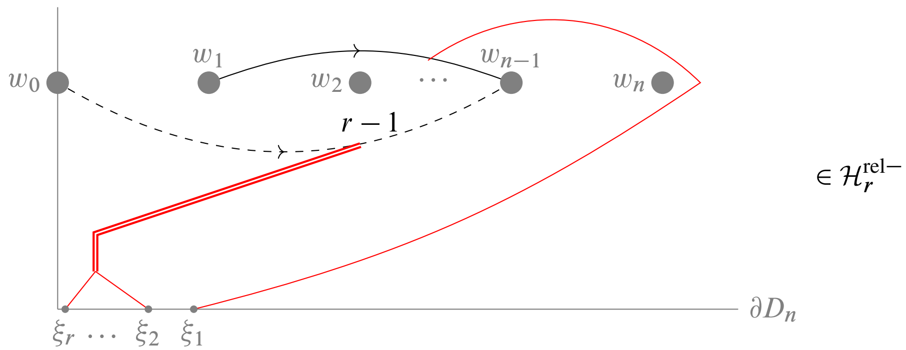
Remark 3.3 If ϕ^{k_1} and ϕ^{k_2} correspond to chains with disjoint supports, there exists an associated chain $\{\phi^{k_1}, \phi^{k_2}\} \in X_{k_1+k_2}$.

Remark 3.4 The map ϕ^k for any k factors through $\mathrm{Conf}_k(D_n)$; namely, it is the composition of a map to $\mathrm{Conf}_k(D_n)$ and the quotient to X_r, as follows:

$$\phi^k\colon \Delta^k \to \mathrm{Conf}_k(D_n) \to X_k.$$

In what follows it will sometimes be more convenient to think about the image of ϕ^k as a submanifold of $\mathrm{Conf}_k(D_n)$ before considering the quotient by permutations.

Example 3.5 By analogy, there is a natural class in $\mathcal{H}_r^{\mathrm{rel}-}$ associated with the diagram



When we draw a plain arc, it corresponds to the image of a 1*–dimensional simplex with one configuration point embedded*, while a dashed arc indexed by $r-1$ corresponds to an $(r-1)$–simplex, so to $r-1$ configuration points embedded. Red handles are considered, defining a cycle with local coefficients as above.

3.2 Structural result

We now study the algebraic structure of $\mathcal{H}_r^{\mathrm{rel}-}$.

Proposition 3.6 *For $r \in \mathbb{N}$, the module $\mathcal{H}_r^{\mathrm{rel}-}$ is a free $\mathcal{R}_{\max}$–module of dimension $\binom{n+r-1}{r}$, generated by the family $\mathcal{U}$ of code sequences. Moreover, it is the only nonvanishing module of the complex $\mathrm{H}_\bullet^{\mathrm{lf}}(X_r, X_r^-; L_r)$.*

Proof Throughout the proof, the local ring of coefficients will remain L_r, so we omit it in the notation. Let $X_r^{\mathbb{R}}$ be the set $\{x_1, \dots, x_r\} \in X_r$ such that $x_1, \dots, x_r$ lie in the segment $[w_0, w_n)$. Set $X_r^{\mathbb{R},-} = X_r^{\mathbb{R}} \cap X_r^-$. We use these simpler spaces to compute the homology, thanks to the following lemma, which can be seen as a Bigelow interpretation of the Salvetti retract complex associated with a hyperplanes arrangement [22]. This method is adapted from [4, Lemma 3.1].

Lemma 3.7 (Bigelow's trick) *The map*

$$\mathrm{H}_\bullet^{\mathrm{lf}}(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r) \to \mathrm{H}_\bullet^{\mathrm{lf}}(X_r, X_r^-; L_r) \tag{2}$$

induced by inclusion is an isomorphism.

Proof Let $\epsilon > 0$ and A_ϵ be the set of $\{x_1, \dots, x_r\} \in X_r$ such that $|x_i - x_j| \geq \epsilon$ and $|x_i - w_k| \geq \epsilon$ for all distinct $i, j = 1, \dots, r$ and $k = 1, \dots, n$. This family of compact sets yields a basis of compact sets for X_r, so it suffices to show that, for all sufficiently small ϵ, the map

$$H_\bullet(X_r^{\mathbb{R}}, (X_r^{\mathbb{R}} \setminus A_\epsilon) \cup X_r^{\mathbb{R},-}) \to H_\bullet(X_r, (X_r \setminus A_\epsilon) \cup X_r^-)$$

induced by inclusion is an isomorphism. This is sufficient by means of the inductive limit over compact sets definition of Borel–Moore homology; see Remark A.3 in the appendix.

Let $D_n' \subset D_n$ be a closed $\frac{\epsilon}{2}$–neighborhood of the interval $[w_0, w_n)$. Let X_r' be the configuration space of r points in D_n', and $X_r'^- = X_r' \cap X_r^-$ be the ones with a coordinate in w_0. The map

$$H_\bullet(X_r', (X_r' \setminus A_\epsilon) \cup X_r'^-) \to H_\bullet(X_r, (X_r \setminus A_\epsilon) \cup X_r^-) \tag{3}$$

induced by inclusion is an isomorphism. To see this, note that the obvious homotopy shrinking X_r to X_r' is a homotopy of the pairs involved. In other words, points in $X_r \setminus A_\epsilon$ corresponding to close points stay in it because the homotopy is a contraction. We will refer to this process — proving that (3) is an isomorphism — as the *compressing trick* later on.

Let V be the set of $\{x_1, \dots, x_r\} \in X_r$ with either $\Re(x_i) = \Re(x_j)$ for some $i, j \in \{1, \dots, r\}$ or $\Re(x_i) = w_k$ for some $i \in \{1, \dots, r\}$ and $k \in \{1, \dots, n\}$. Let $U = X_r' \setminus V$. Note that V is a closed subset contained in $X_r' \setminus A_\epsilon$, which is the interior of $(X_r' \setminus A_\epsilon) \cup X_r'^-$. This shows that V satisfies the required hypothesis to perform the excision of the pair, so the map

$$H_\bullet(U, (U \setminus A_\epsilon) \cup (X_r'^- \cap U)) \to H_\bullet(X_r', (X_r' \setminus A_\epsilon) \cup X_r'^-)$$

induced by inclusion is an isomorphism by the excision theorem.

Finally, there is an obvious *vertical line* deformation retraction that sends U to $X_r^{\mathbb{R}}$, taking $\{x_1, \dots, x_r\}$ to $\{\Re(x_1), \dots, \Re(x_r)\}$. This is again a contraction homotopy, so $U \setminus A_\epsilon$ is preserved and $X_r' \cap U$ is sent to $X_r^{\mathbb{R},-}$. This retraction guarantees that the map

$$H_\bullet(X_r^{\mathbb{R}}, (X_r^{\mathbb{R}} \setminus A_\epsilon) \cup X_r^{\mathbb{R},-}) \to H_\bullet(U, (U \setminus A_\epsilon) \cup (X_r'^- \cap U))$$

induced by inclusion is an isomorphism, and concludes the proof of Lemma 3.7. □

To prove the proposition, it remains to compute the complex $\mathrm{H}_\bullet^{\mathrm{lf}}(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r)$. Let $A_\epsilon^{\mathbb{R}} \in X_r^{\mathbb{R}}$ be the set of configurations $\{x_1, \dots, x_r\}$ of $X_r^{\mathbb{R}}$ such that $|x_i - x_j| \geq \epsilon$

and $|x_i - w_k| \geq \epsilon$ for $i, j = 1, \dots, r$ and $k = 1, \dots, n$. Let $A_\epsilon^{\mathbb{R},w_0}$ be $A_\epsilon^{\mathbb{R}}$ with the additional condition that $|x_i - w_0| \geq \epsilon$ for $i = 1, \dots, r$. We are going to show that, for sufficiently small ϵ, the complex

$$H_\bullet(X_r^{\mathbb{R}}, (X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}) \cup X_r^{\mathbb{R},-}; L_r)$$

is isomorphic to the Borel–Moore one of a disjoint union of open simplexes defined by code sequences. This will end the computation of $\mathrm{H}_\bullet^{\mathrm{lf}}(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r)$ by definition of Borel–Moore homology. To do so, first we note that the following spaces are homotopically equivalent:

$$\begin{aligned}(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}) \cup X_r^{\mathbb{R},-} &= \left\{ \{x_1, \dots, x_r\} \in X_r^{\mathbb{R}} \;\middle|\; \begin{array}{c} |x_i - x_j| < \epsilon \text{ for } i, j = 1, \dots, r, \\ |x_i - w_k| < \epsilon \text{ for } k = 1, \dots, n, \\ \text{or } x_i = w_0 \end{array} \right\} \\ &\simeq \left\{ \{x_1, \dots, x_r\} \in X_r^{\mathbb{R}} \;\middle|\; \begin{array}{c} |x_i - x_j| < \epsilon \text{ for } i, j = 1, \dots, r, \\ |x_i - w_k| < \epsilon \text{ for } k = 1, \dots, n, \\ \text{or } |x_i - w_0| < \epsilon \end{array} \right\} \\ &= X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}.\end{aligned}$$

This shows that

$$H_\bullet(X_r^{\mathbb{R}}, (X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}) \cup X_r^{\mathbb{R},-}; L_r) \simeq H_\bullet(X_r^{\mathbb{R}}, X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}; L_r).$$

Then one notes that $X_r^{\mathbb{R},-}$ is closed in $A_\epsilon^{\mathbb{R},w_0}$, so we can perform the excision and the map

$$H_\bullet(X_r^{\mathbb{R}} \setminus X_r^{\mathbb{R},-}, (X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}) \setminus X_r^{\mathbb{R},-}; L_r) \to H_\bullet(X_r^{\mathbb{R}}, X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}; L_r)$$

induced by inclusion is an isomorphism. Let $X_r^{\mathbb{R}}(w_0) \subset X_r^{\mathbb{R}}$ be the space of configurations without any coordinate in w_0. The space $X_r^{\mathbb{R}}(w_0)$ is exactly the space of configurations of r points in (w_0, w_n) such that every coordinate is different from w_k for $k = 0, \dots, n$. For sufficiently small ϵ, we have shown that the two homologies

$$H_\bullet(X_r^{\mathbb{R}}, (X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}) \cup X_r^{\mathbb{R},-}; L_r) \simeq H_\bullet(X_r^{\mathbb{R}}(w_0), X_r^{\mathbb{R}}(w_0) \setminus A_\epsilon^{\mathbb{R},w_0}; L_r)$$

are isomorphic. Then, as the family of $A_\epsilon^{\mathbb{R},w_0}$ is a compact set basis for $X_r^{\mathbb{R}}(w_0)$, we end up with the homologies

$$\mathrm{H}_\bullet^{\mathrm{lf}}(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r) \simeq \mathrm{H}_\bullet^{\mathrm{lf}}(X_r^{\mathbb{R}}(w_0); L_r)$$

being isomorphic. To conclude the computation we take Bigelow's decomposition of $X_r^{\mathbb{R}}(w_0)$ using code sequences as follows and as was done in [4]. For $\boldsymbol{k} \in E^0_{n,r}$, the set of all $\{x_1, \dots, x_r\} \in X_r$ such that $x_1, \dots, x_r \in (w_0, w_n)$ and

$$\#\big(\{x_1, \dots, x_r\} \cap (w_i, w_{i+1})\big) = k_i$$

for $i = 0, \dots, n-1$ is exactly $U(k_0, \dots, k_{n-1})$, and one notes that

$$X_r^{\mathbb{R}}(w_0) = \bigsqcup_{\boldsymbol{k} \in E^0_{n,r}} U_{\boldsymbol{k}}.$$

From this disjoint union of open simplexes, we deduce that $\mathrm{H}^{\mathrm{lf}}_r(X_r^{\mathbb{R}}(w_0); L_r)$ is the direct sum of $\# E^0_{n,r} = \binom{n+r-1}{r}$ copies of $\mathcal{R}_{\max}$ while all other $\mathrm{H}^{\mathrm{lf}}_k(X_r^{\mathbb{R}}(w_0); L_r)$ for $k \neq r$ vanish. The homology $\mathrm{H}^{\mathrm{lf}}_\bullet(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r)$ has the same decomposition, which concludes the proof. □

Bigelow's trick was initially used to show the following:

Proposition 3.8 [4, Lemma 3.1] *The morphism*

$$\mathrm{H}^{\mathrm{lf}}_\bullet(X_r^{\mathbb{R}}(w_0); L_r) \to \mathrm{H}^{\mathrm{lf}}_\bullet(X_r(w_0); L_r)$$

induced by inclusion is an isomorphism of homologies.

From this and from the proof of Proposition 3.6, one gets the following corollary:

Corollary 3.9
- *The morphism $\mathrm{H}^{\mathrm{lf}}_\bullet(X_r(w_0); L_r) \to \mathrm{H}^{\mathrm{lf}}_\bullet(X_r, X_r^-; L_r)$ induced by inclusion is an isomorphism.*
- *The family $\mathcal{U} = (U_{\boldsymbol{k}})_{\boldsymbol{k} \in E^0_{n,r}}$ yields a basis of $\mathcal{H}_r^{\mathrm{rel}-}$ as an $\mathcal{R}_{\max}$–module.*

We conclude this part with two remarks about the proof of Proposition 3.6.

Remark 3.10
- The proof of Proposition 3.6 is constructive in the sense that it provides a process to express homology classes in the $\mathcal{U}$–basis. This will be used in the next sections.
- Throughout the proof of Proposition 3.6, the local system does not change; no morphism of the latter is needed. The proof relies only on topological operations such as excisions and homotopy equivalences. In some sense the proof is rigid regarding the local ring of coefficients, and should be adaptable with another one.

4 Computation rules

4.1 Homology techniques

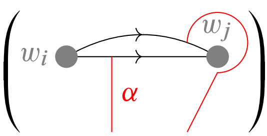
Remark 4.1 (handle rule) Let B be a singular locally finite r–cycle of $C_r(X_r, X_r^-, \mathbb{Z})$. We've seen a process to choose a lift of B to the homology with local coefficients in L_r, using a handle which is a path joining $\boldsymbol{\xi}$ to $x \in B$. Let α and β be two different paths joining $\boldsymbol{\xi}$ and B. Let $\widehat{B}^\alpha$ and $\widehat{B}^\beta$ be the lifts of B chosen using α and β, respectively. By the *handle rule*, we have the relation, in $\mathcal{H}_r^{\mathrm{rel}-}$,

$$\widehat{B}^\alpha = \rho_r(\alpha\beta^{-1})\widehat{B}^\beta,$$

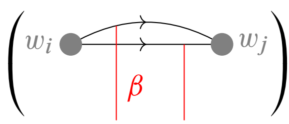
where ρ_r is the representation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ used to construct L_r in Definition 2.5. This expresses how the local system coordinate of a homological class is translated after a change of handle.

Remark 4.2 One must be careful while permuting red arcs of a multiarcs or code sequence–like class (see Section 3.1). Indeed, as the parametrization of the underlying simplex is ruled by the order of relating arcs to the basepoint, such a permutation of red handles multiplies the class by its signature. We show one example:

We have the following equality between these two classes in $\mathcal{H}_2^{\mathrm{rel}-}$:



$$= \mathrm{sign}(\alpha\beta^{-1})\rho_r(\alpha\beta^{-1})$$



with $\rho_r(\alpha\beta^{-1}) = t^{-1}q^{-2\alpha_j}$ and $\mathrm{sign}(\alpha\beta^{-1}) = -1$. Indeed, we suppose that the drawing is empty everywhere outside the parentheses besides the red handles α and β that join the basepoint $\boldsymbol{\xi}$ in the boundary. We suppose also that α and β follow exactly the same paths outside parentheses. This allows us to draw the braid $\alpha\beta^{-1}$ in Figure 3.

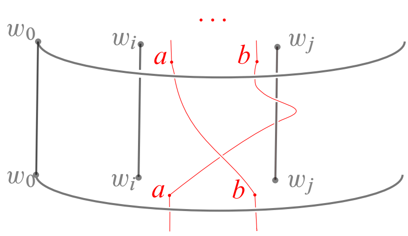


Figure 3: The braid $\alpha\beta^{-1}$.

The figure continues outside of the box but, as the path to the basepoint is the same for α and β, the path of the upper box is the inverse of the lower one. As the local system is abelian, the out box parts of the braid won't contribute to $\rho_r(\alpha\beta^{-1})$ (nor to $\operatorname{sign}(\alpha\beta^{-1})$). Considering the definition of ρ_r (Definition 2.5), one sees that the local system coordinate of the above path is $t^{-1}q^{-2\alpha_j}$ and so is the one of $\alpha\beta^{-1}$.

One should also notice that the signature of the braid is $(-1)^{e(t)}$, where $e(t)$ is the exponent of t in the local coefficient. This last remark is equivalent to the following very useful remark for what follows:

$$\operatorname{sign}(\alpha\beta^{-1})\rho_r(\alpha\beta^{-1}) = \rho_r(\alpha\beta^{-1})_{|t=\mathtt{t}},$$

where $\mathtt{t} := -t$. This will be extensively used in all following computations.

We reformulate the compressing trick used in the proof of Proposition 3.6 in a local version.

Proposition 4.3 (compressing trick) *Let $D_p \subset D_n$ (and $D_p^0 \subset D_n$, respectively) be a topological punctured disk with punctures $w_{n_1}, \dots, w_{n_p}$ and $n_i \in \{1, \dots, n\}$ for $i = 1, \dots, p$ (resp. D_p^0 contains also w_0). Let $X_r(D_p)$ (resp. $X_r(D_p^0)$) be the space of configuration of r points inside D_p (resp. D_p^0). Let D'_p (resp. ${D'}_p^0$) be an ϵ–neighborhood of a segment in D_p (resp. D_p^0) joining the points $w_{n_1}, \dots, w_{n_p}$ (resp. having an end in w_0) and contained in the real axis, with ϵ small enough to have $D'_p \subset D_p$. Then the morphisms*

$$H_\bullet(X_r(D'_p)) \to H_\bullet(X_r(D_p))$$

and

$$H_\bullet(X_r({D'}_p^0), X_r({D'}_p^0)^-) \to H_\bullet(X_r(D_p^0), X_r(D_p^0)^-)$$

induced by inclusion are isomorphisms (the module $X_r({D'}_p^0)^-$ stands for configurations with one point in w_0). All the homology modules are Borel–Moore ones (or equivalently of locally finite chains) and considered with coefficients in the local system L_r restricted to the space of interest, so we omit it in the notation.

Proof The proof is exactly the same as the one of (3) being an isomorphism, in the proof of Lemma 3.7, but performed inside D_p (resp. D_p^0). □

Proposition 4.4 (combing process) *Let $M = M(D_1^{k_1}, \dots, D_d^{k_d})$ be a class associated with a drawing made of disjoint dashed arcs D_1 indexed by k_1, D_2 indexed by k_2 and so on, all of them related to the basepoint $\boldsymbol{\xi}$ by red handles. Suppose the*

(k_1)–handle reaches $D_1^{k_1}$ in a point x. Let $D_1^{k_1} = D_1^- \cup_x D_1^+$ be a subdivision of the arc D_1 following its orientation. Let D be an arc joining x to some $w \in \{w_0, \dots, w_n\}$ and such that D is disjoint from all the $D_i^{k_1}$'s. Let $l \in \{0, \dots, k_1\}$, and M^l be the class obtained from M by modifying its drawing by

$$M^l = M((D_1^- \star D)^l, (D^{-1} \star D_1^+)^{k_1 - l}, D_2^{k_2}, \dots, D_d^{k_d}),$$

so that the initial arc D_1 is divided into two, one indexed by l and the other one by $k_1 - l$. Handles are preserved from M, except for the (k_1)–handle, which is divided into two: one (l)–handle joining $(D_1^- \star D)^l$ in x and one $(k_1 - l)$–handle joining $((D_1^+ \star D)^{-1})^{k_1 - l}$ in x. There is the homological relation

$$M = \sum_{l=0}^{k_1} M^l.$$

(See Examples 4.5 and 4.6 of such combing, which should help the understanding of the statement.)

Proof Suppose the class $M = M(D_1^{k_1})$ is made of only one dashed arc. Let

$$\phi^{k_1} : \Delta^{k_1} \to X_{k_1}, \quad (t_1, \dots, t_{k_1}) \mapsto \{\phi(t_i) \mid i = 1, \dots, k_1\},$$

be the chain naturally associated with the index k_1 dashed arc of the considered class, where ϕ is a parametrization of D^{k_1}. We subdivide the simplex: for $l \in \{0, \dots, k_1\}$ let $\Delta^{k_1, l}$ be defined as

$$\Delta^{k_1, l} = \{(t_1, \dots, t_{k_1}) \in \Delta^k \mid t_l < \phi^{-1}(x) < t_{l+1}\},$$

whose image by ϕ^{k_1} corresponds to configurations for which the handle together with D arrives between the images of t_l and t_{l+1}. Let $\phi^{k_1, l}$ be the restriction of ϕ^{k_1} to $\Delta^{k_1, l}$. Let

$$h_t : I \to D_n$$

be an isotopy (rel endpoints) sending the arc D^{k_1} to the right one of Figure 4 (arcs oriented from left to right).

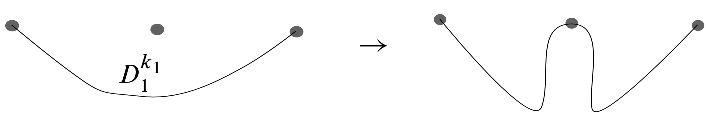


Figure 4: The isotopy h_t.

For all t in I, let $\phi_t^{k_1}$ be the map

$$\phi_t^{k_1} : \Delta^{k_1} \to X_{k_1}, \quad (t_1, \ldots, t_{k_1}) \mapsto \{h_t \circ \phi(t_i) \mid i = 1, \ldots, k_1\},$$

and let $\phi_t^{k_1,l}$ be the map

$$\phi_t^{k_1,l} : \Delta^{k_1,l} \to X_{k_1}, \quad (t_1, \ldots, t_{k_1}) \mapsto \{h_t \circ \phi(t_i) \mid i = 1, \ldots, k_1\},$$

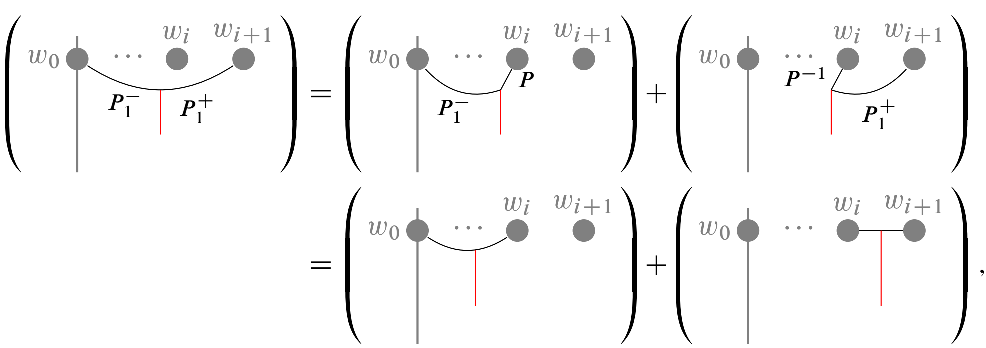
namely the restriction to $\Delta^{k_1,l}$. Let $[\phi_t^{k_1}]$ and $[\phi_t^{k_1,l}]$ be the corresponding chains. One notes that $\phi_0^{k_1,l} = \phi^{k_1,l}$ and $\phi_0^{k_1} = \phi^{k_1}$. In terms of chains, we have, for all $t \in I$,

$$[\phi_t^{k_1}] = \sum_l [\phi_t^{k_1,l}];$$

this is because $\{\Delta^{k_1,l} \mid l = 0, \ldots, k_1\}$ is a subdivision of Δ^{k_1}. For $t = 0$, this chain is $[\phi^{k_1}]$, while, for $t = 1$, terms of the sum are Borel–Moore cycles homologous to M^l. This shows that $[\phi^{k_1}]$ and $\sum_l M^l$ are homotopic, so the relation $M = \sum_{l=0}^{k_1} M^l$ holds in $\mathrm{H}_r^{\mathrm{lf}}(X_r, X_r^-, \mathbb{Z})$. Then the lifting process is unchanged as handles are preserved. This proves the proposition for a class composed by one dashed arc, and it generalizes to all classes with disjoint dashed arcs, as only the first component is involved in the combing. □

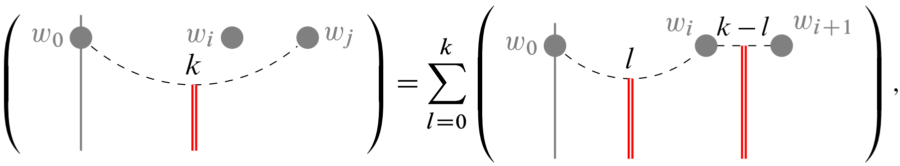
Two examples of combings that will be used many times follow next.

Example 4.5 (breaking a plain arc) By considering a path joining the red handle to w_i, one can check the relations between homology classes (all arcs oriented from left to right)



where drawings are the same outside boxes. To obtain the second line we have applied small isotopies without changing the homology class. One notes that, before the small isotopies are applied, handles are unchanged.

Example 4.6 (breaking a dashed arc) By considering a path joining the red handle to w_i one can check the relations between homology classes



where the drawings are the same outside boxes.

4.2 Diagram rules

We use homology techniques presented in the previous section to set diagram rules between homology classes. These rules expressed with coefficients in the ring $\mathcal{R}_{\max}$ involve quantum numbers that we introduce now.

Definition 4.7 Let i be a positive integer. We define elements of $\mathbb{Z}[t^{\pm 1}] \subset \mathcal{R}_{\max}$,

$$(i)_t := (1 + t + \cdots + t^{i-1}) = \frac{1-t^i}{1-t}, \quad (k)_t! := \prod_{i=1}^{k} (i)_t, \quad \binom{k}{l}_t := \frac{(k)_t!}{(k-l)_t!\,(l)_t!}.$$

Notation In what follows we will use extensively the variable $-t$ instead of t, so we fix notation for it,

$$\mathtt{t} := -t.$$

Notation Since we work with Borel–Moore homology with local coefficients, one can think of it as the complex

$$H_\bullet(X_r, (X_r \setminus A_\epsilon) \cup X_r^-; L_r)$$

for a small ϵ, with A_ϵ defined as in the proof of Proposition 3.6. A dashed arc indexed by $k > 1$ corresponds to an embedding of k points (a k–simplex) inside the arc.

As the order of points does not matter — working in X_r — one can think of the dashed arc as in Figure 5.

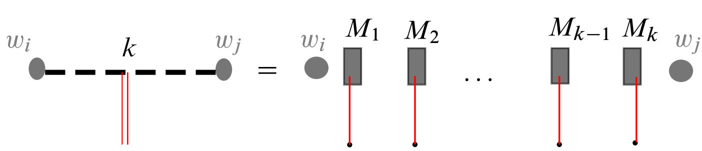


Figure 5: Dashed arc model.

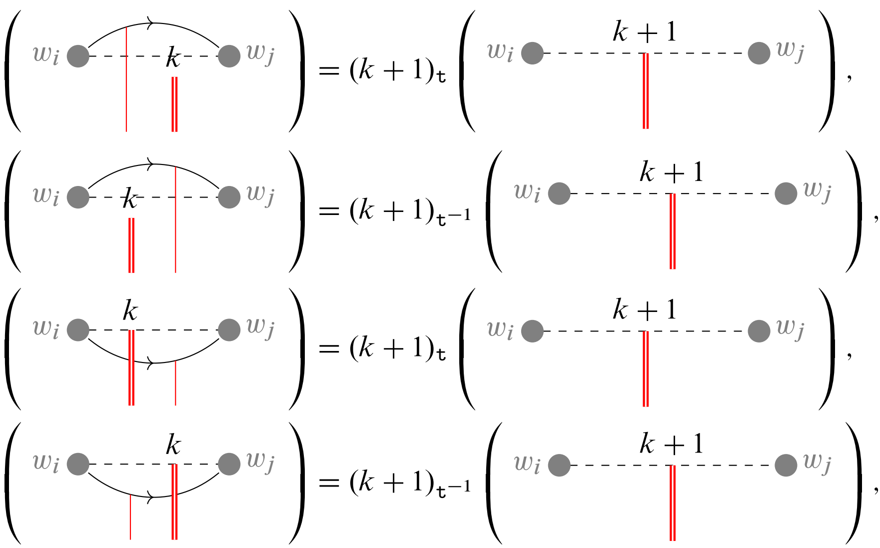
On the left side, we see a standard piece of an element of $\mathcal{U}$ and, on the right side, one can think of this element as the image of one point by the embedding

$$\Delta^k \to [w_i, w_j],$$

where M_i is the image of t_i, the i^{th} coordinate of Δ^k. The M_i are represented by gray boxes to keep in mind that we work relatively to $X_r \setminus A_\epsilon$. Every point is lifted to the maximal abelian cover $(\widehat{X}_r)$ using the red handle reaching it. A first diffeomorphism of D_n has been applied, allowing one to imagine this picture with w_i facing w_j. This diffeomorphism does not change homology classes.

The above picture will be useful to deal with the proof of the following crucial homological relations, showing a first appearance of quantum numbers.

Lemma 4.8 *Let $k > 1$ be an integer. The following equalities hold in $\mathcal{H}_\bullet^{\text{rel}-}$:*



where we suppose that the classes are the same everywhere outside parentheses, with red handles joining the same basepoints and following the same paths.

Proof We prove the first equality; the last three correspond to symmetric situations so they are proved similarly. The idea of the proof is an application of the compressing trick from Proposition 4.3, which consists in applying a homotopy compressing the disk until points cannot approach each other vertically anymore without meeting. Namely, let D be the disk depicted in the parentheses. While compressing D to an open $\frac{\epsilon}{2}$–neighborhood D' of (w_i, w_j), the plain arc from the top will approach the dashed arc.

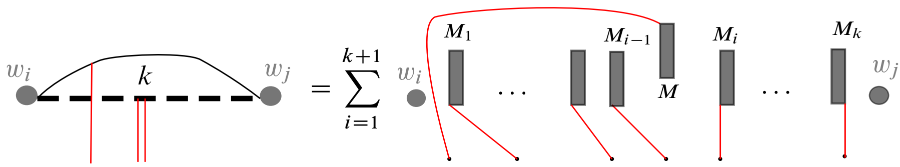


Figure 6: Homological relation.

As we work in Borel–Moore homology, so relatively to $X_r \setminus A_\epsilon$ for a small ϵ, at some points the point lying on the plain arc will cut the dashed arc to put its ϵ–neighborhood in. As there are k points lying on the dashed arc, there are $k+1$ possibilities of cuts (between $(w_0, M_1), (M_1, M_2), \dots, (M_{k-1}, M_k)$ or (M_k, w_j)). The situation may be summed up as the equality of Figure 6. In the figure, we distinguish the point M from the plain arc coming between M_{i-1} and M_i in the sum.

To be more precise, let ϕ^k be the chain

$$\Delta^k \to X_k$$

associated with the index k dashed arc. Let $\psi\colon I \to D_n$ be the one associated with the plain one. Then

$$\Psi = \{\psi, \phi^k\}\colon I \times \Delta^k \to X_{k+1}$$

is the chain associated with the left object of the equality we are studying. For $i = 1, \dots, k+1$, let Δ_i be

$$\Delta_i = \{(t, t_1, \dots, t_k) \in I \times \Delta^k \mid t_{i-1} < t < t_i\}$$

and Ψ_i be the restriction of Ψ to Δ_i. In terms of chains we have

$$[\Psi] = \sum_i [\Psi_i],$$

as the set $\{\Delta_i \mid i = 1, \dots, k+1\}$ is a subdivision of $I \times \Delta^k$. Every Δ_i is naturally homeomorphic to the standard simplex Δ^{k+1}, but it involves a permutation of coordinates in the parametrization of the simplex

$$\tau_i\colon (t, t_1, \dots, t_k) \mapsto (t_1, \dots, t_{i-1}, t, t_i, \dots, t_k),$$

where τ_i can be seen as an element of $\mathfrak{S}_{k+1}$. By homotoping the plain arc to the dashed one, one obtains a homotopy from Ψ_i to $\phi^{k+1} \circ \tau_i$ for all $i \in \{1, \dots, k+1\}$, considered as chains of $\mathrm{Conf}_k(D_n)$. Then

$$[\Psi] = \sum_{i=1}^{k+1} \mathrm{sign}(\tau_i)[\phi^{k+1}] = \sum_{i=1}^{k+1} (-1)^{i-1}[\phi^{k+1}].$$

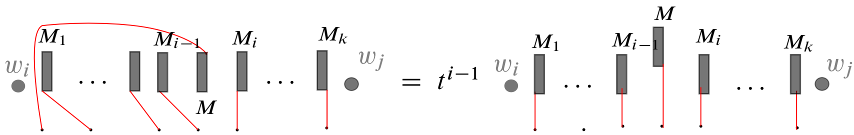
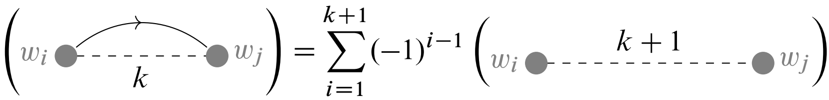


Figure 7: Local system relation.

This shows that the relation

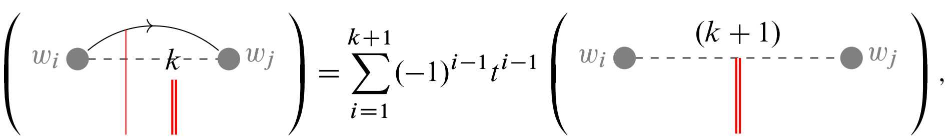
(4) 

holds in $H(X_r, X_r^-, \mathbb{Z})$. This can be seen as Figure 6 without handles. (A drawing without handles corresponds to an unlifted homology class.)

Now it's just a matter of reorganizing the handles in the elements of the sum in Figure 6 to get a dashed arc model. Using the handle rule, one can check that for $i \in \{1, \dots, k+1\}$ we have the equality of Figure 7 in the local system homology.

To see this, we draw the braid associated with this change of handle in Figure 8, so one verifies its local coordinate to be t^{i-1} (as $i-1$ red strands are passing successively in front of the i^{th} one).

Again, in the picture, one has to imagine that the red handles are going back to the basepoint before and after this box following the same paths so that these parts do not contribute to the local system coefficient. Now we can conclude



where the term t^{i-1} comes from what we've just noted and $(-1)^{i-1}$ comes from (4). This concludes the proof of the equality we were looking for, remembering the notation $\mathtt{t} = -t$. □

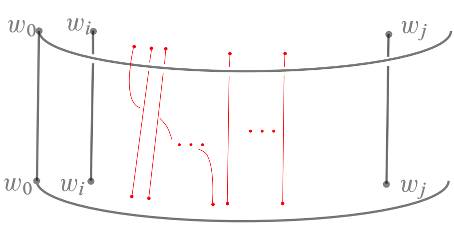


Figure 8: Handle rule.

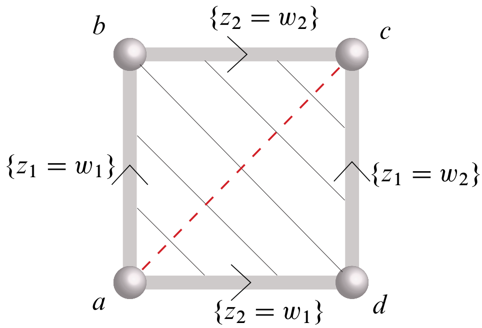
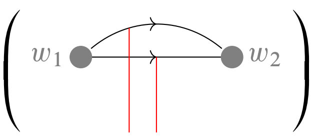
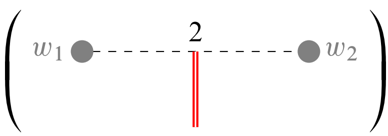


Figure 9: The square corresponding to the left term.

We study one simple example for illustrating the above lemma.

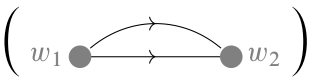
Example 4.9 Let's study the case $k = 1$ of the above lemma, which gives

(5) 

$= (1-t)$ 

.

Let's first consider the case $t = q^{\alpha_1} = q^{\alpha_2} = 1$ consisting in working with $\mathbb{Z}$–homology, namely without considering the cover. As $t = 1$, the above relation becomes



$= 0.$

The left term corresponds to an embedding of a square in $\mathrm{Conf}_2(D_2)$ (considering only two punctures w_1 and w_2 in the disk), itself defining a cycle in X_2 and a class in $\mathrm{H}_2^{\mathrm{lf}}(X_2; \mathbb{Z})$. This square is represented in Figure 9, where points a, b, c and d are respectively (w_1, w_1), (w_1, w_2), (w_2, w_2) and (w_2, w_1), and gray tubes are parts of hyperplanes (equations written in the figure). The arrows recall the orientation of embedded intervals.

The red dashed diagonal serves to help the reader figuring out how to decompose the square as the gluing of two simplices. These two simplices are identified after taking the quotient by the permutation group but with opposite orientations, which justifies that this chain is zero, so it is as a homology class. Now we remove the conditions $t = q^{\alpha_1} = q^{\alpha_2} = 1$, and we go back to (5). Now a, b, c, d and the hyperplanes from Figure 9 are lifted to $\widehat{\mathrm{Conf}_2(D_2)}$, the cover associated with ρ_2. In Figure 10, the square from Figure 9 is seen from another side, so as to make the hyperplane $\{z_1 = z_2\}$ appear. More precisely, Figure 9 can be thought as a top view of Figure 10.

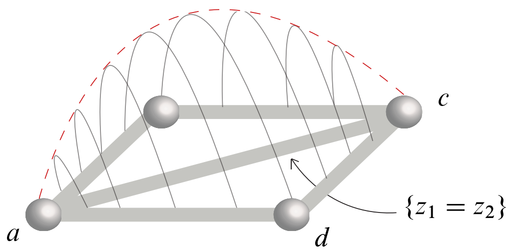


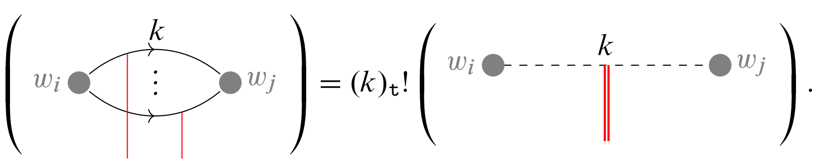
Figure 10: The square corresponding to the left term.

The isotopy crashing the plain arc on the dashed arc in the proof of the above Lemma 4.8 corresponds to an isotopy crashing the dashed surface from Figure 10 on the ground square (containing the hyperplane $\{z_1 = z_2\}$). Figure 11 shows a movie of such an isotopy viewed from the top.

The red circle shows where the path corresponding to red handles arrives. The last step of the movie shows the sum of two simplices glued from either side of the hyperplane $\{z_1 = z_2\}$. The red tube shows how the red handle path has to bypass this hyperplane (involving the t coefficient). All this isotopy happens in $\widehat{\mathrm{Conf}_2(D_2)}$. Working in X_2, upper and lower simplices are identified (with different lifts and orientation), so that in $\widehat{X}_2$ the end of the isotopy as given in Figure 12 is a class in $\mathcal{H}_2^{\mathrm{abs}}$. This is the right term of (5).

From Lemma 4.8 we deduce several corollaries. A first straightforward consequence of Lemma 4.8 is the following:

Corollary 4.10 *Let $k > 1$ be an integer; the following equality holds in $\mathcal{H}_\bullet^{\mathrm{rel}-}$:*



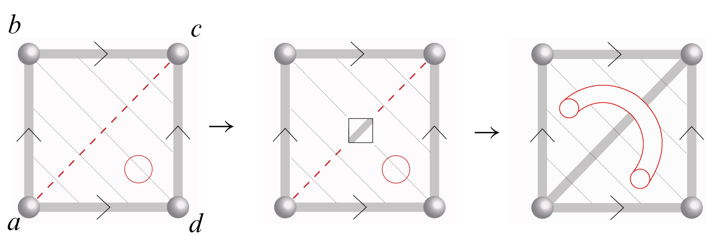


Figure 11: The movie of the isotopy.

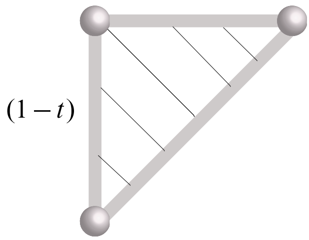
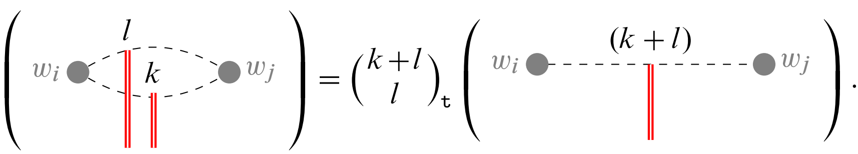


Figure 12: The end of the isotopy considered in $\mathcal{H}_2^{\mathrm{abs}}$.

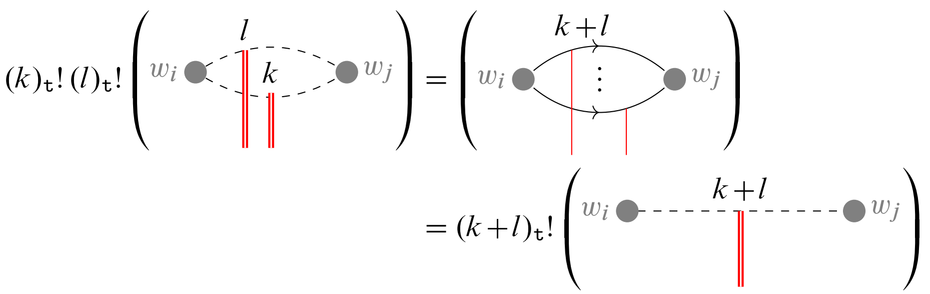
Proof The proof is by recursion on k. The recursion property is given by Lemma 4.8. □

Lemma 4.8 also allows one to compute the fusion between two dashed arcs.

Corollary 4.11 *For integers $k, l > 1$, there is the following relation between classes*:



Proof The two following equalities are direct consequences of Lemma 4.8:



One conclude using the integral equality

$$(k+l)_{\mathtt{t}}! = (k)_{\mathtt{t}}!\,(l)_{\mathtt{t}}! \binom{k+l}{l}_{\mathtt{t}}$$

and simplification by $(k)_{\mathtt{t}}!\,(l)_{\mathtt{t}}!$. □

4.3 Basis of multiarcs

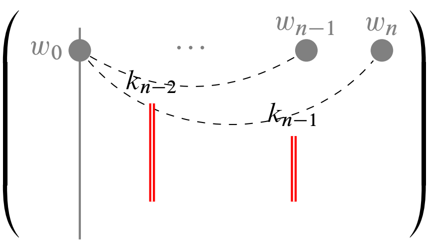
We recall that $\mathcal{A}'$ and $\mathcal{U}$ are respectively families of code sequences and of multiarcs, and that $\mathcal{U}$ was shown to be a basis of $\mathcal{H}_r^{\mathrm{rel}-}$ as an $\mathcal{R}_{\max}$–module; see Corollary 3.9. We prove a proposition relating multiarcs with code sequences.

Proposition 4.12 *Let $\boldsymbol{k} \in E^0_{n,r}$. There is the following expression for the standard multiarc in terms of code sequences:*

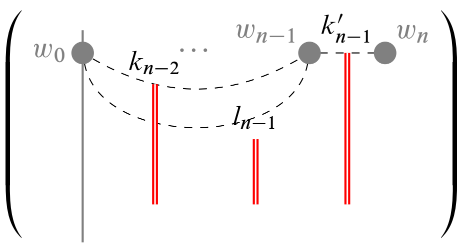
$$A'(k_0,\ldots,k_{n-1}) = \sum_{l_{n-1}=0}^{k_{n-1}} \sum_{l_{n-2}=0}^{k_{n-2}+l_{n-1}} \cdots \sum_{l_1=0}^{k_1+l_2} \left(\prod_{i=0}^{n-2} \binom{k_i+l_{i+1}}{l_{i+1}}_{\mathfrak{t}} U(k'_0, k''_1, \ldots, k''_{n-2}, k'_{n-1}) \right),$$

where $k'_0 = k_0 + l_1$, $k'_{n-1} = k_{n-1} - l_{n-1}$ and $k''_i = k_i + l_{i+1} - l_i$ for $i = 1,\ldots,n-2$.

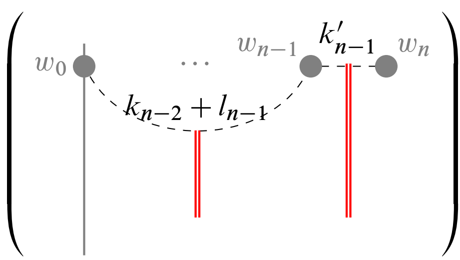
Proof Let $\boldsymbol{k} \in E^0_{n,r}$ and A' be its associated multiarcs. We treat one by one the dashed arcs of A', starting with the one ending at w_n, then the one ending at w_{n-1}, and so on. The first step is the following:



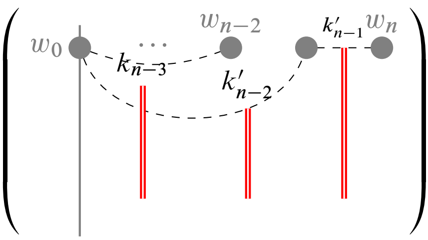
$$= \sum_{l_{n-1}=0}^{k_{n-1}}$$



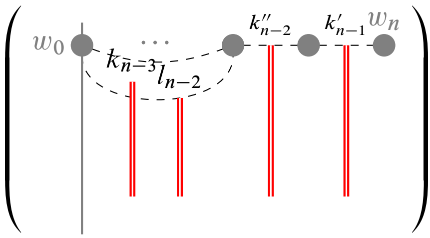
$$= \sum_{l_{n-1}=0}^{k_{n-1}} \binom{k_{n-2}+l_{n-1}}{l_{n-1}}_{\mathfrak{t}}$$



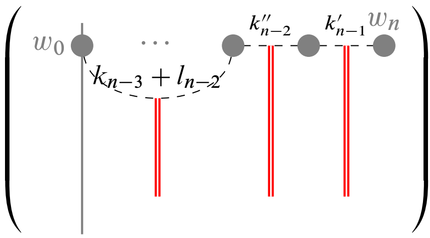
with $k'_{n-1} = k_{n-1} - l_{n-1}$. The first equality is a breaking of a dashed arc; see Example 4.6. The second equality is a direct application of Corollary 4.11. The end of the proof is an iteration of this process. The next step is the following, with $k'_{n-2} = k_{n-2} + l_{n-1}$:



$$= \sum_{l_{n-2}=0}^{k'_{n-2}}$$



$$= \sum_{l_{n-2}=0}^{k'_{n-2}} \binom{k_{n-3}+l_{n-2}}{l_{n-2}}_{\mathtt{t}}$$



,

where $k''_{n-2} = k'_{n-2} - l_{n-2}$. A complete iteration of this process gives the formula of the proposition. □

By looking at the diagonal terms of the matrix expressing multiarcs in the code sequence basis, one gets the following corollary:

Corollary 4.13 (basis of multiarcs) *The family $\mathcal{A}'$ of multiarcs is a basis of $\mathcal{H}_r^{\mathrm{rel}-}$ as an $\mathcal{R}_{\max}$–module.*

Proof Let $E^0_{n,r}$ be given the lexical order. This yields an order on the families $\mathcal{A}'$ and $\mathcal{U}$. One can see from Proposition 4.12 that, with this order, the matrix expressing multiarcs in the code sequence basis is upper-triangular. The determinant of this matrix is given by the product of diagonal terms. The diagonal terms are the binomial in the sum of the formula from Proposition 4.12 corresponding to $l_i = 0$ for all $i \in \{1, \dots, n-1\}$. In these cases, the binomials are equal to 1, so the determinant of the matrix is 1. As $\mathcal{U}$ is a basis and the change of basis determinant is invertible, the proof is complete. □

The family of multiarcs will play a central role in this work as it is a basis of the homology thanks to this last result.

5 Quantum algebra

This section is independent from the previous ones. We fix notation for quantum algebra objects that will be recovered by the above introduced homological modules.

The most standard definition of the quantum algebra $U_q\mathfrak{sl}(2)$ is as a vector space over a rational field.

Definition 5.1 The algebra $U_q\mathfrak{sl}(2)$ is the algebra over $\mathbb{Q}(q)$ generated by elements E, F and $K^{\pm 1}$, satisfying the relations

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, \quad KK^{-1} = K^{-1}K = 1.$$

The algebra $U_q\mathfrak{sl}(2)$ is endowed with a coalgebra structure defined by Δ and ϵ by

$$\begin{aligned} \Delta(E) &= 1\otimes E + E\otimes K, & \Delta(F) &= K^{-1}\otimes F + F\otimes 1,\\ \Delta(K) &= K\otimes K, & \Delta(K^{-1}) &= K^{-1}\otimes K^{-1},\\ \epsilon(E) &= \epsilon(F) = 0, & \epsilon(K) &= \epsilon(K^{-1}) = 1, \end{aligned}$$

and an antipode is defined by

$$S(E) = EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

This provides a *Hopf algebra* structure (neither commutative nor cocommutative), so the category of modules over $U_q\mathfrak{sl}(2)$ is monoidal; namely, there is a natural action over tensor products of modules given by the coproduct.

Remark 5.2 (specialization issue) The *specialization* process of the parameter q is algebraically the following. Let $\xi \in \mathbb{C}$ be a complex number. By specialization of q to the parameter ξ, one considers the morphism

$$\mathrm{eval}\colon \mathbb{Q}(q) \to \mathbb{C}, \quad q \mapsto \xi,$$

and the complex vector space

$$U_\xi = \mathbb{C} \otimes_{\mathrm{eval}} U_q\mathfrak{sl}(2).$$

By working with q as a variable, one can encounter problems — if ξ is not transcendental, for instance. To define quantum topological invariants from $U_q\mathfrak{sl}(2)$–modules, we are sometimes interested in q being a root of unity, for which the ground ring $\mathbb{Q}(q)$ is not appropriate.

The above remark justifies the definition of integral versions for $U_q\mathfrak{sl}(2)$, the aim of next subsection.

Definition 5.3 (integral version [5, Section 9.2]) Let $\mathcal{R}_0 = \mathbb{Z}[q^{\pm 1}]$ be the ring of Laurent polynomials in the single variable q. An *integral version* for $U_q\mathfrak{sl}(2)$ is an $\mathcal{R}_0$–subalgebra $U_{\mathcal{R}_0}$ of $U_q\mathfrak{sl}(2)$ such that the natural map

$$U_{\mathcal{R}_0} \otimes_{\mathcal{R}_0} \mathbb{Q}(q) \to U_q\mathfrak{sl}(2)$$

is an isomorphism of $\mathbb{Q}(q)$–algebras.

Then, for $\xi \in \mathbb{C}^*$, the specialization of $U_{\mathcal{R}_0}$ to ξ means the vector space

$$U_\xi = \mathbb{C} \otimes_{\mathrm{eval}} U_{\mathcal{R}_0} \quad \text{with } \mathrm{eval}\colon \mathcal{R}_0 \to \mathbb{C}.$$

We introduce another version of quantum numbers. We will relate them to those from Definition 4.7 in Remark 6.1.

Definition 5.4 Let i be a positive integer. We define elements of $\mathbb{Z}[q^{\pm 1}]$,

$$[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}, \quad [k]_q! := \prod_{i=1}^{k} [i]_q, \quad \begin{bmatrix} k \\ l \end{bmatrix}_q := \frac{[k]_q!}{[k-l]_q!\,[l]_q!}.$$

5.1 An integral version

In this section, we define an integral version for $U_q\mathfrak{sl}(2)$ that will be central for the present work. This integral version is similar to the one introduced by Lusztig [19]. The difference is that we consider only the divided powers of F as generators, not those of E. This version is introduced in [10; 12] (with subtle differences in the definitions of divided powers for F). We follow the one of [12], so we first define the divided powers, presenting a minor difference from the original ones of Lusztig. Let

$$F^{(n)} = \frac{(q - q^{-1})^n}{[n]_q!} F^n.$$

Let $\mathcal{R}_0 = \mathbb{Z}[q^{\pm 1}]$ be the ring of integral Laurent polynomials in the variable q.

Definition 5.5 (half-integral algebra [10; 12]) Let $U_q^{L/2}\mathfrak{sl}(2)$ be the $\mathcal{R}_0$–subalgebra of $U_q\mathfrak{sl}(2)$ generated by E, $K^{\pm 1}$ and $F^{(n)}$ for $n \in \mathbb{N}^*$. We call it a *half-integral version* for $U_q\mathfrak{sl}(2)$, the word "half" to illustrate that we consider only half the divided powers as generators.

Remark 5.6 (relations in $U_q^{L/2}\mathfrak{sl}(2)$ [12, (16)–(17)]) The relations among generators involving divided powers are

$$KF^{(n)}K^{-1} = q^{-2n}F^{(n)},$$
$$[E, F^{(n+1)}] = F^{(n)}(q^{-n}K - q^nK^{-1}),$$
$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}.$$

Together with the relations from Definition 5.1, they complete the presentation of $U_q^{L/2}\mathfrak{sl}(2)$.

$U_q^{L/2}\mathfrak{sl}(2)$ inherits a Hopf algebra structure, making its category of modules monoidal. The coproduct is given by

$$\Delta(K) = K \otimes K,$$
$$\Delta(E) = E \otimes K + 1 \otimes E,$$
$$\Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-j(n-j)}K^{j-n}F^{(j)} \otimes F^{(n-j)}.$$

Proposition 5.7 *The algebra $U_q^{L/2}\mathfrak{sl}(2)$ admits as an $\mathcal{R}_0$–basis the set*

$$\{K^lE^mF^{(n)} \mid l \in \mathbb{Z}, m, n \in \mathbb{N}\}.$$

5.2 Verma modules and braiding

Now we define a special family of universal objects in the category of $U_q\mathfrak{sl}(2)$–modules, we express their presentation in the special case of $U_q^{L/2}\mathfrak{sl}(2)$ and we give a braiding for this family of modules. Namely, the *Verma modules* are infinite-dimensional modules which have a universal (among quantum groups) definition, and which depend on a parameter. Again, we work with this parameter as a variable with an integral ring, letting $\mathcal{R}_1 := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$. In [12], they give an explicit presentation for the integral Verma module of $U_q^{L/2}\mathfrak{sl}(2)$, which we recall here.

Definition 5.8 (Verma modules for $U_q^{L/2}\mathfrak{sl}(2)$ [12, (18)]) Let V^s be the Verma module of $U_q^{L/2}\mathfrak{sl}(2)$. It is the infinite $\mathcal{R}_1$–module, generated by vectors $\{v_0, v_1, \dots\}$, and endowed with an action of $U_q^{L/2}\mathfrak{sl}(2)$, generators acting as

$$K \cdot v_j = sq^{-2j}v_j,$$
$$E \cdot v_j = v_{j-1},$$
$$F^{(n)}v_j = \left(\begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{j+k}) \right) v_{j+n}.$$

Remark 5.9 (weight vectors) We will often make implicitly the change of variable $s := q^\alpha$ and denote V^s by V^α. This choice is made to use a practical and usual denomination for eigenvalues of the K–action (which is diagonal in the given basis). Namely, we say that the vector v_j is *of weight* $\alpha - 2j$, as $K \cdot v_j = q^{\alpha-2j} v_j$. The notation with s shows an integral Laurent polynomials structure, strictly speaking.

Definition 5.10 (R–matrix [12, (21)]) Let $s = q^\alpha$ and $t = q^{\alpha'}$. The operator $q^{H\otimes H/2}$ is

$$q^{H\otimes H/2} \colon V^s \otimes V^t \to V^s \otimes V^t, \quad v_i \otimes v_j \mapsto q^{(\alpha-2i)(\alpha'-2j)} v_i \otimes v_j.$$

We define the R–matrix (see [13, Part 2] for general definitions)

$$R := q^{H\otimes H/2} \sum_{n=0}^{\infty} q^{n(n-1)/2} E^n \otimes F^{(n)}.$$

It is not yet a well-defined object as the sum is infinite, but it will be well defined as an operator on Verma modules; see the following proposition (the sum always cuts off when applied to tensor product of Verma vectors).

Proposition 5.11 [12, Theorem 7] *Let V^s and V^t be Verma modules of $U_q^{L/2}\mathfrak{sl}(2)$ (with $s = q^\alpha$ and $t = q^{\alpha'}$). Let* R *be the operator*

$$\mathrm{R} := q^{-\alpha\alpha'/2} T \circ R,$$

where T is the twist defined by $T(v \otimes w) = w \otimes v$. Then R *provides a braiding for $U_q^{L/2}\mathfrak{sl}(2)$ integral Verma modules. Namely, the morphism*

$$Q \colon \mathcal{R}_1[\mathcal{B}_n] \to \mathrm{End}_{\mathcal{R}_1, U_q^{L/2}\mathfrak{sl}(2)}(V^{s\otimes n}), \quad \sigma_i \mapsto 1^{\otimes i-1} \otimes \mathrm{R} \otimes 1^{\otimes n-i-2},$$

is an $\mathcal{R}_1$–algebra morphism. It provides a representation of $\mathcal{B}_n$ whose action commutes with that of $U_q^{L/2}\mathfrak{sl}(2)$.

Remark 5.12 One can consider a braid action over $V^{s_1} \otimes \cdots \otimes V^{s_n}$, so that the morphism Q is well defined but becomes multiplicative (ie an algebra morphism) only when restricted to the pure braid group $\mathcal{PB}_n$, so as to be an endomorphism.

5.3 Finite-dimensional braid representations

Although braid group representations over products of Verma modules are infinite-dimensional, we find finite-dimensional subrepresentations using the commutativity of the braid action with the quantum structure.

Remark 5.13 For $r \in \mathbb{N}$:

- The subspace $W_{n,r} = \operatorname{Ker}(K - s^n q^{-2r})$ of $(V^s)^{\otimes n}$ provides a subrepresentation of $\mathcal{B}_n$.
- The subspace $Y_{n,r} = W_{n,r} \cap \operatorname{Ker} E \subset W_{n,r}$ provides a subrepresentation of $\mathcal{B}_n$.

We usually call $W_{n,r}$ the space of *subweight* r vectors, while $Y_{n,r}$ is called the space of *highest-weight* vectors.

Using the definition of the coproduct, the following remark is easily checked:

Remark 5.14 (weight structure) The weight structure is determined by actions of generators: the action of $F^{(1)}$ sends an element in $W_{n,r}$ to an element in $W_{n,r+1}$ while the action of E sends an element in $W_{n,r}$ to an element in $W_{n,r-1}$. Moreover, the tensor product of Verma modules is graded by weights:

$$(V^s)^{\otimes n} = \bigoplus_{r \in \mathbb{N}} W_{n,r}.$$

Theorem 5.15 (irreducibility of highest-weight modules [12, Theorem 21]) *The $\mathcal{B}_n$–representations $Y_{n,r}$ are irreducible over the fraction field $\mathbb{Q}(q,s)$.*

6 Homological model for $U_q^{L/2}\mathfrak{sl}(2)$ Verma modules

In this section we recover quantum algebra representations in homological modules.

6.1 Homological action of $U_q^{L/2}\mathfrak{sl}(2)$

We recall the scheme for the Verma module grading that is explained in Remark 5.14:

$$W_{n,r} \underset{F^{(1)}}{\overset{E}{\leftrightarrows}} W_{n,r+1}.$$

The goal of this section is to construct homological operators E, $K^{\pm 1}$ and $F^{(k)}$ such that they mimic the weight structure existing on quantum Verma modules. Namely, we want homological operators to fit with the scheme

$$\mathcal{H}_r^{\mathrm{rel}-} \underset{F^{(1)}}{\overset{E}{\leftrightarrows}} \mathcal{H}_{r+1}^{\mathrm{rel}-}.$$

The definitions for homological operators were inspired by [9]. In their article, the authors define such operators acting upon a topological module built from configuration space X_r. The fact that their module has a homological definition remains conjectural, namely [9, Conjectures 6.1 and 6.2].

The following remark, relating the two types of quantum numbers we have introduced in Definitions 4.7 and 5.4, will be useful for computations:

Remark 6.1 Let $\mathtt{t} = q^{-2}$; the following relations hold in $\mathbb{Z}[q^{\pm 1}]$:

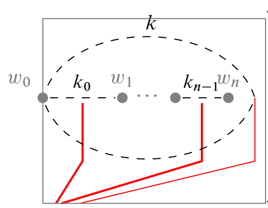
$$(i)_{\mathtt{t}} = q^{1-i}[i]_q, \quad (k)_{\mathtt{t}}! = q^{-k(k-1)/2}[k]_q!, \quad \binom{k+l}{l}_{\mathtt{t}} = q^{-kl}\begin{bmatrix} k+l \\ l \end{bmatrix}_q.$$

6.1.1 Action of $F^{(1)}$ and its divided powers We want the operator $F^{(1)}$ to go from $\mathcal{H}_r^{\mathrm{rel}-}$ to $\mathcal{H}_{r+1}^{\mathrm{rel}-}$; it has to increase by 1 the degree of a chain while passing from X_r to X_{r+1} for the topological space. By extension, we will build operators $F^{(k)}$ for $k > 1$ going from $\mathcal{H}_r^{\mathrm{rel}-}$ to $\mathcal{H}_{r+k}^{\mathrm{rel}-}$. We define them using the family $\mathcal{U}$ shown to be an $\mathcal{R}_{\max}$–basis of the homology — it is not difficult to define the operator without a basis, but it complicates notation.

Definition 6.2 (divided powers of F) We define the family of homological operators

$$F^{(k)} \colon \mathcal{H}_r^{\mathrm{rel}-} \to \mathcal{H}_{r+k}^{\mathrm{rel}-},$$

$$U(k_0, \dots, k_{n-1}) \mapsto q^{k(1-k)/2} q^{k\sum_{i=1}^n \alpha_i} \left(\quad \right).$$

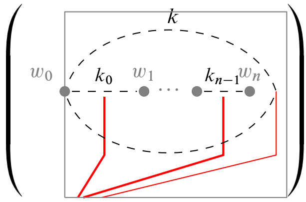


Remark 6.3

- In terms of homology classes with coefficients in $\mathbb{Z}$, involved by the union of dashed arcs corresponding to a product of simplexes, the operator $F^{(k)}$ simply adds an index k dashed arc that goes once around the boundary in counterclockwise direction.
- For the local coefficient definition, we chose to simplify the drawing by adding a straight handle, but it costs a coefficient $q^{k\sum_{i=1}^n \alpha_i}$ that one can remove using another more complicated family of handles. We will work with the simpler

drawing and will add the coefficient ad hoc in the following computations, so we define an intermediate operator

$$(F')^{(k)}: \mathcal{H}_r^{\mathrm{rel}-} \to \mathcal{H}_{r+k}^{\mathrm{rel}-}, \quad U(k_0, \dots, k_{n-1}) \mapsto$$



such that $F'^{(k)} = q^{k(1-k)/2} q^{k \sum_{i=1}^{n} \alpha_i} F^{(k)}$.

The following proposition justifies the *divided powers* denomination:

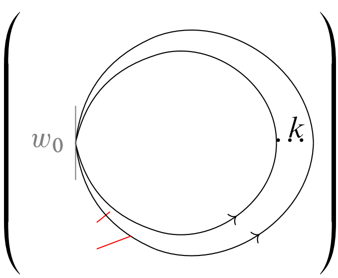
Proposition 6.4 (divided powers of F) *There is the relation between elements of* $\mathrm{Hom}_{\mathcal{R}_{\max}}(\mathcal{H}_r^{\mathrm{rel}-}, \mathcal{H}_{r+k}^{\mathrm{rel}-})$,

$$(F^{(1)})^k = q^{k(k-1)/2} (k)_{\mathtt{t}}!\, F^{(k)}.$$

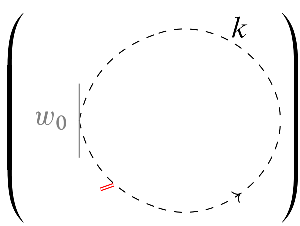
Let $\mathtt{t} = q^{-2}$*; then*

$$(F^{(1)})^k = [k]_q!\, F^{(k)}.$$

Proof This is a direct consequence of the equality of classes



$$= (k)_{\mathtt{t}}!$$



which can be proved as Corollary 4.10, and whatever stands inside the circles. On the right, there are k parallel arcs rounding along the boundary counterclockwise, while on the left there is one dashed arc rounding along the boundary. This shows that $F'^k = (k)_{\mathtt{t}}!\, F'^{(k)}$, and the first statement follows. To get the second equality, for $\mathtt{t} = q^{-2}$ one uses directly Remark 6.1. □

6.1.2 Actions of E and K

To define the action of $E \in \mathrm{Hom}_{\mathcal{R}_{\max}}(\mathcal{H}_r^{\mathrm{rel}-}, \mathcal{H}_{r-1}^{\mathrm{rel}-})$, we need a way to remove one configuration point. This is the purpose of the morphisms defined next:

Definition 6.5 • Let ψ^r be the homeomorphism

$$\psi^r : X_r \setminus X_r^- \to X_{r+1}^-, \qquad Z \mapsto Z \cup w_0, \quad \boldsymbol{\xi}^r \mapsto \{\xi_1, \dots, \xi_r, w_0\}.$$

- ψ^r induces
$$\psi^r_* : \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) \to \pi_1(X_{r+1}^-, \{\boldsymbol{\xi}^r, w_0\}).$$
We provide a natural way to transport the basepoint on the right to $\boldsymbol{\xi}^{r+1}$, namely we move w_0 along ∂D_n through a path φ^r defined as
$$\varphi^r : I \to X_{r+1}, \quad t \mapsto \varphi^r(t) = \{\varphi_1(t), \xi_r, \dots, \xi_1\},$$
where φ_1 goes from ξ_{r+1} to w_0 along ∂D_n in the clockwise direction, while other coordinates remain fixed in $\boldsymbol{\xi}^r$.
- We then let Φ^r be the composition of the above ψ^r_* with the morphism induced by the change of basepoint through conjugation by φ^r,
$$\Phi^r : \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) \to \pi_1(X_{r+1}, \boldsymbol{\xi}^{r+1}).$$

In what follows we will often omit the indices r in φ^r, $\boldsymbol{\xi}^r$ and Φ^r, to simplify notation when no confusion is possible.

Lemma 6.6 *The morphism Φ^r lifted to the local system level,*

$$\Phi^r : L_r \restriction_{X_r \setminus X_r^-} \to L_{r+1} \restriction_{X_{r+1}^-},$$

is an isomorphism of local systems.

Proof The underlying spaces are homeomorphic through ψ (addition of w_0). Let ρ_r be the representation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ providing the local system L_r. The diagram

$$\begin{array}{ccc} \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) & \xrightarrow{\Phi^r} & \pi_1(X_{r+1}, \boldsymbol{\xi}^{r+1}) \\ \downarrow \rho_r & & \downarrow \rho_{r+1} \\ \mathbb{Z}^{n+1} = \bigoplus_{i \in \{1,\dots,n\}} \mathbb{Z}\langle q^{\alpha_i} \rangle \oplus \mathbb{Z}\langle t \rangle & \xrightarrow[\mathrm{Id}]{} & \bigoplus_{i \in \{1,\dots,n\}} \mathbb{Z}\langle q^{\alpha_i} \rangle \oplus \mathbb{Z}\langle t \rangle \end{array}$$

is commutative, which proves the lemma. The commutation is easy to verify, thinking of the representation of $\pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r)$ given in Remark 2.2. The morphism Φ^r simply adds a straight strand to the braid, without modifying its image by ρ_r. □

Remark 6.7 We formulate the above Lemma 6.6 for homologies. In other words, the choice of path φ in Definition 6.5 yields the isomorphism

$$\Phi^r : \mathrm{H}_r^{\mathrm{lf}}(X_r \setminus X_r^-; L_r) \to \mathrm{H}_r^{\mathrm{lf}}(X_{r+1}^-; L_{r+1}).$$

Let an element in $\mathrm{H}_r^{\mathrm{lf}}(X_r \setminus X_r^-; L_r)$ be given by a pair $([\Delta], h)$, where $[\Delta]$ is the class of a chain in $\mathrm{H}_r^{\mathrm{lf}}(X_r \setminus X_r^-; \mathbb{Z})$ and h a path relating $\boldsymbol{\xi}^r$ to Δ (the case of interest). Then its image by Φ^r is determined by the pair $\big([\{\Delta, w_0\}], \{h, w_0\} \circ \varphi^r\big)$. One must pay attention to the fact that the isomorphism between homologies depends on the choice of φ, as does the operator E defined in Definition 6.9 below.

Remark 6.8 Recall that

$$H_{r-1}(X_{r-1} \setminus X_{r-1}^-; L_{r-1}) = H_{r-1}(X_{r-1}(w_0); L_{r-1}) = \mathcal{H}_{r-1}^{\mathrm{rel}-},$$

where $X_r(w_0)$ is the space of configurations of X_r without coordinate in w_0. The first equality is the fact that $X_{r-1} \setminus X_{r-1}^-$ and $X_{r-1}(w_0)$ are canonically homeomorphic. The second one is Corollary 3.9.

From this identification one is able to define an operator E as in the following definition:

Definition 6.9 (action of E) Let E be the operator defined by

$$E : \mathcal{H}_r^{\mathrm{rel}-} \xrightarrow{\partial_*} H_{r-1}(X_r^-; L_r) \xrightarrow{(\Phi^r)^{-1}} H_{r-1}(X_{r-1} \setminus X_{r-1}^-; L_{r-1}) = \mathcal{H}_{r-1}^{\mathrm{rel}-}.$$

The arrow ∂_* is the boundary map of the exact sequence of the pair (X_r, X_r^-). The arrow $(\Phi^r)^{-1}$ is the inverse isomorphism provided by Lemma 6.6 (see Remark 6.7) and the last equality is the above Remark 6.8.

Remark 6.10 The definition of E is the boundary map of the relative exact sequence of the pair involved; the rest are just isomorphic identifications of homology modules. Namely, the operator E reads the part of the boundary that lies in X_r^-.

We give a first example of a computation with a standard code sequence.

Example 6.11 (action of E on a code sequence) Let $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E_{n,r}^0$ and $U_{\boldsymbol{k}}$ be its associated standard code sequence. One can check that

$$E \cdot U_{\boldsymbol{k}} = U(k_0 - 1, \dots, k_{n-1}).$$

Consider first $U(k_0, 0, \dots, 0)$ and let ϕ^{k_0} be the chain associated with the index k_0 dashed arc. We recall our parametrization of the standard simplex,

$$\Delta^{k_0} = \{0 \le t_1 \le \cdots \le t_{k_0} \le 1\},$$

so that its only boundary part sent to configurations with one coordinate in w_0 is $\{t_1 = 0\} \in \Delta^k$. We note that ϕ^{k_0} restricted to $\{t_1 = 0\}$ is ϕ^{k_0-1} (the chain associated with same dashed arc but indexed by $k_0 - 1$), by shifting left the parametrization: $(0, t_2, \dots, t_{k_0}) \mapsto (t_2, \dots, t_{k_0})$ (which does not involve any permutation which could change orientation). Consequently, one sees that the equality holds at the level of homology over $\mathbb{Z}$.

To deal with the handle rule lifting process, we note that only the leftmost configuration point embedded in $U(k_0, \dots, k_{n-1})$ can join w_0. This is saying that the only part of the boundary of $U(k_0, \dots, k_{n-1})$ lying in X_r^- corresponds to the leftmost point being in w_0. No local coefficient appears while applying $(\Phi^r)^{-1}$ (Lemma 6.6) thanks to the fact that the handle joining the leftmost configuration point is the leftmost handle, and it joins ξ_r, namely the leftmost basepoint's coordinate. Another way to see this is by noting that the path following the leftmost handle, then going to w_0 along $U_{\boldsymbol{k}}$, then back to ξ_r along the boundary, can be homotoped to w_0 without perturbing other handles. In other words, composition (of the path corresponding to red handles) with the inverse of the path φ^r (Definition 6.5) does not involve any change of local coordinate.

The action of the operator K is a diagonal action encoding the value of r.

Definition 6.12 (action of K) For $r \in \mathbb{N}$, the operator K is the diagonal action, over $\mathcal{H}_r^{\text{rel}-}$,

$$K = q^{\sum_i \alpha_i} \, \mathtt{t}^r \, \mathrm{Id}_{\mathcal{H}_r^{\text{rel}-}}.$$

We define the operator K^{-1} to be the inverse of K.

6.1.3 Homological $U_q\mathfrak{sl}(2)$ representation Let $\mathcal{H} = \bigoplus_{r \in \mathbb{N}} \mathcal{H}_r^{\text{rel}-}$; the actions of E, $F^{(1)}$ and K are endomorphisms of $\mathcal{H}$. We have the following proposition:

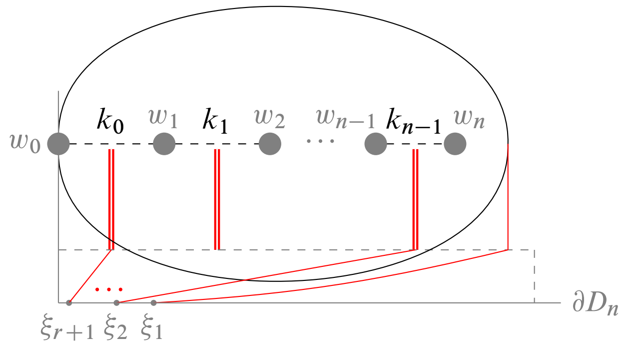
Proposition 6.13 *The operators E, $F^{(1)}$ and K satisfy the relations*

$$KE = \mathtt{t}^{-1} EK, \quad KF^{(1)} = \mathtt{t}\, F^{(1)} K \quad \textit{and} \quad [E, F^{(1)}] = K - K^{-1}.$$

Proof The first two relations are direct consequences of the facts that $F^{(1)}$ increases r by 1 and E decreases it by 1 and the (diagonal) definition of K. It remains to prove the last one. The proof can be performed without considering a basis of $\mathcal{H}$, although we do it here using the basis of code sequences for easier reading. Let $r \in \mathbb{N}$; we recall that $\mathcal{U} = (U_{\boldsymbol{k}})_{\boldsymbol{k} \in E_{n,r}^0}$ is a basis of $\mathcal{H}_r^{\text{rel}-}$ as an $\mathcal{R}_{\max}$–module. Let $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E_{n,r}^0$.

First we compute the commutation between E and F' before renormalizing F' to $F^{(1)}$. The class $F' \cdot (U_{\boldsymbol{k}})$ corresponds to

$F'^{(1)} \cdot U_{\boldsymbol{k}} =$

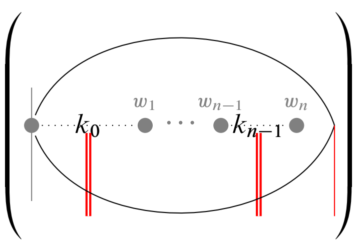


Applying E to this class gives the part of its boundary lying in X_r^-. There are $r+1$ points embedded in this class, r of them in the dashed arcs and the last one in the plain arc. The part of the boundary lying in X_r^- is the sum of

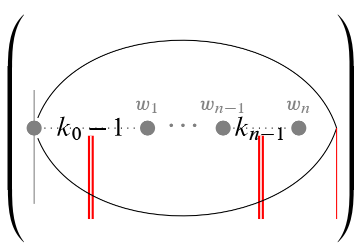
- the leftmost point of dashed arcs going to w_0 (ie given by one boundary component from the simplex defined by the leftmost dashed arc, the image of $\{t_1 = 0\}$, where t_1 is the first parameter of the latter simplex), and
- the two boundary parts corresponding to "back and front faces" parametrized by the plain arc (the image of $\{t = 0\}$ and $\{t = 1\}$, where t is the coordinate sent to the plain arc).

This corresponds to the equality

$E \cdot$

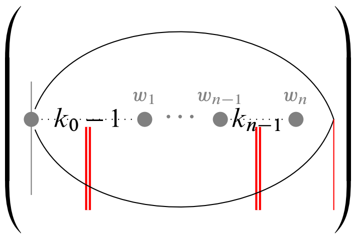


$=$



$+ C \times U(k_0, \ldots, k_{n-1}),$

where the coefficient C is the computation of the relative boundary part coming from the plain arc. One sees that

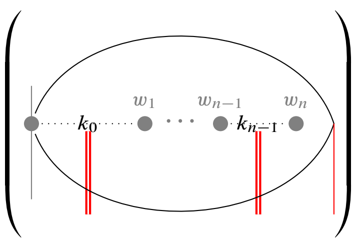


$$= F' \cdot (E \cdot U(k_0, \dots, k_{n-1}))$$

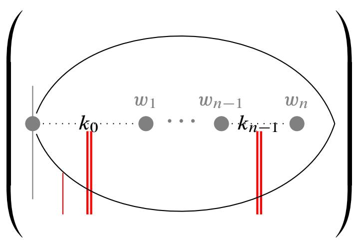
using Example 6.11. We also mention that this term is zero if $k_0 = 0$. This gives

$$[E, F'] \cdot U(k_0, \dots, k_{n-1}) = C \times U(k_0, \dots, k_{n-1}),$$

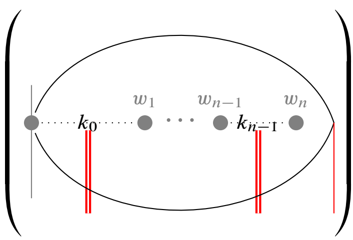
so it remains to compute the coefficient C. The coefficient C is the difference $C_1 - C_2$, where C_1 and C_2 satisfy the equations



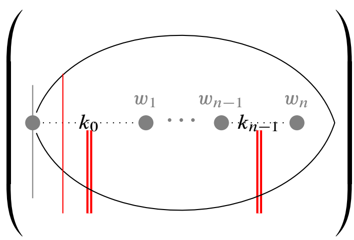
$$= C_1$$



,



$$= C_2$$



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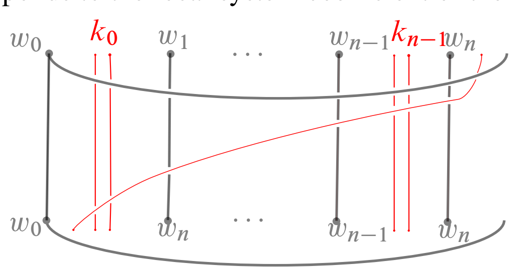
This is by the same handle argument as in Example 6.11 (application of the path φ^{-1} from Definition 6.5). We compute these coefficients using the handle rule. The coefficient C_1 corresponds to the local system coefficient of the braid



while C_2 corresponds to the same braid but with the red front strand passing in the back. We emphasize that in the braid picture we got rid of parts of red handles lying outside parentheses. Outside the parentheses, the paths consist in going to the basepoint without crossing each other, staying in front of the w_i, so that, above and below the box, the contributions to the handle local system coefficient balance each other. Then it is straightforward to compute the local system coefficient of these braids; we get

$$C_1 = \mathtt{t}^{\sum_{i=0}^{n-1} k_i} = \mathtt{t}^r, \quad C_2 = \mathtt{t}^{-r} q^{-2\sum_{i=1}^{n} \alpha_i},$$

so that

$$[E, F'] \cdot U(k_0, \dots, k_{n-1}) = (\mathtt{t}^r - \mathtt{t}^{-r} q^{-2\sum_{i=1}^{n} \alpha_i}) \times U(k_0, \dots, k_{n-1}).$$

We recall that

$$[E, F^{(1)}] = q^{\sum \alpha_i} [E, F'],$$

which concludes the proof. □

Theorem 1 *Let $q^{-2} = \mathtt{t}$. The infinite module $\mathcal{H}$ together with the above-described action of E, $F^{(1)}$, $K^{\pm 1}$ and $F^{(k)}$ for $k \geq 2$ yields a representation of the integral algebra $U_q^{L/2}\mathfrak{sl}(2)$.*

Proof The algebra $U_q^{L/2}\mathfrak{sl}(2)$ is presented in Definition 5.5. We use the same notation (from Section 5.1) for generators and we recover the same relations. Namely, the relations between E, $F^{(1)}$ and $K^{\pm 1}$ are recovered using Proposition 6.13, while the fact that $F^{(k)}$ are the so-called divided powers of $F^{(1)}$ — see Proposition 6.4 — ensures that the relations involving them hold. □

Remark 6.14 Even if it is not necessary to prove them knowing Proposition 6.4 (divided power property), we can check homologically the relations involving the divided powers of $F^{(1)}$ (relations introduced in Remark 5.6). Namely,

$$[E, F^{(n+1)}] = F^{(n)} (q^{-n} K - q^n K^{-1})$$

is a simple computation of the relative boundary of a class as in the proof of Proposition 6.13, while

$$F^{(n)} F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}$$

is a direct consequence of the homological Corollary 4.11.

We have a complete homological description of the relations holding in $U_q^{L/2}\mathfrak{sl}(2)$.

Remark 6.15 Using Proposition 6.13, one sees that we have a representation of the *simply connected* rational version of $U_q\mathfrak{sl}(2)$, for which are introduced generators that correspond to square roots of K and K^{-1}. See [6, Section 9; 1, Remark 2.2; 5, Section 9.1] for information about this version of $U_q\mathfrak{sl}(2)$.

6.2 Computation of the $U_q^{L/2}\mathfrak{sl}(2)$–action

In this section we compute the action of the operators E, $F^{(1)}$ and K in the basis of multiarcs, in order to recognize the representation of $U_q\mathfrak{sl}(2)$ obtained over $\mathcal{H}$. First we define a normalized version of the multiarc basis.

Definition 6.16 (normalized multiarcs) Let $\boldsymbol{k} \in E^0_{n,r}$ and let $A(k_0,\dots,k_{n-1})$ be the element of $\mathcal{H}_r^{\mathrm{rel}-}$

$$A(\boldsymbol{k}) = q^{\alpha_1(k_1+\cdots+k_{n-1})+\alpha_2(k_2+\cdots+k_{n-1})+\cdots+\alpha_{n-1}k_{n-1}} A'(\boldsymbol{k}).$$

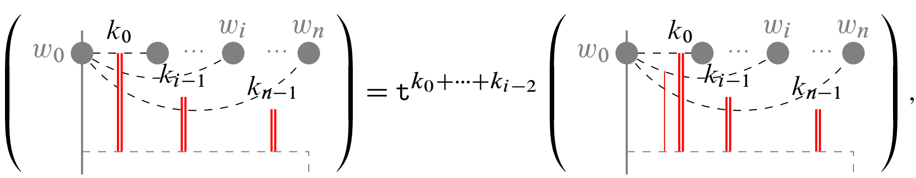
Let $\mathcal{A} = (A(\boldsymbol{k}))_{\boldsymbol{k}\in E^0_{n,r}}$ be the corresponding family indexed by $E^0_{n,r}$. By convention, $A(k_0,\dots,k_{n-1})$ is defined to be $0 \in \mathcal{H}_r^{\mathrm{rel}-}$ whenever $k_i = -1$ for some $i \in \{0,\dots,n-1\}$.

Remark 6.17 The family $\mathcal{A}$ is obtained from $\mathcal{A}'$ by a diagonal matrix of invertible coefficients in $\mathcal{R}_{\max}$, so $\mathcal{A}$ is still a basis of $\mathcal{H}_r^{\mathrm{rel}-}$ as an $\mathcal{R}_{\max}$–module. As for the definition of divided powers of F, we could have chosen to avoid the normalization coefficient but would have to draw more complicated handles. In the following computations we will work with $\mathcal{A}'$ drawings and add the coefficient ad hoc to work with the family $\mathcal{A}$.

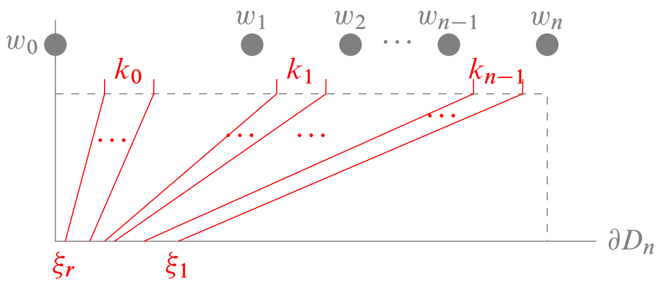
We are going to compute the action of operators in this basis, and we will see that it recovers the basis of $U_q^{L/2}\mathfrak{sl}(2)$ Verma modules.

6.2.1 Action of E First we need a lemma to reorganize handles.

Lemma 6.18 *For $(k_0,\dots,k_{n-1}) \in E^0_{n,r}$, let $A'(k_0,\dots,k_{n-1})$ be the standard multiarc. For $i = 1,\dots,n$, there is the relation, in $\mathcal{H}_r^{\mathrm{rel}-}$,*



where, in the right term, only one component of the red tube indexed by k_i has been moved to the extreme left of the other red handles. Namely, only the leftmost handle *composing the (k_i)–handle (tube of k_i parallel handles) has been pushed to the left of the (k_0)–handle. Down the parentheses, red handles join the basepoint following a usual dashed box, without crossing with each other. The left class follows this box as in*



while the right one has the leftmost single handle following the leftmost path of the above dashed box. All other handles are right-shifted.

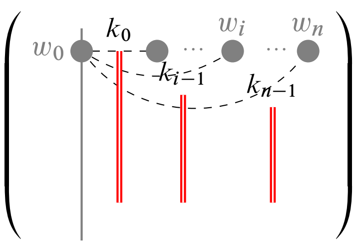
Proof This is a straightforward consequence of the handle rule. The braid involved is drawn in Figure 13, so that one sees the local system coefficient (we did not draw the punctures as they don't play any role). □

Lemma 6.19 *For any $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E^0_{n,r}$, the action of E over the standard multiarcs is*

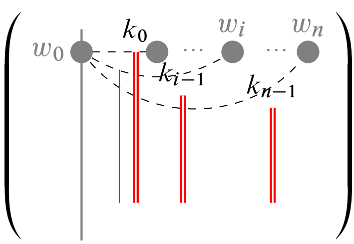
$$E \cdot A'(k_0, \dots, k_{n-1}) = \sum_{i=0}^{n-1} \mathtt{t}^{k_0+\dots+k_{i-1}} A'(k_0, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_{n-1}).$$

Proof The leftmost component of every dashed component of $A'(k_0, \dots, k_{n-1})$ has one end in w_0. For $i = 1, \dots, n-1$, we have, from the above lemma,

$A'(k_0, \dots, k_{n-1}) =$



$= \mathtt{t}^{k_0+\dots+k_{i-2}}$



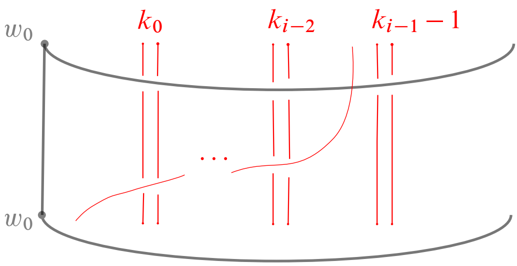
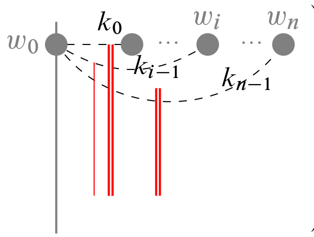
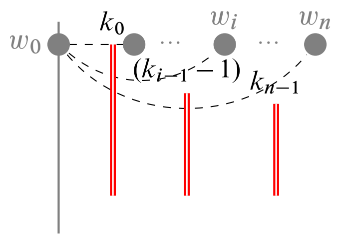


Figure 13: Handle rule.

Using exactly the same arguments as in Example 6.11, we have

$\partial_*\Bigg($ 

$\Bigg) = \Bigg($ 

$\Bigg) + \cdots,$

where the rest of the terms concern boundary terms coming from other arcs (different from the k_{i-1}–indexed one). Every dashed arc indexed by k_i for $i = 0, \dots, n-1$ can be treated the same way. The boundary of $A(k_0, \dots, k_{n-1})$ relative to w_0 is then the sum of these terms, and one gets the statement of the lemma. □

One has the following action over the normalized multiarcs:

Proposition 6.20 (action of E over multiarcs) *For any* $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E^0_{n,r}$*, the action of* E *over the* (*normalized*) *multiarc is*

$$E \cdot A(k_0, \dots, k_{n-1}) = \sum_{i=0}^{n-1} q^{\alpha_1 + \cdots + \alpha_i}\, \mathtt{t}^{k_0 + \cdots + k_{i-1}}\, A(k_0, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_{n-1}).$$

Proof This is a simple computation:

$$\begin{aligned} E \cdot A(k_0, \dots, k_{n-1}) &= q^{\alpha_1(k_1 + \cdots + k_{n-1}) + \cdots + \alpha_{n-1} k_{n-1}} \sum_{i=0}^{n-1} \mathtt{t}^{k_0 + \cdots + k_{i-1}}\, A'(k_0, \dots, k_i - 1, \dots, k_{n-1}) \\ &= \sum_{i=0}^{n-1} q^{\alpha_1 + \cdots + \alpha_i}\, \mathtt{t}^{k_0 + \cdots + k_{i-1}}\, A(k_0, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_{n-1}). \end{aligned}$$

□

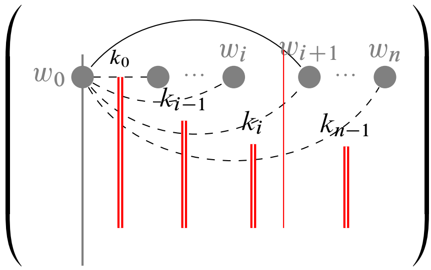
We emphasize the action in the case of one puncture.

Corollary 6.21 $(n = 1)$ *Let $n = 1$ and $k \in \mathbb{N}$, and $A(k)$ be the associated element of $\mathcal{H}$. Then*

$$E \cdot A(k) = A(k-1).$$

6.2.2 Action of $F^{(k)}$ Let $i \in \{1, \dots, n\}$ and S_i be the class

$S_i =$



$\in \mathcal{H}_r^{\mathrm{rel}-}$.

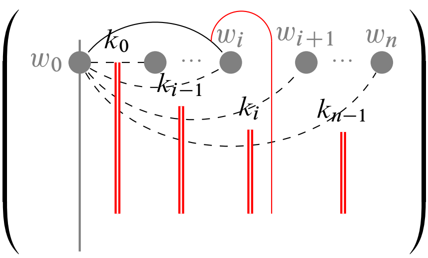
Namely, one recognizes a standard $(k_0, \dots, k_{n-1})$–multiarc to which a plain arc as in the picture has been added. To compute the action of $F^{(1)}$ we need the following lemma, allowing us to deal with S_i by recursion:

Lemma 6.22 *For $i \in \{2, \dots, n\}$, the following equality holds in $\mathcal{H}_r^{\mathrm{rel}-}$:*

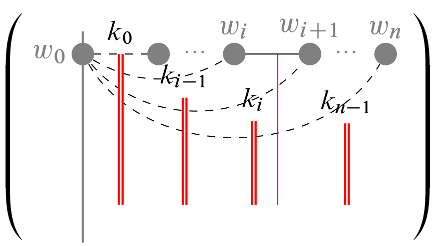
$$S_i = (k_i + 1)_{\mathtt{t}^{-1}} A'(k_0, \dots, k_i + 1, k_{i+1}, \dots, k_{n-1}) \\ - \mathtt{t}^{-k_i} (k_{i-1} + 1)_{\mathtt{t}} A'(k_0, \dots, k_{i-1} + 1, k_i, \dots, k_{n-1}) + q^{-2\alpha_i}\, \mathtt{t}^{-k_i}\, S_{i-1}.$$

Proof By a breaking of a plain arc (see Example 4.5), one gets the decomposition

$S_i =$

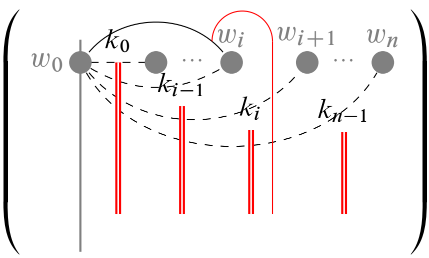


$+$



.

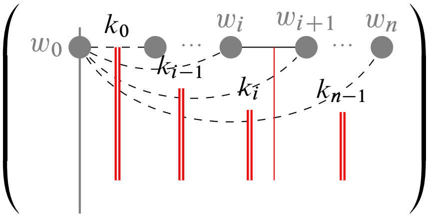
We treat both terms on the right independently. From the first one we get

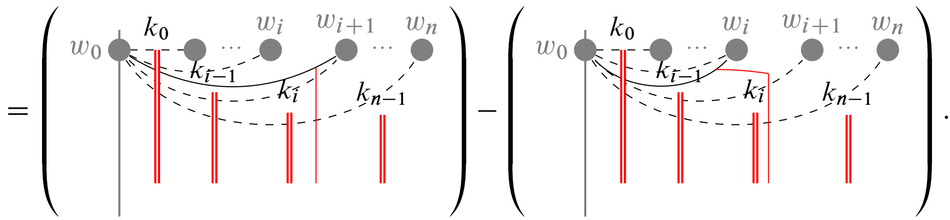


$= q^{-2\alpha_i}\, \mathtt{t}^{-k_i}\, S_{i-1},$

which follows from the handle rule.

Again, to treat the second term, breaking the plain arc (Example 4.5) leads to





To decompose these two terms in the standard multiarc basis, one must apply Lemma 4.8 to crash a plain arc over a dashed one, after a simple application of the handle rule to reorganize the handles of the right term. This recovers the lemma. □

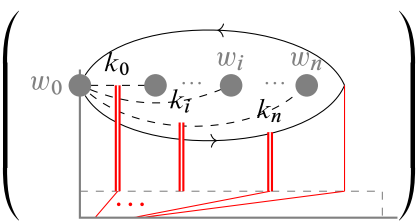
We use this lemma to compute the action of $F^{(1)}$ in the multiarcs basis.

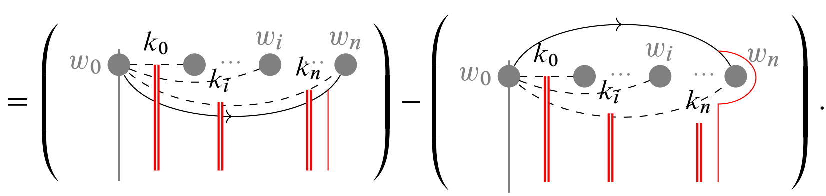
Lemma 6.23 *Let $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E^0_{n,r}$; the action of $F^{(1)}$ over the associated standard multiarc is*

$$F^{(1)} \cdot A'(\boldsymbol{k}) = \sum_{i=0}^{n-1} q^{\sum_{j=1}^{i+1} \alpha_j} q^{-\sum_{j=i+2}^{n} \alpha_j} \mathtt{t}^{-\sum_{j=i+1}^{n-1} k_j} (k_i+1)_{\mathtt{t}} (1 - q^{-2\alpha_{i+1}} \mathtt{t}^{-k_i}) A'(\boldsymbol{k})_i,$$

where $A'(\boldsymbol{k})_i$ means $A'(k_0, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_{n-1})$.

Proof First, we compute the element $F' \cdot A'(k_0, \dots, k_{n-1})$ of $\mathcal{H}_r^{\mathrm{rel}-}$. This corresponds to the following class, for which we give a decomposition in $\mathcal{H}_r^{\mathrm{rel}-}$:





This decomposition follows from a breaking of a plain arc (Example 4.5). The minus sign is due to the reverse of the orientation of the right term's plain arc. The first term of the decomposition is in position to apply Lemma 4.8, while, after a handle rule, one recognizes S_{n-1} in the second term. Finally, we get

$$F' \cdot A'(k_0, \dots, k_{n-1}) = (k_{n-1}+1)_{\mathtt{t}} A'(k_0, \dots, k_{n-1}+1) - q^{-2\alpha_n} S_{n-1}.$$

Thanks to the recursive property of S_{n-1}, the proof is achieved using Lemma 6.22, so one gets

$$F' \cdot A'(\boldsymbol{k}) = \sum_{i=0}^{n-1} q^{-2\sum_{j=i+2}^{n} \alpha_j} \, \mathtt{t}^{-\sum_{j=i+1}^{n-1} k_j} (k_i+1)_{\mathtt{t}} (1 - q^{-2\alpha_{i+1}} \, \mathtt{t}^{-k_i}) A'(\boldsymbol{k})_i.$$

By multiplication of the action by $q^{\sum \alpha_i}$, one obtains the expected action for $F^{(1)}$ over the multiarc basis. □

Proposition 6.24 (action of $F^{(1)}$ over multiarcs) *Let $\boldsymbol{k} = (k_0, \dots, k_{n-1}) \in E^0_{n,r}$; the action of $F^{(1)}$ over the associated standard* (*normalized*) *multiarc is*

$$F^{(1)} \cdot A(\boldsymbol{k}) = \sum_{i=0}^{n-1} q^{-\sum_{j=i+2}^{n} \alpha_j} \, \mathtt{t}^{-\sum_{j=i+1}^{n-1} k_j} \, q^{\alpha_{i+1}} (k_i+1)_{\mathtt{t}} (1 - q^{-2\alpha_{i+1}} \, \mathtt{t}^{-k_i}) A(\boldsymbol{k})_i.$$

Proof This is a straightforward consequence of the previous lemma and of the normalization sending the family $\mathcal{A}'$ to $\mathcal{A}$. □

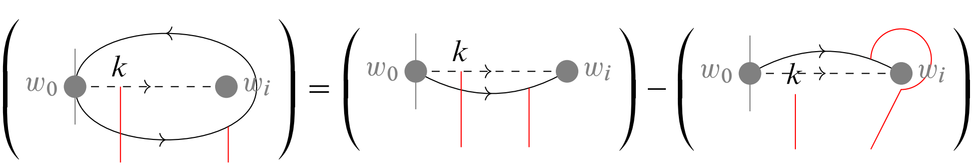
We emphasize again the case $n = 1$.

Corollary 6.25 ($n = 1$) *Let $n = 1$ and $k \in \mathbb{N}$, and $A(k)$ be the associated element of $\mathcal{H}$. Then*

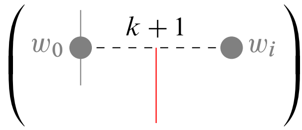
$$F^{(1)} \cdot A(k) = q^{\alpha_1} (k+1)_{\mathtt{t}} (1 - q^{-2\alpha_1} \, \mathtt{t}^{-k}) A(k+1).$$

We end this section by giving the action of the divided powers $F^{(l)}$ but only in the case of one puncture. We need the following remark:

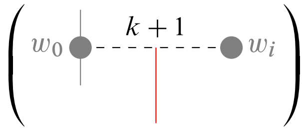
Remark 6.26 We observe the relations between homology classes



$$= ((k+1)_{\mathtt{t}} - q^{-2\alpha_i}(k+1)_{\mathtt{t}^{-1}})$$



$$= (k+1)_{\mathtt{t}}(1 - q^{-2\alpha_i}\,\mathtt{t}^{-k})$$



,

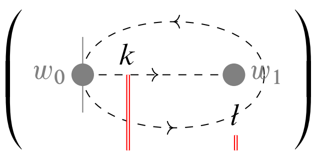
where everything stands inside a small neighborhood of the picture, without perturbing the rest of the class contained outside of it. The first equality comes from a breaking of a plain arc; see Example 4.5. The second one is a consequence first of the application of a handle rule to get vertical handles, and then relations from Lemma 4.8.

Proposition 6.27 (action of $F^{(l)}$, $n = 1$) *Let $n = 1$ and $k \in \mathbb{N}$, and $A(k)$ be the associated element of $\mathcal{H}$. Let $l \in \mathbb{N}$; then*

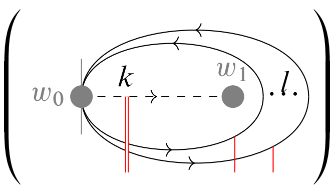
$$F^{(l)} \cdot A(k) = q^{-l(l-1)/2} q^{l\alpha_1} \binom{k+l}{k}_{\mathtt{t}} \prod_{m=0}^{l} (1 - q^{-2\alpha_1}\,\mathtt{t}^{-m}) A(k+l).$$

Proof We have

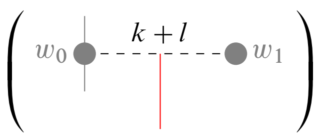
$$(l)_{\mathtt{t}}!\, F'^{(l)} \cdot A(k) = (l)_{\mathtt{t}}!$$



$$=$$



$$= \prod_{m=0}^{l} (k+m)_{\mathtt{t}}!\,(1 - q^{-2\alpha_1}\,\mathtt{t}^{-m})$$



.

The second equality comes from Corollary 4.10 and the last one is an iteration of the relations from Remark 6.26. Finally, we have

$$F'^{(l)} \cdot A(k) = \binom{k+l}{k}_{\mathtt{t}} \prod_{m=0}^{l} (1 - q^{-2\alpha_1}\,\mathtt{t}^{-m}) A(k+l).$$

The proposition is proved after the normalization passing from $F'^{(l)}$ to $F^{(l)}$. □

6.2.3 Recovering monoidality of Verma modules for $U_q^{L/2}\mathfrak{sl}(2)$ Since in this section the number n of punctures is particularly important, we denote by $\mathcal{H}^{\alpha_1,\dots,\alpha_n}$ the module $\mathcal{H}$ built from $X_r(w_0,\dots,w_n)$ with coefficients in $\mathcal{R}_{\max}=\mathbb{Z}[\mathtt{t}^{\pm 1},q^{\pm\alpha_1},\dots,q^{\pm\alpha_n}]$.

Remark 6.28 We recall the action of K. We distinguish the cases whether n is greater than 1 or not:

- Let $n=1$, so $\mathcal{R}_{\max}=\mathbb{Z}[\mathtt{t}^{\pm 1},q^{\pm\alpha_1}]$. Let $k\in\mathbb{N}$ and let $A(k)$ be the associated element of $\mathcal{H}^{\alpha_1}$. Then
$$K\cdot A(k)=q^{\alpha_1}\,\mathtt{t}^k\,A(k).$$
- Let $n>1$ and $\boldsymbol{k}\in E^0_{n,r}$, and let $A(\boldsymbol{k})$ be the associated element of $\mathcal{H}^{\mathrm{rel}-}_r\in\mathcal{H}^{\alpha_1,\dots,\alpha_n}$. Then
$$K\cdot A(\boldsymbol{k})=q^{\sum_{i=1}^n\alpha_i}\,\mathtt{t}^r\,A(\boldsymbol{k}).$$

Proposition 6.29 *Let $\mathtt{t}=q^{-2}$. The module $\mathcal{H}^{\alpha_1}$ is a Verma module for $U_q^{L/2}\mathfrak{sl}(2)$ of highest weight α_1.*

Proof The presentation of the action over a Verma module is given in [12] (see relations (18)) and is recalled in Definition 5.8. Using Corollaries 6.21 and 6.25 and the above remark in the case $n=1$, one recognizes the presentation of the Verma module. Namely, let $\mathtt{t}=q^{-2}$ and let $s=q^{\alpha_1}$. Then
$$K\cdot A(k)=q^{\alpha_1}\,\mathtt{t}^k\,A(k)=sq^{-2k}A(k),\qquad E\cdot A(k)=A(k-1)$$
and
$$F^{(1)}\cdot A(k)=q^{\alpha_1}(k+1)_{\mathtt{t}}(1-q^{-2\alpha_1}\,\mathtt{t}^{-k})A(k+1)=[k+1]_q(sq^{-k}-s^{-1}q^k)A(k+1).$$
The last equality uses Remark 6.1.

These expressions ensure that the isomorphism of $\mathbb{Z}[s^{\pm 1},q^{\pm 1}]$–modules
$$\mathcal{H}^{\alpha_1}\to V^{\alpha_1},\qquad A(k)\mapsto v_k\quad\text{for } k\in\mathbb{N},$$
is $U_q^{L/2}\mathfrak{sl}(2)$–equivariant. □

Remark 6.30 There is an isomorphism of $\mathcal{R}_{\max}$–modules
$$\mathrm{tens}\colon\mathcal{H}^{\alpha_1,\dots,\alpha_n}\to\mathcal{H}^{\alpha_1}\otimes\cdots\otimes\mathcal{H}^{\alpha_n},\quad A(k_0,\dots,k_{n-1})\mapsto A(k_0)\otimes\cdots\otimes A(k_{n-1}).$$

Theorem 2 (monoidality of Verma modules) *For* $\mathtt{t} = q^{-2}$, *the morphism*

$$\operatorname{tens}: \mathcal{H}^{\alpha_1,\dots,\alpha_n} \to \mathcal{H}^{\alpha_1} \otimes \cdots \otimes \mathcal{H}^{\alpha_n}$$

is an isomorphism of $U_q^{L/2}\mathfrak{sl}(2)$*–modules.*

Proof From Proposition 6.24 one notes that the formulas satisfy

$$\begin{aligned}
&\operatorname{tens}(F^{(1)} \cdot A(\boldsymbol{k}))\\
&\quad= \operatorname{tens}\left(\sum_{i=0}^{n-1} q^{-\sum_{j=i+2}^{n}\alpha_j}\, \mathtt{t}^{-\sum_{j=i+1}^{n-1}k_j}\, q^{\alpha_{i+1}}(k_i+1)_{\mathtt{t}}(1-q^{-2\alpha_{i+1}}\,\mathtt{t}^{-k_i})A(\boldsymbol{k})_i\right)\\
&\quad= \sum_{i=0}^{n-1} A(k_0)\otimes\cdots\otimes(q^{\alpha_{i+1}}(k_i+1)_{\mathtt{t}}(1-q^{-2\alpha_{i+1}}\,\mathtt{t}^{-k_i}))A(k_i+1)\\
&\qquad\qquad\otimes q^{-\alpha_{i+2}}\,\mathtt{t}^{-k_{i+1}}\,A(k_{i+1})\otimes\cdots\otimes q^{-\alpha_n}\,\mathtt{t}^{-k_{n-1}}\,A(k_{n-1})\\
&\quad= \sum_{i=0}^{n-1}(1\otimes 1\otimes\cdots\otimes F^{(1)}\otimes K^{-1}\otimes\cdots\otimes K^{-1})A(k_0)\otimes\cdots\otimes A(k_{n-1}),
\end{aligned}$$

where the $F^{(1)}$ in the sum is in the $(i+1)^{\text{st}}$ position, and one recognizes the expression of $\Delta^n(F^{(1)})$.

We do the same for E: from Proposition 6.20 we have

$$\begin{aligned}
&\operatorname{tens}(E\cdot A(k_0,\dots,k_{n-1}))\\
&\qquad= \operatorname{tens}\left(\sum_{i=0}^{n-1} q^{\alpha_1+\cdots+\alpha_i}\,\mathtt{t}^{k_0+\cdots+k_{i-1}}\,A(k_0,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_{n-1})\right)\\
&\qquad= \sum_{i=0}^{n-1}(K\otimes\cdots\otimes K\otimes E\otimes 1\otimes\cdots\otimes 1)A(k_0)\otimes\cdots\otimes A(k_{n-1}),
\end{aligned}$$

which proves that the action of E over $\mathcal{H}^{\alpha_1,\dots,\alpha_n}$ corresponds to the action of $\Delta^n(E)$ over the tensor product. The same proof works for the action of K, so the theorem holds. □

Remark 6.31 The above theorem suggests that there should exist a homological interpretation of the $U_q^{L/2}\mathfrak{sl}(2)$ coproduct, probably in terms of gluing once-punctured disks along arcs of their boundary. The morphism tens should then be the involved homological operator.

We note that

$$\operatorname{tens}(A'(\boldsymbol{k})) = q^{\alpha_1(k_1+\cdots+k_{n-1})+\alpha_2(k_2+\cdots+k_{n-1})+\cdots+\alpha_{n-1}k_{n-1}}\,A'(k_0)\otimes\cdots\otimes A'(k_{n-1}),$$

so multiarcs are divided into tensor products of single arcs, with coefficients appearing from the gluing operation. If one is able to draw the handles corresponding to the normalization coefficient, one should know how to glue once-punctured disks.

Remark 6.32 Theorem 2 answers [9, Conjecture 6.2]. In fact, the isomorphism was suggested by the conjecture while the topological basis was not the one that fits with the integral coefficients setting. See Corollary 7.5 for details.

6.3 Homological braid action

6.3.1 Definition of the action In this section we present an extension of Lawrence representations [18] for braid groups, following her ideas. The starting point is the fact that the braid group is the mapping class group of D_n.

Remark Recall that the braid group on n strands is the mapping class group of D_n,

$$\mathcal{B}_n = \mathrm{Mod}(D_n) = \mathrm{Homeo}(D_n, \partial D)/\mathrm{Homeo}_0(D_n, \partial D).$$

This definition is isomorphic to the Artin presentation of the braid group (Definition 1.1) by sending generator σ_i to the mapping class of the half Dehn twist swapping punctures w_i and w_{i+1}. The *pure braid group* $\mathcal{PB}_n$ is defined to be those braids fixing the punctures pointwise.

The action of a homeomorphism of D_n can be generalized to X_r as homeomorphisms extend to the configuration space coordinate by coordinate. Namely, if ϕ is a homeomorphism of D_n, the map

$$X_r \to X_r, \quad \{z_1, \dots, z_r\} \mapsto \{\phi(z_1), \dots, \phi(z_r)\},$$

is a homeomorphism. We show that the action of half Dehn twists passes to homology with local coefficients in L_r, treating the unicolored case ($\alpha_1 = \cdots = \alpha_n$) separately from the general one. In the unicolored case, we get a representation of the standard braid group.

Lemma 6.33 (representation of the braid group) *Let $\alpha = \alpha_1 = \cdots = \alpha_n$, so that $\mathcal{R}_{\max} = \mathbb{Z}[\mathtt{t}^{\pm 1}, q^{\pm\alpha}]$. Let $\beta \in \mathcal{B}_n$ be a braid and $\widehat{\beta}$ a homeomorphism representing β. The action of $\widehat{\beta}$ on X_r described above lifts to $\mathcal{H}_r^{\mathrm{rel}-}$, so it yields a homological representation of the braid group,*

$$\mathrm{R}^{\mathrm{hom}}\colon \mathcal{R}_{\max}[\mathcal{B}_n] \to \mathrm{End}_{\mathcal{R}_{\max}}(\mathcal{H}^{\alpha,\dots,\alpha}).$$

Proof Let $\sigma \in \mathcal{B}_n$ be one of the Artin braid generators; the lemma is a direct consequence of the invariance of the local system representation under the braid action, namely the commutativity of the diagram

$$\begin{array}{ccc} \pi_1(X_r, \boldsymbol{\xi}^r) & \xrightarrow{\quad\widehat{\sigma}_*\quad} & \pi_1(X_r, \boldsymbol{\xi}^r) \\ \Big\downarrow{\scriptstyle \rho_r} & & \Big\downarrow{\scriptstyle \rho_r} \\ \mathbb{Z}^2 = \mathbb{Z}\langle q^\alpha\rangle \oplus \mathbb{Z}\langle t\rangle & \xrightarrow[\quad\mathrm{Id}\quad]{} & \mathbb{Z}\langle q^\alpha\rangle \oplus \mathbb{Z}\langle t\rangle \end{array}$$

where $\widehat{\sigma}$ is a half Dehn twist of X_r associated with σ and $\widehat{\sigma}_*$ its lift to the fundamental group. It is easy to see that, for $l \in \{1, \dots, r-1\}$ and $k \in \{1, \dots, n\}$,

$$\rho_r(\widehat{\sigma}_*(\sigma_l)) = \rho_r(\sigma_l) \quad \text{and} \quad \rho_r(\widehat{\sigma}_*(B_{r,k})) = \rho_r(B_{r,k}),$$

considering the generators σ_l and $B_{r,k}$ of $\pi_1(X_r, \boldsymbol{\xi}^r)$ introduced in Remark 2.2. This ensures that the action lifts to the maximal abelian cover, and that it commutes with deck transformations, so that the action on homology with local coefficients is well defined. The action is invariant under isotopies, so the action of $\mathcal{B}_n$ is well defined. □

To deal with different colors, we need a morphism to follow the change of colors in $\mathcal{R}_{\max}$.

Definition 6.34 Let $s \in \mathfrak{S}_n$ be a permutation. We define the morphism

$$\hat{s}\colon \mathcal{R}_{\max} \to \mathcal{R}_{\max}, \qquad q^{\alpha_i} \mapsto q^{\alpha_{s(i)}}, \quad t \mapsto t.$$

Lemma 6.35 (representation of the pure braid group) *In the general case, let σ_i be an Artin generator of $\mathcal{B}_n$, with $i \in \{1, \dots, n-1\}$ and $s \in \mathfrak{S}_n$. There exists a well-defined action of σ_i lifted to homology,*

$$\mathrm{R}^{\mathrm{hom}}(\sigma_i) \in \mathrm{Hom}_{\mathcal{R}_{\max}}(\mathcal{H}^{s(\alpha_1)} \otimes \cdots \otimes \mathcal{H}^{s(\alpha_n)}, \mathcal{H}^{s\tau_i(\alpha_1)} \otimes \cdots \otimes \mathcal{H}^{s\tau_i(\alpha_n)}),$$

where $\tau_i = (i, i+1) \in \mathfrak{S}_n$. There is a well-defined action of $\mathcal{PB}_n$ over $\mathcal{H}^{\alpha_1,\dots,\alpha_n}$,

$$\mathrm{R}^{\mathrm{hom}}\colon \mathcal{R}_{\max}[\mathcal{PB}_n] \to \mathrm{End}_{\mathcal{R}_{\max}}(\mathcal{H}^{\alpha_1,\dots,\alpha_n}).$$

This action commutes with the $\mathcal{R}_{\max}$ structure.

Proof The proof is almost the same as the one for Lemma 6.33. Namely, it is a consequence of the fact that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X_r,\boldsymbol{\xi}^r) & \xrightarrow{\widehat{\sigma}_*} & \pi_1(X_r,\boldsymbol{\xi}^r) \\ \downarrow{\rho_r} & & \downarrow{\rho_r} \\ \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\dots,n\}} & \xrightarrow[\widehat{\tau}_k]{} & \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\dots,n\}} \end{array}$$

The fact that this diagram commutes comes from the observation

$$(\widehat{\sigma}_i)_*(B_{r,k}) = B_{r,k+1},$$

while other generators of $\pi_1(X_r,\boldsymbol{\xi}^r)$ are not perturbed by the action of σ_i. For a pure braid β, we have

$$\mathrm{R}^{\mathrm{hom}}(\beta) \in \mathrm{End}_{\mathcal{R}_{\max}}(\mathcal{H}^{\alpha_1}\otimes\cdots\otimes\mathcal{H}^{\alpha_n}),$$

and, as β can be written as a composition of generators σ_i, by composition of diagrams, one obtains the commutative diagram

$$\begin{array}{ccc} \pi_1(X_r,\boldsymbol{\xi}^r) & \xrightarrow{\widehat{\beta}_*} & \pi_1(X_r,\boldsymbol{\xi}^r) \\ \downarrow{\rho_r} & & \downarrow{\rho_r} \\ \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\dots,n\}} & \xrightarrow[\mathrm{Id}]{} & \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\dots,n\}} \end{array}$$

with the identity morphism on the second line coming from the pureness of β. This ends the proof as for Lemma 6.33. □

6.3.2 Computation of the action In the case of two punctures w_1 and w_2, we can perform the computation of the action of the single braid generator of $\mathcal{B}_2$, and first we recall the classical operators necessary to define the R–matrix of $U_q\mathfrak{sl}(2)$.

Definition 6.36 Let $q^{(H\otimes H/2)}$ be the $\mathcal{R}_{\max}$–linear map

$$\begin{aligned} q^{(H\otimes H/2)}\colon \mathcal{H}^{\alpha_1}\otimes\mathcal{H}^{\alpha_2} &\to \mathcal{H}^{\alpha_1}\otimes\mathcal{H}^{\alpha_2}, \\ A^{\alpha_1}(k)\otimes A^{\alpha_2}(k') &\mapsto q^{(\alpha_1-2k)(\alpha_2-2k')/2} A^{\alpha_1}(k)\otimes A^{\alpha_2}(k'), \end{aligned}$$

and T be

$$T\colon \mathcal{H}^{\alpha_1}\otimes\mathcal{H}^{\alpha_2}\to\mathcal{H}^{\alpha_2}\otimes\mathcal{H}^{\alpha_1}, \quad A^{\alpha_1}(k)\otimes A^{\alpha_2}(k') \mapsto A^{\alpha_2}(k')\otimes A^{\alpha_1}(k),$$

where $A(k')^{\alpha_1}$ and $A(k)^{\alpha_2}$ are multiarcs of $\mathcal{H}^{\alpha_1}$ and $\mathcal{H}^{\alpha_2}$, respectively.

Lemma 6.37 (braid action with two punctures) *Let* $\mathtt{t} = q^{-2}$. *Let* $k, k' \in \mathbb{N}$ *and* σ_1 *be the standard generator of the braid group on two strands. Its action can be expressed as*

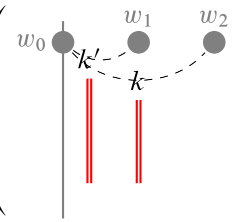
$$\operatorname{tens}\big(\mathrm{R}^{\mathrm{hom}}(\sigma_1)(A(k',k)^{\alpha_1,\alpha_2})\big) = \left[q^{-\alpha_1\alpha_2/2} q^{(H\otimes H/2)} \circ \sum_{l=0}^{k} (q^{l(l-1)/2} E^l \otimes F^{(l)}) \circ T\right] A(k')^{\alpha_1} \otimes A(k)^{\alpha_2},$$

where $A(k',k)^{\alpha_1,\alpha_2}$, $A(k')^{\alpha_1}$ *and* $A(k)^{\alpha_2}$ *are vectors of* $\mathcal{H}^{\alpha_1,\alpha_2}$, $\mathcal{H}^{\alpha_1}$, *and* $\mathcal{H}^{\alpha_2}$, *respectively.*

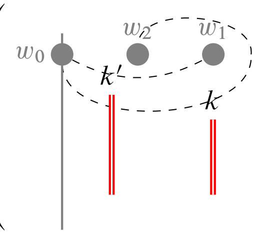
Proof We have the relations between homology classes

$\mathrm{R}^{\mathrm{hom}}(\sigma_1)(A'(k',k)^{\alpha_1,\alpha_2})$

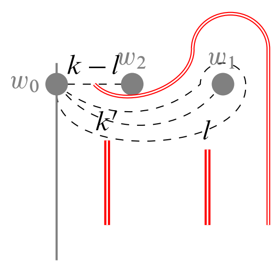
$$= \mathrm{R}^{\mathrm{hom}}(\sigma_1)\left(\vphantom{\int}\right.$$



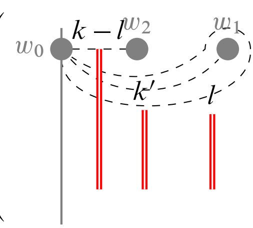
$$\left.\vphantom{\int}\right) = \left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \sum_{l=0}^{k}\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \sum_{l=0}^{k} \mathtt{t}^{-k'(k-l)}\, \mathtt{t}^{-l(k-l)}\, q^{-2(k-l)\alpha_1} \left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right).$$

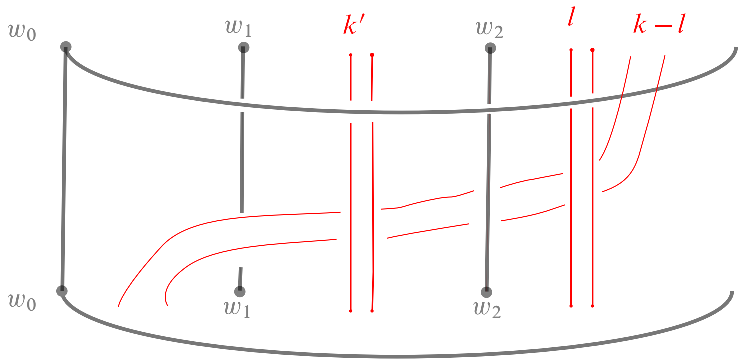


Figure 14: Braided handle rule.

The second equality comes from a breaking of a dashed arc (Example 4.6); the last one is a handle rule, for which we draw the corresponding braid in Figure 14.

The bands represent a $(k-l)$–handle, an (l)–handle and a (k')–handle. On the top and on the bottom of this box there is the part of the path corresponding to the dashed box. Namely, red arcs are going back to $\boldsymbol{\xi}$ without crossing themselves, passing in the front of w_1 and w_2. As this braid has $k-l$ strands passing successively in the back of k' strands, l strands and finally w_2, its local coefficient is $\mathtt{t}^{-(k-l)(k'+l)}\, q^{-2\alpha_2}$. From the local coefficient of this braid we deduce the coefficient appearing in the last term.

Finally, applying the proof of Proposition 6.27 to crash a dashed loop on the index k dashed arc, we get

$$\begin{aligned}&\mathrm{R}^{\mathrm{hom}}(\sigma_1)(A'(k',k)^{\alpha_1,\alpha_2})\\ &=\sum_{l=0}^{k}\mathtt{t}^{-(k'+l)(k-l)}\, q^{-2(k-l)\alpha_1}\binom{k'+l}{l}_{\mathtt{t}}\prod_{m=0}^{l}(1-q^{-2\alpha_1}\,\mathtt{t}^{-m})A'(k-l,k'+l)^{\alpha_2,\alpha_1},\end{aligned}$$

so that

$$\begin{aligned}\mathrm{R}^{\mathrm{hom}}(\sigma_1)(A(k',k)^{\alpha_1,\alpha_2})=\sum_{l=0}^{k}\mathtt{t}^{-(k'+l)(k-l)}\, q^{(-k+2l)\alpha_1}q^{-(k'+l)\alpha_2}\binom{k'+l}{l}_{\mathtt{t}}\\ \cdot\prod_{m=0}^{l}(1-q^{-2\alpha_1}\,\mathtt{t}^{-m})A(k-l,k'+l)^{\alpha_2,\alpha_1}.\end{aligned}$$

Let $\mathtt{t}=q^{-2}$; passing the above expression to tens, we get the following expression for $\mathrm{tens}\big(\mathrm{R}^{\mathrm{hom}}(\sigma_1^{\tau_1})(A(k',k)^{\alpha_1,\alpha_2})\big)$:

$$\sum_{l=0}^{k} q^{2(k'+l)(k-l)+(-k+2l)\alpha_1-(k'+l)\alpha_2} \binom{k'+l}{l}_{\mathtt{t}} \cdot \prod_{m=0}^{l} (1-q^{-2\alpha_1}\mathtt{t}^{-m}) A(k-l)^{\alpha_2} \otimes A(k'+l)^{\alpha_1}.$$

By use of the expression of the action of $F^{(l)}$ in Proposition 6.27, one recognizes

$$\left(\sum_{l=0}^{k} q^{2(k'+l)(k-l)+(-k+2l)\alpha_1-(k'+l)\alpha_2} E^l \otimes F'^{(l)}\right) A(k)^{\alpha_2} \otimes A(k')^{\alpha_1}.$$

Finally, passing from $F'^{(l)}$ to $F^{(l)}$, we get

$$\begin{aligned}\operatorname{tens}&\big(\mathrm{R}^{\mathrm{hom}}(\sigma_1)(A(k',k)^{\alpha_1,\alpha_2})\big)\\ &= \left(\sum_{l=0}^{k} q^{2(k'+l)(k-l)-(k-l)\alpha_1-(k'+l)\alpha_2} q^{l(l-1)/2} E^l \otimes F^{(l)}\right) A(k)^{\alpha_2} \otimes A(k')^{\alpha_1}\\ &= \left[q^{-\alpha_1\alpha_2/2} q^{(H\otimes H/2)} \circ \sum_{l=0}^{k} (q^{l(l-1)/2} E^l \otimes F^{(l)}) \circ T\right] A(k')^{\alpha_1} \otimes A(k)^{\alpha_2}. \qquad \square\end{aligned}$$

Theorem 3 (recovering the R–matrix of $U_q\mathfrak{sl}(2)$) *The representation of the pure braid group over $\mathcal{H}^{\alpha_1,\dots,\alpha_n}$ (resp. the one of $\mathcal{B}_n$ over $\mathcal{H}^{\alpha,\dots,\alpha}$) is isomorphic to the R–matrix representation over the product of $U_q^{L/2}\mathfrak{sl}(2)$ Verma modules $V^{\alpha_1}\otimes\cdots\otimes V^{\alpha_n}$ (resp. over the product $(V^{\alpha})^{\otimes n}$) from Definition 5.10 (and Remark 5.12 for the colored version).*

Proof From Lemma 6.37, the diagram

$$\begin{array}{ccc} \mathcal{H}^{\alpha_1,\alpha_2} & \xrightarrow{\ \operatorname{tens}\ } & \mathcal{H}^{\alpha_1}\otimes\mathcal{H}^{\alpha_2} \\ {\scriptstyle \mathrm{R}^{\mathrm{hom}}(\sigma_1)}\downarrow & & \downarrow{\scriptstyle q^{-\alpha_1\alpha_2/2}R\circ T} \\ \mathcal{H}^{\alpha_2,\alpha_1} & \xrightarrow[\ \operatorname{tens}\ ]{} & \mathcal{H}^{\alpha_2}\otimes\mathcal{H}^{\alpha_1} \end{array}$$

commutes. The action of a braid generator σ_i over a multiarc is contained in a disk that contains the dashed arcs reaching w_i and w_{i+1} and no other, so that the action does not perturb the other arcs. This last fact shows that the proof with two punctures guarantees the general case, and that the diagram

$$\begin{array}{ccc}
\mathcal{H}^{\alpha_1,\dots,\alpha_n} & \xrightarrow{\text{tens}} & \mathcal{H}^{\alpha_1}\otimes\cdots\otimes\mathcal{H}^{\alpha_n} \\
\Big\downarrow {\scriptstyle \mathrm{R}^{\mathrm{hom}}(\sigma_i)} & & \Big\downarrow {\scriptstyle Q(\sigma_i)} \\
\mathcal{H}^{\tau_i\{\alpha_1,\dots,\alpha_n\}} & \xrightarrow[\text{tens}]{} & \mathcal{H}^{\alpha_{\tau_i(1)}}\otimes\cdots\otimes\mathcal{H}^{\alpha_{\tau_i(n)}}
\end{array}$$

commutes, where $Q(\sigma_i) = \mathrm{Id}^{\otimes i-1}\otimes q^{-\prod\alpha_i/2}R\circ T\otimes\mathrm{Id}^{\otimes n-i-1}$. Moreover, all the morphisms involved commute with the $U_q\mathfrak{sl}(2)$ structure. This proves the theorem. □

7 Links with previous works

7.1 Integral version for Kohno's theorem

The following corollary recovers Kohno's theorem [16; 11, Theorem 4.5], recalled in Theorem 1.2(iii), in an integral version, namely with coefficients in $\mathcal{R}_{\max}$. It relates the action of the braid group over elements in the kernel of the action of E to the action over absolute homology modules.

Corollary 7.1 *Under the condition $q^{-2} = \mathtt{t}$, the restriction of the representation of the braid group $\mathcal{B}_n$ to the kernel of the homological action of E yields a subrepresentation of $\mathcal{B}_n$ in $\mathcal{H}$ isomorphic to $\mathcal{H}^E = \bigoplus_{r\in\mathbb{N}^*} H_r^{\mathrm{BM}}(X_r; L_r)$.*

Proof For $r\in\mathbb{N}$, the relative long exact sequence of pairs gives the exact sequence of morphisms

$$H_r(X_r^-; L_r)\to H_r(X_r; L_r)\to\mathcal{H}_r^{\mathrm{rel}-}\xrightarrow{\partial_*}H_{r-1}(X_r^-; L_r)\to H_{r-1}(X_r; L_r),$$

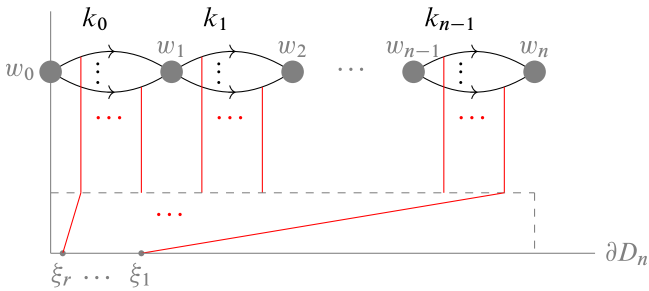
where we have avoided the notation BM as everything is Borel–Moore homology here. Using [4, Lemma 3.1], one gets that $H_{r-1}(X_r; L_r)$ vanishes, while Remark 6.8 above implies that $H_r(X_r^-; L_r)$ vanishes. This provides a short exact sequence

$$0\to H_r(X_r; L_r)\to\mathcal{H}_r^{\mathrm{rel}-}\xrightarrow{\partial_*}H_{r-1}(X_r^-; L_r)\to 0.$$

The kernel of the action of E is exactly the kernel of the map ∂_*. This implies the corollary, as the kernel of the action of E is isomorphic to the module of absolute homology. □

Kohno's theorem [16] holds only for generic choice of parameters, while in the above corollary all morphisms are defined over the ring of Laurent polynomials. Kohno's theorem in terms of bases, as it is stated for instance in [11], uses the basis of *multiforks* that we recall next:

Notation For $\boldsymbol{k} \in E^0_{n,r}$, we let the multifork $F(k_0, \dots, k_{n-1})$ be the class in $\mathcal{H}^{\mathrm{rel}-}_r$ assigned to



This recovers the consequences of Kohno's theorem that can be found in [11], stating that the family of multiforks is generically a basis of $H_r(X_r(w_0); L_r)$. We state precise genericity conditions in the following corollary:

Proposition 7.2 *Let $\boldsymbol{k} \in E^0_{n,r}$; there is the following relation between the standard fork and the code sequence associated with $\boldsymbol{k}$:*

$$F(k_0, \dots, k_{n-1}) = \left(\prod_{i=0}^{n-1} (k_i)_{\mathfrak{t}}! \right) U(k_0, \dots, k_{n-1}).$$

This shows that the family $\mathcal{F}$ is a basis of $\mathcal{H}^{\mathrm{rel}-}_r$ whenever one works over a ring R where all the $(i)_{\mathfrak{t}}!$ are invertible for i an integer at most r.

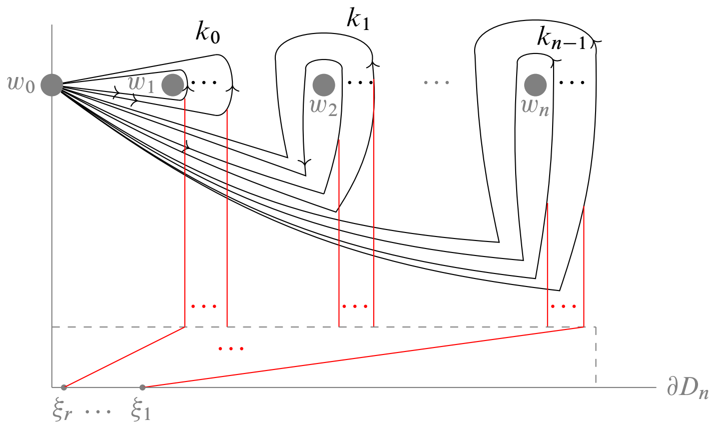
Proof The proof for the relation between multiforks and code sequences is a direct consequence of Corollary 4.10. □

Remark 7.3 Multiforks in the kernel of the action of E are multiforks not reaching w_0, namely of type $F(0, k_1, \dots, k_{n-1})$. This is the link with the multiforks basis used in [11, Theorem 4.5].

7.2 Felder and Wieczerkowski's conjectures

In [9], the authors use as a basis for their module elements called *r–loops*, for which we give a homological definition as follows:

Notation For $\boldsymbol{k} \in E^0_{n,r}$, we call $L(k_0,\dots,k_{n-1})$ an r–*loop*, the class in $\mathcal{H}^{\mathrm{rel}-}_r$ assigned to the drawing



Proposition 7.4 *Let $\boldsymbol{k} \in E^0_{n,r}$. There is the relation between the standard multiarc and the r–loop associated with $\boldsymbol{k}$*

$$L(k_0,\dots,k_{n-1}) = \left(\prod_{i=0}^{n-1}(k_i)_{\mathtt{t}}!\prod_{k=0}^{k_i}(1-q^{-2\alpha_i}\,\mathtt{t}^{-k})\right)A'(k_0,\dots,k_{n-1}).$$

Proof To prove the proposition, one treats separately the loops winding around w_1 from those winding around w_2, etc. Every case is a straightforward recursion, using Remark 6.26, and leads to the formula of the proposition. □

This answers [9, Conjecture 6.1]. In fact, it is a more precise statement, saying exactly under which conditions the family of r–loops is a basis of the homology.

Corollary 7.5 [9, Conjecture 6.1] *If R is a ring in which $(1-q^{-2\alpha_i}\,\mathtt{t}^{-k})$ is invertible for all $i=1,\dots,n$ and so is $(k)_{\mathtt{t}}!$ (for $k \le r$), then $\mathcal{H}^{\mathrm{rel}-}_r$ is a free R–module admitting the family $\mathcal{L}$ of r–loops as a basis.*

Actually, the lifts of the r–loops chosen in [9] are not exactly the same as ours; namely, the handles we've chosen do not correspond to their choice of lift. As a change of lift corresponds to multiplication by an invertible monomial of $\mathcal{R}_{\max}$, the conditions to be a basis are the same.

Appendix

A.1 Local coefficients

Remark A.1 We recall that the representation ρ_r defining the local system L_r is canonically equivalent to the construction of a covering map over X_r. Namely, one can consider the universal cover $\widetilde{X}_r$ of X_r, upon which there is an action of $\pi_1(X_r)$. By taking the quotient of $\widetilde{X}_r$ by the action of $\operatorname{Ker}\rho_r \subset \pi_1(X_r)$, one gets a cover $\widehat{X}_r$ of X_r. The group of deck transformations is then isomorphic to $\operatorname{Im}(\rho_r) = \mathbb{Z}^{n+1}$. There are three equivalent ways to build the chain complex with local coefficients in L_r:

$$C_\bullet(X_r; L_r) \simeq C_\bullet(\widetilde{X}_r, \mathbb{Z}) \otimes_{\pi_1(X_r)} \mathcal{R}_{\max} \simeq C_\bullet(\widehat{X}_r).$$

The first one corresponds to a complex with coefficients in a locally trivial bundle. In the middle one, the action of $\pi_1(X_r)$ is the one over the universal cover on the left, and given by ρ_r on the right. The last one corresponds to the singular chain complex of $\widehat{X}_r$ with the deck transformations action of $\mathcal{R}_{\max}$.

We use L_r or ρ_r to designate both the representation of $\pi_1(X_r)$ or the covering $\widehat{X}_r$ together with the deck transformations group action.

A.2 Locally finite chains

In this work we use the locally finite version for singular homology, which is isomorphic in our case to the Borel–Moore homology. This version controls the noncompact phenomena arising at punctures. We give general ideas and definitions of these homologies in this section. Let X be a locally compact topological space.

Definition A.2 (locally finite homology) The *locally finite chain complex* associated with X is the chain complex for which we allow infinite chains under the condition that their geometrical realization in X is locally finite (for the topology of X). The latter guarantees that the boundary map is well defined.

Let $Y \subset X$. The *relative to Y* locally finite chain complex corresponds to the locally finite chain complex of X modded out by the one of Y. The homology of locally finite chains is the homology corresponding to this definition of chain complexes. We use the notation $\mathrm{H}^{\mathrm{lf}}_\bullet(X)$ to denote the locally finite homology.

Remark A.3 [4] The homology of locally finite chains is isomorphic to the Borel–Moore homology, which can be defined as

$$H_\bullet^{\mathrm{BM}}(X) = \varprojlim H_\bullet(X, X \setminus A),$$

where the inverse limit is taken over all compact subsets A of X. The relative case is then

$$H_\bullet^{\mathrm{BM}}(X, Y) = \varprojlim H_\bullet(X, (X \setminus A) \cup Y)$$

for $Y \subset \partial X$.

The above fact that Borel–Moore homology consists in a limit of homology over compact spaces allows generalizations of many compact singular homology properties.

Locally finite homology have very different properties than the usual ones when the space is noncompact. We give first examples:

Example A.4 We give the example of the real line being a locally finite cycle, and a related example.

Real line Any 0–chain is null homologous (so that the 0–homology does not encode connectedness). Let p be a point; the chain

$$\sigma = \sum_{i=0}^{\infty} [p+i, p+i+1)$$

has p as Borel–Moore boundary, while the chain

$$\sum_{-\infty<k<\infty} [k, k+1)$$

has no boundary and hence is a cycle. This shows that $H_k^{\mathrm{BM}}(\mathbb{R}) = \mathbb{Z}$ if $k = 1$ and is 0 otherwise, and can be generalized to $H_k^{\mathrm{BM}}(\mathbb{R}^n) = \mathbb{Z}$ if $k = n$ and is 0 otherwise.

Punctures Let D_n be the punctured disk and c be a small circle running once around a puncture p. Then c is a cycle using the same kind of telescopic infinite chain as in the previous point.

We emphasize that the previous example generalizes.

Remark A.5 There are the following facts:

Compact space If X is compact, then the singular and locally finite homologies are identical.

Submanifold In the spirit of the previous example, any closed oriented submanifold defines a class in Borel–Moore homology, but not in ordinary homology unless the submanifold is compact.

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