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Georgios Kydonakis

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Georgios Kydonakis

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

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FROM HYPERBOLIC DEHN FILLING TO SURGERIES IN REPRESENTATION VARIETIES

GEORGIOS KYDONAKIS

Abstract. In this semi-expository article we describe a gluing method developed for constructing certain model objects in representation varieties $\mathrm{Hom}\left(\pi_1\left(\Sigma\right), G\right)$ for a topological surface Σ and a semisimple Lie group G. Explicit examples are demonstrated in the case of Θ-positive representations lying in the $p\cdot(2g-2)-1$ many exceptional connected components of the $\mathrm{SO}\left(p, p+1\right)$-character variety for $p>2$.

1. Introduction

A Dehn surgery on a 3-manifold M containing a link $L \subset S^3$ is a 2-step process involving the removal of an open tubular neighborhood of the link (drilling) and then gluing back a solid torus using a homeomorphism from the boundary of the solid torus to each of the torus boundary components of M (filling). Of particular interest are the many inequivalent ways one can perform the filling step of the operation, thus providing a way to represent certain examples of 3-dimensional manifolds. In fact, the so-called *fundamental theorem of surgery theory* by W. Lickorish and A. Wallace implies that every closed orientable and connected 3-manifold can be obtained by performing a Dehn surgery on a link in a 3-sphere.

William Thurston introduced hyperbolic geometry to this operation, thus opening the way to certain breakthroughs in 3-manifold theory. His *hyperbolic Dehn filling theorem* implies that the complete hyperbolic structure on the interior of a compact 3-manifold with boundary has a space of hyperbolic deformations parameterized by the generalized Dehn filling coefficients describing the metric completion of the ends of the interior. Among the various and deep advances marked by this result, we highlight here the fact that using hyperbolic Dehn surgery theory one can also obtain examples of non-Haken manifolds, whose hyperbolicity cannot be shown by the uniformization theorem; in fact, the proof of Thurston's theorem does not depend on uniformization. In general, such examples of non-Haken manifolds are not easy to construct otherwise. Deformations of hyperbolic cone structures can, moreover, be better understood when viewed through this prism. In the course of proving Thurston's theorem, one shows not only the existence of a 1-parameter family of cone 3-manifold structures, but can also obtain a path of corresponding holonomies in the representation variety $\mathrm{Hom}\left(\pi_1\left(M\right), \mathrm{SL}\left(2, \mathbb{C}\right)\right)$.

Hyperbolic Dehn surgery exists only in dimension 3. The purpose of this semi-expository article, however, is to describe a set of similar ideas of surgery techniques in representation varieties $\mathrm{Hom}\left(\pi_1\left(M\right), G\right)$, where M this time is a closed connected and oriented topological surface of genus $g \geq 2$ and G is a semisimple Lie group. The Teichmüller space, viewed as the moduli space of marked hyperbolic structures on Σ, can be realized as a connected component of the representation variety $\mathrm{Hom}\left(\pi_1\left(\Sigma\right), \mathrm{PSL}\left(2, \mathbb{R}\right)\right)$. The recently-emerged field of

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higher Teichmüller theory involves the study of certain connected components of the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), G)$, which share essential geometric, topological and dynamical properties as the classical Teichmüller space.

We describe here a gluing construction in $\mathrm{Hom}(\pi_1(\Sigma), G)$ "in the tradition" of Thurston's hyperbolic Dehn filling procedure. The parameters involved in this construction are the genus of the surface Σ and the holonomy of a surface group representation along the boundary of Σ.

The non-abelian Hodge correspondence referring to a homeomorphism between representation varieties and moduli spaces of Higgs bundles over a Riemann surface (with underlying topological surface Σ as above) allows us to develop a gluing procedure for the corresponding holomorphic objects, and this is making it easier to determine the connected component these newly constructed model objects lie, using an explicit computation of appropriate topological invariants that emerge for their holomorphic counterparts.

This way, one can construct specific models in certain subsets of $\mathrm{Hom}(\pi_1(\Sigma), G)$, that are hard to be obtained otherwise; in particular, model representations that do not factor as $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{R}) \to G$. These models can be used in turn to describe their deformations in the representation variety and use them as a means to study open subsets (or connected components) of objects with certain geometric properties. As an example, we study here model Θ-positive representations that lie in the $p \cdot (2g-2) - 1$ many exceptional components of the $\mathrm{SO}(p, p+1)$-character variety for $p > 2$.

2. Hyperbolic Dehn Surgery

In this section we review the basic concepts involved in the hyperbolic Dehn surgery operation. Even though the results of the technique summarized here do not apply for the case of character varieties that we will study next, these provide a motivation and an interesting analogy of the fundamental ideals behind these surgery methods.

2.1. **Dehn surgery.** Dehn surgery is a method that has found profound relevance in 3-manifold topology and knot theory. It provides a way to represent 3-dimensional manifolds using a "drilling and filling" process. At first, a solid torus is removed from a 3-manifold (drilling) and then it is re-attached in many inequivalent ways (filling). This two-stage operation was introduced by Max Dehn in Kapitel II of his 1910 article *Über die Topologie des dreidimensionales Raumes* [19] as a method for constructing *Poincaré spaces*, that is, non-simply connected 3-manifolds with the same topology as the 3-sphere. The texts of S. Boyer [9], C. Gordon [35], [34], J. Luecke [65] offer a broad survey on this construction with numerous references for further study.

The basic parameter of the Dehn surgery operation, in particular referring to the filling stage of the operation, is that of a *slope* on a torus; we briefly introduce this next. Let M be a closed connected orientable 3-manifold. Take a link $L \subset S^3$. For $N(K) \subset \mathrm{int}(W)$ a closed tubular neighborhood of a knot K, the manifold $M_K = W \backslash \mathrm{int}(N(K))$ is referred to as the exterior of K. Let now $T \subset \partial M$ be a toral boundary component of M. Given any homeomorphism $f : \partial\left(S^1 \times D^2\right) \to T$, take the identification space $M(T; f) = \left(S^1 \times D^2\right) \cup_f M$ obtained by identifying the points of $\partial\left(S^1 \times D^2\right)$ with their images by f. We shall call $M(T; f)$ a *Dehn filling* of M along T. A Dehn surgery on a knot K is then a filling of M_K along $\partial N(K)$. Note that a filling $M(T; f)$ depends only on the isotopy class of the attaching homeomorphism $f : \partial\left(S^1 \times D^2\right) \to T$. In fact the dependence of f is much weaker, for if

$C_0 = \{pt\} \times \partial D^2 \subset \partial \left(S^1 \times D^2\right)$, then $M(T; f)$ depends only on the isotopy class of the curve $f(C_0)$ in T.

Definition 2.1. A *slope* on a torus T is the isotopy class of an essential unoriented simple closed curve on T. The set of slopes on T will be denoted by $s(T)$. Moreover, two slopes r_1, r_2 on T are called *dual* if they have representative curves which intersect exactly once and transversely. Finally, if K is a knot in a 3-manifold W, then a slope of K is any slope on $\partial N(K)$.

One has the following proposition:

Proposition 2.2. *A Dehn filling of M along a torus $T \subseteq \partial M$ is determined up to orientation preserving homeomorphism, by a slope on T. Furthermore, any slope on T arises as the slope of a Dehn filling of M.*

The next problem involves the existence and uniqueness of a surgery presentation of a given closed connected orientable 3-manifold by surgery on a finite number of knots in the 3-sphere. By a set of surgery data $(L; r_1, \ldots, r_n)$ we shall mean a link $L = K_1 \cup \cdots \cup K_n$ lying in the interior of a 3-manifold W together with a slope r_i for each of its components K_i. We denote by $L(r_1, \ldots, r_n)$ the manifold obtained by performing the Dehn surgeries prescribed by the surgery data . In the special case when $W = S^3$ and each r_i is an integral slope, the surgery data $(L; r_1, \ldots, r_n)$ is often called a *framed link*.

The following result is known as the *fundamental theorem of surgery theory*; it was proved using different and independent approaches by W. Lickorish and A. Wallace:

Theorem 2.3 (W. Lickorish [63], A. Wallace [92])**.** . *Let W be a closed connected orientable 3-manifold. There exists a framed link $(L; r_1, \ldots, r_n)$ in S^3 such that W is homeomorphic to $L(r_1, \ldots, r_n)$.*

For the problem of uniqueness of a surgery presentation of a given manifold, R. Kirby [51] introduced two moves on (integrally) framed links which do not alter the presented manifold; he also proved that two framed links present manifolds which are orientation preserving homeomorphic if and only if they are related by a finite sequence of these moves, nowadays called *Kirby moves*. This problem was completely analyzed by D. Rolfsen in [78].

2.2. **Hyperbolic Dehn surgery.** A breakthrough in 3-manifold theory as well as in knot theory was signified by the introduction by William Thurston of hyperbolic geometry into the Dehn surgery operation. Necessary and sufficient conditions for the complete gluing of a hyperbolic 3-manifold were given by H. Seifert in [86]. The concept of the link of a cusp point of a hyperbolic 3-manifold was introduced by W. Thurston in his 1979 lecture notes [89].

The celebrated *hyperbolic Dehn filling theorem* of Thurston (Theorem 5.9 in [89]) provides that the complete hyperbolic structure on the interior of a compact 3-manifold with boundary has a space of hyperbolic deformations parameterized by the generalized Dehn filling coefficients describing the metric completion of the ends of the interior.

Among the various and deep advances in 3-manifold theory marked by this result, we will highlight here the fact that using hyperbolic Dehn surgery theory one can also obtain examples of non-Haken manifolds, whose hyperbolicity cannot be shown by the uniformization theorem; in fact, the proof of Thurston's theorem does not depend on uniformization. Deformations of hyperbolic cone structures can, moreover, be better understood when viewed through this prism. Another important aspect to be stressed next is the role the generalized Dehn filling coefficients play in the perception of the spaces of hyperbolic deformations parameterized by these coefficients.

The Theorem was first proven in Thurston's notes [89] in the manifold case and has been later extended in the case of orbifolds by W. Dunbar and R. Meyerhoff [22]. A detailed review of the proof in both these cases can be found in Appendix B of [8] using (in the manifold case) an argument of Q. Zhou [97]. We will follow next the description from [8] for our purposes. The statement of the theorem is the following:

Theorem 2.4 (Hyperbolic Dehn filling theorem, W. Thurston [89])**.** *Let M be a compact 3-manifold with boundary $\partial M = T_1^2 \cup \cdots \cup T_k^2$ a non-empty union of tori, whose interior $int(M)$ is complete hyperbolic with finite volume. There exists a neighborhood of $\{\infty, \ldots \infty\}$ in $S^2 \times \cdots \times S^2$, such that the complete hyperbolic structure on $int(M)$ has a space of hyperbolic deformations parameterized by the generalized Dehn filling coefficients in this neighborhood.*

The first major step in the proof involves the construction of the algebraic deformation of the holonomies around each boundary component of the manifold M. By the Mostow rigidity theorem the character of the holonomy is unique and deformations of the holonomy imply deformations of the complete hyperbolic structure on $\operatorname{int}(M)$. The second step is to associate generalized Dehn filling coefficients to the aforementioned deformation. Let us see this more closely:

For each boundary component T_j^2 of M, where $j = 1, \ldots, k$, fix two oriented simple closed curves μ_j and λ_j generating the fundamental group $\pi_1\left(T_j^2\right)$. The holonomy of μ_j and λ_j can be viewed as affine transformations of $\mathbb{C} = \partial \mathbb{H}^3 \setminus \{\infty\}$ (∞ being a point fixed by μ_j and λ_j). Then, one can introduce holomorphic parameters u_j and v_j, for each $j = 1, \ldots, k$, to be the branches of the logarithm of the linear part of the holonomy around μ_j and λ_j respectively. For $U \subset \mathbb{C}^k$ a neighborhood of the origin, associate to each $u \in U$ a point $\rho_u \in \mathcal{X}(M) = \operatorname{Hom}(\pi_1(M), \operatorname{SL}(2,\mathbb{C}))//\operatorname{SL}(2,\mathbb{C})$ in the $\operatorname{SL}(2,\mathbb{C})$-character variety; this can be done by considering an analytic section

$$s : V \subset \mathcal{X}(M) \to \operatorname{Hom}(\pi_1(M), \operatorname{SL}(2,\mathbb{C})),$$

such that $s(\chi_0) = \rho_0$, where ρ_0 is a lift of the holonomy representation of $\operatorname{int}(M)$ and $\chi_0 \in \mathcal{X}(M)$ its character. Then, one has the following important lemma:

Lemma 2.5 (Lemma B.1.6 in [8])**.** *For $j = 1, \ldots, k$, there is an analytic map $A_j : U \to SL(2,\mathbb{C})$ such that for every $u \in U$:*

$$\rho_u(\mu_j) = \varepsilon_j A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1}, \qquad \textit{with } \varepsilon_j = \pm 1,$$

while the commutativity between λ_j and μ_j implies the following:

Lemma 2.6 (Lemma B.1.7 in [8])**.** *There exist unique analytic functions $v_j, \tau_j : U \to \mathbb{C}$ such that $v_j(0) = 0$ and, for every $u \in U$,*

$$\rho_u(\lambda_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

(1) $\tau_j(0) \in \mathbb{C} - \mathbb{R}$
(2) $sinh(v_j/2) = \tau_j sinh(u_j/2)$
(3) v_j is odd in u_j and even in u_l, for $l \neq j$

(4) $v_j(u) = u_j\left(\tau_j(u) + O\left(|u|^2\right)\right)$.

We are finally set to define the *Dehn filling coefficients*:

Definition 2.7 (W. Thurston [89])**.** For $u \in U$ we define the *generalized Dehn filling coefficients* of the j-th cusp $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula

$$\begin{cases} (p_j, q_j) & = \infty, & \text{if } u_j = 0 \\ p_j u_j + q_j v_j & = 2\pi\sqrt{-1} & \text{if } u_j \neq 0. \end{cases}$$

These coefficients are well-defined and the map

$$\begin{aligned} U &\to S^2 \times \cdots \times S^2 \\ u &\mapsto ((p_1, q_1), \ldots, (p_k, q_k)) \end{aligned}$$

defines a homeomorphism between U and a neighborhood of $\{\infty, \ldots, \infty\}$.

Remark 2.8. If $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion at the j-th torus is a non-singular hyperbolic 3-manifold, which topologically is the Dehn filling with surgery meridian $p_j\mu_j + q_j\lambda_j$. One may also perform (p,q)-Dehn surgery also when p and q are not necessarily coprime integers; this refers to *orbifold Dehn surgery*, as in [22]. For instance, $(p,0)$-Dehn surgery on a knot $K \subset S^3$ provides an orbifold with base S^3 and singular set the knot K with cone angle $2\pi/p$.

The third step in the proof of Theorem 2.4 involves the construction of the developing maps with the given holonomies. In particular, for $D_0 : \widetilde{\mathrm{int}\,(M)} \to \mathbb{H}^3$, the developing map for the complete structure on $\mathrm{int}\,(M)$ with holonomy ρ_0, then for each $u \in U$ there is a developing map $D_u : \widetilde{\mathrm{int}\,(M)} \to \mathbb{H}^3$ with holonomy ρ_u, such that the completion of $\mathrm{int}\,(M)$ is given by the generalized Dehn filling coefficients of u; we refer to §B.1.3 of [8] for a full proof of the statement.

We remark here that the family of maps $\{D_u\}_{u\in U}$ is continuous on u in the compact $\mathcal{C}^1$-topology and that the result above shows not only the existence of a 1-parameter family of cone 3-manifold structures, but also gives a path of corresponding holonomies in the representation variety $\mathrm{Hom}\,(\pi_1(M), \mathrm{SL}\,(2,\mathbb{C}))$.

2.3. **Haken manifolds and Thurston's Uniformization.** The notion of a *Haken manifold* involves a large class of closed 3-manifolds which plays an important role in the study of the topology of 3-manifolds. These were introduced by Wolfgang Haken [41] as a class of compact irreducible 3-manifolds containing incompressible surfaces, for which he showed in [42] that they admit a hierarchy to a union of 3-balls by cutting along essential embedded surfaces. This property allows one to produce certain statements for Haken manifolds using an induction process. Let us next state these definitions more rigorously:

Definition 2.9. Let M be a 3-manifold. A properly embedded surface $\Sigma \subset M$ is called *incompressible* if the map between their fundamental groups $\pi_1(\Sigma) \to \pi_1(M)$ is injective. Otherwise, the surface is called *compressible* and a torus in an irreducible 3-manifold is compressible if and only if it bounds a solid torus.

Definition 2.10. A compact orientable 3-manifold M is called a *Haken manifold*, if it is irreducible and contains an orientable, incompressible surface $\Sigma \subset M$.

In [42], W. Haken associated a notion of complexity to a Haken manifold, which decreases when one cuts the Haken manifold along an incompressible 3-manifold; this can be iterated in order to reduce the complexity until we obtain 3-balls. This approach was a key ingredient in the proof of the Waldhausen theorem showing that closed Haken manifolds are topologically characterized by their fundamental groups:

Theorem 2.11 (F. Waldhausen, Corollary 6.5 in [90]). *Let M and M' be two Haken manifolds with $\pi_1(M) \to \pi_1(M')$. Then M and M' are diffeomorphic.*

An algorithm to determine whether a 3-manifold is Haken was given by W. Jaco and U. Oertel [49]. Thurston's studies on various examples of 3-manifolds admitting complete hyperbolic metrics lead to his proof of a "uniformization theorem" satisfied by this large class of Haken manifolds:

Theorem 2.12 (Uniformization Theorem for Haken manifolds [89]). *Any atoroidal Haken manifold M admits a hyperbolic structure. By atoroidal here is meant that any embedded incompressible torus is boundary parallel, that is, it can be isotoped into a boundary component of M.*

Thurston's proof was using the hierarchy property of Haken manifolds. By the Waldhausen theorem, (a Haken manifold) M can be decomposed into a finite sum of closed balls B^3 by incompressible surfaces; in other words, there exists a sequence of manifolds with boundary

$$M \mapsto M_1 \mapsto \ldots \mapsto B^3 \cup \ldots \cup B^3.$$

Then, starting with hyperbolic structures on the balls B^3 we may get a hyperbolic structure by gluing at each step in this sequence from these balls back to M. A full proof of this theorem was never published by W. Thurston; fairly detailed outlines of the proof can be found in the articles by J. Morgan [70] or C. Wall [91]. It also follows from G. Perelman's proof of the more general geometrization conjecture of Thurston constructing the Ricci flow with surgeries on 3-manifolds [72]; see also [4], [71].

The geometrization conjecture evolved from W. Thurston's considerations that a similar uniformization theorem as for Haken manifolds should hold for all closed 3-manifolds. An important fact considered was that non-Haken manifolds do not contain incompressible surfaces, thus it is impossible to decompose those into simpler pieces. One way by which Thurston proved that non-Haken atoroidal 3-manifolds can be equipped with a hyperbolic structure was by deforming the structure of a cone manifold by increasing its cone angle.

However, using hyperbolic Dehn surgery it is possible to obtain *non-Haken manifolds, whose hyperbolicity cannot be shown by the uniformization theorem.* Such examples are not easy to construct otherwise; see A. Reid [77] for explicit examples of non-Haken hyperbolic 3-manifolds with a finite cover which fibers over the circle. Moreover, deformations of hyperbolic structures can be described more concretely using the framework of hyperbolic Dehn surgery.

In [46] C. Hodgson and S. Kerckhoff established a universal upper bound on the number of non-hyperbolic Dehn surgeries per boundary torus, thus giving a quantitative version of Thurston's hyperbolic Dehn filling theorem; see also the later article of M. Lackenby and R. Meyerhoff [60] on the maximal number of exceptional Dehn surgeries, providing a proof to Gordon's conjecture [34] on the number of the exceptional slopes. For example, Dehn surgeries on the figure-eight knot produce non-Haken, hyperbolic 3-manifolds except in ten cases. For the exterior of the figure-eight knot in S^3 the exceptional surgeries, that is, the ones which do not result in a hyperbolic structure are

$$\{(1,0), (0,1), \pm(1,1), \pm(2,1), \pm(3,1), \pm(4,1)\}.$$

3. Higher Teichmüller Theory

The newly-emerged field of higher Teichmüller theory concerns the study of connected components of character varieties for semisimple real Lie groups, that are entirely consisted of discrete and faithful representations. We summarize here some of the very basic topological and geometric properties of these spaces, as well as a recent unified approach to the subject by O. Guichard and A. Wienhard, which seems to be identifying all those cases when such components are apparent.

3.1. **The holonomy principle (à la Ehresmann-Weil-Thurston).** The geometrization program of Thurston for 3-manifolds has been significantly influenced by the theory of locally homogeneous geometric structures on manifolds pioneered in Charles Ehresmann's article *Sur les espaces localement homogènes* published in 1936 [24]. This is where the origins of what we call today a (G,X)-structure on manifolds can be traced, for a Lie group G and a homogeneous space X. A (G,X)-manifold inherits all of the local geometry of X invariant under the action of the group G.

In [24] C. Ehresmann studies Riemannian manifolds of constant curvature, which are locally modeled on the n-Euclidean space, the n-sphere or the n-hyperbolic space, depending on whether the curvature is zero, positive or negative, respectively. These manifolds were called *Clifford-Klein space forms.* Later on, he also studies in [23] locally homogeneous structures on manifolds M which are not necessarily Riemannian. In this context, a fiber bundle with structure group G and fiber X is associated to a (G,X)-structure on M. This fiber bundle structure corresponds to what is today called a *Cartan connection*, and now the locally homogeneous structures on manifolds which are Riemannian, are precisely those Cartan connections which are, in fact, flat.

It is in this context that Thurston's geometrization program was suggesting that every closed 3-manifold can be canonically decomposed into pieces which have locally homogeneous Riemannian structures, each of one of eight local models. Deformations of the locally homogeneous structures of C. Ehresmann can be described by the following fundamental principle:

Theorem 3.1 (Ehresmann-Weil-Thurston Holonomy Principle, [89])**.** *Let X be a manifold and G a Lie group acting transitively on X. Let also M be a compact (G,X)-manifold with holonomy representation $\rho : \pi(M) \to G$. Then, the following hold:*

1. *If ρ' is sufficiently close to ρ in the space of representations $Hom(\pi_1(M), G)$, then there exists a (nearby) (G,X)-structure on M with holonomy representation ρ'.*
2. *If M' is a (G,X)-manifold near M with both M and M' having the same holonomy ρ, then there exists an isomorphism $i : M \xrightarrow{\cong} M'$ isotopic to the identity.*

As a Corollary to this theorem, for a closed manifold X, the set of holonomy representations of (G,X)-structures on M is open in $\mathrm{Hom}(\pi_1(M), G)$ with respect to the classical topology. The main ideas for these results were described in Thurston's notes [89]; full proofs regarding the holonomy principle can be found in the works of W. Lok [64], R. Canary-D. Epstein-P. Green [15] and W. Goldman [31]. A detailed historical survey on the ideas of C. Ehresmann and the holonomy principle can be found in W. Goldman's article [30].

The study of surgery techniques in certain subsets of $\mathrm{Hom}(\pi_1(M), G)/G$ of all group homomorphisms of the fundamental group $\pi(M)$ into G, up to conjugation by G, in the case when M is a closed connected and oriented topological *surface* of genus $g \geq 2$ will be the main subject of interest in the sequel of this chapter.

3.2. **The Teichmüller space.** Let Σ be a closed connected and oriented topological surface with negative Euler characteristic $\chi(\Sigma) = 2 - 2g < 0$, for g the genus of Σ. The *Teichmüller space* $\mathcal{T}(\Sigma)$ of the surface Σ is defined as the space of marked conformal classes of Riemannian metrics on Σ. The uniformization theorem of Riemann-Poincaré-Koebe (see [18] for a complete account) guarantees the existence of a unique hyperbolic metric with constant curvature -1 in each conformal class. The Teichmüller space can be thus identified with the moduli space of marked hyperbolic structures. Moreover, the mapping class group $\mathrm{Mod}(\Sigma)$, that is, the group of all diffeomorphisms of Σ modulo the ones which are isotopic to the identity, acts naturally on $\mathcal{T}(\Sigma)$ by changing the marking; this action is properly discontinuous and the quotient is the moduli space $\mathcal{M}(\Sigma)$ of Riemann surfaces of topological type given by Σ.

A well-known fact about the Teichmüller space is that it is homeomorphic to $\mathbb{R}^{6g-6}$. There are several ways to see this. One direct way is by parameterizing $\mathcal{T}(\Sigma)$ by Fenchel-Nielsen coordinates - a complete proof may be found in [76], Theorem 9.7.4. Another method is to use Teichmüller's theorem to identify $\mathcal{T}(\Sigma)$ with the unit ball in the vector space $Q(M)$ of holomorphic quadratic differentials on a Riemann surface M homeomorphic to Σ - a detailed proof can be found in [48], Theorem 7.2.1. In fact, $\mathcal{T}(\Sigma)$ can be identified with the entire vector space $Q(M)$ using Hopf differentials of harmonic maps from M to a Riemann surface of topological type given by Σ - see the article of M. Wolf [96] for this approach. An application of the Riemann-Roch theorem finally provides that $\dim_{\mathbb{R}} Q(M) = 6g - 6$, for genus $g \geq 2$; see, for example, Corollary 5.4.2 in [50] for a proof.

However, what opens the way from the classical Teichmüller theory to what is today called *Higher Teichmüller Theory* is the algebraic realization of the space $\mathcal{T}(\Sigma)$ as a subspace of the moduli space of representations of the fundamental group of Σ into the isometry group of the hyperbolic plane. This algebraic realization is conceived through the holonomy representation of a hyperbolic structure. Indeed, for (M, f) a hyperbolic structure over Σ, the orientation preserving homeomorphism $f : \Sigma \to M$ induces an isomorphism of fundamental groups $f_* : \pi_1(\Sigma) \to \pi_1(M)$ and $\pi_1(M)$ acts as the group of deck transformations by isometries on $\tilde{M} \cong \mathbb{H}^2$. But, since $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$, the orientation preserving isometries, it follows that this action induces a homomorphism $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$, which is well-defined up to conjugation by $\mathrm{PSL}(2, \mathbb{R})$. This homomorphism is called the *holonomy* of the hyperbolic structure (M, f). The *representation variety*

$$\mathcal{R}(\mathrm{PSL}(2, \mathbb{R})) := \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{R}))//\mathrm{PSL}(2, \mathbb{R})$$

is the largest Hausdorff quotient of all group homomorphisms $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ modulo conjugation by $\mathrm{PSL}(2, \mathbb{R})$. Furthermore, representations induced by equivalent hyperbolic structures using the above approach are conjugate by an element in $\mathrm{PSL}(2, \mathbb{R})$ and conversely; we can therefore realize

$$\mathcal{T}(\Sigma) \subset \mathcal{R}(\mathrm{PSL}(2, \mathbb{R})).$$

A. Weil in [93] (see also Theorem 6.19 in [75]) proved that the set of discrete such embeddings $\{\pi_1(\Sigma) \hookrightarrow \mathrm{PSL}(2, \mathbb{R})\}$ is open in the quotient space $\mathcal{R}(\mathrm{PSL}(2, \mathbb{R}))$. This open subset is called the *Fricke space* $\mathcal{F}(\Sigma)$ of the topological surface Σ. Fricke spaces first appeared in the work of R. Fricke and F. Klein [27] defined in terms of Fuchsian groups (see [3] for an expository account).

The connected components of the representation variety $\mathcal{R}(\mathrm{PSL}(2, \mathbb{R}))$ are distinguished in terms of the Euler class $e(\rho)$ of a representation ρ; such a topological invariant for a representation ρ can be considered in realm of the Riemann-Hilbert correspondence and the associated flat $\mathrm{PSL}(2, \mathbb{R})$-bundle.

In [33], W. Goldman showed that this Euler class distinguishes the connected components and takes values in $\mathbb{Z}\cap[\chi(\Sigma), -\chi(\Sigma)]$. In particular, the Fricke space $\mathcal{F}(\Sigma)$ identifies with the component maximizing this characteristic class (consisting of representations that correspond to holonomies of hyperbolic structures on Σ).

To conclude this discussion about the Teichmüller space, the uniformization theorem implies that $\mathcal{F}(\Sigma)$ and $\mathcal{T}(\Sigma)$ can be identified, therefore the Teichmüller space is a *connected component* of the representation variety $\mathcal{R}(\mathrm{PSL}(2,\mathbb{R}))$. In fact, it is one of the two connected components entirely consisting of discrete and faithful representations $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$; the other such component is $\mathcal{T}\left(\bar{\Sigma}\right)$, that is, the Teichmüller space of the surface $\bar{\Sigma}$ with the opposite orientation.

Since the representation variety can be considered for any reductive Lie group G, it is natural to ask whether there are special connected components of it for *higher rank Lie groups* G than $\mathrm{PSL}(2,\mathbb{R})$, which consist entirely of representations related to significant geometric or dynamical structures on the fixed topological surface. This question leads to the introduction of higher Teichmüller spaces as we shall see next.

3.3. **Higher Teichmüller spaces.** Let Σ be a closed oriented (topological) surface of genus g. The fundamental group of Σ is described by

$$\pi_1(\Sigma) = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod [a_i, b_i] = 1 \right\rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator. The set of all representations of $\pi_1(\Sigma)$ into a connected reductive real Lie group G, $\mathrm{Hom}(\pi_1(\Sigma), G)$, can be naturally identified with the subset of G^{2g} consisting of $2g$-tuples $(A_1, B_1, \ldots, A_g, B_g)$ satisfying the algebraic equation $\prod [A_i, B_i] = 1$. The group G acts on the space $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugation

$$(g \cdot \rho) = g\rho(\gamma)\, g^{-1},$$

where $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(\Sigma), G)$ and $\gamma \in \pi_1(\Sigma)$, and the restriction of this action to the subspace $\mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma), G)$ of reductive representations provides that the orbit space is Hausdorff. Here, by a reductive representation we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. When G is algebraic, this is equivalent to the Zariski closure of the image of $\pi_1(\Sigma)$ in G being a reductive group. Define the *moduli space of reductive representations of* $\pi_1(\Sigma)$ *into* G to be the orbit space

$$\mathcal{R}(G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma), G)/G$$

The following theorem of W. Goldman [32] provides this space is a real analytic variety and so $\mathcal{R}(G)$ is usually called the *character variety*:

Theorem 3.2 (W. Goldman [32])**.** *The moduli space $\mathcal{R}(G)$ has the structure of a real analytic variety, which is algebraic if G is algebraic and is a complex variety if G is complex.*

Higher Teichmüller Theory is concerned with the study of the properties of fundamental group representations lying in certain subsets of the character variety $\mathcal{R}(G)$, for simple real groups G. An abundance of methods from geometry, gauge theory, algebraic geometry and dynamics is used to approach these certain subsets, which is apparent (not only) due to the non-abelian Hodge theory for the moduli space $\mathcal{R}(G)$. The term *higher Teichmüller space* has been introduced by V. Fock and A. Goncharov in [25], who showed that there exist subsets entirely consisted of discrete and faithful representations in $\mathcal{R}(G)$ in the case when G is a split real Lie group. Today, the term refers to connected components having this property, but in a broader sense:

Definition 3.3. Let Σ be a closed connected oriented topological surface of genus $g \geq 2$ and G a semisimple real Lie group. A *higher Teichmüller space* is a connected component of the character variety $\mathcal{R}(G)$ that is entirely consisted of faithful representations with discrete image.

Several essential features of higher Teichmüller spaces can be traced back to the ideas and work of W. Thurston. For instance, Thurston's shear coordinates have been extended in this setting by F. Labourie and G. McShane [59] (see also [25]). Generalizations of the McShane identities for higher Teichmüller spaces were obtained by Y. Huang and Z. Sun in [47]; these are expressed in terms of simple root lengths, triple ratios and edge functions. I. Le in [61] gave a definition of a higher lamination in the spirit of Thurston for the space of framed G-local systems over Σ and showed that this coincides with the approach of V. Fock and A. Goncharov [25] as the tropical points of a higher Teichmüller space. Another example is the pressure metric for higher Teichmüller spaces from [11], [12], which can be viewed as a generalization of the Weil-Peterson metric on the Teichmüller space as seen by W. Thurston.

Examples, however, of such connected components appeared long before the term was invented. For an adjoint split real semisimple Lie group G, there exists a unique embedding $\pi : \mathrm{SL}(2,\mathbb{R}) \to G$, which is the associated Lie group homomorphism to a principal 3-dimensional subalgebra of $\mathfrak{g}$, Kostant's principal subalgebra $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{g}$ (see [53]). For a fixed discrete embedding $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$, N. Hitchin in [44] showed that the subspace containing $\pi \circ \iota : \pi_1(\Sigma) \to G$ is a connected component and, in fact, topologically trivial of dimension $(2g-2)\dim G$. In the special case when the group is $G = \mathrm{PSL}(2,\mathbb{R})$, this component is the Teichmüller space.

Following the work of N. Hitchin, it became apparent that the spaces identified, now called *Hitchin components*, include representations with important geometric features. For instance, F. Labourie introduced in [57] the notion of an *Anosov representation* and used techniques from dynamical systems to prove (among other essential geometric properties) that representations lying inside the component of Hitchin for $G = \mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$ or $\mathrm{PO}(n,n+1)$ are faithful with discrete image; we refer the reader to [11], [37], [38], [58], [59], [62], [73] for subsequent works on the geometric and dynamical properties of representations in the Hitchin components.

The second family of Lie groups G that components of discrete and faithful representations have been detected is the family of Hermitian Lie groups of non-compact type, that is, the symmetric space associated to G is an irreducible Hermitian symmetric space of non-compact type. In this case, a characteristic number called the *Toledo invariant* of a representation $\rho : \pi_1(\Sigma) \to G$ can be defined as the integer

$$T_\rho := \langle \rho^*(\kappa_G), [\Sigma] \rangle,$$

where $\rho^*(\kappa_G)$ is the pullback of the Kähler class $\kappa_G \in H_c^2(G,\mathbb{R})$ of G and $[\Sigma] \in H_2(\Sigma,\mathbb{R})$ is the orientation class. The absolute value of the Toledo invariant has an upper bound of Milnor-Wood type

$$|T_\rho| \leq (2g-2)\,\mathrm{rk}(G) \tag{3.4}$$

and a representation $\rho : \pi_1(\Sigma) \to G$ is called *maximal* when this upper bound is, in fact, achieved. Subspaces of maximal representations also have interesting geometric and dynamical properties and, in particular, are entirely consisted of discrete and faithful representations as seen in [13] and [14].

It is also interesting to note at this point that in the case when the group G is the group $\mathrm{PSL}\,(2,\mathbb{R})$, the Toledo invariant is actually the Euler class, inequality (3.4) is the Milnor-Wood inequality for the Euler class and the space of maximal representations in this case is identified with the Teichmüller space, as in [33].

We refer the reader to the survey articles of A. Wienhard [94] and B. Pozzetti [74] for a broader presentation of the geometric properties of higher Teichmüller spaces, as well as for an overview of the similarities and differences of these spaces compared to the Teichmüller space.

3.4. **Θ-positive representations.** The special connected components introduced for the two families of Lie groups above, namely the adjoint split real semisimple Lie groups and the Hermitian Lie groups on non-compact type share (among many other fundamental properties) a common characterization that relates to the existence of a continuous equivariant map sending positive triples in $\mathbb{RP}^1$ to positive triples in certain flag varieties associated to the Lie group G. This property was identified by F. Labourie [57], O. Guichard [37] and V. Fock-A. Goncharov [25] in the case of split semisimple real Lie groups, while by M. Burger-A. Iozzi-A. Wienhard [14] for Hermitian Lie groups of non-compact type.

This in turn provided the motivation to propose in [39] that the characterization above in terms of positivity can, in fact, distinguish *all* higher Teichmüller spaces. We next include more details about this general conjectural picture; for complete reference the reader is directed to the original article of O. Guichard and A. Wienhard [39].

The definition of a Θ-positive structure for a real semisimple Lie group G is a generalization of G. Lusztig's total positivity condition in [66] and is given in regards to properties of the Lie algebra of parabolic subgroups $P_\Theta < G$ defined by a subset of simple positive roots $\Theta \subset \Delta$. In these terms, let $\mathfrak{u}_\Theta := \sum_{\alpha\in\Sigma_\Theta^+} \mathfrak{g}_\alpha$, for $\Sigma_\Theta^+ = \Sigma^+ \backslash \mathrm{Span}\,(\Delta - \Theta)$, where Σ^+ denotes the set of positive roots, and then the standard parabolic subgroup P_Θ associated to $\Theta \subset \Delta$ is the normalizer in G of $\mathfrak{u}_\Theta$. Denote the Levi factor of P_Θ by L_Θ, which acts on $\mathfrak{u}_\Theta$ via the adjoint action, and by L_Θ^0 the component of L_Θ containing the identity.

For $\mathfrak{z}_\Theta$, the center of the Lie algebra $\mathfrak{l}_\Theta := \mathrm{Lie}\,(L_\Theta)$, $\mathfrak{u}_\Theta$ decomposes into weight spaces

$$\mathfrak{u}_\Theta = \sum_{\beta\in\mathfrak{z}_\Theta^*} \mathfrak{u}_\beta,$$

where $\mathfrak{u}_\beta := \{N \in \mathfrak{u}_\Theta \,|\mathrm{ad}\,(Z)\,N = \beta\,(Z)\,N, \text{ for every } Z \in \mathfrak{z}_\Theta\}$.

Definition 3.5. Let G be a semisimple Lie group with finite center and $\Theta \subset \Delta$ a subset of simple roots. The group G admits a Θ-*positive structure* if for all $\beta \in \Theta$, there exists an L_Θ^0-invariant sharp convex cone in $\mathfrak{u}_\beta$.

A central result in [39] provides that the semisimple Lie groups G that can admit a Θ-positive structure are classified as follows:

Theorem 3.6 (O. Guichard-A. Wienhard, Theorem 4.3 in [39])**.** *A semisimple Lie group G admits a Θ-positive structure if and only if the pair (G, Θ) belongs to one of the following four cases:*

(1) G is a split real form and $\Theta = \Delta$.
(2) G is a Hermitian symmetric Lie group of tube type and $\Theta = \{\alpha_r\}$.
(3) G is locally isomorphic to a group $SO\,(p,q)$, for $p \neq q$, and $\Theta = \{\alpha_1, \ldots, \alpha_{p-1}\}$.

(4) G is a real form of the groups F_4, E_6, E_7, E_8 with restricted root system of type F_4, and $\Theta = \{\alpha_1, \alpha_2\}$.

For $U_\Theta := \exp\left(_\Theta\right)$ and for P_Θ^{opp} the normalizer in G of $\mathfrak{u}_\Theta^{opp} := \sum_{\alpha\in\Sigma_\Theta^+} \mathfrak{g}_{-\alpha}$, one may consider positive triples in the generalized flag variety G/P_Θ as follows:

Definition 3.7. Fix E_Θ and F_Θ to be the standard flags in G/P_Θ such that $\mathrm{Stab}_G\left(F_\Theta\right) = P_\Theta$ and $\mathrm{Stab}_G\left(E_\Theta\right) = P_\Theta^{opp}$. For any $S_\Theta \in G/P_\Theta$ transverse to F_Θ, there exists $u_{S_\Theta} \subset U_\Theta$ such that $S_\Theta = u_{S_\Theta} E_\Theta$. The triple $\left(E_\Theta, S_\Theta, F_\Theta\right)$ in the generalized flag variety G/P_Θ will be called Θ-*positive*, if $u_{S_\Theta} \in U_\Theta^{>0}$, for $U_\Theta^{>0}$ the Θ-positive semigroup of U_Θ (see p.11 in [39] for the precise notion).

The definition of a Θ-positive fundamental group representation is now the following:

Definition 3.8 (O. Guichard-A. Wienhard [39])**.** Let Σ be a closed connected and oriented topological surface of genus $g \geq 2$ and let G be a semisimple Lie group admitting a Θ-positive structure. A representation of the fundamental group of Σ into G will be called Θ-*positive*, if there exists a ρ-equivariant positive map $\xi : \partial\pi_1\left(\Sigma\right) = \mathbb{RP}^1 \to G/P_\Theta$ sending positive triples in $\mathbb{RP}^1$ to Θ-positive triples in G/P_Θ .

A positive response to the following conjecture would imply that higher Teichmüller spaces emerge only as subsets of the character varieties for the four families of Lie groups G listed in Theorem 3.6 above:

Conjecture 3.9 (Conjecture 5.4 in [39])**.** *For Σ and G as above, any Θ-positive representation $\rho : \pi_1\left(\Sigma\right) \to G$ is P_Θ-Anosov. The set of Θ-positive representations $\rho : \pi_1\left(\Sigma\right) \to G$ is open and closed in the character variety $\mathcal{R}\left(G\right)$.*

4. Non-abelian Hodge theory

A major contribution to the various methods available in order to study higher Teichmüller spaces involves fixing a complex structure J on the topological surface Σ, thus transforming this to a Riemann surface $X = (\Sigma, J)$, therefore opening the way to holomorphic techniques and the theory of *Higgs bundles*, as initiated by Nigel Hitchin in his article *The self duality equations on a Riemann surface* published in 1987 [45]. The non-abelian Hodge theory correspondence provides a real-analytic isomorphism between the character variety $\mathcal{R}\left(G\right)$ and the moduli space of polystable G-Higgs bundles, which we briefly introduce next.

4.1. **Moduli spaces of G-Higgs bundles.**

Let X be a compact Riemann surface and let G be a real reductive group. The latter involves considering *Cartan data* (G, H, θ, B), where $H \subset G$ is a maximal compact subgroup, $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution and B is a non-degenerate bilinear form on $\mathfrak{g}$, which is $\mathrm{Ad}\left(G\right)$-invariant and θ-invariant. The Cartan involution θ gives a decomposition (called the *Cartan decomposition*)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

into its ± 1-eigenspaces, where $\mathfrak{h}$ is the Lie algebra of H.

Let $H^{\mathbb{C}}$ be the complexification of H and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ be the complexification of the Cartan decomposition. The adjoint action of G on $\mathfrak{g}$ restricts to give a representation (the isotropy representation) of H on $\mathfrak{m}$. This is independent of the choice of Cartan decomposition, since any two Cartan decompositions of G are related by a conjugation using also that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, while the same is true for the complexified isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$. This introduces the following definition:

Definition 4.1. Let $K \cong T^*X$ be the canonical line bundle over a compact Riemann surface X. A *G-Higgs bundle* is a pair (E, φ) where

- E is a principal holomorphic $H^{\mathbb{C}}$-bundle over X and
- φ is a holomorphic section of the vector bundle $E\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K = \left(E \times_\iota \mathfrak{m}^{\mathbb{C}}\right) \otimes K$.

The section φ is called the *Higgs field*. Two G-Higgs bundles (E, φ) and (E', φ') are said to be *isomorphic* if there is a principal bundle isomorphism $E \cong E'$ which takes φ to φ' under the induced isomorphism $E\left(\mathfrak{m}^{\mathbb{C}}\right) \cong E'\left(\mathfrak{m}^{\mathbb{C}}\right)$.

To define a moduli space of G-Higgs bundles we need to consider a notion of semistability, stability and polystability. These notions are defined in terms of an antidominant character for a parabolic subgroup $P \subseteq H^{\mathbb{C}}$ and a holomorphic reduction σ of the structure group of the bundle E from $H^{\mathbb{C}}$ to P (see [29] for the precise definitions).

When the group G is connected, principal $H^{\mathbb{C}}$-bundles E are topologically classified by a characteristic class $c(E) \in H^2\left(X, \pi_1\left(H^{\mathbb{C}}\right)\right) \cong \pi_1\left(H^{\mathbb{C}}\right) \cong \pi_1(H) \cong \pi_1(G)$.

Definition 4.2. For a fixed class $d \in \pi_1(G)$, the *moduli space of polystable G-Higgs bundles* with respect to the group of complex gauge transformations is defined as the set of isomorphism classes of polystable G-Higgs bundles (E, φ) such that $c(E) = d$. We will denote this set by $\mathcal{M}_d(G)$.

Using the general GIT constructions of A. Schmitt for decorated principal bundles in the case of a real form of a complex reductive algebraic Lie group, it is shown that the moduli space $\mathcal{M}_d(G)$ is an algebraic variety. The expected dimension of the moduli space of G-Higgs bundles is $(g-1)\dim G^{\mathbb{C}}$, in the case when G is a connected semisimple real Lie group; see [29], [80], [81] for details.

4.2. ***G*-Hitchin equations.** Let (E, φ) be a G-Higgs bundle over a compact Riemann surface X. By a slight abuse of notation we shall denote the underlying smooth objects of E and φ by the same symbols. The Higgs field can be thus viewed as a $(1,0)$-form $\varphi \in \Omega^{1,0}\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$. Given a reduction h of structure group to H in the smooth $H^{\mathbb{C}}$-bundle E, we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E. Let $\tau_h : \Omega^{1,0}\left(E\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \to \Omega^{0,1}\left(E\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$ be defined by the compact conjugation of $\mathfrak{g}^{\mathbb{C}}$ which is given fiberwise by the reduction h, combined with complex conjugation on complex 1-forms. The next theorem was proved in [29] for an arbitrary reductive real Lie group G.

Theorem 4.3 (Hitchin-Kobayashi correspondence, Theorem 3.21 in [29])**.** *There exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = 0$$

if and only if (E, φ) is polystable.

From the point of view of moduli spaces it is convenient to fix a C^∞ principal H-bundle $\mathbf{E}_H$ with fixed topological class $d \in \pi_1(H)$ and study the moduli space of solutions to Hitchin's equations for a pair (A, φ) consisting of an H-connection A and $\varphi \in \Omega^{1,0}\left(X, \mathbf{E}_H\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$ with

$$\begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0 \end{aligned} \qquad (*)$$

where d_A is the covariant derivative associated to A and $\bar{\partial}_A$ is the $(0,1)$-part of d_A, defining the holomorphic structure on $\mathbf{E}_H$. Also, τ is defined by the fixed reduction of structure group

$\mathbf{E}_H \hookrightarrow \mathbf{E}_H\left(H^{\mathbb{C}}\right)$. The gauge group $\mathcal{G}_H$ of $\mathbf{E}_H$ acts on the space of solutions by conjugation and the moduli space of solutions is defined by

$$\mathcal{M}_d^{\text{gauge}}(G) := \{(A,\varphi) \text{ satisfying equations } (*)\}/\mathcal{G}_H .$$

Now, Theorem 4.3 implies that there is a homeomorphism

$$\mathcal{M}_d(G) \cong \mathcal{M}_d^{\text{gauge}}(G) .$$

Using the one-to-one correspondence between H-connections on $\mathbf{E}_H$ and $\bar{\partial}$-operators on $\mathbf{E}_{H^{\mathbb{C}}}$, the homeomorphism in the above theorem can be interpreted as saying that in the $\mathcal{G}_H^{\mathbb{C}}$-orbit of a polystable G-Higgs bundle $\left(\bar{\partial}_{E_0},\varphi_0\right)$ we can find another Higgs bundle $\left(\bar{\partial}_E,\varphi\right)$ whose corresponding pair (d_A,φ) satisfies the equation $F_A - [\varphi,\tau(\varphi)] = 0$, and this is unique up to H-gauge transformations.

4.3. **The non-abelian Hodge correspondence.** We can assign a topological invariant to a representation $\rho \in \mathcal{R}(G)$ by considering its corresponding flat G-bundle on Σ defined as $E_\rho = \tilde{\Sigma}\times_\rho G$. Here $\tilde{\Sigma} \to \Sigma$ is the universal cover and $\pi_1(\Sigma)$ acts on G via ρ. A topological invariant is then given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G) \simeq \pi_1(H)$, for $H \subseteq G$ a maximal compact subgroup of G. For a fixed $d \in \pi_1(G)$ the moduli space of reductive representations with fixed topological invariant d is now defined as the subvariety

$$\mathcal{R}_d(G) := \{[\rho] \in \mathcal{R}(G)\,|\,c(\rho) = d\} .$$

A reductive fundamental group representation corresponds to a solution to the Hitchin equations. This is seen using that any solution h to Hitchin's equations defines a flat reductive G-connection

$$D = D_h + \varphi - \tau(\varphi) , \tag{4.4}$$

where D_h is the unique H-connection on E compatible with its holomorphic structure. Conversely, given a flat reductive connection D on a G-bundle E_G, there exists a harmonic metric, in other words, a reduction of structure group to $H \subset G$ corresponding to a harmonic section of $E_G/H \to X$. This reduction produces a solution to Hitchin's equations such that Equation 4.4 holds.

In summary, equipping the surface Σ with a complex structure J, a reductive representation of $\pi_1(\Sigma)$ into G corresponds to a polystable G-Higgs bundle over the Riemann surface $X = (\Sigma, J)$; this is the content of *non-abelian Hodge correspondence*; its proof is based on combined work by N. Hitchin [45], C. Simpson [83], [85], S. Donaldson [20] and K. Corlette [17]:

Theorem 4.5 (Non-abelian Hodge correspondence)**.** *Let G be a connected semisimple real Lie group with maximal compact subgroup $H \subseteq G$ and let $d \in \pi_1(G) \simeq \pi_1(H)$. Then there exists a homeomorphism*

$$\mathcal{R}_d(G) \cong \mathcal{M}_d(G) .$$

The introduction of holomorphic techniques via the non-abelian Hodge correspondence allows the description of a theory of higher Teichmüller spaces from the Higgs bundle point of view. In [10], the authors obtain a parameterization of special components of the moduli space of Higgs bundles on a compact Riemann surface using the decomposition data for a complex simple Lie algebra $\mathfrak{g}$. The possible decompositions of $\mathfrak{g}$ are defined by a newly introduced class of $\mathfrak{sl}(2,\mathbb{R})$-triples and the classification of these triples is shown to be in bijection with the classification of the Θ-positive structures of O. Guichard and A. Wienhard (Theorem 3.6). We refer to [10] for the precise statements; see also the survey article of O. García-Prada [28] for a broader description of the results for higher Teichmüller spaces that can be obtained using the theory of Higgs bundles.

5. Surgeries in representation varieties-General theory

We next describe a gluing construction for points lying inside the moduli spaces appearing in the non-abelian Hodge correspondence. In particular, this technique can be used to obtain specific model objects of the moduli spaces which is hard to be constructed otherwise. Such models can help improve our understanding of the geometric properties of the subsets of the character variety they live in.

5.1. **Topological gluing construction.** For a closed oriented surface Σ of genus g, let $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ be a decomposition of Σ along one simple closed oriented separating geodesic into two subsurfaces, say Σ_l and Σ_r. Let now $\rho_l : \pi_1(\Sigma) \to G$ and $\rho_r : \pi_1(\Sigma) \to G$ be two representations into a semisimple Lie group G.

One could amalgamate the restriction of ρ_l to Σ_l with the restriction of ρ_r to Σ_r, however the holonomies of those along γ do not have to agree a priori. However, if the holonomies do agree (possibly after applying a deformation of at least one of the two representations for the holonomies to match up), then one can introduce new representations by gluing with a use of the van Kampen theorem at the level of topological surfaces, as follows.

Definition 5.1. A *hybrid representation* is defined as the amalgamated representation

$$\rho := \rho_l \big|_{\pi_1(\Sigma_l)} * \rho_r \big|_{\pi_1(\Sigma_r)} : \pi_1(\Sigma) \simeq \pi_1(\Sigma_l) *_{\langle\gamma\rangle} \pi_1(\Sigma_r) \to G.$$

Remark 5.2. The assumption that the holonomies agree over the boundary is crucial. In §3.3.1 of [40] O. Guichard and A. Wienhard provide an explicit example of hybrid representations in the case when the group is the symplectic group $\mathrm{Sp}(4,\mathbb{R})$. Special attention is paid there in order to establish this assumption via an appropriate deformation argument.

The above construction/definition can be generalized to the case when the subsurfaces Σ_l and Σ_r are not necessarily connected (cf. §3.3.2 of [40]). For Σ as earlier, let $\Sigma_1 \subset \Sigma$ denote a subsurface with Euler characteristic $\chi(\Sigma_1) \leq -1$. The (nonempty) boundary of Σ_1 is a union of disjoint circles

$$\partial\Sigma_1 = \coprod_{d \in \pi_0(\partial\Sigma_1)} \gamma_d.$$

The circles γ_d are oriented so that for each d, the surface Σ_1 lies on the left of γ_d. Now, write

$$\Sigma \backslash \partial\Sigma_1 = \bigcup_{c \in \pi_0(\Sigma \backslash \partial\Sigma_1)} \Sigma_c.$$

Then, for any $d \in \pi_0(\partial\Sigma_1)$, the curve γ_d bounds exactly two connected components of $\Sigma\backslash\partial\Sigma_1$, namely, one is included in Σ_1 and denoted by $\Sigma_{l(d)}$ with $l(d) \in \pi_0(\Sigma_1)$, while the other is included in the complement of Σ_1 and is denoted by $\Sigma_{r(d)}$ with $r(d) \in \pi_0(\Sigma\backslash\Sigma_1)$. This way, we have $l(d), r(d) \in \pi_0(\Sigma\backslash\partial\Sigma_1)$, but it can be that $l(d) = l(d')$ or that $r(d) = r(d')$, for $d \neq d'$.

Assume now that the graph with vertex set $\pi_0(\Sigma\backslash\Sigma_1)$ and edges given by the pairs $\{l(d), r(d)\}_{d \in \pi_0(\partial\Sigma_1)}$ is a tree. This allows us to apply a generalized van Kampen theorem argument and write the fundamental group $\pi_1(\Sigma)$ as the amalgamated product of the groups $\pi_1(\Sigma_c)$, for all $c \in \pi_0(\Sigma\backslash\partial\Sigma_1)$ over the groups $\pi_1(\gamma_d)$, for all $d \in \pi_0(\partial\Sigma_1)$. The assumption about the graph being a tree guarantees that no HNN-extension appears in the construction of the fundamental group.

Now choose representations $\rho_c : \pi_1(\Sigma) \to G$ for every $c \in \pi_0(\Sigma_1)$ and consider their restrictions $\tilde{\rho}_c := \rho_c \left|_{\pi_1(\Sigma_c)}\right.$. Assuming that is is possible to choose elements $g_c \in G$ for each $c \in \pi_0(\Sigma \backslash \partial\Sigma_1)$ such that for every $d \in \pi_0(\partial\Sigma_1)$ the following condition on the holonomies is satisfied

$$g_{l(d)}\rho_{l(d)}(\gamma_d)\, g_{l(d)}^{-1} = g_{r(d)}\rho_{r(d)}(\gamma_d)\, g_{r(d)}^{-1},$$

then one may construct a hybrid representation $\rho : \pi_1(\Sigma) \to G$ by amalgamating the representations $g_c\rho_c g_c^{-1}$, for each $c \in \pi_0(\Sigma \backslash \partial\Sigma_1)$.

5.2. **Gluing in exceptional components of the moduli space.** Motivated by the amalgamation construction for representations and in the realm of the non-abelian Hodge correspondence, one may seek for an analogous gluing construction from a holomorphic point of view. The benefit from establishing this method in the Higgs bundle moduli space is that it is easier to compute the Higgs bundle invariants for any models constructed in order to identify in which connected component these new objects lie. Indeed, for the cases when the Lie group is the group $\mathrm{Sp}(4,\mathbb{R})$ or $\mathrm{SO}(p,p+1)$ the moduli space has a number of exceptional components in terms of their topological and geometric properties; these exceptional components do, in fact, fall in the class of higher Teichmüller spaces. It is for such components that a gluing construction for Higgs bundles can provide good models that are not easily obtained otherwise, thus allowing us to study more closely the components themselves. Examples of models in the case of the group $\mathrm{Sp}(4,\mathbb{R})$ were obtained in [55] (see also [54]), while for $G = \mathrm{SO}(p,p+1)$ we will demonstrate some examples in §6.

5.2.1. *Parabolic $GL(n,\mathbb{C})$-Higgs bundles.* Remember that the amalgamation method involved fundamental group representations defined over a surface with boundary. The appropriate analog to a surface group representation into a reductive Lie group G for a surface with boundary is a *parabolic G-Higgs bundle* over a Riemann surface with a divisor. This involves an extra layer of structure encoded by a weighted filtration on each fiber of the bundle over a collection of finitely-many distinct points of the surface. We include next basic definitions for a parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle; concrete examples of such pairs will be studied later on in §6.

Parabolic vector bundles over Riemann surfaces with marked points were introduced by C. Seshadri in [82] and similar to the Narasimhan-Seshadri correspondence, there is an analogous correspondence between stable parabolic bundles and unitary representations of the fundamental group of the punctured surface with fixed holonomy class around each puncture [68]. Later on, C. Simpson in [84] proved a non-abelian Hodge correspondence over a *non-compact* curve.

Definition 5.3. Let X be a closed, connected, smooth Riemann surface of genus $g \geq 2$ with s-many marked points $x_1, \ldots, x_s$ and let a divisor $D = \{x_1, \ldots, x_s\}$. A *parabolic vector bundle* E over X is a holomorphic vector bundle $E \to X$ of rank n with *parabolic structure* at each $x \in D$ (*weighted flag* on each fiber E_x):

$$\begin{gathered} E_x = E_{x,1} \supset E_{x,2} \supset \ldots \supset E_{x,r(x)+1} = \{0\} \\ 0 \leq \alpha_1(x) < \ldots < \alpha_{r(x)}(x) < 1. \end{gathered}$$

The real numbers $\alpha_i(x) \in [0,1)$ for $1 \leq i \leq r(x)$ are called the *weights* of the subspaces E_x and we usually write (E,α) to denote a parabolic vector bundle equipped with a parabolic

structure determined by a system of weights $\alpha(x) = \left(\alpha_1(x), \ldots, \alpha_{r(x)}(x)\right)$ at each $x \in D$; whenever the system of weights is not discussed in the context, we will be omitting the notation α to ease exposition. Moreover, let $k_i(x) = \dim(E_{x,i}/E_{x,i+1})$ denote the *multiplicity* of the weight $\alpha_i(x)$ and notice that $\sum_i k_i(x) = n$. A weighted flag shall be called *full*, if $k_i(x) = 1$ for every $1 \leq i \leq r(x)$ and every $x \in D$.
The *parabolic degree* and *parabolic slope* of a vector bundle equipped with a parabolic structure are the real numbers

$$\operatorname{par}\deg(E) = \deg E + \sum_{x \in D} \sum_{i=1}^{r(x)} k_i(x)\, \alpha_i(x),$$

$$\operatorname{par}\mu(\mathrm{E}) = \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)}.$$

Definition 5.4. Let K be the canonical bundle over X and E a parabolic vector bundle. The bundle morphism $\Phi : E \to E \otimes K(D)$ will be called a *parabolic Higgs field*, if it preserves the parabolic structure at each point $x \in D$:

$$\Phi|_x(E_{x,i}) \subset E_{x,i} \otimes K(D)|_x .$$

In particular, we call Φ *strongly parabolic*, if

$$\Phi|_x(E_{x,i}) \subset E_{x,i+1} \otimes K(D)|_x ,$$

in other words, Φ is a meromorphic endomorphism valued 1-form with simple poles along the divisor D, whose residue at $x \in D$ is nilpotent with respect to the filtration.

After these considerations we define parabolic Higgs bundles as follows.

Definition 5.5. Let K be the canonical bundle over X and E a parabolic vector bundle over X. A *parabolic Higgs bundle* is a pair (E, Φ), where E is a parabolic vector bundle and $\Phi : E \to E \otimes K(D)$ is a strongly parabolic Higgs field.

Analogously to the non-parabolic case, we may define a notion of stability as follows:

Definition 5.6. A parabolic Higgs bundle will be called *stable* (resp. *semistable*) if for every Φ-invariant parabolic subbundle $F \leq E$ it is $\operatorname{par}\mu(F) < \operatorname{par}\mu(E)$ (resp. $\leq$). Furthermore, it will be called *polystable* if it is the direct sum of stable parabolic Higgs bundles of the same parabolic slope.

5.3. **Complex connected sum of Riemann surfaces.** In order to describe how two parabolic Higgs bundles can be glued to a (non-parabolic) Higgs bundle, the first step is to glue their underlying surfaces with boundary as follows.

Take annuli $\mathbb{A}_1 = \{z \in \mathbb{C}\,|r_1 < |z| < R_1\}$ and $\mathbb{A}_2 = \{z \in \mathbb{C}\,|r_2 < |z| < R_2\}$ on the complex plane, and consider the Möbius transformation $f_\lambda : \mathbb{A}_1 \to \mathbb{A}_2$ with $f_\lambda(z) = \frac{\lambda}{z}$, where $\lambda \in \mathbb{C}$ with $|\lambda| = r_2 R_1 = r_1 R_2$. This is a conformal biholomorphism (equivalently bijective, angle-preserving and orientation-preserving) between the two annuli and the continuous extension of the function $z \mapsto |f_\lambda(z)|$ to the closure of $\mathbb{A}_1$ reverses the order of the boundary components.

Let two compact Riemann surfaces X_1, X_2 of respective genera g_1, g_2. Choose points $p \in X_1$, $q \in X_2$ and local charts around these points $\psi_i : U_i \to \Delta(0, \varepsilon_i)$ on X_i, for $i = 1, 2$. Now fix positive real numbers $r_i < R_i < \varepsilon_i$ such that the following two conditions are satisfied:

- $\psi_i^{-1}\left(\overline{\Delta(0,R_i)}\right) \cap U_j \neq \varnothing$, for every $U_j \neq U_i$ from the complex atlas of X_i. In other words, we are considering an annulus around each of the p and q contained entirely in the neighborhood of a single chart, and
- $\frac{R_2}{r_2} = \frac{R_1}{r_1}$.

Set now

$$X_i^* = X_i \backslash \psi_i^{-1}\left(\overline{\Delta(0,r_i)}\right).$$

Choosing the biholomorphism $f_\lambda : \mathbb{A}_1 \to \mathbb{A}_2$ as above, f_λ is used to glue the two Riemann surfaces X_1, X_2 along the inverse image of the annuli $\mathbb{A}_1, \mathbb{A}_2$ on the surfaces, via the biholomorphism

$$g_\lambda : \Omega_1 = \psi_1^{-1}(\mathbb{A}_1) \to \Omega_2 = \psi_2^{-1}(\mathbb{A}_2)$$

with $g_\lambda = \psi_2^{-1} \circ f_\lambda \circ \psi_1$.

Define $X_\lambda = X_1 \#_\lambda X_2 = X_1^* \coprod X_2^* / \sim$, where the gluing of Ω_1 and Ω_2 is performed through the equivalence relation which identifies $y \in \Omega_1$ with $w \in \Omega_2$ iff $w = g_\lambda(y)$. For collections of s-many distinct points D_1 on X_1 and D_2 on X_2, this procedure is assumed to be taking place for annuli around each pair of points (p,q) for $p \in D_1$ and $q \in D_2$.

If X_1, X_2 are orientable and orientations are chosen for both, since f_λ is orientation preserving we obtain a natural orientation on the connected sum $X_1 \# X_2$ which coincides with the given ones on X_1^* and X_2^*.

Therefore, $X_\# = X_1 \# X_2$ is a Riemann surface of genus g_1+g_2+s-1, the *complex connected sum*, where g_i is the genus of the X_i and s is the number of points in D_1 and D_2. Its complex structure however is heavily dependent on the parameters p_i, q_i, λ.

5.4. **Gluing at the level of solutions to Hitchin's equations.** For gluing two parabolic G-Higgs bundles over a complex connected sum $X_\#$ of Riemann surfaces, we choose to switch to the language of solutions to Hitchin's equations and make use of the analytic techniques of C. Taubes for gluing instantons over 4-manifolds [88] in order to control the stability condition. These techniques have been applied to establish similar gluing constructions for solutions to gauge-theoretic equations, as for instance in [21], [26], [43], [79], and pertain first to finding good *local model solutions* of the gauge-theoretic equations. Then one has to put, using appropriate gauge transformations, the initial data into these model forms, which are identified locally over annuli around the marked points, thus allowing a construction of a new pair over $X_\#$ that combines the original data from X_1 and X_2. This produces, however, an *approximate solution* of the equations, which then has to be corrected to an exact solution via a gauge transformation. The argument providing the existence of such a gauge is translated into a Banach fixed point theorem argument and involves the study of the linearization of a relevant elliptic operator. We briefly describe these steps in the sequel; for complete proofs we refer to [54] and [55].

5.4.1. *The local model.* Local SL(2,$\mathbb{R}$)-model solutions to the Hitchin equations can be obtained by studying the behavior of the harmonic map between a surface X with a given complex structure and the surface X with the corresponding Riemannian metric of constant curvature -4, under degeneration of the domain Riemann surface X to a nodal surface; cf. [87], [95].

For a stable SL(2,$\mathbb{R}$)-Higgs bundle (E,Φ) on X with $E = L \oplus L^{-1}$ for L a holomorphic square root of the canonical line bundle over X endowed with an auxiliary hermitian metric h_0, and $\Phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \in H^0(X, \mathfrak{sl}(E))$ for q a holomorphic quadratic differential, there is an induced hermitian metric $H_0 = h_0 \oplus h_0^{-1}$ on E and $A = A_L \oplus A_L^{-1}$ the associated Chern connection with respect to h. The stability condition implies that there exists a complex gauge transformation g unique up to unitary gauge transformations, such that $(A_{1,s}, \Phi_{1,s}) := g^* (A, \Phi)$ is a solution to the Hitchin equations. Calculations in [87] considering the hermitian metric on L and a complex gauge giving rise to an exact solution $(A_{1,s}, \Phi_{1,s})$ of the self-duality equations imply that

$$A_{1,s} = O\left(|\zeta|^s\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{d\zeta}{\zeta} - \frac{d\bar{\zeta}}{\bar{\zeta}} \right), \ \Phi_{1,s} = \left(1 + O\left(|\zeta|^s\right)\right) \begin{pmatrix} 0 & \frac{s}{2} \\ \frac{s}{2} & 0 \end{pmatrix} \frac{d\zeta}{i\zeta}$$

for local coordinates ζ. Therefore, after a unitary change of frame, the Higgs field $\Phi_{1,s}$ is asymptotic to the model Higgs field $\Phi_s^{\text{mod}} = \begin{pmatrix} \frac{s}{2} & 0 \\ 0 & -\frac{s}{2} \end{pmatrix} \frac{d\zeta}{i\zeta}$, while the connection $A_{1,s}$ is asymptotic to the trivial flat connection.

In conclusion, the *model solution* to the SL(2,$\mathbb{R}$)-Hitchin equations we will be considering is described by

$$A^{\text{ mod }} = 0, \ \Phi^{\text{ mod }} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z}$$

over a punctured disk with z-coordinates around the puncture with the condition that $C \in \mathbb{R}$ with $C \neq 0$ and that the meromorphic quadratic differential $q := \det \Phi^{\text{ mod }}$ has at least one simple zero. That this is indeed the generic case, is discussed in [67].

5.4.2. *Approximate solutions of the SL(2,$\mathbb{R}$)-Hitchin equations.* Let X be a compact Riemann surface and $D := \{p_1, \ldots, p_s\}$ be a collection of s-many distinct points on X. Moreover, let (E, h) be a hermitian vector bundle on E. Choose an initial pair $\left(A^{\text{ mod }}, \Phi^{\text{ mod }}\right)$ on E, such that in some unitary trivialization of E around each point $p \in D$, the pair coincides with the local model from §5.4.1; of course, on the interior of each region $X \setminus \{p\}$ the pair $\left(A^{\text{ mod }}, \Phi^{\text{ mod }}\right)$ need not satisfy the Hitchin equations.

One can then define *global Sobolev spaces on* X as the spaces of admissible deformations of the model unitary connection and the model Higgs field $\left(A^{\text{ mod }}, \Phi^{\text{ mod }}\right)$ and introduce the moduli space $\mathcal{M}\left(X^\times\right)$ of solutions to the Hitchin equations modulo unitary gauge transformation, which are close to the model solution over a punctured Riemann surface $X^\times := X - D$ for some fixed parameter $C \in \mathbb{R}$; this moduli space was explicitly constructed by H. Konno in [52] as a hyperkähler quotient.

In fact, as was shown by O. Biquard and P. Boalch (Lemma 5.3 in [5]) and later improved by J. Swoboda (Lemma 3.2 in [87]), a pair $(A, \Phi) \in \mathcal{M}\left(X^\times\right)$ is asymptotically close to the model $\left(A^{\text{mod}}, \Phi^{\text{mod}}\right)$ near each puncture in D. In particular, there exists a complex gauge transformation $g = \exp(\gamma)$, such that $g^* (A, \Phi)$ coincides with $\left(A_p^{\text{mod}}, \Phi_p^{\text{mod}}\right)$ on a sufficiently small neighborhood of the point p, for each $p \in D$.

We shall now use this complex gauge transformation as well as a smooth cut-off function to obtain an approximate solution to the SL(2,$\mathbb{R}$)-Hitchin equations. For fixed local coordinates z around each puncture p and the positive function $r = |z|$ around the puncture, fix a constant $0 < R < 1$ and choose a smooth cut-off function $\chi_R : [0, \infty) \to [0, 1]$ with $\text{supp}\chi \subseteq [0, R]$ and $\chi_R(r) = 1$ for $r \leq \frac{3R}{4}$. We impose the further requirement on the growth rate of this cut-off function:

$$|r\partial_r \chi_R| + \left|(r\partial r)^2 \chi_R\right| \leq k \tag{5.7}$$

for some constant k not depending on R.

The map $x \mapsto \chi_R(r(x)) : X^\times \to \mathbb{R}$ gives rise to a smooth cut-off function on the punctured surface $X^\times$ which by a slight abuse of notation we shall still denote by χ_R. We may use this function χ_R to glue the two pairs (A, Φ) and $\left(A_p^{\text{mod}}, \Phi_p^{\text{mod}}\right)$ into an *approximate solution*

$$\left(A_R^{app}, \Phi_R^{app}\right) := \exp\left(\chi_R \gamma\right)^* (A, \Phi).$$

The pair $\left(A_R^{app}, \Phi_R^{app}\right)$ is a smooth pair and is by construction an exact solution of the Hitchin equations away from each punctured neighborhood $\mathcal{U}_p$, while it coincides with the model pair $\left(A_p^{\text{mod}}, \Phi_p^{\text{mod}}\right)$ near each puncture. More precisely, we have:

$$\left(A_R^{app}, \Phi_R^{app}\right) = \begin{cases} (A, \Phi), \text{ over } X \setminus \bigcup\limits_{p \in D} \left\{z \in \mathcal{U}_p \left| \frac{3R}{4} \leq |z| \leq R\right.\right\} \\ \left(A_p^{\text{mod}}, \Phi_p^{\text{mod}}\right), \text{ over } \left\{z \in \mathcal{U}_p \left| 0 < |z| \leq \frac{3R}{4}\right.\right\}, \text{ for each } p \in D. \end{cases}$$

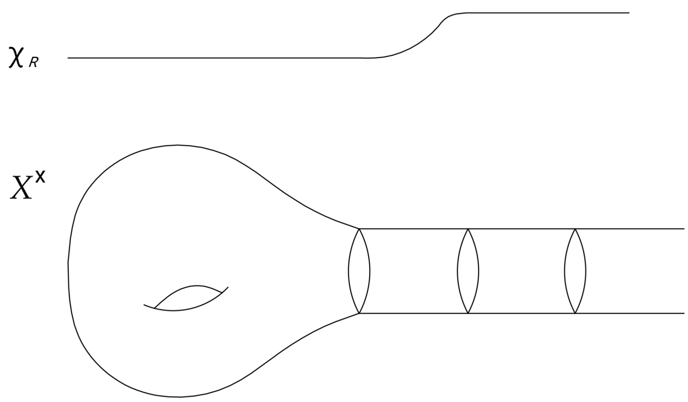

Figure 1. Constructing an approximate solution over the punctured surface $X^\times$.

Since $\left(A_R^{app}, \Phi_R^{app}\right)$ is complex gauge equivalent to an exact solution (A, Φ) of the Hitchin equations, the Higgs field Φ_R^{app} is holomorphic with respect to the holomorphic structure $\bar{\partial}_{A_R^{app}}$, in other words, one has $\bar{\partial}_{A_R^{app}} \Phi_R^{app} = 0$. Moreover, assumption (5.7) on the growth rate of the bump function χ_R provides us with a good estimate of the error up to which $\left(A_R^{app}, \Phi_R^{app}\right)$ satisfies the first among the Hitchin equations, $F(A) + [\Phi, \Phi^*] = 0$.

5.5. **Approximate solutions to the G-Hitchin equations.** We now wish to obtain an approximate G-Higgs pair by extending the SL(2,$\mathbb{C}$)-data via an embedding

$$\phi : \text{SL}(2,\mathbb{R}) \hookrightarrow G,$$

for a reductive Lie group G. It is important that copies of a maximal compact subgroup of SL(2,$\mathbb{R}$) are mapped via ϕ into copies of a maximal compact subgroup of G and that the norm of the infinitesimal deformation ϕ_* on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ satisfies a Lipschitz condition. Assuming that this is indeed the case for an embedding ϕ (examples can be found in [55] and will be demonstrated in §6), one gets by extension via the embedding ϕ a $G^{\mathbb{C}}$-pair satisfying the G-Hitchin equations up to an error, which we have good control of.

For $i = 1, 2$, let X_i be a closed Riemann surface of genus g_i and let $D_1 = \{p_1, \ldots, p_s\}$, $D_2 = \{q_1, \ldots, q_s\}$ a divisor of s-many distinct points on X_1 (X_2 respectively). Choose local coordinates z near the points in D_1 and local coordinates w near the points in D_2. Assume that we get via an embedding as was described above approximate solutions (A_1, Φ_1), (A_2, Φ_2), which agree over neighborhoods around the points in the divisors D_1 and D_2, with $A_1 = A_2 = 0$ and with $\Phi_1(z) = -\Phi_2(w)$. Then, there is a suitable frame for the connections over which the hermitian metrics are both described by the identity matrix and so they are constant in particular. Set $\left(A_{p,q}^{\mathrm{mod}}, \Phi_{p,q}^{\mathrm{mod}}\right) := \left(A_{1,p}^{\mathrm{mod}}, \Phi_{1,p}^{\mathrm{mod}}\right) = -\left(A_{2,q}^{\mathrm{mod}}, \Phi_{2,q}^{\mathrm{mod}}\right)$. We can glue the pairs $(A_1, \Phi_1), (A_2, \Phi_2)$ together to get an *approximate solution* of the G-Hitchin equations over the complex connected sum $X_\# := X_1 \# X_2$:

$$\left(A_R^{app}, \Phi_R^{app}\right) := \begin{cases} (A_1, \Phi_1), & \text{over } X_1 \backslash X_2 \\ \left(A_{p,q}^{\mathrm{mod}}, \Phi_{p,q}^{\mathrm{mod}}\right), & \text{over } \Omega \text{ around each pair of points } (p,q) \\ (A_2, \Phi_2) & \text{over } X_2 \backslash X_1. \end{cases}$$

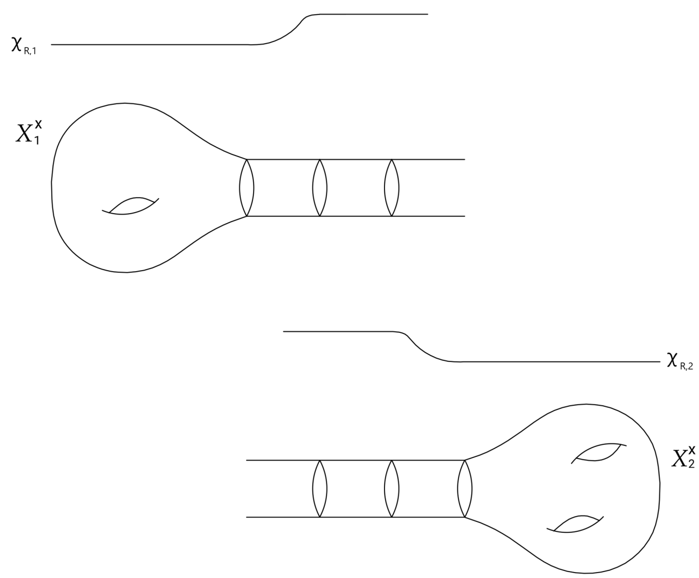

Figure 2. Constructing approximate solutions over $X_1^\times$ and $X_2^\times$.

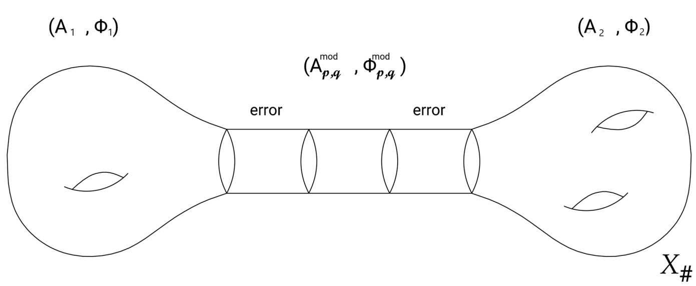

Figure 3. $(A_R^{app}, \Phi_R^{app})$ over the complex connected sum $X_\#$.

By construction, $\left(A_R^{app}, \Phi_R^{app}\right)$ is a smooth pair on $X_\#$, complex gauge equivalent to an exact solution of the Hitchin equations by a smooth gauge transformation defined over all of $X_\#$. It satisfies the second Hitchin equation (holomorphicity), while the first equation is satisfied up to an error which we have good control of.

5.6. **The contraction mapping argument.** A standard strategy, due largely to C. Taubes [88], for correcting an approximate solution to an exact solution of gauge-theoretic equations involves studying the linearization of a relevant elliptic operator. In the Higgs bundle setting, the linearization of the Hitchin operator was first described in [67] and furthermore in [87] for solutions to the SL(2,$\mathbb{C}$)-self duality equations over a nodal surface. We are going to use this analytic machinery to correct our approximate solution to an exact solution over the complex connected sum of Riemann surfaces. We next summarize this strategy.

Let G be a connected, semisimple Lie group. For the complex connected sum $X_\#$ consider the nonlinear G-Hitchin operator at a pair $(A, \Phi) \in \Omega^1\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \oplus \Omega^{1,0}\left(X_\#, E_H\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$:

$$\mathcal{H}\left(A, \Phi\right) = \left(F\left(A\right) - \left[\Phi, \tau\left(\Phi\right)\right], \bar{\partial}_A \Phi\right).$$

Moreover, consider the orbit map

$$\gamma \mapsto \mathcal{O}_{(A,\Phi)}\left(\gamma\right) = g^*\left(A, \Phi\right) = \left(g^* A, g^{-1}\Phi g\right),$$

for $g = \exp\left(\gamma\right)$ and $\gamma \in \Omega^0\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right)$, where $H \subset G$ is a maximal compact subgroup.

Therefore, correcting the approximate solution $\left(A_R^{app}, \Phi_R^{app}\right)$ to an exact solution of the G-Hitchin equations accounts to finding a point γ in the complex gauge orbit of $\left(A_R^{app}, \Phi_R^{app}\right)$, for which $\mathcal{H}\left(g^*\left(A_R^{app}, \Phi_R^{app}\right)\right) = 0$. However, since we have seen that the second equation is satisfied by the pair $\left(A_R^{app}, \Phi_R^{app}\right)$ and since the condition $\bar{\partial}_A \Phi = 0$ is preserved under the action of the complex gauge group $\mathcal{G}_H^{\mathbb{C}}$, we actually seek a solution γ to the following equation

$$\mathcal{F}_R\left(\gamma\right) := pr_1 \circ \mathcal{H} \circ \mathcal{O}_{\left(A_R^{app}, \Phi_R^{app}\right)}\left(\exp(\gamma)\right) = 0.$$

For a Taylor series expansion of this operator

$$\mathcal{F}_R\left(\gamma\right) = pr_1 \mathcal{H}\left(A_R^{app}, \Phi_R^{app}\right) + L_{\left(A_R^{app}, \Phi_R^{app}\right)}\left(\gamma\right) + Q_R\left(\gamma\right),$$

where Q_R includes the quadratic and higher order terms in γ, we can then see that $\mathcal{F}_R\left(\gamma\right) = 0$ if and only if γ is a fixed point of the map

$$\begin{aligned} T : H_B^2\left(X_\#\right) &\to H_B^2\left(X_\#\right) \\ \gamma &\mapsto -G_R\left(\mathcal{H}\left(A_R^{app}, \Phi_R^{app}\right) + Q_R(\gamma)\right), \end{aligned}$$

where we denoted $G_R := L^{-1}_{\left(A_R^{app}, \Phi_R^{app}\right)}$ and $H_B^2\left(X_\#\right)$ is the Hilbert space defined by

$$H_B^2\left(X_\#\right) := \left\{\gamma \in L^2\left(X_\#\right) \middle| \nabla_B \gamma, \nabla_B^2 \gamma \in L^2\left(X_\#\right)\right\},$$

for a fixed background connection ∇_B defined as a smooth extension to $X_\#$ of the model connection $A_{p,q}^{\mathrm{mod}}$ over the cylinder for each pair of points (p, q).

The problem then reduces to showing that the mapping T is a contraction of the open ball B_{ρ_R} of radius ρ_R in $H_B^2\left(X_\#\right)$, since then from Banach's fixed point theorem there will exist a unique γ such that $T\left(\gamma\right) = \gamma$, in other words, such that $\mathcal{F}_R\left(\gamma\right) = 0$. In particular, one needs to show that:

(1) T is a contraction defined on B_{ρ_R} for some ρ_R, and
(2) T maps B_{ρ_R} to B_{ρ_R}.

In order to complete the above described contraction mapping argument, we need to show the following:

i: The linearized operator at the approximate solution $L_{\left(A_R^{app},\Phi_R^{app}\right)}$ is invertible.

ii: There is an upper bound for the inverse operator $G_R = L^{-1}_{\left(A_R^{app},\Phi_R^{app}\right)}$ as an operator $L^2\left(rdrd\theta\right) \to L^2\left(rdrd\theta\right)$.

iii: There is an upper bound for the inverse operator $G_R = L^{-1}_{\left(A_R^{app},\Phi_R^{app}\right)}$ also when viewed as an operator $L^2\left(rdrd\theta\right) \to H_B^2\left(X_\#, rdrd\theta\right)$.

iv: We can control a Lipschitz constant for Q_R, that means there exists a constant $C > 0$ such that

$$\|Q_R\left(\gamma_1\right) - Q_R\left(\gamma_0\right)\|_{L^2} \leq C\rho\|\gamma_1 - \gamma_0\|_{H_B^2}$$

for all $0 < \rho \leq 1$ and $\gamma_0, \gamma_1 \in B_\rho$, the closed ball of radius ρ around 0 in $H_B^2\left(X_\#\right)$.

Once these estimates are computed, one can correct the approximate solution constructed into an exact solution of the G-Hitchin equations.

Theorem 5.8. *There exists a constant $0 < R_0 < 1$, and for every $0 < R < R_0$ there exist a constant $\sigma_R > 0$ and a unique section $\gamma \in H_B^2\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right)$ satisfying $\|\gamma\|_{H_B^2\left(X_\#\right)} \leq \sigma_R$, so that, for $g = \exp\left(\gamma\right)$,*

$$\left(A_\#, \Phi_\#\right) = g^*\left(A_R^{app}, \Phi_R^{app}\right)$$

is an exact solution of the G-Hitchin equations over the closed surface $X_\#$.

Theorem 5.8 now implies that for $\bar{\partial} := A_\#^{0,1}$, the Higgs bundle $\left(E_\# := \left(\mathbb{E}_\#, \bar{\partial}\right), \Phi_\#\right)$ is a polystable G-Higgs bundle over the complex connected sum $X_\#$. Collecting the steps from the previous subsections one has the following:

Theorem 5.9. *Let X_1 be a closed Riemann surface of genus g_1 and $D_1 = \{p_1, \ldots, p_s\}$ be a collection of s-many distinct points on X_1. Let also G be a subgroup of $GL\left(n,\mathbb{C}\right)$. Consider respectively a closed Riemann surface X_2 of genus g_2 and a collection of also s-many distinct points $D_2 = \{q_1, \ldots, q_s\}$ on X_2. Let $\left(E_1, \Phi_1\right) \to X_1$ and $\left(E_2, \Phi_2\right) \to X_2$ be parabolic polystable G-Higgs bundles with corresponding solutions to the Hitchin equations $\left(A_1, \Phi_1\right)$ and $\left(A_2, \Phi_2\right)$. Assume that these solutions agree with model solutions $\left(A_{1,p_i}^{\mathrm{mod}}, \Phi_{1,p_i}^{\mathrm{mod}}\right)$ and $\left(A_{2,q_j}^{\mathrm{mod}}, \Phi_{2,q_j}^{\mathrm{mod}}\right)$ near the points $p_i \in D_1$ and $q_j \in D_2$, and that the model solutions satisfy $\left(A_{1,p_i}^{\mathrm{mod}}, \Phi_{1,p_i}^{\mathrm{mod}}\right) = -\left(A_{2,q_j}^{\mathrm{mod}}, \Phi_{2,q_j}^{\mathrm{mod}}\right)$, for s-many possible pairs of points $\left(p_i, q_j\right)$. Then there is a polystable G-Higgs bundle $\left(E_\#, \Phi_\#\right) \to X_\#$, constructed over the complex connected sum of Riemann surfaces $X_\# = X_1 \# X_2$, which agrees with the initial data over $X_\# \backslash X_1$ and $X_\# \backslash X_2$.*

Definition 5.10. We call an G-Higgs bundle constructed by the procedure developed above a *hybrid G-Higgs bundle.*

5.7. **Topological invariants.** In order to identify the connected component of the moduli space $\mathcal{M}\left(G\right)$ a hybrid Higgs bundle lies, one needs to look at how the Higgs bundle topological invariants behave under the complex connected sum operation. The next two propositions show that there is an additivity property for topological invariants over the connected sum operation, both from the Higgs bundle and the surface group representation point of view.

When the group G is a subgroup of $\mathrm{GL}\left(n,\mathbb{C}\right)$, the data of a parabolic G-Higgs bundle (defined in full generality in [6]) reduce to the data of a parabolic Higgs bundle as seen in

§5.2.1. Moreover, the basic topological invariant of a parabolic (resp. non-parabolic) pair is the parabolic degree (resp. degree) of some underlying parabolic (resp. non-parabolic) bundle in the Higgs bundle data. We refer to [56] for a detailed description of this data and the corresponding topological invariants for a number of cases of parabolic G-Higgs bundles.

The following proposition now describes an additivity property for the degrees:

Proposition 5.11 (Proposition 8.1 in [55]). *Let $X_\# = X_1 \# X_2$ be the complex connected sum of two closed Riemann surfaces X_1 and X_2 with divisors D_1 and D_2 of s-many distinct points on each surface, and let V_1, V_2 be parabolic vector bundles over X_1 and X_2 respectively. Then, if the parabolic bundles V_1, V_2 glue to a bundle $V_1 \# V_2$ over $X_\#$, the following identity holds*

$$\deg\left(V_1 \# V_2\right) = par\deg\left(V_1\right) + par\deg\left(V_2\right).$$

Considering the connected sum of the underlying topological surfaces $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ along a loop γ, a notion of Toledo invariant is defined for representations over these subsurfaces with boundary; see [14] for a detailed definition in this context. Moreover, the authors in [14] have established an additivity property for the Toledo invariant over a connected sum of surfaces. In particular:

Proposition 5.12 (Proposition 3.2 in [14]). *If $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ is the connected sum of two subsurfaces Σ_i along a simple closed separating loop γ, then*

$$T_\rho = T_{\rho_1} + T_{\rho_2},$$

where $\rho_i = \rho\left|_{\pi_1(\Sigma_i)}\right.$, for $i = 1, 2$.

The above propositions allow one to determine the topological invariants of the hybrid Higgs bundles, respectively fundamental group representations from the topological invariants of the underlying objects that were deformed and glued together. Note, in particular, that this property implies that the amalgamated product of two maximal representations is again a maximal representation defined over the compact surface Σ.

6. Examples: Model Higgs bundles in exceptional components of orthogonal groups

We now exhibit specific examples where the previous gluing construction can provide model objects lying inside higher Teichmüller spaces of particular geometric importance.

When the Lie group is $G = \mathrm{Sp}(4, \mathbb{R})$, hybrid Higgs bundles in the exceptional connected components of the maximal $G = \mathrm{Sp}(4, \mathbb{R})$-Higgs bundles identified by P. Gothen in [36] were obtained in [55]. We next provide such examples in the case of the group $G = \mathrm{SO}(p, p+1)$, which involves an extra parameter compared to the $\mathrm{Sp}(4, \mathbb{R})$-case; note, however, that a maximality property is not apparent in this case (apart from when $p = 2$).

6.1. **$\mathbf{SO}(p, q)$-Higgs bundle data.** The connected components of the $\mathrm{SO}(p, q)$-character variety $\mathcal{R}(\mathrm{SO}(p, q))$ can be more explicitly described using the theory of Higgs bundles. Let X be a compact Riemann surface with underlying topological surface Σ. Under the non-abelian Hodge correspondence, fundamental group representations into the group $\mathrm{SO}(p, q)$ correspond to holomorphic tuples (V, Q_V, W, Q_W, η) over X, where:

- (V, Q_V) and (W, Q_W) are holomorphic orthogonal bundles of rank p and q respectively with the additional condition that $\wedge^p(V) \cong \wedge^q(W)$.
- $\eta : W \to V \otimes K$ is a holomorphic section of $\mathrm{Hom}(W, V) \otimes K$.

Using Higgs bundle methods, in particular a real valued proper function defined by the L^2-norm of the Higgs field and a natural holomorphic $\mathbb{C}^*$-action, the authors in [2] classify *all* polystable local minima of the Hitchin function in $\mathcal{M}\left(\mathrm{SO}\left(p,q\right)\right)$, for $2<p\leq q$. For these moduli spaces, not all local minima occur at fixed points of the $\mathbb{C}^*$-action and additional connected components of $\mathcal{M}\left(\mathrm{SO}\left(p,q\right)\right)$ emerge by constructing a map

$$\Psi:\mathcal{M}_{K^p}\left(\mathrm{SO}\left(1,q-p+1\right)\right)\times\bigoplus_{j=1}^{p-1}H^0\left(X,K^{2j}\right)\to\mathcal{M}\left(\mathrm{SO}\left(p,q\right)\right),$$

which is an isomorphism onto its image, open and closed. In the description above, the term $\mathcal{M}_{K^p}\left(\mathrm{SO}\left(1,q-p+1\right)\right)$ denotes the moduli space of K^p-twisted $\mathrm{SO}\left(1,q-p+1\right)$-Higgs bundles on the Riemann surface X, where K is the canonical line bundle over X, and $\bigoplus\limits_{j=1}^{p-1}H^0\left(X,K^{2j}\right)$ denotes the vector space of holomorphic differentials of degree $2j$. Note that a K^p-twisted $\mathrm{SO}\left(1,n\right)$-Higgs bundle is defined by a triple $\left(I,\hat{W},\hat{\eta}\right)$, where $\left(\hat{W},Q_{\hat{W}}\right)$ is a rank n orthogonal bundle, $I=\wedge^n\hat{W}$ and $\hat{\eta}\in H^0\left(\mathrm{Hom}\left(\hat{W},I\right)\otimes K^p\right)$. A point in the image of the map Ψ is then described by

$$\Psi\left(\left(I,\hat{W},\hat{\eta}\right),q_2,\ldots,q_{2p-2}\right)=\left(V,W,\eta\right),\tag{6.1}$$

where

$$\begin{aligned}V&:=I\otimes\left(K^{p-1}\oplus K^{p-3}\oplus\ldots\oplus K^{3-p}\oplus K^{1-p}\right),\\ W&:=\hat{W}\oplus I\otimes\left(K^{p-2}\oplus K^{p-4}\oplus\ldots\oplus K^{4-p}\oplus K^{2-p}\right),\\ \eta&:=\begin{pmatrix}\hat{\eta}&q_2&q_4&\cdots&q_{2p-2}\\0&1&q_2&\cdots&q_{2p-4}\\\vdots&&\ddots&&\vdots\\\vdots&&&1&q_2\\0&0&\cdots&0&1\end{pmatrix}.\end{aligned}\tag{6.2}$$

Moreover, an $\mathrm{SO}\left(p,q\right)$-Higgs bundle $\left(V,W,\eta\right)$ is (poly)stable if and only if the K^p-twisted $\mathrm{SO}\left(1,n\right)$-Higgs bundle $\left(I,\hat{W},\hat{\eta}\right)$ is (poly)stable (see Lemma 4.4 in [2]).

The case when $q=p+1$ is even more special, because the relevant K^p-twisted $\mathrm{O}\left(q-p+1\right)$-Higgs bundles in the pre-image of Ψ are now rank 2 orthogonal bundles. In this case, when the first Stiefel-Whitney class $w_1\left(\hat{W},Q_{\hat{W}}\right)$ vanishes, then the structure group of $\hat{W}$ reduces to $\mathrm{SO}\left(2,\mathbb{C}\right)\cong\mathbb{C}^*$ and thus

$$\left(\hat{W},Q_{\hat{W}}\right)\cong\left(M\oplus M^{-1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right),$$

for a degree d holomorphic line bundle $M\in\mathrm{Pic}^d\left(X\right)$, while for stability reasons d is an integer in the interval $[0,p\left(2g-2\right)]$. This degree is a new topological invariant, which distinguishes extra components of the moduli space $\mathcal{M}\left(\mathrm{SO}\left(p,p+1\right)\right)$, and in [16] is proven the following:

Theorem 6.3 (Theorem 4.1 in [16]). *For each integer $d\in\left(0,p\left(2g-2\right)-1\right]$ there is a smooth connected component $\mathcal{R}_d\left(SO\left(p,p+1\right)\right)$ of the moduli space $\mathcal{R}\left(SO\left(p,p+1\right)\right)$, which does not contain representations with compact Zariski closure.*

Since all points in these $p\,(2g-2)-1$ many components are smooth, all corresponding fundamental group representations are *irreducible representations*. In fact, these representations are conjectured in [16] to have Zariski dense image. For this reason we shall call these components *exceptional* to distinguish them among the rest of the components of the character varieties $\mathcal{R}\,(\mathrm{SO}\,(p,q))$ that are not detected by the fixed points of the $\mathbb{C}^*$-action.

Definition 6.4. The connected components of the moduli space $\mathcal{M}\,(\mathrm{SO}\,(p,p+1))$, which are smooth, will be called the *exceptional components* of the moduli space $\mathcal{M}\,(\mathrm{SO}\,(p,p+1))$.

For each integer $0 < d \leq p\,(2g-2)-1$, the Higgs bundles (V,W,η) in the exceptional components are described by the map Ψ from (6.1) as follows:

$$(V,Q_V)=\left(K^{p-1}\oplus K^{p-3}\oplus\ldots\oplus K^{3-p}\oplus K^{1-p},\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right),$$

$$(W,Q_W)=\left(M\oplus K^{p-2}\oplus K^{p-4}\oplus\ldots\oplus K^{4-p}\oplus K^{2-p}\oplus M^{-1},\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right), \tag{6.5}$$

$$\eta=\begin{pmatrix} 0 & 0 & \ldots & 0 & \nu\\ 1 & q_2 & q_4 & \cdots & q_{2p-2}\\ 0 & 1 & q_2 & & q_{2p-4}\\ 0 & 0 & \ddots & & \vdots\\ \vdots & & \ddots & 1 & q_2\\ 0 & 0 & \ldots & 0 & \mu \end{pmatrix}: V\to W\otimes K,$$

for $M\in \mathrm{Pic}^d\,(X)$, and sections $\mu\in H^0\left(M^{-1}K^p\right)\setminus\{0\}$ and $\nu\in H^0\,(MK^p)$ with $0\neq\mu\neq\lambda\nu$.

In the case when $d=p\,(2g-2)$, then (V,W,η) lies in the *Hitchin component* of $\mathcal{M}\,(\mathrm{SO}\,(p,p+1))$ with data

$$(V,Q_V)=\left(K^{p-1}\oplus K^{p-3}\oplus\ldots\oplus K^{3-p}\oplus K^{1-p},\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right),$$

$$(W,Q_W)=\left(K^{p}\oplus K^{p-2}\oplus K^{p-4}\oplus\ldots\oplus K^{4-p}\oplus K^{2-p}\oplus K^{-p},\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right),$$

$$\eta=\begin{pmatrix} q_2 & q_4 & \ldots & q_{2p-2} & q_{2p}\\ 1 & q_2 & q_4 & \cdots & q_{2p-2}\\ 0 & 1 & q_2 & & q_{2p-4}\\ 0 & 0 & \ddots & & \vdots\\ \vdots & & \ddots & 1 & q_2\\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}: V\to W\otimes K.$$

6.2. **Hitchin equations for orthogonal groups.** The moduli space of polystable $\mathrm{SO}(p,q)$-Higgs bundles is alternatively viewed as the moduli space of polystable pairs $\left(\bar{\partial}_E, \Phi\right)$ modulo the gauge group $\mathcal{G}(\mathbf{E})$, where $\bar{\partial}_E$ is a Dolbeault operator on a principal $\mathrm{SO}(p,\mathbb{C}) \times \mathrm{SO}(q,\mathbb{C})$-bundle $\mathbf{E}$ and $\Phi \in \Omega^{1,0}\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$ satisfying $\bar{\partial}_E(\Phi) = 0$, for the (-1)-eigenspace $\mathfrak{m}$ in the Cartan decomposition of the Lie algebra of the group $\mathrm{SO}(p,q)$.

For the principal $\mathrm{SO}(p,\mathbb{C}) \times \mathrm{SO}(q,\mathbb{C})$-bundle $\mathbf{E}$ equipped with a Dolbeault operator $\bar{\partial}_E$, the gauge group

$$\mathcal{G}(\mathbf{E}) \cong \Omega^0\left(E_{\mathrm{SO}(p,\mathbb{C})}(\mathrm{SO}(p,\mathbb{C}))\right) \times \Omega^0\left(E_{\mathrm{SO}(q,\mathbb{C})}(\mathrm{SO}(q,\mathbb{C}))\right)$$

acts on the operators $\bar{\partial}_E$ by conjugation, where $\mathbf{E} = E_{\mathrm{SO}(p,\mathbb{C})} \times E_{\mathrm{SO}(q,\mathbb{C})}$. Now a Dolbeault operator on $\mathbf{E}$ corresponds to a connection A on the reduction V of $\mathbf{E}$ to $\mathrm{SO}(p,\mathbb{C}) \times \mathrm{SO}(q,\mathbb{C})$ and consider a Higgs field $\Phi \in \Omega^{1,0}\left(V\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$.

The group $G = \mathrm{SO}(p,q)$ is a real form of $\mathrm{SO}(p+q,\mathbb{C})$. It coincides with the compact real form when $p = q = 0$ and with the split real form when $p = q$ for $p+q$ even, or when $q = p+1$ for $p+q$ odd. Matrix conjugation $\tau(X) = \bar{X}$ defines the compact real form; indeed, we check

$$\begin{aligned}\mathfrak{so}(p+q) &= \left\{X \in \mathfrak{so}(p+q,\mathbb{C}) \,\middle|\, X = \bar{X}\right\} \\ &= \left\{X \in \mathfrak{so}(p+q,\mathbb{R}) \,\middle|\, X + X^T = 0\right\}.\end{aligned}$$

If we locally write $\Phi = \varphi dz$, then a calculation shows that

$$[\Phi, \tau(\Phi)] = \begin{pmatrix} -\varphi\varphi^* - \bar{\varphi}\varphi^T & \\ & -\varphi^T\bar{\varphi} - \varphi^*\varphi \end{pmatrix}.$$

The Hitchin-Kobayashi correspondence for $G = \mathrm{SO}(p,q)$ provides that if an $\mathrm{SO}(p,q)$-Higgs bundle (V, Q_V, W, Q_W, η) is polystable, then and only then the pair (A,Φ) as considered above satisfies the Hitchin equation

$$\begin{cases} F_A - [\Phi, \tau(\Phi)] = 0 \\ \quad \bar{\partial}_A(\Phi) = 0, \end{cases}$$

where F_A denotes the curvature of the unique connection compatible with the structure group reduction and the holomorphic structure. For a local description of the connection $A = (A_1, A_2)$ the equation $F_A - [\Phi, \tau(\Phi)] = 0$ becomes

$$\begin{aligned} F_{A_1} + \varphi\varphi^* + \bar{\varphi}\varphi^T &= F_{A_1} + 2\mathrm{Re}\left(\varphi\varphi^*\right) = 0 \\ F_{A_2} + \varphi^T\bar{\varphi} + \varphi^*\varphi &= F_{A_2} + 2\mathrm{Re}\left(\varphi^T\bar{\varphi}\right) = 0. \end{aligned}$$

6.3. **Model parabolic $\mathbf{SL}(2,\mathbb{R})$-Higgs bundles.** Parabolic $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles corresponding via the non-abelian Hodge correspondence to Fuchsian representations of the fundamental group of a punctured surface into the group $\mathrm{PSL}(2,\mathbb{R})$ were first identified by I. Biswas, P. Arés-Gastesi and S. Govindarajan in [7]; see also the article of G. Mondello [69] for a complete topological description of the relevant representation space. We next investigate these pairs more closely.

Let $D = \{x_1, \ldots, x_s\}$ be a finite collection of s-many points on a closed genus g Riemann surface X, such that $2g-2+s > 0$. Let K denote the canonical line bundle over the Riemann surface X. Consider the pair (E,Φ), where:

(1) $E := (L \otimes \iota)^* \oplus L$,
where L is a line bundle with $L^2 = K$ and $\iota := \mathcal{O}_X(D)$ denotes the line bundle over the divisor D; we equip the bundle E with a parabolic structure given by a trivial flag $E_{x_i} \supset \{0\}$ and weight $\frac{1}{2}$ for every $1 \leq i \leq s$.

(2) $\Phi := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in H^0\left(X, \mathrm{End}\left(E\right) \otimes K \otimes \iota\right)$.

Then, the pair (E, Φ) is a stable parabolic $\mathrm{SL}\left(2, \mathbb{R}\right)$-Higgs bundle with parabolic degree $\mathrm{pardeg}(E) = 0$. Therefore, from the non-abelian Hodge correspondence on non-compact curves [84], the vector bundle E supports a tame harmonic metric; the local estimate for this hermitian metric on E restricted to the line bundle L is

$$r^{\frac{1}{2}} |\log r|^{\frac{1}{2}},$$

for $r = |z|$. Indeed, if $\beta \in \mathbb{R}$ denotes in general the weights in the filtration of the filtered local system F corresponding to a parabolic Higgs bundle with weights α, for $0 \leq \alpha < 1$, then, if W_k is the span of vectors of weights $\leq k$, the weight filtration of $\mathrm{Res}_x\left(F\right)$ describes the behavior of the tame harmonic map under the local estimate

$$C r^{\beta} |\log r|^{\frac{k}{2}}.$$

In our case, the weight is $\alpha = \frac{1}{2} = \beta$ and the residue at each point $x_i \in D$ is $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, an upper triangular 2×2 nilpotent matrix. Thus, its weight filtration is $W_{-2} = 0$, $W_{-1} = W_0 = \mathrm{Im}\left(N\right) = \ker\left(N\right)$, and $W_1 =$ the whole space. Therefore, in the notation of C. Simpson from [84] we have that $L \subset W_1$ and $L \not\subset W_0 = W_{-1}$, while the hermitian metric on the line bundle L is locally

$$r^{\alpha} |\log r|^{\frac{k}{2}} = r^{\frac{1}{2}} |\log r|^{\frac{1}{2}}.$$

For the parabolic dual $(L \otimes \iota)^*$, the weight is by construction equal to $1 - \frac{1}{2}$ and in the weight filtration for the residue it is $(L \otimes \iota)^* \subset W_{-1}$ and $L \not\subset W_1$. Thus, the hermitian metric on $(L \otimes \iota)^*$ is locally

$$r^{\alpha} |\log r|^{\frac{k}{2}} = r^{1-\frac{1}{2}} |\log r|^{-\frac{1}{2}} = r^{\frac{1}{2}} |\log r|^{-\frac{1}{2}}.$$

In conclusion, the metric on $\mathrm{Hom}\left(L, (L \otimes \iota)^*\right)$ is induced by the restricted tame harmonic metric of E on the line bundles L and $(L \otimes \iota)^*$, as a section of $L^* \otimes (L \otimes \iota)^*$ and is locally described by

$$r^{-\frac{1}{2}} |\log r|^{-\frac{1}{2}} \cdot r^{\frac{1}{2}} |\log r|^{-\frac{1}{2}} = |\log r|^{-1},$$

for $r = |z|$. Subsequently, the metric on the tangent bundle L^{-2} is locally

$$r^{-\frac{1}{2}} |\log r|^{-\frac{1}{2}} \cdot r^{-\frac{1}{2}} |\log r|^{-\frac{1}{2}} = r^{-1} |\log r|^{-1}$$

and is therefore the Poincaré metric of the punctured disk on $\mathbb{C}$; we refer the interested reader to [7] and [84] for further information.

6.4. **Parabolic $\mathrm{SO}\left(p, p+1\right)$-models.** We next construct model parabolic $\mathrm{SO}\left(p, p+1\right)$-Higgs bundles which shall be later on used in providing the desired (non-parabolic) $\mathrm{SO}\left(p, p+1\right)$-models in the exceptional components over the complex connected sum of Riemann surfaces. Of critical importance to this construction are the parabolic $\mathrm{SL}\left(2, \mathbb{R}\right)$-Higgs bundles (E, Φ) of I. Biswas, P. Arés-Gastesi and S. Govindarajan from [7] described earlier. As we have seen in §5.4.1, from the gauge theoretic viewpoint, a model solution to the $\mathrm{SL}\left(2, \mathbb{C}\right)$-Hitchin equations that corresponds to the polystable pair (E, Φ) is given by a pair $\left(A^{\text{mod}}, \Phi^{\text{mod}}\right)$, where

$$A^{\text{mod}} = 0, \ \Phi^{\text{mod}} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z}$$

over a punctured disk with z-coordinates around the puncture with the condition that $C \in \mathbb{R}$ with $C \neq 0$, and that the meromorphic quadratic differential $q := \det \Phi^{\text{ mod}}$ has at least one simple zero.

6.4.1. *Models via the irreducible representation* $SL(2,\mathbb{R}) \hookrightarrow SO(p,p+1)$. We next construct model parabolic $\mathrm{SO}(p,p+1)$-Higgs bundles lying inside the parabolic Teichmüller component for $\mathrm{SO}(p,p+1)$. The general construction of this component was carried out in [56], while in the non-parabolic case, a detailed construction of models can be found in [1].

The connected component $\mathrm{SO}_0(p,p+1)$ of the special orthogonal group containing the identity is a split real form of $\mathrm{SO}(2p+1,\mathbb{C})$. The Lie algebra of $\mathrm{SO}(p,p+1)$ is

$$\begin{aligned}\mathfrak{so}(p,p+1) &= \left\{X \in \mathfrak{sl}(2p+1,\mathbb{R}) \,\middle|\, X^t I_{p,p+1} + I_{p,p+1} X = 0\right\} \\ &= \left\{\begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} \middle| X_1, X_3 \text{ real skew-sym. of rank } p, p+1 \text{ resp.;}\right. \\ &\qquad \left. X_2 \text{ real } (p \times (p+1))\text{-matrix}\right\}.\end{aligned}$$

The Lie algebra $\mathfrak{so}(p,p+1)$ admits a Cartan decomposition $\mathfrak{so}(p,p+1) = \mathfrak{h} \oplus \mathfrak{m}$ into its (± 1)-eigenspaces, where

$$\mathfrak{h} = \mathfrak{so}(p) \times \mathfrak{so}(p+1) = \left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix} \middle| X_1 \in \mathfrak{so}(p), X_3 \in \mathfrak{so}(p+1)\right\},$$

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & X_2 \\ X_2^t & 0 \end{pmatrix} \middle| X_2 \text{ real } (p \times (p+1))\text{-matrix}\right\},$$

and

$$\mathfrak{m}^{\mathbb{C}} = \left\{\begin{pmatrix} 0 & X_2 \\ X_2^t & 0 \end{pmatrix} \middle| X_2 \text{ complex } (p \times (p+1))\text{-matrix}\right\}.$$

If $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{so}(p,p+1)$ and Δ is the set of the corresponding roots, then the element

$$\sum_{\alpha \in \Delta} c_\alpha X_\alpha \in \mathfrak{so}(2p+1,\mathbb{C}),$$

is regular nilpotent, for $c_\alpha \neq 0$, $\alpha \in \Pi$ and X_α a root vector for α, where

$$\Delta^+ = \{e_i \pm e_j, \text{ with } 1 \leq i < j \leq p\} \cup \{e_i, 1 \leq i \leq p\},$$

$$\Pi = \{a_i = e_i - e_{i+1}, \ 1 \leq i \leq p-1\} \cup \{a_p = e_p\}.$$

The corresponding root vectors are
$X_{e_i - e_j} = E_{i,j} - E_{p+j,p+i}$
$X_{e_i + e_j} = E_{i,p+j} - E_{j,p+i}$
$X_{e_i} = E_{i,2p+1} - E_{2p+1,p+i}$
$X_{-e_i} = E_{p+i,2p+1} - E_{2p+1,i}$.

Now, let $x := \sum_{i=1}^{p} 2(p+1-i)(E_{i,i} - E_{p+i,p+i})$ and take $e := \sum_{a \in \Pi} X_a$. From this choice it is then satisfied that $[x,e] = 2e$, for the semisimple element x and the regular nilpotent element e. Moreover, the conditions $[x,\tilde{e}] = -2\tilde{e}$ and $[e,\tilde{e}] = x$ determine another nilpotent element $\tilde{e}$, thus the triple $\langle x, e, \tilde{e}\rangle \cong \mathfrak{sl}(2,\mathbb{C})$ defines a principal 3-dimensional Lie subalgebra of $\mathfrak{so}(p,p+1)$.

The adjoint action $\langle x, e, \tilde{e} \rangle \cong \mathfrak{so}(2, \mathbb{C}) \to \mathrm{End}(\mathfrak{so}(2p+1, \mathbb{C}))$ of this subalgebra decomposes $\mathfrak{so}(p, p+1)$ as a direct sum of irreducible representations

$$(2p+1, \mathbb{C}) = \overset{p}{\underset{i=1}{\oplus}} V_i,$$

with $\dim V_i = 4i-1$, for $1 \leq i \leq p$. Therefore, $V_i = S^{4i-2}\mathbb{C}^2$, $1 \leq i \leq p$ with eigenvalues $4i-2$, $4i-4$,..., $-4i+4$, $-4i+2$ for the action of $\mathrm{ad}x$, and the highest weight vectors are $e_1, \ldots, e_p$, where e_i has eigenvalue $4i-2$, for $1 \leq i \leq p$.

Considering the representation

$$\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{so}(2p+1, \mathbb{C}),$$

for $\mathfrak{so}(2p+1, \mathbb{C}) = S^2\mathbb{C}^2 + S^6\mathbb{C}^2 + \ldots + S^{4p-2}\mathbb{C}^2 = \Lambda^2\left(S^{2p}\mathbb{C}^2\right)$, we may next deduce the defining data (E_1, Φ_1) for a parabolic $\mathrm{SO}(p, p+1)$-Higgs bundle inside the parabolic Teichmüller component for the split real form $G^r = \mathrm{SO}_0(p, p+1)$. The parabolic vector bundle is obtained from the $(2p)$-th symmetric power of the parabolic $\mathrm{SL}(2, \mathbb{R})$-bundle in the Teichmüller component, as follows.

Let X_1 be a compact Riemann surface of genus g_1, $D_1 = \{p_1, \ldots, p_s\}$ a collection of s-many distinct points on X_1 and let $L_1 \to X_1$ with $L_1^2 \cong K_{X_1}$ and $\iota_1 = \mathcal{O}_{X_1}(D_1)$. Consider the parabolic vector bundle $(L_1 \otimes \iota_1)^* \oplus L_1$ over (X_1, D_1), equipped with a trivial flag and weight $\frac{1}{2}$. Then, the vector bundle E_1 of a model parabolic $\mathrm{SO}(p, p+1)$-Higgs bundle in the parabolic Teichmüller component is

$$\begin{aligned}
E_1 :&= S^{2p}\left((L_1 \otimes \iota_1)^* \oplus L_1\right) \\
&= L_1^{-2p} \otimes \mathcal{O}(-pD_1) \oplus L_1^{-2p+2} \otimes \mathcal{O}((1-p)D_1) \oplus \ldots \\
&\ldots \oplus L_1^{2p-2} \otimes \mathcal{O}((p-1)D_1) \oplus L_1^{2p} \otimes \mathcal{O}(pD_1) \\
&= K_1^{-p} \otimes \mathcal{O}(-pD_1) \oplus K_1^{-(p-1)} \otimes \mathcal{O}((1-p)D_1) \oplus \ldots \\
&\ldots \oplus K_1^{p-1} \otimes \mathcal{O}((p-1)D_1) \oplus K_1^{p} \otimes \mathcal{O}(pD_1),
\end{aligned}$$

equipped with a trivial parabolic flag and weight 0.

Remark 6.6. Note that in the above description we have included the consideration for the parabolic structure in a symmetric power of a parabolic bundle. In fact, restricting attention on the first original term $(L_1 \otimes \iota_1)^*$ with weight $\frac{1}{2}$, then the symmetric power $S^{2p}\left((L_1 \otimes \iota_1)^*\right)$ is the line bundle $L_1^{-2p} \otimes \mathcal{O}(-2pD_1)$ with weight $2p \cdot \frac{1}{2} = p$. However, we obtain a well-defined parabolic bundle by reducing the weight to a number within the interval $[0, 1)$, this means, by tensoring $L_1^{-2p} \otimes \mathcal{O}(-2pD_1)$ by $\mathcal{O}(pD_1)$. We thus get $K_1^{-p} \otimes \mathcal{O}(-pD_1)$ with weight 0, as appears in the first term of the parabolic bundle E_1 above.

The Higgs field in the parabolic $\mathrm{SO}(p, p+1)$-Teichmüller component is given by

$$\tilde{e} + q_1 e_1 + \ldots + q_p e_p,$$

for $(q_1, \ldots, q_p) \in \overset{p}{\underset{i=1}{\oplus}} H^0\left(K_1^{2i} \otimes \iota_1^{2i-1}\right)$ and $e_1, \ldots, e_p$ are the highest weight vectors. From the set of simple roots of $\mathfrak{so}(p, p+1)$,

$$\Pi = \{e_i - e_{i+1},\ 1 \leq i \leq p-1\} \cup \{e_p\},$$

we obtain the 3-dimensional subalgebra $\langle x, e, \tilde{e} \rangle \cong \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{so}(p, p+1)$, with

$$x=\left(\begin{array}{cccc|cccc|c}
2p &&&&&&&& \\
& 2(p-1) &&&&&&& \\
&& \ddots &&&&&& \\
&&& 2 &&&&& \\
\hline
&&&& -2p &&&& \\
&&&&& -2(p-1) &&& \\
&&&&&& \ddots && \\
&&&&&&& -2 & \\
\hline
&&&&&&&& 0
\end{array}\right), \tag{6.7}$$

$$e=\left(\begin{array}{cccc|cccc|c}
0 & 1 &&&&&&& \\
& \ddots & \ddots &&&&&& \\
&&& 1 &&&&& \\
&&& 0 &&&&& \\
\hline
&&&& 0 &&&& \\
&&&& -1 &&&& \\
&&&&& \ddots &&& \\
&&&&&& -1 & 0 & \\
\hline
&&&&&&& -1 & 0
\end{array}\right),$$

the semisimple and regular nilpotent element respectively.

From the analysis above we deduce that a model parabolic Higgs pair lying inside the parabolic $\mathrm{SO}_0(p,p+1)$-Hitchin component which is a local minimum of the Hitchin functional, when viewed as an $\mathrm{SL}(2p+1,\mathbb{C})$-pair, it is a pair (E_1,Φ_1) with

- $E_1=K_1^{-p}\otimes\mathcal{O}(-pD_1)\oplus K_1^{-(p-1)}\otimes\mathcal{O}((1-p)D_1)\oplus\ldots\oplus K_1^{(p-1)}\otimes\mathcal{O}((p-1)D_1)\oplus K_1^{p}\otimes\mathcal{O}(pD_1)$
  a parabolic vector bundle of rank $2p+1$ over (X_1,D_1) equipped with a parabolic structure given by a trivial flag and weight 0,
- $\Phi_1=\begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots &&&&& \\ 0 & \cdots &&& 0 & 1 \end{pmatrix}: E_1\to E_1\otimes K_1\otimes\iota_1$
  as a $p\times(p+1)$-matrix.

The next lemma is analogous to Lemma 2.1 in [7].

Lemma 6.8. *The parabolic Higgs bundle (E_1,Φ_1) above is a parabolic stable Higgs bundle of parabolic degree zero.*

Proof. The proof that $\operatorname{pardeg}(E_1)=0$ is immediate, following the properties of the parabolic degree on a direct sum and the dual of a parabolic bundle. The Φ_1-invariant proper subbundles of E_1 are of the form

$$K_1^{-p}\otimes\mathcal{O}(-pD_1)\oplus K_1^{-(p-1)}\otimes\mathcal{O}(-(p-1)D_1)\oplus\ldots\oplus K_1^{m-p}\otimes\mathcal{O}((m-p)D_1),$$

for $0\le m\le 2p-1$. One now checks that these all have negative parabolic degree, that is, strictly less than $\operatorname{pardeg}(E_1)$. $\square$

Therefore, from the punctured-surface version of the non-abelian Hodge correspondence [84], there is a tame harmonic metric on the vector bundle E_1. Let A_1 denote the associated Chern connection. Parabolic stability implies the existence of a complex gauge transformation, unique up to modification by a unitary gauge, such that (A_1, Φ_1) solves the Hitchin equations.

In a suitably chosen local holomorphic trivialization of E_1, the pair (A_1, Φ_1) is asymptotic to a model solution, which after a unitary change of frame can be written locally over a punctured neighborhood around a point $p_i \in D_1$ as

$$A_1^{\mathrm{mod}} = 0, \ \Phi_1^{\mathrm{mod}} = Cx\frac{dz}{z},$$

where x denotes the semisimple element from (6.7) and z the local coordinates around the point $p_i \in D_1$.

6.4.2. *Models via the general map Ψ.* Let X_2 be a compact Riemann surface of genus g_2 and $D_2 = \{q_1, \ldots, q_s\}$ a collection of s-many points on X_2. Let $\iota_2 = \mathcal{O}_{X_2}(D_2)$. The second family of model parabolic $\mathrm{SO}(p, p+1)$-Higgs bundles is obtained via the more general map

$$\Psi^{par} : \mathcal{M}^{par}_{K_2^p \otimes \iota_2^{p-1}}(\mathrm{SO}(1,2)) \times \bigoplus_{j=1}^{p-1} H^0\left(X_2, K_2^{2j} \otimes \iota_2^{2j-1}\right) \to \mathcal{M}^{par}(\mathrm{SO}(p,p+1))$$

defined as in (6.1), but considering also the relevant parabolic structures. Take $\left(I, \hat{W}, \hat{\eta}\right) \in \mathcal{M}^{par}_{K_2^p \otimes \iota_2^{p-1}}(\mathrm{SO}(1,2))$, the moduli space of K_2^p-twisted parabolic $\mathrm{SO}(1,2)$-Higgs bundles, for

- $\hat{W} := \tilde{M} \oplus \tilde{M}^{\vee}$, for $\tilde{M} \cong \mathcal{O}((2k-1-p)D_2)$ with $k = 1, \ldots, p$ an integer.
- $I := \wedge^2_{par} \hat{W} \cong \wedge \tilde{M} \otimes \wedge \tilde{M}^{\vee} \cong \tilde{M} \otimes \tilde{M}^{\vee} \cong \mathcal{O}$.
- $\hat{\eta} = 0$.

Then, one gets by the definition of the map Ψ^{par} the triple $\Psi^{par}\left(\left(I, \hat{W}, \hat{\eta}\right), (0, \ldots, 0)\right) =: (V, W, \eta)$, where

- $V = K_2^{p-1} \otimes \mathcal{O}((p-1)D_2) \oplus \ldots \oplus K_2^{1-p} \otimes \mathcal{O}((1-p)D_2)$
- $W = \tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{p-2} \otimes \mathcal{O}((p-2)D_2) \oplus \ldots \oplus K_2^{2-p} \otimes \mathcal{O}((2-p)D_2)$
- $\eta = \begin{pmatrix} \hat{\eta}=0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$.

From the description of the Higgs bundle data we see that since $\hat{\eta} = 0$, the triple (V, W, η) reduces to an $\mathrm{SO}(p, p-1) \times \mathrm{SO}(2)$-Higgs bundle whose $\mathrm{SO}(p, p-1)$-factor lies in the parabolic Hitchin component. We rather define this as an $\mathrm{SL}(2p+1, \mathbb{C})$-pair (E_2, Φ_2), where

- $E_2 = V \oplus W = \tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{-(p-1)} \otimes \mathcal{O}((1-p)D_2) \oplus \ldots \oplus K_2^{p-1} \otimes \mathcal{O}((p-1)D_2)$,
- $\Phi_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & & \ddots & & \\ & & & & & & & 1 \\ 0 & 0 & 0 & \cdots & & & & 0 \end{pmatrix} : E_2 \to E_2 \otimes K_2 \otimes \iota_2$.

The Φ_2-invariant proper subbundles of E_2 are

$$\begin{aligned}
&\tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{-(p-1)} \otimes \mathcal{O}\left((1-p)\, D_2\right) \\
&\tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{-(p-1)} \otimes \mathcal{O}\left((1-p)\, D_2\right) \oplus K_2^{-(p-2)} \otimes \mathcal{O}\left((2-p)\, D_2\right) \\
&\vdots \\
&\tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{-(p-1)} \otimes \mathcal{O}\left((1-p)\, D_2\right) \oplus \ldots \oplus K_2^{(p-2)} \otimes \mathcal{O}\left((p-2)\, D_2\right),
\end{aligned}$$

or, in general, these are of the form

$$\tilde{M} \oplus \tilde{M}^{\vee} \oplus K_2^{-(p-1)} \otimes \mathcal{O}\left((1-p)\, D_2\right) \oplus \ldots \oplus K_2^{(l-p)} \otimes \mathcal{O}\left((l-p)\, D_2\right),$$

for each $1 \leq l \leq 2p-2$. As in the previous lemma, one sees that all proper Φ_2-invariant subbundles of E_2 have negative parabolic degree, while $\operatorname{pardeg}(E_2) = 0$. Therefore, the models (E_2, Φ_2) for every $k = 1, \ldots, p$ are all parabolic stable. For A_2 be the Chern connection with respect to a tame harmonic metric on E_2, in a suitably chosen local holomorphic trivialization of E_2, the pair (A_2, Φ_2) is, after conjugation by a unitary gauge, asymptotic to a model solution which locally over a punctured neighborhood around a point $q_j \in D_2$ is written as

$$A_2^{\text{mod}} = 0, \;\; \Phi_2^{\text{mod}} = \begin{pmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 2\,(p-1)\,C &&&&&& \\ &&& \ddots &&&&& \\ &&&& 2C &&&& \\ &&&&& -2(p-1)C &&& \\ &&&&&& \ddots && \\ &&&&&&& -2C & \\ &&&&&&&& 0 \end{pmatrix} \frac{dw}{w},$$

for coordinates w around each puncture $q_j \in D_2$.

6.5. **Gauge-theoretic gluing of parabolic SO$(p, p+1)$-Higgs bundles.** We have described parabolic SO$(p, p+1)$-models (E_i, Φ_i), $i = 1, 2$, which are parabolic stable. Model solutions to the Hitchin equations over punctured disks corresponding to the pairs (E_i, Φ_i)

are respectively of the form

$$A_1^{\text{mod}} = 0, \ \Phi_1^{\text{mod}} = \begin{pmatrix} 2pC & & & & & & & \\ & 2(p-1)C & & & & & & \\ & & \ddots & & & & & \\ & & & 2C & & & & \\ & & & & -2pC & & & \\ & & & & & \ddots & & \\ & & & & & & -2C & \\ & & & & & & & 0 \end{pmatrix} \frac{dz}{z},$$

$$A_2^{\text{mod}} = 0, \ \Phi_2^{\text{mod}} = \begin{pmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 2(p-1)C & & & & & & \\ & & & \ddots & & & & & \\ & & & & 2C & & & & \\ & & & & & -2(p-1)C & & & \\ & & & & & & \ddots & & \\ & & & & & & & -2C & \\ & & & & & & & & 0 \end{pmatrix} \frac{dw}{w}.$$

In order to glue the above parabolic $\mathrm{SO}(p, p+1)$-Higgs bundles over the complex connected sum of Riemann surfaces $X_\# := X_1 \# X_2$ of genus $g = g_1 + g_2 + s - 1$ we shall use the gauge-theoretic gluing construction summarized in §5. To this end, the initial model data $\left(A_i^{\text{mod}}, \Phi_i^{\text{mod}}\right)$ should identify locally over the annuli around the points in the divisors of s-many points D_i, for $i = 1, 2$. This is achieved using the perturbation argument described next.

Consider the embedding

$$\Psi_i^{par} : \mathcal{M}^{par}_{K_i^p \otimes \iota_i^{p-1}}(\mathrm{SO}(1,2)) \times \bigoplus_{j=1}^{p-1} H^0\left(X_i, K_i^{2j} \otimes \iota_i^{2j-1}\right) \to \mathcal{M}^{par}(\mathrm{SO}(p, p+1)),$$

for $i = 1, 2$. Over the pair (X_1, D_1), take a parabolic $\mathrm{SO}(1,2)$-Higgs bundle defined by the triple $\left(\hat{W}_1, I_1, \hat{\eta}_1\right)$ with

$$\begin{aligned} \hat{W}_1 &:= K_1^\vee \oplus K_1 \\ I_1 &\cong \mathcal{O} \\ \hat{\eta}_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Let $\left(\tilde{A}_1, \tilde{\Phi}_1\right)$ be the corresponding solution to the Hitchin equations. There is a complex gauge transformation which locally puts $\left(\tilde{A}_1, \tilde{\Phi}_1\right)$ into the model form

$$\tilde{A}_1^{\text{mod}} = 0, \ \tilde{\Phi}_1^{\text{mod}} = \begin{pmatrix} 2C & 0 & 0 \\ 0 & -2C & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{dz}{z}$$

over a disk centered at the points in D_1 with coordinates z.

Remark 6.9. The existence of this complex gauge transformation is provided for the local $\mathrm{SL}(2,\mathbb{R})$-model solution $\left(A^{\text{ mod }}=0,\ \Phi^{\text{ mod }}=\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}\frac{dz}{z}\right)$; then we embed into $\mathrm{SO}(1,2)$.

Using the map Ψ_1^{par} above, the parabolic stable $\mathrm{SO}(p,p+1)$-pair (E_1,Φ_1) over (X_1,D_1) corresponds to an *approximate solution* (A_1,Φ_1) of the Hitchin equations, which near each point of D_1 has the form $\left(A_1^{\text{mod}},\Phi_1^{\text{mod}}\right)$ with

$$A_1^{\text{mod}}=0,\ \ \Phi_1^{\text{mod}}=\begin{pmatrix} 2pC & & & & & & & & \\ & 2(p-1)C & & & & & & & \\ & & \ddots & & & & & & \\ & & & 2C & & & & & \\ & & & & -2pC & & & & \\ & & & & & \ddots & & & \\ & & & & & & -2C & \\ & & & & & & & 0 \end{pmatrix}\frac{dz}{z},$$

for $p>2$ and $C\in\mathbb{R}$ nonzero.

Over the pair (X_2,D_2), take the triple $\left(\hat{W}_2,I_2,\hat{\eta}_2\right)$ with

$$\begin{aligned}&\hat{W}_2:=\tilde{M}\oplus\tilde{M}^{\vee},\ \text{where } \tilde{M}\cong\mathcal{O}\left((2k-1-p)\,D_2\right),\ \text{for } k=1,\dots,p\\ &I_2\cong\mathcal{O}\\ &\hat{\eta}_2\in H^0\left(\operatorname{Hom}\left(\hat{W}_2,I_2\right)\otimes K_2^p\otimes\iota_2^{p-1}\right).\end{aligned}$$

Applying a similar argument as above, we may perturb the relevant $\mathrm{SL}(2,\mathbb{R})$-pair and extend our data to $\mathrm{SO}(p,p+1)$ to finally get an *approximate* solution (A_2,Φ_2), which near each point of D_2 has the form $\left(A_2^{\text{mod}},\Phi_2^{\text{mod}}\right)$ with

$$A_2^{\text{mod}}=0,\ \ \Phi_2^{\text{mod}}=\begin{pmatrix} -2pC & & & & & & & \\ & -2(p-1)C & & & & & & \\ & & \ddots & & & & & \\ & & & -2C & & & & \\ & & & & 2pC & & & \\ & & & & & \ddots & & \\ & & & & & & 2C & \\ & & & & & & & 0 \end{pmatrix}\frac{dw}{w},$$

for $p>2$ and $C\in\mathbb{R}$ nonzero, as above.

The complex connected sum of Riemann surfaces $X_\#=X_1\#X_2$ is realized along the curve $zw=\lambda$ for a parameter $\lambda\in\mathbb{C}$, and so $\frac{dz}{z}=-\frac{dw}{w}$ for coordinates on annuli around each puncture which are glued using a biholomorphism for each pair of points (p_i,q_j) from the divisors D_1 and D_2. Let $\Omega\subset X_\#$ denote the result of gluing these pairs of annuli and set $\left(A_{p_i,q_j}^{\text{mod}},\Phi_{p_i,q_j}^{\text{mod}}\right):=\left(A_1^{\text{mod}},\Phi_1^{\text{mod}}\right)=-\left(A_2^{\text{mod}},\Phi_2^{\text{mod}}\right)$. We can glue the pairs (A_1,Φ_1), (A_2,Φ_2) together to get an *approximate solution* of the $\mathrm{SO}(p,p+1)$-Hitchin equations:

$$\left(A^{app},\Phi^{app}\right):=\begin{cases}\left(A_1,\Phi_1\right), & \text{over } X_1\backslash X_2\\ \left(A^{\text{mod}}_{p_i,q_j},\Phi^{\text{mod}}_{p_i,q_j}\right), & \text{over } \Omega \text{ around each pair of points } \left(p_i,q_j\right)\\ \left(A_2,\Phi_2\right) & \text{over } X_2\backslash X_1,\end{cases}$$

over the connected sum bundle over $X_{\#}$.

By construction, (A^{app},Φ^{app}) is a smooth pair on $X_{\#}$, complex gauge equivalent to an exact solution of the Hitchin equations by a smooth gauge transformation defined over all of $X_{\#}$. The next step is to correct the approximate solution (A^{app},Φ^{app}) to an exact solution of the $\mathrm{SO}(p,p+1)$-Hitchin equations. We follow the contraction mapping argument for the nonlinear G-Hitchin operator from §5.6 developed for a general connected semisimple Lie group G. For the proof of the fact that the linearization operator is invertible one easily adapts the proofs in [55] in the case $G=\mathrm{SO}(p,p+1)$ for the compact real form $\tau(\Phi)=\bar{\varphi}dz$ on a Higgs field $\Phi=\varphi dz$ and for the model parabolic $\mathrm{SO}(p,p+1)$-Higgs bundles we have considered with $\Phi_1^{\text{mod}}=\varphi_1^{\text{mod}}\frac{dz}{z}$, for

$$\varphi_1^{\text{mod}}=\begin{pmatrix}2pC &&&&&&&\\ & 2(p-1)C &&&&&&\\ && \ddots &&&&&\\ &&& 2C &&&&\\ &&&& -2pC &&&\\ &&&&& \ddots &&\\ &&&&&& -2C &\\ &&&&&&& 0\end{pmatrix}$$

and $p>2$, $C\in\mathbb{R}$ nonzero.

Thus, Theorem 5.9 adapts in the case $G=\mathrm{SO}(p,p+1)$ to provide the following:

Theorem 6.10. *Let X_1 be a closed Riemann surface of genus g_1 and $D_1=\{p_1,\dots,p_s\}$ be a collection of s-many distinct points on X_1. Consider respectively a closed Riemann surface X_2 of genus g_2 and a collection of also s-many distinct points $D_2=\{q_1,\dots,q_s\}$ on X_2. Let $(E_1,\Phi_1)\to X_1$ and $(E_2,\Phi_2)\to X_2$ be parabolic polystable $SO(p,p+1)$-Higgs bundles, one from each of the families described in §6.4.1 and §6.4.2 with corresponding solutions to the Hitchin equations (A_1,Φ_1) and (A_2,Φ_2). Then there is a polystable $SO(p,p+1)$-Higgs bundle $(E_{\#},\Phi_{\#})\to X_{\#}$ over the complex connected sum of Riemann surfaces $X_{\#}=X_1\#X_2$, which agrees with the initial data over $X_{\#}\backslash X_1$ and $X_{\#}\backslash X_2$.*

Definition 6.11. We call such an $\mathrm{SO}(p,p+1)$-Higgs bundle constructed by the theorem above a *hybrid $SO(p,p+1)$-Higgs bundle.*

6.6. **Model representations in the exceptional components of $\mathcal{R}(\mathbf{SO}(p,p+1))$.** We now show that the specific hybrid $\mathrm{SO}(p,p+1)$-Higgs bundles constructed in the previous section lie inside the $p(2g-2)-1$ exceptional components of the character variety $\mathcal{R}(\mathrm{SO}(p,p+1))$. In fact, by varying the parameters in the construction, namely, the genera g_1, g_2 of the Riemann surfaces X_1, X_2, the number of points s in the divisors D_1, D_2, and the weight $\alpha=2k-1-p$ for the line bundle $\tilde{M}\cong\mathcal{O}((2k-1-p)D_2)$, one obtains models in *all exceptional components.* This is seen by an explicit computation of the degree of the line bundle

M appearing in the description (6.5) of the Higgs bundle data; the exceptional components are fully distinguished by the degree of this line bundle. We have considered:

$$\begin{aligned} E_1 = & K_1^{-p} \otimes \mathcal{O}(-pD_1) \oplus K_1^{-(p-1)} \otimes \mathcal{O}((1-p)D_1) \oplus \dots \\ & \dots \oplus K_1^{(p-1)} \otimes \mathcal{O}((p-1)D_1) \oplus K_1^p \otimes \mathcal{O}(pD_1), \text{ and} \end{aligned}$$

$$\begin{aligned} E_2 = V \oplus W = & \tilde{M}^\vee \oplus \tilde{M} \oplus K_2^{-(p-1)} \otimes \mathcal{O}((1-p)D_2) \oplus \dots \\ & \dots \oplus K_2^{p-1} \otimes \mathcal{O}((p-1)D_2), \end{aligned}$$

with $\tilde{M} \cong \mathcal{O}((2k-1-p)D_2)$ and $\operatorname{pardeg}\left(\tilde{M}\right) = (2k-1-p)s$, for $k = 1, \dots, p$. We now use Proposition 5.11, which asserts an additivity property for the parabolic degree of the bundle over the connected sum operation. We thus have that for each $j \in \{1-p, \dots, p-1\}$ the bundle $K_1^{\otimes_{par} j} \# K_2^{\otimes_{par} -j}$ has degree

$$\begin{aligned} \deg\left(K_1^{\otimes_{par} j} \# K_2^{\otimes_{par} j}\right) &= \operatorname{pardeg}\left(K_1^j \otimes \mathcal{O}(jD_1)\right) + \operatorname{pardeg}\left(K_2^j \otimes \mathcal{O}(jD_2)\right) \\ &= j(2g_1 - 2 + s) + j(2g_2 - 2 + s) \\ &= 2j(g_1 + g_2 + s - 1 - 1) \\ &= 2j\left(g_{X_\#} - 1\right) \\ &= \deg K_{X_\#}^{\otimes j}. \end{aligned}$$

It is thus a line bundle isomorphic to $K_{X_\#}^{\otimes j}$.

Moreover, gluing the parabolic line bundles $K_1^p \otimes \mathcal{O}(pD_1)$ and $\tilde{M}$ provides a line bundle $M \in \operatorname{Pic}(X_\#)$ with degree

$$\begin{aligned} \deg(M) &= \operatorname{pardeg}\left(K_1^p \otimes \mathcal{O}(pD_1)\right) + \operatorname{pardeg}\left(\tilde{M}\right) \\ &= p(2g_1 - 2 + s) + (2k-1-p)s \\ &= 2p(g_1 - 1) + (2k-1)s. \end{aligned}$$

We deduce that the result of the construction is a Higgs bundle (V, W_k, η) with data V and η as in (6.5) and

$$W_k := M \oplus K_{X_\#}^{p-2} \oplus \dots \oplus K_{X_\#}^{2-p} \oplus M^{-1}$$

with $d = \deg(M) = 2p(g_1 - 1) + (2k-1)s$, for $k = 1, \dots, p$. One can now check that varying the values of the parameters g_1, s and k, we can obtain model $\mathrm{SO}(p, p+1)$-Higgs bundles by gluing, which exhaust all the exceptional smooth $p(2g-2) - 1$ components of $\mathcal{M}(\mathrm{SO}(p, p+1))$.

Remark 6.12. Notice that the case when $p = 1$ actually describes the $\mathrm{Sp}(4, \mathbb{R})$-case from [55]. Indeed, we then have $k = 1$ and so $\tilde{M} \cong \mathcal{O}$ with $d = \deg(M) = 2(g_1 - 1) + s = -\chi(\Sigma_l)$. The case $p > 2$ thus involves an *extra parameter* on the non-trivial line bundle $\tilde{M}$ given by the parabolic structure on a trivial flag.

7. The gluing of positive representations

The choice of parabolic $\mathrm{SO}(p, p+1)$-models we made involves an extra property apart from the convenient topological data of the underlying bundles. In fact, the model fundamental group representations corresponding to those are *positive*, as is implied by the fact that we chose a Hitchin representation into $\mathrm{SO}(p, p+1)$, and a representation which factors through $\mathrm{SO}(p-1, p) \times \mathrm{SO}(2)$ with $\mathrm{SO}(p-1, p)$-factor in the relative Hitchin component. In this subsection, we verify that the Θ-positivity condition is preserved by the above gluing construction, as long as the holonomy of the representations agrees around each pair of punctures on the two surfaces.

For the model parabolic $\mathrm{SO}(p, p+1)$-Higgs bundles $(E_{\bullet}, \Phi_{\bullet})$ considered, let $\rho_{\bullet} : \pi_1(X_{\bullet}) \to \mathrm{SO}(p, p+1)$ be the corresponding fundamental group representations via the non-abelian Hodge correspondence, for $\bullet = l, r$. For these representations, it holds that

$$\rho_l(\gamma_i) = \rho_r(\gamma_i)$$

for a curve γ_i around each boundary components for $i = 1, \ldots, s$. Moreover, since $\rho_{\bullet}$ is Θ-positive there exist boundary maps

$$\beta_{\bullet} : C_{\bullet} \to \mathcal{F},$$

which are positive and $\rho_{\bullet}$-equivariant, where $C_{\bullet} = \Gamma \partial_{\infty} \pi_1 \left(\tilde{X}_{\bullet}\right)$, $\mathcal{F}$ denotes a generalized flag variety and $\tilde{X}_{\bullet}$ is the universal cover of the surface $X_{\bullet}$ for $\bullet = l, r$. The aim is to show that under the condition that $\rho_l(\gamma_i) = \rho_r(\gamma_i)$, there is a ρ-equivariant boundary map

$$\beta_{\#} : \Omega \to \mathcal{F},$$

where $\rho = \rho_l * \rho_r$ is the amalgamation of the representations ρ_l, ρ_r and Ω denotes the boundary at infinity of $\pi_1\left(\tilde{X}_{\#}\right)$.

We first describe the gluing at the level of infinity of π_1 (cf. also p. 95-100 in [25] for a similar gluing construction of the V. Fock-A. Goncharov positivity condition in the case of split real Lie groups). Let

$$\partial_{\infty} \pi_1(X_{\bullet}) = \partial_{\infty} \tilde{X}_{\bullet} = S^1 \setminus \coprod \text{intervals},$$

for $\bullet = l, r$. In particular, let

$$\partial_{\infty} \pi_1\left(\tilde{X}_l\right) = S^1 \setminus \coprod_{\eta \in \pi_1 \tilde{X}_l / \langle \gamma \rangle} \eta_{(\gamma_-, \gamma_+)} \subset [\gamma_+, \gamma_-] = S^1 \setminus (\gamma_-, \gamma_+),$$

that is, we choose the infinitely many green intervals inside the left hand side of the disk and $\partial_{\infty} \pi_1\left(\tilde{X}_r\right)$ are the complementary ones (see Figure 4).

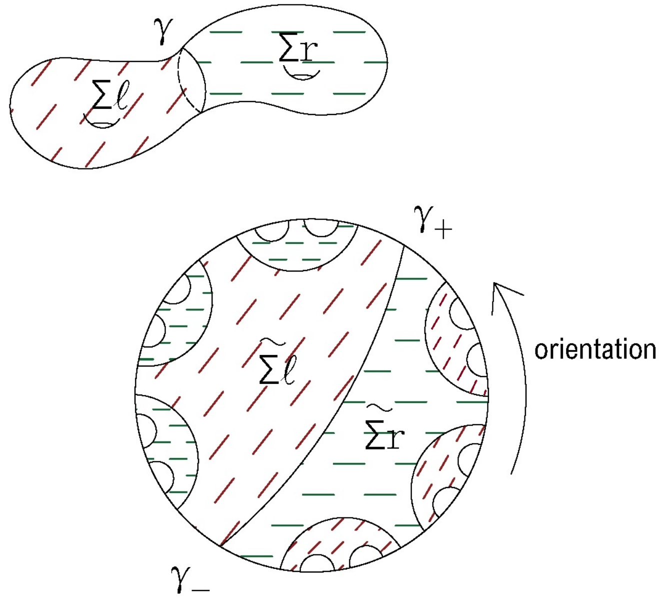

Figure 4. Gluing of geodesic boundaries.

Now we map the geodesic γ to each marked boundary geodesic on one side and iterate the process. This has a *tree structure,* where:

$$\text{vertices} = \pi_0\left(\mathbb{N}_2\backslash\pi_1 X_{\#}\mathrm{Axis}\left(\gamma\right)\right) = \pi_0\left(\left(\bigcup\Gamma\tilde{X}_l\right)\cup\left(\bigcup\Gamma\tilde{X}_r\right)\right)$$
$$\text{edges} = \text{if they share a copy of } \mathrm{Axis}\left(\gamma\right).$$

Therefore,

$$S^1 \supset \Omega := \left(\bigcup\Gamma\partial_\infty\pi_1\left(\tilde{X}_r\right)\right)\cup\left(\bigcup\Gamma\partial_\infty\pi_1\left(\tilde{X}_l\right)\right),$$

where $C_{\bullet} := \bigcup\Gamma\partial_\infty\pi_1\left(\tilde{X}_{\bullet}\right)$, for $\bullet = l, r$. We need to understand the intersection $gC_l \cap g'C_r$. Notice that then and only then it is $gC_l = g'C_r$. Similarly, $gC_l \cap g'C_r \neq \varnothing$ iff $g = g' \mod \langle\gamma\rangle$, thus $gC_l \cap g'C_r = g\left\{\gamma_+, \gamma_-\right\}$.

Assume now that there exist boundary maps $\beta_{\bullet}$, for $\bullet = l, r$ as described earlier. Then we want to define a boundary map $\beta_{\#} : \Omega \to \mathcal{F}$, ρ-equivariant and positive. It is enough to argue that there is a map like the one described in Figure 5 for every $x \in C_{\bullet}\backslash\left\{\gamma_+, \gamma_-\right\}$ (we can assume that one is the identity element so let g_0 be the identity). In that figure we mean $x_+ \in g_+C_{\bullet}$, $x_- \in g_-C_{\bullet}$ and $x_0 \in g_0C_{\bullet}$, for $\bullet = l, r$.

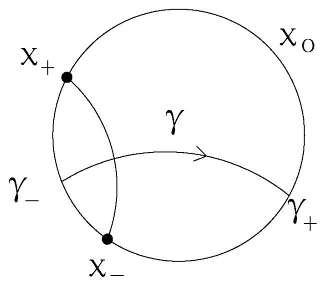

Figure 5. Gluing of boundary curves.

Moreover, for the map $\beta_{\#} : \Omega \to \mathcal{F}$, let $g \cdot x \mapsto \rho(g)\, \beta_l(x)$ and $g' \cdot x \mapsto \rho(g')\, \beta_l(x')$. Then, $gx = g'x'$, when there is $h \in \pi_1 X_l$ such that $g' = gh$ and $hx' = x$. From this, and using the fact that the holonomies of ρ_l and ρ_r agree on the boundary, it is implied that

$$\rho(g)\, \beta_l(x) = \rho\left(g'\right) \beta_l\left(x'\right).$$

Thus, the boundary curve $\beta_{\#}$ is also ρ-equivariant.

In conclusion, the amalgamated representations on the connected sum constructed are *model Θ-positive representations* exhausting the $p(2g-2)-1$ exceptional components of the character variety $\mathcal{R}(\mathrm{SO}(p, p+1))$.

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Max-Planck-Institut für Mathematik
Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `kydonakis@mpim-bonn.mpg.de`