Scattering Angles in Kerr Metrics

Poul H. Damgaard[1], Jitze Hoogeveen[1], Andres Luna[1], and Justin Vines[2]

[1] *Niels Bohr International Academy*
Niels Bohr Institute, University of Copenhagen
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
[2] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute)*
Am Muhlenberg 1, Potsdam 14476, Germany

(Dated: August 24, 2022)

Scattering angles for probes in Kerr metrics are derived for scattering in the equatorial plane of the black hole. We use a method that naturally resums all orders in the spin of the Kerr black hole, thus facilitating comparisons with scattering-angle computations based on the Post-Minkowskian expansion from scattering amplitudes or worldline calculations. We extend these results to spinning black-hole probes up to and including second order in the probe spin and any order in the Post-Minkowskian expansion, for probe spins aligned with the Kerr spin. When truncating to third Post-Minkowskian order, our results agree with those obtained by amplitude and worldline methods.

INTRODUCTION

The gravitational bending of light around the Sun famously provided one of the earliest observational checks on predictions from Einstein's general theory of relativity. Since then, gravitational optics has become one of the common tools of observational astronomy, in fact now used inversely to infer mass distributions of massive objects partly obstructing light in the direct line of sight. The light-like bending angles of general relativity thus have a central position in modern physics.

Recently, scattering angles of massive objects have attracted renewed attention from an entirely different direction. For gravitational-wave predictions of black-hole mergers one needs the effective Hamiltonian that governs the dynamics of two massive bodies in general relativity. It was suggested in Ref. [1] that an improvement of traditional analytical approaches based on Post-Newtonian expansions could come from the Post-Minkowskian expansion of the scattering regime. This suggests that modern amplitude methods of the quantized theory to great advantage may be used to infer the effective two-body interactions of general relativity [2–4] after properly removing all non-classical contributions at loop level [5]. In a short span of time there has been enormous progress in this direction, with results to third Post-Minkowskian order now fully under control [6–15]. Even amplitude computations to fourth Post-Minkowskian order [16] and, in the probe limit, fifth Post-Minkowskian order [17] have now been considered. A parallel track based on effective field theory in the worldline formalism offers results at similar high orders in the Post-Minkowskian expansion [18–27]. For recent reviews, see, *e.g.*, refs. [28–30].

Adding classical spin to the Post-Minkowskian expansion leads to interesting challenges in the amplitude approach due to the traditional barrier at spin-2 in quantum field theory (although recent progress in describing massive higher spin states has been made in Refs. [31–34]). Results at the first Post-Minkowskian order and all orders in the spins were derived by solving Einstein's field equations directly [35]. Amplitude-based and worldline approaches have since made substantial progress towards obtaining Post-Minkowskian results with spin [36–61]. In order to have known limits in which to compare amplitude-based results for scattering angles with those computed directly from general relativity, we here reconsider the classical problem of scattering in the equatorial plane of a Kerr black hole. We restrict the spin of the black hole to be parallel with the orbital angular momentum and the motion is therefore restricted to the equatorial plane. A single scattering angle can then describe the asymptotic motion and the situation is rather similar to that of scattering around a Schwarzschild black hole except for the fact that the scattering angle will depend on whether the black hole spin is pointing in the same direction as the orbital angular momentum, or opposite. We will be working with metrics of signature $(-,+,+,+)$ throughout.

One of the interesting observations of the first-order Post-Minkowskian result of ref. [35] was that the spins, to that order in the Post-Minkowskian expansion, could be provided in an exact (resummed) form. The same resummed form naturally appears also from amplitude calculations to the same order in the Post-Minkowskian expansion [38] and remnants of such a structure can be found also at second Post-Minkowskian order, at least up to fourth order in one of the spins [38]. It turns out that this structure of resummed spin is a general feature of the probe limit: if the probe is spinless, we can show this to any order in the Post-Minkowskian expansion. Taking the lightlike limit, and expanding in the black hole spin, we recover the Kerr results for the bending of light [62]. As we shall detail below, there are several other checks on our results as well.

Finally, an interesting and challenging problem is that of adding spin to the probe. We shall derive expressions for the Kerr scattering angle for a spinning probe, with the probe spin aligned with both the orbital angular momentum and the Kerr spin, valid up to (and including) second order in the black-hole probe spin. In principle, these calculations can be carried through to arbitary Post-Minkowskian order and we illustrate that below by providing analytical expressions up to and including $\mathcal{O}(G^5)$. Truncating to third Post-Minkowskian order our results agree with those of refs. [25, 37, 57]. The general expressions we present here for the probe limit both without and with spin may be useful for checks on amplitude computations at higher orders in the Post-Minkowskian expansion.

WARM-UP: SCATTERING IN SCHWARZSCHILD METRICS

Before we turn to the main subject, it is instructive to describe our method in a far simpler setting that still retains the important features. We therefore first consider the computationally easier problem of scattering around a Schwarzschild black hole. This will highlight the importance of choosing suitable variables to simplify the calculation.

Consider first a scattering problem in a spherically symmetric effective potential $V_{\rm eff}(r)$ for which the radial mo-

mentum reads

$$p_r \;=\; \sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{\rm eff}(r)}\;, \tag{1}$$

where p_∞ is the three-momentum at radial infinity and L is the conserved angular momentum. As is well known from analytical mechanics (say, fom Hamilton-Jacobi theory), the scattering angle χ in such a situation follows from the relation

$$\frac{\chi}{2} = -\frac{\partial}{\partial L}\int_{r_m}^{\infty} dr \sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{\rm eff}(r)} - \frac{\pi}{2}, \tag{2}$$

where r_m is the turning point of the orbit. This is determined by the condition $p_r(r_m) = 0$, i.e. at the (real and positive) point where the integrand vanishes. One may legitimately move the derivative with respect to L inside the integral since the boundary term at r_m vanishes by definition. The scattering angle can thus be computed from

$$\frac{\chi}{2} = L\int_{r_m}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{p_\infty^2 - \frac{L^2}{r^2} - V_{\rm eff}(r)}} - \frac{\pi}{2}\;, \tag{3}$$

which not only appears to depend on r_m but even seems to be singular due to the integrand diverging at the endpoint. As is well known, these subtleties are only apparent and the whole expression is completely well defined. In reality, though, except for a very small set of integrable potentials $V_{\rm eff}(r)$, we wish to find the scattering angle as a perturbative series in the strength of the potential. A very compact solution to this problem was recently provided in ref. [63], where the scattering angle is given in terms of a series of finite integrals, with one new integral appearing for each order in perturbation theory. The final result reads

$$\chi = \sum_{k=1}^{\infty} \frac{2b}{k!} \int_0^{\infty} du \left(\frac{d}{du^2}\right)^k \frac{[V_{\rm eff}(r)]^k r^{2(k-1)}}{p_\infty^{2k}}\;. \tag{4}$$

Here $r^2 = u^2 + b^2$ and the impact parameter b has been introduced in the usual way by $b = L/p_\infty$. Note that all integrals now run along the full positive line, and they thus become elementary for power-law potentials. One important example which immediately fits into this framework is that of scattering in a Schwarzschild metric expressed in isotropic coordinates, and thus with line element

$$ds^2 = -\left(\frac{1+\frac{GM}{2r}}{1-\frac{GM}{2r}}\right)^2 dt^2 + \left(1+\frac{GM}{2r}\right)^4 \left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{5}$$

This translates into the effective potential [64]

$$V_{\rm eff}(r) = m^2(\gamma^2 - 1) - m^2\left(1+\frac{GM}{2r}\right)^4 \left(\gamma^2 \left(\frac{1+\frac{GM}{2r}}{1-\frac{GM}{2r}}\right)^2 - 1\right)\;. \tag{6}$$

Here $\gamma = 1/\sqrt{1-v^2}$ is the usual Lorentz contraction factor and we have chosen the scattering to take place in the equatorial plane of $\theta = \pi/2$. Writing down the Schwarzschild scattering angle to any order in G is thus straightforward upon expansion of the potential in a power series and subsequent use of eq. (4).

We now wish to generalize the derivation of ref. [63] so that it is amenable to more general metrics. We will follow the standard approach based on solving for the radial momentum p_r. However, for general metrics, and in particular also for the Schwarzschild metric in coordinates different from isotropic, this will not lead to an expression of the simple form (1). In order to retain as many as possible of the simplifying features of the approach followed in ref. [63] we will make a suitable change of variables to a metric which in the limit of no interactions ($G \to 0$) reduces to the metric of flat Minkowski space in spherical coordinates. As a consequence, we recover the simple relation

$$p_r = \sqrt{p_\infty^2 - L^2/r^2} \tag{7}$$

in that limit. We will refer to metrics with this property as being in *normal form*. An example will best illustrate

what we mean. To this end, let us consider the Schwarzschild metric, but now written in standard Schwarzschild coordinates

$$ds^2 = -\left(1+\frac{r_s}{r}\right)^2 dt^2 + \left(1-\frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{8}$$

where we have defined $r_s \equiv 2GM$. For the obvious choice $\theta = \pi/2$ the metric leads to

$$p_r^2 = \frac{(E^2-m^2)r^3 + m^2 r^2 r_s - L^2(r-r_s)}{r(r-r_s)^2}. \tag{9}$$

In the free case, the radial momentum eq. (9) reduces to $p_r^2 = p_\infty^2 - L^2/r^2$. It is thus possible to separate out this part and write, equivalently,

$$p_r^2 = p_\infty^2 - \frac{L^2}{r^2} - r_s\left(\frac{m^2(r-r_s) - E^2(2r-r_s) + \frac{L^2}{r^2}(r-r_s)}{(r-r_s)^2}\right), \tag{10}$$

where the last term involving the bracket clearly vanishes as $G \to 0$, and we have simply added and subtracted L^2/r^2. The advantage of this rewriting is that it makes it natural to interpret the remainder

$$U(r,L) \;\equiv\; r_s\frac{m^2(r-r_s) - E^2(2r-r_s) + \frac{L^2}{r^2}(r-r_s)}{(r-r_s)^2} \tag{11}$$

formally as an L-dependent potential. The Schwarzschild metric in these coordinates is therefore already of normal form, and the scattering angle can thus still be written as

$$\frac{\chi}{2} = -\frac{\partial}{\partial L}\int_{r_m}^{\infty} dr \sqrt{p_\infty^2 - \frac{L^2}{r^2} - U(r,L)} - \frac{\pi}{2}, \tag{12}$$

but the derivative will now also act on $U(r,L)$. We hence need to generalize the derivation of ref. [63] to this new situation. Moreover, we discover that the L-derivative of eq. (12), which is so natural from the canonical formalism, can be disposed of so as to open up for more general situations including the spin of the probe. Let us jump ahead to the final result which turns out to be surprisingly simple. In order to introduce it, we write the scattering angle in the form

$$\frac{\chi}{2} = \int_{r_m}^{\infty} dr \frac{d\phi}{dr} - \frac{\pi}{2} = -\int_{r_m}^{\infty} \frac{h(r)}{p_\infty}\left(1 - b^2/r^2 - \frac{U(r,b)}{p_\infty^2}\right)^{-1/2} - \frac{\pi}{2}\ , \tag{13}$$

where

$$h(r) \;\equiv\; -\frac{d\phi}{dr}p_r. \tag{14}$$

This rather trivial rewriting in fact anticipates, in simple cases, a first order derivative representation of χ in terms of the radial action as in eq. (12). Moreover, it allows for greater flexibility regarding the effective potentials we can handle. In terms of these quantities, the scattering angle is given by

$$\chi = -2\sum_{n=0}^{\infty}\int_0^{\infty} du \left(\frac{d}{du^2}\right)^n h(r)\frac{r^{2n}U(r,b)^n}{n!p_\infty^{2n+1}} - \pi, \qquad r^2 = u^2 + b^2\ . \tag{15}$$

The function $h(r)$ needs to be determined for each specific scattering situation but it often takes very simple forms. As an example, for the Schwarzschild metric in isotropic coordinates it is simply $h(r) = -bp_\infty/r^2$. The identification of $h(r)$ is useful for both non-spinning and spinning probes but it is particularly suited for the latter, where there may be no obvious way in which to relate the integrand of the scattering angle to a first-order derivative. As detailed in our derivation below, the formula (43) is valid for any $h(r)$ which is real analytic on the interval $r \in [r_m, \infty[$, and falls off as $\lim_{r\to\infty} h(r) \sim 1/r^n$, with $n \geq 2$. These conditions are always met for the cases considered in this paper.

A compact formula for the scattering angle in metrics of normal form

Although the final result eq. (43) is surprisingly simple, the steps leading to it are involved and we display them now with a fair amount of detail. Let us first introduce some general notation. For any non-trivial metric, p_r will depend on G and this dependence carries all of the information about the scattering angle. We define $T \equiv p_r^2\big|_{G\to 0}$ so that we can write

$$p_r^2 = T(r) - U(r) \tag{16}$$

where, by construction, $U(r)$ carries all the G-dependence. Both T and U depend on the radial coordinate r as indicated but may also in general depend on orbital angular momentum L and any other parameters of the metric. The function U is a close analogue of a classical effective potential associated with the given metric (for some choice of coordinates). If U carries no L-dependence the method used in ref. [63] can straightforwardly be used to derive the scattering angle in perturbation theory. Here we consider its generalization to the L-dependent setting, focusing on a formulation that will encompass the case of spinning probes.

After having introduced this notation, we now return to the case of a metric which we assume is already in normal form. As explained above, this means that $T(r)$ takes the simple form

$$T(r) \;=\; p_\infty^2 - \frac{L^2}{r^2} \tag{17}$$

in those coordinates. We recall that we can then write the scattering angle as

$$\frac{\chi}{2} = \int_{r_m}^{\infty} dr \frac{d\phi}{dr} - \frac{\pi}{2} = -\int_{r_m}^{\infty} \frac{h(r)}{p_\infty}\left(1 - b^2/r^2 - \frac{U(r,b)}{p_\infty^2}\right)^{-1/2} - \frac{\pi}{2}, \qquad \frac{d\phi}{dr} = -\frac{h(r)}{p_r}, \tag{18}$$

with $h(r)$ assumed to obey the analyticity and fall-off requirements listed above. This will be the starting point for our derivation.

Now, using the condition $p_r(r_m) = 0$, and following the derivation of ref. [63], we find it convenient to isolate

$$\frac{b^2}{r^2} = \frac{r_m^2}{r^2} - \frac{r_m^2}{r^2}\frac{U(r_m,b)}{p_\infty^2}, \tag{19}$$

and insert this into eq. (18). This gives

$$\chi/2 = -\int_{r_m}^{\infty} dr\, \frac{h(r)}{p_\infty}\left(1 - \frac{r_m^2}{r^2} - W(r,b)\right)^{-1/2} - \pi/2, \tag{20}$$

where

$$W(r,b) \equiv \frac{1}{p_\infty^2}\left(U(r,b) - \frac{r_m^2}{r^2}U(r_m,b)\right). \tag{21}$$

Changing integration variable to u through $r^2 = u^2 + r_m^2$ (where $u \geq 0$), we get

$$\frac{\chi}{2} = -\int_0^{\infty} du\, \frac{h(r)}{p_\infty}\left(1 - \frac{r^2}{u^2}W(r,b)\right)^{-1/2} - \frac{\pi}{2}, \tag{22}$$

where r just stands for $r = \sqrt{u^2 + r_m^2}$. Use of the binomial expansion

$$(1+x)^{-1/2} = 1 + \sum_{n=0}^{\infty}\binom{-1/2}{n+1}x^{n+1} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!!}{2^{n+1}(n+1)!}x^{n+1}, \tag{23}$$

yields the following expression for the angle

$$\frac{\chi}{2} = F_0(r_m) - \sum_{n=0}^{\infty}\frac{(2n+1)!!}{2^{n+1}(n+1)!}\int_0^{\infty} du\, \frac{1}{u^{2(n+1)}}\left(\frac{h(r)}{p_\infty}r^{2(n+1)}W(r,b)^{n+1}\right) - \frac{\pi}{2}, \tag{24}$$

where we have defined the function

$$F_0(r_m) \equiv -\frac{1}{p_\infty}\int_0^\infty du\ h(r) \qquad r^2 = u^2 + r_m^2\ . \tag{25}$$

Although this integral is often elementary (such as for the Schwarzschild metric in isotropic coordinates), we do not need to evaluate it explicitly. This will become clear below. In fact, this function, being dependent on r_m must disappear in the end since the scattering angle should not depend on r_m. The remaining terms above can be rewritten by means of the integration-by-parts identity [65],

$$\int_0^\infty \frac{du}{u^{2(n+1)}} f(u) = \frac{1}{(2n+1)!!}\int_0^\infty du \left(\frac{1}{u}\frac{d}{du}\right)^{n+1} f(u) = \frac{2^{n+1}}{(2n+1)!!}\int_0^\infty du \left(\frac{d}{du^2}\right)^{n+1} f(u), \tag{26}$$

valid for any $\mathcal{C}^\infty$ function f for which $f(u)/u^{2n+1}$ vanishes at zero and infinity. On account of our assumptions about $h(r)$, eq. (26) may be applied to eq. (24) to obtain

$$\frac{\chi}{2} = F_0(r_m) - \sum_{n=0}^{\infty}\Delta_n - \frac{\pi}{2}, \tag{27}$$

where we have defined

$$\Delta_n \ \equiv\ \frac{1}{(n+1)!}\int_0^\infty du \left(\frac{d}{du^2}\right)^{n+1}\left[\frac{h(r)}{p_\infty} r^{2(n+1)} W(r,b)^{n+1}\right]. \tag{28}$$

Furthermore, writing

$$W(r,b)^{n+1} = \frac{U(r,b)^{n+1}}{p_\infty^{2(n+1)}}(1-x)^{n+1} \quad \text{with} \quad x \equiv -\frac{r_m^2}{r^2}\frac{U(r_m,b)}{U(r,b)}, \tag{29}$$

we can again Taylor expand, this time in powers of x, to get

$$\Delta_n = \int_0^\infty du \left(\frac{d}{du^2}\right)^{n+1} \sum_{k=0}^{n+1}\frac{1}{(n-k+1)!k!}\frac{h(r)}{p_\infty}\frac{r^{2(n+1)}U(r,b)^{n-k+1}}{p_\infty^{2(n+1)}}\left[-\frac{r_m^2}{r^2}U(r_m,b)\right]^k, \tag{30}$$

and once again we can use eq. (19) to substitute the $U(r_m,b)$ in the square brackets. This results in

$$\Delta_n = \sum_{k=0}^{n+1}\frac{(b^2-r_m^2)^k}{k!}\int_0^\infty du \left(\frac{d}{du^2}\right)^{n+1} \frac{h(r)}{p_\infty}\frac{r^{2(n-k+1)}U(r,b)^{n-k+1}}{(n-k+1)!\,p_\infty^{2(n-k+1)}}. \tag{31}$$

Note that the only explicit r_m-dependence in the integrand occurs through $r = \sqrt{u^2+r_m^2}$. Since r is symmetric in r_m^2 and u^2, we can exchange derivatives in u^2 for derivatives in r_m^2, and consider the identity

$$\left(\frac{d}{du^2}\right)^{n+1} = \left(\frac{d}{dr_m^2}\right)^k\left(\frac{d}{du^2}\right)^{n-k+1}\ . \tag{32}$$

Applying this to the sum in eq. (31), we find

$$\Delta_n = \sum_{k=0}^{n+1}\Delta_{n,k}(r_m)\ , \tag{33}$$

where we have defined

$$\Delta_{n,k} \equiv \frac{(b^2-r_m^2)^k}{k!}\left(\frac{d}{dr_m^2}\right)^k\int_0^\infty du\ \left(\frac{d}{du^2}\right)^{n-k+1}\ \frac{h(r)}{p_\infty}\frac{r^{2(n-k+1)}U(r,b)^{n-k+1}}{(n-k+1)!\,p_\infty^{2(n-k+1)}}\ . \tag{34}$$

We observe that the term with $k = n+1$ is U-independent. Crucially, as we shall demonstrate next, this fact will make the apparent r_m-dependence disappear, cancelling the r_m-dependent piece $F_0(r_m)$. We start by evaluating the

$k = n + 1$ and $F_0(r_m)$ terms together, and introduce (the reason for the factor $1/2$ on the left hand side will become clear shortly),

$$\frac{1}{2}\zeta_{-1} \equiv F_0(r_m) - \sum_{n=0}^{\infty} \Delta_{n,n+1}(r_m). \tag{35}$$

Consider now the Taylor expansion of $F_0(r_m)$ around $r_m = b$. This reads

$$F_0(b) = -\sum_{n=0}^{\infty} \frac{(b^2 - r_m^2)^n}{n!} \left(\frac{d}{dr_m^2}\right)^n \int_0^{\infty} du\ \frac{h(r)}{p_\infty}. \tag{36}$$

Furthermore, we note that the sum $\sum_{n=0}^{\infty} \Delta_{n,n+1}$ can be rewritten as

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta_{n,n+1} &= \sum_{n=0}^{\infty} \frac{(b^2 - r_m^2)^{n+1}}{(n+1)!} \left(\frac{d}{dr_m^2}\right)^{n+1} \int_0^{\infty} du\ \frac{h(r)}{p_\infty}, \\ &= -\int_0^{\infty} du \frac{h(r)}{p_\infty} + \sum_{n=0}^{\infty} \frac{(b^2 - r_m^2)^n}{n!} \left(\frac{d}{dr_m^2}\right)^n \int_0^{\infty} du\ \frac{h(r)}{p_\infty}, \end{aligned} \tag{37}$$

where in the second line we have added and subtracted $F_0(r_m)$. Making use of eq. (36) and the definition of $F_0(r_m)$ we find

$$\sum_{n=0}^{\infty} \Delta_{n,n+1} = F_0(r_m) - F_0(b)\ . \tag{38}$$

Inserting this into eq. (35) results in $\frac{1}{2}\zeta_{-1} = F_0(b)$. The r_m-dependence has explicitly disappeared from this term. It follows from the above that the scattering angle can be written in the form

$$\chi - \zeta_{-1} + \pi = -2\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(b^2 - r_m^2)^k}{k!} \left(\frac{d}{dr_m^2}\right)^k \int_0^{\infty} du \left(\frac{d}{du^2}\right)^{n-k+1} \frac{h(r)}{p_\infty} \frac{r^{2(n-k+1)} U(r,b)^{n-k+1}}{(n-k+1)!\, p_\infty^{2(n-k+1)}} \tag{39}$$

To simplify our notation, we now define

$$\zeta_n(x) \equiv -2\int_0^{\infty} du \left(\frac{d}{du^2}\right)^{n+1} \frac{h(r)}{p_\infty} \frac{r^{2(n+1)} U(r,b)^{n+1}}{(n+1)! p_\infty^{2(n+1)}}, \qquad r^2 = u^2 + x^2 \tag{40}$$

so that

$$\begin{aligned} \chi - \zeta_{-1} + \pi &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(b^2 - r_m^2)^k}{k!} \left(\frac{d}{dr_m^2}\right)^k \zeta_{n-k}(r_m) \\ &= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(b^2 - r_m^2)^\ell}{\ell!} \left(\frac{d}{dr_m^2}\right)^\ell \zeta_n(r_m) \end{aligned} \tag{41}$$

which we recognize as the Taylor expansion of $\zeta_n(b)$ around the point r_m. In this way, the turning point r_m has explicitly disappeared from all terms of the scattering angle, as it should. No regularization of the involved integrals and no use of *ad hoc* rules has been needed. The final formula for the scattering angle thus becomes

$$\chi + \pi = \sum_{n=0}^{\infty} \zeta_n(b) + \zeta_{-1}, \tag{42}$$

where $\zeta_n(b)$ is given in eq. (40) evaluated, as we see, at $r^2 = u^2 + b^2$. The choice of notation for ζ_{-1} is now clear, as this is precisely ζ_n from eq. (40) evaluated at $x = b$ and $n = -1$. Thus we can write eq. (42) as

$$\chi = -2\sum_{n=0}^{\infty} \int_0^{\infty} du \left(\frac{d}{du^2}\right)^n h(r) \frac{r^{2n} U(r,b)^n}{n! p_\infty^{2n+1}} - \pi, \qquad r^2 = u^2 + b^2 \tag{43}$$

This is a very general result, valid for any well-behaved $h(r)$ as stipulated in precise terms above. As we shall see next, it will apply to both scalar and spinning test bodies up to cubic order in the spin of the test particle S and we see no obstacle towards it being applicable to any order in the spin of the probes. The essential ingredient is that the spin of the test body is considered in perturbation theory. Note that when interactions are turned off, the function $h(r)$ reduces to $h(r) = -bp_\infty/r^2$ and the $n = 0$ term in the sum above thus produces zero scattering angle up to terms that vanish when interactions are set to zero.

Having derived this general expression for the scattering angle, it is of interest to see the form it takes when the scattering angle can be expressed in terms of an L-derivative of the radial action as in eq. (12). We will then also see how it relates to the formula derived in ref. [63] for the special case where $U(r, b)$ does not depend on b. Carrying out the derivative in eq. (12), we see that

$$\frac{d\phi}{dr} = -\frac{\partial}{\partial L}p_r = \frac{1}{p_r}\left(\frac{bp_\infty}{r^2} + \frac{1}{2p_\infty}\frac{\partial}{\partial b}U(r, b)\right), \tag{44}$$

and thus

$$h(r) = -\left(\frac{bp_\infty}{r^2} + \frac{1}{2p_\infty}\frac{\partial}{\partial b}U(r, b)\right) . \tag{45}$$

Let us define

$$\Upsilon(r, b) \equiv \frac{\partial}{\partial b}U(r, b) . \tag{46}$$

Inserting $h(r)$ from eq. (45) into (43), we get

$$\begin{aligned}\chi &= 2\sum_{n=0}^{\infty}\int_0^\infty du\left(\frac{d}{du^2}\right)^n\left[\frac{bp_\infty}{r^2} + \frac{1}{2p_\infty}\Upsilon(r, b)\right]\frac{r^{2n}U(r, b)^n}{n!p_\infty^{2(n+1)-1}} - \pi \\ &= 2b\sum_{n=0}^{\infty}\int_0^\infty du\left(\frac{d}{du^2}\right)^{n+1}\frac{r^{2n}U(r, b)^{n+1}}{(n+1)!p_\infty^{2(n+1)}} + \sum_{n=0}^{\infty}\int_0^\infty du\left(\frac{d}{du^2}\right)^n\Upsilon(r, b)\frac{r^{2n}U(r, b)^n}{n!p_\infty^{2(n+1)}}\end{aligned} \tag{47}$$

where we recall the notation $r^2 = u^2 + b^2$. Note that the $n = 0$ term from the free part has cancelled the explicit π, and we have relabelled the remaining terms accordingly. This is a valid and compact form for the scattering angle but we can simplify it further by use of the identity (which is valid for $r^2 = u^2 + b^2$),

$$\frac{d}{db}\left[r^{2n}U(r, b)^{n+1}\right] = (n + 1)r^{2n}\Upsilon(r, b)U(r, b)^n + 2b\frac{d}{du^2}\left[r^{2n}U(r, b)^{n+1}\right] . \tag{48}$$

Substituting this into eq. (47) we obtain the compact expression

$$\chi = \sum_{n=0}^{\infty}\int_0^\infty du\left(\frac{d}{du^2}\right)^n\frac{d}{db}\left[\frac{r^{2n}U(r, b)^{n+1}}{(n+1)!p_\infty^{2(n+1)}}\right], \qquad r^2 = u^2 + b^2 \tag{49}$$

In the special case of a b-independent $U(r, b)$ this is seen to reduce to the formula (4). Remarkably, whether based on the general formula (43) or on the special case (49), we have found that these final results for the scattering angle are almost as simple as those given in ref. [63] even though $U(r, b)$, which effectively acts as a potential, here can depend on angular momentum L (or $b = L/p_\infty$). The more general representation (43) is valid even when we cannot obviously write the scattering angle as an L-derivative of the radial action.

If we substitute the specific form of $U(r, L)$ for Schwarzschild coordinates as in eq. (11) we do indeed recover the correct Schwarzschild scattering angles from eq. (43). To illustrate this point, we perform the Taylor expansion of the $U(r, b)$ in eq. (11) to second order in G. This results in

$$U(r, b) = U_1(r, b) + U_2(r, b) + \mathcal{O}(G^3) = -\frac{(2E^2 - m^2)r^2 - L^2}{r^3}r_s - \frac{(3E^2 - m^2)r^2 - L^2}{r^4}r_s^2 + \mathcal{O}(G^3) . \tag{50}$$

To leading order in G only the single term $U_1(r,b)$ contributes to the scattering angle and we thus get

$$\chi_1 = \int_0^\infty du \frac{d}{db}\frac{1}{p_\infty^2}U_1(r,b) = \frac{2GM(2E^2-m^2)}{bp_\infty^2} = \frac{2(2\gamma^2-1)GMm^2}{bp_\infty^2}, \tag{51}$$

where $E^2=\gamma^2 m^2$ and $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz factor of the test-particle with velocity v at infinity. This is the well-known leading order result. At second order in G there are two contributions: the U_1 term from $n=1$ in the sum (49) and the U_2 term from $n=0$ part. The second-order contribution χ_2 to the scattering angle is thus

$$\begin{aligned}\chi_2 &= \int_0^\infty du \frac{\partial}{\partial b}\left(\frac{d}{du^2}\right)\frac{r^2U_1^2(r,b)}{2p_\infty^4} + \int_0^\infty du\frac{\partial}{\partial b}\frac{U_2(r,b)}{p_\infty^2} = r_s^2\left(\frac{\pi(6E^2-2m^2-p_\infty^2)}{4p_\infty^2b^2} - \frac{\pi(8E^2-4m^2-3p_\infty^2)}{16b^2p_\infty^2}\right)\\ &= 4G^2M^2m^2\pi\left(\frac{5\gamma^2-1}{4p_\infty^2b^2} - \frac{5\gamma^2-1}{16p_\infty^2b^2}\right) = \frac{3G^2M^2m^2\pi(5\gamma^2-1)}{4p_\infty^2b^2}\end{aligned} \tag{52}$$

which is the known answer. In Table 1 below we list the contributions up to and including tenth order in G computed straightforwardly in this manner, but expressed in terms of velocity v rather than γ for the sake of compactness.

n	$\chi_n/\frac{G^nM^n}{b^nv^{2n}}$
1	$2\left(v^2+1\right)$
2	$(3\pi/4)v^2\left(v^2+4\right)$
3	$(2/3)\left(5v^6+45v^4+15v^2-1\right)$
4	$(105\pi/64)v^4\left(v^4+16v^2+16\right)$
5	$(2/5)\left(21v^{10}+525v^8+1050v^6+210v^4-15v^2+1\right)$
6	$(1155\pi/256)v^6\left(v^6+36v^4+120v^2+64\right)$
7	$(2/35)\left(429v^{14}+21021v^{12}+105105v^{10}+105105v^8+15015v^6-1001v^4+91v^2-5\right)$
8	$(45045\pi/16384)v^8\left(5v^8+320v^6+2240v^4+3584v^2+1280\right)$
9	$(2/63)(2431v^{18}+196911v^{16}+1837836v^{14}+4288284v^{12}$ $+2756754v^{10}+306306v^8-18564v^6+1836v^4-153v^2+7)$
10	$(2909907\pi/65536)v^{10}\left(v^{10}+100v^8+1200v^6+3840v^4+3840v^2+1024\right)$

TABLE I. Scattering angle of a non-spinning test particle in a Schwarzschild background up to 10th order in G. χ_n is the G^n contribution to the full scattering angle $\chi=\sum_n\chi_n$.

SCATTERING IN KERR METRICS

We next consider applying the formula we found in the previous section to the scattering of a small non-spinning probe around a Kerr black hole. A standard choice for the metric is Boyer-Lindquist coordinates (t,r,ϕ), for which, when restricted to the equatorial $\theta=\pi/2$ plane, the metric reads

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}\right) & 0 & -\frac{r_sa}{r} \\ 0 & \frac{r^2}{r^2-r_sr+a^2} & 0 \\ -\frac{r_sa}{r} & 0 & \frac{(r+r_s)a^2+r^3}{r}\end{pmatrix}. \tag{53}$$

Letting a test body orbit in this $\theta=\pi/2$ plane, it will have its orbital angular momentum L conserved, and it will therefore remain in that plane. This allows for a well-defined scattering angle and it will also allow us to rewrite the

n	$\chi_n/\frac{G^n M^n}{v^{2n}(b^2-a^2)^{(3n-1)/2}}$
1	$2\big[-2av+b(1+v^2)\big]$
2	$(\pi/2a^2)\Big[(b^2-a^2)^{5/2}v^4+(a-bv)^3\big[-4a^2v+3ab+b^2v\big]\Big]$
3	$(2/3)\big[2a^5v\left(3v^4-10v^2-9\right)-3a^4b\left(v^6-15v^4-45v^2-5\right)-4a^3b^2v\left(15v^4+70v^2+27\right)$ $+2a^2b^3\left(11v^6+135v^4+105v^2+5\right)-18ab^4v\left(5v^4+10v^2+1\right)+b^5\left(5v^6+45v^4+15v^2-1\right)\big]$
4	$(3\pi/16a^4)\Big[2(b^2-a^2)^{11/2}v^8+(a-bv)^5\big[-8a^6v\left(14v^2+5\right)+5a^5b\left(72v^2+7\right)$ $+a^4b^2v\left(16v^2-305\right)-5a^3b^3\left(11v^2-14\right)-a^2b^4v\left(11v^2-30\right)+10ab^5v^2+2b^6v^3\big]\Big]$
5	$(2/15)\big[-2a^9v\left(15v^8-60v^6+378v^4+900v^2+175\right)$ $+15a^8b\left(v^{10}-15v^8+210v^6+1050v^4+525v^2+21\right)+8a^7b^2v\left(45v^8-780v^6-6426v^4-6300v^2-875\right)$ $-140a^6b^3\left(v^{10}-45v^8-630v^6-1050v^4-315v^2-9\right)-84a^5b^4v\left(45v^8+1020v^6+2814v^4+1500v^2+125\right)$ $+14a^4b^5\left(67v^{10}+3375v^8+15750v^6+14070v^4+2295v^2+27\right)-1400a^3b^6v\left(9v^8+84v^6+126v^4+36v^2+1\right)$ $+36a^2b^7\left(29v^{10}+875v^8+2450v^6+1190v^4+65v^2-1\right)-50ab^8v\left(63v^8+420v^6+378v^4+36v^2-1\right)$ $+3b^9\left(21v^{10}+525v^8+1050v^6+210v^4-15v^2+1\right)\big]$
6	$(5\pi/128a^6)\Big[8(b^2-a^2)^{17/2}v^{12}+(a-bv)^7\big[-4a^{10}v\left(1584v^4+1540v^2+189\right)$ $+7a^9b\left(5720v^4+2808v^2+99\right)-a^8b^2v\left(2200v^4+85232v^2+17829\right)+14a^7b^3\left(260v^4+5391v^2+330\right)$ $-2a^6b^4v\left(334v^4-1491v^2+14070\right)+21a^5b^5\left(85v^4-272v^2+176\right)+a^4b^6v\left(255v^4-1904v^2+1680\right)$ $-28a^3b^7v^2\left(17v^2-24\right)-4a^2b^8v^3\left(17v^2-56\right)+56ab^9v^4+8b^{10}v^5\big]\Big]$

TABLE II. Scattering angle of a non-spinning test particle in a Kerr background up to 6th order in G. χ_n is the G^n contribution to the full scattering angle $\chi=\sum_n\chi_n$.

Kerr metric in normal form. We find the radial momentum p_r from the Hamilton-Jacobi equation

$$p_r^2=\frac{r\left(p_\infty^2r^3+m^3r^2r_s+(a^2p_\infty^2-L^2)r+r_s(Ea-L)^2\right)}{(a^2+r^2-r_sr)^2},\tag{54}$$

and we can write it in the form $p_r^2=T(r,L,a)-U(r,L,a)$, with

$$T(r,L,a)\equiv\frac{r^2}{r^2+a^2}\left(p_\infty^2-\frac{L^2}{r^2+a^2}\right),\tag{55}$$

which indeed is independent of G, and

$$\begin{aligned}U(r,L,a)\equiv&-\frac{\left[(2E^2-m^2)r^6+(m^2-E^2)r^5r_s+((4E^2-m^2)a^4-4ELa^3)r^2\right]rr_s}{(a^2+r^2-r_sr)^2(a^2+r^2)^2}\\&-\frac{\left[((5E^2-2m^2)a^2-2ELa-L^2)r^4-r_s((E^2-m^2)a^2-L^2)r^3+a^4(Ea-L)^2\right]rr_s}{(a^2+r^2-r_sr)^2(a^2+r^2)^2},\end{aligned}\tag{56}$$

which carries all G-dependence. Although well separated into T and U pieces, we notice that T is not of the free kind shown in eq. (17). Thus, Boyer-Lindquist coordinates are not of normal form and we need to choose different coordinates in order for our formalism to be applicable. As noted in the previous section, the needed change of integration variables in the radial action can equivalently be viewed as a coordinate transformation away from Boyer-Lindquist coordinates, thus leading to a different metric.

Indeed, in the $G\to0$ limit the Boyer-Lindquist metric eq. (53) takes the form

$$g_{\mu\nu}=\begin{pmatrix}-1&0&0\\0&\frac{r^2}{r^2+a^2}&0\\0&0&a^2+r^2\end{pmatrix}\tag{57}$$

which does not correspond to flat Minkowski space in ordinary polar coordinates. Since the Kerr metric is diagonal and only depends on the radial coordinate r in this limit, we can find a coordinate change $r \to \rho(r)$ which allows us to recover the free structure of T. This change is given simply by

$$\rho^2 = r^2 + a^2 \ . \tag{58}$$

For this new radial coordinate ρ the Kerr metric takes the form

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} \tag{59}$$

in the $G \to 0$ limit, corresponding to a metric in normal form. The transformation eq. (58), also automatically produces the needed

$$T(\rho, L, a) = p_\infty^2 - \frac{L^2}{r^2} \ . \tag{60}$$

Note that the free part of the Kerr metric becomes independent of the black-hole spin a in these coordinates. The radial momentum p_r transforms like

$$p_\rho = \frac{dr}{d\rho} p_r \tag{61}$$

under this coordinate change, and so we obtain the new effective potential

$$\tilde{U}(\rho, b, a) = \left(\frac{dr}{d\rho}\right)^2 U(r, b, a). \tag{62}$$

We may thus write the formula for the scattering angle (49) in terms of ρ as

$$\chi = \sum_{n=0}^{\infty} \int_0^\infty du \frac{\partial}{\partial b} \left(\frac{d}{du^2}\right)^n \frac{\rho^{2n} \tilde{U}(\rho, b)^{n+1}}{(n+1)! p_\infty^{2(n+1)}}, \qquad \rho^2 = u^2 + b^2. \tag{63}$$

As a first quick check, we compute the scattering angle χ_1 to leading order in G using eq. (63). We find

$$\chi_1 = \frac{2GM\left(\gamma^2(2b - 2av) - b\right)}{\gamma^2 v^2 \left(b^2 - a^2\right)} \tag{64}$$

where v is the asymptotic velocity of the test-particle. This agrees with the scattering angle computed in ref. [35] when restricted to the test-body limit. We emphasize that as in ref. [35] our result gives the scattering angle to all orders in the spin of the black hole a. It is easy to verify that in the light-like limit $v \to 1, \gamma \to \infty$, the result above reproduces the terms of the expansion provided in ref. [62].

Although the integrals are slightly more involved than those of Schwarzschild scattering, the final results for massive-probe scattering are relatively simple. In table II we list results up to and including sixth order in G (this table can readily be extended based on our general formula). Again, expanding in powers of a and taking the massless limit this reproduces the well-known light-bending formulas for Kerr metrics.

We note that the resulting scattering angle contribution χ_n to any order in G displays some simple patterns. First, the scattering angle naturally has emerged in a form that re-sums all orders in a. Second, to order n one may identify the prefactor

$$c_n \equiv \frac{G^n M^n}{(b^2 - a^2)^{(3n-1)/2} v^{2n}}. \tag{65}$$

which accounts for the all-order-in-spin behaviour. We also note that even orders in powers of G are relatively simpler, and one can easily identify more structural patterns in them. To be concrete, we observe that after factorising the

term in eq. (65), the angle takes the form

$$\chi_n/c_n = (d_{1,n}\pi/a^{2n})\Big[d_{2,n}v^{2n}(b^2-a^2)^{(3n-1)/2} + (a-bv)^{n+1}\sum_{\ell=0}^{2n-2} a^\ell b^{2n-2-\ell} f_{n,\ell}(v)\Big] \tag{66}$$

where $d_{i,n}$ are numerical constants, and $f_{n,k}(v)$ is a polynomial in v of degree $n-1$ for even ℓ or $n-2$ for odd ℓ.

ADDING SPIN TO THE PROBE

It is well known that it is possible to consider a probe limit in which the mass is negligible but the (rescaled) spin of the probe is finite. In this section we extend the scattering angle calculation to the case of a spinning probe in a Kerr metric. We will be able to derive results up to and including second order in the probe spin.

The description of the motion of extended bodies in general relativity is a complicated problem, and one usually needs to resort to the use of approximation schemes. For example, one may utilize a multipolar approximation method originally devised by Tulczyjew [66], to explicitly work out the equations of motion. This method was applied by Steinhoff and Puetzfeld in ref. [67], using a multipole approximation up to the quadrupolar order, i.e. keeping the covariant monopole $t^{\mu\nu}$, dipole $t^{\mu\nu\alpha}$ and quadrupole $t^{\mu\nu\alpha\beta}$ moments. They write the stress-energy tensor $T^{\mu\nu}$ in the manifestly covariant form

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} - \nabla_\alpha(t^{\mu\nu\alpha}\delta_{(4)}) + \frac{1}{2!}\nabla_{\alpha\beta}(t^{\mu\nu\alpha\beta}\delta_{(4)})\right]. \tag{67}$$

Here τ is the proper time of the worldline $x^\rho(\tau)$, and $\delta_{(4)} \equiv \delta(y^\rho - x^\rho(\tau))$. The dynamics of the multipolar test body follow from demanding that the stress-energy tensor (67) is covariantly conserved

$$\nabla_\nu T^{\mu\nu} = 0\,. \tag{68}$$

This is sometimes referred to as Mathisson's variational equations of mechanics [68, 69] and imposes certain conditions on the multipole moments. Building on Tulczyjew's method [66], eq. (68) is explicitly evaluated in ref. [67] where it was found that the multipole tensors $t^{\mu\nu\cdots}$ can be expressed in terms of a vector p^μ, an antisymmetric tensor $S^{\mu\nu}$, and Dixon's reduced moment $J^{\mu\nu\alpha\beta}$, which has the same symmetries as the Riemann tensor. In terms of them, the stress-energy tensor becomes [67]

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[\dot{x}^{(\mu}p^{\nu)}\delta_{(4)} + \frac{1}{3}R_{\alpha\beta\rho}{}^{(\mu}J^{\nu)\rho\beta\alpha}\delta_{(4)} + \nabla_\alpha\left(\dot{x}^{(\mu}S^{\nu)\alpha}\delta_{(4)}\right) - \frac{2}{3}\nabla_\alpha\nabla_\beta\left(J^{\mu\alpha\beta\nu}\delta_{(4)}\right)\right], \tag{69}$$

where $\dot{x}^\mu$ is the tangent to the worldline, $R_{\mu\nu\rho\sigma}$ is the Riemann tensor (defined via $2\nabla_{[\mu}\nabla_{\nu]}w_\rho = R_{\mu\nu\rho}{}^\sigma w_\sigma$), where the (square) brackets denote (anti-) symmetrization of enclosed indices (e.g. $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$). The vector p^μ and tensor $S^{\mu\nu}$ are then identified as the linear momentum vector and spin tensor of the object (which now play the role of monopole and dipole moment). The motion of a multipolar test body (or probe) in a generic curved background spacetime is described by the two equations governing the evolution of its momentum and spin along the worldline, which are also obtained by evaluating eq. (68) in ref. [67]. Through the quadrupolar order in the multipole expansion, they read

$$\frac{Dp_\mu}{d\tau} + \frac{1}{2}R_{\mu\nu\rho\sigma}\dot{x}^\nu S^{\rho\sigma} = -\frac{1}{6}\nabla_\mu R_{\kappa\lambda\rho\sigma}J^{\kappa\lambda\rho\sigma}, \tag{70a}$$

$$\frac{DS^{\mu\nu}}{d\tau} - 2p^{[\mu}\dot{x}^{\nu]} = \frac{4}{3}R^{[\mu}{}_{\lambda\rho\sigma}J^{\nu]\lambda\rho\sigma}. \tag{70b}$$

These are the Mathisson-Papapetrou-Dixon equations [68–70]. They may also be derived from (68) without the assumption of a distributional $T^{\mu\nu}$ (see *e.g.* ref. [71]), or alternatively from an effective action (see refs. [72, 73] for recent examples). For a specific quadrupole tensor given as a function of p^μ and $S^{\mu\nu}$, a closed set of evolution equations is completed by the imposition of a "spin supplementary condition". We will here employ the Tulczyjew-Dixon choice [66, 70],

$$p_\mu S^{\mu\nu} = 0, \tag{71}$$

(n,k)	$\chi_{n,k}/\frac{G^n M^n (S/m)^k}{v^{2n}(b^2-a^2)^{(3n+2k-1)/2}}$
$(1,1)$	$-4(av-b)(a-bv)$
$(1,2)$	$-4a^3v+6a^2b\left(v^2+1\right)-12ab^2v+2b^3\left(v^2+1\right)$
$(2,1)$	$(3\pi/2)(av-b)(a-bv)\big[a^2\left(-2v^2-3\right)+10abv-b^2\left(3v^2+2\right)\big]$
$(2,2)$	$(3\pi/4)\big[-10a^5v\left(v^2+1\right)+a^4b\left(12v^4+71v^2+12\right)-90a^3b^2v\left(v^2+1\right)$ $+a^2b^3\left(21v^4+128v^2+21\right)-40ab^4v\left(v^2+1\right)+b^5\left(2v^4+11v^2+2\right)\big]$
$(3,1)$	$8(av-b)(a-bv)\big[a^4\left(-v^4-10v^2-5\right)+8a^3bv\left(3v^2+5\right)$ $-2a^2b^2\left(5v^4+38v^2+5\right)+8ab^3v\left(5v^2+3\right)-b^4\left(5v^4+10v^2+1\right)\big]$
$(3,2)$	$4\big[-2a^7v\left(7v^4+30v^2+11\right)+5a^6b\left(3v^6+55v^4+65v^2+5\right)$ $-6a^5b^2v\left(51v^4+190v^2+63\right)+5a^4b^3\left(17v^6+265v^4+275v^2+19\right)$ $-10a^3b^4v\left(53v^4+170v^2+49\right)+3a^2b^5\left(19v^6+255v^4+225v^2+13\right)$ $-10ab^6v\left(11v^4+30v^2+7\right)+b^7\left(3v^6+35v^4+25v^2+1\right)\big]$
$(4,1)$	$(105\pi/16)(av-b)(a-bv)^3\big[a^4\left(-8v^4-20v^2-5\right)+12a^3bv\left(6v^2+5\right)$ $-2a^2b^2\left(10v^4+79v^2+10\right)+12ab^3v\left(5v^2+6\right)-b^4\left(5v^4+20v^2+8\right)\big]$
$(4,2)$	$(15\pi/32)(a-bv)\big[-2a^8v\left(24v^6+320v^4+485v^2+95\right)$ $+7a^7b\left(248v^6+1100v^4+635v^2+30\right)-a^6b^2v\left(760v^6+15808v^4+25345v^2+4980\right)$ $+105a^5b^3\left(92v^6+466v^4+276v^2+13\right)-15a^4b^4v\left(106v^6+2272v^4+3860v^2+769\right)$ $+21a^3b^5\left(405v^6+2050v^4+1258v^2+60\right)-a^2b^6v\left(585v^6+12140v^4+20270v^2+4196\right)$ $+7ab^7\left(160v^6+775v^4+460v^2+24\right)-5b^8v\left(4v^6+79v^4+124v^2+24\right)\big]$
$(5,1)$	$4(av-b)(a-bv)\big[a^8\left(v^8-36v^6-378v^4-420v^2-63\right)$ $+64a^7bv\left(v^6+27v^4+63v^2+21\right)-4a^6b^2\left(9v^8+668v^6+3222v^4+2268v^2+105\right)$ $+64a^5b^3v\left(27v^6+289v^4+405v^2+63\right)-2a^4b^4\left(189v^8+6444v^6+18094v^4+6444v^2+189\right)$ $+64a^3b^5v\left(63v^6+405v^4+289v^2+27\right)-4a^2b^6\left(105v^8+2268v^6+3222v^4+668v^2+9\right)$ $+64ab^7v\left(21v^6+63v^4+27v^2+1\right)-b^8\left(63v^8+420v^6+378v^4+36v^2-1\right)\big]$
$(5,2)$	$-2\big[2a^{11}v\left(11v^8+500v^6+2114v^4+1652v^2+203\right)$ $-7a^{10}b\left(3v^{10}+467v^8+4214v^6+6734v^4+2087v^2+63\right)$ $+2a^9b^2v\left(1665v^8+35036v^6+110726v^4+73052v^2+8001\right)$ $-21a^8b^3\left(51v^{10}+3491v^8+22358v^6+28910v^4+7703v^2+207\right)$ $+12a^7b^4v\left(2845v^8+41836v^6+103726v^4+56812v^2+5341\right)$ $-42a^6b^5\left(133v^{10}+6517v^8+32410v^6+33922v^4+7489v^2+169\right)$ $+84a^5b^6v\left(829v^8+9580v^6+19054v^4+8428v^2+637\right)$ $-6a^4b^7\left(1029v^{10}+40789v^8+164122v^6+137410v^4+23617v^2+393\right)$ $+42a^3b^8v\left(795v^8+7604v^6+12194v^4+4148v^2+219\right)$ $-a^2b^9\left(1449v^{10}+49049v^8+162722v^6+105770v^4+12437v^2+93\right)$ $+14ab^{10}v\left(203v^8+1652v^6+2114v^4+500v^2+11\right)$ $-b^{11}\left(35v^{10}+1043v^8+2870v^6+1358v^4+71v^2-1\right)\big]$
$(6,1)$	$(3465\pi/128)(av-b)(a-bv)^5\big[a^6\left(-32v^6-112v^4-70v^2-7\right)$ $+26a^5bv\left(16v^4+28v^2+7\right)-a^4b^2\left(112v^6+1796v^4+1337v^2+70\right)$ $+52a^3b^3v\left(14v^4+57v^2+14\right)-a^2b^4\left(70v^6+1337v^4+1796v^2+112\right)$ $+26ab^5v\left(7v^4+28v^2+16\right)-b^6\left(7v^6+70v^4+112v^2+32\right)\big]$

TABLE III. Scattering angle of a spinning probe in a Kerr background up to 5th order in G. $\chi_{n,k}$ is the $G^n S^k$ contribution to the full scattering angle $\chi=\sum_{n,k}\chi_{n,k}$.

which, together with (70), determines the worldline tangent $\dot{x}^\mu$ in terms of the other quantities. Given a Killing vector

field ξ^μ of the background space-time, and regardless of the choice of the spin supplementary condition, an important property of the equations (70) is that the quantity

$$\mathcal{P}_\xi = \xi^\mu p_\mu + \frac{1}{2} S^{\mu\nu} \nabla_\mu \xi_\nu \tag{72}$$

is conserved along the worldline, *i.e.*, $D\mathcal{P}_\xi/d\tau = 0$. This holds to all orders in the multipole expansion [71]. The system of equations (70)–(71) is explicitly invariant under reparametrizations of the worldline, but for simplicity we will here adopt the condition $\dot{x}^2 \equiv \dot{x}_\mu \dot{x}^\mu = -1$, making τ the proper time.

A form of the quadrupole tensor J appropriate to describe a spin-induced quadrupole, quadratic in the spin, and assuming eq. (71) is given by

$$J^{\mu\nu\rho\sigma} = \frac{-3}{(-p^2)^{3/2}} p^{[\mu} S^{\nu]} p^{[\rho} S^{\sigma]}, \tag{73}$$

for the case of a black-hole probe [74]. We will restrict ourselves to such probes here but stress that probes with internal and finite-size structure can be treated in this formalism as well. Here, S^μ is the Pauli-Lubanski spin vector,

$$S^\mu = -\frac{1}{2} \epsilon^\mu{}_{\nu\rho\sigma} \frac{p^\nu}{\sqrt{-p^2}} S^{\rho\sigma} \qquad \Leftrightarrow \qquad S^{\mu\nu} = \epsilon^{\mu\nu}{}_{\rho\sigma} \frac{p^\rho}{\sqrt{-p^2}} S^\sigma, \tag{74}$$

with $p_\mu S^\mu = 0$. It has invariant magnitude

$$S^2 \equiv S_\mu S^\mu = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}. \tag{75}$$

We next solve for the worldline tangent $\dot{x}^\mu$ by covariantly differentiating (71) with respect to τ and inserting equations (70). For our case of a black-hole probe with its associated spin-induced quadrupole (73), and working perturbatively in the test body's spin S, one finds after a remarkable cancellation the simple relation

$$\dot{x}^\mu = \frac{p^\mu}{\sqrt{-p^2}} + \mathcal{O}(S^3), \tag{76}$$

as noted in ref. [75]. That is, the tangent is still proportional to the momentum through this order, for a black hole. Finally, one can verify from (70) with (73)–(74) that the quantity

$$m^2 \equiv p^2 + R_{\mu\nu\rho\sigma} \frac{p^\mu p^\rho}{-p^2} S^\nu S^\rho + \mathcal{O}(S^3) \tag{77}$$

is conserved to the order shown. Taking the flat space limit, we identify m with the mass of the scattered probe. All of this holds in a general curved background.

We now restrict ourselves to the background of a Kerr spacetime outside a black hole of mass M and spin Ma in Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \phi)$. We again consider the motion in the equatorial plane $\theta = \pi/2$, and with the probe spin aligned (or anti-aligned) with the symmetry axis, $S^\mu = -S e_\theta{}^\mu$ where $e_\theta{}^\mu$ is the unit vector in the θ direction. We take the constant scalar S to carry a sign: positive when the probe spin is aligned with the Kerr spin, and negative when anti-aligned. Note that the motion will remain in the equatorial plane only when the spin is aligned, and the spin will remain aligned only when the motion is in the equatorial plane. In this case, the evolution equation (70b) for $S^{\mu\nu}$ is automatically satisfied, and the content of evolution equation (70a) for p^μ is equivalent to the conservation equations for three constants of motion: the invariant mass m of (77) and the two constants (72) from the two Killing vectors of the Kerr background. The timelike Killing vector $t^\mu \equiv (\partial_t)^\mu$ gives the conserved energy $E \equiv \mathcal{P}_t$, and the axial Killing vector $\phi^\mu \equiv (\partial_\phi)^\mu$ gives in this aligned-spin/equatorial case the total angular momentum $J \equiv \mathcal{P}_\phi$,

$$E = -p_a t^a - \frac{1}{2} S^{ab} \nabla_a t_b \qquad\qquad J = p_a \phi^a + \frac{1}{2} S^{ab} \nabla_a \phi_b \tag{78}$$

$$= -p_t + \frac{GMS}{r^3 \sqrt{-p^2}} (p_\phi + a p_t), \qquad\qquad = p_\phi + \frac{S}{\sqrt{-p^2}} \Big[-p_t + \frac{GMa}{r^3} (p_\phi + a p_t) \Big], \tag{79}$$

where the second line has evaluated in terms of the momentum components $p_\mu = (p_t, p_r, p_\theta = 0, p_\phi)$ in Boyer-Lindquist

coordinates with $\theta = 0, p_\theta = 0$ and with the spin tensor as specified above. Similarly evaluating (77) yields

$$m^2 = -p^2 + \frac{GMS^2}{r^3}\left[1 + 3\frac{(p_\phi + ap_t)^2}{r^2(-p^2)}\right] + \mathcal{O}(S^3), \tag{80}$$

with

$$-p^2 = -g^{\mu\nu}p_\mu p_\nu = \frac{[(r^2+a^2)p_t + ap_\phi]^2}{r^2\Delta} - \frac{(p_\phi + ap_t)^2}{r^2} - \frac{\Delta}{r^2}p_r^2, \tag{81}$$

and $\Delta \equiv r^2 + a^2 - 2GMr$. Now the system (78)–(81) can be solved, working perturbatively in S, for the momentum components p_t, p_r and p_ϕ as functions of only the Boyer-Lindquist radial coordinate r and the constants M, a, E, J and m.

From the relation (76) for the tangent vector, with $p^\mu = g^{\mu\nu}p_\nu$, evaluating the r and ϕ components yields

$$\dot{\phi} = \frac{1}{\Delta\sqrt{-p^2}}\Big[p_\phi - \frac{2GM}{r}(p_\phi + ap_t)\Big] + \mathcal{O}(S^3), \tag{82}$$

$$\dot{r} = \frac{\Delta\, p_r}{r^2\sqrt{-p^2}} + \mathcal{O}(S^3), \tag{83}$$

which can each be expressed as functions of r and the constants of motion from the results above. Then the scattering angle χ can be computed from

$$\chi = 2\int_{r_m}^{\infty} dr\, \frac{\dot{\phi}}{\dot{r}} - \pi\,. \tag{84}$$

From this expression we can immediately make contact with our general formula (18). Note that $d\phi/dr$ up to and including $\mathcal{O}(S^2)$ has the correct form to readily identify $h(r)$. First,

$$\frac{d\phi}{dr} = \frac{r(lr - 2G\kappa M)}{p_r\left(a^2 + r(r-2GM)\right)^2} - \frac{aG\kappa MS}{mrp_r\left(a^2 + r(r-2GM)\right)^2} + \frac{G\kappa MS^2(r-2GM)}{mr^2p_r\left(a^2 + r(r-2GM)\right)^2} + \mathcal{O}(S^3) \tag{85}$$

where for simplicity we have introduced $\gamma = E/m$ and $l \equiv L/m$, where $L \equiv J - \gamma S$ is the orbital angular momentum (cf. eq. (79) as $r \to \infty$), and $\kappa \equiv l - \gamma a$. The radial momentum p_r is the positive root of eq. (81),

$$p_r = \sqrt{\frac{r^2}{\Delta}\left[\frac{[(r^2+a^2)p_t + ap_\phi]^2}{r^2\Delta} - \frac{(p_\phi + ap_t)^2}{r^2} + p^2\right]}\,. \tag{86}$$

This identifies

$$h(r) = -\frac{r(lr - 2G\kappa M)}{\left(a^2 + r(r-2GM)\right)^2} + \frac{aG\kappa MS}{mr\left(a^2 + r(r-2GM)\right)^2} - \frac{G\kappa MS^2(r-2GM)}{m^2r^2\left(a^2 + r(r-2GM)\right)^2} + \mathcal{O}(S^3)\,. \tag{87}$$

We can readily use the expression above in normal coordinate systems by means of the transformation (58). Furthermore, $h(r)$ obeys our requirements of being real analytic on the interval $r \in [r_m, \infty[$ and with falls-off as $\lim_{r\to\infty} h(r) \sim 1/r^n$, with $n \geq 2$. Therefore, eq. (43) can be used. Note that to each order in S, the radial momentum p_r must be expanded correspondingly. An equivalent form of this integrand was first derived (by the same methods) in ref. [76].

Results up to sixth order in G and up to second order in probe spin S are given in Table III. We note that the pattern of resummation in the Kerr black hole spin is generalized to an overall prefactor of

$$c_{n,k} \equiv \frac{G^n M^n (S/m)^k}{v^{2n}(b^2 - a^2)^{(3n+2k-1)/2}} \tag{88}$$

to first ($k = 1$) and second ($k = 2$) order in the probe spin. It is tempting to conjecture that this pattern will hold to higher orders ($k > 2$) in the probe spin. Furthermore, we observe that the remainders after factorising eq. (88) again

shows remarkable structures to linear order in the spin of the probe S, i.e. for $k = 1$

$$\chi_{n,1}/c_{n,1} = (av - b)(a - bv) \sum_{\ell=0}^{2n-2} a^\ell b^{2n-2-\ell} f_{n,1,\ell}(v) \qquad \text{for odd } n, \tag{89}$$

$$\chi_{n,1}/c_{n,1} = (av - b)(a - bv)^{n-1} \sum_{\ell=0}^{n} a^\ell b^{n-\ell} f_{n,1,\ell}(v) \qquad \text{for even } n, \tag{90}$$

where $f_{n,1,\ell}(v)$ are polynomials in v of order n for even n and order $2n - 2$ for odd n. We have not found any discernible structure for the results at quadratic order in S.

CONCLUSION

We have derived a simple formula for the scattering angle of massless probes in external black hole metrics. Building on the compact formula presented in ref. [63], we have found a scattering angle expression that straightforwardly handles metrics in any choice of coordinates belonging to a class we have denoted as normal. In such coordinates the metric enjoys the property of reducing to flat Minkowski metric in polar coordinates when one takes the limit $G \to 0$. To illustrate, we have derived the scattering angles of massive and massless probes in the metric of a Schwarzschild black hole in Schwarzschild coordinates. The final scattering angle formula is manifestly free of any dependence on the turning point r_m of the orbit without any need of regularization or prescription.

While of interest in itself, the existence of such a compact formula for the scattering angle becomes more important in the case of scattering in the equatorial plane of Kerr black hole metrics. Choosing standard Boyer-Lindquist coordinates, one notices that the Kerr metric is not in normal form in those coordinates. We show that a simple transformation of the radial coordinate brings the Kerr metric to normal form and we are then able to rather effortlessly calculate the scattering angle in this Kerr metric to any desired order in G. Interestingly, we find that the resulting expressions all re-sum the dependence on the black hole spin a to all orders, for any fixed order in G. Finally, we have extended these scattering angle calculations to the case of spinning black-hole probes in the aligned (or anti-aligned) case of spins in the equatorial plane of the Kerr metric. Our results display regularities up to second order in the probe spin that may lead to a better understanding of all-order results in the case of scattering of spinning black holes. We expect the resulting expressions to be useful for the community presently computing scattering angles from gravitational scattering amplitudes in the Post-Minkowskian expansion.

Acknowledgements: The work of P.H.D. and A.L. was supported in part by Independent Research Fund Denmark, grant number 0135-00089B. This research was also supported in part by the National Science Foundation under Grant No. NSF PHY-1748958. AL is supported partially by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 847523 INTERACTIONS. We thank the Kavli Institute of theoretical Physics for hospitality during the program "High-Precision Gravitational Waves" where part of this work was done.

[1] T. Damour, Phys. Rev. D **94**, no.10, 104015 (2016) doi:10.1103/PhysRevD.94.104015 [arXiv:1609.00354 [gr-qc]]; Phys. Rev. D **97**, no.4, 044038 (2018) doi:10.1103/PhysRevD.97.044038 [arXiv:1710.10599 [gr-qc]].

[2] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, Phys. Rev. Lett. **121**, no. 17, 171601 (2018) doi:10.1103/PhysRevLett.121.171601 [arXiv:1806.04920 [hep-th]].

[3] C. Cheung, I. Z. Rothstein and M. P. Solon, Phys. Rev. Lett. **121**, no. 25, 251101 (2018) doi:10.1103/PhysRevLett.121.251101 [arXiv:1808.02489 [hep-th]].

[4] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. D **100**, no.8, 084040 (2019) doi:10.1103/PhysRevD.100.084040 [arXiv:1906.01579 [hep-th]].

[5] D. A. Kosower, B. Maybee and D. O'Connell, JHEP **02** (2019), 137 doi:10.1007/JHEP02(2019)137 [arXiv:1811.10950 [hep-th]].

[6] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon and M. Zeng, Phys. Rev. Lett. **122**, no.20, 201603 (2019) doi:10.1103/PhysRevLett.122.201603 [arXiv:1901.04424 [hep-th]]; JHEP **10**, 206 (2019) doi:10.1007/JHEP10(2019)206 [arXiv:1908.01493 [hep-th]].

[7] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent and J. Vines, Phys. Rev. D **99** (2019) no.10, 104004 [arXiv:1901.07102 [gr-qc]].

[8] J. Parra-Martinez, M. S. Ruf and M. Zeng, JHEP **11**, 023 (2020) doi:10.1007/JHEP11(2020)023 [arXiv:2005.04236 [hep-th]].

[9] S. Mougiakakos, M. M. Riva and F. Vernizzi, Phys. Rev. D **104**, no.2, 024041 (2021) doi:10.1103/PhysRevD.104.024041 [arXiv:2102.08339 [gr-qc]].

[10] E. Herrmann, J. Parra-Martinez, M. S. Ruf and M. Zeng, JHEP **10**, 148 (2021) doi:10.1007/JHEP10(2021)148 [arXiv:2104.03957 [hep-th]].

[11] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, Phys. Lett. B **811**, 135924 (2020) doi:10.1016/j.physletb.2020.135924 [arXiv:2008.12743 [hep-th]]; Phys. Lett. B **818** (2021), 136379 [arXiv:2101.05772 [hep-th]]; JHEP **07**, 169 (2021) doi:10.1007/JHEP07(2021)169 [arXiv:2104.03256 [hep-th]].

[12] T. Damour, Phys. Rev. D **102** (2020) no.12, 124008 doi:10.1103/PhysRevD.102.124008 [arXiv:2010.01641 [gr-qc]].

[13] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté and P. Vanhove, Phys. Rev. D **104** (2021) no.2, 026009 [arXiv:2104.04510 [hep-th]]; JHEP **08**, 172 (2021) doi:10.1007/JHEP08(2021)172 [arXiv:2105.05218 [hep-th]].

[14] P. H. Damgaard, L. Plante and P. Vanhove, JHEP **11**, 213 (2021) doi:10.1007/JHEP11(2021)213 [arXiv:2107.12891 [hep-th]].

[15] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, JHEP **10**, 118 (2021) doi:10.1007/JHEP10(2021)118 [arXiv:2108.04216 [hep-th]].

[16] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C. H. Shen, M. P. Solon and M. Zeng, Phys. Rev. Lett. **126**, no.17, 171601 (2021) doi:10.1103/PhysRevLett.126.171601 [arXiv:2101.07254 [hep-th]]; [arXiv:2112.10750 [hep-th]].

[17] N. E. J. Bjerrum-Bohr, L. Planté and P. Vanhove, JHEP **03**, 071 (2022) doi:10.1007/JHEP03(2022)071 [arXiv:2111.02976 [hep-th]].

[18] G. Kälin and R. A. Porto, JHEP **11**, 106 (2020) doi:10.1007/JHEP11(2020)106 [arXiv:2006.01184 [hep-th]].

[19] G. Kälin, Z. Liu and R. A. Porto, Phys. Rev. Lett. **125**, no.26, 261103 (2020) doi:10.1103/PhysRevLett.125.261103 [arXiv:2007.04977 [hep-th]]; Phys. Rev. D **102**, 124025 (2020) doi:10.1103/PhysRevD.102.124025 [arXiv:2008.06047 [hep-th]].

[20] Z. Liu, R. A. Porto and Z. Yang, JHEP **06**, 012 (2021) doi:10.1007/JHEP06(2021)012 [arXiv:2102.10059 [hep-th]].

[21] C. Dlapa, G. Kälin, Z. Liu and R. A. Porto, [arXiv:2106.08276 [hep-th]]; [arXiv:2112.11296 [hep-th]].

[22] G. Kälin, J. Neef and R. A. Porto, [arXiv:2207.00580 [hep-th]].

[23] G. Mogull, J. Plefka and J. Steinhoff, JHEP **02**, 048 (2021) doi:10.1007/JHEP02(2021)048 [arXiv:2010.02865 [hep-th]].

[24] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, Phys. Rev. Lett. **126**, no.20, 201103 (2021) doi:10.1103/PhysRevLett.126.201103 [arXiv:2101.12688 [gr-qc]]; Phys. Rev. Lett. **128** (2022) no.1, 011101 doi:10.1103/PhysRevLett.128.011101 [arXiv:2106.10256 [hep-th]]; JHEP **01** (2022), 027 doi:10.1007/JHEP01(2022)027 [arXiv:2109.04465 [hep-th]].

[25] G. U. Jakobsen and G. Mogull, Phys. Rev. Lett. **128**, no.14, 141102 (2022) doi:10.1103/PhysRevLett.128.141102 [arXiv:2201.07778 [hep-th]].

[26] G. U. Jakobsen, G. Mogull, J. Plefka and B. Sauer, [arXiv:2207.00569 [hep-th]].

[27] M. V. S. Saketh and J. Vines, [arXiv:2208.03170 [gr-qc]].

[28] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plante and P. Vanhove, [arXiv:2203.13024 [hep-th]].

[29] D. A. Kosower, R. Monteiro and D. O'Connell, [arXiv:2203.13025 [hep-th]].

[30] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M. P. Solon and M. Zeng, [arXiv:2204.05194 [hep-th]].

[31] Y. F. Bautista, A. Guevara, C. Kavanagh and J. Vines, [arXiv:2107.10179 [hep-th]].

[32] M. Chiodaroli, H. Johansson and P. Pichini, JHEP **02**, 156 (2022) doi:10.1007/JHEP02(2022)156 [arXiv:2107.14779 [hep-th]].

[33] L. Cangemi and P. Pichini, [arXiv:2207.03947 [hep-th]].

[34] A. Ochirov and E. Skvortsov, [arXiv:2207.14597 [hep-th]].

[35] J. Vines, Class. Quant. Grav. **35**, no.8, 084002 (2018) doi:10.1088/1361-6382/aaa3a8 [arXiv:1709.06016 [gr-qc]].

[36] A. Guevara, JHEP **04**, 033 (2019) doi:10.1007/JHEP04(2019)033 [arXiv:1706.02314 [hep-th]].

[37] J. Vines, J. Steinhoff and A. Buonanno, Phys. Rev. D **99**, no.6, 064054 (2019) doi:10.1103/PhysRevD.99.064054 [arXiv:1812.00956 [gr-qc]].

[38] A. Guevara, A. Ochirov and J. Vines, JHEP **09**, 056 (2019) doi:10.1007/JHEP09(2019)056 [arXiv:1812.06895 [hep-th]]; Phys. Rev. D **100**, no.10, 104024 (2019) doi:10.1103/PhysRevD.100.104024 [arXiv:1906.10071 [hep-th]].

[39] N. Siemonsen and J. Vines, Phys. Rev. D **101**, no.6, 064066 (2020) doi:10.1103/PhysRevD.101.064066 [arXiv:1909.07361 [gr-qc]].

[40] Y. F. Bautista and A. Guevara, [arXiv:1903.12419 [hep-th]].

[41] B. Maybee, D. O'Connell and J. Vines, JHEP **12**, 156 (2019) doi:10.1007/JHEP12(2019)156 [arXiv:1906.09260 [hep-th]].

[42] N. Arkani-Hamed, Y. t. Huang and D. O'Connell, JHEP **01**, 046 (2020) doi:10.1007/JHEP01(2020)046 [arXiv:1906.10100 [hep-th]].

[43] A. Guevara, B. Maybee, A. Ochirov, D. O'connell and J. Vines, JHEP **03**, 201 (2021) doi:10.1007/JHEP03(2021)201 [arXiv:2012.11570 [hep-th]].

[44] A. Cristofoli, R. Gonzo, N. Moynihan, D. O'Connell, A. Ross, M. Sergola and C. D. White, [arXiv:2112.07556 [hep-th]].

[45] M. Z. Chung, Y. T. Huang, J. W. Kim and S. Lee, JHEP **04**, 156 (2019) doi:10.1007/JHEP04(2019)156 [arXiv:1812.08752 [hep-th]]; JHEP **05**, 105 (2020) doi:10.1007/JHEP05(2020)105 [arXiv:2003.06600 [hep-th]].

[46] M. Z. Chung, Y. T. Huang and J. W. Kim, JHEP **09**, 074 (2020) doi:10.1007/JHEP09(2020)074 [arXiv:1908.08463 [hep-th]];

JHEP **12** (2020), 103 doi:10.1007/JHEP12(2020)103 [arXiv:1911.12775 [hep-th]].
[47] W. M. Chen, M. Z. Chung, Y. t. Huang and J. W. Kim, [arXiv:2111.13639 [hep-th]]; W. M. Chen, M. Z. Chung, Y. t. Huang and J. W. Kim, [arXiv:2205.07305 [hep-th]].
[48] J. W. Kim, [arXiv:2207.04970 [hep-th]].
[49] P. H. Damgaard, K. Haddad and A. Helset, JHEP **11**, 070 (2019) doi:10.1007/JHEP11(2019)070 [arXiv:1908.10308 [hep-ph]].
[50] K. Haddad and A. Helset, Phys. Rev. Lett. **125**, 181603 (2020) doi:10.1103/PhysRevLett.125.181603 [arXiv:2005.13897 [hep-th]].
[51] K. Haddad, Phys. Rev. D **105**, no.2, 026004 (2022) doi:10.1103/PhysRevD.105.026004 [arXiv:2109.04427 [hep-th]].
[52] R. Aoude, K. Haddad and A. Helset, JHEP **05**, 051 (2020) doi:10.1007/JHEP05(2020)051 [arXiv:2001.09164 [hep-th]]; JHEP **07** (2022), 072 doi:10.1007/JHEP07(2022)072 [arXiv:2203.06197 [hep-th]]; [arXiv:2205.02809 [hep-th]].
[53] Z. Bern, A. Luna, R. Roiban, C. H. Shen and M. Zeng, Phys. Rev. D **104**, no.6, 065014 (2021) doi:10.1103/PhysRevD.104.065014 [arXiv:2005.03071 [hep-th]].
[54] D. Kosmopoulos and A. Luna, JHEP **07**, 037 (2021) doi:10.1007/JHEP07(2021)037 [arXiv:2102.10137 [hep-th]].
[55] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban and F. Teng, [arXiv:2203.06202 [hep-th]].
[56] L. de la Cruz, [arXiv:2207.03452 [hep-th]].
[57] F. Febres Cordero, M. Kraus, G. Lin, M. S. Ruf and M. Zeng, [arXiv:2205.07357 [hep-th]].
[58] F. Alessio and P. Di Vecchia, doi:10.1016/j.physletb.2022.137258 [arXiv:2203.13272 [hep-th]].
[59] T. Adamo, A. Cristofoli and P. Tourkine, [arXiv:2112.09113 [hep-th]].
[60] G. Menezes and M. Sergola, [arXiv:2205.11701 [hep-th]].
[61] M. M. Riva, F. Vernizzi and L. K. Wong, Phys. Rev. D **106**, no.4, 044013 (2022) doi:10.1103/PhysRevD.106.044013 [arXiv:2205.15295 [hep-th]].
[62] S. V. Iyer and E. C. Hansen, [arXiv:0908.0085 [gr-qc]].
[63] N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, JHEP **08**, 038 (2020) doi:10.1007/JHEP08(2020)038 [arXiv:1910.09366 [hep-th]].
[64] P. H. Damgaard and P. Vanhove, Phys. Rev. D **104**, no.10, 104029 (2021) doi:10.1103/PhysRevD.104.104029 [arXiv:2108.11248 [hep-th]].
[65] S. J. Wallace, Annals Phys. **78**, 190-257 (1973) doi:10.1016/0003-4916(73)90008-0
[66] W. M. Tulczyjew, "Motion of multipole particles in general relativity theory," *Acta Phys. Pol.* **18** (1959) 393–409.
[67] J. Steinhoff and D. Puetzfeld, Phys. Rev. D **81**, 044019 (2010) doi:10.1103/PhysRevD.81.044019 [arXiv:0909.3756 [gr-qc]].
[68] M. Mathisson, "Neue Mechanik materieller Systeme," *Acta Phys. Pol.* **6** (1937) 163–200.
[69] M. Mathisson, "Republication of: New mechanics of material systems," *Gen. Relativ. Gravit.* **42** (2010) 1011–1048.
[70] W. G. Dixon, "Dynamics of extended bodies in general relativity III. Equations of motion" Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **277** (1974) 59–119
[71] A. I. Harte, Fund. Theor. Phys. **179**, 327-398 (2015) doi:10.1007/978-3-319-18335-0_12 [arXiv:1405.5077 [gr-qc]].
[72] J. Vines, D. Kunst, J. Steinhoff and T. Hinderer, Phys. Rev. D **93**, no.10, 103008 (2016) [erratum: Phys. Rev. D **104**, no.2, 029902 (2021)] doi:10.1103/PhysRevD.104.029902 [arXiv:1601.07529 [gr-qc]].
[73] S. Marsat, Class. Quant. Grav. **32**, no.8, 085008 (2015) doi:10.1088/0264-9381/32/8/085008 [arXiv:1411.4118 [gr-qc]].
[74] J. Steinhoff, Annalen Phys. **523**, 296-353 (2011) doi:10.1002/andp.201000178 [arXiv:1106.4203 [gr-qc]].
[75] D. Bini, G. Faye and A. Geralico, Phys. Rev. D **92**, no.10, 104003 (2015) doi:10.1103/PhysRevD.92.104003 [arXiv:1507.07441 [gr-qc]].
[76] D. Bini, A. Geralico and J. Vines, Phys. Rev. D **96**, no.8, 084044 (2017) doi:10.1103/PhysRevD.96.084044 [arXiv:1707.09814 [gr-qc]].