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Proving Fixed-Point Theorems Employing Fuzzy (σ, \mathcal{Z}) -Contractive-Type Mappings

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Abstract: In this article, the concept of fuzzy (σ, \mathcal{Z}) -contractive mappings is introduced in the setting of fuzzy metric spaces. Thereafter, we utilize our newly introduced concept to prove some existence and uniqueness theorems in \mathcal{M} -complete fuzzy metric spaces. Our obtained theorems extend and generalize the corresponding results in the existing literature. Moreover, some examples are adopted to exhibit the utility of the newly obtained results.

Keywords: fuzzy (σ, \mathcal{Z}) -contractive mappings; fuzzy metric spaces; fuzzy \mathcal{Z} -contractive mappings



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1. Introduction and Motivation

In the theory of fuzzy sets and systems, many researchers have attempted to formulate an appropriate definition of fuzzy metric space (e.g., [1–3]). The most natural and widely acceptable definition is essentially due to Kramosil and Michálek [4]. Grabiec [5] is one of the earliest mathematicians to study the theory of the fixed point in fuzzy metric spaces. In doing so, he introduced the notions of \mathcal{G} -Cauchyness and the \mathcal{G} -completeness of fuzzy metric spaces and extended the fixed-point theorems of Banach and Edelstein from metric spaces to fuzzy metric spaces introduced by Kramosil and Michálek. It has been observed that the notions of \mathcal{G} -Cauchy sequences and \mathcal{G} -completeness are relatively strong. With a view toward having a Hausdorff topology on a fuzzy metric space, George and Veeramani [6] modified the definition of the fuzzy metric space due to Kramosil and Michálek [4] and also established some valuable related results.

In 2002, Gregori and Sapena [7] initiated a class of mappings called fuzzy contractive mappings and proved a fuzzy version of the Banach contraction theorem in the sense of George and Veeramani. Thereafter, employing a control function satisfying suitable properties, Miheţ [8] and Wardowski [9] generalized the class of fuzzy contractive mappings by introducing the concepts of the fuzzy ψ -contractive mapping and fuzzy \mathcal{H} -contractive mappings, respectively. For such kind of work, we refer the reader to [10–25]. Very recently, Shukla et al. gave the concept of fuzzy \mathcal{Z} -contractive mappings (see Definition 4, given later), which unifies all the classes of mappings mentioned earlier.

On the other hand, the concept of the σ -admissible mappings was introduced by Samet et al. (see [26], Definition 2.2) in metric spaces. In [27], Gopal and Vetro extended this notion to the setting of fuzzy metric spaces (see Definition 8, given later). Employing this notion, they introduced the concept of σ - ψ -fuzzy contractive mappings and proved a theorem that ensures the existence of a fixed point for this types of mappings. Their presented theorem extends, generalizes, and improves the corresponding results given in the literature.

This article aims to enlarge the class of fuzzy \mathcal{Z} -contractive mappings by introducing the family of fuzzy (σ, \mathcal{Z}) -contractive mappings to cover all of the concepts introduced

in [7–9,28,29], besides extending a result due to Gopal and Vetro [27]. Our newly introduced notion is utilized to prove some results in \mathcal{M} -complete fuzzy metric spaces. Finally, some examples are adopted to demonstrate that our newly presented results are a proper extension of Shukla et al.’s results [28].

2. Mathematical Preliminaries

In this section, we present some introductory material from the theory of fuzzy metric spaces needed to prove our results.

Definition 1 ([30]). Let $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ be a binary operation. We say that $*$ is a continuous t-norm if the following assumptions are fulfilled:

- (N1) $*$ is associative and commutative;
- (N2) $*$ is continuous;
- (N3) $r_1 * r_2 \leq r_3 * r_4$ whenever $r_1 \leq r_3$ and $r_2 \leq r_4$;
- (N4) $r_1 * 1 = r_1$;

for all $r_1, r_2, r_3, r_4 \in [0, 1]$.

Three primary continuous t-norms examples are: $r_1 * r_2 = r_1 \cdot r_2$, $r_1 * r_2 = \min\{r_1, r_2\}$, and $r_1 * r_2 = \max\{r_1 + r_2 - 1, 0\}$, which are known as the product, minimum, and Lukasiewicz t-norms, respectively.

By modifying the concept of fuzzy metric space introduced in [4], George and Veeramani attempted the following definition:

Definition 2 ([6]). Let \mathcal{K} be a non-empty set and $\mathcal{M} : \mathcal{K}^2 \times (0, \infty) \rightarrow [0, 1]$ be a fuzzy set. The ordered triple $(\mathcal{K}, \mathcal{M}, *)$ is called a fuzzy metric space (FMS), where $*$ is a continuous t-norm if the following assumptions are fulfilled (for all $\alpha, \beta, \gamma \in \mathcal{K}$ and $t, s > 0$):

- (G1) $\mathcal{M}(\alpha, \beta, t) > 0$;
- (G2) $\mathcal{M}(\alpha, \beta, t) = 1$ if and only if $\alpha = \beta$;
- (G3) $\mathcal{M}(\alpha, \beta, t) = \mathcal{M}(\beta, \alpha, t)$;
- (G4) $\mathcal{M}(\alpha, \gamma, t) * \mathcal{M}(\gamma, \beta, s) \leq \mathcal{M}(\alpha, \beta, t + s)$;
- (G5) $\mathcal{M}(\alpha, \beta, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Let $(\mathcal{K}, \mathcal{M}, *)$ be a fuzzy metric space. For $t > 0$, the open ball $\mathfrak{B}(x, r, t)$ with center $x \in \mathcal{K}$ and radius $r \in (0, 1)$ is defined by

$$\mathfrak{B}(x, r, t) = \{y \in \mathcal{K} : \mathcal{M}(x, y, t) > 1 - r\}.$$

A subset $A \subset \mathcal{K}$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in (0, 1)$ such that $\mathfrak{B}(x, r, t) \subset A$. The family of all open subsets of \mathcal{K} is a topology on \mathcal{K} , called the topology induced by the fuzzy metric \mathcal{M} .

Definition 3 ([5,6]). A sequence $\{\alpha_n\}$ in an FMS, $(\mathcal{K}, \mathcal{M}, *)$ is called:

- (a) Convergent and converges to $\alpha \in \mathcal{K}$, if $\lim_{n \rightarrow \infty} \mathcal{M}(\alpha_n, \alpha, t) = 1$, for each $t > 0$;
- (b) \mathcal{M} -Cauchy, if for each $\epsilon \in (0, 1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $\mathcal{M}(\alpha_m, \alpha_n, t) > 1 - \epsilon$, for each $m, n \geq n_0$;
- (c) \mathcal{G} -Cauchy, if $\lim_{n \rightarrow \infty} \mathcal{M}(\alpha_n, \alpha_{n+p}, t) = 1$, for each $t > 0$ and $p \in \mathbb{N}$.

Lemma 1 ([5,6]). Let $(\mathcal{K}, \mathcal{M}, *)$ be a fuzzy metric space:

- (1) \mathcal{M} is continuous on $\mathcal{K}^2 \times (0, \infty)$;
- (2) $\mathcal{M}(\alpha, \beta, \cdot)$ is a non-decreasing function on $(0, \infty)$, for each $\alpha, \beta \in \mathcal{K}$;
- (3) The limit of a convergent sequence in $(\mathcal{K}, \mathcal{M}, *)$ is unique.

A fuzzy metric space $(\mathcal{K}, \mathcal{M}, *)$ is called an \mathcal{M} -complete (\mathcal{G} -complete) FMS, if every \mathcal{M} -Cauchy (\mathcal{G} -Cauchy) sequence of \mathcal{K} converges in \mathcal{K} .

Let \mathcal{Z} be the set of all $\zeta : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$, which satisfy the condition:

$$\zeta(l, s) > s, \forall l, s \in (0, 1).$$

Example 1 ([28]). Consider the functions $\zeta_i : (0, 1] \times (0, 1] \rightarrow \mathbb{R}, i = 1, 2, 3$ which are defined by:

1. $\zeta_1(l, s) = \frac{s}{l}$;
2. $\zeta_2(l, s) = \frac{1}{s+l} + l$;
3. $\zeta_3(l, s) = \begin{cases} l, & \text{if } l > s, \\ \sqrt{s} & \text{if } s \geq l, \end{cases}$;

for all $l, s \in (0, 1]$. Then, $\zeta_i \in \mathcal{Z}, i = 1, 2, 3$.

Remark 1 ([28]). From the definition of ζ , it is obvious that $\zeta(t, t) > t$, for all $t \in (0, 1)$.

Employing the function ζ that satisfies the above condition, Shukla et al. unified and extended the contractive-type mappings introduced in [7–9,29] by introducing the following interesting class of mappings:

Definition 4 ([28]). Let T be a self-mapping of an FMS $(\mathcal{K}, \mathcal{M}, *)$. The mapping T is said to be fuzzy \mathcal{Z} -contractive if there is $\zeta \in \mathcal{Z}$ such that

$$\mathcal{M}(T\alpha, T\beta, t) \geq \zeta(\mathcal{M}(T\alpha, T\beta, t), \mathcal{M}(\alpha, \beta, t)),$$

for each $\alpha, \beta \in \mathcal{K}$ with $T\alpha \neq T\beta$ and $t > 0$.

Let \mathcal{K} be a nonempty set, $x_0 \in \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{K}$. A sequence $\{x_n\} \subseteq \mathcal{K}$ is called a Picard sequence of T based at x_0 if

$$x_n = Tx_{n-1} = T^n x_0, \forall n \in \mathbb{N}.$$

Definition 5 ([28]). Let T be a self-mapping of an FMS $(\mathcal{K}, \mathcal{M}, *)$ and $\zeta \in \mathcal{Z}$. Assume that $\{\alpha_n\}$ is any Picard sequence for all $n \in \mathbb{N}$. The quadruple $(\mathcal{K}, \mathcal{M}, T, \zeta)$ is said to have the property (S) if for each $n \in \mathbb{N}$ and $t > 0$ with

$$\inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) \leq \inf_{m>n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t)$$

implies $\lim_{n \rightarrow \infty} \inf_{m>n} \zeta(\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t), \mathcal{M}(\alpha_n, \alpha_m, t)) = 1.$

Definition 6 ([28]). Let T be a self-mapping of an FMS $(\mathcal{K}, \mathcal{M}, *)$ and $\zeta \in \mathcal{Z}$. Assume that $\{\alpha_n\}$ is any Picard sequence for all $n \in \mathbb{N}$. The quadruple $(\mathcal{K}, \mathcal{M}, T, \zeta)$ is said to have the property (S') if for each $n \in \mathbb{N}$ and $t > 0$ with $0 < \lim_{n \rightarrow \infty} \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) < 1$ and

$$\inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) \leq \inf_{m>n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t)$$

implies $\lim_{n \rightarrow \infty} \inf_{m>n} \zeta(\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t), \mathcal{M}(\alpha_n, \alpha_m, t)) = 1.$

Notice that the condition (S') is weaker than the condition (S) (see [28], Example 3.18). Shukla et al. [28] proved the following theorem as a consequence of their study.

Theorem 1. Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $T : \mathcal{K} \rightarrow \mathcal{K}$ be a fuzzy \mathcal{Z} -contractive mapping with respect to $\zeta \in \mathcal{Z}$. If the quadruple $(\mathcal{K}, \mathcal{M}, T, \zeta)$ has the property (S), then T admits a unique fixed point.

Let Ψ denote the set of all $\psi : (0, 1] \rightarrow (0, 1]$, which have the following properties:

- (Ψ1) ψ is a left continuous and non-decreasing function;
- (Ψ2) $\psi(r) > r$, for all $r \in (0, 1)$.

Lemma 2 ([31]). *If $\psi \in \Psi$, then $\psi(1) = 1$.*

Inspired by the work of Samet et al. [26], Gopal and Vetro [27] employed a function $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ instead of $\sigma : \mathcal{K} \times \mathcal{K} \rightarrow [0, \infty)$ and introduced the following definition:

Definition 7 ([27]). *Let T be a self-mapping of an FMS $(\mathcal{K}, \mathcal{M}, *)$. We say that T is a σ - ψ -fuzzy contractive mapping if there exist two functions $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ and $\psi \in \Psi$ such that*

$$\mathcal{M}(\alpha, \beta, t) > 0 \implies \sigma(\alpha, \beta, t)\mathcal{M}(T\alpha, T\beta, t) \geq \psi(\mathcal{M}(\alpha, \beta, t)),$$

for all $\alpha, \beta \in \mathcal{K}, t > 0$ with $\alpha \neq \beta$.

Definition 8 ([27]). *Let T be a self-mapping of an FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$. A mapping T is said to be σ -admissible if*

$$\forall \alpha, \beta \in \mathcal{K}, t > 0, [\sigma(\alpha, \beta, t) \leq 1 \implies \sigma(T\alpha, T\beta, t) \leq 1]. \tag{1}$$

For examples of the σ -admissible mapping of fuzzy metric spaces, we refer the reader to [27,32]. Now, we add another two examples of σ -admissible mappings.

Example 2. *Consider a mapping $T : \mathcal{K} \rightarrow \mathcal{K}$, where $\mathcal{K} = [0, \infty)$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ defined by*

$$T\alpha = \sqrt{\alpha}, \forall \alpha \in \mathcal{K} \quad \text{and} \quad \sigma(\alpha, \beta, t) = \begin{cases} e^{\frac{\beta-\alpha}{t}} & \alpha \geq \beta, \\ 2 & \alpha < \beta. \end{cases}$$

Then, T is a σ -admissible mapping.

Example 3. *Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$, where $\mathcal{K} = [0, \infty)$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ defined by*

$$T\alpha = \ln \alpha, \forall \alpha \in \mathcal{K} \quad \text{and} \quad \sigma(\alpha, \beta, t) = \begin{cases} \frac{1}{t} & \alpha \geq \beta, \\ 2 & \text{otherwise.} \end{cases}$$

Then, T is a σ -admissible mapping.

Based on the above two definitions, the authors in [27] proved the following theorem:

Theorem 2 ([27]). *Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow [0, \infty)$. Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$ is a σ - ψ -fuzzy contractive mapping satisfying the following assumptions:*

- (i) T is σ -admissible;
- (ii) There exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$;
- (iii) For each sequence $\{\alpha_n\}$ of \mathcal{K} with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0, t > 0$;
- (iv) If $\{\alpha_n\}$ is a sequence in \mathcal{K} such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha \in \mathcal{K}$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, then $\sigma(\alpha_n, \alpha, t) \leq 1$.

Then, T admits a fixed point.

3. Main Results

Throughout this article, $(\mathcal{K}, \mathcal{M}, *)$ is a fuzzy metric space in the George and Veeramani sense. First of all, we start by introducing the notion of fuzzy (σ, \mathcal{Z}) -contractive mappings, which include many existing and familiar concepts as special cases.

Definition 9. Let T be a self-mapping of an FMS $(\mathcal{K}, \mathcal{M}, *)$. We say that T is a fuzzy (σ, \mathcal{Z}) -contractive with respect to $\xi \in \mathcal{Z}$ if there is $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\sigma(\alpha, \beta, t)\mathcal{M}(T\alpha, T\beta, t) \geq \xi(\mathcal{M}(T\alpha, T\beta, t), \mathcal{M}(\alpha, \beta, t)), \tag{2}$$

for all $\alpha, \beta \in \mathcal{K}, t > 0$ with $T\alpha \neq T\beta$.

Remark 2. By adopting the functions ξ and σ suitably in Definition 9, we deduce some well-known contractions as demonstrated below (for all $\alpha, \beta \in \mathcal{K}$ and $t > 0$):

- (a) If $\sigma(\alpha, \beta, t) = 1$, for each $\alpha, \beta \in \mathcal{K}$ and $t > 0$, then Definition 9 reduces to Definition 4.
- (b) Taking $\xi(l, s) = \psi(s)$, for each $l, s \in (0, 1]$ and $\psi \in \Psi$ in Definition 9, we deduce Definition 7.

It is worth mentioning here that every fuzzy \mathcal{Z} -contractive is a fuzzy (σ, \mathcal{Z}) -contractive mapping, but the reverse is not in general true, as demonstrated by the following example:

Example 4. Let $\mathcal{K} = [0, 1]$ and $d(x, y) = |x - y|$, for all $x, y \in \mathcal{K}$. Let \mathcal{M} be a fuzzy set on $\mathcal{K}^2 \times (0, \infty)$ given by $\mathcal{M}(x, y, t) = \frac{t}{t+d(x,y)}$, $t > 0$. Define a mapping $T : \mathcal{K} \rightarrow \mathcal{K}$ by

$$T(x) = 1 - x, \text{ for all } x \in \mathcal{K}.$$

T is not a fuzzy \mathcal{Z} -contractive mapping. On the contrary, we assume that T is a fuzzy \mathcal{Z} -contractive with respect to some $\xi \in \mathcal{Z}$. Take $x, y \in \mathcal{K}$ such that $Tx \neq Ty$. Since $\mathcal{M}(x, y, t) = \mathcal{M}(Tx, Ty, t) = \frac{t}{t+|x-y|} \in (0, 1)$, using Remark 1, we have

$$\begin{aligned} \frac{t}{t+|x-y|} = \mathcal{M}(x, y, t) = \mathcal{M}(Tx, Ty, t) &\geq \xi(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &> \mathcal{M}(x, y, t) = \frac{t}{t+|x-y|}, \end{aligned}$$

for all $t > 0$, which is a contradiction. Hence, T is not a fuzzy \mathcal{Z} -contractive mapping. To show that T is a fuzzy (σ, \mathcal{Z}) -contractive mapping, we need to define two essential functions: $\xi : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ by

$$\xi(l, s) = s + l \text{ and } \sigma(x, y, t) = 2.$$

It is clear that $\xi \in \mathcal{Z}$. Then, for all $x, y \in \mathcal{K}, t > 0$, we have

$$\begin{aligned} \frac{2t}{t+|x-y|} &= \sigma(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \xi(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \mathcal{M}(x, y, t) + \mathcal{M}(Tx, Ty, t) = \frac{2t}{t+|x-y|}, \end{aligned}$$

which shows that T is a fuzzy (σ, \mathcal{Z}) -contractive mapping.

Now, we are able to formulate our first main result as follows:

Theorem 3. Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$. Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$ is a fuzzy (σ, \mathcal{Z}) -contractive mapping and the following properties hold:

- (a) T is σ -admissible;

- (b) The quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S);
- (c) There exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$;
- (d) For each sequence $\{\alpha_n\}$ of \mathcal{K} with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0, t > 0$;
- (e) T is continuous.

Then, T admits a fixed point.

Proof. Pick out an arbitrary point α_0 in \mathcal{K} such that $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$, and consider a Picard sequence $\{\alpha_n\}$ in \mathcal{K} , that is,

$$\alpha_{n+1} = T\alpha_n, \text{ for all } n \in \mathbb{N}_0.$$

In case $\alpha_{n_0} = \alpha_{n_0+1}$, for some $n_0 \in \mathbb{N}_0$, then the fixed point of the mapping T is nothing but α_{n_0} . Assume that $\alpha_{n+1} \neq \alpha_n$, for each $n \in \mathbb{N}_0$. As T is σ -admissible, we have:

$$\sigma(\alpha_0, \alpha_1, t) = \sigma(\alpha_0, T\alpha_0, t) \leq 1 \implies \sigma(\alpha_1, \alpha_2, t) = \sigma(T\alpha_0, T\alpha_1, t) \leq 1.$$

The induction on n , gives rise to:

$$\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1, \text{ for each } n \in \mathbb{N}_0 \text{ and } t > 0. \tag{3}$$

Moreover, if for some $m > n$, $\alpha_n = \alpha_m$, then the contractive condition (2) and Equation (3) imply that:

$$\begin{aligned} \mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) &\geq \sigma(\alpha_n, \alpha_{n+1}, t) \mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) \\ &\geq \xi(\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t), \mathcal{M}(\alpha_n, \alpha_{n+1}, t)) \\ &> \mathcal{M}(\alpha_n, \alpha_{n+1}, t); \end{aligned}$$

hence,

$$\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) > \mathcal{M}(\alpha_n, \alpha_{n+1}, t).$$

Continuing in this way, one can show that

$$\mathcal{M}(\alpha_m, \alpha_{m+1}, t) > \mathcal{M}(\alpha_{m-1}, \alpha_m, t) > \dots > \mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) > \mathcal{M}(\alpha_n, \alpha_{n+1}, t).$$

Since $\alpha_n = \alpha_m$ for some $m > n$, we have $\alpha_{n+1} = \alpha_{m+1}$. This together with the above relation leads to a contradiction. Therefore, $\alpha_n \neq \alpha_m$ for each $m > n$. In view of the condition (d), there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1, \forall m, n \in \mathbb{N}$ with

$$m > n \geq k_0 \text{ and } t > 0.$$

Applying the contractive condition (2) and making use of the above inequality, we obtain

$$\begin{aligned} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t) &\geq \sigma(\alpha_n, \alpha_m, t) \mathcal{M}(T\alpha_n, T\alpha_m, t) \\ &\geq \xi(\mathcal{M}(T\alpha_n, T\alpha_m, t), \mathcal{M}(\alpha_n, \alpha_m, t)) \\ &> \mathcal{M}(\alpha_n, \alpha_m, t), \end{aligned} \tag{4}$$

and hence,

$$\forall m > n, \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t) > \mathcal{M}(\alpha_n, \alpha_m, t).$$

In the above inequality, taking the infimum over $m (> n)$ and letting $a_n(t) = \inf_{m > n} \mathcal{M}(\alpha_n, \alpha_m, t)$ we obtain that $a_n(t) \leq a_{n+1}(t)$, for each $t > 0$, and hence, $\{a_n(t)\}$ is a nondecreasing and bounded. Therefore, there exists $a(t)$ such that $\lim_{n \rightarrow \infty} a_n(t) = a(t)$. Our claim is to justify that $a(t) = 1$, for each $t > 0$. On the contrary, we assume that

$a(s) > 1$, for some $s > 0$. From the fact that the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S), we obtain

$$\lim_{n \rightarrow \infty} \inf_{m > n} \xi(\mathcal{M}(\alpha_n, \alpha_m, s), \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, s)) = 1. \tag{5}$$

Equation (4) gives rise to

$$\begin{aligned} a_{n+1}(s) &= \inf_{m > n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, s) \geq \inf_{m > n} \xi(\mathcal{M}(T\alpha_n, T\alpha_m, s), \mathcal{M}(\alpha_n, \alpha_m, s)) \\ &\geq \inf_{m > n} \mathcal{M}(\alpha_n, \alpha_m, s) = a_n(s). \end{aligned}$$

Taking $n \rightarrow \infty$ in the above relation and using Equation (5), we obtain

$$\lim_{n \rightarrow \infty} \inf_{m > n} \mathcal{M}(\alpha_n, \alpha_m, s) = a(s) = 1,$$

which is a contradiction to the assumption ($a(s) > 1$ for some $s > 0$). This contradiction concludes that, for each $t > 0$, $\lim_{n, m \rightarrow \infty} \mathcal{M}(\alpha_n, \alpha_m, t) = 1$, that is, $\{\alpha_n\}$ is a Cauchy sequence. Due to the \mathcal{M} -completeness of the fuzzy metric space $(\mathcal{K}, \mathcal{M}, *)$, there is $\gamma \in \mathcal{K}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{M}(\alpha_n, \gamma, t) = 1,$$

for all $t > 0$. The continuity of the mapping T implies that $\lim_{n \rightarrow \infty} \mathcal{M}(T\alpha_n, T\gamma, t) = 1$, for each $t > 0$, and hence,

$$\lim_{n \rightarrow \infty} \mathcal{M}(\alpha_{n+1}, T\gamma, t) = \lim_{n \rightarrow \infty} \mathcal{M}(T\alpha_n, T\gamma, t) = 1,$$

for all $t > 0$. Therefore, $T\gamma = \gamma$, due to the uniqueness of the limit. \square

In order to support the above-obtained result, we provide an example. Precisely, we show that Theorem 3 can be used to cover this example while Theorem 1 is not applicable.

Example 5. Consider $\mathcal{K} = \{A_1 = (0,0), A_2 = (1,0), A_3 = (1,2), A_4 = (0,1), A_5 = (1,3)\} \subseteq \mathbb{R}^2$. Define the fuzzy metric \mathcal{M} as

$$\mathcal{M}(\alpha, \beta, t) = e^{\frac{-2d(\alpha, \beta)}{t}}, \forall \alpha, \beta \in \mathcal{K}, t > 0,$$

where $d(\alpha, \beta)$ is the Euclidean metric on \mathbb{R}^2 . It is obvious that $(\mathcal{K}, \mathcal{M}, *)$ is an \mathcal{M} -complete FMS with respect to the product t -norm. Let $T : \mathcal{K} \rightarrow \mathcal{K}$ be defined by

$$T(\alpha) = \begin{cases} A_1, & \text{if } \alpha \in \{A_1, A_3, A_4, A_5\}, \\ A_5, & \text{if } \alpha = A_2. \end{cases}$$

Furthermore, define $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ by

$$\sigma(\alpha, \beta, t) = \begin{cases} e^{\frac{2}{t}(2\sqrt{10}-3)}, & \text{if } \alpha = A_2 \text{ \& } \beta \in \{A_1, A_3, A_4, A_5\} \\ & \text{or } \alpha = \beta = A_2 \\ 1 & \text{otherwise,} \end{cases}$$

and $\xi : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ by

$$\xi(l, s) = \frac{s}{l}, \text{ for all } l, s \in (0, 1].$$

For all $\alpha, \beta \in \mathcal{K}$, we have

$$\mathcal{M}(T\alpha, T\beta, t) = \begin{cases} e^{-\frac{2}{t}(\sqrt{10})}, & \text{if } \alpha = A_2 \text{ \& } \beta \in \{A_1, A_3, A_4, A_5\} \\ & \text{or } \alpha \in \{A_1, A_3, A_4, A_5\} \text{ \& } \beta = A_2 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\alpha, \beta \in \mathcal{K}$ such that $\sigma(\alpha, \beta, t) \leq 1$. Then, $\alpha, \beta \in \{A_1, A_3, A_4, A_5\}$, and by the definition of T , we conclude that $T\alpha = T\beta \in \{A_1, A_3, A_4, A_5\}$; hence, $\sigma(T\alpha, T\beta) = 1$. Therefore, the mapping T is σ -admissible. Furthermore, $A_4 \in \mathcal{K}$ and $\sigma(A_4, TA_4, t) = \alpha(A_4, A_1, t) = 1$, for each $t > 0$. Further, let $\{\alpha_n\} \subseteq \mathcal{K}$ such that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$ with $k_0 = 1$, for each $n \in \mathbb{N}$. It follows that $\alpha_n \in \{A_1, A_3, A_4, A_5\}$, for each $n \in \mathbb{N}$, and hence, $\sigma(\alpha_n, \alpha_m, t) \leq 1$ for all $m, n \in \mathbb{N}$ and $t > 0$. Furthermore, it is obvious that the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ has the property (S).

Finally, to show that T is a fuzzy (σ, \mathcal{Z}) -contractive mapping, we only need to consider the case $\alpha = A_2$ and $\beta \in \{A_1, A_3, A_4, A_5\}$. In this case, $\sigma(\alpha, \beta, t) = e^{\frac{2}{t}(2\sqrt{10}-1)}$, and hence,

$$\begin{aligned} e^{\frac{2}{t}(\sqrt{10}-3)} &= e^{\frac{2}{t}(2\sqrt{10}-3)} \cdot e^{-\frac{2}{t}(\sqrt{10})} = \sigma(\alpha, \beta, t)\mathcal{M}(T\alpha, T\beta, t) \\ &\geq \xi(\mathcal{M}(T\alpha, T\beta, t), \mathcal{M}(\alpha, \beta, t)) \\ &= \frac{\mathcal{M}(\alpha, \beta, t)}{\mathcal{M}(T\alpha, T\beta, t)} = \frac{e^{-\frac{6}{t}}}{e^{-\frac{2}{t}(\sqrt{10})}} = e^{\frac{2}{t}(\sqrt{10}-3)}, \end{aligned}$$

which shows that T is a fuzzy (σ, \mathcal{Z}) -contractive mapping. Therefore, all the hypotheses of Theorem 3 are satisfied. This ensures that the mapping T admits a fixed point (namely $x = A_1$).

However, T is not a fuzzy \mathcal{Z} -contractive mapping. On the contrary, we assume T is fuzzy \mathcal{Z} -contractive with respect to some $\xi \in \mathcal{Z}$. Take $\alpha = A_2$ and $\beta = A_4$. As $\mathcal{M}(\alpha, \beta, t) = e^{-\frac{2\sqrt{2}}{t}} \in (0, 1)$ and $\mathcal{M}(T\alpha, T\beta, t) = e^{-\frac{2\sqrt{10}}{t}} \in (0, 1)$, from the contractive condition and the definition of ξ , we have

$$\begin{aligned} e^{-\frac{2\sqrt{10}}{t}} &= \mathcal{M}(T\alpha, T\beta, t) \geq \xi(\mathcal{M}(T\alpha, T\beta, t), \mathcal{M}(\alpha, \beta, t)) \\ &> \mathcal{M}(\alpha, \beta, t) = e^{-\frac{2\sqrt{2}}{t}}, \end{aligned}$$

for all $t > 0$, which is a contradiction. Hence, T is not a fuzzy \mathcal{Z} -contractive mapping.

One of the advantages of σ -admissible mappings is that the continuity of the mapping is no longer required for the existence of a fixed point provided that the space under consideration satisfies a suitable condition (namely (e') given in the next theorem). Precisely, we state and prove the following theorem:

Theorem 4. Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow [0, \infty)$. Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$ is a fuzzy (σ, \mathcal{Z}) -contractive mapping satisfy the following assumptions:

- (a) T is σ -admissible;
- (b) The quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S);
- (c) There exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$;
- (d) For each sequence $\{\alpha_n\}$ of \mathcal{K} with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0, t > 0$;
- (e') If $\{\alpha_n\}$ is a sequence in \mathcal{K} such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha \in \mathcal{K}$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, then $\sigma(\alpha_n, \alpha, t) \leq 1$.

Then, T admits a fixed point.

Proof. The frame of the proof is the same as that in the previous theorem (Theorem 3). Therefore, for a Cauchy sequence $\{\alpha_n\}$ in a complete FMS $(\mathcal{K}, \mathcal{M}, *)$, there exists $\gamma \in \mathcal{K}$ such that

$$\lim_{n \rightarrow \infty} (\alpha_n, \gamma, t) = 1, \forall t > 0. \tag{6}$$

Furthermore, we have $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, and hence, as a consequence of the condition (e') , we obtain

$$\sigma(\alpha_n, \gamma, t) \leq 1, \tag{7}$$

for each $n \in \mathbb{N}$ and $t > 0$. Now, we have to show that T admits a fixed point (say γ). On the contrary, assume that $T\alpha_n \neq \gamma$, for all $n \in \mathbb{N}$. Without loss of generality, one can assume that $\alpha_n \neq \gamma$ and $\alpha_n \neq T\alpha_n$. Then, there is $s > 0$ such that

$$\mathcal{M}(\alpha_n, \gamma, s) < 1, \mathcal{M}(\alpha_n, T\alpha_n, s) < 1 \text{ and } \mathcal{M}(T\alpha_n, T\alpha_n, s) < 1, \forall n \in \mathbb{N}. \tag{8}$$

Using (2), (7) and (8), we obtain

$$\begin{aligned} \mathcal{M}(\alpha_{n+1}, T\alpha_n, s) &\geq \sigma(\alpha_n, \gamma, s)\mathcal{M}(T\alpha_n, T\alpha_n, s) \\ &\geq \xi(\mathcal{M}(T\alpha_n, T\alpha_n, s), \mathcal{M}(\alpha_n, \gamma, s)) \\ &> \mathcal{M}(\alpha_n, \gamma, s). \end{aligned} \tag{9}$$

Taking $n \rightarrow \infty$ and making use of (6), we obtain $\mathcal{M}(\gamma, T\alpha_n, s) \geq 1$, a contradiction. Therefore, for all $t > 0$, $\mathcal{M}(\gamma, T\alpha_n, t) = 1$, that is γ is the fixed point of T . \square

Next, we support Theorem 4 by an example in which the mapping T is not continuous. Moreover, we show the applicability of Theorem 4 over Theorems 1 and 3.

Example 6. Let $\mathcal{K} = [0, \infty)$, the set of all nonnegative real numbers, $*$ be a minimum t -norm, and \mathcal{M} be a fuzzy set on $\mathcal{K}^2 \times (0, \infty)$ given by $\mathcal{M}(x, y, t) = e^{-\frac{|x-y|}{t}}$, for all $x, y \in \mathcal{K}$ and all $t > 0$. Then, $(\mathcal{K}, \mathcal{M}, *)$ is an \mathcal{M} -complete fuzzy metric space. Consider the mapping $T : \mathcal{K} \rightarrow \mathcal{K}$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1], \\ x + 1, & \text{if } x \in (1, \infty). \end{cases}$$

It is obvious that T is not continuous at $x = 1$, and hence, Theorem 3 cannot be applied to this example. Define two essential functions $\xi : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ by

$$\xi(l, s) = \begin{cases} l, & \text{if } l > s, \\ \sqrt{s}, & \text{if } s \geq l, \end{cases} \text{ and } \sigma(x, y, t) = \begin{cases} 1, & x, y \in [0, 1], \\ e^{-\frac{|\frac{x}{2}-y-1|}{t}}, & x \in [0, 1] \ \& \ y \in (1, \infty), \\ e^{-\frac{|\frac{y}{2}-x-1|}{t}}, & y \in [0, 1] \ \& \ x \in (1, \infty), \\ e^{-\frac{|x-y|}{t}}, & x, y \in (1, \infty), \end{cases}$$

Let $x, y \in \mathcal{K}$ such that $\sigma(x, y, t) \leq 1$. Then, either $x, y \in [0, 1]$ or $x = y \in (1, \infty)$. In case $x, y \in [0, 1]$, by the definition of T , we have $Tx, Ty \in [0, 1]$, and hence, $\sigma(Tx, Ty) = 1$. In the other case, if $x = y \in (1, \infty)$, then again, by the definition of T , we have $Tx = Ty \in (1, \infty)$, and hence, $\sigma(Tx, Ty) = 1$. Therefore, T is a σ -admissible mapping. Furthermore, $1 \in \mathcal{K}$ and $\sigma(1, T1, t) = \sigma(1, \frac{1}{2}, t) = 1$. Further, let $\{\alpha_n\}$ be a sequence in \mathcal{K} such that $\lim_{n \rightarrow \infty} \alpha_n = x$ with $k_0 = 1$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for all $n \in \mathbb{N}$. From the definition of α , it follows that $\alpha_n \in [0, 1]$, for all $n \in \mathbb{N}$, if we assume that $x \in (1, \infty)$, then we assume

$$\mathcal{M}(\alpha_n, x, t) = e^{-\frac{|\alpha_n-x|}{t}} < 1, \text{ for all } t > 0,$$

which is a contradiction of the assumption that $\lim_{n \rightarrow \infty} \alpha_n = x$. Thus, we have $x \in [0, 1]$. Therefore, $\alpha(\alpha_n, x, t) \leq 1$ and $\alpha(\alpha_n, \alpha_m, t) \leq 1$ for all $m, n \in \mathbb{N}$ and $t > 0$.

Finally, we show that T is a fuzzy (α, \mathcal{Z}) -contractive mapping. To do so, for all $x, y \in x$ with $Tx \neq Ty$, we consider the following four cases.

Case I: If $x, y \in [0, 1]$, then (as $\alpha(x, y, t) = 1$ and $\mathcal{M}(Tx, Ty, t) > \mathcal{M}(x, y, t)$), and we have

$$\begin{aligned} e^{\left(\frac{-|x-y|}{2t}\right)} &= \alpha(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \zeta(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \mathcal{M}(Tx, Ty, t) = e^{\left(\frac{-|x-y|}{2t}\right)}. \end{aligned}$$

Case II: If $x \in [0, 1]$ and $y \in (1, \infty)$, then (as $\alpha(x, y, t) = e^{\left(\frac{|\frac{x}{2}-y-1\right)}{t}\right)}$), and we distinguish two subcases.

Subcase I: If $\mathcal{M}(Tx, Ty, t) < \mathcal{M}(x, y, t)$, then we have

$$\begin{aligned} 1 &= e^{\frac{|\frac{x}{2}-y-1|}{t}} \cdot e^{\frac{-|\frac{x}{2}-y-1|}{t}} = \alpha(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \zeta(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \mathcal{M}(Tx, Ty, t) = e^{\frac{-|\frac{x}{2}-y-1|}{t}}. \end{aligned}$$

Subcase II: If $\mathcal{M}(Tx, Ty, t) \geq \mathcal{M}(x, y, t)$, then we have

$$\begin{aligned} 1 &= e^{\frac{|\frac{x}{2}-y-1|}{t}} \cdot e^{\frac{-|\frac{x}{2}-y-1|}{t}} = \alpha(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \zeta(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \sqrt{\mathcal{M}(x, y, t)} = e^{\frac{-|\frac{x}{2}-y-1|}{2t}}. \end{aligned}$$

Case III: This case is similar to that in **Case II**.

Case IV: If $x, y \in (1, \infty)$, then (as $\alpha(x, y, t) = e^{\left(\frac{|x-y|}{t}\right)}$ and $\mathcal{M}(Tx, Ty, t) = \mathcal{M}(x, y, t)$), and we have

$$\begin{aligned} 1 &= e^{\frac{|x-y|}{t}} \cdot e^{\frac{-|x-y|}{t}} = \alpha(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \zeta(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \sqrt{\mathcal{M}(Tx, Ty, t)} = e^{\frac{-|x-y|}{2t}}. \end{aligned}$$

Hence, in all cases, T is a fuzzy (α, \mathcal{Z}) -contractive mapping. Therefore, all the hypotheses of Theorem 4 are satisfied. Hence, T has a fixed point (namely $x = 0$).

However, T is not a fuzzy \mathcal{Z} -contractive mapping. To see this, we consider the case that $x, y \in (1, \infty)$ and take into account Remark 1; we have $\mathcal{M}(Tx, Ty, t) = \mathcal{M}(x, y, t)$, and hence,

$$\begin{aligned} e^{\frac{-|x-y|}{t}} &= \alpha(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \zeta(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &> \mathcal{M}(x, y, t) = e^{\frac{-|x-y|}{t}}. \end{aligned}$$

which impossible; hence, T is not a fuzzy \mathcal{Z} -contractive mapping.

Now, by an example (see also [28], Example 3.10), we show that the assumption (b) of Theorems 3 and 4 is not superfluous.

Example 7. Let $\mathcal{K} = \mathbb{N}$, and define the fuzzy metric \mathcal{M} by

$$\mathcal{M}(x, y, t) = \min\left\{\frac{x}{y}, \frac{y}{x}\right\} \text{ for all } x, y \in \mathcal{K}, t > 0.$$

Then, $(\mathcal{K}, \mathcal{M}, *)$ is an \mathcal{M} -complete fuzzy metric space where $*$ is the product t -norm. Define a mapping $T : \mathcal{K} \rightarrow \mathcal{K}$ by $Tx = x + 1$ for all $x \in \mathcal{K}$. Then, T is a fuzzy (σ, \mathcal{Z}) -contractive mapping with respect to the functions $\xi : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ by

$$\xi(l, s) = \begin{cases} \frac{l+s}{2}, & \text{if } l > s, \\ 1 & \text{if } s \geq l, \end{cases} \quad \text{and} \quad \sigma(x, y, t) = 1.$$

From the definition of σ , it is very clear that the conditions (a), (c), (d), (e), and (e') of Theorems 3 and 4 are satisfied. Moreover, trivial calculations show that the condition (b) does not hold, that is the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ does not have the property (S). Notice that T does not have a fixed point.

Next, the following example shows that the assumption (c) of Theorems 3 and 4 is not superfluous.

Example 8. Let $\mathcal{K} = [1, \infty)$ and the fuzzy metric be defined by $\mathcal{M}(x, y, t) = \frac{t}{t+d(x,y)}$, $t > 0$, where $d(x, y) = |x - y|$, for all $x, y \in \mathcal{K}$. Then, $(\mathcal{K}, \mathcal{M}, *)$ is an \mathcal{M} -complete fuzzy metric space, where $*$ is a product t -norm. Define a mapping $T : \mathcal{K} \rightarrow \mathcal{K}$ by

$$T(x) = x + 1, \text{ for all } x \in \mathcal{K}.$$

Furthermore, we define two essential functions: $\xi : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$ by

$$\xi(l, s) = \frac{s}{l} \text{ and } \sigma(x, y, t) = 1 + \frac{|x - y|}{t}.$$

Then, for all $x, y \in \mathcal{K}$ such that $Tx \neq Ty$ and $t > 0$, we have

$$\begin{aligned} 1 &= \sigma(x, y, t)\mathcal{M}(Tx, Ty, t) \\ &\geq \xi(\mathcal{M}(Tx, Ty, t), \mathcal{M}(x, y, t)) \\ &= \frac{\mathcal{M}(Tx, Ty, t)}{\mathcal{M}(x, y, t)} = 1, \end{aligned}$$

which shows that T is fuzzy (σ, \mathcal{Z}) -contractive. Moreover, it is easy to show that the conditions (a), (b), (d), (e), and (e') of Theorems 3 and 4 hold. Now, note that there is no x_0 in \mathcal{K} such that $\sigma(x_0, Tx_0, t) \leq 1$ for $t > 0$. Thus, the condition (c) of Theorems 3 and 4 does not hold. Observe that the mapping T does not have a fixed point.

The following theorem enables us to extend the fixed point result for the family of σ - ψ -fuzzy contractive mappings due to Gopal and Vetro [27] with an additional condition.

Theorem 5. Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow (0, \infty)$. Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$ is a fuzzy (σ, \mathcal{Z}) -contractive mapping and the following properties hold:

- (a) T is σ -admissible;
- (b') The quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S');
- (c) There exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$;
- (d) For each sequence $\{\alpha_n\}$ of \mathcal{K} with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0, t > 0$;
- (e') If $\{\alpha_n\}$ is a sequence in \mathcal{K} such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha \in \mathcal{K}$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, then $\sigma(\alpha_n, \alpha, t) \leq 1$.

In addition, assume that $\lim_{n \rightarrow \infty} \inf_{m > n} \mathcal{M}(T^n \alpha, T^m \alpha, t) > 0$, for all $x \in \mathcal{K}$ and $t > 0$. Then, T admits a fixed point.

Proof. Following the same lines of the proof of Theorem 4 and taking into account that the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S') instead of the property (S) with the

fact that $\lim_{n \rightarrow \infty} \inf_{m > n} \mathcal{M}(T^n \alpha, T^m \alpha, t) > 0$, for all $x \in \mathcal{K}$ and $t > 0$, we obtain the required result. \square

Next, we discuss the uniqueness of the fixed point in Theorems 3–5. In order to ensure the uniqueness of the fixed point, we add one more sufficient condition to the hypothesis of the theorems. Precisely, we take into account the following condition:

(h) For each $\alpha, \beta \in \text{Fix}(T)$, we have $\sigma(\alpha, \beta, t) \leq 1$, for all $t > 0$.

Theorem 6. *In addition to the hypothesis of Theorems 3–5, assume that the condition (h) holds. Then, the fixed point of T is unique.*

Proof. Theorems 3–5 ensure the existence of a fixed point of T . Assume that γ_1 and γ_2 are two distinct fixed points, that is, $T\gamma_1 = \gamma_1 \neq \gamma_2 = T\gamma_2$. Then, there exists $s > 0$ such that $\mathcal{M}(\gamma_1, \gamma_2, s) < 1$. As T is a fuzzy (σ, \mathcal{Z}) -contractive mapping, in view of the definition of ξ and condition (h), we have

$$\mathcal{M}(\gamma_1, \gamma_2, s) \geq \sigma(\gamma_1, \gamma_2, s) \mathcal{M}(\gamma_1, \gamma_2, s) \geq \xi(\mathcal{M}(\gamma_1, \gamma_2, s), \mathcal{M}(\gamma_1, \gamma_2, s)) > \mathcal{M}(\gamma_1, \gamma_2, s),$$

a contradiction. Therefore, $\mathcal{M}(\gamma_1, \gamma_2, t) = 1$, for all $t > 0$, that is $\gamma_1 = \gamma_2$. \square

Remark 3. *Observe that the mappings defined in Examples 5 and 6 satisfy the condition (h), and hence, according to Theorem 6, T admits a unique fixed point.*

Corollary 1 ([28]). *Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $T : \mathcal{K} \rightarrow \mathcal{K}$. Assume that T is a fuzzy \mathcal{Z} -contractive mapping and the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S). Then, T possesses a unique fixed point.*

Proof. The existence of the fixed point follows from Remark 2, Part (a), and Theorem 4, and the uniqueness of the fixed point follows from Theorem 6. \square

Corollary 2 ([28]). *Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $T : \mathcal{K} \rightarrow \mathcal{K}$. Assume that T is a fuzzy \mathcal{Z} -contractive mapping and the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S'). In addition, assume that $\lim_{n \rightarrow \infty} \inf_{m > n} \mathcal{M}(\alpha_n, \alpha_m, t) > 0$, for all $x \in \mathcal{K}$ and $t > 0$. Then, T possesses a unique fixed point.*

Proof. The existence of the fixed point follows from Remark 2, Part (a), and Theorem 5, and the uniqueness of the fixed point follows from Theorem 6. \square

Corollary 3. *Let $(\mathcal{K}, \mathcal{M}, *)$ be an \mathcal{M} -complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \rightarrow [0, \infty)$. Assume that $T : \mathcal{K} \rightarrow \mathcal{K}$ is a σ - ψ -fuzzy contractive mapping satisfying the following assumptions:*

- (i) T is σ -admissible;
- (ii) There exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, T\alpha_0, t) \leq 1$, for each $t > 0$;
- (iii) For each sequence $\{\alpha_n\}$ of \mathcal{K} with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0, t > 0$;
- (iv) If $\{\alpha_n\}$ is a sequence in \mathcal{K} such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha \in \mathcal{K}$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, then $\sigma(\alpha_n, \alpha, t) \leq 1$.

In addition, assume that $\lim_{n \rightarrow \infty} \inf_{m > n} \mathcal{M}(\alpha_n, \alpha_m, t) > 0$, for all $x \in \mathcal{K}$ and $t > 0$. Then, T admits a fixed point.

Proof. In view of Theorem 5 and Remark 2 part (b), we need only to show that the quadruple $(\mathcal{K}, \mathcal{M}, T, \xi)$ owns the property (S'), where $\xi(l, s) = \psi(s)$, for each $l, s \in (0, 1]$ and $\psi \in \Psi$. Assume that $\{\alpha_n\}$ is any Picard sequence for all $n \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ and $t > 0$,

$\inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) \leq \inf_{m>n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t)$ with $0 < \lim_{n \rightarrow \infty} \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) = \alpha(t)$. Then, by the definition of ψ , we have

$$\lim_{n \rightarrow \infty} \inf_{m>n} \zeta(\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t), \mathcal{M}(\alpha_n, \alpha_m, t)) = \psi(\alpha(t)).$$

Since T is a σ -admissible mapping, one easily can show that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}_0$ and $t > 0$. Applying Condition (iii), there exists $k_0 \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ with $m > n \geq k_0$, $t > 0$, we have

$$\sigma(\alpha_n, \alpha_m, t) \leq 1.$$

The σ - ψ -fuzzy contractivity of the mapping T gives rise to

$$\begin{aligned} \psi(\alpha(t)) &\leq \lim_{n \rightarrow \infty} \inf_{m>n} \sigma(\alpha_n, \alpha_m, t) \mathcal{M}(T\alpha_n, T\alpha_m, t) \\ &\leq \lim_{n \rightarrow \infty} \inf_{m>n} \mathcal{M}(T\alpha_n, T\alpha_m, t) = \alpha(t), \end{aligned}$$

that is, $\psi(\alpha) \leq \alpha(t)$ which implies that $\alpha(t) = 1, \forall t > 0$, and hence, we have

$$\lim_{n \rightarrow \infty} \inf_{m>n} \zeta(\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t), \mathcal{M}(\alpha_n, \alpha_m, t)) = \psi(\alpha(t)) = \psi(1) = 1.$$

This completes the proof. \square

4. Conclusions

Motivated by the results of Shukla et al. [28] and Gopal et al. [27], we introduced the notion of fuzzy (σ, \mathcal{Z}) -contractive mappings, which enlarge and unify the class of fuzzy \mathcal{Z} -contractive mappings introduced in [28] and the family of σ - ψ -fuzzy contractive mappings obtained in [27]. The new class of mappings covers all the concepts introduced in [7–9,27–29]. Our newly introduced notion was utilized to prove some results in \mathcal{M} -complete fuzzy metric spaces. Finally, some examples were adopted to demonstrate that our newly presented results are a proper extension of Shukla et al.'s results [28].

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Abbreviations

The following abbreviations are used in this manuscript:

FMS Fuzzy metric space

References

1. Deng, Z. Fuzzy pseudo-metric spaces. *J. Math. Anal. Appl.* **1982**, *86*, 74–95. [CrossRef]
2. Erceg, M.A. Metric spaces in fuzzy set theory. *J. Math. Anal. Appl.* **1979**, *69*, 205–230. [CrossRef]
3. Kaleva, O.; Seikkala, S. On fuzzy metric spaces. *Fuzzy Sets Syst.* **1984**, *12*, 215–229. [CrossRef]
4. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
5. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389. [CrossRef]

6. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [[CrossRef](#)]
7. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [[CrossRef](#)]
8. Mihet, D. Fuzzy ψ -contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2008**, *159*, 739–744. [[CrossRef](#)]
9. Wardowski, D. Fuzzy contractive mappings and fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **2013**, *222*, 108–114. [[CrossRef](#)]
10. Saleh, H.N.; Khan, I.A.; Imdad, M.; Alfaqih, W.M. New fuzzy φ -fixed point results employing a new class of fuzzy contractive mappings. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5391–5402. [[CrossRef](#)]
11. Saleh, H.N.; Imdad, M.; Hasanuzzaman, M. A short and sharpened way to approach fixed point results involving fuzzy \mathcal{H} -contractive mappings. *Fixed Point Theory Appl.* **2020**, *2020*, 15. [[CrossRef](#)]
12. Nasr Saleh, H.; Imdad, M.; Khan, I.; Hasanuzzaman, M. Fuzzy Θ F-contractive mappings and their fixed points with applications. *J. Intell. Fuzzy Syst.* **2020**, *39*, 7097–7106. [[CrossRef](#)]
13. Gregori, V.; Miñana, J.J.; Miravet, D. Contractive sequences in fuzzy metric spaces. *Fuzzy Sets Syst.* **2020**, *379*, 125–133. [[CrossRef](#)]
14. Huang, H.; Carić, B.; Došenović, T.; Rakić, D.; Brdar, M. Fixed point theorems in fuzzy metric spaces via fuzzy F-contraction. *Mathematics* **2021**, *9*, 641. [[CrossRef](#)]
15. Melliani, S.; Moussaoui, A. Fixed point theorem using a new class of fuzzy contractive mappings. *J. Univ. Math.* **2018**, *1*, 148–154.
16. Roldán-López-de Hierro, A.F.; Karapinar, E.; Manro, S. Some new fixed-point theorems in fuzzy metric spaces. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2257–2264. [[CrossRef](#)]
17. Abbas, M.; Imdad, M.; Gopal, D. ψ -weak contractions in fuzzy metric spaces. *Iran. J. Fuzzy Syst.* **2011**, *8*, 141–148.
18. Sedghi, S.; Shobkolaei, N.; Došenović, T.; Radenović, S. Suzuki-type of common fixed-point theorems in fuzzy metric spaces. *Math. Slovaca* **2018**, *68*, 451–462. [[CrossRef](#)]
19. Imdad, M.; Ali, J. Some common fixed-point theorems in fuzzy metric spaces. *Math. Commun.* **2006**, *11*, 153–163.
20. Imdad, M.; Ali, J.; Hasan, M. Common fixed-point theorems in fuzzy metric spaces employing common property (EA). *Math. Comput. Model.* **2012**, *55*, 770–778. [[CrossRef](#)]
21. Beg, I.; Gopal, D.; Dosenovic, T.; Rakić, D. α -type fuzzy H-contractive mappings in fuzzy metric spaces. *Fixed Point Theory Appl.* **2016**, *19*, 463–474. [[CrossRef](#)]
22. Beg, I.; Sedghi, S.; Shobe, N. Fixed point theorems in fuzzy metric spaces. *Int. J. Anal.* **2013**, *2013*, 934145. [[CrossRef](#)]
23. Zheng, D.; Wang, P. Meir-Keeler theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2019**, *370*, 120–128. [[CrossRef](#)]
24. Sezen, M.S. Fixed point theorems for new type contractive mappings. *J. Funct. Spaces* **2019**, *2019*, 2153563. [[CrossRef](#)]
25. Mohammadi, B.; Hussain, A.; Parvaneh, V.; Saleem, N.; Shahkoobi, R.J. Fixed point results for generalized fuzzy contractive mappings in fuzzy metric spaces with application in integral equations. *J. Math.* **2021**, *2021*, 9931066. [[CrossRef](#)]
26. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 2154–2165. [[CrossRef](#)]
27. Gopal, D.; Vetro, C. Some new fixed-point theorems in fuzzy metric spaces. *Iran. J. Fuzzy Syst.* **2014**, *11*, 95–107. [[CrossRef](#)]
28. Shukla, S.; Gopal, D.; Sintunavarat, W. A new class of fuzzy contractive mappings and fixed-point theorems. *Fuzzy Sets Syst.* **2018**, *350*, 85–94. [[CrossRef](#)]
29. Tirado Peláez, P. Contraction mappings in fuzzy quasi-metric spaces and $[0, 1]$ -fuzzy posets. *Fixed Point Theory* **2012**, *13*, 273–283.
30. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 313–334. [[CrossRef](#)]
31. Vetro, C. Fixed points in weak non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2011**, *162*, 84–90. [[CrossRef](#)]
32. Gupta, V.; Saini, R.; Verma, M. Fixed point theorems for single valued α - ψ -mappings in fuzzy metric spaces. *Commun. Fac. Sci. Univ. Ank. Ser. Math. Stat.* **2018**, *68*, 392–400. [[CrossRef](#)]