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Various results concerning homogenisation of nematic liquid crystals

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RESUMEN

En esta tesis presentamos varios resultados relativos a la homogeneización de cristales líquidos nemáticos: dos de ellos son en dominios perforados, mientras que el otro se refiere a las tasas de convergencia para la homogeneización de fronteras. El primer trabajo descrito en este tesis es un resultado de Γ-convergencia para el modelo de Landau-de Gennes en dominios 3D con perforaciones conectadas. El objetivo del análisis es encontrar nuevos términos en el funcional de energía que sean independientes del gradiente del tensor Q. El segundo resultado es una estimación de error para un modelo 2D utilizado para describir los efectos de rugosidad en cristales líquidos nemáticos mediante problemas de homogeneización, utilizando de nuevo el modelo de Landau-de Gennes. El último problema es un resultado de convergencia local en L^2 para un problema de homogeneización en $\mathbb{R}^2$ con perforaciones aisladas. Aquí utilizamos el modelo de Oseen-Frank, con el objetivo de encontrar nuevos términos dependientes del gradiente en el funcional de energía.

En las siguientes líneas presentamos las principales ideas de cada capítulo.

- Capítulo 1 - *Introduction*

Comenzamos aquí con una breve introducción a los cristales líquidos nemáticos. Presentamos dos modelos variacionales principales utilizados para describir los cristales líquidos nemáticos: Landau-de Gennes (LdG) y Oseen-Frank (OF). Para la teoría de LdG, que utiliza los tensores Q como parámetro de orden, presentamos las opciones típicas para cada tipo de contribución energética (bulk, elástica y superficial). Para la teoría OF basada en el director nemático $n \in \mathbb{S}^2$ discutimos la energía elástica, que depende del director y de su gradiente. A continuación presentamos un breve resumen de los principales resultados matemáticos obtenidos para LdG y OF.

- Capítulo 2 - *Homogenised bulk terms in a case of the Landau-de Gennes model*

En este capítulo, analizamos un problema de homogeneización en $\mathbb{R}^3$ utilizando el modelo de Landau-de Gennes en el que las perforaciones forman una microlátice cúbica. Por microlátice cúbica entendemos una familia de paralelepípedos interconectados de escala muy pequeña (ver Figura 2) y a veces nos referimos a ella simplemente como andamio. Este tipo de geometría para las inclusiones se utiliza principalmente en la industria, donde tales andamios se denominan matriz sólida porosa bicontinua o BPSM (como en [22], [69] o [68]) y tales objetos pueden construirse mediante la técnica de polimerización de dos fotones (two-photon polymerisation), también llamada 2PP o TPP. Una visión general de esta técnica de impresión 3D se puede encontrar en [9].

El trabajo de capítulo 2 continúa en la dirección de estudiar el material homogeneizado y se basa en el trabajo de [28] y [29], que también se basó en [13, 16, 23]. La idea general de estos trabajos es demostrar que el límite de homogeneización de un cristal líquido nemático con inclusiones coloidales de una geometría específica puede generar un nuevo material, que

se comporta como un nuevo cristal líquido nemático, pero ahora con diferentes parámetros del material. En [28] y [29], se desconecta el conjunto de partículas de inclusión, obtenidas a partir de partículas modelo diferentes o idénticas, de forma que la distancia entre las partículas es considerablemente mayor que el tamaño de las mismas, lo que se denomina régimen diluido. Además, en este régimen, la fracción de volumen de los coloides tiende a 0. Sin embargo, la configuración geométrica de una microlátice cúbica es más relevante desde el punto de vista físico, ya que en [28] y en [29] no se pueden posicionar *a priori* las partículas coloidales de forma periódica. Aquí la periodicidad se genera automáticamente por la estructura de la microlatícula cúbica.

La construcción matemática de una microlátice cúbica puede verse de la siguiente manera: primero elegimos un pequeño parámetro $\varepsilon > 0$, luego construimos una familia de paralelepípedos idénticos disjuntos colocados de forma periódica (sus centros están a una distancia de ε entre sí), de la siguiente forma:

$$\mathcal{C}^\alpha = \left[-\frac{\varepsilon^\alpha}{2p}, +\frac{\varepsilon^\alpha}{2p} \right] \times \left[-\frac{\varepsilon^\alpha}{2q}, +\frac{\varepsilon^\alpha}{2q} \right] \times \left[-\frac{\varepsilon^\alpha}{2r}, +\frac{\varepsilon^\alpha}{2r} \right],$$

con $p,\ q,\ r \in [1, +\infty)$ y $\alpha \in (1, 2)$. Luego los *interconectamos* con otras 3 familias de paralelípedos idénticos, de forma que conseguimos un andamio conectado, pero, al mismo tiempo, no los conectamos con $\partial\Omega$ (el andamio está incluido en Ω y no toca $\partial\Omega$). Explicamos por qué elegimos α entre 1 y 2 en Remark 2.2.6 y destacamos aquí que, por nuestra construcción del andamiaje, su volumen tiende a 0 a medida que $\varepsilon \to 0$. Al mismo tiempo, si la familia inicial de paralelepípedos idénticos disjuntos son realmente cubos, es decir, $p = q = r$, decimos que el andamio presenta *simetría cúbica*.

En este entorno, podemos demostrar que, en el límite a medida que $\varepsilon \to 0$, la interacción superficial entre el cristal líquido nemático y el andamio se transforma en una energía de tipo bulk y, afinando las longitudes del andamio y eligiendo densidades de energía superficial específicas, podemos conseguir los coeficientes bulk deseados. Por lo tanto, partiendo de un cristal líquido nemático con coeficientes a granel a, b y c (como se presenta en Subsection 1.3.2) confinado en un dominio perforado por una microrred cúbica, entonces, dado unos a', b', c' y eligiendo convenientemente los parámetros del andamiaje y la energía superficial, en el límite, podemos conseguir un nuevo material de tipo cristal líquido con nuevos coeficientes de masa a', b' y c', centrándonos en conseguir a', ya que esto depende de la temperatura (depende de la temperatura a la que el estado isotrópico pierde estabilidad - ver Subsection 1.3.2 para más detalles).

Para ser más precisos, dejemos que Ω sea un dominio acotado y liso y que $\mathcal{N}_\varepsilon$ sea una microlátice cúbica, para $\varepsilon > 0$. Consideramos el siguiente funcional de energía libre de Landau-de Gennes:

$$\mathcal{F}_\varepsilon[Q] := \int_{\Omega_\varepsilon} \big(f_e(\nabla Q) + f_b(Q)\big)\mathrm{d}x + \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)} \int_{\partial\mathcal{N}_\varepsilon} f_s(Q, \nu)\mathrm{d}\sigma,$$

con $\Omega_\varepsilon = \Omega \setminus \mathcal{N}_\varepsilon$ y suponemos que:

- $f_e : \mathcal{S}_0 \otimes \mathbb{R}^3 \to [0, +\infty)$ *es diferenciable, fuertemente convexa*[1] *y existe una constante* $\lambda_e > 0$ *tal que*
$$\lambda_e^{-1}|D|^2 \leq f_e(D) \leq \lambda_e |D|^2, \quad |(\nabla f_e)(D)| \leq \lambda_e(|D| + 1),$$
para cualquier $D \in \mathcal{S}_0 \times \mathbb{R}^3$.

- $f_b : \mathcal{S}_0 \to \mathbb{R}$ *es continua, acotada desde abajo y existe una constante* $\lambda_b > 0$ *tal que* $|f_b(Q)| \leq \lambda_b(|Q|^6 + 1)$ *para cualquier* $Q \in \mathcal{S}_0$.

- $f_s : \mathcal{S}_0 \times \mathbb{S}^2 \to \mathbb{R}$ *es continua y existe una constante estrictamente positiva* λ_s *tal que, para cualquier* $Q_1, Q_2 \in \mathcal{S}_0$ *y* $\nu \in \mathbb{S}^2$*, tenemos*
$$|f_s(Q_1, \nu) - f_s(Q_2, \nu)| \leq \lambda_s |Q_1 - Q_2| \big(|Q_1|^3 + |Q_2|^3 + 1\big).$$

También imponemos "strong anchoring" en la frontera de Ω: sea $g \in H^{1/2}(\partial\Omega, \mathcal{S}_0)$ un dato de frontera y denotamos por $H^1_g(\Omega, \mathcal{S}_0)$ un conjunto de funciones Q de $H^1(\Omega, \mathcal{S}_0)$ tal que $Q = g$ en $\partial\Omega$ en el sentido de la traza. Del mismo modo, definimos $H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$ como $H^1(\Omega_\varepsilon)$ con $Q = g$ en $\partial\Omega$ en el sentido de la traza.

Para presentar el resultado principal, necesitamos introducir la función $f_{hom} : \mathcal{S}_0 \to \mathbb{R}$ como:

$$f_{hom}(Q) := \frac{q+r}{qr} \int_{\mathcal{C}^x} f_s(Q, \nu) \mathrm{d}\sigma + \frac{p+r}{pr} \int_{\mathcal{C}^y} f_s(Q, \nu) \mathrm{d}\sigma + \frac{p+q}{pq} \int_{\mathcal{C}^z} f_s(Q, \nu) \mathrm{d}\sigma,$$

para cualquier $Q \in \mathcal{S}_0$, donde $\mathcal{C} = \big[-1/2,\ 1/2\big]^3$ y $\mathcal{C}^x$, $\mathcal{C}^y$ y $\mathcal{C}^z$ son las uniones de las caras normales a Ox, Oy y Oz.

Además, dentro del andamio, utilizamos el operador de extensión armónica, $E_\varepsilon : H^1_g(\Omega_\varepsilon, \mathcal{S}_0) \to H^1_g(\Omega, \mathcal{S}_0)$, definido de la siguiente manera: en Ω_ε tenemos $E_\varepsilon Q := Q$ y dentro del andamio, $E_\varepsilon Q$ es la solución única del siguiente problema:

$$\begin{cases} \Delta E_\varepsilon Q = 0 & \text{in } \mathcal{N}_\varepsilon \\ E_\varepsilon Q \equiv Q & \text{on } \partial\mathcal{N}_\varepsilon. \end{cases}$$

Utilizando herramientas de Γ-convergencia, podemos demostrar el resultado principal para este situación general:

Teorema. Sea $\mathcal{F}_0 : \mathcal{S}_0 \to [0, +\infty)$ definido como

$$\mathcal{F}_0[Q] := \int_\Omega \big(f_e(\nabla Q) + f_b(Q) + f_{hom}(Q)\big) \mathrm{d}x$$

y dejemos que $Q_0 \in H^1_g(\Omega, \mathcal{S}_0)$ sea un minimizador local aislado de H^1 para $\mathcal{F}_0$, es decir existe $\delta_0 > 0$ tal que $\mathcal{F}_0[Q_0] < \mathcal{F}_0[Q]$ para cualquier $Q \in H^1_g(\Omega, \mathcal{S}_0)$ tal que $\|Q - Q_0\|_{H^1_g(\Omega, \mathcal{S}_0)} \leq \delta_0$ y $Q \neq Q_0$. Entonces, para cualquier ε suficientemente pequeño, existe una secuencia de minimizadores locales Q_ε de $\mathcal{F}_\varepsilon$ tal que $E_\varepsilon Q_\varepsilon \to Q_0$ fuertemente en $H^1_g(\Omega, \mathcal{S}_0)$.

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[1] Digamos que una función $f : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ es fuertemente convexa si existe $\theta > 0$ tal que la función $\tilde{f} : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ definida por $\tilde{f}(D) = f(D) - \theta|D|^2$ es convexa.

Al mismo tiempo, somos capaces de calcular un índice de convergencia para la rapidez con la que las energías superficiales convergen a su funcional homogeneizado. Para ello, primero tenemos que tener en cuenta la geometría del andamio: a partir de la red inicial de paralelepípedos construidos, sólo unos pocos de ellos están en contacto con el cristal líquido nemático - sólo los que están cerca de $\partial\Omega$, mientras que los otros tocan el NLC sólo por sus bordes. En Subsection 2.4.2, , demostramos que estas interacciones no tienen ninguna contribución en el límite a medida que $\varepsilon \to 0$, por lo tanto, el funcional homogeneizado está dado por el límite de las energías superficiales calculadas en las "paredes de los paralelepípedos de conexión". Suponiendo además que f_s es localmente Lipschitz continuo y que g está acotado y Lipschitz, podemos demostrar en Proposition 2.6.1 de Section 2.6 que la tasa de convergencia descrita anteriormente es de orden ε^{m_0}, con

$$m_0 = \min\left\{\frac{\alpha - 1}{3}, 2 - \alpha\right\},$$

donde $\alpha \in (1, 2)$ es el parámetro utilizado para la construcción de la microlátice cúbica.

- Capítulo 3 - *Error estimates for rugosity effects*

En este capítulo, consideramos el caso de un cristal líquido nemático en un dominio con una frontera oscilante. Nos interesa estudiar el caso en el que las superficies onduladas, para las que la longitud de onda es de tamaño comparable a la amplitud, pueden conducir a energías superficiales efectivas en el límite a medida que la amplitud converge a cero. Los problemas de este tipo se han considerado en el lenguaje de la homogeneización de las EDP en un dominio con una frontera oscilante, donde se puede demostrar rigurosamente que ciertos sistemas *rugosas*, escalares, lineales, tienen ciertos comportamientos efectivos en el límite. Estos han sido considerados, por ejemplo, en el contexto de [4, 6, 7, 12, 32] o [48], pero la lista no es en absoluto exhaustiva. Una visión general contemporánea de la literatura de esta dirección se puede encontrar, por ejemplo, en la introducción de [6]. Sin embargo, la naturaleza de las energías superficiales físicamente significativas en el contexto de los cristales líquidos nemáticos proporciona modelos que aún no han sido considerados en la literatura dentro de este marco de homogeneización.

Consideramos un escenario simplificado de una losa bidimensional con rugosidad periódica y una energía libre cuadrática, que proporciona un modelo de juguete de un *paranemático*. Es decir, un sistema de moléculas mesogénicas a alta temperatura que se ha fundido en un estado isotrópico, pero que aún admite algún ordenamiento nemático local inducido por la superficie. En este caso, gracias a la simplicidad del sistema, podemos proporcionar estimaciones cuantitativas sobre cómo se comportan los estados básicos en el límite homogeneizado. Consideramos un parámetro de rugosidad, ε, arbitrariamente pequeño, que se utiliza para describir la frontera oscilante y luego el problema límite describe el comportamiento a medida que este parámetro tiende a cero. En una situación física el parámetro ε es pequeño, pero finito. Si se intenta entonces entender hasta qué punto el problema límite es una buena descripción del problema con ε pequeño, se necesita obtener *tasas de convergencia*. Mientras que un resultado de Γ-convergencia nos da una descripción del sistema en un límite potencialmente no físico, la obtención de una tasa de convergencia permite la comprensión cuantitativa de la aproximación al límite teórico en regímenes de parámetros físicamente razonables.

Se sabe, a partir de la teoría general de la homogeneización, que las tasas de convergencia pueden mejorarse calculando correctores, una manifestación del hecho de que los fenómenos de la capa límite generan diferencias localizadas entre los dos problemas (véase, por ejemplo, el Lema 5.1 frente al Teorema 5.2 en [4]). Un enfoque alternativo, para obtener mejores tasas de convergencia, y sin utilizar correctores, es utilizar normas más débiles que no pongan demasiado peso en lo que ocurre en la frontera. Este enfoque parece no estar estudiado en la literatura de homogeneización estándar y es nuestra principal contribución aquí. Utilizamos un argumento de dualidad en un entorno L^p que, sin embargo, no incluye el punto final $p = +\infty$, que esperamos que sea el óptimo. Además, el uso del argumento de dualidad se basa en la estructura lineal y una extensión al caso no lineal no es inmediata. Para entender estas cuestiones, analizamos el escenario más simplificado posible que sigue teniendo cierta relevancia física.

Consideramos la situación de una losa bidimensional con rugosidad periódica y el modelo de Landau-de Gennes para la descripción del cristal líquido nemático utilizado. Más concretamente, el dominio limitante es de la forma $\Omega_0 = \{(x,y) \mid x \in [0,2\pi),\ y \in (0,R)\}$, donde $R > 0$ es una constante, y el dominio rugoso es de la forma $\Omega_\varepsilon = \{(x,y) \mid x \in [0,2\pi),\ y \in (\varphi_\varepsilon(x),R)\}$, donde $\varphi_\varepsilon(x) = \varepsilon\varphi(x/\varepsilon)$ y $\varphi : \mathbb{R} \to \mathbb{R}$ es una función C^2 2π-periódica con $\varphi \geq 0$. Denotamos con $\Gamma_\varepsilon = \{(x, \varepsilon\varphi(x/\varepsilon)) \mid x \in [0,2\pi)\}$ la frontera rugosa y con $\Gamma_R = \{(x,R) \mid x \in [0,2\pi)\}$ la frontera superior fija de los dominios. Consideramos una energía libre cuadrática de la siguiente forma:

$$\mathcal{F}_\varepsilon[Q] = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}(x,y) + \int_{\Gamma_\varepsilon} \frac{w_0}{2}|Q - Q_\varepsilon^0|^2 \,\mathrm{d}\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2}|Q - Q_R|^2 \,\mathrm{d}\sigma_R,$$

donde $c > 0$ es constante, $w_0 > 0$ es la fuerza de anclaje, $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$ y $Q_R = \nu_R \otimes \nu_R - \frac{1}{2}I$ (ν_ε y ν_R son las normales exteriores a Γ_ε y Γ_R).

En este modelo simplificado, utilizando Proposition 3.2.1, podemos identificar la energía superficial homogeneizada como $\frac{w_{ef}}{2}|Q - Q_{ef}|^2$, con $w_{ef} = w_0\gamma$ y $Q_{ef} = \frac{1}{\gamma}\begin{pmatrix} G_1 & G_2 \\ G_2 & -G_1 \end{pmatrix}$, donde γ, G_1 y G_2 se definen en Definition 3.2.1. La función de energía libre homogeneizada es entonces de la forma

$$\mathcal{F}_0[Q] = \int_{\Omega_0} |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}(x,y) + \int_{\Gamma_0} \frac{w_{ef}}{2}|Q - Q_{ef}|^2 \,\mathrm{d}\sigma_0 + \int_{\Gamma_R} \frac{w_0}{2}|Q - Q_R|^2 \,\mathrm{d}\sigma_R,$$

donde ν_0 es la normal exterior de $\Gamma_0 = \{(x,0) \mid x \in [0,2\pi)\}$.

Sea Q_ε el minimizador de $\mathcal{F}_\varepsilon$ y Q_0 el minimizador de $\mathcal{F}_0$. En [12] y [48], los autores son capaces de demostrar que $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$. Según [32], nuestro modelo simplificado está bajo el caso $0 = \beta = \alpha - 1$, en el que demuestran que $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq K_2(\sqrt{\varepsilon} + 1)$. Tanto en [5] como en [6], se demuestra que $(Q_\varepsilon)_{\varepsilon>0}$ converge fuertemente en $L^2(\Omega_\varepsilon)$ a Q_0, bajo varios supuestos para los dominios. Utilizando “boundary layers”, en [4] los autores son capaces de demostrar que $\|Q_\varepsilon - Q_0 - \varepsilon Q_1\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$, donde Q_1 es un término de frontera de primer orden ("first-order boundary term"). En este trabajo, podemos demostrar la siguiente estimación de error:

Teorema. Para cualquier $p \in (2, +\infty)$, existe una constante ε-independiente C tal que:

$$\|Q_0 - Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \cdot \varepsilon^{\frac{p-1}{p}},$$

donde la constante C depende de c, w_0, p, $\|\varphi\|_{L^\infty([0,2\pi))}$, $\|\varphi'\|_{L^\infty([0,2\pi))}$, Ω_0 y $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

Es fácil observar que la fracción $\frac{p-1}{p}$, con $p \in (2, +\infty)$, nos permite obtener cualquier exponente deseado del intervalo $(1/2, 1)$. Para demostrar este teorema, primero mostramos en Section 3.3 que Q_ε y Q_0 existen y admiten la regularidad $W^{2,p}$, para cualquier $p \in (2, +\infty)$. A continuación, adaptamos en Section 3.4 las pruebas de [12] y [48] al caso de las funciones $W^{1,p}$ para obtener Proposition 3.5.1. El resultado que dicta el exponente de ε de nuestra estimación del error es Lemma 3.4.2. Una estimación similar a este lema representa [6, Lemma 5.1, (15)], donde el exponente obtenido es $\frac{d+2}{2d}$ para $L^{\frac{2d}{d-2}}$ estimaciones, para cualquier $d > 2$. La prueba de nuestra estimación del error también se basa en la construcción de un operador de extensión, de $W^{1,p}(\Omega_\varepsilon)$ a $W^{1,p}(\Omega_0)$, que se define en Definition 3.5.3 y tiene límites independientes de ε. Con todos estos ingredientes, podemos entonces demostrar el resultado principal de esta parte, en Section 3.5.

- Capítulo 4 - *Homogenised elastic terms in a case of the Oseen-Frank model*

Consideramos una energía elástica general en un caso 2D para el modelo de Oseen-Frank con perforaciones aisladas. El objetivo de este estudio es analizar cómo se podría obtener un cristal líquido nemático con propiedades elásticas novedosas mediante procedimientos de homogeneización. Bajo condiciones adecuadas podemos analizar el problema de homogeneización de valores en $\mathbb{S}^1$ a través de un problema escalar obtenido mediante el procedimiento de lifting. También demostramos un resultado de convergencia local en L^2.

En este capítulo, consideramos un cristal líquido nemático en un dominio acotado, liso y simplemente conectado $\Omega \subset \mathbb{R}^2$ y consideramos una versión generalizada en $\mathbb{R}^2$ de la energía de Oseen-Frank:

$$E[n] = \int_\Omega K_1(n)\big(\text{div } n\big)^2 + K_2(n)\big(\text{div } n\big)\big(\text{curl } n\big) + K_3(n)\big(\text{curl } n\big)^2 \,\text{d}x + \mu \int_\Omega \big(n \cdot n_0\big)^2 \,\text{d}x,$$

donde los coeficientes elásticos K_1, K_2 y K_3 ya no son necesariamente constantes, sino que ahora dependen de n. La razón de considerar esta generalización es que el tipo de homogeneización que vamos a considerar, mediante coloides, proporciona un funcional de esta forma. Así, en concreto, partiendo de las constantes K_1, K_2 y K_3obtendremos, mediante la homogeneización coloidal, una funcional de este tipo. Además, hemos añadido un nuevo término, en el que μ es una constante positiva y $n_0 \in \mathbb{S}^1$ is also constant. también es constante. Imponemos condiciones para K_1, K_2 y K_3 tal que, para $\mu = 0$, tenemos $E[n] \geq 0$, para cualquier $n \in \mathbb{S}^1$, y $E[n] = 0$, para cualquier n constante. El término que contiene μ también intenta imitar, de forma muy simplificada, un campo magnético constante externo aplicado al cristal líquido nemático, que obliga a competir entre la minimización de la energía elástica del material y el deseo de alinearse perpendicularmente al campo magnético.

Ahora perforamos el dominio de forma periódica, de la siguiente manera. Consideramos una partícula modelo T, formada por N_T componentes mutuamente disjuntos que denotamos T^i, donde $i \in \{1, 2, \ldots, N_T\}$. Suponemos que cada componente T^i es un conjunto compacto, acotado, liso y simplemente conectado de la celda periódica $Y = (0,1)^2$. Consideramos un parámetro pequeño $\varepsilon > 0$ y construimos una látice de puntos X_ε tal que en cada punto $\xi \in X_\varepsilon$, tenemos $\varepsilon\big(\xi + Y\big) \subset \Omega$. Denotamos el número de tales puntos por N_ε y después, en cada punto $x_\varepsilon^j \in X_\varepsilon$, con $j \in \{1, N_\varepsilon\}$, perforamos el dominio con el conjunto $T_\varepsilon^{i,j} = \varepsilon\big(x_\varepsilon^j + T^i\big)$. Denotamos por T_ε la unión de todos los $T_\varepsilon^{i,j}$ y por $\Omega_\varepsilon := \Omega \setminus T_\varepsilon$ the perforated domain. el dominio perforado. Por nuestra construcción, los agujeros están suficientemente alejados de $\partial\Omega$.

Consideramos la siguiente función de energía:

$$\mathbf{F}_\varepsilon(u) = \int_{\Omega_\varepsilon} \kappa_1(u)\big(\text{curl } u\big)^2 + \kappa_2(u)\big(\text{curl } u\big)\big(\text{div } u\big) + \kappa_3(u)\big(\text{div } u\big)^2 + \mu\big(u \cdot \overline{u}\big)^2 \, \mathrm{d}x,$$

donde κ_1, κ_2 y κ_3 se suponen en $C^2(\mathbb{S}^1; \mathbb{R})$, $\mu > 0$ es una constante positiva y $\overline{u} \in \mathbb{S}^1$ también es constante. Despreciamos, por ahora, el espacio al que pertenece u.

Nos interesa estudiar el siguiente problema de homogeneización: dados los coeficientes elásticos iniciales κ_1, κ_2 y κ_3 y las partículas modelo T^i, nos gustaría obtener, a medida que $\varepsilon \to 0$, un nuevo material, que se comporta también como un cristal líquido nemático, pero ahora con nuevos coeficientes elásticos: κ_1^*, κ_2^* y κ_3^*. Como nuestro objetivo es generar nuevos coeficientes elásticos, despreciamos cualquier tipo de energía superficial típica (como la de Rapini-Papoular, por ejemplo) e imponemos, por simplicidad, que $u = (1,0)$ en $\partial\Omega$ y no imponemos condiciones de contorno en las perforaciones. De este modo, consideramos $\mathbf{F}_\varepsilon : \mathbf{V}_\varepsilon \to [0, +\infty)$, donde

$$\mathbf{V}_\varepsilon = \{u \in H^1(\Omega_\varepsilon; \mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}.$$

Nuestra elección del modelo de Oseen-Frank da lugar a algunos retos interesantes, debido a que trabajamos con funciones con valores en $\mathbb{S}^1$, como sigue. En primer lugar, teniendo $u \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$, existe una extensión $E_\varepsilon u \in H^1(\Omega; \mathbb{R})$ siempre que los agujeros sean suficientemente regulares, pero no necesariamente en $H^1(\Omega; \mathbb{S}^1)$. En segundo lugar, dado $u \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$, no podemos esperar a priori tener una función $\varphi \in H^1(\Omega_\varepsilon; \mathbb{R})$ tal que $u = (\cos\varphi, \sin\varphi)$. Para superar los problemas mencionados anteriormente, hacemos uso de varios resultados de [21] que nos dan conexiones entre el grado topológico de una función, la posibilidad de extender una función con valores en $\mathbb{S}^1$ y la existencia de una elevación φ.

La hipótesis principal de nuestro trabajo se basa en que podemos tener estados energéticos del material lo suficientemente bajos como para que exista una secuencia $\big(u_\varepsilon\big)_{\varepsilon > 0} \subset \mathbf{V}_\varepsilon$ de puntos críticos de $\mathbf{F}_\varepsilon$ con la propiedad de que su grado topológico computado en la frontera de los agujeros $T_\varepsilon^{i,j}$ debe ser 0. De este modo, demostramos que existe una función de elevación

$\varphi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$, para cada u_ε dado por el argumento anterior, tal que $u_\varepsilon = (\cos\varphi_\varepsilon, \sin\varphi_\varepsilon)$. Además, dado que $u_\varepsilon = (1,0)$ on $\partial\Omega_\varepsilon$, definimos el espacio

$$V_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon;\mathbb{R}) \ : \ \varphi = 0 \text{ on } \partial\Omega\}$$

y observamos que $u_\varepsilon \in \mathbf{V}_\varepsilon$ implica $\varphi_\varepsilon \in V_\varepsilon$.

Observamos que, en este entorno, el problema de homogeneización escalar representa un caso particular del trabajo realizado en [34] y es de la forma:

$$\begin{cases} -\text{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = \mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon) & \text{in } \Omega_\varepsilon \\ A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu = 0 & \text{on } \partial T_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

donde A es una función de valor matricial que depende de un parámetro que contiene toda la información relacionada con los coeficientes elásticos iniciales y $\mathscr{B}$ tiene crecimiento cuadrático en la segunda variable y depende de la derivada de A, es decir A'.

El resultado principal de [34] afirma que existe $\varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ tal que $E_0\varphi_\varepsilon \rightharpoonup \varphi_0$ débilmente en $L^2(\Omega)$ (donde E_0 es la extensión por 0 en los agujeros) y que resuelve la siguiente EDP:

$$\begin{cases} -\text{div}(A_0(\varphi_0)\nabla\varphi_0) = \mathscr{B}_0(\varphi_0, \nabla\varphi_0) & \text{in } \Omega \\ \varphi_0 = 0 & \text{on } \partial\Omega \end{cases}$$

donde A_0 y $\mathscr{B}_0$ son los componentes homogeneizados obtenidos de A y $\mathscr{B}$.

Entonces, por Proposition 4.3.6, podemos decir que $u_0 = (\cos\varphi_0, \sin\varphi_0)$ es un punto crítico de la siguiente funcional de energía homogeneizada $\mathbf{F}_0 : \mathbf{V}_0 \to [0, +\infty)$:

$$\mathbf{F}_0(u) = \int_\Omega \kappa_1^*(u)\big(\text{curl } u\big)^2 + \kappa_2^*(u)\big(\text{curl } u\big)\big(\text{div } u\big) + \kappa_3^*(u)\big(\text{div } u\big)^2 + \theta_0\mu\big(u\cdot\overline{u}\big)^2 \, d\mathbf{x},$$

donde θ_0 representa la fracción de volumen entre la parte de cristal líquido nemático y la celda periódica y $\mathbf{V}_0 = \{u \in H^1(\Omega;\mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}$.

Las funciones κ_1^*, κ_2^* y κ_3^* representan los nuevos coeficientes elásticos para el material homogeneizado. Su dependencia de los coeficientes elásticos iniciales κ_1, κ_2 y κ_3 se da en Subsection 4.5.5 y se basa en el uso de la misma matriz correctora que en, por ejemplo [14, 15, 34, 39, 40].

Nos gustaría ahora expresar la dependencia entre la secuencia elegida de puntos críticos u_ε y la función construida u_0. En primer lugar, observamos que, en [33], los autores son capaces de demostrar que las soluciones φ_ε están uniformemente acotadas en V_ε. Entonces, utilizando también [5, Lemma 2.3], podemos demostrar el siguiente resultado:

Teorema. A lo largo de una subsecuencia de $(u_\varepsilon)_{\varepsilon>0}$, todavía denotada con el subíndice ε:

para cualquier conjunto abierto ω tal que $\overline{\omega} \subset \Omega$, tenemos $\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon\cap\omega;\mathbb{S}^1)} = 0$.

Como se indica en [34], no hay que esperar una fuerte convergencia de φ_ε a φ_0 en $L^2(\Omega)$, ni tampoco en casi todas partes en Ω. Sin embargo, si consideramos que los coeficientes elásticos iniciales son constantes, entonces tenemos

$$\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega_\varepsilon)} \to 0, \text{ as } \varepsilon \to 0,$$

ya que nuestro problema es un caso particular de [36], en el que consideran agujeros aislados en cada celda, o, en cierta medida, esto puede verse como [5, Theorem A.1], donde consideran la situación más generalizada de agujeros conectados. Además, se podría demostrar de forma muy similar a la del [5, Appendix] que podemos extender el resultado de convergencia local hasta la frontera de Ω, ya que imponemos condiciones homogéneas de contorno de Dirichlet.

ABSTRACT

In this thesis we present various results concerning homogenisation of nematic liquid crystals: two of them are in perforated domains, while the other one concerns rates of convergence for boundary homogenisation. The first work described in this thesis is a Γ-convergence result for the Landau-de Gennes model in $3D$ domains with connected perforations. The goal of the analysis is to find new terms in the energy functional that are independent of the gradient. The second result is an error estimate for a 2D toy model used to describe rugosity effects in nematic liquid crystals via homogenisation problems, using once again the Landau-de Gennes model. The last problem is a local L^2-convergence result for a homogenisation problem in $\mathbb{R}^2$ with isolated perforations. Here we use the Oseen-Frank model, with the goal of finding new gradient-dependent terms in the energy functional.

We start, in Chapter 1, with a brief introduction to nematic liquid crystals. We introduce two major variational models used to describe nematic liquid crystals: Landau-de Gennes (LdG) and Oseen-Frank (OF). For LdG theory, which uses Q-tensors as the order parameter, we present typical choices for each type of energy contribution (bulk, elastic and surface). For OF theory based on the order parameter $\mathbf{n} \in \mathbb{S}^2$ we discuss the elastic energy, that depends on the director and its gradient. We then present a short summary of the main mathematical results obtained for LdG and OF.

In Chapter 2, we analyse a homogenisation problem in $\mathbb{R}^3$ using the Landau-de Gennes model in which the perforations form a cubic microlattice. We assume a dillute regime, that is the volume of the cubic microlattice tends to 0 as its characteristic length scale tends to 0. The goal of this problem is to show that, given this geometrical setting, by choosing various types of surface energies one can obtain a new material in the limit of vanishing characteristic size of the microlattice. This material also behaves like a nematic liquid crystal, but now with different bulk coefficients. At the end of this chapter, we discuss a rate of convergence of the approximating surface energies to a homogenised term.

In Chapter 3, we concentrate on achieving and improving error estimates in homogenisation problems, since they can give us crucial information for manufacturing processes. Here, we consider a simplified 2D model in which we highlight how one could replace a rugose boundary with the imposed homeotropic alignment by a flat boundary with an effective alignment depending on the initial geometry of the rugosity. We are able to improve an L^2 error estimate for a class of linear nonhomogeneous Robin problems.

In Chapter 4, we consider a general elastic energy in a 2D case for the Oseen-Frank model in domains with isolated perforations. The goal of this study is to analyse how one could obtain a nematic liquid crystal with novel elastic properties via homogenisation procedures. Under suitable conditions we can analyse the $\mathbb{S}^1$-valued homogenisation problem via a scalar problem obtained through the lifting procedure. We also prove a local L^2 convergence result.

CONTENTS

1

INTRODUCTION

Liquid crystals are materials which, beside having the possibility of being in the conventional states of matter (solid, liquid and gas), can enter a new, intermediary, state of matter, also called mesophase, between the solid and liquid states of matter. The discovery of liquid crystals is traditionally assigned to the publication of Friedrich R. K. Reinitzer's work in 1888 [66] (later translated in English in [67] and also in [70]), called *Contributions to the knowledge of cholesterol*. Reinitzer was an Austrian botanist who, while analysing the properties of *cholesteryl benzoate*, a material which is solid at room temperature, observed an interesting phenomena. At 145.5°C, the material turns into a cloudy liquid, hence presenting turbidity, but if one were to increase the temperature of the material and exceed the value of 178.5°C, only then it would become a clear liquid. Reinitzer sent his observations to the German physicist Otto Lehmann, who was using the newly invented technique of polarized microscophy. Lehmann became interested in these materials and became a leading figure in the early study of liquid crystals. In 1889, in [57], Lehmann creates the term "flowing crystals" to describe these new materials and, according to [73], by 1900 he started using the terminology "liquid crystals".

One would think that the discovery of such new materials and, especially, of a new state of matter would be welcomed and appreciated by the whole academic world, but that was definitely not the case here. Gustav H. J. A. Tammann was a physical chemist and represented one of the main figures who would challenge this discovery: initially, by simply stating that the turbidity (the cloudiness of a fluid) observed in cholesteryl benzoate happens mainly because of impurities, later by publishing two articles with rather interesting titles (*On the so-called liquid crystal phases* - [74] - in 1901 and *On the so-called liquid crystal phases II* - [75] - in 1902). This culminates in 1905 by publicly challenging Lehmann over the authenticity of liquid crystals. More details related to the rather turbid first years of the development, recognition and acceptance of liquid crystals by the academic community can be found in [42], where the authors present a very detailed line of events from the history of liquid crystals by not only highlighting the events, but also illustrating various factors that help the reader understand better what lead to such events.

Going back to the double melting phenomena observed by Reinitzer for cholesteryl benzoate nowadays, the temperature at which the material enters the LC phase is called *melting point* and the one at which enters the isotropic liquid state is called *clearing point*. In 1907, Daniel Vorländer, published [77] (translated in English in [70]), analysed the importance of molecular shape in liquid crystal materials. By constructing the *ortho*, *meta* and *para* isomers of PAA, he was able to prove that only the *para* isomer is a liquid crystal, the one which has elongated molecules.

There are two important classifications of liquid crystals: the first one is with respect to which property of the material needs to be changed such that it can achieve a mesophase and the other one is described with respect to what mesophases it can achieve. *Thermotropic* liquid crystals are materials which can enter a mesophase due to the change of their temperature, while *lyotropic* liquid crystals are the ones which can enter a mesophase due to the change of their concentration of particles. In 1922, Georges Friedel was able to identify in Lehmann's liquid crystals new mesophases and he introduced, in [47], the following classification of liquid crystals: nematics, smectics and cholesterics. The term nematic comes from the Greek *nema*, which means thread, and in a nematic phase, the particles tend to align locally to a preferred direction. The cholesteric phase is similar with the nematic one, but now the preferred orientational configuration is helical. Friedel used the term cholesteric due to the amount of cholesterol products that presented this property. A smectic liquid crystal has a layered structure, therefore it is more ordered than a nematic phase. The term smectic comes from the Greek *smegma*, which means soap, and Friedel used this terminology due to the amount of soap-like products that presented this layered structure.

The development of new materials always gives rise to the following natural question: where can we use them? According to [42], at the beginning of the 20th century, the overall perspective was that liquid crystals are very interesting new materials, but with no future possible applications (see, for example, Vorländer's quote from [42, page 193]). However, with the launch and development of television and TVs, as time passed, a new possible idea emerged during 1960-1970: a flat-screen display using liquid crystal technology that will be hanged on the wall. Various types of liquid crystal display technologies were developed, most important of which we would like to mention [55] (the dynamic scattering LCD) and the patents [54] and [45] (the twisted nematic displays - which we still use to this date). But most of them had initially the following problem: if one were to use a portable liquid crystal display, then this device should not consume too much power (such that it can last days, not just hours) and, most importantly, the liquid crystal material should be stable in the nematic phase for a wide range of temperatures. One of the initial compounds used in LCDs was MBBA, which was stable at room temperature, but if the temperature was below a value around $20^\circ\ C$, then the LCD would need a heater in order to function properly. This severely affected the portability idea of the device and the necessity of creating new liquid crystalline materials emerged once again. In 1973, George Gray and his team publishes [51], where they present a new class of liquid crystalline materials which are very suited for the use in LCDs. By 1974, the range of temperatures at which the liquid crystal material from an LCD is still in the nematic phase was between $-10^\circ C$ and $60^\circ C$, which allowed LCDs to be broadly used in various applications. Since then, the LCD technology has constantly evolved: the rather old flat screen LCD TVs have now been replaced by curved ones, to offer the viewer a better cinema experience, while their size has grown significantly throughout time. Also, mobile phones with foldable LCDs have been patented and are slowly rising in popularity.

As mentioned in the previous paragraph, one of the contributing factors to creating a global presence and a wide spread of applications of LCDs has been the development of new liquid crystalline materials, by mixing various liquid crystals either among them or with other types of substances. Hence, the study of homogenisation problems for liquid crystals represents an important direction of research in the liquid crystal community, since it might offer an insight on how to create or design liquid crystals with desired properties, such as new elastic properties, new optical properties or thermotropic properties (as it was the case for the early years of development of LCDs). In this work, the focus is on thermotropic nematic liquid crystals and we start, in Section 1.1, to offer a better description of these materials. At the same time, we have consciously neglected in this section the presentation of the mathematical models that can describe the alignment of the liquid crystal particles, but, in Section 1.3 and Section 1.2, we present two of them: the Landau-de Gennes theory and the Oseen-Frank one, which are later going to be deployed in the results present in this work.

1.1 NEMATIC LIQUID CRYSTALS

The liquid crystal state of matter is an intermediary state of matter between the conventional solid and isotropic liquid states of matter. We recall here that the particles of a nematic liquid crystal (NLC) are rod-like structures, meaning that they have an elongated shape and they present a head-to-tail symmetry. While the particles of such of a material can translate freely, meaning that we have no positional ordering - just like in the isotropic liquid phase, however their specific feature is that they tend to align locally to a preferred direction, meaning that there is a local orientational ordering, mimicking the solid state of matter. In the following figure, we offer a schematic representation of the alignment of the particles of a thermotropic nematic liquid crystal with respect to the change of temperature.

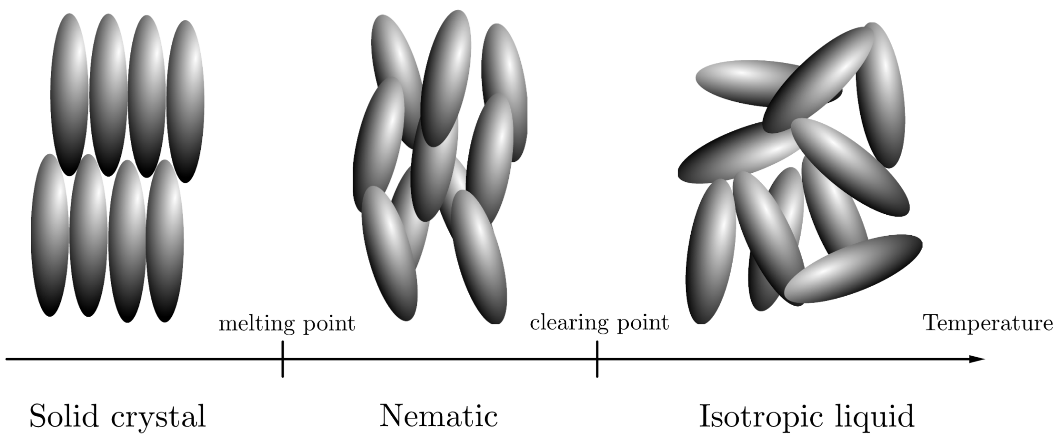

Figure 1: Thermotropic nematic liquid crystals. Image courtesy of J. M. Taylor.

The first discovered nematic liquid crystal is due to the organic chemist Ludwig Gattermann and it is para-azoxyanisole, also known as PAA. The melting point of this material is around 118°C and the clearing point is around 135°C, according to, for example, [79]. The first known nematic liquid crystal that can enter the nematic phase at room temperature (around 21°C) is MBBA, or methoxybenzylidenebutylaniline, and was first prepared by H. Kelker and B. Scheurle in 1969, according to [42]. More details about this material and others related to it (the family of p-alkyloxybenzilidene-p-n-alkylanilines) can be found, for example, in [63].

In order to measure the local orientation of the particles, at a macroscopic/mesoscopic scale, we need first to take into account their special geometry. In classical mechanics, continuous bodies are formed from material points, while here all the NLC particles have a microstructure for which their properties have, on the macroscopic scale, a mechanical significance. We proceed in the following fashion.

Let us consider that $\Omega \subset \mathbb{R}^3$ is a domain that contains NLC and let $x \in \Omega$. We assume that at the point x, we have a preferred direction of the molecules and we can continue in two different ways:

- a more macroscopic model, the Oseen-Frank model, where we assume that the preffered direction of the particles at the point x is described by $n(x) \in \mathbb{S}^2$, with $\mathbb{S}^2$ being the unit sphere from $\mathbb{R}^3$. We use, in this case, functions of the type $n : \Omega \to \mathbb{S}^2$.
- a more mesoscopic model, capable also of describing the phase transition between isotropic and nematic states of matter, the Landau-de Gennes model, in which we use a Q-tensor, a symmetric traceless 3×3 real matrix, that is going to store some information related to the orientation of the particles at the point x. We use, in this case, functions of the type $Q : \Omega \to \mathcal{S}_0$, where $\mathcal{S}_0$ is the set of all Q-tensors.

1.2 OSEEN-FRANK THEORY

Let us assume that the orientation of the particles contained at the point x is described by a single unit vector $n \in \mathbb{S}^2$. Hence, for the entire nematic liquid crystal material, we construct a function $n : \Omega \to \mathbb{S}^2$ for which $n(x)$ represents the preferred direction of the particles contained at the point x.

It is assumed that the local energy of the material is described by a free energy density, also called the free energy integrand, of the following form:

$$w = w(n, \nabla n),$$

where the dependency on ∇n is considered due to the spatial distortions of the material. Moreover, we construct the total elastic free energy, described as:

$$W = \int_\Omega w(n(x), \nabla n(x))\, \mathrm{d}x.$$

It is generally assumed that for a *relaxed* configuration (one without any external influences) we have $w = 0$ and it is supposed that any other configuration would imply a higher energy of the material, hence, we impose the condition:

$$w(n, \nabla n) \geq 0.$$

Recalling now that a nematic liquid crystal has rod-like particles, implying a head-to-tail symmetry, we must also impose that:

$$w(n, \nabla n) = w(-n, -\nabla n).$$

At the same time, the free energy of the material per unit volume has to be the same when computed with respect to two frames of references. Thus, a *frame-indifference* condition is necessary, which takes the form:

$$w(n, \nabla n) = w(Rn, R\nabla n R^T),$$

for any rotation $R \in \mathcal{O}(3)$.

The construction of such free energy integrand w is based on the work of F. C. Frank from 1958 [46], which was build upon the work of H. Zocher from 1925 [81] and later on the work of C. W. Oseen in 1933 [62]. The reader can also consult [73] and [76] for more information.

The representation formula for w is the following:

$$\begin{aligned} 2w(n, \nabla n) :=& K_1 \big(\text{div } n\big)^2 + K_2 \big(n \cdot \text{curl } n\big)^2 + K_3 \big|n \times \text{curl } n\big|^2 + \qquad (1.2.1) \\ &+ (K_2 + K_4)\big(\text{tr}(\nabla n)^2 - (\text{div } n)^2\big), \qquad (1.2.2) \end{aligned}$$

where K_1, K_2, K_3 and K_4 are often referred as *Frank's constants* or *moduli*. Each of the coefficients from the previous equation are related to a specific type of deformation of the material: K_1 is the *splay* constant, K_2 is the *twist* constant, K_3 is the *bend* constant and $(K_2 + K_4)$ is the *saddle-splay* constant. Splay, twist and bend names refer to specific types of deformation and can be visualised, for example, in [73, Figure 2.1] or in [60, Fig. 1.4]. Specific values for *Frank's* constants can be found at [73, (2.60) and (2.61)] for PAA and MBBA, while in [60, Table 1.1] are presented the values for 5CB and 8CB, other two nematic liquid crystals.

The previous constrains that we have imposed for w as in (1.2.1) generate the following inequalities, due to the work of Ericksen [43]:

$$K_1 \geq 0,\ K_2 \geq 0,\ K_3 \geq 0,\ K_2 \geq |K_4| \text{ and } 2K_1 \geq K_2 + K_4,$$

which are known as *Ericksen inequalities*.

Sometimes, another free energy integrand is considered, often referred as the *one-constant approximation*, which assumes

$$K := K_1 = K_2 = K_3 \text{ and } K_4 = 0.$$

In this case, one could prove that the energy integrand (1.2.1) becomes:

$$w(n, \nabla n) = \frac{1}{2} K |\nabla n|^2,$$

(see, for example, [73, p. 23]).

1.3 LANDAU-DE GENNES THEORY

The following introduction follows the same lines as in [38], [59] or [76].

Let us assume that, for the molecules that we imagine being at the point $x \in \Omega$, their orientation is described by a probability density function $f_x : \mathbb{S}^2 \to \mathbb{R}_+$ with

$$\int_{\mathbb{S}^2} f_x(p) \, \mathrm{d}A = 1, \tag{1.3.1}$$

where $\mathrm{d}A$ represents the area measure on $\mathbb{S}^2$. The last equality describes that the probability of finding a molecule at the point x however oriented in $\mathbb{S}^2$ is 1. For $M \subset \mathbb{S}^2$, the probability $p[M]$ of finding a molecule at the point x oriented in M is defined as:

$$p[M] := \int_M f_x(p) \, \mathrm{d}A.$$

Since a nematic liquid crystal has particles for which the probability of finding the head or the tail in the direction of p are the same, we have:

$$f_x(p) = f_x(-p), \ \forall p \in \mathbb{S}^2. \tag{1.3.2}$$

Let m be the first order moment of f_x at x, defined as:

$$m(x) := \int_{\mathbb{S}^2} p \ f_x(p) \, \mathrm{d}A.$$

Using (1.3.2), we obtain:

$$m(x) = \int_{\mathbb{S}^2} p f_x(p) \, \mathrm{d}A = \int_{\mathbb{S}^2} (-p) f_x(-p) \, \mathrm{d}A = -\int_{\mathbb{S}^2} -p f_x(p) \, \mathrm{d}A = -m(x),$$

hence $m(x) = 0$.

Therefore, the information about the orientation of the particles at the point x is contained in the higher order moments of f_x. Let M be the following second order tensor:

$$M(x) := \int_{\mathbb{S}^2} (p \otimes p) \ f_x(p) \, \mathrm{d}A,$$

where for $p = (p_1, p_2, p_3) \in \mathbb{S}^2$ we denote by $p \otimes p$ the 3×3 matrix with components $(p_{ij})_{i,j\in\{1,2,3\}}$ defined as $p_{i,j} = p_i p_j$, for any $i, j \in \{1, 2, 3\}$. From its definition, it is easy to see that M satisfies $\mathrm{tr}(M) = 1$ and $M^T = M$, meaning that M is a symmetric tensor with trace equal to 1.

Let now e be a unit vector from $\mathbb{R}^3$. Since M can be seen as a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^3$, one can easily deduce the following equality:

$$e \cdot Me = \int_{\mathbb{S}^2} (p \cdot e)^2 f_x(p) \, \mathrm{d}A.$$

Moreover, we have the following inequalities:

$$0 \le e \cdot Me \le \int_{\mathbb{S}^2} f_x(p) \, \mathrm{d}A = 1,$$

due to (1.3.1) and since, for any $p, e \in \mathbb{S}^2$, we have $(p \cdot e)^2 \le 1$. The lower bound is achieved whenever nearly all molecules from x are perpendicular to e and the upper bound when they are parralel to e.

Let us consider now the case in which the probability density function f is constant. This and (1.3.1) implies that

$$f_0 := f_x \equiv \frac{1}{4\pi}.$$

Such a setting actually implies that the particles can orient themselves, with equal probability, in any possible direction from $\mathbb{S}^2$, which represents the case in which the nematic liquid crystal is in the isotropic liquid state of matter. We denote by M_0 the second order tensor associated to f_0:

$$M_0 = \frac{1}{4\pi} \int_{\mathbb{S}^2} p \otimes p \, \mathrm{d}A.$$

It can be proved that M_0 commutes with any rotation R and this, together with the constraint that M_0 does not vanish, implies that M_0 must be a multiple of the identity (the proof can be found, for example, in [76]). Since $\mathrm{tr}(M_0) = 1$, we get that:

$$M_0 = \frac{1}{3} \mathbb{I}_3,$$

where $\mathbb{I}_3$ is the 3×3 identity tensor.

We now introduce the *order tensor* defined as

$$Q := M - M_0,$$

which measures the deviation of second order moments M associated to a probability density function f_x from the second order tensor M_0 which characterises the isotropic liquid state of matter. This can also be written as:

$$Q(x) = \int_{\mathbb{S}^2} f_x(p)\left(p \otimes p - \frac{1}{3}\mathbb{I}_3\right) \mathrm{d}A. \tag{1.3.3}$$

Due to the properties of M and M_0, one obtains that Q is a symmetric traceless 3×3 real matrix. We call such a matrix a *Q-tensor* and we denote the set of all Q-tensors as:

$$\mathcal{S}_0 := \{Q \in \mathcal{M}_3(\mathbb{R}) \mid Q = Q^T,\ \mathrm{tr}(Q) = 0\},$$

where $\mathcal{M}_3(\mathbb{R})$ denotes the set of all 3×3 real matrices.

We classify now the Q-tensors based on their eigenvalues. Let Q be a Q-tensor, λ_1, λ_2 and λ_3 its eigenvalues and e_1, e_2 and e_3 the corresponding eigenvectors. Since $\mathrm{tr}(Q) = 0$, we have $\lambda_3 = -\lambda_1 - \lambda_2$.

For $\lambda_1 = \lambda_2 = \lambda_3$, we obtain that they are all equal to 0, which implies that $Q = 0$. This corresponds to the isotropic state.

For $\lambda_1 = \lambda_2$, using the Spectral Theorem as in [76], it can be proved that:

$$Q = s\left(n \otimes n - \frac{1}{3}\mathbb{I}_3\right),$$

where $s = -3\lambda_1$ and $n = e_3$. We say, in this case, that Q is *uniaxial*. Moreover, s is called the *scalar order parameter* and n is called the *director*.

For $\lambda_1 \neq \lambda_2$, one obtains, in a similar fashion as in the previous case, that:

$$Q = -\left(s_1 n_1 \otimes n_1 + s_2 n_2 \otimes n_2\right) + \frac{1}{3}(s_1 + s_2)\mathbb{I}_3,$$

where $n_1 = e_1$, $n_2 = e_2$, $s_1 = -2\lambda_1 - \lambda_2$ and $s_2 = -\lambda_1 - 2\lambda_2$. We say, in this case, that Q is *biaxial*.

We also make here the observation that Q is computed at a point x, hence the previous classification holds for $Q(x)$, so a nematic liquid crystal can be uniaxial at some points and biaxial at others.

Moreover, any Q-tensor defined via the "microscopic" definition (1.3.3) must satisfy the eigenvalues constraint

$$-\frac{1}{3} < \lambda_i < \frac{2}{3},\ \forall i \in \{1, 2, 3\},$$

which is often called the *physicality constraint* and the interval $\left(-\frac{1}{3}, \frac{2}{3}\right)$ as the *physical regime* (such as in, for example, [10]).

In this work, we consider $Q : \Omega \to \mathcal{S}_0$ as modelling NLC configurations and focus on studying critical points of the free energy

$$\int_{\Omega} \mathcal{F}_{el}(Q(x), \nabla Q(x)) + \mathcal{F}_b(Q(x))\, \mathrm{d}x + \int_{\partial\Omega} \mathcal{F}_s(Q(x), \nu(x))\, \mathrm{d}S(x),$$

where we have taken into consideration an elastic energy density (given by $\mathcal{F}_{el}$), a bulk energy density ($\mathcal{F}_b$) and a surface energy density ($\mathcal{F}_s$). We continue our work with the presentation of each of these energy densities.

1.3.1 ELASTIC ENERGY

The elastic energy density $\mathcal{F}_{el}$ measures the spatial distortions of a nematic liquid crystal inside of the domain Ω, hence it is a function of Q and ∇Q. The typical choice for the elastic energy density is given by

$$\mathcal{F}_{el}(Q, \nabla Q) := L_1\, \partial_k Q_{ij} \partial_k Q_{ij} + L_2\, \partial_j Q_{ij} \partial_k Q_{ik} + L_3\, \partial_j Q_{ik} \partial_k Q_{ij},$$

where L_1, L_2 and L_3 are material constants and the Einstein's summation convention was used. In a similar fashion as in the case of the Oseen-Frank elastic energy, this choice of $\mathcal{F}_{el}$ agrees with the physical invariances of the material and, moreover, we need to impose the following conditions:

$$L_1 > 0,\ -L_1 < L_3 < 2L_1 \text{ and } -\frac{3}{5}L_1 - \frac{1}{10}L_3 < L_2,$$

similar to Ericksen inequalities, in order for the elastic energy to be coercive and bounded from below.

The one constant approximation of the Landau-de-Gennes elastic energy is frequently used to simplify the analysis. This approximation has the form:

$$\mathcal{F}_{el}(Q, \nabla Q) := L|\nabla Q|^2,$$

where L is a positive constant.

1.3.2 BULK ENERGY

The bulk energy density $\mathcal{F}_b$ models the phase transition from the isotropic liquid state of matter to the nematic state. Due to the geometry of the particles, a frame-indifferent condition is usually imposed on $\mathcal{F}_b$, in the following mathematical sense: $\mathcal{F}_b(Q) = \mathcal{F}_b(RQR^T)$, for any $Q \in \mathcal{S}_0$ and any rotation $R \in \mathcal{O}(3)$.

The typical choice for $\mathcal{F}_b$ is a quartic polynomial in Q, as in, for example, [38] or [76], and it is of the following form:

$$\mathcal{F}_b(Q) = \frac{1}{2}A \cdot Q_{ij}Q_{ji} + \frac{1}{3}B \cdot Q_{ij}Q_{jk}Q_{ki} + \frac{1}{4}C \cdot \left(Q_{ij}Q_{ji}\right)^2,$$

where Einstein's summation convention was used. This can also be written as:

$$\mathcal{F}_b(Q) = \frac{1}{2}A \cdot \mathrm{tr}(Q^2) + \frac{1}{3}B \cdot \mathrm{tr}(Q^3) + \frac{1}{4}C \cdot \left(\mathrm{tr}(Q^2)\right)^2.$$

The constant A is temperature dependent and is of the form $A = \alpha(T - T^*)$, where α is a material constant, T is the absolute temperature and T^* is a characteristic liquid crystal temperature (it is the temperature at which the isotropic phase loses stability), while B and C are also material constants. Moreover, the coefficient B is negative by the frame-indifferent condition and C is positive, otherwise the energy functional will have no lower bounds.

This choice of a quartic polynomial in Q for the bulk energy is used because it is the lowest order term in a Taylor expansion of the bulk energy that in suitable regimes predicts a uniaxial phase as a global minimiser. For large enough values of A, the bulk energy is globally minimised at $Q = 0$, which corresponds to the isotropic phase, and for small enough values of A, the global minimisers are uniaxial Q-tensors for which the corresponding scalar order parameters are explicitly computable, as in, for example, [59].

Note that higher order polynomials can be used, such as the sextic Landau-de Gennes potential, which can be relevant for obtaining biaxial Q-tensors as global minimisers (see [38, Sect. 2.3.3]) and is of the following form:

$$\mathcal{F}_b(Q) = a_2\,\mathrm{tr}(Q^2) - a_3\,\mathrm{tr}(Q^3) + a_4\left(\mathrm{tr}(Q^2)\right)^2 + a_5\,\mathrm{tr}(Q^2)\,\mathrm{tr}(Q^3) + a_6\left(\mathrm{tr}(Q^2)\right)^3 + a_6'\left(\mathrm{tr}(Q^3)\right)^2,$$

with $a_6 > 0$ and $6a_6 + a_6' > 0$.

1.3.3 SURFACE ENERGY

The interaction at an interface between a nematic liquid crystal and another material, which can be either a solid, a liquid or a vapour, is a crucial component in the development of liquid crystal-display technologies. One typically distinguishes between two situations: *strong anchoring* and *weak anchoring*.

In the *strong anchoring* case, $\mathcal{F}_s$ is neglected and only Dirichlet boundary conditions are imposed, such as $Q = Q_0$, where $Q_0 \in \mathcal{S}_0$ is the prescribed desired alignment on the interface. By chemically treating the surface of the material, one can achieve a *homeotropic* alignment of the NLC particles, meaning that they are perpendicular to the interface. At the same time, by rubbing techniques, one can achieve *homogeneous* alignment of the NLC particles, meaning that they lie parallel to the surface.

In the weak anchoring case, we consider a surface energy described by a surface energy density $\mathcal{F}_s$, which is generally assumed that it depends on Q and ν, where ν is the exterior normal to the interface $\partial\Omega$. As in the previous sections, due to the physical invariances of the system, we need to impose the condition:

$$\mathcal{F}_s(Q,\nu) = \mathcal{F}_s(RQR^T, R\nu),$$

for any rotation $R \in \mathcal{O}_3$.

In [28, Prop. 2.6], the authors are able to prove that such $\mathcal{F}_s$ can be described by a function $\widetilde{\mathcal{F}_s} : \mathbb{R}^4 \to \mathbb{R}$ in the following way:

$$\mathcal{F}_s(Q,\nu) = \widetilde{\mathcal{F}_s}(\mathrm{tr}(Q^2),\ \mathrm{tr}(Q^3),\ \nu \cdot Q\nu,\ \nu \cdot Q^2\nu).$$

The typical choice for such $\mathcal{F}_s$ is the Rapini-Papoular surface energy density [64]:

$$\mathcal{F}_s(Q,\nu) = \frac{W_{\text{surface}}}{2}\big(Q - Q_\nu\big)^2,$$

where $W_{\text{surface}} > 0$ is constant and it is called *anchoring strength* and $Q_\nu = s_0\big(\nu \otimes \nu - \mathbb{I}_3\big)$ is a uniaxial Q-tensor derived from ν with constant scalar order parameter s_0. Higher values of W_{surface} correspond to a higher penalisation of deviations from the preferred state and the sign of s_0 describes the preferred type of alignment of NLC molecules on $\partial\Omega$: for $s_0 > 0$, the alignment is parallel to $\partial\Omega$, while for $s_0 < 0$ the alignment will be perpendicular on $\partial\Omega$. Typical values for W_{surface} can be found in [60, Table 1.2].

Another choice for $\mathcal{F}_s$ that satisfies the physical invariances is represented by:

$$\mathcal{F}_s(Q,\nu) = a(\nu \cdot Q^2\nu) + b(\nu \cdot Q\nu)(\nu \cdot Q^2\nu) + c(\nu Q^2\nu)^2,$$

where a, b and c are positive constants, which is similar to an expression proposed by T. J. Sluckin & A. Poniewierski in [71], based on an idea of W. J. A. Goossens [50].

1.4 CONTENTS OF THE THESIS

In this thesis we present various results concerning homogenisation of nematic liquid crystals: two projects are posed in perforated domains, while the third project concerns rates of convergence for boundary homogenisation. First we present a Γ-convergence result in $\mathbb{R}^3$ for domains with connected perforations. We consider the Landau-de Gennes model with the goal of finding new terms without gradients in the energy functional. The second result is an error estimate for a 2D toy model used to describe rugosity effects in nematic liquid crystals via homogenisation problems. Here we once again use the Landau-de Gennes model. The last result establishes local L^2-convergence for a homogenisation problem in $\mathbb{R}^2$ in domains with isolated perforations. This result for the Oseen-Frank model leads us to novel effective elastic terms.

In Chapter 2, we analyse a homogenisation problem in $\mathbb{R}^3$ using the Landau-de Gennes model in which the perforations form a cubic microlattice and we assume to work in a dilute regime, that is the volume of the cubic microlattice tends to 0 as its characteristic length scale tends to 0. The goal of this problem is to show that, given this geometrical setting, by choosing various types of surface energies one can obtain a new material in the limit, which also behaves like a nematic liquid crystal, but now with different bulk coefficients. At the end of this chapter, we also establish a rate of convergence for how fast the approximating surface energies converge to a homogenised term.

In Chapter 3, the emphasis is on achieving and improving error estimates in homogenisation problems. These estimates can give us crucial information for manufacturing processes. We consider a simplified 2D model in which we highlight how one could replace a rugose boundary with an imposed homeotropic alignment by a flat boundary with an effective alignment depending on the initial geometry of the rugosity. We are able here to improve an L^2 error estimate for a class of linear nonhomogeneous Robin problems.

In Chapter 4, we consider a general elastic energy in a 2D case for the Oseen-Frank model in domains with isolated perforations. The goal of this study is to analyse how one could produce a new nematic liquid crystal with novel elastic properties via homogenisation procedures. Under suitable conditions we can analyse the $\mathbb{S}^1$-valued homogenisation problem via a scalar problem obtained through the lifting procedure and we prove a local L^2 convergence result.

1.4.1 HOMOGENISED BULK TERMS IN A CASE OF THE LANDAU-DE GENNES MODEL

In this chapter, we consider a cubic microlattice scaffold within a nematic liquid crystal described by a Landau-de Gennes model. By cubic microlattice scaffold we understand a family of inter-connected parallelepipeds of very small scale, as in Figure 2, and we sometimes refer to it simply as scaffold. This type of geometry for the inclusions is mainly used in industry, where such scaffolds are called *bicontinuous porous solid matrix* or *BPSM* (such as in [22], [69] or [68]) and such objects can be constructed via *two-photon polymerisation* technique, also called (2PP or TPP). A general overview of this 3D printing technique can be found in [9].

This work continues in the direction of studying the homogenised material and it is built on the work from [28] and [29], which was also based on [13, 16, 23]. The general thrust of these papers is to prove that the homogenisation limit of a nematic liquid crystal with colloidal inclusions of a specific geometry can generate a new material, which behaves like a new nematic liquid crystal, but now with different material parameters. In [28] and [29], the set of inclusion particles is disconnected, obtained from different or identical model particles, in such a way that the distance between the particles is considerable larger than the size of them, which is called the dilute regime. Also, in this regime, the volume fraction of colloids tends to zero. However, the geometric configuration of a cubic microlattice is more relevant from the physical point of view, since in [28] and in [29] one cannot position *a priori* the colloidal particles in a periodic fashion. Here the periodicity is automatically generated by the structure of the cubic microlattice.

The mathematical construction of a cubic microlattice can be seen in the following way: we first choose a small parameter $\varepsilon > 0$, then we construct a family of disjoint identical parallelepipeds placed in a periodic fashion (their centers are at a distance of ε between each other), of the following form:

$$\mathcal{C}^\alpha = \left[-\frac{\varepsilon^\alpha}{2p}, +\frac{\varepsilon^\alpha}{2p}\right] \times \left[-\frac{\varepsilon^\alpha}{2q}, +\frac{\varepsilon^\alpha}{2q}\right] \times \left[-\frac{\varepsilon^\alpha}{2r}, +\frac{\varepsilon^\alpha}{2r}\right],$$

with $p,\ q,\ r \in [1, +\infty)$ and $\alpha \in (1, 2)$. Then we *inter-connect* them with other 3 families of identical parallelipipeds, such that we achieve a connected scaffold, but, at the same time, we do not connect them with $\partial\Omega$ (the scaffold is included in Ω and does not touch $\partial\Omega$). We explain why we choose α between 1 and 2 in Remark 2.2.6 and we emphasize here that, by our construction of the scaffold, its volume tends to 0 as $\varepsilon \to 0$. At the same time, if the initial family of disjoint identical parallelepipeds are actually cubes, that is $p = q = r$, we say that the scaffold presents *cubic symmetry*.

In this setting, we are able to prove that, in the limit as $\varepsilon \to 0$, the surface interaction between the nematic liquid crystal and the scaffold transforms into a bulk-type of energy and, by tuning the lengths of the scaffold and by choosing specific surface energy densities, we can achieve desired bulk coefficients. Hence, starting from a nematic liquid crystal with bulk coefficients a, b and c (as presented in Subsection 1.3.2) confined in a domain perforated by a cubic microlattice, then, given some a', b', c' and suitably choosing parameters of the scaffolding and the surface energy, in the limit, we can achieve a new liquid crystal-type of material, with new bulk coefficients a', b' and c', with the focus on achieving a', since this is temperature-dependent (it depends on the temperature at which the isotropic state loses stability - see Subsection 1.3.2 for more details).

To be more precise, let Ω be a bounded and smooth domain and let $\mathcal{N}_\varepsilon$ be a cubic microlattice scaffold, for $\varepsilon > 0$. We consider the following Landau-de Gennes free energy functional:

$$\mathcal{F}_\varepsilon[Q] := \int_{\Omega_\varepsilon} \big(f_e(\nabla Q) + f_b(Q)\big)\mathrm{d}x + \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)} \int_{\partial\mathcal{N}_\varepsilon} f_s(Q, \nu)\mathrm{d}\sigma,$$

where $\Omega_\varepsilon = \Omega \setminus \mathcal{N}_\varepsilon$ and we assume that:

- $f_e : \mathcal{S}_0 \otimes \mathbb{R}^3 \to [0, +\infty)$ *is differentiable, strongly convex*[1] *and there exists a constant* $\lambda_e > 0$ *such that*

  $$\lambda_e^{-1}|D|^2 \le f_e(D) \le \lambda_e|D|^2, \quad |(\nabla f_e)(D)| \le \lambda_e(|D| + 1),$$

  for any $D \in \mathcal{S}_0 \times \mathbb{R}^3$.

- $f_b : \mathcal{S}_0 \to \mathbb{R}$ *is continuous, bounded from below and there exists a constant* $\lambda_b > 0$ *such that* $|f_b(Q)| \le \lambda_b(|Q|^6 + 1)$ *for any* $Q \in \mathcal{S}_0$.

- $f_s : \mathcal{S}_0 \times \mathbb{S}^2 \to \mathbb{R}$ *is continuous and there exists a strictly positive constant* λ_s *such that, for any* $Q_1, Q_2 \in \mathcal{S}_0$ *and any* $\nu \in \mathbb{S}^2$, *we have*

  $$|f_s(Q_1, \nu) - f_s(Q_2, \nu)| \le \lambda_s|Q_1 - Q_2|\big(|Q_1|^3 + |Q_2|^3 + 1\big).$$

[1] We say that a function $f : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ is strongly convex if there exists $\theta > 0$ such that the function $\tilde{f} : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ defined by $\tilde{f}(D) = f(D) - \theta|D|^2$ is convex.

We also impose strong anchoring on the boundary of Ω: let $g \in H^{1/2}(\partial\Omega, \mathcal{S}_0)$ be a boundary datum and we denote by $H^1_g(\Omega, \mathcal{S}_0)$ the set of maps Q from $H^1(\Omega, \mathcal{S}_0)$ such that $Q = g$ on $\partial\Omega$ in the trace sense. Similarly, we define $H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$ to be $H^1(\Omega_\varepsilon)$ with $Q = g$ on $\partial\Omega$ in the trace sense.

In order to present the main result, we need to introduce the function $f_{hom} : \mathcal{S}_0 \to \mathbb{R}$ as:

$$f_{hom}(Q) := \frac{q+r}{qr} \int_{\mathcal{C}^x} f_s(Q,\nu)\mathrm{d}\sigma + \frac{p+r}{pr} \int_{\mathcal{C}^y} f_s(Q,\nu)\mathrm{d}\sigma + \frac{p+q}{pq} \int_{\mathcal{C}^z} f_s(Q,\nu)\mathrm{d}\sigma,$$

for any $Q \in \mathcal{S}_0$, where $\mathcal{C} = \left[-1/2,\ 1/2\right]^3$ and $\mathcal{C}^x$, $\mathcal{C}^y$ and $\mathcal{C}^z$ are the unions of the faces normal to Ox, Oy and Oz.

Moreover, inside the scaffold, we use the harmonic extension operator, $E_\varepsilon : H^1_g(\Omega_\varepsilon, \mathcal{S}_0) \to H^1_g(\Omega, \mathcal{S}_0)$, defined in the following way: in Ω_ε we have $E_\varepsilon Q := Q$ and inside the scaffold, $E_\varepsilon Q$ is the unique solution of the following problem:

$$\begin{cases} \Delta E_\varepsilon Q = 0 & \text{in } \mathcal{N}_\varepsilon \\ E_\varepsilon Q \equiv Q & \text{on } \partial\mathcal{N}_\varepsilon. \end{cases}$$

Using Γ-convergence tools, we are able to prove the main result for this general framework:

Theorem 1.4.1. Let $\mathcal{F}_0 : \mathcal{S}_0 \to [0, +\infty)$ be defined as

$$\mathcal{F}_0[Q] := \int_\Omega \big(f_e(\nabla Q) + f_b(Q) + f_{hom}(Q)\big)\mathrm{d}x$$

and let $Q_0 \in H^1_g(\Omega, \mathcal{S}_0)$ be an isolated H^1-local minimiser for $\mathcal{F}_0$, that is, there exists $\delta_0 > 0$ such that $\mathcal{F}_0[Q_0] < \mathcal{F}_0[Q]$ for any $Q \in H^1_g(\Omega, \mathcal{S}_0)$ such that $\|Q - Q_0\|_{H^1_g(\Omega,\mathcal{S}_0)} \leq \delta_0$ and $Q \neq Q_0$. Then for any ε small enough, there exists a sequence of H^1-local minimisers Q_ε of $\mathcal{F}_\varepsilon$ such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1_g(\Omega, \mathcal{S}_0)$.

At the same time, we are able to compute a rate of convergence for how fast the surface energies converge to their homogenised functional. In order to achieve this, we first need to take into account the geometry of the scaffold: from the initial lattice of parallelepipeds constructed, only few of them are in contact with the nematic liquid crystal - only those which are close to $\partial\Omega$, while the others touch the NLC only by their edges. In Subsection 2.4.2, we prove that these interactions have no contribution in the limit as $\varepsilon \to 0$, hence, the homogenised functional is given by the limit of the surface energies computed on the "walls of the connecting parallelepipeds". By assuming further that f_s is locally Lipschitz continuous and that g is bounded and Lipschitz, we are able to prove in Proposition 2.6.1 from Section 2.6 that the previously described rate of convergence is of order ε^{m_0}, with

$$m_0 = \min\left\{\frac{\alpha - 1}{3}, 2 - \alpha\right\},$$

where $\alpha \in (1, 2)$ is the parameter used for the construction of the cubic microlattice.

We now present some applications of Theorem 1.4.1 for some particular cases of interest for the Landau-de Gennes model. We distinguish first two cases: the scaffold chosen presents or not cubic symmetry (that is, $p = q = r$ or not). Then we split the discussion depending on the bulk energy density chosen and, correspondingly, the surface energy density chosen.

1) Let us assume first that $p = q = r$. In this case, f_{hom} becomes:

$$f_{hom}(Q) := \frac{2}{p} \int_{\partial C} f_s(Q, \nu) \mathrm{d}\sigma.$$

a) We consider first the typical choice for the Landau-de Gennes bulk energy density:

$$f_b(Q) = a \operatorname{tr}(Q^2) - b \operatorname{tr}(Q^3) + c \operatorname{tr}(Q^2)^2,$$

where $a, b, c \in \mathbb{R}$, with $b, c > 0$.

In this case, in order to achieve a new material with the parameters a', b' and c', that we simply write $(a, b, c) \rightsquigarrow (a', b', c')$, we choose

$$f_s^{LDG}(Q, \nu) = \frac{p}{4}\Big((a' - a)(\nu \cdot Q^2\nu) - (b' - b)(\nu \cdot Q^3\nu) + 2(c' - c)(\nu \cdot Q^4\nu)\Big)$$

and we obtain

$$f_{hom}^{LDG}(Q) = (a' - a) \operatorname{tr}(Q^2) - (b' - b) \operatorname{tr}(Q^3) + (c' - c) \left(\operatorname{tr}(Q^2)\right)^2.$$

In this way, the functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ become:

$$\mathcal{F}_\varepsilon^{LDG}[Q_\varepsilon] := \int_{\Omega_\varepsilon} \left(f_e(\nabla Q_\varepsilon) + a \operatorname{tr}(Q_\varepsilon^2) - b \operatorname{tr}(Q_\varepsilon^3) + c \left(\operatorname{tr}(Q_\varepsilon^2)\right)^2\right) \mathrm{d}x + \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial \mathcal{N}_\varepsilon} f_s^{LDG}(Q_\varepsilon, \nu) \mathrm{d}\sigma$$

and

$$\mathcal{F}_0^{LDG}[Q] := \int_\Omega \left(f_e(\nabla Q) + a' \operatorname{tr}(Q^2) - b' \operatorname{tr}(Q^3) + c' \left(\operatorname{tr}(Q^2)\right)^2\right) \mathrm{d}x,$$

and we are able to present the main result for this subcase.

Theorem 1.4.2. Let (a, b, c) and (a', b', c') be two set of parameters with $c > 0$ and $c' > 0$. Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{LDG}$, and for $\varepsilon > 0$ small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{LDG}$, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega, \mathcal{S}_0)$.

b) We also consider the following bulk energy density

$$f_b^{RP}(Q) = a \operatorname{tr}(Q^2)$$

and we choose f_s to be given by the Rapini-Papoular form (2.3.6):

$$f_s^{RP}(Q, \nu) = \frac{p}{12}(a' - a) \operatorname{tr}(Q - Q_\nu)^2,$$

where $Q_\nu = \nu \otimes \nu - \mathbb{I}_3/3$ and $\mathbb{I}_3$ is the 3×3 identity matrix. In this way, we have $(a, 0, 0) \rightsquigarrow (a', 0, 0)$,

$$f_{hom}^{RP}(Q) = (a' - a)\ \mathrm{tr}(Q^2)$$

and the free energy functionals become

$$\mathcal{F}_\varepsilon^{RP}[Q_\varepsilon] := \int_{\Omega_\varepsilon} \left(f_e(\nabla Q_\varepsilon) + a\,\mathrm{tr}(Q_\varepsilon^2)\right)\mathrm{d}x + \frac{p}{2} \cdot (a' - a) \cdot \left(\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon} \mathrm{tr}(Q_\varepsilon - Q_\nu)^2 \mathrm{d}\sigma\right)$$

and

$$\mathcal{F}_0^{RP}[Q] := \int_\Omega \left(f_e(\nabla Q) + a'\,\mathrm{tr}(Q^2)\right)\mathrm{d}x.$$

Theorem 1.4.3. Let a and a' be two parameters. Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{RP}$, and for $\varepsilon > 0$ small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{RP}$, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega, \mathcal{S}_0)$.

c) Since the typical choice for the Landau-de Gennes bulk energy density represents the lowest order polynomial from a Taylor series expansion in Q that admits a uniaxial state as a global minimiser, we highlight also the case in which

$$f_b^{gen}(Q) = \sum_{k=2}^{N} a_k\,\mathrm{tr}(Q^k),$$

where $N \in \mathbb{N}$, $N \geq 4$ is fixed, with the coefficients $a_k \in \mathbb{R}$ chosen such that the polynomial $h : \mathbb{R} \to \mathbb{R}$, defined by $h(x) = \sum_{k=2}^{N} a_k x^k$, for any $x \in \mathbb{R}$, admits at least one local minimum over $\mathbb{R}$.

Here, we choose

$$f_s^{gen}(Q, \nu) = \frac{p}{4} \sum_{k=2}^{M} b_k(\nu \cdot Q^k \nu),$$

where $(b_k)_{k \in \overline{2,M}}$ are the coefficients of the polynomial $i : \mathbb{R} \to \mathbb{R}$ of degree $M \in \mathbb{N}$, $M \geq 4$, defined by $i(x) = \sum_{k=2}^{M} b_k x^k$, for any $x \in \mathbb{R}$, with the property that i admits at least one local minimum over $\mathbb{R}$.

In this way,

$$f_{hom}^{gen}(Q) = \sum_{k=2}^{\max\{M,N\}} c_k\ \mathrm{tr}(Q^k),$$

where, for any $k \in \overline{2, \max\{M,N\}}$, we have

$$c_k = \begin{cases} a_k + b_k, \text{ if } 2 \leq k \leq \min\{M,N\} \\ a_k, \text{ if } \min\{M,N\} < k \leq \max\{M,N\} \text{ and } M \leq N \\ b_k, \text{ if } \min\{M,N\} < k \leq \max\{M,N\} \text{ and } M \geq N. \end{cases}$$

The free energy functionals become

$$\mathcal{F}_\varepsilon^{gen}[Q_\varepsilon] := \int_{\Omega_\varepsilon} \left(f_e(\nabla Q_\varepsilon) + \sum_{k=2}^{N} a_k \operatorname{tr}(Q_\varepsilon^k) \right) \mathrm{d}x + \frac{p}{4} \cdot \sum_{k=2}^{M} b_k \cdot \left(\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial \mathcal{N}_\varepsilon} (\nu \cdot Q_\varepsilon^k \nu) \mathrm{d}\sigma \right)$$

and

$$\mathcal{F}_0^{gen}[Q] = \int_\Omega \left(f_e(\nabla Q) + \sum_{k=2}^{\max\{M,N\}} c_k \operatorname{tr}(Q^k) \right) \mathrm{d}x.$$

Theorem 1.4.4. Let $(a_k)_{k \in \overline{2,N}}$ and $(b_k)_{k \in \overline{2,M}}$ be such that the polynomials h and i defined earlier admit at least one local minimum over $\mathbb{R}$. Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{gen}$, and for $\varepsilon > 0$ small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{gen}$, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega, \mathcal{S}_0)$.

2) For the loss of cubic symmetry case, we only highlight the case in which f_b is the typical choice of Landau-de Gennes bulk energy density:

$$f_b(Q) = a \operatorname{tr}(Q^2) - b \operatorname{tr}(Q^3) + c \operatorname{tr}(Q^4) = a \operatorname{tr}(Q^2) - b \operatorname{tr}(Q^3) + \frac{c}{2} \left(\operatorname{tr}(Q^2)\right)^2,$$

with $c > 0$. Similar results can be obtained for the other cases in which we modify the form of f_b.

In order to describe f_{hom}, we introduce

$$A = \frac{1}{3} \begin{pmatrix} -\frac{2}{p} + \frac{1}{q} + \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{p} - \frac{2}{q} + \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{p} + \frac{1}{q} - \frac{2}{r} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{q} + \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{p} + \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{p} + \frac{1}{q} \end{pmatrix}.$$

and $\omega = \frac{2}{3}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)$. Note that A, B and ω are constants depending only on the choice of p, q and r. Moreover, we have $\operatorname{tr}(A) = 0$ and $B = \omega \mathbb{I}_3 + A$, where $\mathbb{I}_3$ is the 3×3 identity matrix.

We consider the following surface energy density

$$f_s^{asym}(Q, \nu) = \frac{1}{2\omega} \left((a' - a)(\nu \cdot Q^2 \nu) - (b' - b)(\nu \cdot Q^3 \nu) + (c' - c)(\nu \cdot Q^4 \nu) \right),$$

with a', b' and c' real parameters such that $c' > 0$ and the associated free energy functional:

$$\mathcal{F}_\varepsilon^{asym}[Q_\varepsilon] := \int_\Omega \big(f_e(\nabla Q_\varepsilon) + a\ \mathrm{tr}(Q_\varepsilon^2) - b\ \mathrm{tr}(Q_\varepsilon^3) + c\ \mathrm{tr}(Q_\varepsilon^4)\big)\mathrm{d}x + \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon} f_s^{asym}(Q_\varepsilon, \nu)\mathrm{d}\sigma.$$

In this case, the function f_{hom} becomes

$$\begin{aligned} f_{hom}^{asym}(Q) = \big((a'-a)\mathrm{tr}(Q^2) - (b'-b)\mathrm{tr}(Q^3) + (c'-c)\mathrm{tr}(Q^4)\big) + \\ + \frac{1}{\omega}\big((a'-a)\mathrm{tr}(A\cdot Q^2) - (b'-b)\mathrm{tr}(A\cdot Q^3) + (c'-c)\mathrm{tr}(A\cdot Q^4)\big). \end{aligned}$$

Theorem 1.4.5. Let (a,b,c) and (a',b',c') be two set of parameters with $c>0$ and $c'>0$. Then, for $\varepsilon > 0$ small enough and for any isolated H^1-local minimiser Q_0 of the functional:

$$\begin{aligned} \mathcal{F}_0^{asym}[Q] := \int_\Omega \big(f_e(\nabla Q) + a'\mathrm{tr}(Q^2) - b'\mathrm{tr}(Q^3) + c'\big(\mathrm{tr}(Q^2)\big)^2\big)\mathrm{d}x + \\ + \frac{1}{\omega}\int_\Omega \big((a'-a)\mathrm{tr}(A\cdot Q^2(x)) - (b'-b)\mathrm{tr}(A\cdot Q^3(x)) + (c'-c)\mathrm{tr}(A\cdot Q^4(x))\big)\mathrm{d}x \end{aligned}$$

there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{asym}$, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega, \mathcal{S}_0)$.

The terms of the form $\mathrm{tr}(A\cdot Q^k)$ describe a new preferred alignment of the liquid crystal particles inside of the domain, given by the loss of the cubic symmetry of the scaffold.

1.4.2 ERROR ESTIMATES FOR RUGOSITY EFFECTS

In this chapter, we consider the case of a nematic liquid crystal in a domain with an oscillating boundary. We are interested to study the case in which undulated surfaces, for which the wavelength is of comparable size to the amplitude, can lead to effective surface energies in the limit as the amplitude converges to zero. Problems of this flavour have been considered in the language of homogenisation of PDEs in a domain with an oscillating boundary, where certain scalar, linear, *rugose* systems may be rigorously proven to have certain effective behaviours in the limit. These have been considered, for example, in the context of [4, 6, 7, 12, 32] or [48], but the list is not by any means exhaustive. A contemporary overview of the literature from this direction can be found, for example, in the introduction of [6]. The nature of physically meaningful surface energies in the context nematic liquid crystals however provides models that have yet to be considered in the literature within this homogenisation framework.

We consider a simplified setting of a two-dimensional slab with periodic rugosity and a quadratic free energy, which provides a toy model of a *paranematic*. That is, a high-temperature system of mesogenic molecules which has melted into an isotropic state, but still admits some local nematic ordering induced by the surface. In this case, by the simplicity of the system, we are able to provide quantitative estimates on how ground states behave in the homogenised limit. We consider a rugosity parameter, ε, arbitrarily small, that is used to describe the oscillating boundary and then the limit problem describes the behaviour as this parameter tends to zero. In a physical situation the parameter ε is small, but finite. If one then attempts to understand to what extent the limit problem is a good description of the problem with ε small, one needs to obtain *convergence rates*. While a Γ-convergence result gives us a description of the system in a potentially unphysical limit, obtaining a convergence rate allows quantitative understanding of the approach to the theoretical limit in physically reasonable parameter regimes.

It is known from the general theory of homogenisation that convergence rates can be improved by calculating correctors, a manifestation of the fact that the boundary layer phenomena generate localized differences between the two problems (see for instance Lemma 5.1 versus Theorem 5.2 in [4]). An alternative approach, in order to obtain improved convergence rates, and without using correctors, is to use weaker norms that do not put too much weight on what happens at the boundary. This approach seems not to be studied in the standard homogenisation literature and is our main contribution here. We use a duality argument in an L^p setting that however does not include the endpoint $p = +\infty$, which we expect to be the optimal one. Also the use of duality argument builds on the linear structure and an extension to the the nonlinear case is not immediate. In order to understand these issues we analyse the most simplified setting possible that still has a certain physical relevance.

We consider the situation of a two-dimensional slab with periodic rugosity and the Landau-de Gennes model for the description of the nematic liquid crystal used. More specifically, the limiting domain is of the form $\Omega_0 = \{(x,y) \mid x \in [0,2\pi),\ y \in (0,R)\}$, where $R > 0$ is a constant, and the rugose domain is of the form $\Omega_\varepsilon = \{(x,y) \mid x \in [0,2\pi),\ y \in (\varphi_\varepsilon(x),R)\}$, where $\varphi_\varepsilon(x) = \varepsilon\varphi(x/\varepsilon)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2 2π-periodic function with $\varphi \geq 0$. We denote with $\Gamma_\varepsilon = \{(x, \varepsilon\varphi(x/\varepsilon)) \mid x \in [0,2\pi)\}$ the rugose boundary and with $\Gamma_R = \{(x,R) \mid x \in [0,2\pi)\}$ the fixed upper boundary of the domains. We consider a quadratic free energy of the following form:

$$\mathcal{F}_\varepsilon[Q] = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}(x,y) + \int_{\Gamma_\varepsilon} \frac{w_0}{2}|Q - Q_\varepsilon^0|^2 \,\mathrm{d}\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2}|Q - Q_R|^2 \,\mathrm{d}\sigma_R,$$

where $c > 0$ is constant, $w_0 > 0$ is the anchoring strength, $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$ and $Q_R = \nu_R \otimes \nu_R - \frac{1}{2}I$ $\big(\nu_\varepsilon$ and ν_R are the outward normals to Γ_ε and $\Gamma_R\big)$.

In this simplified model, using Proposition 3.2.1, we are able to identify the homogenised surface energy as $\frac{w_{ef}}{2}|Q - Q_{ef}|^2$, with $w_{ef} = w_0\gamma$ and $Q_{ef} = \frac{1}{\gamma}\begin{pmatrix} G_1 & G_2 \\ G_2 & -G_1 \end{pmatrix}$, where γ, G_1 and G_2 are defined in Definition 3.2.1. The homogenised free energy functional is then of the form

$$\mathcal{F}_0[Q] = \int_{\Omega_0} |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}(x,y) + \int_{\Gamma_0} \frac{w_{ef}}{2}|Q - Q_{ef}|^2 \,\mathrm{d}\sigma_0 + \int_{\Gamma_R} \frac{w_0}{2}|Q - Q_R|^2 \,\mathrm{d}\sigma_R,$$

where ν_0 is the outward normal to $\Gamma_0 = \{(x,0) \mid x \in [0,2\pi)\}$.

Let Q_ε be the minimiser of $\mathcal{F}_\varepsilon$ and Q_0 the minimiser of $\mathcal{F}_0$. In [12] and [48], the authors are able to prove that $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$. According to [32], our simplified model is under the case $0 = \beta = \alpha - 1$, in which they prove that $\|Q_\varepsilon - Q_0\|_{H^1(\Omega_\varepsilon)} \leq K_2(\sqrt{\varepsilon} + 1)$. Both in [5] and [6], it is proved that $(Q_\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2(\Omega_\varepsilon)$ to Q_0, under various assumptions for the domains. Using boundary layers, in [4] the authors are able to prove that $\|Q_\varepsilon - Q_0 - \varepsilon Q_1\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}$, where Q_1 is a first-order boundary term. In this work, we are able to prove the following L^2 error estimate:

Theorem 1.4.6. For any $p \in (2, +\infty)$, there exists an ε-independent constant C such that:

$$\|Q_0 - Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \cdot \varepsilon^{\frac{p-1}{p}},$$

where the constant C depends on c, w_0, p, $\|\varphi\|_{L^\infty([0,2\pi))}$, $\|\varphi'\|_{L^\infty([0,2\pi))}$, Ω_0 and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

It is easy to observe that the fraction $\frac{p-1}{p}$, with $p \in (2, +\infty)$, allows us to obtain any desired exponent from the interval $(1/2, 1)$. In order to prove this theorem, we first show in Section 3.3 that Q_ε and Q_0 exist and admit $W^{2,p}$ regularity, for any $p \in (2, +\infty)$. Then, we adapt in Section 3.4 the proofs from [12] and [48] to the case of $W^{1,p}$ functions in order to obtain Proposition 3.5.1. The result that dictates the exponent of ε from our error estimate is Lemma 3.4.2. A similar estimate to this lemma represents [6, Lemma 5.1, (15)], where the exponent obtained is $\frac{d+2}{2d}$ for $L^{\frac{2d}{d-2}}$ estimates, for any $d > 2$. The proof of our error estimate is also based on the construction of an extension operator, from $W^{1,p}(\Omega_\varepsilon)$ to $W^{1,p}(\Omega_0)$, which is defined in Definition 3.5.3 and has ε-independent bounds. With all of these ingredients, we are able then to prove the main result of this part, in Section 3.5.

At the end of this subsection, we would also like to mention that one could achieve more general results for rugosity effects. In [31], a Γ-convergence result in $\mathbb{R}^n$ is achieved in a sufficiently general setting such that one could consider either the Landau-de Gennes setting or the Oseen-Frank one. For more details, the reader can consult [31].

1.4.3 HOMOGENISED ELASTIC TERMS IN A CASE OF THE OSEEN-FRANK MODEL

We consider a nematic liquid crystal in a bounded, smooth and simply connected domain $\Omega \subset \mathbb{R}^2$ and we consider a generalised version in $\mathbb{R}^2$ of the Oseen-Frank energy introduced in (1.2.1):

$$E[n] = \int_\Omega K_1(n)\big(\text{div } n\big)^2 + K_2(n)\big(\text{div } n\big)\big(\text{curl } n\big) + K_3(n)\big(\text{curl } n\big)^2 \,\mathrm{d}x + \mu \int_\Omega \big(n \cdot n_0\big)^2 \,\mathrm{d}x,$$

where the elastic coefficients K_1, K_2 and K_3 are not necessarily constants any more, but they now depend on n. The reason for considering this generalisation is that the type of homogenisation we will consider, using colloids, provides a functional of this form. So, in particular, starting from K_1, K_2 and K_3 constants, we will get, through colloidal homogenisation, a functional of this type. Moreover, we have added a new term, in which μ is a positive constant and $n_0 \in \mathbb{S}^1$ is also constant. We impose conditions on K_1, K_2 and K_3 such that, for $\mu = 0$, we have $E[n] \geq 0$, for any $n \in \mathbb{S}^1$, and $E[n] = 0$, for any n constant. The term containing μ also tries to mimic, in a very simplified fashion, an external constant magnetic field applied to the nematic liquid crystal, which forces a competition between minimising the elastic energy of the material and the desire to align perpendicular to the magnetic field.

We now perforate the domain in a periodic fashion, in the following way. We consider a model particle T, made up from N_T mutually disjoint components which we denote T^i, where $i \in \{1, 2, \ldots, N_T\}$. We assume that each component T^i is a bounded, smooth and simply connected compact set from the periodic cell $Y = (0,1)^2$. We consider a small parameter $\varepsilon > 0$ and we construct a lattice of points X_ε such that in each point $\xi \in X_\varepsilon$, we have $\varepsilon\big(\xi + Y\big) \subset \Omega$. We denote the number of such points by N_ε and then, in each point $x_\varepsilon^j \in X_\varepsilon$, with $j \in \{1, N_\varepsilon\}$, we perforate the domain with the set $T_\varepsilon^{i,j} = \varepsilon\big(x_\varepsilon^j + T^i\big)$. We denote by T_ε the union of all $T_\varepsilon^{i,j}$s and by $\Omega_\varepsilon := \Omega \setminus T_\varepsilon$ the perforated domain. By our construction, the holes are sufficiently far away from $\partial\Omega$.

We consider the following energy functional:

$$\mathbf{F}_\varepsilon(u) = \int_{\Omega_\varepsilon} \kappa_1(u)\big(\text{curl } u\big)^2 + \kappa_2(u)\big(\text{curl } u\big)\big(\text{div } u\big) + \kappa_3(u)\big(\text{div } u\big)^2 + \mu\big(u \cdot \overline{u}\big)^2 \, \mathrm{d}x,$$

where κ_1, κ_2 and κ_3 are assumed to be in $C^2(\mathbb{S}^1; \mathbb{R})$, $\mu > 0$ is a positive constant and $\overline{u} \in \mathbb{S}^1$ is also constant. We neglect, for now, the space from which u belongs.

We are interested to study the following homogenisation problem: given initial elastic coefficients κ_1, κ_2 and κ_3 and the model particles T^i, we would like to obtain, as $\varepsilon \to 0$, a new material, which behaves also like a nematic liquid crystal, but now with new elastic coefficients: κ_1^*, κ_2^* and κ_3^*. Since our goal is to generate new elastic coefficients, we neglect any sort of typical surface energy (such as Rapini-Papoular, for example) and we impose, for simplicity, that $u = (1,0)$ on $\partial\Omega$ and we impose no boundary conditions on the perforations. In this way, we consider $\mathbf{F}_\varepsilon : \mathbf{V}_\varepsilon \to [0, +\infty)$, where

$$\mathbf{V}_\varepsilon = \{u \in H^1(\Omega_\varepsilon; \mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}.$$

Our choice of the Oseen-Frank model gives rise to some interesting challenges, due to the fact that we work with $\mathbb{S}^1$-valued functions, as follows. First, having $u \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$, there exists an extension $E_\varepsilon u \in H^1(\Omega; \mathbb{R})$ as long as the holes are sufficiently regular, but not necessarily in $H^1(\Omega; \mathbb{S}^1)$. Secondly, given $u \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$, we can not a priori expect to have a function $\varphi \in H^1(\Omega_\varepsilon; \mathbb{R})$ such that $u = (\cos\varphi, \sin\varphi)$. In order to overcome the previously mentioned issues, we make use of various results from [21] that give us connections between the topological degree of a function, the possibility of extending an $\mathbb{S}^1$-valued function and the existence of a lifting φ.

The main assumption of our work is based on the fact that we can have low enough energy states of the material such that there exists a sequence $\big(u_\varepsilon\big)_{\varepsilon > 0} \subset \mathbf{V}_\varepsilon$ of critical points of $\mathbf{F}_\varepsilon$ with the property that their topological degree computed on the boundary of the holes $T_\varepsilon^{i,j}$ must be 0. In this way, we prove that there exists a lifting function $\varphi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$, for each u_ε given by the previous argument, such that $u_\varepsilon = (\cos\varphi_\varepsilon, \sin\varphi_\varepsilon)$. Moreover, since $u_\varepsilon = (1,0)$ on $\partial\Omega_\varepsilon$, we define the space

$$V_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon; \mathbb{R}) \ : \ \varphi = 0 \text{ on } \partial\Omega\}$$

and we note that $u_\varepsilon \in \mathbf{V}_\varepsilon$ implies $\varphi_\varepsilon \in V_\varepsilon$.

We observe that, in this setting, the scalar homogenisation problem represents a particular case of the work done in [34] and is of the form:

$$\begin{cases} -\text{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = \mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon) & \text{in } \Omega_\varepsilon \\ A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu = 0 & \text{on } \partial T_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4.1}$$

where A is a matrix-valued function depending on one parameter which contains all the information related to the initial elastic coefficients and $\mathscr{B}$ has quadratic growth in the second variable and it depends on the derivative of A, namely A'.

The main result from [34] states that there exists $\varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $E_0\varphi_\varepsilon \rightharpoonup \varphi_0$ weakly in $L^2(\Omega)$ (where E_0 is the extension by 0 in the holes) and that it solves the following PDE:

$$\begin{cases} -\text{div}(A_0(\varphi_0)\nabla\varphi_0) = \mathscr{B}_0(\varphi_0, \nabla\varphi_0) & \text{in } \Omega \\ \varphi_0 = 0 & \text{on } \partial\Omega \end{cases}$$

where A_0 and $\mathscr{B}_0$ are the homogenised components obtained from A and $\mathscr{B}$.

Then, by Proposition 4.3.6, we are able to say that $u_0 = (\cos\varphi_0, \sin\varphi_0)$ is a critical point of the following homogenised energy functional $\mathbf{F}_0 : \mathbf{V}_0 \to [0, +\infty)$:

$$\mathbf{F}_0(u) = \int_\Omega \kappa_1^*(u)(\text{curl } u)^2 + \kappa_2^*(u)(\text{curl } u)(\text{div } u) + \kappa_3^*(u)(\text{div } u)^2 + \theta_0\mu(u \cdot \overline{u})^2 \, \mathrm{d}\mathbf{x}, \tag{1.4.2}$$

where θ_0 represents the volume fraction between the nematic liquid crystal part and the periodic cell and $\mathbf{V}_0 = \{u \in H^1(\Omega; \mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}$.

The functions κ_1^*, κ_2^* and κ_3^* from (1.4.2) represent the new elastic coefficients for the homogenised material. Their dependency on the initial elastic coefficients κ_1, κ_2 and κ_3 is given in Subsection 4.5.5 and it is based on using the same corrector matrix as in, for example [14, 15, 34, 39, 40].

We would like now to express the dependency between the chosen sequence of critical points u_ε and the constructed function u_0. We first note that, in [33], the authors are able to prove that the solutions φ_ε of (1.4.1) are uniformly bounded in V_ε. Then, by also using [5, Lemma 2.3], we are able to prove the following result:

Theorem 1.4.7. Along a subsequence of $(u_\varepsilon)_{\varepsilon>0}$, still denoted with subscript ε:

$$\text{for any open set } \omega \text{ such that } \overline{\omega} \subset \Omega, \text{ we have } \lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon\cap\omega;\mathbb{S}^1)} = 0.$$

As stated in [34], one should not expect strong convergence of φ_ε to φ_0 in $L^2(\Omega)$, nor almost everywhere in Ω. However, if we were to consider the initial elastic coefficients as being constants, then we have

$$\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega_\varepsilon)} \to 0, \text{ as } \varepsilon \to 0,$$

since our problem is a particular case of [36], in which they consider holes that are isolated in each cell, or, by some extent, this can be seen as [5, Theorem A.1], where they consider the more generalised situation of connected holes. Moreover, one could prove in a very similar fashion as in [5, Appendix] that we can extend the local convergence result up to the boundary of Ω, since we impose homogeneous Dirichlet boundary conditions.

2

HOMOGENISED BULK TERMS IN A CASE OF THE LANDAU-DE GENNES MODEL

Abstract

We consider a Landau-de Gennes model for a connected cubic lattice scaffold in a nematic host, in a dilute regime. We analyse the homogenised limit for both cases in which the lattice of embedded particles presents or not cubic symmetry and then we compute the free effective energy of the composite material.

In the cubic symmetry case, we impose different types of surface anchoring energy densities, such as quartic, Rapini-Papoular or more general versions, and, in this case, we show that we can tune any coefficient from the corresponding bulk potential, especially the phase transition temperature.

In the case with loss of cubic symmetry, we prove similar results in which the effective free energy functional has now an additional term, which describes a change in the preferred alignment of the liquid crystal particles inside the domain.

Moreover, we compute the rate of convergence for how fast the surface energies converge to the homogenised one in terms of the H^1 norm of the difference between a minimiser of the homogenised free energy and a correspoding strongly converging sequence of minimisers of the approximating free energies.

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2.1 INTRODUCTION

We consider a cubic microlattice scaffold constructed of connected particles of micrometer scale, within a nematic liquid crystal. In this article, we treat the particles of the cubic microlattice as being inclusions from the mathematical point of view, while they might be interpreted as colloids from the physical point of view, even though they do not possess all of their properties. The cubic microlattice scaffold is also called a bicontinuous porous solid matrix (BPSM) in the physics literature (for example, see [22], [68] or [69]). By cubic microlattice scaffold we understand a connected family of parallelepipeds or cubes of different sizes, placed in a periodic fashion, as in Figure 2, where only the embedded particles have been shown. For simplicity, we might refer to this object as being a scaffold or a cubic microlattice. This type of scaffold is usually obtained using the *two-photon polymerization* (TPP or 2PP) process, which represents a technique of 3D-manufacturing structures and which can generate stand-alone objects. An overview of the field of TPP processes can be found in [9]. There are numerous experiments, theory and computer simulations regarding embedding microparticles into nematic liquid crystals (for example, see [41], [61] and [65]).

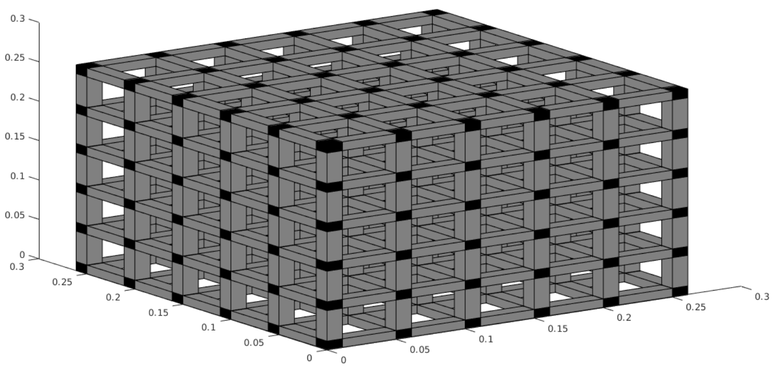


Figure 2: Example of a cubic microlattice.

The system bears mathematical similarities to that of colloids embedded into nematic liquid crystals. The mathematical studies of nematic colloids (the mixture of colloidal particles embedded into nematic liquid crystals) are split into two broad categories:

- one is dealing with the effect produced by a small number of particles in this mixture, with a focus on the defect patterns that arise in the alignment of the nematic particles induced by the interaction at the boundary of the colloid between the two combined materials (see, for example, [2, 3, 24, 27, 25, 26, 78]);
- the other one treats the study of the collective effects, that is the homogenisation process (see, for example, [13, 16, 23, 28] and [29]).

This work continues within the second direction, that is studying the homogenised material, and it is built on the work from [28] and [29], which was also based on [13, 16, 23]. The general thrust of these papers is to prove that the homogenisation limit of a nematic liquid crystal with colloidal inclusions of a specific geometry can generate a new material, which behaves like a new nematic liquid crystal, but now with different material parameters.

In [28] and [29], the set of inclusion particles is disconnected, obtained from different or identical model particles, in such a way that the distance between the particles is considerable larger than the size of them, which is called the dilute regime. Also, in this regime, the volume fraction of colloids tends to zero.

In this article, we are going to consider the case of a cubic microlattice scaffold, as shown in Figure 2. The idea of using such a particular geometry for the scaffold comes from the work done in [68]. At the same time, this geometric configuration is more relevant from the physical point of view, since in [28] and in [29] one cannot position *a priori* the colloidal particles in a periodic fashion. Here the periodicity is automatically generated by the structure of the cubic microlattice. We construct two types of scaffolds: one with identical cubes centered in a periodic 3D lattice of points, cubes which are inter-connected by parallelepipeds, and one where we replace the cube with a parallelepiped with three different length sides. If by *cubic symmetry* we understand the family of rotations that leave a cube invariant, then the first case is when the scaffold particles have cubic symmetry and the second one is with the loss of this type of symmetry.

The main new aspects of this work are:

- the set of all the inclusion particles is now replaced with an individual inclusion particle, which can be seen as a connected union of smaller particles
- the model particle that we use (that is, a parallelepiped or a cube) grants us the possibility to compute the surface contribution for arbitrarily high order terms in the surface energy density - hence, a generalisation has been done for higher order polynomials in the bulk energy potential that admit at least one local minimiser (see Theorem 2.3.4);
- in the case where the cubic symmetry is lost, we obtain a new term into the homogenised limit that can be seen as a change in the preferred alignment of the liquid crystal particles inside the domain (see Theorem 2.3.5);
- we obtain a rate of convergence for how fast the surface energies converge to the homogenised one (more details in Proposition 2.6.1); in remark 2.6.1, we also obtain a rate of convergence for how fast the sequence of minimisers of the free energies tend to a minimiser of the homogenised free energy;

Liquid crystal materials, which typically consist of either rod-like or disc-like molecules, can achieve a state of matter which has properties between those of conventional liquids and those of solid crystals. The liquid crystal state of matter is one where there exists a long range orientational order for the molecules. In order to quantify the local preferred alignment of the rod-like molecules, we use the theory of Q-tensors (for more details, see [59]). A background of the field of liquid crystal materials can be found in [38].

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain from $\mathbb{R}^3$. For every $\varepsilon > 0$, we construct a cubic microlattice $\mathcal{N}_\varepsilon$ inside of Ω, such that, as $\varepsilon \to 0$, the volume of the scaffold tends to 0.

More details regarding the construction of the cubic microlattice can be found in Section 2.2 and in Subsection 2.7.1.

Let $\Omega_\varepsilon = \Omega \setminus \mathcal{N}_\varepsilon$, which represents the space where only liquid crystal particles can be found. We use functions $Q : \Omega_\varepsilon \to \mathcal{S}_0$ to describe the orientation of the liquid crystal particles, where:

$$\mathcal{S}_0 = \{Q \in \mathbb{R}^{3\times 3} : \; Q = Q^T, \; \mathrm{tr}(Q) = 0\},$$

is denoted as the set of Q-tensors. In the space $\mathcal{S}_0$, if we define $|Q| = \left(\mathrm{tr}(Q^2)\right)^{1/2}$, for any $Q \in \mathcal{S}_0$, we can see that $\mathcal{S}_0$ is a normed linear space and the so-called Frobenius norm is induced by the scalar product $Q \cdot P = \mathrm{tr}(Q \cdot P)$.

We consider the following Landau-de Gennes free energy functional:

$$\mathcal{F}_\varepsilon[Q] := \int_{\Omega_\varepsilon} \big(f_e(\nabla Q) + f_b(Q)\big)\mathrm{d}x + \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)} \int_{\partial\mathcal{N}_\varepsilon} f_s(Q, \nu)\mathrm{d}\sigma, \tag{2.1.1}$$

where f_e represents the *elastic energy*, f_b the *bulk energy*, f_s the *surface density energy*, α is a real parameter and $\partial\mathcal{N}_\varepsilon$ the *surface of the scaffold*. The coefficient in front of the surface energy term is chosen such that the denominator $\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)$ balances the effect given by the surface terms from $\partial\mathcal{N}_\varepsilon$, in the limit $\varepsilon \to 0$.

The *elastic energy*, also called the distortion energy, penalises the distortion of Q in the space and, in the Landau-de Gennes theory, it is usually considered to be a positive definite quadratic form in ∇Q. More details regarding the elastic energy used can be found in Subsection 2.3.2.

The *bulk energy* in our case consists only of the thermotropic energy, which is a potential function that describes the preferred state of the liquid crystal, that is either uniaxial, biaxial or isotropic[1]. For large values of the temperature, the minimum of this energy is obtained in the isotropic case, that is $Q = 0$, and for small values, the minimum set is a connected set of the form $s(\nu \otimes \nu - \mathbb{I}_3/3)$, with $\nu \in \mathbb{S}^2$ and $\mathbb{I}_3$ the identity 3×3 matrix, and this is a connected set diffeomorphic with the real projective plane. The simplest form that we can take for the *bulk energy* in our case is the quartic expansion:

$$f_b(Q) = a\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\,\mathrm{tr}(Q^2)^2, \tag{2.1.2}$$

where the coefficient a depends on the temperature of the liquid crystal and b and c depend on the properties of the liquid crystal material, with $b, c > 0$. The coefficient of $\mathrm{tr}(Q^2)$ depends on the temperature at which the phase transition occurs. More specifically, a from (2.1.2) is of the form $a := a_*(T - T_*)$, in which a_* is a material parameter and T_* is the characteristic temperature of the nematic liquid crystal material (the temperature where the isotropic state starts losing *local* stability). More details regarding the bulk energy used can be found in Subsection 2.3.2.

The *surface energy* describes the interaction between the liquid crystal material and the boundary of the scaffold. We assume, for simplicity, that it depends only on Q and on ν, where

1 The isotropic case corresponds to the case in which $Q = 0$. The uniaxial case corresponds to the one in which two of the eigenvalues of Q are equal and the third one has a different value. The biaxial case corresponds to the case in which all the eigenvalues have different values.

ν is the outward normal at the boundary of the cubic microlattice. Throughout this work, we choose several versions for the surface energy, depending on the bulk energy used and on whether the scaffold presents cubic symmetry or not. More details regarding the surface energies used can be found in Subsection 2.3.2.

We are interested in studying the behaviour of the whole material when $\varepsilon \to 0$. We will show that in our dilute regime we obtain for the homogenised material an energy functional of the following form

$$\mathcal{F}_0[Q] := \int_\Omega \big(f_e(\nabla Q) + f_b(Q) + f_{hom}(Q)\big)\mathrm{d}x, \tag{2.1.3}$$

where f_{hom} is defined in (2.3.1) and in (2.3.2), depending on the choice of f_b.

Our focus will be on *a priori* designing the f_{hom}, in terms of the available parameters of the system. More specifically, if (a, b, c) are the parameters from (2.1.2) of the nematic liquid crystal used in the homogenisation process and (a', b', c') are the desired parameters for the homogenised material, our goal is to choose the lenghts of the model particle used for constructing the scaffold and a surface energy density f_s such that if, for example, the bulk energy chosen is the one from (2.1.2), then, in the limit $\varepsilon \to 0$, we want to obtain a f_{hom} with the following property:

$$f_b(Q) + f_{hom}(Q) = a' \operatorname{tr}(Q^2) - b' \operatorname{tr}(Q^3) + c' \operatorname{tr}(Q^2)^2.$$

The article is organised in the following manner:

- in Section 2.2 we present the technical assumptions of the problem;
- in Section 2.3 we present the main results of this work: a general result together with its applications to the Landau-de Gennes model;
- in Section 2.4 we present the study of the properties of the functional $\mathcal{F}_\varepsilon$ for a fixed value of $\varepsilon > 0$;
- in Section 2.5 we glue together the properties studied in the previous section and analyse the Γ-limit of $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$ and we prove the main theorems stated in Section 2.3;
- in Section 2.6 we analyse the rate of convergence of the sequence of surface energies to the homogenised surface functional, where the main result is Proposition 2.6.1, but we also analyse the rate of convergence of the sequence of minimisers of the free energies to a minimiser of the homogenised free energy (see remark 2.6.1)

and

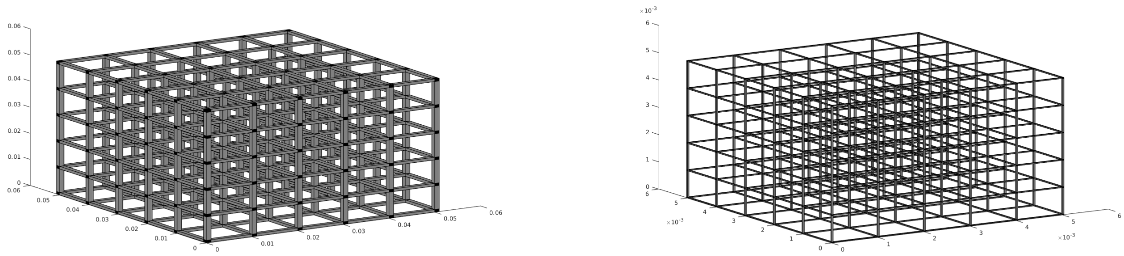
- in Section 2.7 we prove various results, the most important of which is the proposition regarding the explicit extension function that we use in Subsection 2.4.1.

2.2 NOTATIONS AND TECHNICAL ASSUMPTIONS

Let $\Omega \subset \mathbb{R}^3$ be a bounded, Lipschitz domain, that models the ambient liquid crystal, and let $\mathcal{C} \subset \mathbb{R}^3$ be the model particle for the cubic microlattice. Since Ω is bounded in $\mathbb{R}^3$, then:

$$\exists\ L_0,\ l_0,\ h_0 \in [0, +\infty) \text{ such that } \overline{\Omega} \subseteq [-L_0, L_0] \times [-l_0, l_0] \times [-h_0, h_0]. \tag{2.2.1}$$

In Figure 3, we illustrate some examples of cubic microlattices, where the "connecting" boxes (which can be seen better in Figure 2 as being the black cubes) are cubes of size ε^α, with $\alpha = 1.4999$ [2] and ε has a positive value close to 0, since we desire to work in the dilute regime. The distance between two closest black cubes is equal to ε, therefore the length of the black cubes is significantly smaller than the distance between them, by using the exponent α. [3] For Figure 2, we used $\varepsilon = 0.05$, $\alpha = 1.4999$ and $l = 0.25$, so we keep the same ratio between ε and l as in Figure 3.



(a) $\varepsilon = 0.01$ and $l = 0.05$; (b) $\varepsilon = 0.001$ and $l = 0.005$.

Figure 3: Cubic microlattices constructed in the box $[0, l]^3$ with $\alpha = 1.4999$.

In order to construct such a scaffold, we use as a model particle the cube:

$$\mathcal{C} = \left[-\frac{1}{2}, \frac{1}{2} \right]^3. \tag{2.2.2}$$

We denote by $\partial\mathcal{C}$ the surface of the cube $\mathcal{C}$, which we also write it as:

$$\partial\mathcal{C} = \mathcal{C}^x \ \cup\ \mathcal{C}^y \ \cup\ \mathcal{C}^z, \tag{2.2.3}$$

where $\mathcal{C}^x$ is the union of the two faces of the cube that are perpendicular to the x direction and in the same way are defined $\mathcal{C}^y$ and $\mathcal{C}^z$.

Then, for a fixed value of $\varepsilon > 0$ and an ε-independent positive constant α, we define

$$\mathcal{C}^\alpha = \left[-\frac{\varepsilon^\alpha}{2p}, +\frac{\varepsilon^\alpha}{2p} \right] \times \left[-\frac{\varepsilon^\alpha}{2q}, +\frac{\varepsilon^\alpha}{2q} \right] \times \left[-\frac{\varepsilon^\alpha}{2r}, +\frac{\varepsilon^\alpha}{2r} \right], \tag{2.2.4}$$

[2] We choose α close to the value 3/2 in order to make the difference between the lengths of the sides of the black cubes and the gray parallelepipeds from Figure 2 more visible, for relatively "large" values of ε (0.01, 0.05 or 0.001).

[3] The reason why we represent the lattice only in the box $[0, l]^3$, with $l = 5\varepsilon$, is that if we keep the same l and shrink ε, then the number of boxes appearing in the image would be significantly larger, hence, as we make ε smaller, we also zoom in to have a better picture of what is happening for small values of ε.

with p, q, $r \in [1, +\infty)$.

Remark 2.2.1. Using the notion of cubic symmetry described in introduction, we call the scaffold *symmetric* whenever $p = q = r$. In Figures 2 and 3, we illustrate only the symmetric case $p = q = r = 1$.

We construct now the lattice

$$\mathcal{X}_\varepsilon = \{x \in \Omega \ : \ x = (x_1, x_2, x_3),\ \text{dist}(x, \partial\Omega) \geq \varepsilon \text{ and } x_k/\varepsilon \in \mathbb{Z} \text{ for } k \in \overline{1,3}\}, \tag{2.2.5}$$

which we rewrite it as:

$$\mathcal{X}_\varepsilon = \{x_\varepsilon^i \ : \ i \in \overline{1, N_\varepsilon}\},\ \text{where } N_\varepsilon = \text{card}\big(\mathcal{X}_\varepsilon\big). \tag{2.2.6}$$

Hence, the first part of the scaffold is the family of parallelepipeds

$$\mathcal{C}_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \mathcal{C}_\varepsilon^i,\ \text{where } \mathcal{C}_\varepsilon^i = x_\varepsilon^i + \mathcal{C}^\alpha,\ \text{for every } i \in \overline{1, N_\varepsilon}, \tag{2.2.7}$$

which represents the union of all black parallelepipeds from Figure 2.

Remark 2.2.2. We call throughout this work the black parallelepipeds constructed in (2.2.7) the "inner particles" of the scaffold or the "inner parallelepipeds" or the "inner boxes". In this way, $\mathcal{C}_\varepsilon$ represents the set of all "inner parallelepipeds". We choose the term "inner" because most of these particles will interact with the nematic liquid crystal material only on their edges and most of them are not visible, as shown in Figure 2, except for those which will be close to the boundary of the domain Ω.

We add now the lattice

$$\mathcal{Y}_\varepsilon := \left\{y_\varepsilon \in \Omega \ : \ \exists i, j \in \overline{1, N_\varepsilon} \text{ such that } \left|x_\varepsilon^i - x_\varepsilon^j\right| = \varepsilon \text{ and } y_\varepsilon = \frac{1}{2}\big(x_\varepsilon^i + x_\varepsilon^j\big)\right\},\ \text{card}(\mathcal{Y}_\varepsilon) = M_\varepsilon. \tag{2.2.8}$$

In each of the points from the lattice $\mathcal{Y}_\varepsilon$ we construct a gray parallelepiped, as shown in Figure 2.

Remark 2.2.3. We call throughout this work the gray parallelepipeds from Figure 2 the "connecting parallelepipeds" of the scaffold or the "connecting particles" or the "connecting boxes". The reason why we use this notation is because any single "connecting particle" joins two different "inner particles", for which their centers are at ε distance apart from each other.

Remark 2.2.4. We can interpret now more easily the "inner parallelepipeds" which are close to the $\partial\Omega$ by observing that it has less than 6 adjacent "connecting parallelepipeds". If it has 6, then that "inner parallelepiped" will not be "visible" (in Figure 2) and further away from $\partial\Omega$. Moreover, we prove in Subsection 2.4.2 that all the "inner parallelepipeds" have **no contribution** to the limiting problem, regardless whether they are close to the boundary of Ω or not. This is mainly because the "inner parallelepipeds" which have 6 adjacent "connecting parallelepipeds" touch the nematic liquid crystal only on their edges (which are of measure zero) and the "inner parallelepipeds" which have less than 6, their contribution becomes negligible due to their "small" number. More details can be found in Subsection 2.4.2.

We split this lattice into three parts, since the "connecting parallelepipeds" are elongated into three different directions, granted by the axes of the Cartesian coordinate system in $\mathbb{R}^3$. We denote by $\mathcal{P}_\varepsilon$ the union of all of "connecting parallelepipeds".

Since the scaffold is the union between the "inner particles" and the "connecting particles", we denote the scaffold as $\mathcal{N}_\varepsilon = \mathcal{C}_\varepsilon \cup \mathcal{P}_\varepsilon$ with $\partial\mathcal{N}_\varepsilon$ its surface.

More details regarding the construction of these objects can be found in Subsection 2.7.1.

Remark 2.2.5. In this paper, we use the notation $A \lesssim B$ for two real numbers A and B whenever there exists an ε-independent constant C such that $A \leq C \cdot B$.

We assume furthermore that:

(A_1) *$\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain;*

(A_2) *$1 < \alpha < 2$;*

Remark 2.2.6. The condition $1 < \alpha$ ensures that the "connecting particles" exist and also the dilute regime of the homogenisation problem. This is explained more in Subsection 2.7.1 and in remark 2.7.1. The reason why we impose other bound comes from the fact that if $\alpha > 2$, then equicoercivity may be lost. An example of this situation is described in Lemma 2.4.7 and we follow the same directions as in Lemma 3.6 from [28]. Another way of understanding why we impose this upper bound is given by the factor in front of the surface energy: $\dfrac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)}$, which can be seen as $\dfrac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}$. Since $\alpha > 1$, $1 - \varepsilon^{\alpha-1} \to 1$ as $\varepsilon \to 0$, hence the factor in front of the surface energy behaves like $\varepsilon^{2-\alpha}$. As a comparison, in [28], the factor in front of the surface energy is $\varepsilon^{3-2\alpha}$ and the upper bound used there is $\alpha < 3/2$.

(A_3) *There exists a constant $\lambda_\Omega > 0$ such that*

$$\mathrm{dist}(z_\varepsilon^i, \partial\Omega) + \frac{1}{2}\inf_{j\neq i}|z_\varepsilon^j - z_\varepsilon^i| \geq \lambda_\Omega \varepsilon$$

for any $\varepsilon > 0$ and any center z_ε^i of an object (either a "inner" or "connecting" parallelepiped) that is contained within the cubic microlattice, where $i \in \overline{1,(N_\varepsilon + M_\varepsilon)}$.

(A_4) *As $\varepsilon \to 0$, the measures*

$$\mu_\varepsilon^X := \varepsilon^3 \sum_{k=1}^{X_\varepsilon} \delta_{y_\varepsilon^{x,k}},\ \mu_\varepsilon^Y := \varepsilon^3 \sum_{l=1}^{Y_\varepsilon} \delta_{y_\varepsilon^{y,l}} \text{ and } \mu_\varepsilon^Z := \varepsilon^3 \sum_{m=1}^{Z_\varepsilon} \delta_{y_\varepsilon^{z,m}} \tag{2.2.9}$$

converge weakly (as measures in $\mathbb{R}^3$) to the Lebesgue measure restricted on Ω, denoted $\mathrm{d}x \llcorner \Omega$.*

(A_5) *$f_e : \mathcal{S}_0 \otimes \mathbb{R}^3 \to [0, +\infty)$ is differentiable, strongly convex*[4] *and there exists a constant $\lambda_e > 0$ such that*

$$\lambda_e^{-1}|D|^2 \le f_e(D) \le \lambda_e |D|^2, \quad |(\nabla f_e)(D)| \le \lambda_e(|D| + 1),$$

for any $D \in \mathcal{S}_0 \times \mathbb{R}^3$.

(A_6) *$f_b : \mathcal{S}_0 \to \mathbb{R}$ is continuous, bounded from below and there exists a constant $\lambda_b > 0$ such that $|f_b(Q)| \le \lambda_b(|Q|^6 + 1)$ for any $Q \in \mathcal{S}_0$.*

(A_7) *$f_s : \mathcal{S}_0 \times \mathbb{S}^2 \to \mathbb{R}$ is continuous and there exists a strictly positive constant λ_s such that, for any $Q_1, Q_2 \in \mathcal{S}_0$ and any $\nu \in \mathbb{S}^2$, we have*

$$|f_s(Q_1, \nu) - f_s(Q_2, \nu)| \le \lambda_s |Q_1 - Q_2| \big(|Q_1|^3 + |Q_2|^3 + 1\big).$$

It is easy to see from here that f_s has a quartic growth in Q.

2.3 MAIN RESULTS

2.3.1 GENERAL CASE

Let $f_{hom} : \mathcal{S}_0 \to \mathbb{R}$ be the function defined as:

$$f_{hom}(Q) := \frac{q+r}{qr} \int_{\mathcal{C}^x} f_s(Q, \nu) \mathrm{d}\sigma + \frac{p+r}{pr} \int_{\mathcal{C}^y} f_s(Q, \nu) \mathrm{d}\sigma + \frac{p+q}{pq} \int_{\mathcal{C}^z} f_s(Q, \nu) \mathrm{d}\sigma, \tag{2.3.1}$$

for any $Q \in \mathcal{S}_0$, where $\mathcal{C}^x$, $\mathcal{C}^y$ and $\mathcal{C}^z$ are defined in (2.2.3). From (A_7), we can deduce that f_{hom} is also continuous and that it has a quartic growth. If we work in the symmetric case, that is $p = q = r$, then relation (2.3.1) becomes:

$$f_{hom}(Q) := \frac{2}{p} \int_{\partial \mathcal{C}} f_s(Q, \nu) \mathrm{d}\sigma. \tag{2.3.2}$$

Remark 2.3.1. Throughout this paper, the function f_{hom} is sometimes referred to as being **the homogenised functional**, simply because it represents the effect that arises from the surface energy term in the limiting free energy functional.

The main results of these notes concerns the asymptotic behaviour of local minimisers of the functional $\mathcal{F}_\varepsilon$, as $\varepsilon \to 0$.

Let $g \in H^{1/2}(\partial\Omega, \mathcal{S}_0)$ be a boundary datum. We denote by $H^1_g(\Omega, \mathcal{S}_0)$ the set of maps Q from $H^1(\Omega, \mathcal{S}_0)$ such that $Q = g$ on $\partial\Omega$ in the trace sense. Similarly, we define $H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$ to be $H^1(\Omega_\varepsilon)$ with $Q = g$ on $\partial\Omega$ in the trace sense.

4 We say that a function $f : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$ is strongly convex if there exists $\theta > 0$ such that $\tilde{f} : \mathcal{S}_0 \otimes \mathbb{R}^3 \to \mathbb{R}$, defined by $\tilde{f}(D) = f(D) - \theta|D|^2$, is convex.

We use the harmonic extension operator, $E_\varepsilon : H^1_g(\Omega_\varepsilon, \mathcal{S}_0) \to H^1_g(\Omega, \mathcal{S}_0)$, defined in the following way: $E_\varepsilon Q := Q$ on Ω_ε and inside the scaffold, $E_\varepsilon Q$ is the unique solution of the following problem:

$$\begin{cases} \Delta E_\varepsilon Q = 0 & \text{in } \mathcal{N}_\varepsilon \\ E_\varepsilon Q \equiv Q & \text{on } \partial\mathcal{N}_\varepsilon. \end{cases}$$

Using this framework, we can produce the main result of this work:

Theorem 2.3.1. Suppose that the assumptions (A_1)-(A_7) are satisfied. Let $Q_0 \in H^1_g(\Omega, \mathcal{S}_0)$ be an isolated H^1-local minimiser for $\mathcal{F}_0$, defined in (2.1.3), that is, there exists $\delta_0 > 0$ such that $\mathcal{F}_0[Q_0] < \mathcal{F}_0[Q]$ for any $Q \in H^1_g(\Omega, \mathcal{S}_0)$ such that $\|Q - Q_0\|_{H^1_g(\Omega,\mathcal{S}_0)} \leq \delta_0$ and $Q \neq Q_0$. Then for any ε sufficiently small enough, there exists a sequence of H^1-local minimisers Q_ε of $\mathcal{F}_\varepsilon$ such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1_g(\Omega, \mathcal{S}_0)$.

2.3.2 APPLICATIONS TO THE LANDAU-DE GENNES MODEL

In this subsection, we particularise Theorem 2.3.1 to the case of the Landau-de Gennes model. Before doing this, let us introduce first some of the energies used in this model for nematic liquid crystals.

- ***The elastic energy***

We consider the following form for the *elastic energy*:

$$f_e(\nabla Q) := \sum_{i,j,k\in\{1,2,3\}} \left[\frac{L_1}{2}\left(\frac{\partial Q_{ij}}{\partial x_k}\right)^2 + \frac{L_2}{2}\frac{\partial Q_{ij}}{\partial x_j}\frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2}\frac{\partial Q_{ik}}{\partial x_j}\frac{\partial Q_{ij}}{\partial x_k} \right],$$

where Q_{ij} is the $(ij)^{th}$ component of Q, (x_1, x_2, x_3) represents the usual cartesian coordinates and e_{ijk} represents the Levi-Civita symbol.

In order to fulfill assumption (A_5), we take as in [58]:

$$L_1 > 0, \quad -L_1 < L_3 < 2L_1, \quad -\frac{3}{5}L_1 - \frac{1}{10}L_3 < L_2. \tag{2.3.3}$$

- ***The bulk energy***

For the *bulk energy density*, we use several versions of it. The first one is the classical quartic polynomial in the scalar invariants of Q, defined in (2.1.2), which verifies the conditions of assumption (A_4):

$$f_b(Q) = a\,\mathrm{tr}(Q^2) - b\,\mathrm{tr}(Q^3) + c\,\mathrm{tr}(Q^2)^2.$$

We also prove similar results for a general polynomial in the scalar invariants of Q, that is:

$$f_b^{gen}(Q) = \sum_{k=2}^{N} a_k\,\mathrm{tr}(Q^k), \tag{2.3.4}$$

where $N \in \mathbb{N}$, $N \geq 4$ is fixed, with the coefficients $a_k \in \mathbb{R}$ chosen such that the polynomial $h : \mathbb{R} \to \mathbb{R}$, defined by $h(x) = \sum_{k=2}^{N} a_k x^k$, for any $x \in \mathbb{R}$, admits at least one local minimum over $\mathbb{R}$.

• ***The surface energy***

For each of the versions of the *bulk energy densities*, we choose suitable *surface energy densities*, such that, in the homogenised functional, the surface terms grant an effect f_{hom} which has the same form as the initial bulk energy chosen, but with different coefficients, most important of which the coefficient of $\mathrm{tr}(Q^2)$ is now different. Therefore, our choice of the surface energy density has a strong connection with the bulk energy density chosen.

Moreover, since Theorem 2.3.1 holds for any values of p, q and r, that is, for any type of parallelepiped chosen for the construction of the scaffold, and since, in reality, 2PP (two-photon polymerization) materials with cubic symmetry properties, in the sense from remark 2.2.1, have been obtained (for example, in [68]), then we also split our work on whether the scaffold is symmetric or not.

Hence, our choices of surface energy densities will depend on the bulk energy density chosen and if the scaffold is symmetric or not.

I) If the scaffold is symmetric, as described in remark 2.2.1, then the physical invariances require

$$f_s(UQU^T, Uu) = f_s(Q, u),\ \forall (Q, u) \in \mathcal{S}_0 \times \mathbb{R}^3,\ U \in \mathcal{O}(3)$$

and this leads, according to Proposition 2.6 from [28], to a *surface energy* of the form

$$f_s(Q, \nu) = \tilde{f}_s(\mathrm{tr}(Q^2), \mathrm{tr}(Q^3), \nu \cdot Q\nu, \nu \cdot Q^2\nu),\ \forall (Q, \nu) \in \mathcal{S}_0 \times \mathbb{R}^3. \tag{2.3.5}$$

1) Let us consider the case in which the bulk energy is the classical Landau-de Gennes quartic polynomial in Q, described by (2.1.2). In this case, we use one of the most common forms for the *surface energy*, which is the Rapini-Papoular energy:

$$f_s(Q, \nu) = W\ \mathrm{tr}\big(Q - s_+\big(\nu \otimes \nu - \mathbb{I}_3/3\big)\big)^2, \tag{2.3.6}$$

where W is a coefficient measuring the strength of the anchoring, s_+ is measuring the deviation from the homeotropic (perpendicular) anchoring to the boundary and $\mathbb{I}_3$ is the 3×3 identity matrix.

Another *surface energy density* that we use, which verifies (2.3.5), is the following:

$$f_s(Q, \nu) = k_a(\nu \cdot Q^2\nu) + k_b(\nu \cdot Q\nu)(\nu \cdot Q^2\nu) + k_c(\nu \cdot Q^2\nu)^2 + a'\ \mathrm{tr}(Q^2) + \frac{2b'}{3}\mathrm{tr}(Q^3) + \frac{c'}{2}\mathrm{tr}(Q^2)^2, \tag{2.3.7}$$

where k_a, k_b, k_c, a', b' and c' are constants.

Remark 2.3.2. If our choice of f_s contains terms of the form $\mathrm{tr}(Q^2)$ or $\mathrm{tr}(Q^3)$, then these terms very easily generate in f_{hom} terms similar with the one from the bulk energy defined in (2.1.2), since they are exactly the same. Our goal in the paragraphs is to use the other terms from

(2.3.7) for the surface energy densities, which are of the form $\nu \cdot Q^k \nu$ and for which the previous implication is not that immediate.

2) For the bulk energy density in (2.3.4), we choose a more general form for $f_s(Q, \nu)$, depending only on terms of the form $\nu \cdot Q^k \nu$. In this situation, the function f_{hom} can be computed easily, due to the geometry of the scaffold. More specifically, according to Proposition 2.7.4, we obtain in the homogenised functional terms of the form $\text{tr}(Q^k)$, with $k \geq 4$, but they only depend on $\text{tr}(Q^2)$ and $\text{tr}(Q^3)$, since $\text{tr}(Q) = 0$. In order to prove this statement, let λ_1, λ_2 and λ_3 the eigenvalues of Q. Then they satisfy the system:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \text{tr}(Q^2) \\ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \text{tr}(Q^3) \end{cases}$$

and, by solving the system, we can see that λ_1, λ_2 and λ_3 can be viewed as functions of $\text{tr}(Q^2)$ and $\text{tr}(Q^3)$. Since $\text{tr}(Q^k) = \lambda_1^k + \lambda_2^k + \lambda_3^k$, for any $k \in \mathbb{N}$, $k \geq 1$, then it is easy to see from here that $\text{tr}(Q^k)$, for $k \geq 4$, is depending only on $\text{tr}(Q^2)$ and $\text{tr}(Q^3)$. Indeed, by Cayley-Hamilton theorem, the identity:

$$Q^3 - \frac{1}{2}\text{tr}(Q^2)Q - \frac{1}{3}\text{tr}(Q^3)\mathbb{I}_3 = 0$$

becomes valid for any Q-tensor Q, where $\mathbb{I}_3$ is the 3×3 identity matrix. Multiplying this identity succcessively by Q, Q^2, Q^3 and so on and taking the trace we obtain the claim.

II) If the scaffold does not present cubic symmetry, in the sense of remark 2.2.1, we only illustrate the case in which the bulk energy density is the one from (2.1.2) and the surface energy density is a variation of (2.3.7).

The goal of the next subsubsections is to analyse each particular case described above and to obtain similar results as Theorem 2.3.1 for each of it.

I. The symmetric case: $p = q = r$

Assuming $p = q = r$ implies that the "inner" parallelepipeds constructed in (2.2.7) are actually cubes.

1. We analyse first the case in which f_b is defined in (2.1.2), that is:

$$f_b(Q) = a\,\text{tr}(Q^2) - b\,\text{tr}(Q^3) + c\,\text{tr}(Q^2)^2.$$

a) We analyse the case when $(a, b, c) \rightsquigarrow (a', b'c')$, where all the parameters are non-zero and c and c' are positive, which by "$\rightsquigarrow$" we mean that from a nematic liquid crystal with the parameters (a, b, c) we want to generate a new homogenised material, which also behaves like a NLC, but with parameters (a', b', c').

We choose f_s in this case to be:

$$f_s^{LDG}(Q,\nu) = \frac{p}{4}\Big((a'-a)(\nu\cdot Q^2\nu) - (b'-b)(\nu\cdot Q^3\nu) + 2(c'-c)(\nu\cdot Q^4\nu)\Big) \tag{2.3.8}$$

where a', b' and c' are the desired coefficients in the homogenised bulk potential, such that in the homogenised material, we have:

$$f_{hom}^{LDG}(Q) = (a'-a)\,\mathrm{tr}(Q^2) - (b'-b)\,\mathrm{tr}(Q^3) + (c'-c)\left(\mathrm{tr}(Q^2)\right)^2. \tag{2.3.9}$$

We are interested in studying the behaviour of the whole material when $\varepsilon \to 0$, that is, studying the following functionals:

$$\mathcal{F}_\varepsilon^{LDG}[Q_\varepsilon] := \int_{\Omega_\varepsilon} \left(f_e(\nabla Q_\varepsilon) + a\,\mathrm{tr}(Q_\varepsilon^2) - b\,\mathrm{tr}(Q_\varepsilon^3) + c\left(\mathrm{tr}(Q_\varepsilon^2)\right)^2\right)\mathrm{d}x + \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon} f_s^{LDG}(Q_\varepsilon,\nu)\mathrm{d}\sigma \tag{2.3.10}$$

and

$$\mathcal{F}_0^{LDG}[Q] := \int_\Omega \left(f_e(\nabla Q) + a'\,\mathrm{tr}(Q^2) - b'\,\mathrm{tr}(Q^3) + c'\left(\mathrm{tr}(Q^2)\right)^2\right)\mathrm{d}x. \tag{2.3.11}$$

Theorem 2.3.2. Let (a,b,c) and (a',b',c') be two set of parameters with $c > 0$ and $c' > 0$. Suppose that the assumptions (A_1)-(A_7) are satisfied and also the inequalities from (2.3.3). Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{LDG}$ defined by (2.3.11), and for $\varepsilon > 0$ sufficiently small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{LDG}$, defined by (2.3.10), such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega,\mathcal{S}_0)$.

Proof. This theorem is a particular case of Theorem 2.3.1. It is sufficient to prove that relation (2.3.9) can be obtained via (2.3.2), that is:

$$f_{hom}^{LDG}(Q) = \frac{2}{p}\int_{\partial C} f_s^{LDG}(Q,\nu)\mathrm{d}\sigma = (a'-a)\,\mathrm{tr}(Q^2) - (b'-b)\,\mathrm{tr}(Q^3) + (c'-c)\left(\mathrm{tr}(Q^2)\right)^2.$$

Using Proposition 2.7.4, we have:

$$\int_{\partial C}\nu\cdot Q^2\nu\mathrm{d}\sigma = 2\mathrm{tr}(Q^2),\quad \int_{\partial C}\nu\cdot Q^3\nu\mathrm{d}\sigma = 2\mathrm{tr}(Q^3)\quad\text{and}\quad \int_{\partial C}\nu\cdot Q^4\nu\mathrm{d}\sigma = 2\mathrm{tr}(Q^4),$$

from which we get

$$\frac{2}{p}\int_{\partial C} f_s^{LDG}(Q,\nu)\mathrm{d}\sigma = \frac{2}{p}\cdot\frac{p}{4}\left((a'-a)\cdot 2\mathrm{tr}(Q^2) - (b'-b)\cdot 2\mathrm{tr}(Q^3) + 2(c'-c)\cdot 2\mathrm{tr}(Q^4)\right) \Rightarrow$$
$$\Rightarrow f_{hom}^{LDG}(Q) = (a'-a)\mathrm{tr}(Q^2) - (b'-b)\mathrm{tr}(Q^3) + (c'-c)\cdot 2\mathrm{tr}(Q^4).$$

Since $Q \in \mathcal{S}_0$, then, by Cayley-Hamilton theorem, if λ_1, λ_2 and λ_3 are the eigenvalues of Q, we have:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(Q) = 0 \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \frac{1}{2}\big((\mathrm{tr}(Q))^2 - \mathrm{tr}(Q^2)\big) = -\frac{1}{2}\mathrm{tr}(Q^2) \end{cases}$$

and

$$\begin{aligned} \mathrm{tr}(Q^4) &= \lambda_1^4 + \lambda_2^4 + \lambda_3^4 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2 - 2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2) \\ &= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2 - 2\big((\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)^2 - 2\lambda_1\lambda_2\lambda_3(\lambda_1 + \lambda_2 + \lambda_3)\big) \\ &= \big(\mathrm{tr}(Q^2)\big)^2 - 2\Big(-\frac{1}{2}\mathrm{tr}(Q^2)\Big)^2 \\ &= \frac{1}{2}\big(\mathrm{tr}(Q^2)\big)^2 \end{aligned}$$

from which we get the relation $2\mathrm{tr}(Q^4) = \big(\mathrm{tr}(Q^2)\big)^2$.

Hence, we conclude that:

$$f_{hom}^{LDG}(Q) = (a' - a)\mathrm{tr}(Q^2) - (b' - b)\mathrm{tr}(Q^3) + (c' - c)\big(\mathrm{tr}(Q^2)\big)^2.$$

□

b) We analyse now the case in which we want $(a, 0, 0) \rightsquigarrow (a', 0, 0)$, with a and a' non-zero. In this situation, we have

$$f_b^{RP}(Q) = a\ \mathrm{tr}(Q^2)$$

and we choose f_s to be given by the Rapini-Papoular form (2.3.6):

$$f_s^{RP}(Q, \nu) = \frac{p}{12}(a' - a)\ \mathrm{tr}(Q - Q_\nu)^2, \tag{2.3.12}$$

where $Q_\nu = \nu \otimes \nu - \mathbb{I}_3/3$ and $\mathbb{I}_3$ is the 3×3 identity matrix.

In this case, we have:

$$\mathcal{F}_\varepsilon^{RP}[Q_\varepsilon] := \int_{\Omega_\varepsilon} \big(f_e(\nabla Q_\varepsilon) + a\,\mathrm{tr}(Q_\varepsilon^2)\big)\mathrm{d}x + \frac{p}{2}\cdot(a' - a)\cdot\left(\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon} \mathrm{tr}(Q_\varepsilon - Q_\nu)^2\mathrm{d}\sigma\right) \tag{2.3.13}$$

and we prove that

$$f_{hom}^{RP}(Q) = (a' - a)\ \mathrm{tr}(Q^2), \tag{2.3.14}$$

and

$$\mathcal{F}_0^{RP}[Q] := \int_\Omega \big(f_e(\nabla Q) + a'\,\mathrm{tr}(Q^2)\big)\mathrm{d}x. \tag{2.3.15}$$

Theorem 2.3.3. Let a and a' be two parameters. Suppose that the assumptions (A_1)-(A_7) are satisfied and also the inequalities from (2.3.3). Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{RP}$ defined by (2.3.15), and for $\varepsilon > 0$ sufficiently small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{RP}$, defined by (2.3.13), such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1_g(\Omega, \mathcal{S}_0)$.

Proof. The proof follows the same steps as in the proof of Theorem 2.3.2, using Proposition 2.7.5. We only have to prove that relation (2.3.14) can be obtained using (2.3.2), knowing that (2.3.12) holds.

From (2.3.2) and Proposition 2.7.5, we have:

$$\begin{aligned} f_{hom}^{RP}(Q) &= \frac{2}{p}\int_{\partial C} f_s^{RP}(Q,\nu)\mathrm{d}\sigma = \frac{2}{p}\cdot\frac{p}{12}(a'-a)\int_{\partial C}\mathrm{tr}(Q-Q_\nu)\mathrm{d}\sigma \\ &= \frac{(a'-a)}{6}(6\mathrm{tr}(Q^2)+4) = (a'-a)\mathrm{tr}(Q^2) + \frac{2}{3}(a'-a). \end{aligned}$$

We can eliminate the constant $\frac{2}{3}(a'-a)$ from f_{hom}^{RP}, since it does not influence the minimisers of the functional $\mathcal{F}_\varepsilon^{RP}$, so we obtain: $f_{hom}^{RP}(Q) = (a'-a)\mathrm{tr}(Q^2)$. □

2. We now analyse the situation in which f_b is of the form given by (2.3.4). In this situation, we choose:

$$f_s^{gen}(Q,\nu) = \frac{p}{4}\sum_{k=2}^{M} b_k(\nu\cdot Q^k\nu), \tag{2.3.16}$$

where $(b_k)_{k\in\overline{2,M}}$ are the coefficients of the polynomial $i:\mathbb{R}\to\mathbb{R}$ of degree $M\in\mathbb{N}$, $M\geq 4$, defined by $i(x)=\sum_{k=2}^{M} b_k x^k$, for any $x\in\mathbb{R}$, with the property that i admits at least one local minimum over $\mathbb{R}$.

In the same manner, we have

$$f_{hom}^{gen}(Q) = \sum_{k=2}^{\max\{M,N\}} c_k\,\mathrm{tr}(Q^k),$$

where, for any $k\in\overline{2,\max\{M,N\}}$, we have

$$c_k = \begin{cases} a_k + b_k, & \text{if } 2\leq k\leq \min\{M,N\} \\ a_k, & \text{if } \min\{M,N\} < k \leq \max\{M,N\} \text{ and } M\leq N \\ b_k, & \text{if } \min\{M,N\} < k\leq \max\{M,N\} \text{ and } M\geq N. \end{cases}$$

In this case, $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ become:

$$\mathcal{F}_\varepsilon^{gen}[Q_\varepsilon] := \int_{\Omega_\varepsilon}\left(f_e(\nabla Q_\varepsilon) + \sum_{k=2}^{N} a_k\,\mathrm{tr}(Q_\varepsilon^k)\right)\mathrm{d}x + \frac{p}{4}\cdot\sum_{k=2}^{M} b_k\cdot\left(\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon}(\nu\cdot Q_\varepsilon^k\nu)\mathrm{d}\sigma\right) \tag{2.3.17}$$

and

$$\mathcal{F}_0^{gen}[Q] = \int_\Omega \left(f_e(\nabla Q) + \sum_{k=2}^{\max\{M,N\}} c_k \operatorname{tr}(Q^k) \right) \mathrm{d}x. \tag{2.3.18}$$

Theorem 2.3.4. Let $(a_k)_{k\in\overline{2,N}}$ and $(b_k)_{k\in\overline{2,M}}$ be such that the polynomials h and i defined earlier admit at least one local minimum over $\mathbb{R}$. Suppose that the assumptions (A_1)-(A_7) are satisfied and also the inequalities from (2.3.3). Then, for any isolated H^1-local minimiser Q_0 of the functional $\mathcal{F}_0^{gen}$ defined by (2.3.18), and for $\varepsilon > 0$ sufficiently small enough, there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}_\varepsilon^{gen}$, defined by (2.3.17), such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H_g^1(\Omega, \mathcal{S}_0)$.

Proof. This theorem is a particular case of Theorem 2.3.1. Using once again Proposition 2.7.4, the proof is finished. □

II. The asymmetric case $p \neq q \neq r \neq p$

We now assume that p, q and r are three different real values, each greater than or equal to 1. In this situation, the "inner particles" are not cubes anymore, but simple parallelepipeds.

We only illustrate how to proceed for the case in which we have

$$f_b(Q) = a\operatorname{tr}(Q^2) - b\operatorname{tr}(Q^3) + c\operatorname{tr}(Q^4) = a\operatorname{tr}(Q^2) - b\operatorname{tr}(Q^3) + \frac{c}{2}\left(\operatorname{tr}(Q^2)\right)^2,$$

with $c > 0$. Similar results can be obtained for the other cases in which we modify the form of f_b.

Let

$$A = \frac{1}{3}\begin{pmatrix} -\frac{2}{p}+\frac{1}{q}+\frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{p}-\frac{2}{q}+\frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{p}+\frac{1}{q}-\frac{2}{r} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{q}+\frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{p}+\frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{p}+\frac{1}{q} \end{pmatrix}.$$

and $\omega = \frac{2}{3}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$. Note that A, B and ω are constants depending only on the choice of p, q and r. Moreover, we have $\operatorname{tr}(A) = 0$ and $B = \omega\mathbb{I}_3 + A$, where $\mathbb{I}_3$ is the 3×3 identity matrix.

Consider now

$$f_s^{asym}(Q,\nu) = \frac{1}{2\omega}\big((a'-a)(\nu\cdot Q^2\nu) - (b'-b)(\nu\cdot Q^3\nu) + (c'-c)(\nu\cdot Q^4\nu)\big), \tag{2.3.19}$$

with a', b' and c' real parameters such that $c' > 0$ and the associated free energy functional:

$$\mathcal{F}_\varepsilon^{asym}[Q_\varepsilon] := \int_\Omega \left(f_e(\nabla Q_\varepsilon) + a\operatorname{tr}(Q_\varepsilon^2) - b\operatorname{tr}(Q_\varepsilon^3) + c\operatorname{tr}(Q_\varepsilon^4)\right)\mathrm{d}x + \\ + \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon} f_s^{asym}(Q_\varepsilon,\nu)\,\mathrm{d}\sigma. \tag{2.3.20}$$

We prove in the next theorem that the homogenised functional is:

$$f^{asym}_{hom}(Q) = \big((a'-a)\mathrm{tr}(Q^2) - (b'-b)\mathrm{tr}(Q^3) + (c'-c)\mathrm{tr}(Q^4)\big) + \\ + \frac{1}{\omega}\big((a'-a)\mathrm{tr}(A\cdot Q^2) - (b'-b)\mathrm{tr}(A\cdot Q^3) + (c'-c)\mathrm{tr}(A\cdot Q^4)\big). \tag{2.3.21}$$

Theorem 2.3.5. Let (a,b,c) and (a',b',c') be two set of parameters with $c>0$ and $c'>0$. Suppose that the assumptions (A_1)-(A_7) are satisfied and also the inequalities from (2.3.3). Then, for $\varepsilon > 0$ sufficiently small enough and for any isolated H^1-local minimiser Q_0 of the functional:

$$\mathcal{F}^{asym}_0[Q] := \int_\Omega \big(f_e(\nabla Q) + a' tr(Q^2) - b' tr(Q^3) + c'\big(tr(Q^2)\big)^2\big)\mathrm{d}x + \\ + \frac{1}{\omega}\int_\Omega \big((a'-a)tr(A\cdot Q^2(x)) - (b'-b)tr(A\cdot Q^3(x)) + (c'-c)tr(A\cdot Q^4(x))\big)\mathrm{d}x$$

there exists a sequence of local minimisers Q_ε of the functionals $\mathcal{F}^{asym}_\varepsilon$, defined by (2.3.20), such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1_g(\Omega, \mathcal{S}_0)$.

Proof. We follow the same steps as in Theorem 2.3.2 and in Theorem 2.3.3, that is, we prove that relation (2.3.21) can be obtained using (2.3.1) and (2.3.19).

In the proof of Proposition 2.7.4, we obtain that:

$$\int_{\mathcal{C}^x} \nu\cdot Q^k \nu \mathrm{d}\sigma = 2q_{11,k}, \quad \int_{\mathcal{C}^y} \nu\cdot Q^k \nu \mathrm{d}\sigma = 2q_{22,k} \quad \text{and} \quad \int_{\mathcal{C}^z} \nu\cdot Q^k \nu \mathrm{d}\sigma = 2q_{33,k},$$

for any $k\in\mathbb{N}$, $k\neq 0$, where $q_{ij,k}$ is the ij-th component of Q^k, from which we get:

$$\int_{\mathcal{C}^x} f^{asym}_s(Q,\nu)\mathrm{d}\sigma = \frac{1}{\omega}((a'-a)q_{11,2} - (b'-b)q_{11,3} + (c'-c)q_{11,4})$$
$$\int_{\mathcal{C}^y} f^{asym}_s(Q,\nu)\mathrm{d}\sigma = \frac{1}{\omega}((a'-a)q_{22,2} - (b'-b)q_{22,3} + (c'-c)q_{22,4})$$
$$\int_{\mathcal{C}^z} f^{asym}_s(Q,\nu)\mathrm{d}\sigma = \frac{1}{\omega}((a'-a)q_{33,2} - (b'-b)q_{33,3} + (c'-c)q_{33,4}).$$

Using now (2.3.1), we obtain:

$$f^{asym}_{hom}(Q) = \frac{1}{\omega}(a'-a)\left(q_{11,2}\left(\frac{1}{q}+\frac{1}{r}\right) + q_{22,2}\left(\frac{1}{p}+\frac{1}{r}\right) + q_{33,2}\left(\frac{1}{p}+\frac{1}{q}\right)\right) - \\ - \frac{1}{\omega}(b'-b)\left(q_{11,3}\left(\frac{1}{q}+\frac{1}{r}\right) + q_{22,3}\left(\frac{1}{p}+\frac{1}{r}\right) + q_{33,3}\left(\frac{1}{p}+\frac{1}{q}\right)\right) + \\ + \frac{1}{\omega}(c'-c)\left(q_{11,4}\left(\frac{1}{q}+\frac{1}{r}\right) + q_{22,4}\left(\frac{1}{p}+\frac{1}{r}\right) + q_{33,4}\left(\frac{1}{p}+\frac{1}{q}\right)\right)$$

which we can see as:

$$f_{hom}^{asym}(Q) = \frac{1}{\omega}\big((a'-a)\mathrm{tr}(B\cdot Q^2) - (b'-b)\mathrm{tr}(B\cdot Q^3) + (c'-c)\mathrm{tr}(B\cdot Q^4)\big)$$

and since $B = \omega\mathbb{I}_3 + A$, we obtain:

$$\begin{aligned} f_{hom}^{asym}(Q) = &\big((a'-a)\mathrm{tr}(Q^2) - (b'-b)\mathrm{tr}(Q^3) + (c'-c)\mathrm{tr}(Q^4)\big) + \\ &+ \frac{1}{\omega}\big((a'-a)\mathrm{tr}(A\cdot Q^2) - (b'-b)\mathrm{tr}(A\cdot Q^3) + (c'-c)\mathrm{tr}(A\cdot Q^4)\big), \end{aligned}$$

from which we conclude. □

Remark 2.3.3. We have obtained in this case a part which is exactly the same as in the case in which we have cubic symmetry, but also three terms of the form $tr(A \cdot Q^k)$ which describe a new preferred alignment of the liquid crystal particles inside of the domain, given by the loss of the cubic symmetry of the scaffold.

2.4 PROPERTIES OF THE FUNCTIONAL $\mathcal{F}_\varepsilon$

2.4.1 ANALYTICAL TOOLS: TRACE AND EXTENSION

The main result of this subsection consists on a L^p inequality, which is adapted from lemma 3.1. from [28], because our scaffold now consists on inter-connected particles and the interaction between the liquid crystal and the cubic microlattice happens only up to five faces of the particles of the scaffold.

In the following, given a set $\mathcal{P} \subset \mathbb{R}^2$ and a real number $a > 0$, we define $a\mathcal{P} = \{ax : x \in \mathcal{P}\}$.

Lemma 2.4.1. Let $\mathcal{P} \subseteq \mathbb{R}^2$ be a compact, convex set whose interior contains the origin. Let a and b be positive numbers such that $a < b$. Then there exists a bijective, Lipschitz map $\phi : b\mathcal{P} \setminus a\mathcal{P} \to \overline{B}_b \setminus \overline{B}_a$ that has a Lipschitz inverse and satisfies

$$\|\nabla\phi\|_{L^\infty(b\mathcal{P}\setminus a\mathcal{P})} + \|\nabla(\phi^{-1})\|_{L^\infty(\overline{B}_b\setminus\overline{B}_a)} \le C(\mathcal{P}),$$

where $C(\mathcal{P})$ is a positive constant that depends only on $\mathcal{P}$ and neither on a nor b.

The proof of Lemma 2.4.1 follows the same steps as Lemma 3.2. from [28], the only difference being that now we are in the case of $\mathbb{R}^2$ instead of $\mathbb{R}^3$.

Lemma 2.4.2. Let $\mathcal{P} \subseteq \mathbb{R}^2$ be a compact, convex set whose interior contains the origin and $n \in [2,4]$. Then, there exists $C = C(\mathcal{P}, \phi) > 0$, such that for any $0 < a \le b$ and any $u \in H^1(b\mathcal{P} \setminus a\mathcal{P})$, there holds

$$\oint_{\partial(a\mathcal{P})} |u|^n \mathrm{d}s \lesssim \frac{2aC}{b^2-a^2}\int_{b\mathcal{P}\setminus a\mathcal{P}} |u|^n \mathrm{d}x + \frac{nC}{2}\int_{b\mathcal{P}\setminus a\mathcal{P}} (|u|^{2n-2} + |\nabla u|^2)\mathrm{d}x,$$

where $\oint$ represents the curvilinear integral in $\mathbb{R}^2$.

Proof. Using Lemma 2.4.1, we can restrict without loss of generality to the case in which $\mathcal{P} = \overline{B}_1$, which is the two dimensional unit disk, centered in origin. Then $\partial B_\tau = \{x \in \mathbb{R}^2 : |x| = \tau\} = \{(\rho, \theta) : \rho = \tau,\ \theta \in [0, 2\pi]\}$, for any $\tau > 0$, and we can write, for any $\rho \in [a, b]$ and any $\theta \in [0, 2\pi]$:

$$\begin{aligned}
|u|^n(a,\theta) &= |u|^n(\rho,\theta) - \int_a^\rho \partial_\tau\big(|u|^n\big)(\tau,\theta)\mathrm{d}\tau \\
&\le |u|^n(\rho,\theta) + n\int_a^\rho \big(|u|^{n-1}\cdot|\partial_\tau u|\big)(\tau,\theta)\mathrm{d}\tau \\
&\le |u|^n(\rho,\theta) + \frac{n}{2}\int_a^\rho \big(|u|^{2n-2} + |\partial_\tau u|^2\big)(\tau,\theta)\mathrm{d}\tau \\
|u|^n(a,\theta) &\le |u|^n(\rho,\theta) + \frac{n}{2}\int_a^b \big(|u|^{2n-2} + |\nabla u|^2\big)(\tau,\theta)\mathrm{d}\tau
\end{aligned}$$

If we multiply both sides by ρ and integrate over $[a, b]$ with respect to ρ, we get:

$$\begin{aligned}
|u|^n(a,\theta)\int_a^b \rho\,\mathrm{d}\rho &\le \int_a^b |u|^n(\rho,\theta)\cdot\rho\,\mathrm{d}\rho + \frac{n}{2}\int_a^b \rho\,\mathrm{d}\rho\int_a^b \big(|u|^{2n-2} + |\nabla u|^2\big)(\tau,\theta)\mathrm{d}\tau \\
\frac{b^2-a^2}{2}|u|^n(a,\theta) &\le \int_a^b |u|^n(\rho,\theta)\cdot\rho\,\mathrm{d}\rho + \frac{n(b^2-a^2)}{4}\int_a^b \big(|u|^{2n-2} + |\nabla u|^2\big)(\tau,\theta)\mathrm{d}\tau.
\end{aligned}$$

Since for any $\tau \in [a, b]$ we have $\tau > a$, then:

$$\frac{b^2-a^2}{2a}|u|^n(a,\theta)\cdot a \le \int_a^b |u|^n(\rho,\theta)\cdot\rho\,\mathrm{d}\rho + \frac{n(b^2-a^2)}{4a}\int_a^b \big(|u|^{2n-2} + |\nabla u|^2\big)(\tau,\theta)\cdot\tau\,\mathrm{d}\tau.$$

Now we integrate with respect to θ over $[0, 2\pi]$ and we get:

$$\begin{aligned}
&\frac{b^2-a^2}{2a}\int_0^{2\pi} |u|^n(a,\theta)\cdot a\,\mathrm{d}\theta \le \\
&\quad\le \int_0^{2\pi}\int_a^b |u|^n(\rho,\theta)\cdot\rho\,\mathrm{d}\rho\mathrm{d}\theta + \frac{n(b^2-a^2)}{4a}\int_0^{2\pi}\int_a^b \big(|u|^{2n-2} + |\nabla u|^2\big)(\tau,\theta)\cdot\tau\,\mathrm{d}\tau\mathrm{d}\theta,
\end{aligned}$$

which implies

$$\frac{b^2-a^2}{2a}\oint_{\partial B_a} |u|^n\mathrm{d}s \le \int_{B_b\setminus B_a} |u|^n\mathrm{d}x + \frac{n(b^2-a^2)}{4a}\int_{B_b\setminus B_a} \big(|u|^{2n-2} + |\nabla u|^2\big)\mathrm{d}x,$$

therefore

$$\oint_{\partial B_a} |u|^n\mathrm{d}s \le \frac{2a}{b^2-a^2}\int_{B_b\setminus B_a} |u|^n\mathrm{d}x + \frac{n}{2}\int_{B_b\setminus B_a} \big(|u|^{2n-2} + |\nabla u|^2\big)\mathrm{d}x.$$

If we apply now the Lipschitz homeomorphism ϕ defined by Lemma 2.4.1, the conclusion follows. □

Lemma 2.4.3. For any $Q \in H^1(\Omega_\varepsilon, \mathcal{S}_0)$ and any $n \in [2,4]$, there holds:

$$\frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon-\varepsilon^\alpha)}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q|^n d\sigma \lesssim \frac{n}{2}\cdot\frac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\int_{\Omega_\varepsilon}(|Q|^{2n-2}+|\nabla Q|^2)dx + \frac{1}{2(1-\varepsilon^{\alpha-1})^2}\int_{\Omega_\varepsilon}|Q|^n dx.$$

Proof. Let $I_\varepsilon[Q] = \dfrac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon-\varepsilon^\alpha)}\displaystyle\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q|^n d\sigma$ and

$$I_\varepsilon^X[Q] = \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon-\varepsilon^\alpha)}\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{T}_x^k}|Q|^n d\sigma.$$

Let $\overline{e_1} = (1,0,0)^T$, $\overline{e_2} = (0,1,0)^T$, $\overline{e_3} = (0,0,1)^T$ and $k \in \overline{1, X_\varepsilon}$. Then, according to the definitions from Subsection 2.7.1, $y_\varepsilon^{x,k}$ is the center of the "connecting parallelepiped" $\mathcal{P}_\varepsilon^{x,k}$ with the "contact" faces $\mathcal{T}_x^k$. If this parallelepiped is sufficiently far away from the boundary of Ω, then Figure 4 shows a cross section of a neighbourhood of $\mathcal{P}_\varepsilon^{x,k}$, surrounding $\mathcal{T}_x^k$, a section which is parallel to the yOz plane and which is passing through $y_\varepsilon^{x,k} + \delta\overline{e_1}$, where $\delta \in I_p := \left[-\dfrac{p\varepsilon-\varepsilon^\alpha}{2p}, \dfrac{p\varepsilon-\varepsilon^\alpha}{2p}\right]$. Nevertheles, if the parallelepiped $\mathcal{P}_\varepsilon^{x,k}$ is close to $\partial\Omega$, then the same argument will work, since we have relations (2.2.5) and (2.2.8).

Let $\mathcal{T}_x^k(\delta)$ be

$$\mathcal{T}_x^k(\delta) = \left\{ y_\varepsilon^{x,k} + \delta\overline{e_1} + y\overline{e_2} + z\overline{e_3} \;\middle|\; -\frac{\varepsilon^\alpha}{2q} \le y \le \frac{\varepsilon^\alpha}{2q};\ -\frac{\varepsilon^\alpha}{2r} \le z \le \frac{\varepsilon^\alpha}{2r} \right\},$$

which represents the centered white rectangle from Figure 4.

Let $\mathcal{V}_x^k(\delta)$ be

$$\mathcal{V}_x^k(\delta) = \left\{ y_\varepsilon^{x,k} + \delta\overline{e_1} + y\overline{e_2} + z\overline{e_3} \;\middle|\; -\varepsilon + \frac{\varepsilon^\alpha}{2q} \le y \le \varepsilon - \frac{\varepsilon^\alpha}{2q};\ -\varepsilon + \frac{\varepsilon^\alpha}{2r} \le z \le \varepsilon - \frac{\varepsilon^\alpha}{2r} \right\} \setminus \mathcal{T}_x^k(\delta),$$

which represents the darker shaded area from Figure 4, containing only liquid crystal particles, that is $\mathcal{V}_x^k(\delta) \subset \Omega_\varepsilon$, for any $\delta \in I_p$.

In our case, $\mathcal{V}_x^k(\delta)$ plays the role of $b\mathcal{P} \setminus a\mathcal{P}$ from Lemma 2.4.3.

If for every $\delta \in I_p$, we apply the translation $y_\varepsilon^k + \delta\overline{e_1}$ to the origin of the system, then for

$$\mathcal{P} = \{0\} \times \left[-\frac{1}{2q}, \frac{1}{2q}\right] \times \left[-\frac{1}{2r}, \frac{1}{2r}\right],$$

we can choose $a = \varepsilon^\alpha$, therefore $\varepsilon^\alpha\mathcal{P} = \mathcal{T}_x^k(\delta)$. In order to choose b, we assume: $\dfrac{b}{2r} \le \varepsilon - \dfrac{\varepsilon^\alpha}{2r}$ and $\dfrac{b}{2q} \le \varepsilon - \dfrac{\varepsilon^\alpha}{2q}$, that is: $b \le 2q\varepsilon - \varepsilon^\alpha$ and $b \le 2r\varepsilon - \varepsilon^\alpha$. Since $p, q, r \ge 1$, we can choose $b = 2\varepsilon - \varepsilon^\alpha$. In this way, we have $b\mathcal{P} \setminus a\mathcal{P} \subset \mathcal{V}_x^k(\delta)$ and we also have $b \ge a \Leftrightarrow 2\varepsilon - \varepsilon^\alpha \ge \varepsilon^\alpha \Leftrightarrow \alpha \ge 1$.

Therefore, we can apply Lemma 2.4.2 for Q with $a = \varepsilon^\alpha$, $b = 2\varepsilon - \varepsilon^\alpha$ and $\mathcal{P}$ defined as before, hence:

$$\oint_{\partial\mathcal{T}_x^k(\delta)}|Q|^n ds \lesssim \frac{2\varepsilon^\alpha}{(2\varepsilon-\varepsilon^\alpha)^2-\varepsilon^{2\alpha}}\int_{b\mathcal{P}\setminus\mathcal{T}_x^k(\delta)}|Q|^n dx + \frac{n}{2}\int_{b\mathcal{P}\setminus\mathcal{T}_x^k(\delta)}(|Q|^{2n-2}+|\nabla Q|^2)dx$$

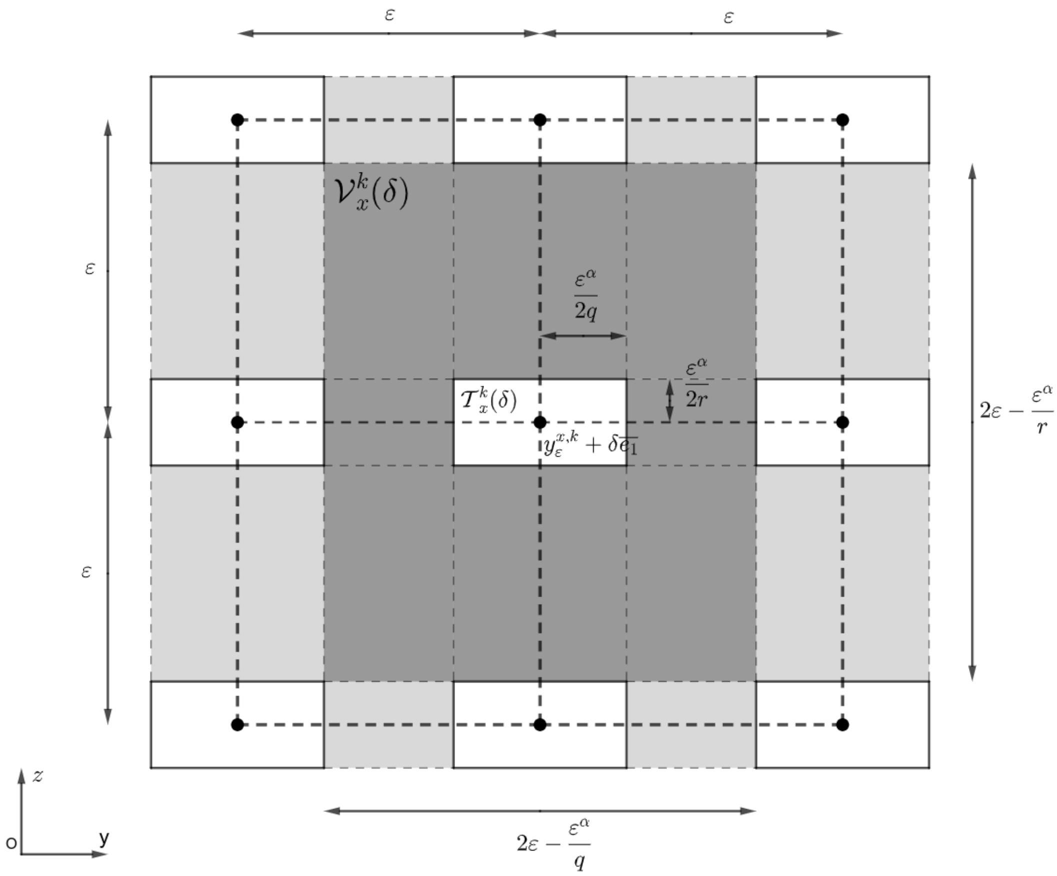


Figure 4: Cross section of the scaffold, parallel to yOz plane, passing through $y_\varepsilon^{x,k} + \delta\overline{e_1}$. The gray shaded areas represent the liquid crystal and the white rectangles represent the sections of the parts of the scaffold nearby.

and since $b\mathcal{P} \subset \mathcal{V}_x^k(\delta)$, we have:

$$\oint_{\partial\mathcal{T}_x^k(\delta)} |Q|^n \mathrm{d}s \lesssim \frac{\varepsilon^\alpha}{2\varepsilon(\varepsilon - \varepsilon^\alpha)} \int_{\mathcal{V}_x^k(\delta)} |Q|^n \mathrm{d}x + \frac{n}{2} \int_{\mathcal{V}_x^k(\delta)} \left(|Q|^{2n-2} + |\nabla Q|^2\right) \mathrm{d}x,$$

for every $\delta \in I_p$. Integrating now with respect to δ over I_p, we get:

$$\begin{aligned}\int_{I_p} \left(\oint_{\partial\mathcal{T}_x^k(\delta)} |Q|^n \mathrm{d}s \right) \mathrm{d}\delta &\lesssim \frac{\varepsilon^\alpha}{2\varepsilon(\varepsilon - \varepsilon^\alpha)} \int_{I_p} \left(\int_{\mathcal{V}_x^k(\delta)} |Q|^n \mathrm{d}x \right) \mathrm{d}\delta + \\ &\quad + \frac{n}{2} \int_{I_p} \left(\int_{\mathcal{V}_x^k(\delta)} \left(|Q|^{2n-2} + |\nabla Q|^2\right) \mathrm{d}x \right) \mathrm{d}\delta \\ \int_{\mathcal{T}_x^k} |Q|^n \mathrm{d}\sigma &\lesssim \frac{\varepsilon^\alpha}{2\varepsilon(\varepsilon - \varepsilon^\alpha)} \int_{\mathcal{U}_x^k} |Q|^n \mathrm{d}x + \frac{n}{2} \int_{\mathcal{U}_x^k} \left(|Q|^{2n-2} + |\nabla Q|^2\right) \mathrm{d}x,\end{aligned}$$

where $\mathcal{U}_x^k := \bigcup_{\delta \in I_p} \mathcal{V}_x^k(\delta) \subset \Omega_\varepsilon$ is now a three dimensional object. Hence:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\mathcal{T}_x^k} |Q|^n \mathrm{d}\sigma \lesssim \frac{n}{2} \cdot \frac{\varepsilon^{2-\alpha}}{1 - \varepsilon^{\alpha-1}} \int_{\mathcal{U}_x^k} \left(|Q|^{2n-2} + |\nabla Q|^2\right) \mathrm{d}x + \frac{1}{2(1 - \varepsilon^{\alpha-1})^2} \int_{\mathcal{U}_x^k} |Q|^n \mathrm{d}x.$$

Repeating the same argument for all the other "connecting parallelepipeds" of the scaffold and considering the fact that parts of $\mathcal{U}_x^k$ are added only up to four times (by constructing the same sets for the nearby "connecting parallelepipeds" from the scaffold), then the conclusion follows. □

Since we are interested in the homogenised material, it is useful to consider maps defined on the entire Ω and for this we use the harmonic extension operator $E_\varepsilon : H_g^1(\Omega_\varepsilon, \mathcal{S}_0) \to H_g^1(\Omega, \mathcal{S}_0)$, defined as follows: for $Q \in H_g^1(\Omega_\varepsilon, \mathcal{S}_0)$, we take $E_\varepsilon Q \equiv Q$ in Ω_ε and inside $\mathcal{N}_\varepsilon$, $E_\varepsilon Q$ solves the following PDE:

$$\begin{cases} \Delta E_\varepsilon Q = 0 & \text{in } \mathcal{N}_\varepsilon \\ E_\varepsilon Q \equiv Q & \text{on } \partial\mathcal{N}_\varepsilon \end{cases} \tag{2.4.1}$$

Since $\mathcal{N}_\varepsilon$ has a Lipschitz boundary, we can apply Theorem 4.19 from [35] and see that there exists a unique solution $E_\varepsilon Q \in H^1(\mathcal{N}_\varepsilon)$ to the problem (2.4.1). Hence the operator E_ε is well defined. Moreover, from (2.4.1), we can see that $E_\varepsilon Q$ verifies:

$$\|\nabla E_\varepsilon Q\|_{L^2(\mathcal{N}_\varepsilon)} = \min\left\{ \|\nabla u\|_{L^2(\mathcal{N}_\varepsilon)} \;\middle|\; u \in H^1(\mathcal{N}_\varepsilon),\ u = Q \text{ on } \partial\mathcal{N}_\varepsilon \right\}. \tag{2.4.2}$$

Our aim is now to prove that the extension operator E_ε is uniformly bounded with respect to $\varepsilon > 0$. More specifically, we prove that the following lemma holds.

Lemma 2.4.4. There exists a constant $C > 0$ such that $\|\nabla E_\varepsilon Q\|_{L^2(\Omega)} \leq C\|\nabla Q\|_{L^2(\Omega_\varepsilon)}$ for any $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is suitably small enough, and for any $Q \in H_g^1(\Omega_\varepsilon, \mathcal{S}_0)$.

Proof. By Subsection 2.7.3, we know that there exists $v \in H^1(\Omega)$ such that:

$$\begin{cases} v \equiv Q \text{ in } \Omega_\varepsilon \\ v = Q \text{ on } \partial\mathcal{N}_\varepsilon \\ \|\nabla v\|_{L^2(\Omega)} \lesssim \|\nabla Q\|_{L^2(\Omega_\varepsilon)}. \end{cases}$$

Using relation (2.4.2), we see that

$$\|\nabla E_\varepsilon Q\|_{L^2(\mathcal{N}_\varepsilon)} \leq \|\nabla v\|_{L^2(\mathcal{N}_\varepsilon)}$$

and because $E_\varepsilon Q \equiv Q$ in Ω_ε, we have $E_\varepsilon Q \equiv v \equiv Q$ in Ω_ε and therefore:

$$\|\nabla E_\varepsilon Q\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} \lesssim \|\nabla Q\|_{L^2(\Omega_\varepsilon)}.$$

□

2.4.2 ZERO CONTRIBUTION FROM THE SURFACE TERMS DEPENDING ON THE "INNER PARALLELEPIPEDS"

In order to describe the *surface energy*, we need a better description of $\partial\mathcal{N}_\varepsilon$, therefore we analyse what faces from every parallelepiped constructed are in contact with the liquid crystal. More precisely, the liquid crystal is in contact with the scaffold:

- on only four of the six faces of the "connecting parallelepipeds", centered in points from $\mathcal{Y}_\varepsilon$, that is, on every $\mathcal{T}_x^k$, $\mathcal{T}_y^l$ and $\mathcal{T}_z^m$, defined in (2.7.6), (2.7.11) and (2.7.16);
- only on the edges of some of the "inner parallelepipeds", centered in some of the points from $\mathcal{X}_\varepsilon$, parallelepipeds which are not close to the boundary of Ω - we are reffering here to the "inner particles" which are not "visible" in Figure 2 - in this case, the interaction is neglected and let

$$N_{\varepsilon,1} \;=\; \text{the total number of parallelepipeds from this case;} \tag{2.4.3}$$

- on at most five of the six faces of some of the "inner parallelepipeds", centered in some of the points from $\mathcal{X}_\varepsilon$, parallelepipeds which are close to the boundary of Ω - we are reffering here to the "inner particles" which are "visible" in Figure 2 - in this case, let

$$N_{\varepsilon,2} \;=\; \text{the total number of parallelepipeds from this case;} \tag{2.4.4}$$

  and let

$$\begin{aligned}\mathcal{S}^i \;=&\; \text{the union of all the rectangles (at most five in this case) that}\\ &\text{are in contact with the liquid crystal material,}\end{aligned} \tag{2.4.5}$$

  for any $i \in \overline{1, N_{\varepsilon,2}}$.

From relations (2.4.3) and (2.4.4), we have $N_\varepsilon = N_{\varepsilon,1} + N_{\varepsilon,2}$. Using (2.7.6), (2.7.11), (2.7.16) and (2.4.5), we can write $\partial\mathcal{N}_\varepsilon = \partial\mathcal{N}_\varepsilon^{\mathcal{S}} \cup \partial\mathcal{N}_\varepsilon^{\mathcal{T}}$, where:

$$\partial\mathcal{N}_\varepsilon^{\mathcal{S}} = \left(\bigcup_{i=1}^{N_{\varepsilon,2}} \mathcal{S}^i\right) \text{ and } \partial\mathcal{N}_\varepsilon^{\mathcal{T}} = \left(\bigcup_{k=1}^{X_\varepsilon} \mathcal{T}_x^k\right) \cup \left(\bigcup_{l=1}^{Y_\varepsilon} \mathcal{T}_y^l\right) \cup \left(\bigcup_{m=1}^{Z_\varepsilon} \mathcal{T}_z^m\right). \tag{2.4.6}$$

Let $J_\varepsilon[Q]$ be the *surface energy* term from (2.1.1) and let us split this term into two parts:

$$J_\varepsilon[Q] = J_\varepsilon^{\mathcal{S}}[Q] + J_\varepsilon^{\mathcal{T}}[Q], \tag{2.4.7}$$

where

$$J_\varepsilon^{\mathcal{S}}[Q] = \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon^{\mathcal{S}}} f_s(Q,\nu)\mathrm{d}\sigma = \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} f_s(Q,\nu)\mathrm{d}\sigma, \tag{2.4.8}$$

using (2.4.6), and

$$J_\varepsilon^{\mathcal{T}}[Q] = \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial \mathcal{N}_\varepsilon^{\mathcal{T}}} f_s(Q, \nu) \mathrm{d}\sigma, \tag{2.4.9}$$

which can be also expressed using (2.4.6) as

$$J_\varepsilon^{\mathcal{T}}[Q] = J_\varepsilon^X[Q] + J_\varepsilon^Y[Q] + J_\varepsilon^Z[Q], \tag{2.4.10}$$

where:

$$\begin{cases} J_\varepsilon^X[Q] &= \dfrac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \displaystyle\sum_{k=1}^{X_\varepsilon} \int_{\mathcal{T}_x^k} f_s(Q, \nu) \mathrm{d}\sigma; \\ J_\varepsilon^Y[Q] &= \dfrac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \displaystyle\sum_{l=1}^{Y_\varepsilon} \int_{\mathcal{T}_y^l} f_s(Q, \nu) \mathrm{d}\sigma; \\ J_\varepsilon^Z[Q] &= \dfrac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \displaystyle\sum_{m=1}^{Z_\varepsilon} \int_{\mathcal{T}_z^m} f_s(Q, \nu) \mathrm{d}\sigma. \end{cases} \tag{2.4.11}$$

In this section, we prove that the surface term $J_\varepsilon^{\mathcal{S}}$ has a negligible contribution to the homogenised material, that is $J_\varepsilon^{\mathcal{S}}[Q] \to 0$ as $\varepsilon \to 0$, for any $Q \in H_g^1(\Omega, \mathcal{S}_0)$, since we can use the extension operator E_ε defined in the previous subsection.

We start by proving if $Q : \overline{\Omega} \to \mathcal{S}_0$ is a bounded, Lipschitz map, then $J_\varepsilon^{\mathcal{S}}[Q] \to 0$ as $\varepsilon \to 0$ and then, by a density argument, for all $Q \in H_g^1(\Omega, \mathcal{S}_0)$.

Lemma 2.4.5. Let $Q : \overline{\Omega} \to \mathcal{S}_0$ be a bounded, Lipschitz map. Then $J_\varepsilon^{\mathcal{S}}[Q] \to 0$, as $\varepsilon \to 0$, where $J_\varepsilon^{\mathcal{S}}$ is defined in (2.4.7) and in (2.4.8).

Proof. By (2.4.8), we have:

$$\begin{aligned} \left| \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} f_s(Q(t), \nu) \mathrm{d}\sigma(t) \right| &\leq \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} \left| f_s(Q(t), \nu) \right| \mathrm{d}\sigma(t) \\ &\leq \frac{C\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} \left(|Q|^4(t) + 1 \right) \mathrm{d}\sigma(t) \\ &\leq \frac{C\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\partial \mathcal{C}_\varepsilon^i} \left(|Q|^4(t) + 1 \right) \mathrm{d}\sigma(t) \\ &\leq \frac{\varepsilon^{3+\alpha}}{\varepsilon - \varepsilon^\alpha} \cdot \frac{2C(p+q+r)}{pqr} \cdot \left(\|Q\|_{L^\infty(\overline{\Omega})}^4 + 1 \right) \cdot \sum_{i=1}^{N_{\varepsilon,2}} \int_{\partial \mathcal{C}} \mathrm{d}\sigma(t) \\ &\leq \frac{\varepsilon^{3+\alpha}}{\varepsilon - \varepsilon^\alpha} \cdot \left(\|Q\|_{L^\infty(\overline{\Omega})}^4 + 1 \right) \cdot \frac{2C(p+q+r)}{pqr} \cdot \sigma(\partial \mathcal{C}) \cdot N_{\varepsilon,2}, \end{aligned}$$

where $\partial \mathcal{C}$ represents the surface of the model particle $\mathcal{C}$ defined in (2.2.2), $\mathcal{C}_\varepsilon^i$ represents the "inner parallelepipeds" constructed in relation (2.2.7), $N_{\varepsilon,2}$ is defined in (2.4.4) and C is the ε-independent constant given from the inequality that states that f_s has a quartic growth in Q, which can be obtained from assumption (A_7). We have also used that Q is bounded on $\overline{\Omega}$. In the proof of Proposition 2.7.3, we obtain $N_{\varepsilon,2} \leq \dfrac{L_0 l_0 + l_0 h_0 + h_0 L_0}{\varepsilon^2}$, hence:

$$\left| \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} f_s(Q(t), \nu) \mathrm{d}\sigma(t) \right| < C' \cdot \frac{\varepsilon^{3+\alpha}}{\varepsilon - \varepsilon^\alpha} \cdot \frac{L_0 l_0 + L_0 h_0 + l_0 h_0}{\varepsilon^2} \Rightarrow$$
$$\Rightarrow \left| \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \sum_{i=1}^{N_{\varepsilon,2}} \int_{\mathcal{S}^i} f_s(Q(t), \nu) \mathrm{d}\sigma(t) \right| < C'' \cdot \frac{\varepsilon^\alpha}{1 - \varepsilon^{\alpha-1}} \to 0 \text{ as } \varepsilon \to 0,$$

since $\alpha \in \left(1, \dfrac{3}{2}\right)$, where L_0, l_0 and h_0 are defined in (2.2.1) and C' and C'' are ε-independent constants.

□

Lemma 2.4.6. For any $Q \in H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$, we have $J^{\mathcal{S}}_\varepsilon[Q] \to 0$ as $\varepsilon \to 0$.

Proof. Let $(Q_j)_{j\geq 1}$ be a sequence of smooth maps that converge strongly in $H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$ to Q. By Lemma 2.4.5, we have $J^{\mathcal{S}}_\varepsilon[Q_j] \to 0$ as $\varepsilon \to 0$, for any $j \geq 1$. By assumption (A_7), we have on $\mathcal{N}^{\mathcal{S}}_\varepsilon$:

$$\begin{aligned} |f_s(Q_j, \nu) - f_s(Q, \nu)| &\leq |Q_j - Q| \left(|Q_j|^3 + |Q|^3 + 1 \right) \\ &\lesssim |Q_j - Q| \left(|Q_j - Q|^3 + |Q|^3 + 1 \right) \\ &\lesssim |Q_j - Q|^4 + |Q_j - Q| \left(|Q|^3 + 1 \right) \end{aligned}$$

Thanks to the continuity of the trace operator from $H^1(\Omega_\varepsilon)$ to $H^{1/2}(\partial\Omega_\varepsilon)$, the Sobolev embedding $H^{1/2}(\partial\Omega_\varepsilon) \hookrightarrow L^4(\partial\Omega_\varepsilon)$ and the strong convergence $Q_j \to Q$ in $H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$, we get that $Q_j \to Q$ a.e. on $\partial\mathcal{N}^{\mathcal{S}}_\varepsilon$, since $\partial\mathcal{N}^{\mathcal{S}}_\varepsilon \subset \partial\Omega_\varepsilon$. Therefore, there exists $\psi \in L^4(\partial\mathcal{N}^{\mathcal{S}}_\varepsilon)$ such that $|Q_j - Q| \leq \psi$ a.e. in $\partial\mathcal{N}^{\mathcal{S}}_\varepsilon$ and we can write:

$$|f_s(Q_j, \nu) - f_s(Q, \nu)| \lesssim \psi^4 + \psi \left(|Q|^3 + 1 \right) \tag{2.4.12}$$

on $\partial\mathcal{N}^{\mathcal{S}}_\varepsilon$, for every $j \geq 1$.

At the same time, we have the compact Sobolev embedding $H^{1/2}(\partial\Omega_\varepsilon) \hookrightarrow L^3(\partial\Omega_\varepsilon)$, therefore $|Q|^3$ is in $L^1(\partial\mathcal{N}^{\mathcal{S}}_\varepsilon)$. Hence, the right hand side from (2.4.12) is in $L^1(\partial\mathcal{N}^{\mathcal{S}}_\varepsilon)$ and we can apply the Lebesgue dominated convergence theorem and get:

$$\lim_{j\to+\infty} \int_{\partial\mathcal{N}^{\mathcal{S}}_\varepsilon} |f_s(Q_j, \nu) - f_s(Q, \nu)| \mathrm{d}\sigma = 0,$$

for any $\varepsilon > 0$ fixed.

Now, because for $\varepsilon \to 0$ we get $|\partial\mathcal{N}^{\mathcal{S}}_\varepsilon| \to 0$ (according to Proposition 2.7.3) and $\dfrac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \to 0$, the conclusion follows. □

Therefore, from now on we omit the term $J_\varepsilon^{\mathcal{S}}$ from the free energy functional and we only study the behaviour of:

$$\mathcal{F}_\varepsilon^{\mathcal{T}}[Q] := \int_{\Omega_\varepsilon} \big(f_e(\nabla Q) + f_b(Q)\big)\mathrm{d}x + \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} f_s(Q,\nu)\mathrm{d}\sigma,$$

which we denote simply by $\mathcal{F}_\varepsilon[Q]$, but we keep the same notation for surfaces generated by the scaffold.

2.4.3 EQUICOERCIVITY OF $\mathcal{F}_\varepsilon$

Proposition 2.4.1. Suppose that the assumptions (A_1)-(A_7) hold and also that there exists $\mu > 0$ such that $f_b(Q) \geq \mu|Q|^6 - C$, for any $Q \in \mathcal{S}_0$. Let $Q \in H_g^1(\Omega_\varepsilon, \mathcal{S}_0)$ satisfy $\mathcal{F}_\varepsilon[Q] \leq M$, for some ε-independent constant. Then there holds

$$\int_{\Omega_\varepsilon} |\nabla Q|^2 \leq C_M$$

for $\varepsilon > 0$ small enough and for some $C_M > 0$ depending only on M, f_e, f_b, f_s and Ω.

Proof. Assumption (A_6) ensures that $|f_s(Q,\nu)| \lesssim |Q|^4 + 1$, therefore:

$$J_\varepsilon^{\mathcal{T}}[Q] \geq -C_1 \cdot \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} (|Q|^4 + 1)\mathrm{d}\sigma \geq -C_1 \cdot \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q|^4 \mathrm{d}\sigma - C_1 \cdot C_s,$$

according to Proposition 2.7.2. Using Lemma 2.4.3 with $n = 4$, we have:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q|^4 \mathrm{d}\sigma \lesssim \frac{2\varepsilon^{2-\alpha}}{(1-\varepsilon^{\alpha-1})} \int_{\Omega_\varepsilon} (|Q|^6 + |\nabla Q|^2)\mathrm{d}x + \frac{1}{2(1-\varepsilon^{\alpha-1})^2} \int_{\Omega_\varepsilon} |Q|^4 \mathrm{d}x,$$

hence

$$J_\varepsilon^{\mathcal{T}}[Q] \geq -C_1 \cdot C_2 \cdot \frac{2\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}} \int_{\Omega_\varepsilon} (|Q|^6 + |\nabla Q|^2)\mathrm{d}x - C_1 \cdot C_2 \cdot \frac{1}{2(1-\varepsilon^{\alpha-1})^2} \int_{\Omega_\varepsilon} |Q|^4 \mathrm{d}x - C_1 \cdot C_s.$$

At the same time, from the generalised version of the Hölder's inequality and from the fact that Ω is bounded, we have

$$\left(\int_{\Omega_\varepsilon} |Q|^4 \mathrm{d}x\right)^{1/4} \leq |\Omega_\varepsilon|^{1/12} \cdot \left(\int_{\Omega_\varepsilon} |Q|^6 \mathrm{d}x\right)^{1/6} \Rightarrow \int_{\Omega_\varepsilon} |Q|^4 \mathrm{d}x < |\Omega|^{1/3} \left(\int_{\Omega_\varepsilon} |Q|^6 \mathrm{d}x\right)^{2/3}$$

and so

$$J_\varepsilon^{\mathcal{T}}[Q] \geq -C_3 \cdot \frac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}} \int_{\Omega_\varepsilon} (|Q|^6 + |\nabla Q|^2)\mathrm{d}x - C_3 \cdot \frac{1}{2(1-\varepsilon^{\alpha-1})^2} \cdot |\Omega|^{1/3} \left(\int_{\Omega_\varepsilon} |Q|^6 \mathrm{d}x\right)^{2/3} - C_3, \tag{2.4.13}$$

where $C_3 = \max\{C_1 \cdot C_2,\ C_1 \cdot C_s\}$.

Since $f_b(Q) \geq \mu|Q|^6 - C$, $f_e(\nabla Q) \geq \lambda_e^{-1}|\nabla Q|^2$ (according to (A_5) and (A_6)) and $|\Omega_\varepsilon| \leq |\Omega|$, we have:

$$\int_{\Omega_\varepsilon} \left(f_b(Q) + f_e(\nabla Q)\right)\mathrm{d}x \geq \mu \int_{\Omega_\varepsilon} |Q|^6 \mathrm{d}x + \lambda_e^{-1} \int_{\Omega_\varepsilon} |\nabla Q|^2 \mathrm{d}x - C|\Omega| \tag{2.4.14}$$

and because $\mathcal{F}_\varepsilon[Q] \leq M$, combining (2.4.13) and (2.4.14), we obtain

$$\left(\lambda_e^{-1} - C_3 \cdot \frac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\right) \int_{\Omega_\varepsilon} |\nabla Q|^2 \mathrm{d}x \leq h_\varepsilon\left(\left(\int_{\Omega_\varepsilon} |Q|^6 \mathrm{d}x\right)^{1/3}\right), \tag{2.4.15}$$

where

$$h_\varepsilon(t) = t^2 \cdot \left(C_4(\varepsilon) + t \cdot C_5(\varepsilon)\right) + C_6,$$

for any $t \geq 0$, with $C_4(\varepsilon) = \dfrac{C_3 \cdot |\Omega|^{1/3}}{2(1-\varepsilon^{\alpha-1})^2}$, $C_5(\varepsilon) = C_3 \cdot \dfrac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}} - \mu$ and $C_6 = \left(M + C_3 + C|\Omega|\right)$. As $\varepsilon \to 0$, we have $C_4(\varepsilon) \searrow \dfrac{C_3 \cdot |\Omega|^{1/3}}{2} > 0$ and $C_5(\varepsilon) \searrow (-\mu) < 0$. Hence, for $\varepsilon > 0$ small enough, we have:

$$\frac{C_3 \cdot |\Omega|^{1/3}}{2} < C_4(\varepsilon) < C_3 \cdot |\Omega|^{1/3} \text{ and } -\mu < C_5(\varepsilon) < -\frac{\mu}{2} < 0. \tag{2.4.16}$$

Let $t_0(\varepsilon)$ be the solution of the equation $C_4(\varepsilon) + t \cdot C_5(\varepsilon) = 0$. We prove that $h_\varepsilon(t)$ is bounded from above on $[0, +\infty)$. Computing the critical points of h_ε, it is easy to check that $2t_0(\varepsilon)/3$ is the point in which the function attains its maximum over $[0, +\infty)$, which is:

$$\max\{h_\varepsilon(t) : t \in [0, +\infty)\} = \frac{4C_4^3(\varepsilon)}{27C_5^2(\varepsilon)} + C_6 < \frac{4}{27} \cdot C_3^3 \cdot |\Omega| \cdot \frac{4}{\mu^2} + C_6,$$

using (2.4.16). Therefore, the function h_ε is bounded from above on $[0, +\infty)$.

Using the same arguments we can see that $\lambda_e^{-1} - C_3 \cdot \dfrac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}$ is also bounded from below, away from 0, for $\varepsilon > 0$ small enough, and from here the conclusion follows, based on relation (2.4.15). □

In the end of this subsection, we present a situation in which if $\alpha > 2$, then the energy becomes unbounded from below. For simplicity, we choose the scaffold to be symmetric with $p = q = r = 1$. Let us consider the following free energy functional:

$$\mathcal{G}_\varepsilon(u) = \int_{\Omega_\varepsilon} \left(|\nabla u|^2 + k|u|^{2l-2}\right) \mathrm{d}x - \delta \cdot \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} \int_{\partial\mathcal{N}_\varepsilon} |u|^l \,\mathrm{d}\sigma,$$

where $k > 0$ and $l \in (2, 4)$, for $u : \Omega \to \mathbb{R}$, $u \in H^1(\Omega_\varepsilon)$.

Lemma 2.4.7. For any $l \in (2, 4)$, $2 < \alpha < \dfrac{4}{l-2}$, $k > 0$, $\delta > 0$, there holds:

$$\inf\{\mathcal{G}_\varepsilon(u) \;:\; u \in H^1(\Omega_\varepsilon), u = 0 \text{ on } \partial\Omega\} \to -\infty \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\mathcal{C}_{3/2} = [-3/2, 3/2]^3$ and we recall that $\mathcal{C} = [-1,1]^3$. Let us consider now $\varphi \in C_c^\infty(\mathcal{C}_{3/2})$ such that $\varphi \equiv 1$ on $\mathcal{C}$.

Let us now choose an "inner parallelepiped" from the scaffold such that it is close to the boundary of Ω and such that it has less than 6 adjacent "connecting parallelepipeds". In this case, some of its faces are in contact with the nematic liquid crystal. Let us denote this "inner parallelepiped" by $\mathcal{C}_\varepsilon^0$ and its center x_ε^0. We can write, based on 2.7.1, that:

$$\mathcal{C}_\varepsilon^0 = x_\varepsilon^0 + \left[-\frac{\varepsilon^\alpha}{2}, \frac{\varepsilon^\alpha}{2}\right]^3 = x_\varepsilon^0 + \frac{\varepsilon^\alpha}{2}\mathcal{C}.$$

Let us define the following function:

$$u_\varepsilon(x) = \varepsilon^{-\alpha/2-\beta}\varphi\left(\frac{2}{\varepsilon^\alpha}(x - x_\varepsilon^0)\right), \ \forall x \in \Omega,$$

where $\beta > 0$ will be chosen later.

Let us also define:

$$\mathcal{R}_\varepsilon^0 = x_\varepsilon^0 + \frac{3}{2}\cdot\frac{\varepsilon^\alpha}{2}\cdot\mathcal{C}.$$

Due to assumption (A_3) and our choice of α, we have that $\mathcal{C}_\varepsilon^0 \subset \mathcal{R}_\varepsilon^0 \subset \Omega$.

From the definition of u_ε, we have that $u_\varepsilon \equiv 1$ in $\mathcal{C}_\varepsilon^0$. Moreover, $u_\varepsilon \equiv 0$ in $\Omega \setminus \mathcal{R}_\varepsilon^0$ and $u_\varepsilon \in H^1(\Omega)$.

Let us define now $\mathcal{T}_\varepsilon^0$ the faces of $\mathcal{C}_\varepsilon^0$ that are in contact with the nematic liquid crystal. Going back to the definition of the free energy functional, we have:

$$\mathcal{G}_\varepsilon(u_\varepsilon) \le \int_{\Omega_\varepsilon} \left(|\nabla u_\varepsilon|^2 + |u_\varepsilon|^{2l-2}\right) \mathrm{d}x - \delta\cdot\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\mathcal{T}_\varepsilon^0} |u_\varepsilon|^l \,\mathrm{d}\sigma,$$

since $\mathcal{T}_\varepsilon^0 \subset \partial\mathcal{N}_\varepsilon$.

Using the properties of u_ε, we obtain that:

$$\begin{aligned}\mathcal{G}_\varepsilon(u_\varepsilon) &\le \int_{\Omega_\varepsilon\cap\mathcal{R}_\varepsilon^0} \left(|\nabla u_\varepsilon|^2 + |u_\varepsilon|^{2l-2}\right) \mathrm{d}x - \delta\cdot\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\mathcal{T}_\varepsilon^0} |u_\varepsilon|^l \,\mathrm{d}\sigma \le \\ &\le \int_{\mathcal{R}_\varepsilon^0\setminus\mathcal{C}_\varepsilon^0} \left(|\nabla u_\varepsilon|^2 + |u_\varepsilon|^{2l-2}\right) \mathrm{d}x - \delta\cdot\varepsilon^{2-\alpha}\int_{\mathcal{T}_\varepsilon^0} |u_\varepsilon|^l \,\mathrm{d}\sigma := \overline{\mathcal{G}_\varepsilon}(u_\varepsilon),\end{aligned}$$

since $\Omega_\varepsilon \cap \mathcal{R}_\varepsilon^0 \subset \mathcal{R}_\varepsilon^0 \setminus \mathcal{C}_\varepsilon^0$ and, due to our choice of α, we have that $\varepsilon/2 < \varepsilon - \varepsilon^\alpha < \varepsilon$, which implies that $-\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha} < -\frac{\varepsilon^{3-\alpha}}{\varepsilon} = -\varepsilon^{2-\alpha}$.

Applying a change of variables, using the definition of u_ε and considering only one face of $\mathcal{T}_\varepsilon^0$, we get:

$$\overline{\mathcal{G}_\varepsilon}(u_\varepsilon) = \frac{1}{2}\cdot\varepsilon^{-2\beta}\int_{\mathcal{C}_{3/2}\setminus\mathcal{C}} |\nabla\varphi|^2 \,\mathrm{d}x + \frac{1}{8}\cdot\varepsilon^{(-\alpha/2-\beta)(2l-2)+3\alpha}\int_{\mathcal{C}_{3/2}\setminus\mathcal{C}} |\varphi|^{2l-2} \,\mathrm{d}x - \frac{\delta}{4}\cdot\varepsilon^{2+\alpha-l(\alpha/2+\beta)}.$$

Since in the last equation the integrals are bounded, in order to prove that $\overline{\mathcal{G}_\varepsilon}$ is unbounded from below, we want to show that there exists $\beta > 0$ such that:

$$\begin{cases} \varepsilon^{-2\beta} < \varepsilon^{2+\alpha-\alpha\cdot\frac{l}{2}-\beta\cdot l} \\ \varepsilon^{\alpha\cdot(4-l)-2\beta\cdot(l-1)} < \varepsilon^{2+\alpha-\alpha\cdot\frac{l}{2}-\beta\cdot l} \end{cases} \Leftrightarrow 2-\alpha\cdot\frac{2-l}{2} < \beta\cdot(l-2) < \alpha\cdot\frac{6-l}{2}-2.$$

But the last inequality is equivalent with choosing $\alpha \in \left(2, \dfrac{4}{l-2}\right)$, hence, $\overline{\mathcal{G}_\varepsilon}$ is unbounded from below and, since $\mathcal{G}_\varepsilon(\cdot) \leq \overline{\mathcal{G}_\varepsilon}(\cdot)$, we conclude our proof. □

Remark 2.4.1. The previous lemma has been adapted from Lemma 3.6 from [28]. In a similar fashion, one can use Lemma 3.7 from [28] to prove similar results for the case in which $\alpha > \dfrac{4}{l-2}$.

2.4.4 LOWER SEMI-CONTINUITY OF $\mathcal{F}_\varepsilon$

Proposition 2.4.2. Suppose that the assumptions (A_1)-(A_7) are satisfied. Then, the following statement holds: for any positive $M > 0$, there exists $\varepsilon_0(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0(M))$ and for any sequence $(Q_j)_{j\in\mathbb{N}}$ from $H^1(\Omega_\varepsilon, \mathcal{S}_0)$ that converges H^1-weakly to a function $Q \in H^1(\Omega_\varepsilon, \mathcal{S}_0)$ and which satisfies $\|\nabla Q_j\|_{L^2(\Omega_\varepsilon)} \leq M$ for any $j \in \mathbb{N}$, then

$$\mathcal{F}_\varepsilon[Q] \leq \liminf_{j\to+\infty} \mathcal{F}_\varepsilon[Q_j].$$

Proof. The proof of Proposition 2.4.2 follows the same steps as in [28]. We prove this proposition on each component of $\mathcal{F}_\varepsilon$. Before that, let

$$\omega_0 = \liminf_{j\to+\infty} \int_{\Omega_\varepsilon} |\nabla Q_j|^2 \mathrm{d}x - \int_{\Omega_\varepsilon} |\nabla Q|^2 \mathrm{d}x.$$

Since $Q_j \rightharpoonup Q$ in H^1, then $\nabla Q_j \rightharpoonup \nabla Q$ in L^2, therefore $\omega_0 \geq 0$. Moreover, up to extracting a subsequence, we can assume that

$$\int_{\Omega_\varepsilon} |\nabla Q_j|^2 \mathrm{d}x \to \int_{\Omega_\varepsilon} |\nabla Q|^2 \mathrm{d}x + \omega_0 \tag{2.4.17}$$

as $j \to +\infty$.

From the assumption (A_5), we have that f_e is strongly convex, that is for $\theta > 0$ small enough, $\tilde{f}_e(D) := f_e(D) - \theta|D|^2$ is a convex function from $\mathcal{S}_0 \otimes \mathbb{R}^3$ to $[0, +\infty)$. In this case, the functional $\int_{\Omega_\varepsilon} \tilde{f}_e(\cdot)\mathrm{d}x$ is lower semicontinuous. Therefore

$$\liminf_{j\to+\infty} \int_{\Omega_\varepsilon} \tilde{f}_e(\nabla Q_j)\mathrm{d}x \geq \int_{\Omega_\varepsilon} \tilde{f}_e(\nabla Q)\mathrm{d}x,$$

from which we get

$$\liminf_{j\to+\infty}\int_{\Omega_\varepsilon} f_e(\nabla Q_j)\mathrm{d}x - \int_{\Omega_\varepsilon} f_e(\nabla Q)\mathrm{d}x \geq$$
$$\geq \left(\liminf_{j\to+\infty}\int_{\Omega_\varepsilon} \tilde{f}_e(\nabla Q_j)\mathrm{d}x - \int_{\Omega_\varepsilon} \tilde{f}_e(\nabla Q)\mathrm{d}x\right) + \theta\omega_0 \geq 0. \tag{2.4.18}$$

Since $Q_j \rightharpoonup Q$ in $H^1(\Omega_\varepsilon)$ and the injection $H^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon)$ is compact, then we can assume, up to extracting a subsequence, that $Q_j \to Q$ a.e. in Ω_ε. Then, from the assumption (A_6), we can see that the sequence $\big(f_b(Q_j)\big)_{j\in\mathbb{N}}$ satisfies all the conditions from Fatou's lemma, therefore:

$$\liminf_{j\to+\infty}\int_{\Omega_\varepsilon} f_b(Q_j)\mathrm{d}x \geq \int_{\Omega_\varepsilon}\liminf_{j\to+\infty} f_b(Q_j)\mathrm{d}x = \int_{\Omega_\varepsilon} f_b(Q)\mathrm{d}x. \tag{2.4.19}$$

Regarding the *surface energy*, we split $\partial\mathcal{N}_\varepsilon^{\mathcal{T}}$ into:

$$A_j = \{x \in \partial\mathcal{N}_\varepsilon^{\mathcal{T}} \;:\; |Q_j(x) - Q(x)| \leq |Q(x)| + 1\}$$
$$B_j = \partial\mathcal{N}_\varepsilon^{\mathcal{T}} \setminus A_j = \{x \in \partial\mathcal{N}_\varepsilon^{\mathcal{T}} \;:\; |Q_j(x) - Q(x)| > |Q(x)| + 1\},$$

for any $j \in \mathbb{N}$.

Using (A_7), we have

$$\int_{A_j} |f_s(Q_j,\nu) - f_s(Q,\nu)|\mathrm{d}\sigma \leq \int_{A_j}\left(|Q_j|^3 + |Q|^3 + 1\right)\cdot|Q_j - Q|\mathrm{d}\sigma$$
$$\leq \int_{A_j}\left((|Q_j - Q| + |Q|)^3 + |Q|^3 + 1\right)\cdot\left(|Q| + 1\right)\mathrm{d}\sigma$$
$$\lesssim \int_{A_j}(|Q|^3 + 1)(|Q| + 1)\mathrm{d}\sigma \lesssim \int_{A_j}(|Q|^4 + 1)\mathrm{d}\sigma.$$

Then due to the continuous embedding of $H^{1/2}(\partial\mathcal{N}_\varepsilon)$ into $L^4(\partial\mathcal{N}_\varepsilon)$:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{A_j}|f_s(Q_j,\nu) - f_s(Q,\nu)|\mathrm{d}\sigma \lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{A_j}(|Q|^4 + 1)\mathrm{d}\sigma < +\infty,$$

according also to Proposition 2.7.2. At the same time, the compact embedding $H^{1/2}(\partial\mathcal{N}_\varepsilon) \hookrightarrow L^2(\partial\mathcal{N}_\varepsilon)$ and the continuity of the trace operator from $H^1(\Omega_\varepsilon)$ into $H^{1/2}(\partial\mathcal{N}_\varepsilon)$ grants that $Q_j \to Q$ a.e. on $\partial\mathcal{N}_\varepsilon$, up to extracting a subsequence. We can now apply the dominated convergence theorem and get:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{A_j}|f_s(Q_j,\nu) - f_s(Q,\nu)|\mathrm{d}\sigma \to 0 \text{ as } j \to +\infty. \tag{2.4.20}$$

Regarding the B_j sets, we have, according to (A_7):

$$\begin{aligned} |f_s(Q_j,\nu) - f_s(Q,\nu)| &\leq \lambda_s |Q_j - Q|\big(|Q_j|^3 + |Q|^3 + 1\big) \\ &\lesssim |Q_j - Q|\big(|Q_j|^3 + |Q_j - Q|^3 + 1\big) \text{ (using } |Q| + 1 < |Q_j - Q|) \\ &\lesssim |Q_j - Q|\big(|Q_j - Q|^3 + |Q|^3 + |Q_j - Q|^3 + 1\big) \\ &\lesssim |Q_j - Q|\big(|Q_j - Q|^3 + 1\big) \lesssim |Q_j - Q|^4. \end{aligned}$$

Using Lemma 2.4.3 for $(Q_j - Q)$ with $n = 4$, we have:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{B_j} |f_s(Q_j,\nu) - f_s(Q,\nu)| d\sigma \lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{B_j} |Q_j - Q|^4 d\sigma \lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q_j - Q|^4 d\sigma$$

$$\lesssim \frac{2\varepsilon^{2-\alpha}}{1 - \varepsilon^{\alpha-1}}\int_{\Omega_\varepsilon} |Q_j - Q|^6 + |\nabla Q_j - \nabla Q|^2 dx + \frac{1}{2(1 - \varepsilon^{\alpha-1})^2}\int_{\Omega_\varepsilon} |Q_j - Q|^4 dx.$$

Since $H^1(\Omega_\varepsilon)$ is compactly embedded into $L^4(\Omega_\varepsilon)$ and $Q_j \rightharpoonup Q$ in $H^1(\Omega_\varepsilon)$, then $Q_j \to Q$ in $L^4(\Omega_\varepsilon)$ and so

$$\frac{1}{2(1 - \varepsilon^{\alpha-1})^2}\int_{\Omega_\varepsilon} |Q_j - Q|^4 dx \to 0, \text{ as } j \to +\infty.$$

For the term containing $|Q_j - Q|^6$, we proceed in the following way:

$$\begin{aligned} \int_{\Omega_\varepsilon} |Q_j - Q|^6 dx &= \int_{\Omega_\varepsilon} |E_\varepsilon(Q_j - Q)|^6 dx \leq \int_\Omega |E_\varepsilon(Q_j - Q)|^6 dx = \|E_\varepsilon(Q_j - Q)\|^6_{L^6(\Omega)} \\ &\lesssim \|E_\varepsilon(Q_j - Q)\|^6_{H^1(\Omega)} \text{ (by the continuous injection } H^1(\Omega) \subset L^6(\Omega)) \\ &\lesssim \|E_\varepsilon(Q_j - Q)\|^6_{H^1_0(\Omega)} \text{ (because } Q_j \equiv Q \text{ on } \partial\Omega) \\ &\lesssim \|\nabla E_\varepsilon(Q_j - Q)\|^6_{L^2(\Omega)} = \left(\int_\Omega |\nabla E_\varepsilon(Q_j - Q)|^2 dx\right)^3 \\ &\lesssim \left(\int_{\Omega_\varepsilon} |\nabla Q_j - \nabla Q|^2 dx\right)^3 \text{ (using Lemma 2.4.4).} \end{aligned}$$

Now, because $\|\nabla Q_j\|_{L^2(\Omega_\varepsilon)} \leq M$, then:

$$\int_{\Omega_\varepsilon} |\nabla Q_j - \nabla Q|^2 dx \leq \int_{\Omega_\varepsilon} (|\nabla Q_j|^2 + |\nabla Q|^2) dx \lesssim M^2$$

and therefore

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\int_{B_j} |f_s(Q_j,\nu) - f_s(Q,\nu)| d\sigma \lesssim \frac{2\varepsilon^{2-\alpha}}{1 - \varepsilon^{\alpha-1}}(1 + M^4)\int_{\Omega_\varepsilon} |\nabla Q_j - \nabla Q|^2 dx + o(1). \quad (2.4.21)$$

Using that $Q_j \rightharpoonup Q$ in $H^1(\Omega_\varepsilon)$ and (2.4.17), we obtain that $\int_{\Omega_\varepsilon} |\nabla Q_j - \nabla Q|^2 dx \to \omega_0$ as $j \to +\infty$ and combining this with (2.4.20) and (2.4.21), we get:

$$\liminf_{j\to+\infty} J_\varepsilon^{\mathcal{T}}[Q_j] - J_\varepsilon^{\mathcal{T}}[Q] \geq -C_M \cdot \omega_0 \cdot \frac{\varepsilon^{2-\alpha}}{1 - \varepsilon^{\alpha-1}}, \quad (2.4.22)$$

where C_M is a constant dependent of M and independent of ε.

According to (2.4.18), (2.4.19) and (2.4.22), we finally obtain that

$$\liminf_{j\to+\infty} \mathcal{F}_\varepsilon[Q_j] - \mathcal{F}_\varepsilon[Q] \geq \left(\theta - C_M \frac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\right)\omega_0$$

and since $\dfrac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}} \to 0$ as $\varepsilon \to 0$, the conclusion follows. □

2.5 CONVERGENCE OF LOCAL MINIMISERS

2.5.1 POINTWISE CONVERGENCE OF THE SURFACE INTEGRAL

The aim of this section is to prove the following statement:

Theorem 2.5.1. Suppose that the assumptions (A_1)-(A_7) are satisfied. Then, for any bounded, Lipschitz map $Q : \overline{\Omega} \to \mathcal{S}_0$, there holds $J_\varepsilon^{\mathcal{T}}[Q] \to J_0[Q]$ as $\varepsilon \to 0$, where

$$J_0[Q] = \int_\Omega f_{hom}(Q)\mathrm{d}x. \tag{2.5.1}$$

Proof. Let us fix a bounded, Lipschitz map $Q : \overline{\Omega} \to \mathcal{S}_0$ and let $\tilde{J}_\varepsilon$ be the following functional:

$$\tilde{J}_\varepsilon[Q] = \frac{\varepsilon^{3-\alpha}}{\varepsilon - \varepsilon^\alpha}\left(\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{T}_x^k} f_s(Q(y_\varepsilon^{x,k}),\nu)\mathrm{d}\sigma + \sum_{l=1}^{Y_\varepsilon}\int_{\mathcal{T}_y^l} f_s(Q(y_\varepsilon^{y,l}),\nu)\mathrm{d}\sigma + \sum_{m=1}^{Z_\varepsilon}\int_{\mathcal{T}_z^m} f_s(Q(y_\varepsilon^{z,m}),\nu)\mathrm{d}\sigma\right), \tag{2.5.2}$$

where $y_\varepsilon^{x,k}$, $y_\varepsilon^{y,l}$ and $y_\varepsilon^{z,m}$ are defined in (2.7.3), (2.7.8) and (2.7.13), $\mathcal{T}_x^k$, $\mathcal{T}_y^l$ and $\mathcal{T}_z^m$ are defined in (2.7.6), (2.7.11) and (2.7.16) and X_ε, Y_ε and Z_ε are defined in (2.7.1), (2.7.7) and (2.7.12).

We prove that $\tilde{J}_\varepsilon[Q] \to J_0[Q]$ and that $\left|J_\varepsilon^{\mathcal{T}}[Q] - \tilde{J}_\varepsilon[Q]\right| \to 0$ as $\varepsilon \to 0$, for any Q with the properties set earlier.

Let

$$\begin{cases} \Psi^X(Q(\tau_0)) = \displaystyle\int_{\mathcal{C}^x} f_s(Q(\tau_0),\nu(\tau))\mathrm{d}\sigma(\tau) \\ \Psi^Y(Q(\tau_0)) = \displaystyle\int_{\mathcal{C}^y} f_s(Q(\tau_0),\nu(\tau))\mathrm{d}\sigma(\tau) \\ \Psi^Z(Q(\tau_0)) = \displaystyle\int_{\mathcal{C}^z} f_s(Q(\tau_0),\nu(\tau))\mathrm{d}\sigma(\tau) \end{cases} \tag{2.5.3}$$

for any $\tau_0 \in \Omega$, where $\mathcal{C}^x$, $\mathcal{C}^y$ and $\mathcal{C}^z$ are defined in (2.2.3). Because f_s is continuous on $\mathcal{S}_0 \times \mathbb{S}^2$, then Ψ^X, Ψ^Y and Ψ^Z are also continuous. In this case, for example, the first sum from (2.5.2), denoted as $\tilde{J}_\varepsilon^X$, becomes:

$$\begin{aligned}\tilde{J}_\varepsilon^X[Q] &= \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{T}_x^k} f_s(Q(y_\varepsilon^{x,k}),\nu)\mathrm{d}\sigma \\ &= \frac{(p-\varepsilon^{\alpha-1})}{pr(1-\varepsilon^{\alpha-1})}\cdot\varepsilon^3\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{C}^y} f_s(Q(y_\varepsilon^{x,k}),\nu)\mathrm{d}\sigma + \frac{(p-\varepsilon^{\alpha-1})}{pq(1-\varepsilon^{\alpha-1})}\cdot\varepsilon^3\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{C}^z} f_s(Q(y_\varepsilon^{x,k}),\nu)\mathrm{d}\sigma \\ &= \frac{(p-\varepsilon^{\alpha-1})}{pr(1-\varepsilon^{\alpha-1})}\cdot\int_\Omega \Psi^Y(Q(\tau))\mathrm{d}\mu_\varepsilon^X(\tau) + \frac{(p-\varepsilon^{\alpha-1})}{pq(1-\varepsilon^{\alpha-1})}\cdot\int_\Omega \Psi^Z(Q(\tau))\mathrm{d}\mu_\varepsilon^X(\tau), \end{aligned} \tag{2.5.4}$$

where μ_ε^X is defined in (2.2.9), that is, assumption (A_4).

According to (A_4), as $\varepsilon \to 0$, μ_ε^X converges weakly* to the Lebesgue measure restricted to Ω and because Ψ^Y and Ψ^Z are continuous, then:

$$\tilde{J}_\varepsilon^X[Q] \to \frac{1}{r}\int_\Omega \Psi^Y(Q(\tau))\mathrm{d}\tau + \frac{1}{q}\int_\Omega \Psi^Z(Q(\tau))\mathrm{d}\tau.$$

Computing in a similar way $\tilde{J}_\varepsilon^y$ and $\tilde{J}_\varepsilon^z$, we get:

$$\tilde{J}_\varepsilon[Q] \to \int_\Omega\left(\frac{q+r}{qr}\Psi^X(Q(\tau)) + \frac{p+r}{pr}\Psi^Y(Q(\tau)) + \frac{p+q}{pq}\Psi^Z(Q(\tau))\right)\mathrm{d}\tau = \int_\Omega f_{hom}(Q(\tau))\mathrm{d}\tau$$

which implies that $\tilde{J}_\varepsilon[Q] \to J_0[Q]$, where J_0 is defined in (2.5.1). For $J_\varepsilon^X[Q]$ and $\tilde{J}_\varepsilon^X[Q]$, we have:

$$\begin{aligned}\left|J_\varepsilon^X[Q]-\tilde{J}_\varepsilon^X[Q]\right| &\leq \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{T}_x^k}\left|f_s(Q(\tau),\nu(\tau)) - f_s(Q(y_\varepsilon^{x,k}),\nu(\tau))\right|\mathrm{d}\sigma(\tau) \\ &\lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\sum_{k=1}^{X_\varepsilon}\int_{\mathcal{T}_x^k}\left(|Q(\tau)|^3+|Q(y_\varepsilon^{x,k})|^3+1\right)|Q(\tau)-Q(y_\varepsilon^{x,k})|\mathrm{d}\sigma(\tau) \\ &\lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\cdot\left(\|Q\|^3_{L^\infty(\overline{\Omega})}+1\right)\cdot\mathrm{Lip}(Q)\cdot\sum_{k=1}^{X_\varepsilon}\sigma(\mathcal{T}_x^k)\cdot\mathrm{diam}(\mathcal{T}_x^k), \end{aligned} \tag{2.5.5}$$

using that Q is bounded on $\overline{\Omega}$ and where $\mathrm{Lip}(Q)$ is the Lipschitz constant of Q, $\sigma(\mathcal{T}_x^k)$ is the total area of $\mathcal{T}_x^k$ and $\mathrm{diam}(\mathcal{T}_x^k)$ is the diameter of $\mathcal{T}_x^k$, which coincides with the diameter of the parallelepiped $\mathcal{P}_\varepsilon^{x,k}$, defined in (2.7.4). Hence

$$\left|J_\varepsilon^X[Q]-\tilde{J}_\varepsilon^X[Q]\right| \lesssim \frac{2(r+q)}{pqr}\cdot\frac{p\varepsilon-\varepsilon^\alpha}{\varepsilon-\varepsilon^\alpha}\cdot X_\varepsilon\cdot\varepsilon^3\cdot\sqrt{\left(\varepsilon-\frac{\varepsilon^\alpha}{p}\right)^2+\left(\frac{\varepsilon^\alpha}{q}\right)^2+\left(\frac{\varepsilon^\alpha}{r}\right)^2} \tag{2.5.6}$$

Now, as $\varepsilon \to 0$, we have: $\sqrt{\left(\varepsilon-\frac{\varepsilon^\alpha}{p}\right)^2+\left(\frac{\varepsilon^\alpha}{q}\right)^2+\left(\frac{\varepsilon^\alpha}{r}\right)^2} \to 0$; $\frac{p\varepsilon-\varepsilon^\alpha}{\varepsilon-\varepsilon^\alpha} = \frac{p-\varepsilon^{\alpha-1}}{1-\varepsilon^{1-\alpha}} \to p$ because $1<\alpha$; $X_\varepsilon\cdot\varepsilon^3 < \left(\frac{L_0}{\varepsilon}-1\right)\cdot\frac{l_0}{\varepsilon}\cdot\frac{h_0}{\varepsilon}\cdot\varepsilon^3 = L_0l_0h_0-\varepsilon l_0h_0$, according to Proposition 2.7.1, where L_0, l_0 and h_0 are defined in (2.2.1). Since X_ε is positive, we see that:

$$0 \leq \lim_{\varepsilon\to 0}\left(X_\varepsilon\cdot\varepsilon^3\right) \leq \lim_{\varepsilon\to 0}\left(L_0l_0h_0-\varepsilon l_0h_0\right) = L_0l_0h_0 < +\infty. \tag{2.5.7}$$

.

Therefore $J_\varepsilon^X[Q] \to \tilde{J}_\varepsilon^X[Q]$ as $\varepsilon \to 0$. We get the same result for the other two components, from which we conclude.

□

Remark 2.5.1. It is easy to see that if we replace the coefficient $\frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon-\varepsilon^\alpha)}$ of the *surface energy* term J_ε with:

- $\frac{\varepsilon^3}{(\varepsilon-\varepsilon^\alpha)^2} \to 0$, then $J_\varepsilon^X[Q] \to 0$ as $\varepsilon \to 0$;
- $\frac{\varepsilon^3}{\varepsilon^{2\alpha}} = \varepsilon^{3-2\alpha} \to 0$, then $J_\varepsilon^X[Q] \to +\infty$ as $\varepsilon \to 0$.

In both cases, we lose the convergence $J_\varepsilon^{\mathcal{T}}[Q] \to J_0[Q]$.

2.5.2 Γ-convergence of the approximating free energies

Lemma 2.5.1. Suppose that the assumption (A_7) is satisfied. Let Q_1 and Q_2 from $H_g^1(\Omega, \mathcal{S}_0)$ be such that

$$\max\{\|\nabla Q_1\|_{L^2(\Omega)}, \|\nabla Q_2\|_{L^2(\Omega)}\} \leq M \tag{2.5.8}$$

for some ε-independent constant M. Then, for ε sufficiently small, we have:

$$\left|J_\varepsilon^{\mathcal{T}}[Q_2] - J_\varepsilon^{\mathcal{T}}[Q_1]\right| \leq C_M\left(\varepsilon^{1/2-\alpha/4} + \|Q_2 - Q_1\|_{L^4(\Omega)}\right) \tag{2.5.9}$$

for some $C_M > 0$ depending only on M, f_s, Ω, $\mathcal{C}$ and g.

Proof. According to (A_7) and Hölder inequality, we have:

$$\left|J_\varepsilon^{\mathcal{T}}[Q_2] - J_\varepsilon^{\mathcal{T}}[Q_1]\right| \lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} \left(|Q_1|^3 + |Q_2|^3 + 1\right)|Q_2 - Q_1|\mathrm{d}\sigma$$

$$\lesssim \frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\left(\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q_2-Q_1|^4\mathrm{d}\sigma\right)^{1/4}\left(\left(\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q_1|^4\mathrm{d}\sigma\right)^{3/4} + \left(\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q_2|^4\mathrm{d}\sigma\right)^{3/4} + |\partial\mathcal{N}_\varepsilon^{\mathcal{T}}|^{3/4}\right).$$

If we make use of Lemma 2.4.3, then:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q_i|^4\mathrm{d}\sigma \lesssim \frac{2\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\int_{\Omega_\varepsilon}\left(|Q_i|^6 + |\nabla Q_i|^2\right)\mathrm{d}x + \frac{1}{2(1-\varepsilon^{\alpha-1})^2}\int_{\Omega_\varepsilon}|Q_i|^4\mathrm{d}x,$$

for any $i \in \{1,2\}$. By the continuous injection $\mathrm{H}^1(\Omega_\varepsilon)$ into $\mathrm{L}^6(\Omega_\varepsilon)$, we have

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}}|Q_i|^4\mathrm{d}\sigma \lesssim \frac{2\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\left(\|\nabla Q_i\|^2_{L^2(\Omega_\varepsilon)} + \|Q_i\|^6_{H^1(\Omega_\varepsilon)}\right) + \frac{1}{2(1-\varepsilon^{\alpha-1})^2}\|Q_i\|^4_{L^4(\Omega_\varepsilon)}. \tag{2.5.10}$$

Using the Poincaré inequality as in Theorem 4.4.7, page 193, from [80], the compact embedding
$H^1(\Omega_\varepsilon) \hookrightarrow L^4(\Omega_\varepsilon)$ and the fact that $\Omega_\varepsilon \subset \Omega$, we get for ε small enough:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q_i|^4 \mathrm{d}\sigma \lesssim \frac{2\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}\left(\|\nabla Q_i\|^2_{L^2(\Omega)} + \|\nabla Q_i\|^6_{L^2(\Omega)}\right) + \frac{1}{2(1-\varepsilon^{\alpha-1})^2}\|\nabla Q_i\|^4_{L^2(\Omega)}$$
$$\lesssim \frac{2\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}}(M^2+M^6) + \frac{1}{2(1-\varepsilon^{\alpha-1})^2}M^4,$$

where we used (2.5.8) and we can see that the right-hand side from the last inequality can be bounded in terms of M, since $\dfrac{\varepsilon^{2-\alpha}}{1-\varepsilon^{\alpha-1}} \searrow 0$ and $\dfrac{1}{(1-\varepsilon^{\alpha-1})^2} \searrow 1$ as $\varepsilon \to 0$. But since $\dfrac{1}{(1-\varepsilon^{\alpha-1})^2} \searrow 1$ as $\varepsilon \to 0$, we can choose $\varepsilon > 0$ such that $\dfrac{1}{(1-\varepsilon^{\alpha-1})^2} < 2$ and we can move the constant 2 under the "$\lesssim$" sign. Hence, the last relation can be written as:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q_i|^4 \mathrm{d}\sigma \lesssim \varepsilon^{2-\alpha}(M^2+M^6) + M^4.$$

In a similar fashion, using the same arguments as before for (2.5.10), we get in the case of $(Q_2 - Q_1)$:

$$\frac{\varepsilon^{3-\alpha}}{\varepsilon-\varepsilon^\alpha}\int_{\partial\mathcal{N}_\varepsilon^{\mathcal{T}}} |Q_2-Q_1|^4 \mathrm{d}\sigma \lesssim \varepsilon^{2-\alpha}(M^2+M^6) + \|Q_2-Q_1\|^4_{L^4(\Omega)}.$$

Using the same bounds as in (2.5.10), we conclude by observing that there exists a constant $C_M > 0$ such that:

$$\left|J_\varepsilon^{\mathcal{T}}[Q_2] - J_\varepsilon^{\mathcal{T}}[Q_1]\right| \le C_M \cdot \left(\left(\varepsilon^{2-\alpha}\right)^{1/4} + \|Q_2-Q_1\|_{L^4(\Omega)}\right)$$

□

Lemma 2.5.2. For any $Q \in H^1_g(\Omega, \mathcal{S}_0)$, there holds $J_\varepsilon^{\mathcal{T}}[Q] \to J_0[Q]$ as $\varepsilon \to 0$.

Proof. Let $(Q_j)_{j\ge1}$ be a sequence of smooth functions that converge strongly to Q in $H^1_g(\Omega, \mathcal{S}_0)$. Then there holds:

$$\left|J_\varepsilon^{\mathcal{T}}[Q] - J_0[Q]\right| \le \left|J_\varepsilon^{\mathcal{T}}[Q] - J_\varepsilon^{\mathcal{T}}[Q_j]\right| + \left|J_\varepsilon^{\mathcal{T}}[Q_j] - J_0[Q_j]\right| + \left|J_0[Q_j] - J_0[Q]\right|.$$

From Lemma 2.5.1, we have that

$$\left|J_\varepsilon^{\mathcal{T}}[Q] - J_\varepsilon^{\mathcal{T}}[Q_j]\right| \lesssim \varepsilon^{1/2-\alpha/4} + \|Q-Q_j\|_{L^4(\Omega)}.$$

and we recall that for $\varepsilon \to 0$ we have $\varepsilon^{1/2-\alpha/4} \to 0$ because $\alpha \in (1,2)$.

Since the $(Q_j)_{j\ge1}$ converge strongly in $H^1_g(\Omega)$, from the compact Sobolev embedding, we get that $Q_j \to Q$ in $L^4(\Omega)$ as $j \to +\infty$, therefore $Q_j \to Q$ a.e. in Ω.

From Theorem 2.5.1, we obtain that $J_\varepsilon^{\mathcal{T}}[Q_j] \to J_0[Q_j]$ as $\varepsilon \to 0$, for any $j \ge 1$.

For the last term, we can write $\left|J_0[Q_j] - J_0[Q]\right| \leq \int_\Omega \left|f_{hom}[Q_j] - f_{hom}[Q]\right| \mathrm{d}x$. In here, we have: f_{hom} is continuous, $Q_j \to Q$ a.e. in Ω and f_{hom} has a quartic growth in Q (because f_s has the same growth), which implies that: $\left|f_{hom}[Q_j]\right| \lesssim |Q_j|^4 + 1$. At the same time, we can assume that there exists $\psi \in L^1(\Omega)$ such that $|Q_j|^4 \leq \psi$, for any $j \geq 1$, a.e. in Ω. Therefore, we can apply the Lebesgue dominated convergence theorem and get that $J_0[Q_j] \to J_0[Q]$ as $j \to +\infty$.

Combining the results from above, we obtain:

$$\limsup_{\varepsilon \to 0} \left|J_\varepsilon^{\mathcal{T}}[Q] - J_\varepsilon^{\mathcal{T}}[Q_j]\right| \lesssim \|Q - Q_j\|_{L^4(\Omega)} + \left|J_0[Q_j] - J_0[Q]\right| \to 0, \quad \text{as } j \to +\infty,$$

from which we conclude.

□

We now prove that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_0$ as $\varepsilon \to 0$, with respect to the weak H^1-topology.

Proposition 2.5.1. Suppose that the assumptions (A_1)-(A_7) are satisfied. Let $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$ be such that $E_\varepsilon Q_\varepsilon \rightharpoonup Q$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. Then:

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] \geq \mathcal{F}_0[Q], \qquad \lim_{\varepsilon \to 0} J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] = J_0[Q].$$

Proof. The proof follows the same steps as in Proposition 4.2. from [28].

Since $E_\varepsilon Q_\varepsilon \rightharpoonup Q$ in $H^1(\Omega)$, then $\left(E_\varepsilon Q_\varepsilon\right)_{\varepsilon > 0}$ is a bounded sequence in $H^1(\Omega)$. Therefore, we can choose a subsequence $\left(E_{\varepsilon_j} Q_{\varepsilon_j}\right)_{j \geq 1} \subset \left(E_\varepsilon Q_\varepsilon\right)_{\varepsilon > 0}$ such that

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] = \lim_{j \to +\infty} \mathcal{F}_{\varepsilon_j}[Q_{\varepsilon_j}].$$

Furthermore, by the compact embeddings $H^1(\Omega) \hookrightarrow L^s(\Omega)$, with $s \in [1, 6)$, we have that $E_{\varepsilon_j} Q_{\varepsilon_j} \to Q$ strongly in $L^s(\Omega)$, for any $s \in [1, 6)$. As a result, we also obtain that $E_{\varepsilon_j} Q_{\varepsilon_j} \to Q$ a.e. in Ω. We denote the subsequence $E_{\varepsilon_j} Q_{\varepsilon_j}$ as $E_\varepsilon Q_\varepsilon$ for the ease of notation.

Now, according to (A_5), we have:

$$\begin{aligned}
\int_{\Omega_\varepsilon} \left(f_e(\nabla Q_\varepsilon) - f_e(\nabla Q)\right) \mathrm{d}x &\geq \int_{\Omega_\varepsilon} \nabla f_e(\nabla Q) : (\nabla Q_\varepsilon - \nabla Q) \mathrm{d}x = \\
&= \int_\Omega \nabla f_e(\nabla Q) : (\nabla Q_\varepsilon - \nabla Q) \mathrm{d}x - \int_{\mathcal{N}_\varepsilon} \nabla f_e(\nabla Q) : (\nabla Q_\varepsilon - \nabla Q) \mathrm{d}x \geq \\
&\geq \int_\Omega \nabla f_e(\nabla Q) : (\nabla Q_\varepsilon - \nabla Q) \mathrm{d}x - \left\|\nabla f_e(\nabla Q)\right\|_{L^2(\mathcal{N}_\varepsilon)} \cdot \left\|\nabla Q_\varepsilon - \nabla Q\right\|_{L^2(\mathcal{N}_\varepsilon)}
\end{aligned} \tag{2.5.11}$$

Because $Q \in H^1(\Omega)$, then $\nabla Q \in L^2(\Omega)$ and, according to (A_5), the relation $\left|\nabla f_e(\nabla Q)\right| \lesssim |\nabla Q| + 1$ implies that $\nabla f_e(\nabla Q) \in L^2(\Omega)$. Therefore, by the weak convergence $E_\varepsilon Q_\varepsilon \rightharpoonup Q$ in $H^1(\Omega)$, the first term from the right hand side in (2.5.11) goes to 0 as $\varepsilon \to 0$. The second term goes to 0 as well thanks additionally to the fact that the volume of the scaffold $\mathcal{N}_\varepsilon$ tends to 0 as $\varepsilon \to 0$, according to Proposition 2.7.1. Hence:

$$\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f_e(\nabla Q_\varepsilon) \mathrm{d}x \geq \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f_e(\nabla Q) \mathrm{d}x = \int_\Omega f_e(\nabla Q) \mathrm{d}x. \tag{2.5.12}$$

For the bulk potential we apply Fatou's lemma, since $f_b(Q_\varepsilon)\chi_{\Omega_\varepsilon} \to f_b(Q)$ a.e. in Ω (because f_b is continuous, $E_\varepsilon Q_\varepsilon \to Q$ a.e. in Ω and $|\mathcal{N}_\varepsilon| \to 0$, according to Proposition 2.7.1) and f_b is bounded from below (according to (A_6)), in order to obtain:

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f_b(Q_\varepsilon)\mathrm{d}x \geq \int_\Omega f_b(Q)\mathrm{d}x. \tag{2.5.13}$$

For the *surface energy*, we first use Lemma 2.5.1 in the following inequality:

$$\begin{aligned} \left|J_\varepsilon^{\mathcal{T}}[E_\varepsilon Q_\varepsilon] - J_0[Q]\right| &\leq \left|J_\varepsilon^{\mathcal{T}}[E_\varepsilon Q_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q]\right| + \left|J_\varepsilon^{\mathcal{T}}[Q] - J_0[Q]\right| \\ &\lesssim \varepsilon^{1/2-\alpha/4} + \left\|E_\varepsilon Q_\varepsilon - Q\right\|_{L^4(\Omega)} + \left|J_\varepsilon^{\mathcal{T}}[Q] - J_0[Q]\right|. \end{aligned}$$

Since we have $\varepsilon^{1/2-\alpha/4} \to 0$ for $\varepsilon \to 0$ (because $\alpha \in (1,2)$), then combining the result from Lemma 2.5.2 with the fact that $E_\varepsilon Q_\varepsilon \to Q$ strongly in $L^4(\Omega)$, we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] = J_0[Q]. \tag{2.5.14}$$

The proof is now complete, considering (2.5.12), (2.5.13) and (2.5.14). □

Proposition 2.5.2. Suppose that the assumptions (A_1)-(A_7) are verified. Then, for any $Q \in H_g^1(\Omega, \mathcal{S}_0)$, there exists a sequence $\left(Q_\varepsilon\right)_{\varepsilon>0}$ such that $Q_\varepsilon \in H^1(\Omega_\varepsilon)$, for any $\varepsilon > 0$, $E_\varepsilon Q_\varepsilon \rightharpoonup Q$ in $H^1(\Omega)$ and:

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] \leq \mathcal{F}_0[Q].$$

The sequence $\left(Q_\varepsilon\right)_{\varepsilon>0}$ is called a recovery sequence.

Proof. Let us define in this case $Q_\varepsilon = Q \cdot \chi_{\Omega_\varepsilon}$. Since $|\mathcal{N}_\varepsilon| \to 0$ as $\varepsilon \to 0$ (according to Proposition 2.7.1), then $\chi_{\Omega_\varepsilon} \to 1$ strongly in $L^1(\Omega)$ and we can apply Lebesgue's dominated converge theorem in order to obtain that:

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f_e(\nabla Q_\varepsilon) + f_b(Q_\varepsilon)\mathrm{d}x &= \int_\Omega f_e(\nabla Q) + f_b(Q)\mathrm{d}x \\ \lim_{\varepsilon\to 0} \left(\mathcal{F}_\varepsilon[Q] - J_\varepsilon^{\mathcal{T}}[Q]\right) &= \mathcal{F}_0[Q] - J_0[Q]. \end{aligned}$$

By Proposition 2.5.1, we have that $\lim_{\varepsilon\to 0} J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] = J_0[Q]$, hence the conclusion follows. □

Proposition 2.5.1 and Proposition 2.5.2 show that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_0$, as $\varepsilon \to 0$, with respect to the weak H^1 topology.

2.5.3 PROOF OF MAIN THEOREMS

Proof of Theorem 2.3.1. Let Q_0 from $H_g^1(\Omega, \mathcal{S}_0)$ be an isolated H^1-local minimiser for $\mathcal{F}_0$, that is, there exists $\delta_0 > 0$ such that $\mathcal{F}_0[Q_0] < \mathcal{F}_0[Q]$, for any $Q \in H_g^1(\Omega, \mathcal{S}_0)$, such that $0 < \left\|Q - Q_0\right\|_{H^1(\Omega)} \leq \delta_0$.

We would like to prove that for any $\varepsilon > 0$, there exists $Q_\varepsilon \in H^1_g(\Omega_\varepsilon, \mathcal{S}_0)$, which is a H^1-local minimiser for $\mathcal{F}_\varepsilon$, such that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1_g(\Omega, \mathcal{S}_0)$ as $\varepsilon \to 0$.

For this, let

$$\mathcal{B}_\varepsilon := \left\{ Q \in H^1_g(\Omega_\varepsilon, \mathcal{S}_0) \ : \ \left\| E_\varepsilon Q - Q_0 \right\|_{H^1(\Omega)} \leq \delta_0 \right\}.$$

Using Mazur's lemma, we can show that the set $\mathcal{B}_\varepsilon$ is sequentially weakly closed in $H^1(\Omega_\varepsilon)$. Then, by Proposition 2.4.2, we can see that, for ε small enough, $\mathcal{F}_\varepsilon$ is lower semicontinuous on $\mathcal{B}_\varepsilon$ and, by Proposition 2.4.1, is also coercive on $\mathcal{B}_\varepsilon$, since any $Q \in \mathcal{B}_\varepsilon$ has $\left\| \nabla Q \right\|_{L^2(\Omega_\varepsilon)} < \left\| \nabla Q_0 \right\|_{L^2(\Omega)} + \delta_0$. Hence, for any ε sufficiently small, the functional $\mathcal{F}_\varepsilon$ admits at least one minimiser Q_ε from $\mathcal{B}_\varepsilon$.

Firstly, we prove that $E_\varepsilon Q_\varepsilon \rightharpoonup Q_0$ weakly in $H^1(\Omega)$, as $\varepsilon \to 0$.

Let $\mathcal{B}_0 := \left\{ Q \in H^1_g(\Omega, \mathcal{S}_0) \ : \ \left\| Q - Q_0 \right\|_{H^1(\Omega)} \leq \delta_0 \right\}$. Because $Q_\varepsilon \in \mathcal{B}_\varepsilon$, then $(E_\varepsilon Q_\varepsilon)_{\varepsilon>0}$ represents a bounded sequence in $H^1(\Omega)$, hence there exists a subsequence, which we still denote $(E_\varepsilon Q_\varepsilon)_{\varepsilon>0}$ for the ease of notation, that converges weakly to a $\tilde{Q} \in \mathcal{B}_0$. We show that $\tilde{Q} = Q_0$.

Since $E_\varepsilon Q_\varepsilon \rightharpoonup \tilde{Q}$ in $H^1_g(\Omega, \mathcal{S}_0)$, we can apply Proposition 2.5.1 and get:

$$\mathcal{F}_0[\tilde{Q}] \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] \leq \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon].$$

But Q_ε is a minimiser of $\mathcal{F}_\varepsilon$ on $\mathcal{B}_\varepsilon$, therefore, since $Q_0\big|_{\Omega_\varepsilon} \in \mathcal{B}_\varepsilon$, we get that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] \leq \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_0] = \lim_{\varepsilon \to 0} \left(\int_{\Omega_\varepsilon} f_e(\nabla Q_0) + f_b(Q) \mathrm{d}x + \mathcal{J}^{\mathcal{T}}_\varepsilon[Q_0] \right) = \mathcal{F}_0[Q_0].$$

Hence, we have $\mathcal{F}_0[\tilde{Q}] \leq \mathcal{F}_0[Q_0]$. Because $\tilde{Q}$ is in $\mathcal{B}_0$, that is $\|\tilde{Q} - Q_0\|_{H^1(\Omega)} \leq \delta_0$, then by the definition of Q_0, we get that $\tilde{Q} = Q_0$.

We now prove that $E_\varepsilon Q_\varepsilon \to Q_0$ strongly in $H^1(\Omega)$, as $\varepsilon \to 0$.

By (A_5), there exists $\theta > 0$ such that the function $\tilde{f}_e(D) = f_e(D) - \theta |D|^2$ is convex. We can repeat the same arguments from Proposition 2.5.1, more specifically, steps (2.5.12) and (2.5.13), to get:

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \tilde{f}_e(\nabla Q_\varepsilon) \mathrm{d}x &\geq \int_\Omega \tilde{f}_e(\nabla Q_0) \mathrm{d}x, \\
\theta \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla Q_\varepsilon|^2 \mathrm{d}x &\geq \theta \int_\Omega |\nabla Q_0|^2 \mathrm{d}x, \\
\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f_b(Q_\varepsilon) \mathrm{d}x &\geq \int_\Omega f_b(Q_0) \mathrm{d}x.
\end{aligned}$$

From Proposition 2.5.1, we have that $J^{\mathcal{T}}_\varepsilon[Q_\varepsilon] \to J_0[Q_0]$ as $\varepsilon \to 0$. Also, from the proof that $\tilde{Q} = Q_0$, we can see that

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon[Q_\varepsilon] = \mathcal{F}_0[Q_0],$$

which implies that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla Q_\varepsilon|^2 \mathrm{d}x = \int_{\Omega}|\nabla Q_0|^2 \mathrm{d}x.$$

This shows us that

$$\nabla(E_\varepsilon Q_\varepsilon)\chi_{\Omega_\varepsilon} \to \nabla Q_0 \text{ strongly in } L^2(\Omega), \tag{2.5.15}$$

where $\chi_{\Omega_\varepsilon}$ is the characteristic function of Ω_ε.

We now show that $\nabla(E_\varepsilon Q_\varepsilon)\chi_{\mathcal{N}_\varepsilon}$ converges strongly to 0 in $L^2(\Omega)$. In order to prove this, in Subsection 2.7.3, we obtain that:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{P}_\varepsilon^{z,m})} \leq C\cdot\|\nabla Q_\varepsilon\|_{L^2(\mathcal{R}_\varepsilon^{z,m})},$$

where $\mathcal{P}_\varepsilon^{z,m}$ is a "connecting parallelepiped" elongated in the Oz direction and $\mathcal{R}_\varepsilon^{z,m}$ is a 3D object that "surrounds" $\mathcal{P}_\varepsilon^{z,m}$. More specifically, we have that:

$$\mathcal{P}_\varepsilon^{z,m} = y_\varepsilon^{z,m} + \left[-\frac{\varepsilon^\alpha}{2p},\frac{\varepsilon^\alpha}{2p}\right]\times\left[-\frac{\varepsilon^\alpha}{2q},\frac{\varepsilon^\alpha}{2q}\right]\times\left[-\frac{r\varepsilon-\varepsilon^\alpha}{2r},\frac{r\varepsilon-\varepsilon^\alpha}{2r}\right]$$

and

$$\mathcal{R}_\varepsilon^{z,m} = y_\varepsilon^{z,m} + \left(\left(\frac{3\varepsilon^\alpha}{4p}[-1,1]\times\frac{3\varepsilon^\alpha}{4q}[-1,1]\right)\Big\backslash\left(\frac{\varepsilon^\alpha}{2p}[-1,1]\times\frac{\varepsilon^\alpha}{2q}[-1,1]\right)\right)\times\left(\frac{r\varepsilon-\varepsilon^\alpha}{2r}[-1,1]\right),$$

where $y_\varepsilon^{z,m}$ is the center of $\mathcal{P}_\varepsilon^{z,m}$ and by $t[-1,1]$, for $t\in\mathbb{R}$, we understand the set $[-t,t]$.

In this way, we have that $|\mathcal{R}_\varepsilon^{z,m}| = \frac{5}{4}\cdot|\mathcal{P}_\varepsilon^{z,m}|$ and $\mathcal{R}_\varepsilon^{z,m}\subset\Omega_\varepsilon$. Moreover, if we take two objects of the type $\mathcal{R}_\varepsilon^{z,m}$, then it is easy to see that they are disjoint: in the Oz direction, $\mathcal{R}_\varepsilon^{z,m}$ has the same height as $\mathcal{P}_\varepsilon^{z,m}$ and in the xOy plane the "surrounding" objects are not touching because we consider $\varepsilon\to 0$ and, since $\alpha\in(1,2)$, we have $\varepsilon^\alpha << \varepsilon$. Applying this technique from Subsection 2.7.3 for all the "connecting parallelepipeds" of the type $\mathcal{R}_\varepsilon^{z,m}$, we obtain that there exists an ε-independent constant such that:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\cup\mathcal{P}_\varepsilon^{z,m})} \leq C\cdot\|\nabla Q_\varepsilon\|_{L^2(\cup\mathcal{R}_\varepsilon^{z,m})},$$

and repeating in the same way for all the other "connecting parallelepipeds", we obtain:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^{\mathcal{T}})} \leq C\cdot\|\nabla Q_\varepsilon\|_{L^2(\mathcal{R}_\varepsilon^{\mathcal{T}})},$$

where $\mathcal{R}_\varepsilon^{\mathcal{T}}$ represents the union of all "surrounding" objects for the "connecting parallelepipeds".

For the "inner parallelepipeds", we can apply the same technique as before. If the "inner parallelepiped" is not "visible" - meaning that is has six adjacent "connecting parallelepipeds" - then the "surrounding" object constructed is included in the union of the six adjacent "connecting parallelipepds" and their respective "surounding" bodies. If the "inner parallelepiped" is close to the boundary - meaning that it has strictly less than 6 adjacent "connecting parallelepipeds" - then the "surrounding" object constructed has a part included in the adjacent "connecting parallelepipeds" and their respective "surrounding" bodies, but also a part which was not taken into account until now. But this "surrounding" object will have its volume only 5/4 times bigger than the "inner parallelepiped" chosen and the same technique as for the "connecting parallelepipeds" can be applied. In this way, one can obtain the following inequality:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^{\mathcal{S}})} \leq C\|\nabla Q_\varepsilon\|_{L^2(\mathcal{R}_\varepsilon^{\mathcal{S}} \cup \mathcal{R}_\varepsilon^{\mathcal{T}})},$$

where the constant used is ε-independent and $\mathcal{R}_\varepsilon^{\mathcal{S}}$ is a 3D object with its volume tending to 0 as $\varepsilon \to 0$ and represents the union of all "surrounding" parts for the "inner parallelepipeds" that are not included in either the "connecting parallelepipeds" or their respective "surrounding" bodies.

Hence, we can obtain that:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{N}_\varepsilon)} \leq C \cdot \|\nabla Q_\varepsilon\|_{L^2(\mathcal{R}_\varepsilon)},$$

where $\mathcal{R}_\varepsilon = \mathcal{R}_\varepsilon^{\mathcal{T}} \cup \mathcal{R}_\varepsilon^{\mathcal{S}} \subset \Omega_\varepsilon$ with $|\mathcal{R}_\varepsilon| \to 0$ as $\varepsilon \to 0$.

Due to our definition of the extension operator E_ε, we have $E_\varepsilon Q_\varepsilon \equiv Q_\varepsilon$ in Ω_ε, so we can write the last inequality as follows:

$$\|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{N}_\varepsilon)} \leq C \cdot \|\nabla E_\varepsilon Q_\varepsilon\|_{L^2(\mathcal{R}_\varepsilon)}. \tag{2.5.16}$$

Now we want to prove that the right hand side term from the last inequality tends to 0 as $\varepsilon \to 0$. For this, we use:

$$\begin{aligned}\int_{\mathcal{R}_\varepsilon} |\nabla E_\varepsilon Q_\varepsilon|^2 \,dx &= \int_{\Omega_\varepsilon} |\nabla E_\varepsilon Q_\varepsilon|^2 \cdot \chi_{\mathcal{R}_\varepsilon} \,dx \leq \int_{\Omega_\varepsilon} 2 \cdot \left(|\nabla E_\varepsilon Q_\varepsilon - \nabla Q_0|^2 + |\nabla Q_0|^2\right) \cdot \chi_{\mathcal{R}_\varepsilon} \,dx \leq \\ &\leq 2\int_{\Omega_\varepsilon} |\nabla E_\varepsilon Q_\varepsilon - \nabla Q_0|^2 \,dx + 2\int_{\Omega} |\nabla Q_0|^2 \cdot \chi_{\mathcal{R}_\varepsilon} \,dx \\ &\leq 2\int_{\Omega} |(\nabla E_\varepsilon Q_\varepsilon)\chi_{\Omega_\varepsilon} - \nabla Q_0|^2 \,dx + 2\int_{\Omega} |\nabla Q_0|^2 \cdot \chi_{\mathcal{R}_\varepsilon} \,dx,\end{aligned}$$

where we have used that $\Omega_\varepsilon \subset \Omega$. Now the right hand side of the last inequality tends to 0: the first integral converges to 0 due to the fact that $(\nabla E_\varepsilon Q_\varepsilon)\chi_{\Omega_\varepsilon} \to \nabla Q_0$ strongly in $L^2(\Omega)$, according to (2.5.15); the second integral converges to 0 because we can apply the dominated convergence theorem, since $|\nabla Q_0|^2 \chi_{\mathcal{R}_\varepsilon}$ converges almost everywhere to 0 in Ω because $|\mathcal{R}_\varepsilon| \to 0$. Going back to (2.5.16), we obtain that $(\nabla E_\varepsilon Q_\varepsilon)\chi_{\mathcal{N}_\varepsilon} \to 0$ strongly in $L^2(\Omega)$, as $\varepsilon \to 0$.

Combining all the results, we obtain that $\nabla E_\varepsilon Q_\varepsilon \to \nabla Q_0$ strongly in $L^2(\Omega)$, hence $E_\varepsilon Q_\varepsilon$ converges strongly to Q_0 in $H^1(\Omega)$, since the weak convergence $E_\varepsilon Q_\varepsilon \rightharpoonup Q_0$ in $H^1(\Omega)$ automatically implies the strong convergence $E_\varepsilon Q_\varepsilon \to Q_0$ in $L^2(\Omega)$.

□

2.6 RATE OF CONVERGENCE

The aim of this section is the study the rate of convergence of the sequence $J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]$ to $J_0[Q_0]$, where $J_\varepsilon^{\mathcal{T}}$ is defined in (2.4.10) and in (2.4.11), J_0 is defined in (2.5.1) and $(Q_\varepsilon)_{\varepsilon>0}$ is a sequence from $H_g^1(\Omega,\mathcal{S}_0)$ that converges H^1-strongly to $Q \in H_g^1(\Omega,\mathcal{S}_0)$. We omit the term $J_\varepsilon^{\mathcal{S}}$ because in Subsection 2.4.2 we proved that this term has no contribution to the homogenised functional.

First, we recall some notations used in the previous sections. For a $Q \in H_g^1(\Omega,\mathcal{S}_0)$, we write $J_0[Q]$ in the following form:

$$\begin{aligned} J_0[Q] = &\int_\Omega \left(\frac{1}{r}\Psi^Y(Q) + \frac{1}{q}\Psi^Z(Q)\right)\mathrm{d}x + \\ &+ \int_\Omega \left(\frac{1}{r}\Psi^X(Q) + \frac{1}{p}\Psi^Z(Q)\right)\mathrm{d}x + \\ &+ \int_\Omega \left(\frac{1}{q}\Psi^X(Q) + \frac{1}{p}\Psi^Y(Q)\right)\mathrm{d}x, \end{aligned} \tag{2.6.1}$$

where Ψ^X, Ψ^Y and Ψ^Z are defined in (2.5.3). We also write $\tilde{J}_\varepsilon[Q]$, defined in (2.5.2), as:

$$\begin{aligned} \tilde{J}_\varepsilon[Q] = &\int_\Omega \left(\frac{p\varepsilon-\varepsilon^\alpha}{pr(\varepsilon-\varepsilon^\alpha)}\Psi^Y(Q) + \frac{p\varepsilon-\varepsilon^\alpha}{pq(\varepsilon-\varepsilon^\alpha)}\Psi^Z(Q)\right)\mathrm{d}\mu_\varepsilon^X + \\ &+ \int_\Omega \left(\frac{q\varepsilon-\varepsilon^\alpha}{qr(\varepsilon-\varepsilon^\alpha)}\Psi^X(Q) + \frac{q\varepsilon-\varepsilon^\alpha}{pq(\varepsilon-\varepsilon^\alpha)}\Psi^Z(Q)\right)\mathrm{d}\mu_\varepsilon^Y + \\ &+ \int_\Omega \left(\frac{r\varepsilon-\varepsilon^\alpha}{qr(\varepsilon-\varepsilon^\alpha)}\Psi^X(Q) + \frac{r\varepsilon-\varepsilon^\alpha}{pr(\varepsilon-\varepsilon^\alpha)}\Psi^Y(Q)\right)\mathrm{d}\mu_\varepsilon^Z, \end{aligned} \tag{2.6.2}$$

using (2.5.4) and the analogous formulae.

We suppose now that:

(H_1) the *surface energy density* f_s is locally Lipschitz continuous.

Using the assumption (A_7), from Section 2.2, we have:

$$|f_s(Q_1,\nu) - f_s(Q_2,\nu)| \lesssim |Q_2 - Q_1|(|Q_1|^3 + |Q_2|^3 + 1), \tag{2.6.3}$$

for any $Q_1, Q_2 \in \mathcal{S}_0$ and any $\nu \in \mathbb{S}^2$, and

$$|f_s(Q,\nu)| \lesssim |Q|^4 + 1, \tag{2.6.4}$$

for any $Q \in \mathcal{S}_0$ and any $\nu \in \mathbb{S}^2$.

We now have the following lemma:

Lemma 2.6.1. For any $K \in \{X, Y, Z\}$, the function Ψ^K is locally Lipschitz continuous and there holds:

$$|\Psi^K(Q)| \lesssim |Q|^4 + 1 \qquad |\nabla\Psi^K(Q)| \lesssim |Q|^3 + 1,$$

for any $Q \in \mathcal{S}_0$. Moreover, the function Ψ^K satisfies:

$$|\Psi^K(Q_1) - \Psi^K(Q_2)| \lesssim |Q_2 - Q_1|(|Q_1|^3 + |Q_2|^3 + 1),$$

for any $Q_1, Q_2 \in \mathcal{S}_0$.

Proof. The proof of this lemma follows immediatly, using the definitions of the functions Ψ^X, Ψ^Y and Ψ^Z from (2.5.3), the assumption (H_1) and the properties of the function f_s from (2.6.3) and (2.6.4). □

We recall now that the measures μ_ε^X, μ_ε^Y and μ_ε^Z, which are defined in (2.2.9), converge weakly*, as measures in $\mathbb{R}^3$, to the Lebesgue measure restricted to Ω, according to (A_4) from Section 2.2. We need to prescribe a rate of convergence and for this we use the $W^{-1,1}$-norm (that is, the dual Lipschitz norm, also known as flat norm in some contexts):

$$\mathbb{F}_\varepsilon := \max_{K\in\{X,Y,Z\}} \sup \left\{ \int_\Omega \varphi \mathrm{d}\mu_\varepsilon^K - \int_\Omega \varphi \mathrm{d}x \ : \ \varphi \in W^{1,\infty}(\Omega), \|\nabla\varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Lemma 2.6.2. There exists a constant $\lambda_{\text{flat}} > 0$ such that $\mathbb{F}_\varepsilon \leq \lambda_{\text{flat}}\varepsilon$ for any $\varepsilon > 0$.

Proof. Let $\varphi \in W^{1,\infty}(\Omega)$. Then, according to the definition of μ_ε^X from (2.2.9), we have:

$$\int_\Omega \varphi \mathrm{d}\mu_\varepsilon^X = \varepsilon^3 \sum_{k=1}^{X_\varepsilon} \varphi(y_\varepsilon^{x,k}) = \sum_{k=1}^{X_\varepsilon} \int_{y_\varepsilon^{x,k}+[-\varepsilon/2,\varepsilon/2]^3} \varphi(y_\varepsilon^{x,k}) \mathrm{d}x,$$

where in the last equality we integrate over the cube with length ε centered in $y_\varepsilon^{x,k}$. Let Ω_ε^X be the following set:

$$\Omega_\varepsilon^X := \bigcup_{k=1}^{X_\varepsilon} \left(y_\varepsilon^{x,k} + [-\varepsilon/2, \varepsilon/2]^3\right).$$

Hence, we can write:

$$\int_\Omega \varphi \mathrm{d}\mu_\varepsilon^X = \int_{\Omega_\varepsilon^X} \varphi(y_\varepsilon^{x,k}) \mathrm{d}x.$$

Then:

$$\begin{aligned} \left| \int_\Omega \varphi \mathrm{d}\mu_\varepsilon^X - \int_\Omega \varphi \mathrm{d}x \right| &\leq \int_{\Omega_\varepsilon^X} |\varphi - \varphi(y_\varepsilon^{x,k})| \mathrm{d}x + \int_{\Omega\setminus\Omega_\varepsilon^X} |\varphi| \mathrm{d}x \\ &\leq \frac{\varepsilon\sqrt{3}}{2} \|\nabla\varphi\|_{L^\infty(\Omega)} \cdot |\Omega_\varepsilon^X| + \|\varphi\|_{L^\infty(\Omega)} \cdot |\Omega \setminus \Omega_\varepsilon^X|, \end{aligned} \tag{2.6.5}$$

where $\frac{\varepsilon\sqrt{3}}{2}$ comes from the largest possible value for $|x - y_\varepsilon^{x,k}|$, with $x \in \left(y_\varepsilon^{x,k} + [-\varepsilon/2, \varepsilon/2]^3\right)$.

If we look now at the definition of the points $y_\varepsilon^{x,k}$ in (2.7.2), hence also at the definition of the points x_ε^i in (2.2.5) and (2.2.6), we observe that $\Omega \setminus \Omega_\varepsilon^X \subset \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$, therefore, we have $|\Omega \setminus \Omega_\varepsilon^X| \leq C \cdot \varepsilon$, where C is an ε-independent constant. At the same time, we have $\Omega_\varepsilon^X \subset \Omega \Rightarrow |\Omega_\varepsilon^X| \leq |\Omega|$, so (2.6.5) becomes:

$$\left| \int_\Omega \varphi \mathrm{d}\mu_\varepsilon^X - \int_\Omega \varphi \mathrm{d}x \right| \leq \frac{\varepsilon\sqrt{3}}{2} \|\nabla\varphi\|_{L^\infty(\Omega)} \cdot |\Omega| + C \cdot \varepsilon \|\varphi\|_{L^\infty(\Omega)} \lesssim \varepsilon \cdot \|\varphi\|_{W^{1,\infty}(\Omega)}.$$

Computing in the same fashion for μ_ε^Y and μ_ε^Z, we obtain the conclusion. □

We also suppose that:

(H_2) g is bounded and Lipschitz, where g represents the prescribed boundary data.

Since Ω is bounded and smooth (by assumption (A_1) from Section 2.2), we can extend the function g to a bounded and Lipschitz map from $\mathbb{R}^3$ to $\mathcal{S}_0$, denoted still as g.

We present an auxiliary result proved in [29]:

Lemma 2.6.3. Let $\Omega \subseteq \mathbb{R}^3$ a bounded, smooth domain, and let $g : \Omega \to \mathcal{S}_0$ be a bounded, Lipschitz map. For any $Q \in H_g^1(\Omega, \mathcal{S}_0)$ and $\sigma \in (0,1)$, there exists a bounded, Lipschitz map $Q_\sigma : \overline{\Omega} \to \mathcal{S}_0$ that satisfies the following properties:

$$Q_\sigma = g \quad \text{on } \partial\Omega$$

$$\|Q_\sigma\|_{L^\infty(\Omega)} \lesssim \sigma^{-1/2} \left(\|Q\|_{H^1(\Omega)} + \|g\|_{L^\infty(\Omega)} \right) \tag{2.6.6}$$

$$\|\nabla Q_\sigma\|_{L^\infty(\Omega)} \lesssim \sigma^{-3/2} \left(\|Q\|_{H^1(\Omega)} + \|g\|_{W^{1,\infty}(\Omega)} \right) \tag{2.6.7}$$

$$\|Q - Q_\sigma\|_{L^2(\Omega)} \lesssim \sigma \|Q\|_{H^1(\Omega)} \tag{2.6.8}$$

$$\|\nabla Q - \nabla Q_\sigma\|_{L^2(\Omega)} \to 0 \quad \text{as } \sigma \to 0. \tag{2.6.9}$$

The main result from this section is the following:

Proposition 2.6.1. Suppose that assumptions (A_1)-(A_7) (from Section 2.2) and (H_1)-(H_2) (from this section) hold. Then, for any $Q \in H_g^1(\Omega, \mathcal{S}_0)$, there exists a sequence $(Q_\varepsilon)_{\varepsilon>0}$ in $H_g^1(\Omega, \mathcal{S}_0)$ that converges $H^1(\Omega)$-strongly to Q and satisfies

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{m_0} \left(\|Q\|_{H^1(\Omega)}^4 + 1 \right),$$

for ε small enough, where $J_\varepsilon^{\mathcal{T}}$ is defined in (2.4.10), J_0 is defined in (2.6.1) and

$$m_0 = \min\left\{ \frac{\alpha - 1}{3}, 2 - \alpha \right\}.$$

The ε-independent constant that is hidden by the use of the sign "$\lesssim$", described as in remark 2.2.5, depends only on the L^∞-norms of g and ∇g, on Ω, f_s and the initial cube $\mathcal{C}$.

Remark 2.6.1. The previous proposition allows us to obtain, as claimed, a rate of convergence for the minimisers $\overline{Q}_\varepsilon$ of $\mathcal{F}_\varepsilon$, given by Theorem 2.3.4, to a minimiser Q of $\mathcal{F}_0$ in terms of $\|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)} = o(1)$ as $\varepsilon \to 0$ (i.e. relation (2.6.10)).

Indeed, this is obtained in the following way. First, let us fix a value for $0 < \varepsilon < 1$ such that equation (2.5.9) holds. Then we use the inequality

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_0[Q]| \leq |J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]| + |J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]|,$$

where Q_ε is the function from $H^1_g(\Omega, \mathcal{S}_0)$ granted by Lemma 2.6.3, with $\sigma = \varepsilon^{m_0}$.

For the first term from the right-hand side from the last inequality, we use relation (2.5.9) and we obtain, for a fixed ε sufficiently small:

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]| \leq C \cdot \left(\varepsilon^{1/2-\alpha/4} + \|E_\varepsilon \overline{Q}_\varepsilon - Q_\varepsilon\|_{L^4(\Omega)}\right),$$

where C is ε-independent.

From the compact Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we obtain:

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]| \leq C \cdot \left(\varepsilon^{1/2-\alpha/4} + \|E_\varepsilon \overline{Q}_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)}\right).$$

Now, we observe that:

$$\begin{aligned}
\|E_\varepsilon \overline{Q}_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} &\leq \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)} + \|Q_\varepsilon - Q\|_{H^1(\Omega)} \\
&\leq \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)} + \|Q_\varepsilon - Q\|_{L^2(\Omega)} + \|\nabla Q_\varepsilon - \nabla Q\|_{L^2(\Omega)} \\
&\leq \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)} + \varepsilon^{m_0}\|Q\|_{H^1(\Omega)} + \|\nabla Q_\varepsilon - \nabla Q\|_{L^2(\Omega)},
\end{aligned}$$

where we have used relation (2.6.8) in the last row. Relation (2.6.9) tells us that $\|\nabla Q_\varepsilon - \nabla Q\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$, hence, by the choice of ε, we can control it with a constant. Since Q is fixed, we can also control $\|Q\|_{H^1(\Omega)}$ with an ε-independent constant. Therefore, we can write:

$$\|E_\varepsilon \overline{Q}_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \lesssim \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)} + \varepsilon^{m_0}.$$

Hence, we have:

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]| \lesssim \varepsilon^{1/2-\alpha/4} + \varepsilon^{m_0} + \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)}.$$

and if we denote by $m_\alpha = \min\{1/2 - \alpha/4, m_0\}$ (which is defined depending whether α is bigger or smaller than $10/7$), we can rewrite:

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_\varepsilon^{\mathcal{T}}[Q_\varepsilon]| \lesssim \varepsilon^{m_\alpha} + \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)},$$

since ε is chosen from $(0,1)$.

For the term $|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]|$, we apply Proposition 2.6.1:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{m_0}(\|Q\|^4_{H^1(\Omega)} + 1)$$

and since Q is fixed, we obtain:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{m_0} \lesssim \varepsilon^{m_\alpha},$$

using the definition of m_α.

If we go back to our initial inequality, we obtain:

$$|J_\varepsilon^{\mathcal{T}}[\overline{Q}_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{m_\alpha} + \|E_\varepsilon \overline{Q}_\varepsilon - Q\|_{H^1(\Omega)}. \tag{2.6.10}$$

where

$$m_\alpha = \begin{cases} \dfrac{\alpha-1}{3}, & 1 < \alpha \leq \dfrac{10}{7}; \\ \dfrac{2-\alpha}{4}, & \dfrac{10}{7} < \alpha < 2. \end{cases}$$

Proof of Proposition 2.6.1. Let us fix a small $\varepsilon \in (0,1)$ such that:

$$\frac{p\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} < 2p, \ \frac{q\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} < 2q \text{ and } \frac{r\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} < 2r. \tag{2.6.11}$$

This is possible since $\dfrac{p\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} \searrow p$, $\dfrac{q\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} \searrow q$ and $\dfrac{r\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} \searrow r$ as $\varepsilon \to 0$ and $p, q, r \geq 1$.

Let now β be a positive parameter, to be chosen later, and let $Q_\varepsilon := Q_{\varepsilon^\beta} \in H^1_g(\Omega, \mathcal{S}_0)$ be the Lipschitz map given by Lemma 2.6.3. Then, we have:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \leq |J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| + |\tilde{J}_\varepsilon[Q_\varepsilon] - J_0[Q_\varepsilon]| + |J_0[Q_\varepsilon] - J_0[Q]|, \tag{2.6.12}$$

where $\tilde{J}_\varepsilon$ is defined in (2.5.2).

We analyse the first term from the right-hand side from (2.6.12). Using the same notations as in Theorem 2.5.1, replacing $\mathrm{Lip}(Q_\varepsilon)$ (the Lipschitz constant) with $\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)}$ and combining relations (2.5.5) and (2.5.6), we obtain:

$$|J_\varepsilon^X[Q_\varepsilon] - \tilde{J}_\varepsilon^X[Q_\varepsilon]| \lesssim (\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1) \cdot \|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \cdot \frac{2(r+q)}{pqr} \cdot \frac{p\varepsilon - \varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} \cdot$$
$$\cdot X_\varepsilon \cdot \varepsilon^3 \cdot \sqrt{\left(\varepsilon - \frac{\varepsilon^\alpha}{p}\right)^2 + \left(\frac{\varepsilon^\alpha}{q}\right)^2 + \left(\frac{\varepsilon^\alpha}{r}\right)^2}.$$

Using (2.5.7) and (2.6.11), we can rewrite the last inequality as follows:

$$|J_\varepsilon^X[Q_\varepsilon] - \tilde{J}_\varepsilon^X[Q_\varepsilon]| \lesssim (\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1) \cdot \|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \cdot \sqrt{\left(\varepsilon - \frac{\varepsilon^\alpha}{p}\right)^2 + \left(\frac{\varepsilon^\alpha}{q}\right)^2 + \left(\frac{\varepsilon^\alpha}{r}\right)^2}, \tag{2.6.13}$$

since the term $\dfrac{2(r+q)}{pqr}$ can be bounded with an ε-independent constant. Because $p, q, r \geq 1$, we have:

$$\frac{\varepsilon^{2\alpha}}{p^2}, \frac{\varepsilon^{2\alpha}}{q^2}, \frac{\varepsilon^{2\alpha}}{r^2} \leq \varepsilon^{2\alpha}$$

and, because $\varepsilon > 0$ and $\alpha \in (1,2)$, we also have:

$$0 < \varepsilon - \varepsilon^\alpha \le \varepsilon - \frac{\varepsilon^\alpha}{k} \le \varepsilon, \text{ for } k \in \{p,q,r\}.$$

Therefore, (2.6.13) becomes:

$$|J_\varepsilon^X[Q_\varepsilon] - \tilde{J}_\varepsilon^X[Q_\varepsilon]| \lesssim \left(\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1\right) \cdot \|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \cdot \sqrt{\varepsilon^2 + 2\varepsilon^{2\alpha}},$$

and using the same arguments for $J_\varepsilon^Y[Q_\varepsilon]$ and $J_\varepsilon^Z[Q_\varepsilon]$, we obtain:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| \lesssim \left(\|Q_\varepsilon\|^3_{L^\infty(\Omega)} + 1\right) \cdot \|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \cdot \sqrt{\varepsilon^2 + 2\varepsilon^{2\alpha}}. \tag{2.6.14}$$

Using Lemma 2.6.3, we have:

$$\|Q_\varepsilon\|^3_{L^\infty(\Omega)} \lesssim \varepsilon^{-3\beta/2}\left(\|Q\|_{H^1(\Omega)} + \|g\|_{L^\infty(\Omega)}\right)^3$$
$$\|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} \lesssim \varepsilon^{-3\beta/2}\left(\|Q\|_{H^1(\Omega)} + \|g\|_{W^{1,\infty}(\Omega)}\right).$$

Now, the constant involved by using the sign "$\lesssim$" is going to depend also on the L^∞-norms of g and ∇g, hence, relation (2.6.14) becomes:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| \lesssim \frac{\sqrt{\varepsilon^2 + \varepsilon^{2\alpha}}}{\varepsilon^{3\beta}}\left(\|Q\|^4_{H^1(\Omega)} + 1\right) \lesssim \sqrt{\varepsilon^{2(1-3\beta)} + \varepsilon^{2(\alpha-3\beta)}}\left(\|Q\|^4_{H^1(\Omega)} + 1\right).$$

Since $\alpha \in (1,2)$, we have $1 - 3\beta < \alpha - 3\beta$. Therefore, we can write the last inequality as follows:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - \tilde{J}_\varepsilon[Q_\varepsilon]| \lesssim \varepsilon^{1-3\beta}\left(\|Q\|^4_{H^1(\Omega)} + 1\right), \tag{2.6.15}$$

since $\varepsilon \in (0,1)$.

In order to analyse better the second term from (2.6.12), which contains $\tilde{J}_\varepsilon[Q_\varepsilon]$ and $J_0[Q_\varepsilon]$, we analyse the first terms from (2.6.1) and (2.6.2):

$$\left|\frac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)}\int_\Omega \Psi^Y(Q_\varepsilon)\mathrm{d}\mu_\varepsilon^X - \frac{1}{r}\int_\Omega \Psi^Y(Q_\varepsilon)\mathrm{d}x\right| \le$$
$$\le \frac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)}\left|\int_\Omega \Psi^Y(Q_\varepsilon)\mathrm{d}\mu_\varepsilon^X - \int_\Omega \Psi^Y(Q_\varepsilon)\mathrm{d}x\right| + \left|\frac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)} - \frac{1}{r}\right| \cdot \left|\int_\Omega \Psi^Y(Q_\varepsilon)\mathrm{d}x\right|.$$

As we have seen before, we have $\dfrac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)} \searrow \dfrac{1}{r}$ and we have chosen $\varepsilon > 0$ such that $\dfrac{1}{r} \le \dfrac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)} < \dfrac{2}{r}$. Moreover, we have $\left|\dfrac{p\varepsilon - \varepsilon^\alpha}{pr(\varepsilon - \varepsilon^\alpha)} - \dfrac{1}{r}\right| = \dfrac{\varepsilon^\alpha(p-1)}{pr(\varepsilon - \varepsilon^\alpha)}$ and we can impose further conditions regarding the choice of ε, such that $\dfrac{\varepsilon^{\alpha}(p-1)}{pr(\varepsilon-\varepsilon^{\alpha})} < \varepsilon^{\alpha-1}$, which is equivalent to choosing ε such that $\varepsilon^{\alpha-1} < 1 - \dfrac{1}{r} + \dfrac{1}{pr}$. Hence, we have:

$$\begin{aligned}\left|\frac{p\varepsilon-\varepsilon^{\alpha}}{pr(\varepsilon-\varepsilon^{\alpha})}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \frac{1}{r}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| \leq \\ \leq \frac{2}{r}\left|\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| + \varepsilon^{\alpha-1}\left|\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right|.\end{aligned}$$

Using the definition of $\mathbb{F}_{\varepsilon}$, we have:

$$\begin{aligned}\left|\frac{p\varepsilon-\varepsilon^{\alpha}}{pr(\varepsilon-\varepsilon^{\alpha})}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \frac{1}{r}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| \leq \\ \leq \frac{2}{r}\cdot\mathbb{F}_{\varepsilon}\cdot\|\Psi^{Y}(Q_{\varepsilon})\|_{W^{1,\infty}(\Omega)} + \varepsilon^{\alpha-1}\|\Psi^{Y}(Q_{\varepsilon})\|_{L^{\infty}(\Omega)}.\end{aligned}$$

Since $Q_{\varepsilon} \in H^{1}_{g}(\Omega, \mathcal{S}_0)$, Lemma 2.6.1 and Lemma 2.6.2, we obtain (also by moving the constant $\dfrac{2}{r}$ under the "$\lesssim$" sign):

$$\begin{aligned}\left|\frac{p\varepsilon-\varepsilon^{\alpha}}{pr(\varepsilon-\varepsilon^{\alpha})}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \frac{1}{r}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| \lesssim \varepsilon\big(\|Q_{\varepsilon}\|^{4}_{L^{\infty}(\Omega)}+1\big)+ \\ +\varepsilon\big(\|Q_{\varepsilon}\|^{3}_{L^{\infty}(\Omega)}+1\big)\cdot\|\nabla Q_{\varepsilon}\|_{L^{\infty}(\Omega)} + \varepsilon^{\alpha-1}\big(\|Q_{\varepsilon}\|^{4}_{L^{\infty}(\Omega)}+1\big),\end{aligned}$$

Applying Lemma 2.6.3, we get:

$$\begin{aligned}\left|\frac{p\varepsilon-\varepsilon^{\alpha}}{pr(\varepsilon-\varepsilon^{\alpha})}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \frac{1}{r}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| \lesssim \varepsilon\Big(\varepsilon^{-2\beta}\big(\|Q\|_{H^{1}(\Omega)}+\|g\|_{L^{\infty}(\Omega)}\big)^{4}+1\Big)+ \\ +\varepsilon\Big(\varepsilon^{-3\beta/2}\big(\|Q\|_{H^{1}(\Omega)}+\|g\|_{L^{\infty}(\Omega)}\big)^{3}+1\Big)\cdot\varepsilon^{-3\beta/2}\big(\|Q\|_{H^{1}(\Omega)}+\|g\|_{W^{1,\infty}(\Omega)}\big)+ \\ +\varepsilon^{\alpha-1}\Big(\varepsilon^{-2\beta}\big(\|Q\|_{H^{1}(\Omega)}+\|g\|_{L^{\infty}(\Omega)}\big)^{4}+1\Big),\end{aligned}$$

Moving the terms $\|g\|_{L^{\infty}(\Omega)}$ and $\|g\|_{W^{1,\infty}(\Omega)}$ under the "$\lesssim$" sign and using the fact that $\beta > 0$ and $\varepsilon \in (0,1)$, we have:

$$\left|\frac{p\varepsilon-\varepsilon^{\alpha}}{pr(\varepsilon-\varepsilon^{\alpha})}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}\mu_{\varepsilon}^{X} - \frac{1}{r}\int_{\Omega}\Psi^{Y}(Q_{\varepsilon})\mathrm{d}x\right| \lesssim \big(\varepsilon^{1-2\beta}+\varepsilon^{1-3\beta}+\varepsilon^{\alpha-2\beta-1}\big)\big(\|Q\|^{4}_{H^{1}(\Omega)}+1\big).$$

Applying the same technique for the other five terms from J_0 and $\tilde{J}_{\varepsilon}$, which are in (2.6.1) and (2.6.2), we obtain:

$$|\tilde{J}_{\varepsilon}[Q_{\varepsilon}] - J_0[Q_{\varepsilon}]| \lesssim \big(\varepsilon^{1-2\beta}+\varepsilon^{1-3\beta}+\varepsilon^{\alpha-2\beta-1}\big)\big(\|Q\|^{4}_{H^{1}(\Omega)}+1\big)$$

and using once again that $\beta > 0$ and $\varepsilon \in (0,1)$, we can write:

$$|\tilde{J}_{\varepsilon}[Q_{\varepsilon}] - J_0[Q_{\varepsilon}]| \lesssim \big(\varepsilon^{1-3\beta}+\varepsilon^{\alpha-2\beta-1}\big)\big(\|Q\|^{4}_{H^{1}(\Omega)}+1\big). \tag{2.6.16}$$

Moving now to the last term from (2.6.12), which is $|J_0[Q_\varepsilon] - J_0[Q]|$, we once again analyse every difference that can be formed with the six terms from the definition of (2.6.1). Hence:

$$\left| \int_\Omega \frac{1}{r} \Psi^Y(Q_\varepsilon) \mathrm{d}x - \int_\Omega \frac{1}{r} \Psi^Y(Q) \mathrm{d}x \right| \leq \frac{1}{r} \int_\Omega \left| \Psi^Y(Q_\varepsilon) - \Psi^Y(Q) \right| \mathrm{d}x.$$

Using Lemma 2.6.1 and moving the constant $\frac{1}{r}$ under the "$\lesssim$" sign, we have:

$$\begin{aligned}
\left| \int_\Omega \frac{1}{r} \Psi^Y(Q_\varepsilon) \mathrm{d}x - \int_\Omega \frac{1}{r} \Psi^Y(Q) \mathrm{d}x \right| &\lesssim \int_\Omega (|Q|^3 + |Q_\varepsilon|^3 + 1) |Q - Q_\varepsilon| \mathrm{d}x \\
&\lesssim \left(\int_\Omega (|Q|^3 + |Q_\varepsilon|^3 + 1)^2 \mathrm{d}x \right)^{1/2} \cdot \left(\int_\Omega |Q - Q_\varepsilon|^2 \mathrm{d}x \right)^{1/2} \\
&\lesssim \left(\int_\Omega (|Q|^6 + |Q_\varepsilon|^6 + 1) \mathrm{d}x \right)^{1/2} \cdot \|Q - Q_\varepsilon\|_{L^2(\Omega)} \\
&\lesssim (\|Q\|^3_{L^6(\Omega)} + \|Q_\varepsilon\|^3_{L^6(\Omega)} + 1) \cdot \|Q - Q_\varepsilon\|_{L^2(\Omega)}.
\end{aligned}$$

The sequence $(Q_\varepsilon)_{\varepsilon>0}$ is bounded in $L^6(\Omega)$, due to the continuous Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and to Lemma 2.6.3. Using once again Lemma 2.6.3 to control $\|Q - Q_\varepsilon\|_{L^2(\Omega)}$, we obtain:

$$\left| \int_\Omega \frac{1}{r} \Psi^Y(Q_\varepsilon) \mathrm{d}x - \int_\Omega \frac{1}{r} \Psi^Y(Q) \mathrm{d}x \right| \lesssim \varepsilon^\beta (\|Q\|^4_{H^1(\Omega)} + 1),$$

hence

$$|J_0[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^\beta (\|Q\|^4_{H^1(\Omega)} + 1). \tag{2.6.17}$$

Combining now relations (2.6.15), (2.6.16) and (2.6.17), we obtain:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \lesssim (\varepsilon^{1-3\beta} + \varepsilon^{\alpha-2\beta-1} + \varepsilon^\beta)(\|Q\|^4_{H^1(\Omega)} + 1).$$

Now we need to find a suitable value for $\beta > 0$ such that we can put the minimum positive value between the exponents $1 - 3\beta$, $\alpha - 2\beta - 1$ and β, in order to obtain the best rate of convergence. This is equivalent to choosing $\beta = \min\left\{\frac{1}{4}, \frac{\alpha-1}{3}, 2-\alpha\right\}$. If $\alpha \in \left(1, \frac{7}{4}\right]$, then $\beta = \frac{\alpha-1}{3}$. If $\alpha \in \left(\frac{7}{4}, 2\right)$, then $\beta = 2 - \alpha$. This implies that:

$$|J_\varepsilon^{\mathcal{T}}[Q_\varepsilon] - J_0[Q]| \lesssim \varepsilon^{m_0} (\|Q\|^4_{H^1(\Omega)} + 1),$$

where $m_0 = \min\left\{\frac{\alpha-1}{3}, 2-\alpha\right\}$ for $\alpha \in (1,2)$. □

2.7 APPENDIX

2.7.1 CONSTRUCTING THE CUBIC MICROLATTICE

In this subsection, we provide more details regarding the construction of the "connecting parallelepipeds", which are the grey parallelepipeds from Figure 2.

In each of the points from $\mathcal{Y}_\varepsilon$ we construct a parallelepiped that connects the parallelepipeds $\mathcal{C}^i_\varepsilon$ and $\mathcal{C}^j_\varepsilon$, where $i,j \in \overline{1,N_\varepsilon}$ such that $|x^i_\varepsilon - x^j_\varepsilon| = \varepsilon$.

If $x^i_\varepsilon - x^j_\varepsilon = \pm(\varepsilon,0,0)^T$, then let:

- $X_\varepsilon = \text{card}\big(\{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x^i_\varepsilon - x^j_\varepsilon = (\varepsilon,0,0)^T,\ i<j\}\big)$ (2.7.1)
- $\Upsilon^x_\varepsilon : \{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x^i_\varepsilon - x^j_\varepsilon = (\varepsilon,0,0)^T,\ i<j\} \to \overline{1,X_\varepsilon}$ a bijection;
- $y^{x,k}_\varepsilon = \dfrac{1}{2}(x^i_\varepsilon + x^j_\varepsilon)$, where $k = \Upsilon^x_\varepsilon(i,j)$; (2.7.2)
- $\mathcal{Y}^x_\varepsilon = \left\{ y^{x,k}_\varepsilon \in \mathcal{Y}_\varepsilon \,\middle|\, y^{x,k}_\varepsilon = \dfrac{1}{2}(x^i_\varepsilon + x^j_\varepsilon),\ k = \Upsilon^x_\varepsilon(i,j) \right\}$; (2.7.3)
- $\mathcal{P}^{x,k}_\varepsilon$ the "connecting parallelepiped" centered in $y^{x,k}_\varepsilon$, defined by $\mathcal{P}^{x,k}_\varepsilon = y^{x,k}_\varepsilon + T_x\mathcal{C}^\alpha$, (2.7.4)

$$\text{where } T_x\mathcal{C}^\alpha = \left[-\frac{p\varepsilon - \varepsilon^\alpha}{2p}, \frac{p\varepsilon - \varepsilon^\alpha}{2p}\right] \times \left[-\frac{\varepsilon^\alpha}{2q}, \frac{\varepsilon^\alpha}{2q}\right] \times \left[-\frac{\varepsilon^\alpha}{2r}, \frac{\varepsilon^\alpha}{2r}\right]; \tag{2.7.5}$$

- $\mathcal{T}^k_x$ be the union of the four transparent faces of $\mathcal{P}^{x,k}_\varepsilon$ that have the length equal to $\dfrac{p\varepsilon - \varepsilon^\alpha}{p}$, which are represented in Figure 5a. (2.7.6)

If $x^i_\varepsilon - x^j_\varepsilon = \pm(0,\varepsilon,0)^T$, then let:

- $Y_\varepsilon = \text{card}\big(\{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x^i_\varepsilon - x^j_\varepsilon = (0,\varepsilon,0)^T,\ i<j\}\big)$ (2.7.7)
- $\Upsilon^y_\varepsilon : \{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x^i_\varepsilon - x^j_\varepsilon = (0,\varepsilon,0)^T,\ i<j\} \to \overline{1,Y_\varepsilon}$ a bijection;
- $y^{y,l}_\varepsilon = \dfrac{1}{2}(x^i_\varepsilon + x^j_\varepsilon)$, where $l = \Upsilon^y_\varepsilon(i,j)$;
- $\mathcal{Y}^y_\varepsilon = \left\{ y^{y,l}_\varepsilon \in \mathcal{Y}_\varepsilon \,\middle|\, y^{y,l}_\varepsilon = \dfrac{1}{2}(x^i_\varepsilon + x^j_\varepsilon),\ l = \Upsilon^y_\varepsilon(i,j) \right\}$; (2.7.8)
- $\mathcal{P}^{y,l}_\varepsilon$ the "connecting parallelepiped" centered in $y^{y,l}_\varepsilon$, defined by $\mathcal{P}^{y,l}_\varepsilon = y^{y,l}_\varepsilon + T_y\mathcal{C}^\alpha$, (2.7.9)

$$\text{where } T_y\mathcal{C}^\alpha = \left[-\frac{\varepsilon^\alpha}{2p}, \frac{\varepsilon^\alpha}{2p}\right] \times \left[-\frac{q\varepsilon - \varepsilon^\alpha}{2q}, \frac{q\varepsilon - \varepsilon^\alpha}{2q}\right] \times \left[-\frac{\varepsilon^\alpha}{2r}, \frac{\varepsilon^\alpha}{2r}\right]; \tag{2.7.10}$$

- $\mathcal{T}^l_y$ be the union of the four transparent faces of $\mathcal{P}^{y,l}_\varepsilon$ that have the length equal to $\dfrac{q\varepsilon - \varepsilon^\alpha}{q}$, which are represented in Figure 5b. (2.7.11)

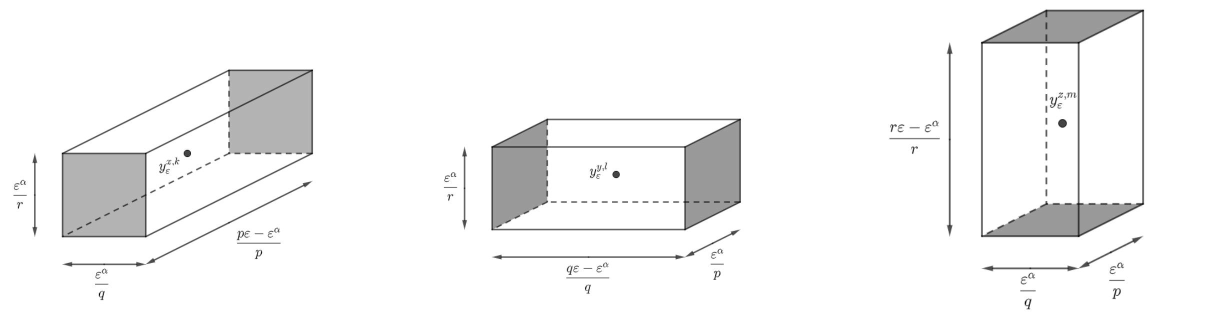
If $x_\varepsilon^i - x_\varepsilon^j = \pm(0,0,\varepsilon)^T$, then let:

- $Z_\varepsilon = \text{card}\big(\{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x_\varepsilon^i - x_\varepsilon^j = (0,0,\varepsilon)^T,\ i<j\}\big)$ (2.7.12)
- $\Upsilon_\varepsilon^y : \{(i,j) \in \overline{1,N_\varepsilon}^2 \mid x_\varepsilon^i - x_\varepsilon^j = (0,0,\varepsilon)^T,\ i<j\} \to \overline{1,Z_\varepsilon}$ a bijection;
- $y_\varepsilon^{z,m} = \dfrac{1}{2}(x_\varepsilon^i + x_\varepsilon^j)$, where $m = \Upsilon_\varepsilon^z(i,j)$;
- $\mathcal{Y}_\varepsilon^z = \left\{ y_\varepsilon^{z,m} \in \mathcal{Y}_\varepsilon \;\middle|\; y_\varepsilon^{z,m} = \dfrac{1}{2}(x_\varepsilon^i + x_\varepsilon^j),\ m = \Upsilon_\varepsilon^z(i,j) \right\}$; (2.7.13)
- $\mathcal{P}_\varepsilon^{z,m}$ the "connecting parallelepiped" centered in $y_\varepsilon^{z,m}$, defined by $\mathcal{P}_\varepsilon^{z,m} = y_\varepsilon^{z,m} + T_z\mathcal{C}^\alpha$, (2.7.14)

$$\text{where } T_z\mathcal{C}^\alpha = \left[-\frac{\varepsilon^\alpha}{2p}, \frac{\varepsilon^\alpha}{2p}\right] \times \left[-\frac{\varepsilon^\alpha}{2q}, \frac{\varepsilon^\alpha}{2q}\right] \times \left[-\frac{r\varepsilon - \varepsilon^\alpha}{2r}, \frac{r\varepsilon - \varepsilon^\alpha}{2r}\right]; \tag{2.7.15}$$

- $\mathcal{T}_z^m$ be the union of the four transparent faces of $\mathcal{P}_\varepsilon^{z,m}$ that have the length equal to $\dfrac{r\varepsilon - \varepsilon^\alpha}{r}$,

which are represented in Figure 5c. (2.7.16)



(a) The "connecting parallelepiped" $\mathcal{P}_\varepsilon^{x,k}$, with the center in $y_\varepsilon^{x,k}$, with lateral transparent faces $\mathcal{T}_x^k$.

(b) The "connecting parallelepiped" $\mathcal{P}_\varepsilon^{y,l}$, with the center in $y_\varepsilon^{y,l}$, with lateral transparent faces $\mathcal{T}_y^l$.

(c) The "connecting parallelepiped" $\mathcal{P}_\varepsilon^{z,m}$, with the center in $y_\varepsilon^{z,m}$, with lateral transparent faces $\mathcal{T}_z^m$.

Figure 5: The three types of "connecting parallelepipeds" with centers from $\mathcal{Y}_\varepsilon$.

Remark 2.7.1. We can see that we need to set $\varepsilon^{\alpha-1} < \min\{p,q,r\}$, otherwise we have $\dfrac{\varepsilon^{\alpha-1}}{p} \geq 1 \Rightarrow \dfrac{\varepsilon^\alpha}{2p} \geq \dfrac{\varepsilon}{2}$ and, in the same way, $\dfrac{\varepsilon^\alpha}{2q} \geq \dfrac{\varepsilon}{2}$ and $\dfrac{\varepsilon^\alpha}{2r} \geq \dfrac{\varepsilon}{2}$, hence the inclusions from the family $\mathcal{C}_\varepsilon$, which represents the set of all "inner parallelepipeds", are not disjoint anymore and they overlap. More specifically, the "connecting parallelepipeds" cannot be constructed anymore. Since the parameters p, q and r are fixed and we are interested in what happens when $\varepsilon \to 0$, then the condition $\varepsilon^{\alpha-1} < \min\{p,q,r\}$ implies that $\alpha \geq 1$. If $\alpha = 1$, then it is easy to see that the volume of the scaffold does not tend to zero as $\varepsilon \to 0$, so we are not in the dilute regime anymore.

2.7.2 VOLUME AND SURFACE AREA OF THE SCAFFOLD

Proposition 2.7.1. The volume of the scaffold $\mathcal{N}_\varepsilon$ tends to 0 as $\varepsilon \to 0$.

Proof. According to (2.2.1), (2.2.5), (2.2.6), (2.2.8), (2.7.1), (2.7.7) and (2.7.12), we have:

- $N_\varepsilon < \dfrac{L_0 l_0 h_0}{\varepsilon^3}$; • $X_\varepsilon < \left(\dfrac{L_0}{\varepsilon} - 1\right) \cdot \dfrac{l_0 h_0}{\varepsilon^2}$; • $Y_\varepsilon < \left(\dfrac{l_0}{\varepsilon} - 1\right) \dfrac{L_0 h_0}{\varepsilon^2}$; • $Z_\varepsilon < \left(\dfrac{h_0}{\varepsilon} - 1\right) \dfrac{L_0 l_0}{\varepsilon^2}$.

Furthermore, we have:

$$|\mathcal{N}_\varepsilon| = N_\varepsilon \cdot \frac{\varepsilon^{3\alpha}}{pqr} + X_\varepsilon \cdot \frac{\varepsilon^{2\alpha}}{pqr}(p\varepsilon - \varepsilon^\alpha) + Y_\varepsilon \cdot \frac{\varepsilon^{2\alpha}}{pqr}(q\varepsilon - \varepsilon^\alpha) + Z_\varepsilon \cdot \frac{\varepsilon^{2\alpha}}{pqr}(r\varepsilon - \varepsilon^\alpha),$$

where $\dfrac{\varepsilon^{3\alpha}}{pqr}$ represents the volume of an "inner parallelepiped" defined in (2.2.7) and $\dfrac{\varepsilon^{2\alpha}}{pqr}(p\varepsilon - \varepsilon^\alpha)$, $\dfrac{\varepsilon^{2\alpha}}{pqr}(q\varepsilon - \varepsilon^\alpha)$ and $\dfrac{\varepsilon^{2\alpha}}{pqr}(r\varepsilon - \varepsilon^\alpha)$ represent the volume of a "connecting parallelepiped" $\mathcal{P}_\varepsilon^{x,k}$, $\mathcal{P}_\varepsilon^{y,l}$ and, respectively, $\mathcal{P}_\varepsilon^{z,m}$, which are defined in (2.7.4), (2.7.9) and (2.7.14). Hence:

$$|\mathcal{N}_\varepsilon| < \frac{L_0 l_0 h_0 (p+q+r)}{pqr}\varepsilon^{2(\alpha-1)} - 2\frac{L_0 l_0 h_0}{pqr}\varepsilon^{3(\alpha-1)} - \left(\frac{L_0 l_0}{pq} + \frac{L_0 h_0}{pr} + \frac{l_0 h_0}{qr}\right)\varepsilon^{2\alpha-1} + \\ + \frac{L_0 l_0 + L_0 h_0 + l_0 h_0}{pqr}\varepsilon^{3\alpha-2}.$$

Because $\alpha > 1$, according to (A_2), then $2(\alpha-1) > 0$, $3(\alpha-1) > 0$, $2\alpha - 1 > 0$ and $3\alpha - 2 > 0$, therefore $|\mathcal{N}_\varepsilon| \to 0$ as $\varepsilon \to 0$. □

Proposition 2.7.2. There exists an ε-independent constant $C_s = C_s(p, q, r, \Omega) > 0$ such that:

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)} |\partial \mathcal{N}_\varepsilon| < C_s.$$

Proof. Using the same tehnique as in Proposition 2.7.1, we have, considering relations (2.7.6), (2.7.11), (2.7.16) and (2.4.6):

$$\begin{aligned} |\partial\mathcal{N}_\varepsilon| &< N_\varepsilon \cdot \varepsilon^{2\alpha} \cdot \frac{2(p+q+r)}{pqr} + X_\varepsilon \cdot \varepsilon^\alpha (p\varepsilon - \varepsilon^\alpha) \cdot \frac{2(q+r)}{pqr} + \\ &+ Y_\varepsilon \cdot \varepsilon^\alpha (q\varepsilon - \varepsilon^\alpha) \cdot \frac{2(p+r)}{pqr} + Z_\varepsilon \cdot \varepsilon^\alpha (r\varepsilon - \varepsilon^\alpha) \cdot \frac{2(p+q)}{pqr} \\ &< C(p, q, r, L_0, l_0, h_0) \cdot \varepsilon^{\alpha-3} \cdot ((p+q+r)\varepsilon - 2\varepsilon^\alpha). \end{aligned}$$

Hence

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^3}{\varepsilon^\alpha(\varepsilon - \varepsilon^\alpha)} |\partial\mathcal{N}_\varepsilon| < C(p, q, r, L_0, l_0, h_0) \cdot \lim_{\varepsilon \to 0} \frac{(p+q+r)\varepsilon - 2\varepsilon^\alpha}{\varepsilon - \varepsilon^\alpha} < +\infty.$$

We denote C_s the constant obtained in the last inequality. □

Proposition 2.7.3. Let $\partial\mathcal{N}_\varepsilon^{\mathcal{S}}$ be the set defined in (2.4.6). Then, for $\varepsilon \to 0$, we have $|\partial\mathcal{N}_\varepsilon^{\mathcal{S}}| \to 0$.

Proof. According to (2.4.6), we have:

$$\partial\mathcal{N}_\varepsilon^{\mathcal{S}} = \left(\bigcup_{i=1}^{N_{\varepsilon,2}} \mathcal{S}^i \right),$$

therefore, we can write:

$$|\partial\mathcal{N}_\varepsilon^{\mathcal{S}}| \leq \sum_{i=1}^{N_{\varepsilon,2}} |\mathcal{S}^i| \leq \sum_{i=1}^{N_{\varepsilon,2}} |\mathcal{C}_\varepsilon^i| \leq \sum_{i=1}^{N_{\varepsilon,2}} \frac{\varepsilon^{2\alpha}(p+q+r)}{pqr} \leq \frac{\varepsilon^{2\alpha}(p+q+r)}{pqr} \cdot N_{\varepsilon,2},$$

where we have used (2.4.5) and (2.2.7). Since $N_{\varepsilon,2}$ counts only the "inner parallelepipeds" that are close to the boundary of Ω (meaning that these objects have less than 6 adjacent "connecting parallelepipeds; also, see (2.4.4)), then we can write:

$$N_{\varepsilon,2} < \frac{L_0 \cdot l_0}{\varepsilon^2} + \frac{L_0 \cdot h_0}{\varepsilon^2} + \frac{l_0 \cdot h_0}{\varepsilon^2},$$

where L_0, l_0 and h_0 are defined in (2.2.1) and they describe the parallelepiped that contains the entire domain Ω. From here, we obtain:

$$|\partial\mathcal{N}_\varepsilon^{\mathcal{S}}| \leq \frac{\varepsilon^{2\alpha}(p+q+r)}{pqr} \cdot \frac{L_0 l_0 + l_0 h_0 + h_0 L_0}{\varepsilon^2} \lesssim \varepsilon^{2(\alpha-1)} \to 0, \quad \text{as } \varepsilon \to 0,$$

since $\alpha > 1$. □

2.7.3 CONSTRUCTING AN EXPLICIT EXTENSION OF Q INSIDE THE SCAFFOLD

The aim of this subsection is to prove that there exists a function $v \in H^1(\Omega)$ such that $v = Q$ on $\partial\mathcal{N}_\varepsilon$, $v = Q$ in Ω_ε and $\|\nabla v\|_{L^2(\Omega)} \lesssim \|\nabla Q\|_{L^2(\Omega_\varepsilon)}$.

In order to prove it, we first construct an explicit extension $u : \Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}} \to \mathcal{S}_0$ such that $u \in H^1(\Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}}, \mathcal{S}_0)$ and there exists a constant C, independent of ε, for which we have

$$\|\nabla u\|_{L^2(\mathcal{N}_\varepsilon^{\mathcal{T}})} \leq C\|\nabla Q\|_{L^2(\Omega_\varepsilon)},$$

which implies $\|\nabla u\|_{L^2(\Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}})} \leq C\|\nabla Q\|_{L^2(\Omega_\varepsilon)}$.

Then we construct $v : \Omega \to \mathcal{S}_0$ such that $v \in H^1(\Omega)$, $v \equiv u$ on $\Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}}$, $v = u$ on $\partial\mathcal{N}_\varepsilon^{\mathcal{T}}$ and there exists a constant c such that:

$$\|\nabla v\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}})},$$

which implies that $\|\nabla v\|_{L^2(\Omega)} \lesssim \|\nabla Q\|_{L^2(\Omega_\varepsilon)}$, using the properties mentioned for u.

We prove first the following result.

Lemma 2.7.1. Let $z_0, a, b \in \mathbb{R}$, $a, b, z_0 > 0$ and let $A_{a,b} = \{(\rho\cos\theta, \rho\sin\theta) \ : \ 0 \leq \theta < 2\pi, \ a < \rho < b\}$ be a two dimensional annulus, $B_a = \{(\rho\cos\theta, \rho\sin\theta) \ : \ 0 \leq \theta < 2\pi, \ 0 \leq \rho < a\}$ be a two dimensional ball with radius a, $\mathcal{A}_{a,b}^{z_0} = A_{a,b} \times (-z_0, z_0) \subset \mathbb{R}^3$ and $\mathcal{B}_a^{z_0} = B_a \times (-z_0, z_0) \subset \mathbb{R}^3$ be a three dimensional cylinder.

Let $Q \in H^1\big(\mathcal{A}^{z_0}_{1,2}, \mathcal{S}_0\big)$. Then the function $u : \mathcal{B}^{z_0}_1 \to \mathcal{S}_0$ defined for any $z \in (-z_0, z_0)$ as

$$u(x,y,z) = \begin{cases} \varphi(\sqrt{x^2+y^2})\, Q\left(\left(\frac{2}{\sqrt{x^2+y^2}} - 1\right)x, \left(\frac{2}{\sqrt{x^2+y^2}} - 1\right)y, z\right) + \\ \quad +(1-\varphi(\sqrt{x^2+y^2}))\fint_{A_{1,3/2}} Q(s,t,z)\mathrm{d}(s,t), \textit{ for } \frac{1}{2} \le \sqrt{x^2+y^2} < 1 \\ \fint_{A_{1,3/2}} Q(s,t,z)\mathrm{d}(s,t), \textit{ for } 0 \le \sqrt{x^2+y^2} \le \frac{1}{2} \end{cases} \tag{2.7.17}$$

is from $H^1(\mathcal{B}^{z_0}_1, \mathcal{S}_0)$, where $\varphi \in C^\infty_c\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ is the following bump function defined as

$$\varphi(\rho) = \begin{cases} \exp\left\{4 - \frac{4}{(2\rho-1)(3-2\rho)}\right\}, \ \forall \rho \in \left(\frac{1}{2}, \frac{3}{2}\right) \\ 0, \ \forall \rho \in \mathbb{R} \setminus \left(\frac{1}{2}, \frac{3}{2}\right) \end{cases},$$

the product $\varphi(\rho)\, Q$ represents product between a scalar and a Q-tensor and $\fint$ represents the average integral sign. Moreover, there exists a constant $c > 0$, independent of z_0, such that: $\|u_t\|_{L^2(\mathcal{B}^{z_0}_1)} \le c\|Q_t\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}$ for any $t \in \{x, y, z\}$, where u_t represents the partial derivative of u with respect to t, and:

$$\|\nabla u\|_{L^2(\mathcal{B}^{z_0}_1)} \le c\|\nabla Q\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

Proof. First of all, we can assume without loss of generality that Q and u are scalar functions, instead of Q-tensors. Hence, we prove this lemma for each of the component of Q and u.

Let $T : A_{1/2,1} \to A_{1,3/2}$ be the reflection defined as:

$$T(x,y) = \left(\left(\frac{2}{\sqrt{x^2+y^2}} - 1\right)x, \left(\frac{2}{\sqrt{x^2+y^2}} - 1\right)y\right) := (x', y'),\ \forall (x,y) \in A_{1/2,1}.$$

Then T is invertible and also bi-Lipschitz.

Let $Q \in H^1(\mathcal{A}^{z_0}_{1,3/2})$ and u defined by (2.7.17). By Theorem 3.17 from [1], we can approximate the function $Q \in H^1(\mathcal{A}^{z_0}_{1,3/2})$ with smooth functions from $C^\infty\big(\overline{\mathcal{A}^{z_0}_{1,3/2}}\big)$.

Let $(Q_k)_{k\ge1} \subset C^\infty\big(\overline{\mathcal{A}^{z_0}_{1,3/2}}\big)$ such that $Q_k \to Q$ strongly in $H^1(\mathcal{A}^{z_0}_{1,3/2})$ and, for any $k \ge 1$, let $u_k : \overline{\mathcal{B}^{z_0}_1} \to \mathbb{R}$ defined for all $z \in (-z_0, z_0)$ as:

$$u_k(x,y,z) = \begin{cases} \varphi(\sqrt{x^2+y^2})\, Q_k(T(x,y),z) + \\ \quad +(1-\varphi(\sqrt{x^2+y^2}))\fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t), \text{ for } \frac{1}{2} \le \sqrt{x^2+y^2} \le 1 \\ \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t), \text{ for } 0 \le \sqrt{x^2+y^2} \le \frac{1}{2} \end{cases}$$

By the above definition, we have that $u_k \in C^0\big(\overline{\mathcal{B}^{z_0}_1}\big)$ and that:

$$\frac{\partial u_k}{\partial x}(x,y,z) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} \cdot \varphi'(\sqrt{x^2+y^2}) \cdot \left(Q_k(x',y',z) - \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right) + \\ +\left(\frac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3} - 1\right) \cdot \varphi(\sqrt{x^2+y^2}) \cdot \left(\frac{\partial Q_k}{\partial x'}(x',y',z)\right) - \\ -\frac{2xy}{\left(\sqrt{x^2+y^2}\right)^3} \cdot \varphi(\sqrt{x^2+y^2}) \cdot \left(\frac{\partial Q_k}{\partial y'}(x',y',z)\right), \text{ for } \frac{1}{2} \leq \sqrt{x^2+y^2} < 1 \\ 0, \text{ for } 0 \leq \sqrt{x^2+y^2} \leq \frac{1}{2}. \end{cases}$$

Since $\varphi\left(\frac{1}{2}\right) = \varphi'\left(\frac{1}{2}\right) = 0$ and $\varphi \in C^\infty\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$, then we have $\frac{\partial u_k}{\partial x} \in C^0\left(\overline{\mathcal{B}_1^{z_0}}\right)$. Moreover, we obtain:

$$\frac{1}{3}\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2(x,y,z)\mathrm{d}(x,y) \leq \int_{A_{1/2,1}} \left|\frac{2xy}{\left(\sqrt{x^2+y^2}\right)^3}\right|^2 |\varphi(\sqrt{x^2+y^2})|^2 \left|\frac{\partial Q_k}{\partial y'}\right|^2(x',y',z)\mathrm{d}(x,y) +$$
$$+\int_{A_{1/2,1}} \left|\frac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3} - 1\right|^2 |\varphi(\sqrt{x^2+y^2})|^2 \left|\frac{\partial Q_k}{\partial x'}\right|^2(x',y',z)\mathrm{d}(x,y) +$$
$$+\int_{A_{1/2,1}} \left|\frac{x}{\sqrt{x^2+y^2}}\right|^2 |\varphi'(\sqrt{x^2+y^2})|^2 \left|Q_k(x',y',z) - \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right|^2 \mathrm{d}(x,y).$$

By the definition of φ, we have $\|\varphi\|_{L^\infty(\mathbb{R})} = 1$ and $\|\varphi'\|_{L^\infty(\mathbb{R})} = 2\sqrt{9+6\sqrt{3}} \cdot e^{1-\sqrt{3}} \approx 4.23 < 5$ (the maximum is obtained for $\rho = 1 - \frac{1}{6}\sqrt{6\sqrt{3}-9}$).

For any (x,y) such that $\frac{1}{2} \leq \sqrt{x^2+y^2} \leq 1$, we have:

- $\left|\frac{x}{\sqrt{x^2+y^2}}\right|^2 = \frac{x^2}{x^2+y^2} \leq 1;$
- $\left|\frac{2xy}{x^2+y^2}\right| \leq 1 \Rightarrow \left|\frac{2xy}{\left(\sqrt{x^2+y^2}\right)^3}\right|^2 \leq \frac{1}{x^2+y^2} \leq 2;$
- $0 \leq \frac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3} \leq \frac{2(x^2+y^2)}{\left(\sqrt{x^2+y^2}\right)^3} \leq 4 \Rightarrow -1 \leq \frac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3} - 1 \leq 3 \Rightarrow$
  $\Rightarrow \left|\frac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3}\right|^2 \leq 9.$

Therefore

$$\frac{1}{3}\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2(x,y,z)\mathrm{d}(x,y) \leq 25\int_{A_{1/2,1}} \left|Q_k(x',y',z) - \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right|^2 \mathrm{d}(x,y) +$$
$$+9\int_{A_{1/2,1}} \left|\frac{\partial Q_k}{\partial x'}\right|^2(x',y',z)\mathrm{d}(x,y) + 4\int_{A_{1/2,1}} \left|\frac{\partial Q_k}{\partial y'}\right|^2(x',y',z)\mathrm{d}(x,y).$$

Using now the change of variables $(x', y') = T(x, y)$, we obtain

$$\mathrm{d}(x,y) = \left(\frac{2}{\sqrt{(x')^2+(y')2}} - 1\right)\mathrm{d}(x',y')$$

and since $(x', y') \in A_{1,3/2}$, we get $1 \geq \dfrac{2}{\sqrt{(x')^2+(y')^2}} - 1 \geq \dfrac{1}{3}$, which implies

$$\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2 (x,y,z)\mathrm{d}(x,y) \leq 75 \int_{A_{1,3/2}} \left| Q_k(x',y',z) - \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right|^2 \mathrm{d}(x',y') + \\ + 27 \int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial x'}\right|^2 (x',y',z)\mathrm{d}(x',y') + 12 \int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial y'}\right|^2 (x',y',z)\mathrm{d}(x',y').$$

For the first term from the right hand side from the last inequality we can apply the Poincaré inequality, since $Q_k(\cdot,\cdot,z) \in H^1(A_{1,3/2})$, for any $z \in (-z_0, z_0)$. Therefore

$$\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2 (x,y,z)\mathrm{d}(x,y) \leq \\ \leq 75 \cdot C_P(A_{1,3/2}) \int_{A_{1,3/2}} \left(\left|\frac{\partial Q_k}{\partial x'}\right|^2 (x',y',z) + \left|\frac{\partial Q_k}{\partial y'}\right|^2 (x',y',z)\right)\mathrm{d}(x',y') + \\ + 27 \int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial x'}\right|^2 (x',y',z)\mathrm{d}(x,y) + 12 \int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial y'}\right|^2 (x',y',z)\mathrm{d}(x,y),$$

where $C_P(A_{1,3/2})$ is the Poincaré constant for the two dimmensional domain $A_{1,3/2}$. Hence, there exists $c_1 > 0$, independent of z_0, such that:

$$\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2 (x,y,z)\mathrm{d}(x,y) \leq c_1 \int_{A_{1,3/2}} \left(\left|\frac{\partial Q_k}{\partial x'}\right|^2 (x',y',z) + \left|\frac{\partial Q_k}{\partial y'}\right|^2 (x',y',z)\right)\mathrm{d}(x',y').$$

Integrating now with respect to $z \in (-z_0, z_0)$, we get:

$$\int_{\mathcal{A}^{z_0}_{1/2,1}} \left|\frac{\partial u_k}{\partial x}\right|^2 (x,y,z)\mathrm{d}(x,y,z) \leq c_1 \int_{\mathcal{A}^{z_0}_{1,3/2}} \left(\left|\frac{\partial Q_k}{\partial x'}\right|^2 (x',y',z) + \left|\frac{\partial Q_k}{\partial y'}\right|^2 (x',y',z)\right)\mathrm{d}(x',y',z) \Rightarrow \\ \Rightarrow \left\|\frac{\partial u_k}{\partial x}\right\|^2_{L^2(\mathcal{A}^{z_0}_{1/2,1})} \leq c_1 \|\nabla Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

Using now the fact that $\dfrac{\partial u_k}{\partial x} \in C^0\left(\overline{\mathcal{B}^{z_0}_1}\right)$ and that $\dfrac{\partial u_k}{\partial x}(x,y,z) = 0$ if $0 \leq \sqrt{x^2+y^2} \leq \dfrac{1}{2}$, then we can write:

$$\left\|\frac{\partial u_k}{\partial x}\right\|^2_{L^2(\mathcal{B}^{z_0}_1)} \leq c_1 \|\nabla Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

In a similar fashion, $\dfrac{\partial u_k}{\partial y}, \dfrac{\partial u_k}{\partial z} \in C^0\big(\overline{\mathcal{B}_1^{z_0}}\big)$, where

$$\frac{\partial u_k}{\partial y}(x,y,z) = \begin{cases} \dfrac{y}{\sqrt{x^2+y^2}} \cdot \varphi'(\sqrt{x^2+y^2}) \cdot \Big(Q_k(x',y',z) - \fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\Big) - \\ \quad -\dfrac{2xy}{\big(\sqrt{x^2+y^2}\big)^3} \cdot \varphi(\sqrt{x^2+y^2}) \cdot \Big(\dfrac{\partial Q_k}{\partial x'}(x',y',z)\Big) + \\ \quad +\Big(\dfrac{2x^2}{\big(\sqrt{x^2+y^2}\big)^3} - 1\Big) \cdot \varphi(\sqrt{x^2+y^2}) \cdot \Big(\dfrac{\partial Q_k}{\partial y'}(x',y',z)\Big), \\ \qquad\qquad \text{for } \dfrac{1}{2} \le \sqrt{x^2+y^2} \le 1 \\ 0, \text{ for } 0 \le \sqrt{x^2+y^2} \le \dfrac{1}{2} \end{cases}$$

with

$$\left\|\frac{\partial u_k}{\partial y}\right\|^2_{L^2(\mathcal{B}_1^{z_0})} \le c_1 \|\nabla Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

and

$$\frac{\partial u_k}{\partial z}(x,y,z) = \begin{cases} \varphi(\sqrt{x^2+y^2})\, \dfrac{\partial Q_k}{\partial z}(x',y',z) + \\ \quad +(1-\varphi(\sqrt{x^2+y^2}))\fint_{A_{1,3/2}} \dfrac{\partial Q_k}{\partial z}(s,t,z)\mathrm{d}(s,t), \text{ for } \dfrac{1}{2} \le \sqrt{x^2+y^2} \le 1 \\ \fint_{A_{1,3/2}} \dfrac{\partial Q_k}{\partial z}(s,t,z)\mathrm{d}(s,t), \text{ for } 0 \le \sqrt{x^2+y^2} \le \dfrac{1}{2} \end{cases}$$

since $Q \in C^\infty\big(\overline{\mathcal{A}^{z_0}_{1,3/2}}\big)$ and $A_{1,3/2}$ is independent of z, so we can move the derivative under the integral.

Then for any $(x,y,z) \in \mathcal{A}^{z_0}_{1/2,1}$, we have:

$$\begin{aligned} \frac{1}{2}\int_{A_{1/2,1}} \left|\frac{\partial u_k}{\partial z}\right|^2 (x,y,z)\mathrm{d}(x,y) &\le \|\varphi\|^2_{L^\infty(\mathbb{R})} \int_{A_{1/2,1}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x,y) + \\ &\quad + \|1-\varphi\|^2_{L^\infty(\mathbb{R})} \int_{A_{1/2,1}} \left|\fint_{A_{1,3/2}} \frac{\partial Q_k}{\partial z}(s,t,z)\mathrm{d}(s,t)\right|^2 \mathrm{d}(x,y) \\ &\le \int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x',y') + \frac{3\pi}{4}\cdot\frac{16}{25\pi^2}\int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x',y') \end{aligned}$$

and integrating with respect to $z \in (-z_0, z_0)$, we obtain

$$\int_{\mathcal{A}^{z_0}_{1/2,1}} \left|\frac{\partial u_k}{\partial z}\right|^2 (x,y,z)\mathrm{d}(x,y,z) \le \left(2 + \frac{24}{25\pi}\right) \int_{\mathcal{A}^{z_0}_{1,3/2}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x',y',z).$$

For any $(x,y,z) \in \mathcal{B}_1^{z_0}$ with $0 \le \sqrt{x^2+y^2} \le \dfrac{1}{2}$ we have:

$$\left|\frac{\partial u_k}{\partial z}\right|^2 (x,y,z) \le \frac{16}{25\pi^2}\int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x',y')$$

which implies

$$\int_{\overline{B_{1/2}}} \left|\frac{\partial u_k}{\partial z}\right|^2 (x,y,z)\mathrm{d}(x,y) \le \frac{4}{25\pi}\int_{A_{1,3/2}} \left|\frac{\partial Q_k}{\partial z}\right|^2 (x',y',z)\mathrm{d}(x',y')$$

and from here we obtain that

$$\left\|\frac{\partial u_k}{\partial z}\right\|^2_{L^2(\mathcal{B}_1^{z_0})} \le \left(3+\frac{3}{25\pi}\right)\|\nabla Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

Now we prove that we can control $\|u_k\|_{L^2(\mathcal{B}_1^{z_0})}$ with $\|Q_k\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}$. For any $(x,y) \in A_{1/2,1}$, we have:

$$|u_k|^2(x,y,z) \le \left|\varphi\left(\sqrt{x^2+y^2}\right)\right|^2 |Q_k(x',y',z)|^2 + \left(1-\varphi\left(\sqrt{x^2+y^2}\right)\right)^2 \left|\fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right|^2$$

which implies

$$\frac{1}{2}\int_{A_{1/2,1}} |u_k|^2(x,y,z)\mathrm{d}(x,y) \le \|1-\varphi\|^2_{L^\infty(\mathbb{R})}\cdot\frac{3}{4\pi}\cdot\frac{16}{25\pi^2}\int_{A_{1,3/2}} |Q_k|^2(x',y',z)\mathrm{d}(x,y)+$$
$$+\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_{A_{1/2,1}} |Q_k|^2(x',y',z)\mathrm{d}(x,y).$$

Using the same change of variables, the same bounds for φ and for $1-\varphi$ and integrating with respect to $z \in (-z_0, z_0)$, we get:

$$\|u_k\|^2_{L^2(\mathcal{A}^{z_0}_{1/2,1})} \le \left(2+\frac{24}{25\pi}\right)\|Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

For any $(x,y) \in \overline{B_{1/2}}$ we have:

$$|u_k|^2(x,y,z) = \left|\fint_{A_{1,3/2}} Q_k(s,t,z)\mathrm{d}(s,t)\right|^2$$

which implies

$$\|u_k\|^2_{L^2(\overline{B_{1/2}}\times(-z_0,z_0))} \le \frac{4}{25\pi}\|Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})},$$

hence

$$\|u_k\|^2_{L^2(\mathcal{B}_1^{z_0})} \le \left(3+\frac{3}{25\pi}\right)\|Q_k\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})}.$$

Combining all the relations that we have obtained, we see that for any $k \ge 1$ we have:

- $u_k \in H^1(\mathcal{B}_1^{z_0})$;
- $\|u_k\|_{L^2(\mathcal{B}_1^{z_0})} \le c_2\|Q_k\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}$, where $c_2 = \sqrt{3+\frac{3}{25\pi}}$;
- $\|\nabla u_k\|_{L^2(\mathcal{B}_1^{z_0})} \le c_3\|\nabla Q_k\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}$, where $c_3 = \max\left\{\sqrt{c_1}, c_2\right\}$.

Now, if we repeat the same argument (as the one used in order to achieve the L^2 control between u_k and Q_k) for the functions $(u_k - u)$, for any $k \geq 1$, we get:

$$\|u_k - u\|_{L^2(\mathcal{B}_1^{z_0})} \leq c_2 \|Q_k - Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})}$$

and since $Q_k \to Q$ strongly in $H^1(\mathcal{A}_{1,3/2}^{z_0})$, hence in $L^2(\mathcal{A}_{1,3/2}^{z_0})$, we obtain that $u_k \to u$ strongly in $L^2(\mathcal{B}_1^{z_0})$.

Because $Q_k \to Q$ strongly in $H^1(\mathcal{A}_{1,3/2}^{z_0})$, then $(Q_k)_{k\geq 1}$ is a bounded sequence in $H^1(\mathcal{A}_{1,3/2}^{z_0})$ and using the inequalities proved before, we get that $(u_k)_{k\geq 1}$ is a bounded sequence in $H^1(\mathcal{B}_1^{z_0})$, therefore there exists a subsequence $(u_{k_j})_{j\geq 1}$ which has the property that $u_{k_j} \rightharpoonup u_0$, with $u_0 \in H^1(\mathcal{B}_1^{z_0})$. From here, we have the following convergences in $L^2(\mathcal{B}_1^{z_0})$: $u_{k_j} \rightharpoonup u_0$ and $u_{k_j} \to u$, so $u = u_0$ a.e. in $\mathcal{B}_1^{z_0}$. However, since $u_0 \in H^1(\mathcal{B}_1^{z_0})$, we obtain that $u \in H^1(\mathcal{B}_1^{z_0})$ with $\nabla u = \nabla u_0$ a.e. in $\mathcal{B}_1^{z_0}$.

Let $\tilde{u}_x : \mathcal{B}_1^{z_0} \to \mathbb{R}$ be the function defined as:

$$\tilde{u}_x(x,y,z) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}} \cdot \varphi'(\sqrt{x^2+y^2}) \cdot \left(Q(x',y',z) - \fint_{A_{1,3/2}} Q(s,t,z) \mathrm{d}(s,t) \right) + \\ \quad + \left(\dfrac{2y^2}{\left(\sqrt{x^2+y^2}\right)^3} - 1 \right) \cdot \varphi(\sqrt{x^2+y^2}) \cdot \left(\dfrac{\partial Q}{\partial x'}(x',y',z) \right) - \\ \quad - \dfrac{2xy}{\left(\sqrt{x^2+y^2}\right)^3} \cdot \varphi(\sqrt{x^2+y^2}) \cdot \left(\dfrac{\partial Q}{\partial y'}(x',y',z) \right), \text{ for } \dfrac{1}{2} \leq \sqrt{x^2+y^2} < 1 \\ 0, \text{ for } 0 \leq \sqrt{x^2+y^2} \leq \dfrac{1}{2}. \end{cases}$$

for every $z \in (-z_0, z_0)$.

Using the same argument as before (we only control the L^2 norm), we can see that:

$$\left\| \frac{\partial u_k}{\partial x} - \tilde{u}_x \right\|_{L^2(\mathcal{B}_1^{z_0})} \leq c_3 \|\nabla Q_k - \nabla Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})}$$

and since $\nabla Q_k \to \nabla Q$ strongly in $L^2(\mathcal{A}_{1,3/2}^{z_0})$, we obtain that $\dfrac{\partial u_k}{\partial x} \to \tilde{u}_x$ strongly in $L^2(\mathcal{B}_1^{z_0})$. But at the same time, we have $\dfrac{\partial u_k}{\partial x} \rightharpoonup \dfrac{\partial u}{\partial x}$ weakly in $L^2(\mathcal{B}_1^{z_0})$, hence $\dfrac{\partial u}{\partial x} = \tilde{u}_x$ a.e. in $L^2(\mathcal{B}_1^{z_0})$ and $\dfrac{\partial u_k}{\partial x} \to \dfrac{\partial u}{\partial x}$ strongly in $L^2(\mathcal{B}_1^{z_0})$. Applying the same argument, we finally prove that $\nabla u_k \to \nabla u$ strongly in $L^2(\mathcal{B}_1^{z_0})$.

In the end, we see that:

$$\begin{aligned} \|\nabla u\|_{L^2(\mathcal{B}_1^{z_0})} &\leq \|\nabla u - \nabla u_k\|_{L^2(\mathcal{B}_1^{z_0})} + \|\nabla u_k\|_{L^2(\mathcal{B}_1^{z_0})} \\ &\leq \|\nabla u - \nabla u_k\|_{L^2(\mathcal{B}_1^{z_0})} + c_3 \|\nabla Q_k\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})} \\ &\leq \|\nabla u - \nabla u_k\|_{L^2(\mathcal{B}_1^{z_0})} + c_3 \|\nabla Q_k - \nabla Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})} + c_3 \|\nabla Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})}. \end{aligned}$$

Because $\nabla u_k \to \nabla u$ strongly in $L^2(\mathcal{B}_1^{z_0})$ and because $\nabla Q_k \to \nabla Q$ strongly in $L^2(\mathcal{A}_{1,3/2}^{z_0})$, we conclude that

$$\|\nabla u\|_{L^2(\mathcal{B}_1^{z_0})} \le c_3 \|\nabla Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})}.$$

□

Now we transform in several steps the sets $\mathcal{B}_1^{z_0}$ and $\mathcal{A}_{1,3/2}^{z_0}$ from the previous lemma into the corresponding regions related to Ω_ε and $\mathcal{N}_\varepsilon^{\mathcal{T}}$, that is, $\mathcal{B}_1^{z_0}$ into $\mathcal{P}_\varepsilon^{z,m}$, which is included in $\mathcal{N}_\varepsilon^{\mathcal{T}}$, and $\mathcal{A}_{1,3/2}^{z_0}$ into a parallelepiped with an interior hole, surrounding $\mathcal{P}_\varepsilon^{z,m}$, which is included in Ω_ε (the hole is exactly the "connecting parallelepiped" $\mathcal{P}_\varepsilon^{z,m}$).

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined as:

$$T_2(x,y,z) = \begin{cases} (0,0,z) & \text{if } x = y = 0, \\ \left(\sqrt{x^2+y^2}, \frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}, z\right) & \text{if } |y| \le x,\ x > 0, \\ \left(-\sqrt{x^2+y^2}, -\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}, z\right) & \text{if } |y| \le -x,\ x < 0, \\ \left(\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}, \sqrt{x^2+y^2}, z\right) & \text{if } |x| \le y,\ y > 0, \\ \left(-\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}, -\sqrt{x^2+y^2}, z\right) & \text{if } |x| \le -y,\ y < 0, \end{cases}$$

with the inverse

$$T_2^{-1}(\xi,\eta,z) = \begin{cases} (0,0,z) & \text{if } \eta = \xi = 0, \\ \left(\xi\cos\frac{\pi}{4}\frac{\eta}{\xi}, \xi\sin\frac{\pi}{4}\frac{\eta}{\xi}, z\right) & \text{if } |\eta| \le |\xi|,\ \xi \neq 0, \\ \left(\eta\sin\frac{\pi}{4}\frac{\xi}{\eta}, \eta\cos\frac{\pi}{4}\frac{\xi}{\eta}, z\right) & \text{if } |\xi| \le |\eta|,\ \eta \neq 0. \end{cases}$$

More specifically, $T_2(x,y,z) = (\Lambda_2(x,y),z)$, where Λ_2 is, according to [52], a bi-Lipschitz continuous map that maps, in $\mathbb{R}^2$, the unit ball into the unit cube and the Jacobian of Λ_2 is constant almost everywhere in $\mathbb{R}^2$. Hence, the transformation T_2 is bi-Lipschitz and the Jacobian of T_2 is constant almost everywhere in $\mathbb{R}^3$.

In our case, we have: $T_2(\mathcal{B}_1^{z_0}) = (-1,1)^2 \times (-z_0,z_0)$ and $T_2(\mathcal{A}_{1,3/2}^{z_0}) = \big((-3/2,3/2)^2 \setminus (-1,1)^2\big) \times (-z_0,z_0)$.

Let $u \in H^1(\mathcal{B}_1^{z_0})$, $Q \in H^1(\mathcal{A}_{1,3/2}^{z_0})$ and the constant $c > 0$ such that $\|\nabla u\|_{L^2(\mathcal{B}_1^{z_0})} \le c\|\nabla Q\|_{L^2(\mathcal{A}_{1,3/2}^{z_0})}$ be given the previous lemma, constant which is independent of z_0. Then we obtain that the functions $\tilde{u} := u \circ T_2^{-1} \in H^1((-1,1)^2 \times (-z_0,z_0))$ and $\tilde{Q} := Q \circ T_2^{-1} \in H^1\big(((-3/2,3/2)^2 \setminus (-1,1)^2) \times (-z_0,z_0)\big)$ and that there exists constants c_j and c_J, which are also independent of z_0, but dependent on the constants given by the Jacobians of T_2 and T_2^{-1}, such that:

$$c_j \|\nabla \tilde{u}\|^2_{L^2(T_2(\mathcal{B}_1^{z_0}))} \le \|\nabla u\|^2_{L^2(\mathcal{B}_1^{z_0})} \le c_J \|\nabla \tilde{u}\|^2_{L^2(T_2(\mathcal{B}_1^{z_0}))}$$

and

$$c_j\|\nabla\tilde{Q}\|^2_{L^2(T_2(\mathcal{A}^{z_0}_{1,3/2}))} \leq \|\nabla Q\|^2_{L^2(\mathcal{A}^{z_0}_{1,3/2})} \leq c_J\|\nabla\tilde{Q}\|^2_{L^2(T_2(\mathcal{A}^{z_0}_{1,3/2}))}.$$

Hence, the inequality $\|\nabla u\|_{L^2(\mathcal{B}^{z_0}_1)} \leq c\|\nabla Q\|_{L^2(\mathcal{A}^{z_0}_{1,3/2})}$ implies that there exists a constant c_0, also independent of z_0, such that:

$$\|\nabla\tilde{u}\|_{L^2(T_2(\mathcal{B}^{z_0}_1))} \leq c_0\|\nabla\tilde{Q}\|_{L^2(T_2(\mathcal{A}^{z_0}_{1,3/2}))}.$$

Now if we use the transformation $T_3(x,y,z) = \varepsilon^\alpha(x,y,z)$ and denote $\overline{u} := \tilde{u}\circ T_3^{-1}$ and $\overline{Q} := \tilde{Q}\circ T_3^{-1}$, we get:

$$\varepsilon^{-\alpha}\|\nabla\overline{u}\|^2_{L^2((T_3\circ T_2)(\mathcal{B}^{z_0}_1))} = \|\nabla\tilde{u}\|^2_{L^2(T_2(\mathcal{B}^{z_0}_1))} \leq c_0^2\|\nabla\tilde{Q}\|^2_{L^2(T_2(\mathcal{A}^{z_0}_{1,3/2}))} = c_0^2\varepsilon^{-\alpha}\|\nabla\overline{Q}\|^2_{L^2((T_3\circ T_2)(\mathcal{B}^{z_0}_1))}$$

which implies that

$$\|\nabla\overline{u}\|_{L^2((T_3\circ T_2)(\mathcal{B}^{z_0}_1))} \leq c_0\|\nabla\overline{Q}\|_{L^2((T_3\circ T_2)(\mathcal{A}^{z_0}_{1,3/2}))}.$$

Since the constant c_0 is independent of the choice of z_0, we can have $z_0 = \dfrac{r\varepsilon - \varepsilon^\alpha}{\varepsilon^\alpha}$.

The final change of variables is based on the mapping $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$ defined as: $T_4(x,y,z) = \left(\dfrac{x}{2p}, \dfrac{y}{2q}, \dfrac{z}{2r}\right)$, where p, q and r are from relation (2.2.4). In this way, if we translate the origin into the center of the parallelepiped $\mathcal{P}^{z,m}_\varepsilon$, we obtain that $(T_4\circ T_3\circ T_2)(\mathcal{B}^{z_0}_1) = \mathcal{P}^{z,m}_\varepsilon$ and we denote by $\mathcal{R}^{z,m}_\varepsilon$ the set $(T_4\circ T_3\circ T_2)(\mathcal{A}^{z_0}_{1,3/2})$, which is the box contained in Ω_ε (for ε small enough) that "surrounds" $\mathcal{P}^{z,m}_\varepsilon$.

The transformation T_4 is bi-Lipschitz and applying the same arguments as before, we obtain that there exists a function $u \in H^1(\mathcal{P}^{z,m}_\varepsilon)$ (u can be seen as $\overline{u}\circ(T_4^{-1})$) such that $u = Q$ on the "contact" faces $\mathcal{T}^m_z$ of $\mathcal{P}^{z,m}_\varepsilon$ and an ε-independent constant $c > 0$ such that

$$\|\nabla u\|_{L^2(\mathcal{P}^{z,m}_\varepsilon)} \leq c\|\nabla Q\|_{L^2(\mathcal{R}^{z,m}_\varepsilon)}.$$

Since the objects $\mathcal{R}^{z,m}_\varepsilon$ are pairwise disjoint (if we look only at the boxes surrounding the "connecting parallelepipeds" with centers in $\mathcal{Y}^z_\varepsilon$), repeating the same argument for every other "connecting parallelepiped" of this type (with centers in $\mathcal{Y}^z_\varepsilon$) and then repeating the same argument for all the others "connecting parallelepipeds" from $\mathcal{N}^\mathcal{T}_\varepsilon$ (that is, with centers in $\mathcal{Y}_\varepsilon$), we obtain $u \in H^1(\Omega_\varepsilon \cup \mathcal{N}^\mathcal{T}_\varepsilon, \mathcal{S}_0)$ an extension of $Q \in H^1(\Omega_\varepsilon, \mathcal{S}_0)$ such that:

$$\begin{cases} u = Q \text{ in } \Omega_\varepsilon \\ u = Q \text{ on } \partial\mathcal{N}^\mathcal{T}_\varepsilon \\ \|\nabla u\|_{L^2(\Omega_\varepsilon\cup\mathcal{N}^\mathcal{T}_\varepsilon)} \leq c\|\nabla Q\|_{L^2(\Omega_\varepsilon)}. \end{cases}$$

Let $\Omega'_\varepsilon = \Omega_\varepsilon \cup \mathcal{N}_\varepsilon^{\mathcal{T}}$. We want now to construct a function $v : \mathcal{N}_\varepsilon^{\mathcal{S}} \to \mathcal{S}_0$ such that $v = u$ on $\partial\Omega'_\varepsilon$ and that there exists a constant $c > 0$, independent of ε such that $\|\nabla v\|_{L^2(\mathcal{N}_\varepsilon^{\mathcal{S}})} \leq c\|\nabla u\|_{L^2(\Omega'_\varepsilon)}$.

But in the case of the family $\mathcal{N}_\varepsilon^{\mathcal{S}}$, these "inner parallelepipeds" are pairwise disjoint for ε small enough, therefore we can construct v in each $\mathcal{C}_\varepsilon^i$, for every $i \in \overline{1, N_\varepsilon}$ and control, independent of ε, $\|\nabla v\|_{L^2(\mathcal{C}_\varepsilon^i)}$ with $\|\nabla u\|_{L^2(\mathcal{P}_\varepsilon^i)}$, where $\mathcal{R}_\varepsilon^i$ is the "surrounding" box for $\mathcal{C}_\varepsilon^i$, constructed in the same way as $\mathcal{R}_\varepsilon^{z,m}$.

Lemma 2.7.2. Let $a, b \in \mathbb{R}_+^*$ with $a < b$, let $\mathcal{B}_a = \{x \in \mathbb{R}^3 \mid |x| < a\}$ and let $\mathcal{A}_{a,b} = \mathcal{B}_b \setminus \overline{\mathcal{B}_a}$.

Let $u \in H^1(\mathcal{A}_{1,2}, \mathcal{S}_0)$. Then the function $v : \mathcal{B}_1 \to \mathcal{S}_0$ defined as

$$v(x,y,z) = \begin{cases} \varphi(\sqrt{x^2+y^2+z^2})\, u\left(\left(\dfrac{2}{\sqrt{x^2+y^2+z^2}} - 1\right)(x,y,z)\right) + \\ \quad +(1-\varphi(\sqrt{x^2+y^2+z^2})) \displaystyle\fint_{\mathcal{A}_{1,3/2}} u(\xi,\eta,\tau)\mathrm{d}(\xi,\eta,\tau), \\ \qquad \textit{for } \dfrac{1}{2} \leq \sqrt{x^2+y^2+z^2} < 1 \\ \displaystyle\fint_{\mathcal{A}_{1,3/2}} u(\xi,\eta,\tau)\mathrm{d}(\xi,\eta,\tau), \textit{ for } 0 \leq \sqrt{x^2+y^2+z^2} \leq \dfrac{1}{2} \end{cases}$$

is from $H^1(\mathcal{B}_1, \mathcal{S}_0)$, where $\varphi \in C_c^\infty\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ is the following bump function defined as

$$\varphi(\rho) = \begin{cases} \exp\left\{4 - \dfrac{4}{(2\rho-1)(3-2\rho)}\right\}, \ \forall \rho \in \left(\dfrac{1}{2}, \dfrac{3}{2}\right) \\ 0, \ \forall \rho \in \mathbb{R} \setminus \left(\dfrac{1}{2}, \dfrac{3}{2}\right) \end{cases},$$

the product $\varphi(\rho)\, u$ represents product between a scalar and a Q-tensor and $\fint$ represents the average integral sign. Moreover, there exists a constant $c > 0$ such that: $\|v_t\|_{L^2(\mathcal{B}_1)} \leq c\|u_t\|_{L^2(\mathcal{A}_{1,3/2})}$ for any $t \in \{x, y, z\}$, where v_t represents the partial derivative of v with respect to t, and:

$$\|\nabla v\|_{L^2(\mathcal{B}_1)} \leq c\|\nabla u\|_{L^2(\mathcal{A}_{1,3/2})}.$$

Remark 2.7.2. Lemma 2.7.2 is just a different version of Lemma 2.7.1. The proof follows the same steps as in Lemma 2.7.1.

Now if we use instead of T_2 the transformation Λ_3, from [52], which is a bi-Lipschitz mapping that transforms the unit ball into the unit cube, and then the transformations T_3 and T_4 as before, we end up with the function v being an extension of u that satisfies:

$$\begin{cases} v \in H^1(\mathcal{C}_\varepsilon^i) \\ v = u \text{ on } \partial\mathcal{C}_\varepsilon^i \\ \|\nabla v\|_{L^2(\mathcal{C}_\varepsilon^i)} \leq c\|\nabla u\|_{L^2(\mathcal{R}_\varepsilon^i)}. \end{cases}$$

Because the objects $\mathcal{R}_\varepsilon^i$ are pairwise disjoint for ε small enough, we construct therefore an extension $v \in H^1(\Omega, \mathcal{S}_0)$ of $u \in H^1(\Omega'_\varepsilon, \mathcal{S}_0)$ such that:

$$\begin{cases} v = u \text{ in } \Omega'_\varepsilon \Rightarrow v = Q \text{ in } \Omega_\varepsilon \text{ and } v = Q \text{ on } \partial\mathcal{N}_\varepsilon^{\mathcal{T}} \\ v = u \text{ on } \partial\mathcal{N}_\varepsilon^{\mathcal{S}} \Rightarrow v = Q \text{ on } \partial\mathcal{N}_\varepsilon^{\mathcal{S}} \\ \|\nabla v\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega'_\varepsilon)} \leq \tilde{c}\|\nabla Q\|_{L^2(\Omega_\varepsilon)}. \end{cases}$$

So we have $v \in H^1(\Omega)$, $v = Q$ in Ω_ε, $v = Q$ on $\partial\mathcal{N}_\varepsilon$ and there exists an ε-independent constant such that:

$$\|\nabla v\|_{L^2(\Omega)} \leq c\|\nabla Q\|_{L^2(\Omega_\varepsilon)}.$$

2.7.4 INTEGRATED ENERGY DENSITIES

In this subsection, we present two propositions that are used in order to prove that using relation (2.3.2), that is:

$$f_{hom}(Q) = \frac{2}{p}\int_{\partial\mathcal{C}} f_s(Q,\nu)\mathrm{d}\sigma,$$

then by using, for example, the choice of the *surface energy density* defined in (2.3.8), which is:

$$f_s^{LDG}(Q,\nu) = \frac{p}{4}\Big((a'-a)(\nu\cdot Q^2\nu) - (b'-b)(\nu\cdot Q^3\nu) + 2(c'-c)(\nu\cdot Q^4\nu)\Big),$$

we can obtain the corresponding homogenised functional defined in (2.3.9), that is:

$$f_{hom}^{LDG}(Q) = (a'-a)\,\mathrm{tr}(Q^2) - (b'-b)\,\mathrm{tr}(Q^3) + (c'-c)\,\big(\mathrm{tr}(Q^2)\big)^2.$$

More specifically, Proposition 2.7.4 treats the case of the classical quartic polynomial in the scalar invariants of Q for the *bulk energy*, defined in (2.1.2), where the choice of the *surface energy density* is in (2.3.8), and the more general version of it, defined in (2.3.4), with the *surface energy density* defined in (2.3.16). Both cases have all of the terms from the picked *surface energy densities* of the form $\nu\cdot Q^k\nu$, with $k \geq 2$. Proposition 2.7.5 treats only the Rapini-Papoular case, where the *surface energy density* is defined in (2.3.12).

Proposition 2.7.4. For any $k \in \mathbb{N}$, $k \geq 2$ and for a fixed matrix $Q \in \mathcal{S}_0$, we have:

$$\mathrm{tr}(Q^k) = \frac{1}{2}\int_{\partial\mathcal{C}} (\nu\cdot Q^k\nu)\mathrm{d}\sigma,$$

where $\partial\mathcal{C}$ is defined in (2.2.3) and ν is the exterior unit normal to $\partial\mathcal{C}$.

Proof. Let $Q^k = \begin{pmatrix} q_{11,k} & q_{12,k} & q_{13,k} \\ q_{21,k} & q_{22,k} & q_{23,k} \\ q_{13,k} & q_{32,k} & q_{33,k} \end{pmatrix}$. According to (2.2.3), we have $\partial\mathcal{C} = \mathcal{C}^x \cup \mathcal{C}^y \cup \mathcal{C}^z$. We compute first the intergral for $\mathcal{C}^x$, on which $\nu = (\pm 1, 0, 0)^T$:

$$\int_{\mathcal{C}^x} (\nu \cdot Q^k \nu) \mathrm{d}\sigma = \int_{\mathcal{C}^x} ((\pm 1, 0, 0)^T \cdot (\pm q_{11,k}, \pm q_{21,k}, \pm q_{31,k})^T) \mathrm{d}\sigma = \int_{\mathcal{C}^x} q_{11,k} \mathrm{d}\sigma = 2q_{11,k},$$

since $\mathcal{C}$ has length 1.

In the same way, we obtain:

$$\int_{\mathcal{C}^y} (\nu \cdot Q^k \nu) \mathrm{d}\sigma = 2q_{22,k} \quad \text{and} \quad \int_{\mathcal{C}^z} (\nu \cdot Q^k(x_0)\nu) \mathrm{d}\sigma = 2q_{33,k},$$

from which we obtain

$$\int_{\partial\mathcal{C}} (\nu \cdot Q^k \nu) \mathrm{d}\sigma = 2\mathrm{tr}(Q^k).$$

□

For the Rapini-Papoular case, we prove that:

Proposition 2.7.5. For a fixed matrix $Q \in \mathcal{S}_0$, we have:

$$6\mathrm{tr}(Q^2) + 4 = \int_{\partial\mathcal{C}} \mathrm{tr}(Q - Q_\nu)^2 \mathrm{d}\sigma,$$

where $\partial\mathcal{C}$ is defined in (2.2.3), $Q_\nu = \nu \otimes \nu - \mathbb{I}_3/3$, ν represents the exterior unit normal to $\partial\mathcal{C}$ and $\mathbb{I}_3$ is the 3×3 identity matrix.

Proof. First of all, we can see that $\mathrm{tr}(Q - Q_\nu)^2 = \mathrm{tr}(Q^2) - \mathrm{tr}(QQ_\nu) - \mathrm{tr}(Q_\nu Q) + \mathrm{tr}(Q_\nu^2)$.

According to (2.2.3), we have $\partial\mathcal{C} = \mathcal{C}_x \cup \mathcal{C}_y \cup \mathcal{C}_z$. On $\mathcal{C}^x$, we have $\nu = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}$ and $Q_\nu = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$. Then $\mathrm{tr}(Q_\nu^2) = \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{2}{3}$. We also obtain $\mathrm{tr}(Q_\nu^2) = \frac{2}{3}$ on $\mathcal{C}^y$ and $\mathcal{C}^z$. Therefore:

$$\int_{\partial\mathcal{C}} \mathrm{tr}(Q_\nu^2) \mathrm{d}\sigma = 6 \cdot \frac{2}{3} = 4,$$

where the constant 6 comes from the total surface of the cube $\mathcal{C}$.

Let $Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & -q_{11} - q_{22} \end{pmatrix}$. Using the computations done earlier for Q_ν on $\mathcal{C}^x$, $\mathcal{C}^y$ and $\mathcal{C}^z$, we get:

$$\begin{aligned}
\int_{\mathcal{C}^x} \big(\mathrm{tr}(Q_\nu Q) + \mathrm{tr}(QQ_\nu)\big) \mathrm{d}\sigma &= 2\left(\frac{2q_{11}}{3} - \frac{q_{22}}{3} + \frac{q_{11} + q_{22}}{3}\right) = 2q_{11} \\
\int_{\mathcal{C}^y} \big(\mathrm{tr}(Q_\nu Q) + \mathrm{tr}(QQ_\nu)\big) \mathrm{d}\sigma &= 2\left(- \frac{q_{11}}{3} + \frac{2q_{22}}{3} + \frac{q_{11} + q_{22}}{3}\right) = 2q_{22} \\
\int_{\mathcal{C}^z} \big(\mathrm{tr}(Q_\nu Q) + \mathrm{tr}(QQ_\nu)\big) \mathrm{d}\sigma &= 2\left(- \frac{q_{11}}{3} - \frac{q_{22}}{3} - \frac{2q_{11} + 2q_{22}}{3}\right) = -2q_{11} - 2q_{22}.
\end{aligned}$$

Combining the last three relations, we get that

$$\int_{\partial\mathcal{C}} \big(\mathrm{tr}(Q_\nu Q) + \mathrm{tr}(QQ_\nu)\big) \mathrm{d}\sigma = 0,$$

from which the conclusion follows, with the observation that the constant 6 in front of $\mathrm{tr}(Q^2)$ appears from the total surface of the cube $\mathcal{C}$, which has the length equal to 1. □

Remark 2.7.3. The constant 4 from Proposition 2.7.5 is neglected when we are studying the asymptotic behaviour of the minimisers of the functional (2.3.13), since adding constants do not influence the form and the existence of the possible minimisers.

3

ERROR ESTIMATES FOR RUGOSITY EFFECTS

Abstract:

We consider a nematic liquid crystal, described by a quadratic free energy in the Landau-de Gennes model, contained in a two-dimensional slab with one periodic oscillating boundary, with the amplitude described by a small parameter $\varepsilon > 0$. We consider the case in which these fine-scale boundary oscillations may be replaced, in the limit as $\varepsilon \to 0$, by an effective homogenised surface energy on a flat boundary, as in [31]. The focus in this chapter is to obtain error estimates for how fast the solutions of the rugose problem converge to the homogenised one, by using duality arguments in L^p spaces, for any $p \in [2, +\infty)$.

Joint work with J. M. Taylor and A. D. Zarnescu. This chapter is part of the preprint [31], which has been accepted for publishing in Communications in Contemporary Mathematics.

3.1 INTRODUCTION OF THE PROBLEM

Consider an ℓ-periodic slab domain in two-dimensions, which represents the typical geometry of liquid crystal experiments, given explicitly as

$$\Omega' = \{(x', y') \in \mathbb{R}^2 : \psi'(x') < y' < R'\},$$

in which $\psi' : \mathbb{R} \to \mathbb{R}$ is an ℓ-periodic function and let $\partial\Omega' = \Gamma'_{\psi'} \cup \Gamma'_R$, where we denote $\Gamma'_{\psi'} = \{(x', \psi(x')) : x' \in \mathbb{R}\}$ and $\Gamma'_R = \{(x', R') : x' \in \mathbb{R}\}$. We consider a toy model, representative of paranematic systems as in [17, 49, 72], over $Q' \in \mathrm{Sym}_0(2) = \{A \in \mathbb{R}^{2\times 2} : A^T = A, \operatorname{Tr}(A) = 0\}$. We consider solutions that respect the symmetry of the domain, that is, $Q' \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathrm{Sym}_0(2))$ such that $Q'(x' + \ell, y') = Q'(x', y')$ for almost every x', y'. The free energy per periodic cell, $C\Omega'$, is to be given as

$$\mathcal{F}'(Q') = \int_{C\Omega'} \frac{L'}{2}|\nabla Q'|^2 + \frac{c'}{2}|Q'|^2 \,\mathrm{d}x'\,\mathrm{d}y' + \int_{C\partial\Omega'} \frac{w'_0}{2}\left|Q' - s'_0\left(\nu\otimes\nu - \frac{1}{2}I\right)\right|^2 \mathrm{d}\sigma(x').$$

Here $C\Omega' = \{(x', y') \in \mathbb{R}^2 : \psi'(x') < y' < R',\ 0 \le x' < \ell\}$, $c' > 0$, $C\partial\Omega' = C\Gamma'_{\psi'} \cup C\Gamma'_R$, with $C\Gamma'_{\psi'} = \{(x', \psi'(x')) : x' \in [0, \ell)\}$ and $C\Gamma'_R = \{(x', R') : x' \in [0, \ell)\}$, $w'_0 > 0$, $s'_0 \in \mathbb{R}$ and ν is the exterior normal. We may non-dimensionalise the system by considering variables $(x, y) = \frac{2\pi}{\ell}(x', y')$, $Q(x, y) = \frac{1}{s'_0}Q'(x', y')$, $\mathcal{F} = \frac{1}{L'(s'_0)^2}\mathcal{F}'$, $c = \frac{c'\ell^2}{4L'\pi^2}$, $w_0 = \frac{w'_0\ell}{2L'\pi}$, $R = \frac{2\pi}{\ell}R'$, $\psi(x) = \psi'(x')$ to give

$$\mathcal{F}(Q) = \int_\Omega |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega} \frac{w_0}{2}\left|Q - \left(\nu\otimes\nu - \frac{1}{2}I\right)\right|^2 \mathrm{d}\sigma(x),$$

with $C\Omega'$ now rescaled as

$$\Omega = \{(x, y) : 0 \le x < 2\pi,\ \psi(x) < y < R\}.$$

Moreover, we can write $\partial\Omega = \Gamma_\psi \cup \Gamma_R$, where $\Gamma_\psi = \{(x, y) : x \in [0, 2\pi),\ y = \psi(x)\}$ and $\Gamma_R = \{(x, R) : x \in [0, 2\pi)\}$, so that we have:

$$\mathcal{F}(Q) = \int_\Omega |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_\psi} \frac{w_0}{2}\left|Q - Q_\psi\right|^2 \mathrm{d}\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2}\left|Q - Q_R\right|^2 \mathrm{d}\sigma_R,$$

with $Q_\psi = \nu_\psi \otimes \nu_\psi - \frac{1}{2}I$ and $Q_R = \nu_R \otimes \nu_R - \frac{1}{2}I$, where ν_ψ and ν_R are the outward normals to Γ_ψ and Γ_R.

3.2 TECHNICAL ASSUMPTIONS AND MAIN RESULT

Assumption 3.2.1. Let $\varepsilon > 0$. We assume that $\varphi_\varepsilon(x) = \varepsilon \cdot \varphi(x/\varepsilon)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2 2π-periodic function with $\varphi \ge 0$.

Let $\Omega_\varepsilon = \{(x, y) : x \in [0, 2\pi),\ \varphi_\varepsilon(x) < y < R\}$ with oscillating boundary $\Gamma_\varepsilon = \{(x, \varphi_\varepsilon(x)) : x \in [0, 2\pi)\}$ and we are interested in studying the following free energy functional:

$$\mathcal{F}_\varepsilon(Q) = \int_{\Omega_\varepsilon} |\nabla Q|^2 + c|Q|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_\varepsilon} \frac{w_0}{2} \left|Q - Q_\varepsilon^0\right|^2 \mathrm{d}\sigma_\varepsilon + \int_{\Gamma_R} \frac{w_0}{2} \left|Q - Q_R\right|^2 \mathrm{d}\sigma_R, \quad (3.2.1)$$

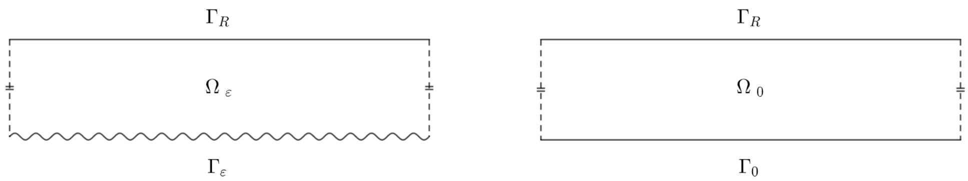
with $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$ and $Q_R = \nu_R \otimes \nu_R - \frac{1}{2}I$, where ν_ε and ν_R are the outward normals to Γ_ε and Γ_R.

We also consider the limit domain

$$\Omega_0 = \{(x, y) : x \in [0, 2\pi),\ 0 < y < R\}$$

with $\partial\Omega_0 = \Gamma_0 \cup \Gamma_R$, where $\Gamma_0 = \{(x, 0) : x \in [0, 2\pi)\}$.

Remark 3.2.1. Using Assumption 3.2.1, we obtain that $\Omega_\varepsilon \subset \Omega_0$ for all $\varepsilon > 0$ and that $\Omega_\varepsilon \to \Omega_0$ as $\varepsilon \to 0$.



(a) The oscillating domain Ω_ε.

(b) The limit domain Ω_0.

Remark 3.2.2. In the next sections, we write the first derivative of φ as φ'. Moreover, we write $\|\varphi\|_\infty$ and $\|\varphi'\|_\infty$ instead of $\|\varphi\|_{L^\infty([0,2\pi))}$ and $\|\varphi'\|_{L^\infty([0,2\pi))}$.

Assumption 3.2.2. We assume that:

$$0 < \varepsilon = \frac{1}{2k} < \|\varphi\|_\infty^{-1} \cdot \frac{R}{2}, \text{ with } k \in \mathbb{N}^*,\ k > \|\varphi\|_\infty \cdot \frac{1}{R}.$$

Remark 3.2.3. Using Assumption 3.2.2, we obtain that $\{(x, y) \mid x \in [0, 2\pi),\ y \in (R/2, R)\} \subset \Omega_\varepsilon$, which tells us that the oscillations of Γ_ε have an amplitude lower than half of the height of the domain Ω_0.

Remark 3.2.4. Using Assumption 3.2.1, the arclength parameter of the curve Γ_ε can be described as the function $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$, defined as:

$$\gamma_\varepsilon(t) = \sqrt{1 + \left(\varphi_\varepsilon{}'(t)\right)^2} = \sqrt{1 + \left(\varphi'(t/\varepsilon)\right)^2},\ \forall t \in \mathbb{R} \quad (3.2.2)$$

and we obtain that:

$$1 \le \gamma_\varepsilon(t) < \sqrt{1 + \|\varphi'\|_\infty^2},\ \forall t \in \mathbb{R}. \quad (3.2.3)$$

Moreover, the outward normal ν_ε to Γ_ε has the following form:

$$\nu_\varepsilon := \nu_\varepsilon(x) = \frac{1}{\gamma_\varepsilon(x)}({\varphi_\varepsilon}'(x), -1) = \frac{1}{\sqrt{1 + \left(\varphi'(x/\varepsilon)\right)^2}}(\varphi'(x/\varepsilon), -1), \tag{3.2.4}$$

for all $x \in [0, 2\pi)$.

Definition 3.2.1. Let γ_0, g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ be real functions defined as

$$\gamma_0(t) = \sqrt{1 + \left(\varphi'(t)\right)^2}, \quad g_1(t) := \frac{\left(\varphi'(t)\right)^2 - 1}{2\left(1 + \left(\varphi'(t)\right)^2\right)} \quad \text{and} \quad g_2(t) := \frac{-2\varphi'(t)}{2\left(1 + \left(\varphi'(t)\right)^2\right)}$$

for all $t \in \mathbb{R}$ and let $\gamma, G_1, G_2 \in \mathbb{R}$ be defined as:

$$\gamma := \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(t)\,\mathrm{d}t, \quad G_1 := \frac{1}{2\pi}\int_0^{2\pi} g_1(t)\gamma_0(t)\,\mathrm{d}t \quad \text{and} \quad G_2 := \frac{1}{2\pi}\int_0^{2\pi} g_2(t)\gamma_0(t)\,\mathrm{d}t.$$

Remark 3.2.5. Since $Q_\varepsilon^0 = \nu_\varepsilon \otimes \nu_\varepsilon - \frac{1}{2}I$, then $Q_\varepsilon^0(x) = \begin{pmatrix} g_1(x/\varepsilon) & g_2(x/\varepsilon) \\ g_2(x/\varepsilon) & -g_1(x/\varepsilon) \end{pmatrix}$, for all $x \in [0, 2\pi)$. Moreover, γ, G_1 and G_2 are constants and $\gamma \geq 1$.

Definition 3.2.2. Let $w_{ef} := \gamma w_0$ and $Q_{ef} := \dfrac{1}{\gamma}\begin{pmatrix} G_1 & G_2 \\ G_2 & -G_1 \end{pmatrix}$.

Remark 3.2.6. In this simplified model, $w_{ef} \in \mathbb{R}$ is constant and, since $\gamma \geq 1$, we have $w_{ef} \geq w_0$ (also observed in [31, Section 3.2]). Moreover, $Q_{ef} \in \mathrm{Sym}_0(2)$ is a constant Q-tensor.

Proposition 3.2.1. We have $\gamma_\varepsilon(\cdot) \rightharpoonup \gamma$ and $\gamma_\varepsilon(\cdot)Q_\varepsilon^0(\cdot) \rightharpoonup \gamma Q_{ef}$ in $L^2([0, 2\pi))$, as $\varepsilon \to 0$.

Proof. Since φ and φ' are 2π-periodic, according to Assumption 3.2.1, and $\varepsilon^{-1} \in \mathbb{N}^*$, according to Assumption 3.2.2, then the functions γ_0, $g_1\gamma_0$ and $g_2\gamma_0$ are also 2π-periodic, which implies that $\gamma_0(\cdot/\varepsilon)$, $(g_1 \cdot \gamma_0)(\cdot/\varepsilon)$ and $(g_2 \cdot \gamma_0)(\cdot/\varepsilon)$ tend to γ, G_1 and, respectively, G_2, as $\varepsilon \to 0$ (see for instance [35, Lemma 9.1]). This implies the conclusion. □

Definition 3.2.3. Let Q_ε be a minimiser of the functional $\mathcal{F}_\varepsilon$. This implies that Q_ε verifies the following Euler-Lagrange equations:

$$\begin{cases} -\Delta Q_\varepsilon + cQ_\varepsilon = 0 & \text{in } \Omega_\varepsilon; \\ \dfrac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} + \dfrac{w_0}{2}Q_\varepsilon = \dfrac{w_0}{2}Q_\varepsilon^0 & \text{on } \Gamma_\varepsilon; \\ \dfrac{\partial Q_\varepsilon}{\partial \nu_R} + \dfrac{w_0}{2}Q_\varepsilon = \dfrac{w_0}{2}Q_R & \text{on } \Gamma_R. \end{cases} \tag{3.2.5}$$

Remark 3.2.7. We prove in Section 3.3 that there exists a unique $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ solution of (3.2.5), for any $1 < p < +\infty$.

Definition 3.2.4. Let $\nu_0(\cdot, 0) = (0, -1)$ be the outward normal of Γ_0. We consider the following PDE:

$$\begin{cases} -\Delta Q_0 + cQ_0 = 0 & \text{in } \Omega_0; \\ \dfrac{\partial Q_0}{\partial \nu_0} + \dfrac{w_{ef}}{2} Q_0 = \dfrac{w_{ef}}{2} Q_{ef} & \text{on } \Gamma_0; \\ \dfrac{\partial Q_0}{\partial \nu_R} + \dfrac{w_0}{2} Q_0 = \dfrac{w_0}{2} Q_R & \text{on } \Gamma_R. \end{cases} \tag{3.2.6}$$

Remark 3.2.8. In Section 3.3, we prove that we have a unique $Q_0 \in W^{2,p}(\Omega_0)$ solution of (3.2.6), for any $p \in (1, +\infty)$.

Remark 3.2.9. Under these assumptions, as a consequence of [31, Theorem 1.1], we have $Q_\varepsilon \overset{R}{\rightarrow} Q_0$, that is, the *rugose* convergence in the sense from [31, Definition 2.5]. To be more specific, let $p \in (1, \infty)$, $D \in \mathbb{R}^2$ such that $\Omega_\varepsilon \subset \Omega_0 \subset\subset D$ and we formally denote by E_D the extension by 0 in $D \setminus \Omega_0$ or $D \setminus \Omega_\varepsilon$. We say that $Q_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ converges to $Q_0 \in W^{1,p}(\Omega_0)$ in a rugose sense, denoted $Q_\varepsilon \overset{R}{\rightarrow} Q_0$, if for any $U \subset\subset \Omega_\varepsilon$ we have $Q_\varepsilon\big|_U \rightharpoonup Q_0\big|_U$ weakly in $W^{1,p}(U)$ and if $E_D Q_\varepsilon \rightharpoonup E_D Q_0$ and $E_D \nabla Q_\varepsilon \rightharpoonup E_D \nabla Q_0$ weakly in $L^p(D)$. However, in this simplified case, we can obtain *quantitative estimates*, which are presented in the main theorem of this chapter:

Theorem 3.2.1. For any $p \in (2, +\infty)$, there exists an ε-independent constant C such that:

$$\|Q_0 - Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \cdot \varepsilon^{\frac{p-1}{p}}, \tag{3.2.7}$$

where the constant C depends on c, w_0, p, $\|\varphi\|_\infty$, $\|\varphi'\|_\infty$, Ω_0 and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

Remark 3.2.10. The constant C from Theorem 3.2.1 can actually be chosen of the following form:

$$C = \max\{1, \|\varphi\|_\infty^{(p-1)/p}\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C(w_0, c, p, \Omega_0, Q_0),$$

where $C(w_0, c, p, \Omega_0, Q_0)$ is an ε-independent constant depending only on w_0, c, p, Ω_0 and $\|Q_0\|_{W^{1,\infty}(\Omega_0)}$.

3.3 REGULARITY OF Q_ε AND Q_0

In this section, we prove that there exists a unique solution Q_ε of (3.2.5) and a unique solution Q_0 of (3.2.6), with $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ and $Q_0 \in W^{2,p}(\Omega_0)$, for any $p \in (1, +\infty)$.

Remark 3.3.1. It is easy to see that problems (3.2.5) and (3.2.6) admit solutions from $W^{1,2}(\Omega_0)$ and $W^{1,2}(\Omega_\varepsilon)$ by using, for example, direct methods, such as in [37], or the approach via Lax-Milgram theorem for elliptic problems, such as in [44] or [53].

Definition 3.3.1. We denote by Φ_{polar} the polar coordinates transform:

$$\Phi_{polar}(x,y) = (y\cos x, y\sin x),\ \forall (x,y) \in [0,2\pi) \times (0,R),$$

by Φ_{tl} the following translation:

$$\Phi_{tl}(x,y) = (x, y+2R),\ \forall (x,y) \in [0,2\pi) \times (0,R)$$

and let $\Phi : \Omega_0 \to \Phi(\Omega_0)$ be defined as $\Phi = \Phi_{polar} \circ \Phi_{tl}$. We define

$$\mathcal{U}_\varepsilon = \Phi(\Omega_\varepsilon) \quad \text{and} \quad \mathcal{U}_0 = \Phi(\Omega_0).$$

Definition 3.3.2. For any $a, b \in \mathbb{R}$ with $a, b > 0$, we denote

$$A_{a,b} = \{(y\cos x, y\sin x) \mid x \in [0,2\pi),\ y \in (a,b)\}.$$

Remark 3.3.2. The transformation $\Phi : \Omega_0 \to \mathcal{U}_0$ is smooth and bi-Lipschitz. Moreover, using Assumption 3.2.1, we have that $\mathcal{U}_\varepsilon$ is a bounded open domain from $\mathbb{R}^2$ with a C^2 boundary and, using Assumption 3.2.2, we have that $A_{5R/2,3R} \subset \mathcal{U}_\varepsilon \subset \mathcal{U}_0 = A_{2R,3R}$.

In order to prove that Q_ε and Q_0 admit $W^{2,p}$ regularity, we use [53, Theorem 2.4.2.6]:

Theorem 3.3.1. Let Ω be a bounded open subset of $\mathbb{R}^n$, with a $C^{1,1}$ boundary. Let $a_{i,j}$ and b_i be uniformly Lipschitz functions in $\overline{\Omega}$ and let a_i be bounded measurable functions in $\overline{\Omega}$. Assume that $a_{i,j} = a_{j,i}$, $1 \leq i,j \leq n$ and that there exists $\alpha > 0$ with

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq -\alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \overline{\Omega}$. Assume in addition that $a_0 \geq \beta > 0$ a.e. in Ω and that

$$b_0 b_\nu = b_0 \sum_{j=1}^{n} b_j \nu^j \geq 0, \quad b_\nu \neq 0$$

on $\Gamma = \partial\Omega$. Then for every $f \in L^p(\Omega)$ and every $g \in W^{1-1/p,p}(\Gamma)$, there exists a unique $u \in W^{2,p}(\Omega)$, which is a solution of

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} D_i(a_{i,j}D_j u) + \sum_{i=1}^{n} a_i D_i u + a_0 u = f & \text{in } \Omega \\ \displaystyle\mathrm{Tr}\left(\sum_{j=1}^{n} b_j D_j u + b_0 u\right) = g & \text{on } \Gamma \end{cases}$$

where Tr is the trace operator.

Corollary 3.3.1. For any $p \in (1, +\infty)$, there exists a unique $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ which solves the problem (3.2.5).

The proof of this corollary can be found in Section 3.6. In a similar way, we can show that:

Corollary 3.3.2. For any $p \in (1, +\infty)$, there exists a unique $Q_0 \in W^{2,p}(\Omega_0)$ which solves the problem (3.2.6).

Remark 3.3.3. Using the method of separation of variables, one can find that Q_0 is of the form:

$$Q_0(x, y) = c_1 \cdot e^{y\sqrt{c}} + c_2 \cdot e^{-y\sqrt{c}},$$

where c_1 and c_2 are two constant Q-tensors that can be found from:

$$\begin{cases} c_1 \cdot \left(\frac{w_{ef}}{2} - \sqrt{c}\right) + c_2 \cdot \left(\frac{w_{ef}}{2} + \sqrt{c}\right) = \frac{w_{ef}}{2} \cdot Q_{ef}; \\ c_1 \cdot e^{R\sqrt{c}} \cdot \left(\frac{w_0}{2} + \sqrt{c}\right) + c_2 \cdot e^{-R\sqrt{c}} \cdot \left(\frac{w_0}{2} - \sqrt{c}\right) = \frac{w_0}{2} \cdot Q_R. \end{cases}$$

3.4 SOME INTEGRAL INEQUALITIES

Definition 3.4.1. Let $p \in (1, +\infty)$. Let us consider the trace operator $\mathrm{Tr} : W^{1,p}(\Omega_0) \to W^{1-1/p,p}(\partial\Omega_0)$. We denote $C_{tr}(p, \Omega_0)$ the constant given by the trace inequality, that is:

$$\|\mathrm{Tr}(\omega)\|_{W^{1-1/p,p}(\partial\Omega_0)} \leq C_{tr}(p, \Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_0)}, \ \forall \omega \in W^{1,p}(\Omega_0).$$

Remark 3.4.1. For notation simplicity, we choose to write $v(\cdot, 0)$ instead of $[\mathrm{Tr}(v)]\big|_{\Gamma_0}(\cdot, 0)$, whenever $v \in W^{1,p}(\Omega_0)$.

Definition 3.4.2. We consider the following bilinear functional on $W^{1,2}(\Omega_\varepsilon) \times W^{1,2}(\Omega_\varepsilon)$:

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + c \cdot u \cdot v) \mathrm{d}(x, y) + \frac{w_0}{2}\left(\int_{\Gamma_\varepsilon} u \cdot v \, \mathrm{d}\sigma_\varepsilon + \int_{\Gamma_R} u \cdot v \, \mathrm{d}\sigma_R\right),$$

for any $u, v \in W^{1,2}(\Omega_\varepsilon)$.

Remark 3.4.2. For notation simplicity, whenever we choose $v \in W^{1,2}(\Omega_0)$, we write $a_\varepsilon(\cdot, v)$ instead of $a_\varepsilon(\cdot, v|_{\Omega_\varepsilon})$.

The goal of this section is to prove the following proposition:

Proposition 3.4.1. Let $p \in (2, +\infty)$. Then there exists an ε-independent constant C_I such that:

$$|a_\varepsilon(Q_0 - Q_\varepsilon, v)| \leq C_I \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)}, \ \forall \ v \in W^{1,p}(\Omega_0).$$

Remark 3.4.3. In Section 3.6, we prove that $a_\varepsilon(\cdot, v)$ is well-defined for all $v \in W^{1,p}(\Omega_0)$. Moreover, in order to obtain Proposition 3.4.1, we split $a_\varepsilon(Q_0 - Q_\varepsilon, v)$ into several parts, which are presented in the following definition.

Definition 3.4.3. Let $v \in W^{1,p}(\Omega_0)$. We denote:

$$
\begin{aligned}
I_1 &= -\int_{\Omega_0\setminus\Omega_\varepsilon} \big(\nabla Q_0 \cdot \nabla v + c \cdot Q_0 \cdot v\big) \mathrm{d}(x,y), \\
I_{21} &= -\frac{w_0}{2}\int_0^{2\pi} \big(v(x,\varepsilon\varphi(x/\varepsilon)) - v(x,0)\big) \cdot Q_\varepsilon^0(x) \cdot \gamma_\varepsilon(x)\, \mathrm{d}x, \\
I_{31} &= \frac{w_0}{2}\int_0^{2\pi} \big(Q_0(x,\varepsilon\varphi(x/\varepsilon)) \cdot v(x,\varepsilon\varphi(x/\varepsilon)) - Q_0(x,0)\cdot v(x,0)\big) \cdot \gamma_\varepsilon(x)\, \mathrm{d}x, \\
I_{22} &= \frac{w_0}{2}\int_0^{2\pi} v(x,0) \cdot \Big(\gamma Q_{ef} - \gamma_\varepsilon(x) Q_\varepsilon^0(x)\Big)\, \mathrm{d}x, \\
I_{32} &= -\frac{w_0}{2}\int_0^{2\pi} Q_0(x,0) \cdot v(x,0) \cdot \Big(\gamma - \gamma_\varepsilon(x)\Big)\, \mathrm{d}x.
\end{aligned}
$$

Proposition 3.4.2. We have

$$
a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 + I_{21} + I_{22} + I_{31} + I_{32},\ \forall\, v \in W^{1,p}(\Omega_0). \tag{3.4.1}
$$

Remark 3.4.4. The proof of Proposition 3.4.2 can be found in Section 3.6. Before proving Proposition 3.4.1, we obtain first estimates for each of the integrals from Definition 3.4.3.

Proposition 3.4.3. Let $p \in (1, +\infty)$. Then for any $v \in W^{1,p}(\Omega_0)$, we have:

$$
|I_1| \le C_1 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0\setminus\Omega_\varepsilon)},\ \forall v \in W^{1,p}(\Omega_0),
$$

where

$$
I_1 = -\int_{\Omega_0\setminus\Omega_\varepsilon} \big(\nabla Q_0 \cdot \nabla v + c \cdot Q_0 \cdot v\big) \mathrm{d}(x,y).
$$

and

$$
C_1 = (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \max\{c, 1\}. \tag{3.4.2}
$$

Proof. We apply Hölder inequality with coefficients p and $p' = \dfrac{p}{p-1}$:

$$
\begin{aligned}
|I_1| &\le \int_{\Omega_0\setminus\Omega_\varepsilon} |\nabla Q_0 \cdot \nabla v| + c|Q_0 \cdot v|\, \mathrm{d}(x,y) \\
&\le \left(\int_{\Omega_0\setminus\Omega_\varepsilon} |\nabla Q_0|^{p'}\, \mathrm{d}(x,y)\right)^{1/p'} \cdot \left(\int_{\Omega_0\setminus\Omega_\varepsilon} |\nabla v|^{p}\, \mathrm{d}(x,y)\right)^{1/p} + \\
&\quad + c \cdot \left(\int_{\Omega_0\setminus\Omega_\varepsilon} |Q_0|^{p'}\, \mathrm{d}(x,y)\right)^{1/p'} \cdot \left(\int_{\Omega_0\setminus\Omega_\varepsilon} |v|^{p}\, \mathrm{d}(x,y)\right)^{1/p} \\
&\le |\Omega_0 \setminus \Omega_\varepsilon|^{1/p'} \Big(\|\nabla Q_0\|_{L^\infty(\Omega_0\setminus\Omega_\varepsilon)} \cdot \|\nabla v\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)} + c \cdot \|Q_0\|_{L^\infty(\Omega_0\setminus\Omega_\varepsilon)} \cdot \|v\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)}\Big).
\end{aligned}
$$

We have:

$$|\Omega_0 \setminus \Omega_\varepsilon| = \int_{\Omega_0\setminus\Omega_\varepsilon} 1 \, dx = \int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} 1 \, dy \, dx$$
$$= \int_0^{2\pi} \varepsilon\varphi(x/\varepsilon) \, dx \leq \varepsilon \cdot 2\pi\|\varphi\|_\infty.$$

In the end, we obtain that:

$$|I_1| \leq C_1 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0\setminus\Omega_\varepsilon)},$$

where

$$C_1 = (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \max\{c, 1\}.$$

□

For the integrals I_{21} and I_{31}, from Definition 3.4.3, we prove first the following lemma:

Lemma 3.4.1. Let $1 < q < p < +\infty$ and $\omega \in W^{1,p}(\Omega_0)$. Then:

$$\left(\int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q \, dx\right)^{1/q} \leq C_{2,q} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)}, \tag{3.4.3}$$

with

$$C_{2,q} = (2\pi)^{\frac{p-q}{pq}} \|\varphi\|_\infty^{\frac{p-1}{p}}. \tag{3.4.4}$$

Proof. We prove the result first for $C^1(\overline{\Omega_0})$ functions.

Let $\omega \in C^1(\overline{\Omega_0})$. Then we have the following inequality:

$$|\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)| \leq \int_0^{\varepsilon\varphi(x/\varepsilon)} \left|\frac{\partial\omega}{\partial y}(x, t)\right| dt,$$

for any $x \in [0, 2\pi]$.

We apply Hölder inequality with exponents q and $q' = \dfrac{q}{q-1}$:

$$|\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)| \leq \left(\int_0^{\varepsilon\varphi(x/\varepsilon)} \left|\frac{\partial\omega}{\partial y}(x, t)\right|^q dt\right)^{1/q} \left(\int_0^{\varepsilon\varphi(x/\varepsilon)} 1^{q/(q-1)} dt\right)^{(q-1)/q}$$
$$\Rightarrow |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q \leq \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1)} \cdot \left|\frac{\partial\omega}{\partial y}(x, t)\right|^q dt,$$

hence

$$\int_0^{2\pi} |\omega(x, \varepsilon\varphi(x/\varepsilon)) - \omega(x, 0)|^q dx \leq \int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1)} \cdot \left|\frac{\partial\omega}{\partial y}(x, t)\right|^q dt \, dx.$$

Let now $k = \dfrac{p}{q} > 1$. We apply Hölder inequality with exponents k and $k' = \dfrac{k}{k-1} = \dfrac{p}{p-q}$ to obtain:

$$\int_0^{2\pi} |\omega(x,\varepsilon\varphi(x/\varepsilon)) - \omega(x,0)|^q \mathrm{d}x \le \left(\int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} (\varepsilon\varphi(x/\varepsilon))^{(q-1)\cdot\frac{p}{p-q}} \mathrm{d}t\, \mathrm{d}x \right)^{\frac{p-q}{p}} \cdot$$
$$\cdot \left(\int_0^{2\pi} \int_0^{\varepsilon\varphi(x/\varepsilon)} \left| \frac{\partial \omega}{\partial y}(x,t) \right|^{q\cdot\frac{p}{q}} \mathrm{d}t\, \mathrm{d}x \right)^{\frac{q}{p}}$$
$$\le \varepsilon^{(q-1)+\frac{p-q}{p}} \left(\int_0^{2\pi} (\varphi(x/\varepsilon))^{1+\frac{pq-p}{p-q}} \mathrm{d}x \right)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|^q_{L^p(\Omega_0\setminus\Omega_\varepsilon)}$$
$$\le \varepsilon^{\frac{q(p-1)}{p}} \left(\int_0^{2\pi} (\varphi(x/\varepsilon))^{\frac{pq-q}{p-q}} \mathrm{d}x \right)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|^q_{L^p(\Omega_0\setminus\Omega_\varepsilon)}$$
$$\le \varepsilon^{\frac{q(p-1)}{p}} \cdot \|\varphi\|_\infty^{\frac{q(p-1)}{p}} \cdot (2\pi)^{\frac{p-q}{p}} \cdot \|\nabla\omega\|^q_{L^p(\Omega_0\setminus\Omega_\varepsilon)}.$$

In the end, we get

$$\left(\int_0^{2\pi} |\omega(x,\varepsilon\varphi(x/\varepsilon)) - \omega(x,0)|^q \mathrm{d}x \right)^{1/q} \le C_{2,q} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla\omega\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)},$$

with $C_{2,q}$ from (3.4.4). We conclude the proof by a classical density argument, using the embeddings $C^1(\overline{\Omega_0}) \hookrightarrow W^{1,p}(\Omega_0) \hookrightarrow L^q(\partial\Omega_0)$. □

Proposition 3.4.4. Let $p \in (1,+\infty)$. For any $v \in W^{1,p}(\Omega_0)$, we have

$$|I_{21}| \le C_{21} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0\setminus\Omega_\varepsilon)},$$

where

$$I_{21} = -\frac{w_0}{2} \int_0^{2\pi} (v(x,\varepsilon\varphi(x/\varepsilon)) - v(x,0)) \cdot Q^0_\varepsilon(x) \cdot \gamma_\varepsilon(x)\, \mathrm{d}x$$

and

$$C_{21} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \sqrt{1+\|\varphi'\|^2_\infty}. \tag{3.4.5}$$

Proof. Let $1 < q < p$ and $q' = \dfrac{q}{q-1}$. Then:

$$\frac{2}{|w_0|}|I_{21}| \le \int_0^{2\pi} \left| v(x,\varepsilon\varphi(x/\varepsilon)) - v(x,0) \right| \cdot \left| Q^0_\varepsilon(x) \cdot \sqrt{1+(\varphi'(x/\varepsilon))^2} \right| \mathrm{d}x$$
$$\le \left(\int_0^{2\pi} \left| v(x,\varepsilon\varphi(x/\varepsilon)) - v(x,0) \right|^q \mathrm{d}x \right)^{1/q} \cdot$$
$$\cdot \left(\int_0^{2\pi} \left| Q^0_\varepsilon(x) \cdot \sqrt{1+(\varphi'(x/\varepsilon))^2} \right|^{q'} \mathrm{d}x \right)^{1/q'}$$

hence

$$
\begin{aligned}
\frac{2}{|w_0|}|I_{21}| \le & \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x,0)|^q \, dx \right)^{1/q} \cdot \\
& \cdot \left(\int_0^{2\pi} \left(\frac{\sqrt{2}}{2} \right)^{q'} \cdot \left(\sqrt{1 + (\varphi'(x/\varepsilon))^2} \right)^{q'} dx \right)^{1/q'} \\
& \le \frac{\sqrt{2}}{2} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x,0)|^q \, dx \right)^{1/q} \cdot \left(\int_0^{2\pi} (\sqrt{1 + \|\varphi'\|_\infty^2})^{q'} \, dx \right)^{1/q'} \\
& \le \frac{\sqrt{2}}{2} \cdot (2\pi)^{1/q'} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x,0)|^q \, dx \right)^{1/q},
\end{aligned}
$$

where we have used (3.2.3) and that $|Q_\varepsilon^0(x)| = \dfrac{\sqrt{2}}{2}$, for all $x \in [0, 2\pi)$.

We apply now Lemma 3.4.1, with the constant $C_{2,q}$ from (3.4.4), in order to obtain:

$$
|I_{21}| \le C_{21} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\nabla v\|_{L^p(\Omega_0 \setminus \Omega_\varepsilon)}.
$$

with

$$
C_{21} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi)^{\frac{q-1}{q}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C_{2,q} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2}.
$$

□

Proposition 3.4.5. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$, we have:

$$
|I_{31}| \le C_{31} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},
$$

where

$$
I_{31} = \frac{w_0}{2} \int_0^{2\pi} \left(Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x,0) \cdot v(x,0) \right) \cdot \gamma_\varepsilon(x) \, dx
$$

and

$$
C_{31} = \frac{|w_0|}{2} \sqrt{1 + \|\varphi'\|_\infty^2} \cdot (2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}} \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \left(|\Omega_0|^{1/p} \cdot C_{tr}(p, \Omega_0) + 1 \right). \tag{3.4.6}
$$

Proof. Let $1 < q < p$ and $q' = \dfrac{q}{q-1}$.

Using (3.2.3), we have:

$$
\begin{aligned}
\frac{2}{|w_0|}|I_{31}| \le & \sqrt{1 + \|\varphi'\|_\infty^2} \int_0^{2\pi} \left| (Q_0 \cdot v)(x, \varepsilon\varphi(x/\varepsilon)) - (Q_0 \cdot v)(x,0) \right| dx \\
\le & \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |Q_0(x, \varepsilon\varphi(x/\varepsilon))| \cdot |v(x, \varepsilon\varphi(x/\varepsilon)) - v(x,0)| \, dx \right) + \\
& + \sqrt{1 + \|\varphi'\|_\infty^2} \cdot \left(\int_0^{2\pi} |v(x,0)| \cdot |Q_0(x, \varepsilon\varphi(x/\varepsilon)) - Q_0(x,0)| \, dx \right)
\end{aligned}
$$

so

$$\frac{2}{|w_0|}|I_{31}| \leq \sqrt{1+\|\varphi'\|_\infty^2}\cdot \\ \cdot\Bigg(\Big(\int_0^{2\pi} |Q_0(x,\varepsilon\varphi(x/\varepsilon))|^{q'}\,\mathrm{d}x\Big)^{1/q'}\Big(\int_0^{2\pi}|v(x,\varepsilon\varphi(x/\varepsilon))-v(x,0)|^q\,\mathrm{d}x\Big)^{1/q}+ \\ +\Big(\int_0^{2\pi}|v(x,0)|^p\,\mathrm{d}x\Big)^{1/p}\cdot\Big(\int_0^{2\pi}\Big|Q_0(x,\varepsilon\varphi(x/\varepsilon))-Q_0(x,0)\Big|^{p'}\,\mathrm{d}x\Big)^{1/p'}\Bigg)$$

where we have applied Hölder inequality with exponents q and q' for the first term and with exponents p and p' for the second one.

For the first term, we apply Lemma 3.4.1 and use the $L^\infty(\Omega_0)$ bounds for Q_0 in order to obtain that:

$$\Big(\int_0^{2\pi}|Q_0(x,\varepsilon\varphi(x/\varepsilon))|^{q'}\,\mathrm{d}x\Big)^{1/q'}\cdot\Big(\int_0^{2\pi}|v(x,\varepsilon\varphi(x/\varepsilon))-v(x,0)|^q\,\mathrm{d}x\Big)^{1/q}\leq \\ \leq (2\pi)^{1/q'}\cdot\|Q_0\|_{L^\infty(\Omega_0)}\cdot(2\pi)^{\frac{p-q}{pq}}\cdot\|\varphi\|_\infty^{\frac{p-1}{p}}\cdot\varepsilon^{\frac{p-1}{p}}\cdot\|\nabla v\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)} \\ \leq \big((2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}}\cdot\|Q_0\|_{L^\infty(\Omega_0)}\big)\cdot\varepsilon^{\frac{p-1}{p}}\cdot\|v\|_{W^{1,p}(\Omega_0)}$$

since $\Omega_0\setminus\Omega_\varepsilon\subset\Omega_0$.

For the other term, we see that we can apply Lemma 3.4.1 with exponents p' and p, since $p>2$ implies that $1<p'<p$, in order to obtain:

$$\Big(\int_0^{2\pi}|v(x,0)|^p\,\mathrm{d}x\Big)^{1/p}\cdot\Big(\int_0^{2\pi}\Big|Q_0(x,\varepsilon\varphi(x/\varepsilon))-Q_0(x,0)\Big|^{p'}\,\mathrm{d}x\Big)^{1/p'}\leq \\ \leq\Big(\int_{\Gamma_0}|v|^p\,\mathrm{d}\sigma_0\Big)^{1/p}\cdot(2\pi)^{\frac{p-p'}{pp'}}\cdot\|\varphi\|_\infty^{\frac{p-1}{p}}\cdot\varepsilon^{\frac{p-1}{p}}\cdot\|\nabla Q_0\|_{L^p(\Omega_0\setminus\Omega_\varepsilon)} \\ \leq\big((2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}}\cdot\|\nabla Q_0\|_{L^\infty(\Omega_0)}\cdot|\Omega_0|^{1/p}\cdot C_{tr}(p,\Omega_0)\big)\cdot\varepsilon^{\frac{p-1}{p}}\cdot\|v\|_{W^{1,p}(\Omega_0)},$$

where we use Definition 3.4.1 and the fact that, if $1<p'<p$, then $\frac{p-p'}{pp'}<\frac{p}{pp'}=\frac{1}{p'}=\frac{p-1}{p}$.

In the end, we obtain that:

$$|I_{31}|\leq C_{31}\cdot\varepsilon^{\frac{p-1}{p}}\cdot\|v\|_{W^{1,p}(\Omega_0)},$$

where

$$C_{31}=\frac{|w_0|}{2}\sqrt{1+\|\varphi'\|_\infty^2}\cdot(2\pi\|\varphi\|_\infty)^{\frac{p-1}{p}}\cdot\|Q_0\|_{W^{1,\infty}(\Omega_0)}\cdot\big(|\Omega_0|^{1/p}\cdot C_{tr}(p,\Omega_0)+1\big).$$

□

Let us now prove the following lemma.

Lemma 3.4.2. Let us consider the case in which Assumption 3.2.2 holds. Let $p \in (2, +\infty)$, $\omega \in W^{1,p}(\Omega_0)$, V be a Banach space and $b : \mathbb{R} \to V$ a 2π-periodic function such that $b \in L^\infty([0, 2\pi))$, for which we write $\|b\|_\infty$ instead of $\|b\|_{L^\infty([0,2\pi))}$. Then:

$$\left| \int_0^{2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, dx \right| \le C_3 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1,p}(\Omega_0)},$$

where $B = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} b(t)\, dt$ and $C_3 = (2\pi)^{\frac{2p-2}{p}} \cdot \|b\|_\infty \cdot C_{tr}(p, \Omega_0)$.

Proof. From Assumption 3.2.2, we have that $\varepsilon^{-1} = 2k \in \mathbb{N}^*$. We write then:

$$\int_0^{2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, dx = \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, dx$$
$$= \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \omega(x,0) b(t)\, dt\, dx - \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \omega(x,0) b(x/\varepsilon)\, dx. \tag{3.4.7}$$

Using the change of variables $x = x' + j\varepsilon \cdot 2\pi$, we obtain:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \omega(x,0) b(x/\varepsilon)\, dx = \sum_{j=0}^{2k-1} \int_0^{\varepsilon\cdot 2\pi} \omega(x' + j\varepsilon\cdot 2\pi, 0) b\left(\frac{x' + j\varepsilon \cdot 2\pi}{\varepsilon}\right) dx'$$

and, since the function b is 2π-periodic and $j \in \mathbb{N}$, we can rewrite the last equality as:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \omega(x,0) b(x/\varepsilon)\, dx = \sum_{j=0}^{2k-1} \int_0^{\varepsilon\cdot 2\pi} \omega(x' + j\varepsilon\cdot 2\pi, 0) b(x'/\varepsilon)\, dx'.$$

Using now the change of variables $x' = \varepsilon t$, we get:

$$\sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \omega(x,0) b(x/\varepsilon)\, dx = \sum_{j=0}^{2k-1} \int_0^{2\pi} \varepsilon \cdot \omega(\varepsilon t + j\varepsilon\cdot 2\pi, 0) b(t)\, dt.$$

Since

$$\varepsilon = \frac{1}{2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} 1\, dx,$$

then (3.4.7) becomes:

$$\int_0^{2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, dx = \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \omega(x,0) b(t)\, dt\, dx -$$
$$- \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \omega(\varepsilon t + j\varepsilon\cdot 2\pi, 0) b(t)\, dt\, dx$$
$$= \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \big(\omega(x,0) - \omega(\varepsilon t + j\varepsilon\cdot 2\pi, 0)\big) \cdot b(t)\, dt\, dx.$$

Then

$$
\begin{aligned}
&\left| \int_0^{2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, \mathrm{d}x \right| \le \\
&\quad \le \frac{1}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \big|\omega(x,0) - \omega(\varepsilon t + j\varepsilon\cdot 2\pi, 0)\big| \cdot \big|b(t)\big| \,\mathrm{d}t\, \mathrm{d}x \\
&\quad \le \frac{\|b\|_\infty}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \big|\omega(x,0) - \omega(\varepsilon t + j\varepsilon\cdot 2\pi, 0)\big| \,\mathrm{d}t\, \mathrm{d}x.
\end{aligned}
$$

We apply now the change of variables $t' = \varepsilon t + j\varepsilon \cdot 2\pi$ and drop the primes to obtain:

$$
\left| \int_0^{2\pi} \omega(x,0)\big(B - b(x/\varepsilon)\big)\, \mathrm{d}x \right| \le \frac{\|b\|_\infty}{2\pi} \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \frac{\big|\omega(x,0) - \omega(t,0)\big|}{\varepsilon} \,\mathrm{d}t\, \mathrm{d}x.
$$

Then:

$$
\begin{aligned}
&\sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \varepsilon^{-1} \cdot \big|\omega(x,0) - \omega(t,0)\big| \,\mathrm{d}t\, \mathrm{d}x \ = \\
&\quad = \sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \frac{\big|\omega(x,0) - \omega(t,0)\big|}{\big|(x,0) - (t,0)\big|} \cdot \frac{\big|(x,0) - (t,0)\big|}{\varepsilon} \,\mathrm{d}t\, \mathrm{d}x \\
&\quad \le \sum_{j=0}^{2k-1} \Bigg[\Bigg(\int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \frac{\big|\omega(x,0) - \omega(t,0)\big|^p}{\big|(x,0) - (t,0)\big|^p} \,\mathrm{d}t\, \mathrm{d}x \Bigg)^{1/p} \cdot \\
&\qquad \cdot \Bigg(\int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \frac{\big|x - t\big|^{p'}}{\varepsilon^{p'}} \,\mathrm{d}t\, \mathrm{d}x \Bigg)^{1/p'} \Bigg],
\end{aligned}
$$

where we have applied Hölder inequality with exponents p and p'. Since $|x - t| \le \varepsilon \cdot 2\pi$, for any $x, t \in [j\varepsilon \cdot 2\pi, (j+1)\varepsilon \cdot 2\pi]$, we obtain:

$$
\begin{aligned}
&\sum_{j=0}^{2k-1} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \varepsilon^{-1} \cdot \big|w(x,0) - w(t,0)\big| \,\mathrm{d}t\, \mathrm{d}x \le \\
&\quad \le (2\pi)^{1+2/p'} \cdot \varepsilon^{2/p'} \cdot \sum_{j=0}^{2k-1} \Bigg(\int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \frac{\big|w(x,0) - w(t,0)\big|^p}{\big|(x,0) - (t,0)\big|^p} \,\mathrm{d}t\, \mathrm{d}x \Bigg)^{1/p} \\
&\quad \le (2\pi)^{1+2/p'} \cdot \varepsilon^{2/p'} \cdot \sum_{j=0}^{2k-1} \Bigg(\int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \frac{\big|w(x,0) - w(t,0)\big|^p}{\big|(x,0) - (t,0)\big|^p} \,\mathrm{d}t\, \mathrm{d}x \Bigg)^{1/p}.
\end{aligned}
$$

Let us denote now:

$$
r_j = \int_{j\varepsilon\cdot 2\pi}^{(j+1)\varepsilon\cdot 2\pi} \int_0^{2\pi} \frac{\big|w(x,0) - w(t,0)\big|^p}{\big|(x,0) - (t,0)\big|^p} \,\mathrm{d}t\, \mathrm{d}x, \ \forall\, j \in \{0, 1, \ldots, 2k-1\}.
$$

We have that $r_j \geq 0$, for all $j \in \{0, 1, \ldots, 2k-1\}$. The function $\mathbb{R} \ni x \to x^{1/p}$ is concave since $p \in (2, +\infty)$, therefore we have the Jensen inequality:

$$\sum_{j=0}^{2k-1} r_j^{1/p} \leq 2k \cdot \left(\frac{1}{2k} \sum_{j=0}^{2k-1} r_j \right)^{1/p}.$$

Since $2k = \varepsilon^{-1}$ and

$$\left(\sum_{j=0}^{2k-1} r_j \right)^{1/p} = \left(\int_0^{2\pi} \int_0^{2\pi} \frac{|\omega(x,0) - \omega(t,0)|^p}{|x-t|^p} \, dt \, dx \right)^{1/p} \leq \|\omega\|_{W^{1-1/p,p}(\Gamma_1)},$$

due to the fact that the left hand side of the last inequality represents the Gagliardo seminorm defined on the space $W^{1-1/p,p}(\Gamma_1)$, we obtain that

$$\begin{aligned}
&\sum_{j=0}^{2k-1} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \int_{j\varepsilon \cdot 2\pi}^{(j+1)\varepsilon \cdot 2\pi} \varepsilon^{-1} \cdot |w(x,0) - w(t,0)| \, dt \, dx \leq \\
&\quad \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{2p-2}{p}} \cdot (2k)^{1-1/p} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)} \\
&\quad \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{2p-2}{p} - 1 + \frac{1}{p}} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)} \\
&\quad \leq (2\pi)^{\frac{3p-2}{p}} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1-1/p,p}(\Gamma_1)}.
\end{aligned}$$

Using Definition 3.4.1, we obtain that:

$$\left| \int_0^{2\pi} \omega(x,0) \big(B - b(x/\varepsilon) \big) \, dx \right| \leq C_3 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\omega\|_{W^{1,p}(\Omega_0)}$$

with

$$C_3 = (2\pi)^{\frac{2p-2}{p}} \cdot \|b\|_\infty \cdot C_{tr}(p, \Omega_0).$$

□

Proposition 3.4.6. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$ we have:

$$|I_{32}| \leq C_{32} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$I_{32} = -\frac{w_0}{2} \int_0^{2\pi} Q_0(x,0) \cdot v(x,0) \cdot \Big(\gamma - \gamma_\varepsilon(x) \Big) \, dx$$

and

$$C_{32} = \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C_{tr}(p, \Omega_0) \cdot \|Q_0\|_{W^{1,\infty}(\Omega_0)}. \tag{3.4.8}$$

Proof. Let $v \in W^{1,p}(\Omega_0)$. Since $Q_0 \in W^{1,\infty}(\Omega_0)$, we have that $Q_0 \cdot v \in W^{1,p}(\Omega_0)$. We apply Lemma 3.4.2 for $\omega = Q_0 \cdot v$ and $b = \gamma_0$, since $\gamma_\varepsilon(x) = \gamma_0(x/\varepsilon)$, with $V = \mathbb{R}$. In this way, we obtain that:

$$|I_{32}| \leq \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \|\gamma_0\|_\infty \cdot C_{tr}(p, \Omega_0) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|Q_0 \cdot v\|_{W^{1,p}(\Omega_0)}.$$

Using now that $\|\gamma_0\|_\infty = \sqrt{1 + \|\varphi'\|_\infty^2}$ and that $\|Q_0 \cdot v\|_{W^{1,p}(\Omega_0)} \leq \|Q_0\|_{W^{1,\infty}(\Omega_0)} \cdot \|v\|_{W^{1,p}(\Omega_0)}$, we obtain the conclusion. □

Proposition 3.4.7. Let $p \in (2, +\infty)$. For any $v \in W^{1,p}(\Omega_0)$ we have:

$$|I_{22}| \leq C_{22} \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)},$$

where

$$I_{22} = \frac{w_0}{2} \int_0^{2\pi} v(x, 0) \cdot \left(\gamma Q_{ef} - \gamma_\varepsilon(x) Q_\varepsilon^0(x) \right) \mathrm{d}x$$

and

$$C_{22} = \frac{|w_0|\sqrt{2}}{4} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C_{tr}(p, \Omega_0). \tag{3.4.9}$$

Proof. Let $v \in W^{1,p}(\Omega_0)$. We apply Lemma 3.4.2 for $\omega = v$ and $b(t) = \gamma_0(t) \begin{pmatrix} g_1(t) & g_2(t) \\ g_2(t) & -g_1(t) \end{pmatrix}$, for all $t \in \mathbb{R}$, with $V = \mathrm{Sym}_0(2)$. In this way, we have:

$$|I_{22}| \leq \frac{|w_0|}{2} \cdot (2\pi)^{\frac{2p-2}{p}} \cdot \|b\|_\infty \cdot C_{tr}(p, \Omega_0) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_0)}.$$

Since $g_1^2(t) + g_2^2(t) = \frac{1}{4}$, for all $t \in \mathbb{R}$, and $\|\gamma_0\|_\infty = \sqrt{1 + \|\varphi'\|_\infty^2}$, we obtain the conclusion. □

We are now able to prove Proposition 3.4.1.

Proof of Proposition 3.4.1. We combine (3.4.1) with Propositions 3.4.3 to 3.4.7 to obtain the conclusion, with, for example,

$$C_I = C_1 + C_{21} + C_{31} + C_{22} + C_{32},$$

where these constants are defined in (3.4.2), (3.4.5), (3.4.6), (3.4.8) and (3.4.9). □

Remark 3.4.5. We can actually choose C_I of the following form:

$$C_I = \max\{1, \|\varphi\|_\infty^{(p-1)/p}\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \cdot C(w_0, c, p, \Omega_0, Q_0),$$

where $C(w_0, c, p, \Omega_0, Q_0)$ is an ε-independent constant depending only on w_0, c, p, Ω_0 and Q_0.

3.5 PROOF OF THE ERROR ESTIMATE

The goal in this section would be to place instead of v in the right-hand side of the inequality from Proposition 3.4.1 something that depends on $(Q_0 - Q_\varepsilon)$, such that we can obtain Theorem 3.2.1.

Throughout this section, we fix $p \in (2, +\infty)$ and let $u_\varepsilon := Q_0 - Q_\varepsilon$.

Remark 3.5.1. The function u_ε solves the following PDE:

$$\begin{cases} -\Delta u_\varepsilon + c u_\varepsilon = 0, \text{ in } \Omega_\varepsilon \\ \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \dfrac{w_0}{2} u_\varepsilon = g_\varepsilon, \text{ on } \Gamma_\varepsilon \\ \\ \dfrac{\partial u_\varepsilon}{\partial \nu_R} + \dfrac{w_0}{2} u_\varepsilon = 0, \text{ on } \Gamma_R \end{cases}$$

where $g_\varepsilon = \dfrac{\partial Q_0}{\partial \nu_\varepsilon} + \dfrac{w_0}{2}(Q_0 - Q_\varepsilon^0)$. Since $\Gamma_\varepsilon \subset \Omega_0$, we have no problems with defining g_ε. By Corollaries 3.3.1 and 3.3.2, we also have that $u_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ and $g_\varepsilon \in W^{1-1/p,p}(\Gamma_\varepsilon)$, for any $1 < p < +\infty$.

We would like now to prove the following proposition (to be compared with Proposition 3.4.1):

Proposition 3.5.1. There exists an ε-independent constant C_0 such that:

$$\left| a_\varepsilon(Q_0 - Q_\varepsilon, v) \right| \le C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_\varepsilon)}, \ \forall v \in W^{1,p}(\Omega_\varepsilon).$$

In order to do so, we need to construct an extension operator $E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega_0)$ such that $E_\varepsilon \omega \equiv \omega$ in Ω_ε for any $\omega \in W^{1,p}(\Omega_\varepsilon)$, with the operator E_ε bounded uniformly in ε. For this, we adapt the bi-Lipschitz maps Φ_ε and Φ_ε^{-1} from [31, Equations (57) and (58)] to this simplified model, where these transformations are only between Ω_ε and Ω_0. In order to construct the desired extension, we introduce the following notation.

Definition 3.5.1. Let $\Omega_1^\varepsilon = \left\{ (x, y) \;\middle|\; x \in [0, 2\pi), \ y \in \left(-\dfrac{R\varepsilon\varphi(x/\varepsilon)}{R - \varepsilon\varphi(x/\varepsilon)}, R \right) \right\}$.

Remark 3.5.2. Using Assumption 3.2.2, it is easy to see that Ω_1^ε is well defined, since $R - \varepsilon\varphi(x/\varepsilon) > 0$, for all $x \in [0, 2\pi)$.

Definition 3.5.2. We define $\Phi_\varepsilon : \Omega_1^\varepsilon \to \Omega_0$ as

$$\Phi_\varepsilon(x, y) = \left(x, y \cdot \frac{R - \varepsilon\varphi(x/\varepsilon)}{R} + \varepsilon\varphi(x/\varepsilon) \right), \ \forall (x, y) \in \Omega_1^\varepsilon$$

and $\Phi_\varepsilon^{-1} : \Omega_0 \to \Omega_1^\varepsilon$ as

$$\Phi_\varepsilon^{-1}(x,y) = \left(x, \frac{R\big(y - \varepsilon\varphi(x/\varepsilon)\big)}{R - \varepsilon\varphi(x/\varepsilon)}\right), \ \forall (x,y) \in \Omega_0.$$

Remark 3.5.3. We have $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$ and, using Assumption 3.2.2, we can prove the following sequence of inclusions:

$$\{(x,y) \mid x \in [0,2\pi),\ y \in (R/2,R)\} \subset \Omega_\varepsilon \subset \Omega_0 \subset \Omega_1^\varepsilon \subset \{(x,y) \mid x \in [0,2\pi),\ y \in (-R,R)\}.$$

Proposition 3.5.2. Φ_ε defines a family of C^2 uniformly bi-Lipschitz maps between Ω_1 and Ω_0. Moreover, there exists an ε-independent constant C_Φ such that:

$$C_\Phi^{-1} \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)} \le \|\tilde{\omega}\|_{W^{1,p}(\Omega_0)} \le C_\Phi \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)}, \tag{3.5.1}$$

for all $\omega \in W^{1,p}(\Omega_\varepsilon)$, where $\tilde{\omega} = \omega \circ \Phi_\varepsilon\big|_{\Omega_0}$, and

$$C_\Phi^{-1} \cdot \|\omega\|_{W^{1,p}(\Omega_0)} \le \|\tilde{\omega}\|_{W^{1,p}(\Omega_1^\varepsilon)} \le C_\Phi \cdot \|\omega\|_{W^{1,p}(\Omega_0)}, \tag{3.5.2}$$

for all $\omega \in W^{1,p}(\Omega_0)$, where $\tilde{\omega} = \omega \circ \Phi_\varepsilon$.

Proof. Since $\varphi \in C^2$, then $\Phi_\varepsilon \in C^2$. Moreover, since the definition of Φ_ε from Definition 3.5.2 is based on [31, Equation (57)], then one can argue similarly as in [31, Proposition 2.5] to obtain that Φ_ε and its inverse are Lipschitz with Lipschitz constants independent of ε. More specifically, using the bound

$$0 < \varepsilon \cdot \|\varphi\|_\infty < \frac{R}{2}$$

from Assumption 3.2.1, we can obtain that

$$\left|\frac{\partial \Phi_\varepsilon}{\partial x}(x,y)\right| \le \max\{2R,1\} \cdot \sqrt{1 + \|\varphi'\|_\infty^2} \quad \text{and} \quad \left|\frac{\partial \Phi_\varepsilon}{\partial y}(x,y)\right| \le 1,\ \forall (x,y) \in \Omega_1^\varepsilon,$$

which implies that Φ_ε is Lipschitz with its Lipschitz constant bounded ε-independent. In the same way, the first order derivatives of Φ_ε^{-1} can be bounded ε-independent (using the same bound as above for ε). To obtain the constant C_Φ, we use the same ε-independent bounds for the first order derivatives of Φ_ε and Φ_ε^{-1} when we apply chain rule in $\tilde{\omega} = \omega \circ \Phi_\varepsilon$ with $\omega \in W^{1,p}(\Omega_0)$. □

Remark 3.5.4. If we were to define $E_\varepsilon \omega = \omega \circ \Phi_\varepsilon\big|_{\Omega_0}$, for any $\omega \in W^{1,p}(\Omega_\varepsilon)$, then we would not have had $E_\varepsilon \omega \equiv \omega$ inside of Ω_ε, so we need the more sophisticated extension that will be provided in Definition 3.5.3 below.

Corollary 3.5.1. Let $\Omega_2 := \{(x,y) \mid x \in [0,2\pi),\ y \in (-R,R)\}$ and let $T : W^{1,p}(\Omega_0) \to W^{1,p}(\Omega_2)$ the following extension operator:

$$T\omega(x,y) = \begin{cases} \omega(x,y), \text{ if } y \in (0,R); \\ \omega(x,-y), \text{ if } y \in (-R,0); \end{cases}$$

for any $\omega \in W^{1,p}(\Omega_0)$. Then there exists an ε-independent constant $C_{ext}(p,\Omega_0) > 0$ such that:

$$\|T\omega\|_{W^{1,p}(\Omega_2)} \le C_{ext}(p,\Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_0)},\ \forall\, \omega \in W^{1,p}(\Omega_0).$$

Remark 3.5.5. The proof is a simple exercise which consists of applying the method of extending a Sobolev function by reflection against a flat boundary, illustrated in [44] for $W^{1,p}(\Omega_0)$ functions.

Definition 3.5.3. Let $E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega_0)$ defined as

$$E_\varepsilon\omega := \left(\left(T(\omega \circ \Phi_\varepsilon|_{\Omega_0})\right)\Big|_{\Omega_1^\varepsilon} \circ \Phi_\varepsilon^{-1} \right),$$

for any $\omega \in W^{1,p}(\Omega_\varepsilon)$. In this way, $E_\varepsilon\omega \equiv \omega$ in Ω_ε, for any $\omega \in W^{1,p}(\Omega_\varepsilon)$.

Proposition 3.5.3. There exists an ε-independent constant C_{ext} such that:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \le C_{ext} \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)},\ \forall\, \omega \in W^{1,p}(\Omega_\varepsilon).$$

Proof. Let $\omega \in W^{1,p}(\Omega_\varepsilon)$ and $\tilde{\omega} = \omega \circ \Phi_\varepsilon|_{\Omega_0}$. Since the transformation Φ_ε^{-1} is bi-Lipschitz with its constants bounded ε-independent, then, using (3.5.2):

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} = \left\| \left((T\tilde{\omega})\Big|_{\Omega_1^\varepsilon} \circ \Phi_\varepsilon^{-1} \right) \right\|_{W^{1,p}(\Omega_0)} \le C_\Phi^{-1} \left\| (T\tilde{\omega})\Big|_{\Omega_1^\varepsilon} \right\|_{W^{1,p}(\Omega_1^\varepsilon)} \le C_\Phi^{-1} \|T\tilde{\omega}\|_{W^{1,p}(\Omega_2)},$$

where in the last inequality we use that $T\tilde{\omega}\Big|_{\Omega_1^\varepsilon}$ is a restriction of $T\tilde{\omega}$ from Ω_2. Then:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \le C_\Phi^{-1} \|T\tilde{\omega}\|_{W^{1,p}(\Omega_2)} \le C_\Phi^{-1} \cdot C_{ext}(p,\Omega_0) \cdot \|\tilde{\omega}\|_{W^{1,p}(\Omega_0)},$$

where we have used Corollary 3.5.1. Since $\tilde{\omega} = \omega \circ \Phi_\varepsilon|_{\Omega_0}$, then, using (3.5.1), we obtain:

$$\|E_\varepsilon\omega\|_{W^{1,p}(\Omega_0)} \le C_{ext}(p,\Omega_0) \cdot \|\omega\|_{W^{1,p}(\Omega_\varepsilon)}.$$

Therefore, we can actually choose $C_{ext} = C_{ext}(p,\Omega_0)$ given by Corollary 3.5.1. □

Proof of Proposition 3.5.1. Let $v \in W^{1,p}(\Omega_\varepsilon)$. Then $E_\varepsilon v \in W^{1,p}(\Omega_0)$ and we can apply Proposition 3.4.1 to obtain:

$$|a_\varepsilon(u_\varepsilon, E_\varepsilon v)| \le C_I \cdot \varepsilon^{\frac{p-2}{p}} \cdot \|E_\varepsilon v\|_{W^{1,p}(\Omega_0)}.$$

Using Definition 3.5.3 and Proposition 3.5.3, we obtain:

$$\left|a_\varepsilon(u_\varepsilon, v)\right| \leq \left(C_I \cdot C_{ext}\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v\|_{W^{1,p}(\Omega_\varepsilon)}.$$

□

Corollary 3.5.2. There exists a unique solution $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ that solves the following PDE:

$$\begin{cases} -\Delta v_\varepsilon + c v_\varepsilon = u_\varepsilon, \text{ in } \Omega_\varepsilon \\ \\ \dfrac{\partial v_\varepsilon}{\partial \nu_\varepsilon} + \dfrac{w_0}{2} v_\varepsilon = 0, \text{ on } \Gamma_\varepsilon \\ \\ \dfrac{\partial v_\varepsilon}{\partial \nu_R} + \dfrac{w_0}{2} v_\varepsilon = 0, \text{ on } \Gamma_R \end{cases} \tag{3.5.3}$$

where $u_\varepsilon = Q_0 - Q_\varepsilon$.

Proof. The proof follows the same steps as in Corollary 3.3.1. □

Proposition 3.5.4. The function v_ε satisfies the following inequality:

$$\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \min\{c^{-1/2}, c^{-1}\} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{3.5.4}$$

where c is the positive constant from the bulk energy defined in (3.2.1).

Proof. Let $w \in W^{1,p}(\Omega_\varepsilon)$. Since $-\Delta v_\varepsilon + c v_\varepsilon = u_\varepsilon$ in Ω_ε, we have:

$$\begin{aligned} \int_{\Omega_\varepsilon} u_\varepsilon \cdot w \, \mathrm{d}(x,y) &= \int_{\Omega_\varepsilon} \big(-\Delta v_\varepsilon + c v_\varepsilon\big) \cdot w \, \mathrm{d}(x,y) \\ &= \int_{\Omega_\varepsilon} \big(\nabla v_\varepsilon \cdot \nabla w + c v_\varepsilon \cdot w\big) \, \mathrm{d}(x,y) - \int_{\partial\Omega_\varepsilon} \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \cdot w \, \mathrm{d}\sigma_\varepsilon \\ &= \int_{\Omega_\varepsilon} \big(c v_\varepsilon \cdot w + \nabla v_\varepsilon \cdot \nabla w\big) \, \mathrm{d}(x,y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} v_\varepsilon \cdot w \, \mathrm{d}\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} v_\varepsilon \cdot w \, \mathrm{d}\sigma_R \end{aligned}$$

Taking $w = u_\varepsilon$, we obtain:

$$\begin{aligned} a_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_{\Omega_\varepsilon} \big(c u_\varepsilon \cdot v_\varepsilon + \nabla u_\varepsilon \cdot \nabla v_\varepsilon\big) \, \mathrm{d}(x,y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} u_\varepsilon \cdot v_\varepsilon \, \mathrm{d}\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} u_\varepsilon \cdot v_\varepsilon \, \mathrm{d}\sigma_R \\ &= \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.5.5}$$

Taking $w = v_\varepsilon$, we obtain:

$$c\|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{w_0}{2}\|v_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \frac{w_0}{2}\|v_\varepsilon\|^2_{L^2(\Gamma_R)} = \int_{\Omega_\varepsilon} u_\varepsilon \cdot v_\varepsilon \, \mathrm{d}x.$$

Now we can see that

$$c\|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq c\|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{w_0}{2}\|v_\varepsilon\|^2_{L^2(\partial\Omega_\varepsilon)} \leq \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

which implies that $\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. In the same way,

$$\|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq c\|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{w_0}{2}\|v_\varepsilon\|^2_{L^2(\partial\Omega_\varepsilon)} \leq$$
$$\leq \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1} \cdot \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}$$

using the last inequality proved. This implies that $\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c^{-1/2} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. In the end, we obtain that:

$$\|v_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \min\{c^{-1/2}, c^{-1}\} \cdot \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

□

Definition 3.5.4. Let $\overline{v}_\varepsilon := v_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0} \circ \Phi^{-1}$, where Φ_ε is introduced in Definition 3.5.2 and Φ in Definition 3.3.1. Since $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$, then $\Omega_\varepsilon = \big(\Phi_\varepsilon\big|_{\Omega_0} \circ \Phi_{-1}\big)(\mathcal{U}_0)$.

Corollary 3.5.3. We have that $\overline{v}_\varepsilon \in W^{2,p}(\mathcal{U}_0)$, for any $p > 2$.

Proof. Since $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$, Φ_ε and Φ are bi-Lipschitz with $\Phi \in C^2(\Omega_0)$ and Φ^{-1} smooth, we obtain the conclusion. □

Remark 3.5.6. Using Definition 3.5.4, we can see that $\overline{v}_\varepsilon$ solves a PDE of the form:

$$\begin{cases} \mathcal{L}\overline{v}_\varepsilon + c\overline{v}_\varepsilon = \overline{u}_\varepsilon, \text{ in } \mathcal{U}_0 \\ \nabla\overline{v}_\varepsilon \cdot \overline{l_1} + \frac{w_0}{2}\overline{v}_\varepsilon = 0, \text{ on } \Phi(\Gamma_0) \\ \nabla\overline{v}_\varepsilon \cdot \overline{l_2} + \frac{w_0}{2}\overline{v}_\varepsilon = 0, \text{ on } \Phi(\Gamma_R) \end{cases} \tag{3.5.6}$$

where $\mathcal{L}$ is an uniformly elliptic operator, $\overline{l_1} \in C^1(\Phi(\Gamma_0))$, $\overline{l_2} \in C^1(\Phi(\Gamma_R))$ and $\overline{u_\varepsilon} := u_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0} \circ \Phi^{-1} \in W^{2,p}(\mathcal{U}_0)$.

Proposition 3.5.5. There exists an ε-independent constant $C_{reg}(\mathcal{U}_0)$ such that $\overline{v}_\varepsilon$ satisfies the following inequality:

$$\|\overline{v}_\varepsilon\|_{H^2(\mathcal{U}_0)} \leq C_{reg}(\mathcal{U}_0) \cdot \big(\|\mathcal{L}\overline{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + \|\overline{v}_\varepsilon\|_{H^{1/2}(\partial\mathcal{U}_0)} + \|\overline{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}\big).$$

Proof. We apply [53, Theorem 2.3.3.2], since all the required conditions are satisfied. □

We can now prove the main result of this chapter.

Proof of Theorem 3.2.1. We apply first Proposition 3.5.1 with $v_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$:

$$\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = a_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)},$$

where we also use (3.5.5). We apply now Proposition 3.5.2:

$$\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le C_0 \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} \le \left(C_0 \cdot C_\Phi\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left\|v_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0}\right\|_{W^{1,p}(\Omega_0)}.$$

We now use the compact embedding $W^{2,2}(\Omega_0) \hookrightarrow\hookrightarrow W^{1,p}(\Omega_0)$ to obtain:

$$\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le \left(C_0 \cdot C_\Phi \cdot C_{emb}(\Omega_0)\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left\|v_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0}\right\|_{W^{2,2}(\Omega_0)},$$

where $C_{emb}(\Omega_0)$ is the constant given by the compact embbeding used.

Recalling Definition 3.3.1, it is easy to see that there exists $C_{polar} > 0$, which is ε-independent, such that:

$$C^{-1}_{polar}\|\omega\|_{W^{2,2}(\Omega_0)} \le \left\|\omega \circ \Phi^{-1}\right\|_{W^{2,2}(\mathcal{U}_0)} \le C_{polar}\|\omega\|_{W^{2,2}(\Omega_0)},\ \forall\, \omega \in W^{2,2}(\Omega_0). \tag{3.5.7}$$

In this way, we have:

$$\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le C_0' \cdot \varepsilon^{\frac{p-1}{p}} \cdot \|\overline{v}_\varepsilon\|_{W^{2,2}(\mathcal{U}_0)},$$

where $C_0' = C_0 \cdot C_\Phi \cdot C_{emb}(\Omega_0) \cdot C_{polar}$.

We apply now Proposition 3.5.5:

$$\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le \left(C_0' \cdot C_{reg}(\mathcal{U}_0)\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left(\|\mathcal{L}\overline{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + \|\overline{v}_\varepsilon\|_{H^{1/2}(\partial\mathcal{U}_0)} + \|\overline{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}\right).$$

Using (3.5.6) and the trace inequality for the trace operator $\mathrm{Tr} : H^1(\mathcal{U}_0) \to H^{1/2}(\partial\mathcal{U}_0)$, we get:

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\le \left(C_0' \cdot C_{reg}(\mathcal{U}_0)\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left(\|\overline{u}_\varepsilon - c\overline{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + (1 + C_{tr}(\mathcal{U}_0))\|\overline{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}\right) \\ &\le \left(C_0' \cdot C_{reg}(\mathcal{U}_0)\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \left(\|\overline{u}_\varepsilon\|_{L^2(\mathcal{U}_0)} + c \cdot \|\overline{v}_\varepsilon\|_{L^2(\mathcal{U}_0)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \|\overline{v}_\varepsilon\|_{H^1(\mathcal{U}_0)}\right)\end{aligned}$$

and, using (3.5.7), we obtain:

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le& \left(C_0' \cdot C_{reg}(\mathcal{U}_0) \cdot C^{-1}_{polar}\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \\ &\cdot \left(\left\|u_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0}\right\|_{L^2(\Omega_0)} + c \cdot \left\|v_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0}\right\|_{L^2(\Omega_0)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \left\|v_\varepsilon \circ \Phi_\varepsilon\big|_{\Omega_0}\right\|_{H^1(\Omega_0)}\right).\end{aligned}$$

Using Proposition 3.5.2, we get:

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le& \left(C_0' \cdot C_{reg}(\mathcal{U}_0) \cdot C^{-1}_{polar} \cdot C^{-1}_\Phi\right) \cdot \varepsilon^{\frac{p-1}{p}} \cdot \\ &\cdot \left(\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + c \cdot \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + (1 + C_{tr}(\mathcal{U}_0)) \cdot \|v_\varepsilon\|_{H^1(\Omega_\varepsilon)}\right).\end{aligned}$$

and then, using Proposition 3.5.4, we obtain:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \left(C_0 \cdot C_{emb}(\Omega_0) \cdot C_{reg}(\mathcal{U}_0)\right) \cdot \left(2 + (1 + C_{tr}(\mathcal{U}_0)) \cdot \min\{c^{-1/2}, c^{-1}\}\right) \cdot \varepsilon^{\frac{p-1}{p}},$$

for any $p > 2$. $\square$

3.6 APPENDIX

In this section, we prove Corollary 3.3.1, that $a_\varepsilon(Q_\varepsilon - Q_0, v)$ is well-defined for any $v \in W^{1,p}(\Omega_0)$ and then we prove Proposition 3.4.2.

Proof of Corollary 3.3.1. We consider $\tilde{Q}_\varepsilon : \mathcal{U}_\varepsilon \to \mathrm{Sym}_0(2)$ such that:

$$Q_\varepsilon(x,y) = \tilde{Q}_\varepsilon\big(\Phi(x,y)\big),\ \forall\ (x,y) \in \Omega_\varepsilon,$$

and we denote $(\tilde{x},\tilde{y}) = \Phi(x,y)$ and $(x,y) = \big(\Phi_1^{-1}(\tilde{x},\tilde{y}), \Phi_2^{-1}(\tilde{x},\tilde{y})\big)$.

Then $\tilde{Q}_\varepsilon$ solves a PDE of the following form:

$$\begin{cases} \sum_{i,j=1}^{2} D_i\big(a_{ij}D_j\tilde{Q}_\varepsilon\big) + \sum_{i=1}^{2} a_i D_i\tilde{Q}_\varepsilon + c\tilde{Q}_\varepsilon = 0, \text{ in } \mathcal{U}_\varepsilon; \\ \sum_{i=1}^{2} b_{1i}D_i\tilde{Q}_\varepsilon + \frac{w_0}{2}\tilde{Q}_\varepsilon = \frac{w_0}{2}\tilde{Q}_\varepsilon^0, \text{ on } \Phi(\Gamma_\varepsilon); \\ \sum_{i=1}^{2} b_{2i}D_i\tilde{Q}_\varepsilon + \frac{w_0}{2}\tilde{Q}_\varepsilon = \frac{w_0}{2}\tilde{Q}_R, \text{ on } \Phi(\Gamma_R). \end{cases} \tag{3.6.1}$$

where $D_1 = \dfrac{\partial}{\partial\tilde{x}}$ and $D_2 = \dfrac{\partial}{\partial\tilde{y}}$. We have that $a_{ij} \in C^\infty(\mathcal{U}_\varepsilon)$ with:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -\tilde{y}^2 - \dfrac{\tilde{x}^2}{\tilde{x}^2+\tilde{y}^2} & \tilde{x}\tilde{y} - \dfrac{\tilde{x}\tilde{y}}{\tilde{x}^2+\tilde{y}^2} \\ \tilde{x}\tilde{y} - \dfrac{\tilde{x}\tilde{y}}{\tilde{x}^2+\tilde{y}^2} & -\tilde{x}^2 - \dfrac{\tilde{y}^2}{\tilde{x}^2+\tilde{y}^2} \end{pmatrix}.$$

The coefficients a_i are also from $C^\infty(\mathcal{U}_\varepsilon)$ with:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \tilde{x} + D_1a_{11} + D_2a_{21} \\ \tilde{y} + D_1a_{12} + D_2a_{22} \end{pmatrix}$$

The coefficients b_{1i} are from $C^1(\Phi(\Gamma_\varepsilon))$ and can be obtained explicitly from $\nabla Q_\varepsilon \cdot \nu_\varepsilon = \sum_{i=1}^2 b_{1i}D_i\tilde{Q}_\varepsilon$. In the same way, b_{2i} are from $C^1(\Phi(\Gamma_R))$ and can be obtained explicitly from $\nabla Q_\varepsilon \cdot (0,1) = \sum_{i=1}^2 b_{2i}D_i\tilde{Q}_\varepsilon$. Moreover, $\tilde{Q}_\varepsilon^0 = Q_\varepsilon^0 \circ \Phi^{-1}$ and $\tilde{Q}_R = Q_R \circ \Phi^{-1}$. Since $Q_\varepsilon^0 \in C^1(\Gamma_\varepsilon)$ and Φ is a smooth bi-Lipschitz transformation, then $\tilde{Q}_\varepsilon^0 \in C^1(\Phi(\Gamma_\varepsilon))$ which implies that $\tilde{Q}_\varepsilon^0 \in W^{1,1-1/p}(\Phi(\Gamma_\varepsilon))$. In the same way, $\tilde{Q}_R \in W^{1,1-1/p}(\Phi(\Gamma_R))$.

Therefore, we can apply Theorem 3.3.1 to obtain that there exists a unique solution $\tilde{Q}_\varepsilon^0 \in W^{2,p}(\mathcal{U}_\varepsilon)$ of the problem (3.6.1). Using now that Φ is smooth, we obtain that there exists a unique solution $Q_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ of the problem (3.2.5). $\square$

Let $p \in (1,+\infty)$ and $v \in W^{1,p}(\Omega_0)$. We recall that $\Omega_\varepsilon \subset \Omega_0$, for all $\varepsilon > 0$, hence we also have that $v|_{\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon)$. Applying Corollary 3.3.1 and Corollary 3.3.2 with exponent

$p' = \dfrac{p-1}{p} \in (1,+\infty)$, we obtain that $Q_\varepsilon \in W^{1,p'}(\Omega_\varepsilon)$ and that $Q_0 \in W^{1,p'}(\Omega_0)$. Using Definition 3.4.2, it is easy to see that $a_\varepsilon(Q_\varepsilon - Q_0, v)$ is well-defined.

Proof of Proposition 3.4.2. In the following paragraphs, we fix $p \in (1,+\infty)$ and $v \in W^{1,p}(\Omega_0)$. We recall now that Q_ε solves weakly (3.2.5), which is the following PDE:

$$\begin{cases} -\Delta Q_\varepsilon + cQ_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon; \\ \dfrac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} + \dfrac{w_0}{2} Q_\varepsilon = \dfrac{w_0}{2} Q_\varepsilon^0 \quad \text{on } \Gamma_\varepsilon; \\ \dfrac{\partial Q_\varepsilon}{\partial \nu_R} + \dfrac{w_0}{2} Q_\varepsilon = \dfrac{w_0}{2} Q_R \quad \text{on } \Gamma_R. \end{cases}$$

Then, using the integration by parts formula, we get:

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} \big(-\Delta Q_\varepsilon + cQ_\varepsilon \big) \cdot v \,\mathrm{d}(x,y) \\ &= \int_{\Omega_\varepsilon} \big(\nabla Q_\varepsilon \cdot \nabla v + cQ_\varepsilon \cdot v \big) \,\mathrm{d}(x,y) - \int_{\Gamma_\varepsilon} \frac{\partial Q_\varepsilon}{\partial \nu_\varepsilon} \,\mathrm{d}\sigma_\varepsilon - \int_{\Gamma_R} \frac{\partial Q_\varepsilon}{\partial \nu_R} \,\mathrm{d}\sigma_R \\ &= \int_{\Omega_\varepsilon} \big(\nabla Q_\varepsilon \cdot \nabla v + cQ_\varepsilon \cdot v \big) \,\mathrm{d}(x,y) + \frac{w_0}{2} \int_{\Gamma_\varepsilon} \big(Q_\varepsilon - Q_\varepsilon^0 \big) \cdot v \,\mathrm{d}\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} \big(Q_\varepsilon - Q_R \big) \cdot v \,\mathrm{d}\sigma_R \\ &= a_\varepsilon(Q_\varepsilon, v) - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \,\mathrm{d}\sigma_\varepsilon - \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \,\mathrm{d}\sigma_R, \end{aligned}$$

according to Definition 3.4.2. Hence

$$a_\varepsilon(Q_\varepsilon, v) = \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \,\mathrm{d}\sigma_\varepsilon + \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \,\mathrm{d}\sigma_R. \tag{3.6.2}$$

For Q_0, we remind the reader that it solves weakly 3.2.6, which is the following PDE:

$$\begin{cases} -\Delta Q_0 + cQ_0 = 0 \quad \text{in } \Omega_0; \\ \dfrac{\partial Q_0}{\partial \nu_0} + \dfrac{w_{ef}}{2} Q_0 = \dfrac{w_{ef}}{2} Q_{ef} \quad \text{on } \Gamma_0; \\ \dfrac{\partial Q_0}{\partial \nu_R} + \dfrac{w_0}{2} Q_0 = \dfrac{w_0}{2} Q_R \quad \text{on } \Gamma_R. \end{cases}$$

Then, using the previous PDE and the integration by parts formula, we get:

$$\begin{aligned}
0 &= \int_{\Omega_0} \big(-\Delta Q_0 + cQ_0 \big) \cdot v \, \mathrm{d}(x,y) \\
&= \int_{\Omega_0} \big(\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v \big) \, \mathrm{d}(x,y) - \int_{\Gamma_0} \frac{\partial Q_0}{\partial \nu_0} \cdot v \, \mathrm{d}\sigma_0 - \int_{\Gamma_R} \frac{\partial Q_0}{\partial \nu_R} \cdot v \, \mathrm{d}\sigma_R \\
&= \int_{\Omega_0} \big(\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v \big) \, \mathrm{d}(x,y) + \frac{w_{ef}}{2} \int_{\Gamma_0} \big(Q_0 - Q_{ef} \big) \cdot v \, \mathrm{d}\sigma_0 + \\
&\quad + \frac{w_0}{2} \int_{\Gamma_R} \big(Q_0 - Q_R \big) \cdot v \, \mathrm{d}\sigma_R \\
&= a_\varepsilon(Q_0, v) + \int_{\Omega_0 \setminus \Omega_\varepsilon} \big(\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v \big) \, \mathrm{d}(x,y) + \frac{w_{ef}}{2} \int_{\Gamma_0} \big(Q_0 - Q_{ef} \big) \cdot v \, \mathrm{d}\sigma_0 - \\
&\quad - \frac{w_0}{2} \int_{\Gamma_R} Q_R \cdot v \, \mathrm{d}\sigma_R - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, \mathrm{d}\sigma_\varepsilon,
\end{aligned}$$

where we have used Definition 3.4.2. Using (3.6.2), we obtain that:

$$\begin{aligned}
a_\varepsilon(Q_0, v) - a_\varepsilon(Q_\varepsilon, v) = &- \int_{\Omega_0 \setminus \Omega_\varepsilon} \big(\nabla Q_0 \cdot \nabla v + cQ_0 \cdot v \big) \, \mathrm{d}(x,y) - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, \mathrm{d}\sigma_\varepsilon - \\
&- \frac{w_{ef}}{2} \int_{\Gamma_0} \big(Q_0 - Q_{ef} \big) \cdot v \, \mathrm{d}\sigma_0 + \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, \mathrm{d}\sigma_\varepsilon
\end{aligned}$$

Using Definition 3.4.3, we already notice that:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 - \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, \mathrm{d}\sigma_\varepsilon - \frac{w_{ef}}{2} \int_{\Gamma_0} \big(Q_0 - Q_{ef} \big) \cdot v \, \mathrm{d}\sigma_0 + \frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, \mathrm{d}\sigma_\varepsilon. \tag{3.6.3}$$

We recall here that $\Gamma_\varepsilon = \{ (x, \varepsilon\varphi(x/\varepsilon)) \mid x \in [0, 2\pi) \}$, $\Gamma_0 = \{ (x, 0) \mid x \in [0, 2\pi) \}$ and $\Gamma_R = \{ (x, R) \mid x \in [0, 2\pi) \}$. Then:

$$\begin{aligned}
\frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_\varepsilon^0 \cdot v \, \mathrm{d}\sigma_\varepsilon &= \frac{w_0}{2} \int_0^{2\pi} Q_\varepsilon^0(x) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) \cdot \gamma_\varepsilon(x) \, \mathrm{d}x, \\
\frac{w_{ef}}{2} \int_{\Gamma_0} \big(Q_0 - Q_{ef} \big) \cdot v \, \mathrm{d}\sigma_0 &= \frac{w_0}{2} \int_0^{2\pi} \big(Q_0(x,0) \cdot \gamma - Q_{ef} \cdot \gamma \big) \cdot v(x,0) \, \mathrm{d}x,
\end{aligned}$$

and

$$\frac{w_0}{2} \int_{\Gamma_\varepsilon} Q_0 \cdot v \, \mathrm{d}\sigma_\varepsilon = \frac{w_0}{2} \int_0^{2\pi} Q_0(x, \varepsilon\varphi(x/\varepsilon)) \cdot v(x, \varepsilon\varphi(x/\varepsilon)) \cdot \gamma_\varepsilon(x) \, \mathrm{d}x,$$

where we have used (3.2.2), (3.2.4) and Remark 3.2.5.

Using the previous equalities in (3.6.3), we obtain that:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 - \frac{w_0}{2}\int_0^{2\pi} \big(v(x,\varepsilon\varphi(x/\varepsilon)) - v(x,0)\big)\cdot\gamma_\varepsilon(x)\cdot Q_\varepsilon^0(x)\,\mathrm{d}x -$$
$$-\frac{w_0}{2}\int_0^{2\pi} v(x,0)\cdot\gamma_\varepsilon(x)\cdot Q_\varepsilon^0(x)\,\mathrm{d}x + \frac{w_0}{2}\int_0^{2\pi} v(x,0)\cdot\gamma\cdot Q_{ef}\,\mathrm{d}x -$$
$$-\frac{w_0}{2}\int_0^{2\pi} Q_0(x,0)\cdot v(x,0)\cdot\gamma\,\mathrm{d}x + \frac{w_0}{2}\int_0^{2\pi} Q_0(x,\varepsilon\varphi(x/\varepsilon))\cdot v(x,\varepsilon\varphi(x/\varepsilon))\cdot\gamma_\varepsilon(x)\,\mathrm{d}x.$$

We see now that the second integral from the right-hand side from the last equality is I_{21}, according to Definition 3.4.3, and that the next two terms generate I_{22}, according to the same definition as before. Hence:

$$a_\varepsilon(Q_0 - Q_\varepsilon, v) = I_1 + I_{21} + I_{22} +$$
$$+\frac{w_0}{2}\int_0^{2\pi} \big(Q_0(x,\varepsilon\varphi(x/\varepsilon))\cdot v(x,\varepsilon\varphi(x/\varepsilon)) - Q_0(x,0)\cdot v(x,0)\big)\cdot\gamma_\varepsilon(x)\,\mathrm{d}x +$$
$$+\frac{w_0}{2}\int_0^{2\pi} Q_0(x,0)\cdot v(x,0)\cdot\gamma_\varepsilon(x)\,\mathrm{d}x - \frac{w_0}{2}\int_0^{2\pi} Q_0(x,0)\cdot v(x,0)\cdot\gamma\,\mathrm{d}x.$$

The last equality proves (3.4.1).

□

4

HOMOGENISED ELASTIC TERMS FOR AN OSEEN-FRANK TYPE OF ENERGY IN $\mathbb{R}^2$

Abstract:

We consider a general formulation for an Oseen-Frank type of elastic energy in a two dimensional setting for a periodically perforated domain, with isolated holes. We impose sufficient conditions such that, for a sequence u_ε of critical points that generate low enough energy states, we can apply the lifting procedure and write $u_\varepsilon = e^{i\varphi_\varepsilon}$ over the entire perforated domain. We study then the scalar homogenisation problem for the phases φ_ε and we prove that we are under the same settings from [34]. By applying [34, Theorem 2.1], we obtain a local L^2 convergence result for the phases φ_ε and we prove that the same L^2 local convergence result holds for the initial $\mathbb{S}^1$-valued homogenisation problem.

Joint work with G. Canevari and A. D. Zarnescu.

4.1 INTRODUCTION OF THE PROBLEM

Nematic liquid crystals are materials for which their particles are elongated rods that, while being in the nematic state, have the local tendency to align to a preferred direction. There are various theories used to describe the orientation of these particles such as Oseen-Frank, Leslie-Ericksen or Landau-de Gennes and the reader can refer to, for example, [73] or [76]. In the Oseen-Frank theory, for the case in which the domain is $\Omega \subset \mathbb{R}^3$, the order parameter is a vector field $\mathbf{n} : \Omega \to \mathbb{S}^2$, usually called the director, which assigns to each point of the domain the preferred direction of alignment. One of the most common choices of an Oseen-Frank energy is of the type:

$$E[\mathbf{n}] = \int_\Omega W(\mathbf{n}, \nabla \mathbf{n}) \, \mathrm{d}V,$$

where

$$\begin{aligned} 2W(\mathbf{n}, \nabla \mathbf{n}) :=& K_1 \big(\mathrm{div}\ \mathbf{n}\big)^2 + K_2 \big(\mathbf{n} \cdot \mathrm{curl}\ \mathbf{n}\big)^2 + K_3 \big|\mathbf{n} \times \mathrm{curl}\ \mathbf{n}\big|^2 + \\ &+ (K_2 + K_4)\big(\mathrm{tr}(\nabla \mathbf{n})^2 - (\mathrm{div}\ \mathbf{n})^2\big), \end{aligned}$$

where the K_i's are called Frank's elastic constants: K_1 is the splay constant, K_2 the twist constant, K_3 the bend constant and $K_{24} := K_2 + K_4$ is the saddle-splay constant. Moreover, they satisfy the Ericksen inequalities, as presented earlier in Section 1.2:

$$K_1 > 0,\ K_2 > 0,\ K_3 > 0,\ K_2 > |K_4|,\ 2K_1 > K_2 + K_4,$$

(see, for example, [43]).

In this chapter, we consider an energy functional which generalizes the one of from the Oseen-Frank theory, written above, but for a 2-dimensional case $\Omega \subset \mathbb{R}^2$:

$$E[\mathbf{n}] = \int_\Omega K_1(\mathbf{n})\big(\mathrm{div}\ \mathbf{n}\big)^2 + K_2(\mathbf{n})\big(\mathrm{div}\ \mathbf{n}\big)\big(\mathrm{curl}\ \mathbf{n}\big) + K_3(\mathbf{n})\big(\mathrm{curl}\ \mathbf{n}\big)^2 \,\mathrm{d}\mathbf{x} + \mu \int_\Omega \big(\mathbf{n} \cdot \mathbf{n}_0\big)^2 \,\mathrm{d}\mathbf{x},$$

where the elastic coefficients are not necessarily constants anymore and we have also added a new term, in which μ is a positive constant and $\mathbf{n}_0 \in \mathbb{S}^1$ is fixed. For $\mu = 0$, we impose conditions on K_1, K_2 and K_3 such that $E[\mathbf{n}] \geq 0$, for all $\mathbf{n} \in \mathbb{S}^1$, and $E[\mathbf{n}] = 0$ for any $\mathbf{n}$ constant. The term containing μ also tries to mimic, in a very simplified fashion, an external constant magnetic field applied to the nematic liquid crystal, hence there is a competition between minimizing the elastic energy of the material and the desire to align perpendicular to the magnetic field.

Starting from this type of energy functional, we analyse the following homogenisation problem: having a nematic liquid crystal with elastic coefficients κ_1, κ_2, κ_3 (for simplicity, we assume they are of class C^2), we would like to obtain, through homogenisation with colloidal inclusions, another material, which behaves also like a nematic liquid crystal, but now with new elastic coefficients - κ_1^*, κ_2^* and κ_3^*. For this, we consider the case in which we perforate the domain Ω in a periodic fashion, where the holes mimic the presence of another material. The periodicity and the size of the holes are comparable to a small parameter $\varepsilon > 0$, which will tend to 0, and we denote the union of all the holes with T_ε and the perforated domain as $\Omega_\varepsilon := \Omega \setminus T_\varepsilon$. Moreover, we assume that the holes do not touch $\partial\Omega$. More details about our assumptions on the perforations can be found in Section 4.2. Since our goal in this chapter is to generate new elastic coefficients, we neglect any sort of typical surface energy (such as Rapini-Papoular, for example) and we impose, for simplicity, that $\mathbf{u} = (1, 0)$ on $\partial\Omega$. That having been said, we consider the energy functional $\mathbf{F}_\varepsilon : \mathbf{V}_\varepsilon \to [0, +\infty)$ defined as

$$\mathbf{F}_\varepsilon(\mathbf{u}) = \int_{\Omega_\varepsilon} \kappa_1(\mathbf{u})\big(\text{curl } \mathbf{u}\big)^2 + \kappa_2(\mathbf{u})\big(\text{curl } \mathbf{u}\big)\big(\text{div } \mathbf{u}\big) + \kappa_3(\mathbf{u})\big(\text{div } \mathbf{u}\big)^2 + \mu\big(\mathbf{u}\cdot\overline{\mathbf{u}}\big)^2 \,\mathrm{d}\mathbf{x}, \tag{4.1.1}$$

for any $\mathbf{u} \in H^1(\Omega_\varepsilon)$, where $\mathbf{V}_\varepsilon = \{\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{S}^1) \;:\; \mathbf{u} = (1, 0) \text{ on } \partial\Omega\}$.

This setting already gives rise to some interesting challenges. First, having a function $\mathbf{u}$ in $H^1(\Omega_\varepsilon; \mathbb{S}^1)$, there exists an extension $\mathbf{E}_\varepsilon \mathbf{u} \in H^1(\Omega; \mathbb{R}^2)$ as long as the holes are sufficiently regular, but not necessarily in $H^1(\Omega; \mathbb{S}^1)$. Secondly, we can not expect a priori that there is a function $\varphi \in H^1(\Omega_\varepsilon; \mathbb{R})$ such that $\mathbf{u} = (\cos\varphi, \sin\varphi)$. In order to overcome these issues, we make use of various results from [21] that give us connections between the topological degree of a function, the possibility of extending an $\mathbb{S}^1$-valued function and the existence of a lifting φ.

The main assumption of our work is based on the fact that we can have low enough energy states of the material such that there exists a sequence $\big(\mathbf{u}_\varepsilon\big)_{\varepsilon>0} \subset H^1(\Omega_\varepsilon; \mathbb{S}^1)$ of critical points of $\mathbf{F}_\varepsilon$ with the property that their topological degree computed on the boundary of each of the holes must be 0. This will imply that we have a lifting function $\varphi_\varepsilon \in H^1(\Omega_\varepsilon; \mathbb{R})$ and we can turn the $\mathbb{S}^1$-valued homogenisation problem into a scalar one. If one were to work without this assumption, then it is possible to prove that such a lifting exists, but only locally in Ω_ε.

The scalar homogenisation problem obtained represents a particular case of the work done in [34] and is of the form:

$$\begin{cases} -\text{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = \mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon) & \text{in } \Omega_\varepsilon \\ A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu = 0 & \text{on } \partial T_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1.2}$$

where A is a matrix-valued function depending on one parameter which contains all the information related to the initial elastic coefficients and $\mathscr{B}$ has quadratic growth in the second variable and it depends on the derivative of A, namely A'. The homogeneous Dirichlet boundary condition on $\partial\Omega$ comes from imposing $\mathbf{u}_\varepsilon = (1, 0)$ on $\partial\Omega$. The main result from [34] states that there exists $\varphi_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $E_0\varphi_\varepsilon \rightharpoonup \varphi_0$ weakly in $L^2(\Omega)$ (where E_0 is the extension by 0 in the holes) and that it solves the following PDE:

$$\begin{cases} -\text{div}(A_0(\varphi_0)\nabla\varphi_0) = \mathscr{B}_0(\varphi_0, \nabla\varphi_0) & \text{in } \Omega \\ \varphi_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1.3}$$

where A_0 and $\mathscr{B}_0$ are the homogenised components obtained from A and $\mathscr{B}$.

When deriving the homogenised energy functional for $\mathbb{S}^1$-valued functions, we prove in Proposition 4.3.6 that $\mathscr{B}_0$ has a similar dependency of A_0' as the connection between $\mathscr{B}$ and A'. In this way, we are able to say that $\mathbf{u}_0 = (\cos\varphi_0, \sin\varphi_0)$ is a critical point of the following homogenised energy functional $\mathbf{F}_0 : \mathbf{V}_0 \to [0, +\infty)$:

$$\mathbf{F}_0(\mathbf{u}) = \int_\Omega \kappa_1^*(\mathbf{u})(\text{curl } \mathbf{u})^2 + \kappa_2^*(\mathbf{u})(\text{curl } \mathbf{u})(\text{div } \mathbf{u}) + \kappa_3^*(\mathbf{u})(\text{div } \mathbf{u})^2 + \theta_0\mu(\mathbf{u}\cdot\overline{\mathbf{u}})^2 \, \mathrm{d}x, \quad (4.1.4)$$

where θ_0 represents the volume fraction between the nematic liquid crystal part and the periodic cell and $\mathbf{V}_0 = \{\mathbf{u} \in H^1(\Omega; \mathbb{S}^1) \ : \ \mathbf{u} = (1,0) \text{ on } \partial\Omega\}$. Moreover, κ_1^*, κ_2^* and κ_3^* are the new elastic coefficients for the homogenised material and they are introduced in Definition 4.3.9.

We want now to obtain a connection between $\mathbf{u}_\varepsilon$ and $\mathbf{u}_0$. In [33], the authors were able to prove that the solutions φ_ε of (4.1.2) are uniformly bounded in $V_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon) \ : \ \varphi = 0 \text{ on } \partial\Omega\}$. Then, by [5, Lemma 2.3], we obtain a local convergence result in the interior Ω, that is, for any ω open such that $\overline{\omega} \subset \Omega$, we have

$$\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega_\varepsilon\cap\omega)} \to 0, \text{ as } \varepsilon \to 0.$$

on a subsequence of φ_ε. Since cos and sin are Lipschitz functions, we are able to prove that, again, on a subsequence of $\mathbf{u}_\varepsilon$, we have:

$$\text{for any open set } \omega \text{ such that } \overline{\omega} \subset \Omega, \text{ we have } \lim_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L^2(\Omega_\varepsilon\cap\omega;\mathbb{S}^1)} = 0.$$

As stated in [34], one should not expect strong convergence of φ_ε to φ_0 in $L^2(\Omega)$, nor almost everywhere in Ω. However, if we were to consider the initial elastic coefficients as being constants, then we have

$$\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega_\varepsilon)} \to 0, \text{ as } \varepsilon \to 0,$$

since our problem is a particular case of [36], in which they consider holes that are isolated in each cell, or, by some extent, this can be seen as [5, Theorem A.1], where they consider the more generalised situation of connected holes. Moreover, one could prove in a very similar fashion as in [5, Appendix] that we can extend the local convergence result up to the boundary of Ω, since we impose homogeneous Dirichlet boundary conditions.

We are also able to describe the dependency between the initial elastic coefficients κ_i and the homogenised ones κ_i^*, using the same corrector matrix as in, for example, [14, 15, 34, 39, 40]. Hence, by computing two solutions on the cell problem, we are able to compute the new elastic coefficients, depending on the initial ones. This, of course, raises the problem of obtaining bounds for how big or how small one could obtain these new elastic coefficients, depending on our choices of holes and initial coefficients. At the same time, the inverse problem is also interesting from the physical point of view: given desired elastic properties, how should one choose the perforations such that, in the limit, a given nematic liquid crystal would achieve those effective properties? Both situations represent interesting directions that the author would like to pursue.

At the same time, the reader should also consult, for example, the works of [8] or [20]. In [8], the authors consider the case of manifold valued Sobolev spaces and they are able to obtain a Γ-convergence result with respect to strong L^p-topology for an energy functional in which the integrand is Carathéodory, 1-periodic in the first variable, but with lower p-bounds a.e. in $\mathbb{R}^N$. However, in the case of periodic holes, we are not able to impose such condition. Moreover, the integrand considered is of the form $f(\mathbf{x}/\varepsilon, \nabla \mathbf{u})$, while our case would be represented by an integrand of the type $f(\mathbf{x}/\varepsilon, \mathbf{u}, \nabla \mathbf{u})$. In [20], the author considers a more general integrand, one in which the dependency on $\mathbf{u}$ is included, but now in the case of real valued Sobolev spaces. Also, the integrand is considered to have lower p-bounds a.e. in $\mathbb{R}^N$, which is in conflict with our choice of perforations. Nevertheless, we do believe that a Γ-convergence result is possible in our case, but this is beyond the scope of this work which should be seen as a preliminary result in this direction.

This chapter is organised as follows: we first present all of our assumptions, we then formulate our main result, we study the consequent scalar homogenisation problem obtained using the lifting procedure and then we present all the necessary information such that we are able to prove the main result. All the other intermediate results can be found in Section 4.5.

In Section 4.2, we present the general assumptions related to the domain chosen, the properties of the holes, the properties of the initial coefficients and the main assumption which implies the existence of a lifting. The proofs of the intermediate results from this section can be found in Subsection 4.5.1. In the end of Section 4.2, we present the main result of this chapter.

In Section 4.3, we explore the scalar problem obtained via the lifting granted in the previous section. We present all the necessary conditions such that we are able to apply [34, Theorem 2.1]. The proofs of the results presented here can be found in Subsection 4.5.2. A more detailed look at the cell problem can be found in Subsection 4.5.3, where we prove that the solutions of the cell problem exist and they depend in a differentiable way with respect to the nonlinear coefficients given by A.

In Section 4.4, we first present the connection between the first order derivative of the homogenised matrix A_0 and the forcing term $\mathscr{B}_0$. The proof of this result can be found in Subsection 4.5.4, where we use all of the properties proved in Subsection 4.5.3. Then, we present the connection between the initial coefficients κ_i and the homogenised ones κ_i^*, for which, once again, the proof can be found in Subsection 4.5.5. In the end of Section 4.4, we prove the main result of this chapter.

4.2 GENERAL ASSUMPTIONS AND MAIN RESULT

Assumption 4.2.1. We assume that $\Omega \subset \mathbb{R}^2$ is a smooth, open, simply connected and bounded set.

Remark 4.2.1. Throughout this chapter, we denote by $\mathbb{S}^1$ the unit circle from $\mathbb{R}^2$.

Assumption 4.2.2. Let $Y = (0,1)^2$ be the reference periodic cell. We assume that the reference hole $T \subset Y$ is of the form

$$T = \bigcup_{i=1}^{N_T} T^i,$$

where T^i is a compact, smooth and simply connected set, for any $i \in \{1, 2, \dots, N_T\}$, where $N_T \in \mathbb{N}^*$, and $T^{i_1} \cap T^{i_2} = \emptyset$, for any $i_1, i_2 \in \{1, 2, \dots, N_T\}$.

Definition 4.2.1. We denote by $\theta_0 = \dfrac{|Y \setminus T|}{|Y|} = |Y \setminus T| = 1 - |T|$ the volume fraction between the nematic liquid crystal part and the periodic cell.

Definition 4.2.2. Let $\varepsilon > 0$, $X_\varepsilon = \{\xi \in \mathbb{Z}^2 \mid \varepsilon(\xi + Y) \subset \Omega\}$ and $N_\varepsilon = \text{card}(X_\varepsilon)$. We define the set of all holes with size of order ε contained in Ω that do not touch $\partial\Omega$ by T_ε, which can be written as:

$$T_\varepsilon = \bigcup_{\mathbf{x}_\varepsilon \in X_\varepsilon} \varepsilon(\mathbf{x}_\varepsilon + T) = \bigcup_{j=1}^{N_\varepsilon} \bigcup_{i=1}^{N_T} \varepsilon\big(\mathbf{x}_\varepsilon^j + T^i\big),$$

where $\mathbf{x}_\varepsilon^j$ is the j-th element of X_ε and T^i is the i-th component of T. We also use the notation Y_ε^j to describe the j-th individual periodic cell, described as $\varepsilon\big(\mathbf{x}_\varepsilon^j + Y\big)$, and $T_\varepsilon^{i,j}$ to describe the i-th component of j-th individual hole, where $i \in \{1, 2, \dots, N_T\}$ and $j \in \{1, 2, \dots, N_\varepsilon\}$. We define now the perforated domain as $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$.

Remark 4.2.2. By the previous construction, the holes $T_\varepsilon^{i,j}$ do not touch $\partial\Omega$. Therefore, we have $\partial\Omega_\varepsilon = \partial\Omega \cup \partial T_\varepsilon$.

We continue with the assumptions for μ, $\overline{\mathbf{u}}$ and κ_i ($i \in \{1, 2, 3\}$) which are used in (4.1.1).

Assumption 4.2.3. We assume that $\mu > 0$ is a constant and that $\overline{\mathbf{u}} \in \mathbb{S}^1$ is also a constant.

Assumption 4.2.4. Let $\alpha > 0$. We assume that $\kappa_1, \kappa_2, \kappa_3 \in C^2(\mathbb{S}^1)$, $\kappa_1(\mathbf{s}) > \alpha$, $\kappa_3(\mathbf{s}) > \alpha$ and $4\big(\kappa_1(\mathbf{s}) - \alpha\big)\big(\kappa_3(\mathbf{s}) - \alpha\big) - \kappa_2^2(\mathbf{s}) > 0$, for all $\mathbf{s} \in \mathbb{S}^1$.

Proposition 4.2.1. Let α and $\kappa_1, \kappa_2, \kappa_3$ be given by Assumption 4.2.4. Let $\mathbf{s} \in \mathbb{S}^1$ fixed. We denote by $f_\mathbf{s}, g_\mathbf{s} : \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined as:

$$\begin{cases} f_\mathbf{s}(x,y) = \kappa_1(\mathbf{s})x^2 + \kappa_2(\mathbf{s})xy + \kappa_3(\mathbf{s})y^2 \\ g_\mathbf{s}(x,y) = \kappa_1(\mathbf{s})x^2 - \kappa_2(\mathbf{s})xy + \kappa_3(\mathbf{s})y^2 \end{cases}, \ \forall (x,y) \in \mathbb{R}^2.$$

Then $f_\mathbf{s}(x,y) \geq \alpha(x^2 + y^2)$ and $g_\mathbf{s}(x,y) \geq \alpha(x^2 + y^2)$, for any $(x,y) \in \mathbb{R}^2$, and the equality sign is achieved only when $x = y = 0$.

Proof. We have, by Assumption 4.2.4, that $\kappa_1(\mathbf{s}) > \alpha$ and $\kappa_3(\mathbf{s}) > \alpha$, hence $\kappa_1(\mathbf{s}) - \alpha > 0$ and $\kappa_3(\mathbf{s}) - \alpha > 0$. Then:

$$\begin{aligned}
&f_{\mathbf{s}}(x,y) = \kappa_1(\mathbf{s})x^2 + \kappa_2(\mathbf{s})xy + \kappa_3(\mathbf{s})y^2\\
&= \alpha(x^2+y^2) + (\kappa_1(\mathbf{s})-\alpha)x^2 + \kappa_2(\mathbf{s})xy + (\kappa_3(\mathbf{s})-\alpha)y^2\\
&= \alpha(x^2+y^2) + \left(\sqrt{\kappa_1(\mathbf{s})-\alpha}\cdot x\right)^2 - 2\cdot\left(\sqrt{\kappa_1(\mathbf{s})-\alpha}\cdot x\right)\cdot\left(\frac{\kappa_2(\mathbf{s})}{2\sqrt{\kappa_1(\mathbf{s})-\alpha}}\cdot y\right)+\\
&\quad + \left(\frac{\kappa_2(\mathbf{s})}{2\sqrt{\kappa_1(\mathbf{s})-\alpha}}\cdot y\right)^2 + \left(\kappa_3(\mathbf{s}) - \frac{\kappa_2^2(\mathbf{s})}{4(\kappa_1(\mathbf{s})-\alpha)}\right)\cdot y^2\\
&= \alpha(x^2+y^2) + \left(\sqrt{\kappa_1(\mathbf{s})-\alpha}\cdot x + \frac{\kappa_2(\mathbf{s})}{2\sqrt{\kappa_1(\mathbf{s})-\alpha}}\cdot y\right)^2 + \frac{4(\kappa_1(\mathbf{s})-\alpha)(\kappa_3(\mathbf{s})-\alpha)-\kappa_2^2(\mathbf{s})}{4(\kappa_1(\mathbf{s})-\alpha)}y^2.
\end{aligned} \tag{4.2.1}$$

This implies that $f_{\mathbf{s}}(x,y) \geq \alpha(x^2+y^2)$, for any $(x,y) \in \mathbb{R}^2$. Moreover, let $(x_0,y_0) \in \mathbb{R}^2$ such that $f_{\mathbf{s}}(x_0,y_0) = \alpha(x_0^2+y_0^2)$. Then, by (4.2.1), we must have $y_0^2 = 0$, since $4(\kappa_1(\mathbf{s})-\alpha)(\kappa_3(\mathbf{s})-\alpha)-\kappa_2^2(\mathbf{s}) > 0$. This also implies that $x_0 = 0$, since $\kappa_1(\mathbf{s}) - \alpha > 0$. For $g_{\mathbf{s}}$, the proof follows the same steps, with the only remark that the second term from (4.2.1) is now the square of a difference. □

Proposition 4.2.2. Under Assumption 4.2.4, one has that:

$$\alpha\|\nabla\mathbf{u}\|^2_{L^2(\Omega_\varepsilon)} \leq \mathbf{F}_\varepsilon(\mathbf{u}),\ \forall\mathbf{u}\in H^1(\Omega_\varepsilon;\mathbb{S}^1).$$

Proof. Let $\mathbf{u} \in H^1(\Omega_\varepsilon;\mathbb{S}^1)$. We observe that:

$$\mu\int_{\Omega_\varepsilon}(\mathbf{u}\cdot\overline{\mathbf{u}})^2\,\mathrm{d}\mathbf{x} \geq 0.$$

Since for any $\mathbf{x}\in\Omega_\varepsilon$, we have curl $\mathbf{u}(\mathbf{x})$, div $\mathbf{u}(\mathbf{x}) \in \mathbb{R}$, we can write, using the same notations as in Proposition 4.2.1:

$$\mathbf{F}_\varepsilon(\mathbf{u}) = \int_{\Omega_\varepsilon} f_{\mathbf{u}(\mathbf{x})}(\operatorname{curl}\mathbf{u}(\mathbf{x}),\operatorname{div}\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

By Proposition 4.2.1, for any $\mathbf{x}\in\Omega_\varepsilon$, we have:

$$f_{\mathbf{u}(\mathbf{x})}(\operatorname{curl}\mathbf{u}(\mathbf{x}),\ \operatorname{div}\mathbf{u}(\mathbf{x})) \geq \alpha\Big(\big(\operatorname{curl}\mathbf{u}(\mathbf{x})\big)^2 + \big(\operatorname{div}\mathbf{u}(\mathbf{x})\big)^2\Big).$$

At the same time, since $\mathbf{u}(\mathbf{x}) \in \mathbb{S}^1$, we have:

$$|\nabla\mathbf{u}(\mathbf{x})|^2 = \big(\operatorname{curl}\mathbf{u}(\mathbf{x})\big)^2 + \big(\operatorname{div}\mathbf{u}(\mathbf{x})\big)^2,$$

from which we conclude the proof. □

Definition 4.2.3. Throughout this chapter, we use the following notations:

- $\mathbf{V}_\varepsilon = \{u \in H^1(\Omega_\varepsilon;\mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}$;
- $\mathbf{V}_0 = \{u \in H^1(\Omega;\mathbb{S}^1) \ : \ u = (1,0) \text{ on } \partial\Omega\}$;
- $V_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon) \ : \ \varphi = 0 \text{ on } \partial\Omega\}$;
- $E_0 : L^2(\Omega_\varepsilon) \to L^2(\Omega)$ the extension operator by 0 in T_ε.

The goal of the next paragraphs is to provide sufficient conditions such that, for a critical point in $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ of $\mathbf{F}_\varepsilon$, there exists a lifting $\varphi_\varepsilon \in V_\varepsilon$ such that we can analyse the $\mathbb{S}^1$-valued homogenisation problem via a scalar homogenisation problem of the same type as in [34].

We first recall some definitions from [21]. For a given $f \in C(\mathbb{S}^1;\mathbb{S}^1)$, we construct the function $h : [0, 2\pi] \to \mathbb{S}^1$ defined as $h(\theta) = f(e^{i\theta})$. By [21, Lemma 1.1], the function h admits a lifting $\psi \in C([0, 2\pi];\mathbb{R})$, that is, $h = e^{i\psi}$. We define the *degree* of f as:

$$\deg f := \frac{\psi(2\pi) - \psi(0)}{2\pi} \in \mathbb{Z}.$$

We recall that the definition of $\deg f$ does not depend on the choice of ψ, by the uniqueness of ψ (mod 2π). The notion of degree can be extended for $W^{1,p}(\mathbb{S}^1;\mathbb{S}^1)$ functions, for any $p \in (1,\infty)$, by the same [21, Lemma 1.1]. Brezis and Nirenberg have extended the concept of degree also for maps $f \in VMO(\mathbb{S}^1;\mathbb{S}^1)$, where VMO stands for the space of functions with vanishing mean oscillations, and, as a consequence of the embedding $W^{1/p,p}(\mathbb{S}^1) \hookrightarrow VMO(\mathbb{S}^1)$, for any $p \in (1,\infty)$, then $\deg f$ is well-defined also for $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ functions, for any $p \in (1,\infty)$, as described in [21, Section 12.0]. There are various definitions of the topological degree of a function, but, for our purposes, we only use the previous definition (this is motivated by [21, Equations (12.23) and (12.24)], where the reader can consult more details).

Remark 4.2.3. Another way in which we can extend the concept of degree for $H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ functions is the following. Let $f \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ and $B_r = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < r\}$, for any $r > 0$. By a similar argument as in, for example, [18] or [19], there exists $\varepsilon = \varepsilon(f) > 0$ and a function $g \in H^1(B_1 \setminus \overline{B_{1-\varepsilon}};\mathbb{S}^1)$ such that $g\big|_{\partial B_1} = f$ in the trace sense. Let now $h : (1-\varepsilon, 1) \times \mathbb{S}^1 \to \mathbb{S}^1$ be defined as $h(\rho, \mathbf{x}) = g(\rho\mathbf{x})$, for any $\mathbf{x} \in \mathbb{S}^1$. Then $h \in H^1((1-\varepsilon, 1) \times \mathbb{S}^1)$ and, by Fubini's theorem, we have $h(\rho, \cdot) \in H^1(\partial B_\rho)$ for almost any $\rho \in (1-\varepsilon, 1)$, hence we can define $\deg h(\rho, \cdot)$. By [21, Proposition 12.14], the function $\rho \to \deg h(\rho, \cdot)$ is constant a.e. in $(1-\varepsilon, 1)$. We define this constant as $\deg h$ and we introduce the degree of f as $\deg f = \deg h$. Moreover, this definition does not depend on the choice of g.

Let now $U \subset \mathbb{R}^2$ be a bounded, smooth and simply connected open set. Let $\Gamma := \partial U$. By the smooth Riemann mapping theorem (see, for example, [11, Theorem A and Corollary]), there exists a diffeomorphism up to the boundary $\Phi : \overline{U} \to \overline{B_1}$ with $\Phi(\Gamma) = \mathbb{S}^1$. For a given $f \in H^{1/2}(\Gamma;\mathbb{S}^1)$, we then have $f\big|_\Gamma \circ \Phi \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$, hence we define:

$$\deg(f, \Gamma) := \deg\left(f \circ \Phi\right)$$

and this degree does not depend on the choice of Φ. For more details, the reader can consult [21, Subsection 12.8.5].

Definition 4.2.4. Let:

$$\mathscr{C}_\varepsilon = \left\{ \mathbf{u} \in \mathbf{V}_\varepsilon \ : \ \deg(\mathbf{u}, \partial T_\varepsilon^{i,j}) = 0, \ \forall i \in \{1,2,\dots,N_T\} \text{ and } \forall j \in \{1,2,\dots,N_\varepsilon\} \right\},$$

and

$$\widetilde{\mathscr{C}}_\varepsilon = \left\{ \mathbf{u} \in \mathbf{V}_\varepsilon \ : \ \exists \varphi \in V_\varepsilon \text{ such that } \mathbf{u} = e^{i\varphi} \text{ a.e. in } \Omega_\varepsilon \right\}.$$

The following proposition shows the connection between the existence of a lifting for functions in $\mathbf{V}_\varepsilon$ and the value of the topological degree on the boundary of each of the holes. The proof can be found in Subsection 4.5.1 and it is based on various results from [21].

Proposition 4.2.3. We have $\mathscr{C}_\varepsilon = \widetilde{\mathscr{C}}_\varepsilon$, for any $\varepsilon > 0$.

We are going to work under the following main assumption:

Assumption 4.2.5. Let $\delta > 0$. We assume there exists a sequence $\left(\mathbf{u}_\varepsilon\right)_{\varepsilon>0} \subset \mathbf{V}_\varepsilon$ of critical points of $\mathbf{F}_\varepsilon$, described in (4.1.1), such that $\mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon) \leq \delta$, for any $\varepsilon > 0$.

Proposition 4.2.4. If $\delta > 0$ from Assumption 4.2.5 is small enough, then any critical point $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ given by Assumption 4.2.5, satisfies $\mathbf{u}_\varepsilon \in \mathscr{C}_\varepsilon$ and, therefore, $\mathbf{u}_\varepsilon \in \widetilde{\mathscr{C}}_\varepsilon$, for any $\varepsilon > 0$.

The proof of Proposition 4.2.4 can be found in Subsection 4.5.1.

Remark 4.2.4. We can always choose $\mu > 0$ such that there are critical points $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ of $\mathbf{F}_\varepsilon$ with $\mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon) \leq \delta$. For this, let $\mathbf{v}_\varepsilon \in \mathbf{V}_\varepsilon$ a minimiser for $\mathbf{F}_\varepsilon$ and let $\mathbf{u}_1(\mathbf{x}) = (1,0)$, for all $\mathbf{x} \in \Omega$. Then $\mathbf{u}_1\big|_{\Omega_\varepsilon} \in \mathbf{V}_\varepsilon$. Since $\mathbf{v}_\varepsilon$ is a minimiser of $\mathbf{F}_\varepsilon$, then:

$$\mathbf{F}_\varepsilon(\mathbf{v}_\varepsilon) \leq \mathbf{F}_\varepsilon\big(\mathbf{u}_1\big|_{\Omega_\varepsilon}\big) = \mu \int_{\Omega_\varepsilon} \left((1,0)\cdot\overline{\mathbf{u}}\right)^2 \mathrm{d}\mathbf{x} \leq \mu|\Omega_\varepsilon| < \mu|\Omega|,$$

since $\overline{\mathbf{u}} \in \mathbb{S}^1$ and $T_\varepsilon \neq \emptyset$. Therefore, the conclusion follows by choosing $\mu < \delta \cdot |\Omega|^{-1}$.

Theorem 4.2.1 (Main result)**.** Under Assumptions 4.2.1 to 4.2.5, if $\left(\mathbf{u}_\varepsilon\right)_{\varepsilon>0} \subset \mathbf{V}_\varepsilon$ is a sequence of critical points of $\mathbf{F}_\varepsilon$, given by Assumption 4.2.5, where $\mathbf{F}_\varepsilon$ is introduced in (4.1.1), then there exists $\mathbf{u}_0 \in \mathbf{V}_0$ a critical point of $\mathbf{F}_0$, described in (4.1.4), such that, along a subsequence, still denoted with subscript ε:

$$\text{for any open set } \omega \text{ such that } \overline{\omega} \subset \Omega, \text{ we have } \lim_{\varepsilon\to 0} \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L^2(\Omega_\varepsilon\cap\omega;\mathbb{S}^1)} = 0. \tag{4.2.2}$$

4.3 THE SCALAR PROBLEM

Let $(\mathbf{u}_\varepsilon)_{\varepsilon>0} \subset \mathbf{V}_\varepsilon$ be the sequence of critical points of $\mathbf{F}_\varepsilon$ given by Assumption 4.2.5. From Proposition 4.2.4, there exists a sequence of functions $(\varphi_\varepsilon)_{\varepsilon>0} \subset V_\varepsilon$ such that:

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \left(\cos\varphi_\varepsilon(\mathbf{x}), \sin\varphi_\varepsilon(\mathbf{x})\right), \ \forall \mathbf{x} \in \Omega_\varepsilon.$$

Therefore, we obtain:

$$\begin{cases} \text{curl } \mathbf{u}_\varepsilon = \cos\varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial x} + \sin\varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial y}, \\ \text{div } \mathbf{u}_\varepsilon = -\sin\varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial x} + \cos\varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial y}. \end{cases}$$

Definition 4.3.1. Let $k_i(t) = \kappa_i(\cos t, \sin t)$, for all $t \in \mathbb{R}$ and all $i \in \{1,2,3\}$.

Remark 4.3.1. It is easy to see that k_i is of class $C^2(\mathbb{R})$ and it is a 2π-periodic function, for any $i \in \{1,2,3\}$. Moreover, by Assumption 4.2.4, we have $k_1(t) > \alpha$, $k_3(t) > \alpha$ and $4(k_1(t)-\alpha)(k_3(t)-\alpha) - k_2^2(t) > 0$, for all $t \in \mathbb{R}$ and α as in Assumption 4.2.4.

Definition 4.3.2. For any $t \in \mathbb{R}$, we define:

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \text{ and } K(t) = \begin{pmatrix} k_1(t) & k_2(t)/2 \\ k_2(t)/2 & k_3(t) \end{pmatrix}.$$

Moreover, let $A(t) = \begin{pmatrix} a(t) & b(t)/2 \\ b(t)/2 & c(t) \end{pmatrix}$ be such that, for any $t \in \mathbb{R}$, we have:

$$A(t) = R(t) \cdot K(t) \cdot R(-t). \tag{4.3.1}$$

Remark 4.3.2. The functions $A, K : \mathbb{R} \to \mathbb{R}^{2\times 2}$ and $a, b, c : \mathbb{R} \to \mathbb{R}$ are 2π-periodic and of class $C^2(\mathbb{R})$, since k_1, k_2, k_3 have these properties. Moreover, $R : \mathbb{R} \to \mathbb{R}^{2\times 2}$ is a smooth map with the following properties: $\det(R(t)) = 1$ and $R^{-1}(t) = R(-t)$, for all $t \in \mathbb{R}$.

Remark 4.3.3. By Definition 4.3.2, we have, for any $t \in \mathbb{R}$:

$$\begin{cases} a(t) = k_1(t)\cos^2 t - k_2(t)\cos t \sin t + k_3(t)\sin^2 t, \\ b(t) = k_1(t) \cdot 2\sin t \cos t + k_2(t) \cdot (\cos^2 t - \sin^2 t) - k_3(t) \cdot 2 \sin t \cos t, \\ c(t) = k_1(t)\sin^2 t + k_2(t)\cos t \sin t + k_3(t)\cos^2 t. \end{cases}$$

Proposition 4.3.1. We have that $a(t)>\alpha$, $c(t)>\alpha$ and $4(a(t)-\alpha)(c(t)-\alpha) - b^2(t) > 0$, for all $t \in \mathbb{R}$, where α is given by Assumption 4.2.4.

Proof. Let $t \in \mathbb{R}$ and $\mathbf{s} = (\cos t, \sin t) \in \mathbb{S}^1$. Then $a(t) = g_{\mathbf{s}}(\cos t, \sin t)$ and $c(t) = f_{\mathbf{s}}(\sin t, \cos t)$, where $f_{\mathbf{s}}$ and $g_{\mathbf{s}}$ are given by Proposition 4.2.1. Applying Proposition 4.2.1 yields:

$$a(t) = g_{\mathbf{s}}(\cos t, \sin t) > \alpha\big(\cos^2 t + \sin^2 t\big) = \alpha,$$

since for any $t \in \mathbb{R}$, we have $(\cos t, \sin t) \neq (0,0)$. In the same way, we have

$$c(t) = f_{\mathbf{s}}(\sin t, \cos t) > \alpha\big(\sin^2 t + \cos^2 t\big) = \alpha.$$

For the last property, we see that:

$$
\begin{aligned}
(a(t)-\alpha)(c(t)-\alpha)-\frac{b^2(t)}{4} &= \det\big(A(t)-\alpha\mathbb{I}_2\big) = \det\big(R(t)K(t)R(-t)-\alpha\mathbb{I}_2\big) \\
&= \det\big(R(t)\cdot(K(t)-\alpha\mathbb{I}_2)\cdot R(-t)\big) \\
&= \det(R(t))\cdot\det(K(t)-\alpha\mathbb{I}_2)\cdot\det(R(-t)) \\
&= \det(K(t)-\alpha\mathbb{I}_2) \\
&= (k_1(t)-\alpha)(k_3(t)-\alpha)-\frac{k_2^2(t)}{4} > 0,
\end{aligned}
$$

where we have used that $\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R(t)R(-t) = \mathbb{I}_2$, $\det(R(t)) = \det(R(-t)) = 1$, for all $t \in \mathbb{R}$ and Remark 4.3.1. $\square$

Let

$$
F_\varepsilon(\varphi_\varepsilon) = \mathbf{F}_\varepsilon\big((\cos\varphi_\varepsilon, \sin\varphi_\varepsilon)\big) = \mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon).
$$

Moreover, since $\overline{\mathbf{u}} \in \mathbb{S}^1$ is constant, by Assumption 4.2.3, let $\overline{\varphi} \in \mathbb{R}$ such that $\overline{\mathbf{u}} = (\cos\overline{\varphi}, \sin\overline{\varphi})$. Then, we can write $F_\varepsilon : V_\varepsilon \to [0,+\infty)$ as:

$$
F_\varepsilon(\varphi) = \int_{\Omega_\varepsilon} a(\varphi)\left(\frac{\partial\varphi}{\partial x}\right)^2 + b(\varphi)\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + c(\varphi)\left(\frac{\partial\varphi}{\partial y}\right)^2 \,\mathrm{d}\mathbf{x} + \mu\int_{\Omega_\varepsilon} \cos^2\left(\varphi_\varepsilon - \overline{\varphi}\right)\,\mathrm{d}\mathbf{x}
$$

or (4.3.2)

$$
F_\varepsilon(\varphi) = \int_{\Omega_\varepsilon} A(\varphi)\nabla\varphi\cdot\nabla\varphi \,\mathrm{d}\mathbf{x} + \mu\int_{\Omega_\varepsilon} \cos^2\left(\varphi_\varepsilon - \overline{\varphi}\right)\,\mathrm{d}\mathbf{x},
$$

for any $\varphi \in V_\varepsilon$.

We prove in Subsection 4.5.6 that if $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ is a critical point of $\mathbf{F}_\varepsilon$, then $\varphi_\varepsilon \in V_\varepsilon$ is a critical point of F_ε. This implies that φ_ε solves the following Euler-Lagrange equation:

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \left(a'(\varphi_\varepsilon)\left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2 + b'(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y} + c'(\varphi_\varepsilon)\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\psi \,\mathrm{d}\mathbf{x} - \mu\int_{\Omega_\varepsilon} \sin\left(2(\varphi_\varepsilon - \overline{\varphi})\right)\psi\,\mathrm{d}\mathbf{x} + \\
&+ 2\int_{\Omega_\varepsilon} a(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi}{\partial x} + b(\varphi_\varepsilon)\left(\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi}{\partial x}\right) + c(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi}{\partial y}\,\mathrm{d}\mathbf{x} = 0,\ \forall\psi \in V_\varepsilon,
\end{aligned}
$$

which can be rewritten as:

$$
\int_{\Omega_\varepsilon} A(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\nabla\psi + \frac{1}{2}(A'(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\nabla\varphi_\varepsilon)\psi\,\mathrm{d}\mathbf{x} - \frac{\mu}{2}\int_{\Omega_\varepsilon} \sin\left(2(\varphi_\varepsilon - \overline{\varphi})\right)\psi\,\mathrm{d}\mathbf{x} = 0,\ \forall\psi \in V_\varepsilon. \tag{4.3.3}
$$

Using the integration by parts formula, we get:

$$
\int_{\Omega_\varepsilon} a(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi}{\partial x}\,\mathrm{d}\mathbf{x} = -\int_{\Omega_\varepsilon}\frac{\partial}{\partial x}\left(a(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial x}\right)\psi\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega_\varepsilon}\left(a(\varphi_\varepsilon)\frac{\partial\varphi_\varepsilon}{\partial x}(\nu\cdot\mathbf{e}_1)\right)\psi\,\mathrm{d}\sigma,
$$

hence

$$\int_{\Omega_\varepsilon} a(\varphi_\varepsilon)\frac{\partial \varphi_\varepsilon}{\partial x}\frac{\partial \psi}{\partial x} + a'(\varphi_\varepsilon)\left(\frac{\partial \varphi_\varepsilon}{\partial x}\right)^2 \psi + a(\varphi_\varepsilon)\frac{\partial^2 \varphi_\varepsilon}{\partial x^2}\psi \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega_\varepsilon} \left(a(\varphi_\varepsilon)\frac{\partial \varphi_\varepsilon}{\partial x}(\nu \cdot \mathbf{e}_1)\right)\psi \, \mathrm{d}\sigma. \quad (4.3.4)$$

Definition 4.3.3. We denote by $L : H^1(\Omega) \to H^{-1}(\Omega)$ the following differential operator:

$$L\varphi = a(\varphi)\frac{\partial^2 \varphi}{\partial x^2} + b(\varphi)\frac{\partial^2 \varphi}{\partial x \partial y} + c(\varphi)\frac{\partial^2 \varphi}{\partial y^2},$$

for any $\varphi \in H^1(\Omega)$.

By computing in the similar fashion for the other components from (4.3.3) as in (4.3.4), by adding them together and by using the fact that since $\partial\Omega_\varepsilon = \partial\Omega \cup \partial T_\varepsilon$ and $\psi \in V_\varepsilon$ implies $\psi = 0$ on $\partial\Omega$, we obtain:

$$\int_{\Omega_\varepsilon} A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nabla\psi + \left(A'(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nabla\varphi_\varepsilon\right)\psi + L\varphi_\varepsilon \cdot \psi \, \mathrm{d}\mathbf{x} = \int_{\partial T_\varepsilon} \left(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu\right)\psi \, \mathrm{d}\sigma. \quad (4.3.5)$$

Remark 4.3.4. We have:

$$\operatorname{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = A'(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nabla\varphi_\varepsilon + L\varphi_\varepsilon.$$

Combining now (4.3.3), (4.3.5) and Remark 4.3.4, we obtain:

$$\int_{\Omega_\varepsilon} \left(\operatorname{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon)\right)\psi \, \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} \left(-\frac{1}{2}A'(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nabla\varphi_\varepsilon + \frac{\mu}{2}\sin(2(\varphi_\varepsilon - \overline{\varphi}))\right)\psi \, \mathrm{d}\mathbf{x} + \quad (4.3.6)$$

$$+ \int_{\partial T_\varepsilon} \left(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu\right)\psi \, \mathrm{d}\sigma. \quad (4.3.7)$$

Definition 4.3.4. Let $B, \mathscr{B} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as $B(t,\xi) = -\frac{1}{2}A'(t)\xi \cdot \xi$ and

$$\mathscr{B}(t,\xi) = -\frac{1}{2}A'(t)\xi \cdot \xi + \frac{\mu}{2}\sin\left(2t - 2\overline{\varphi}\right),$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$.

Remark 4.3.5. Using Remark 4.3.4, Definition 4.3.3 and Definition 4.3.4, we have:

$$\operatorname{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = L\varphi_\varepsilon - 2B(\varphi_\varepsilon, \nabla\varphi_\varepsilon).$$

We can write now (4.3.6) as:

$$\int_{\Omega_\varepsilon} \left(\operatorname{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon - \mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon)\right)\psi \, \mathrm{d}\mathbf{x} = \int_{\partial T_\varepsilon} \left(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu\right)\psi \, \mathrm{d}\sigma, \ \forall \psi \in V_\varepsilon,$$

so φ_ε solves:

$$\begin{cases} -\mathrm{div}(A(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = \mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon) & \text{in } \Omega_\varepsilon \\ A(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nu = 0 & \text{on } \partial T_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the same form presented in the introduction of this chapter, that is, (4.1.2).

We now present some properties of A, B and $\mathscr{B}$, whose proofs can be found in Subsection 4.5.2.

Proposition 4.3.2. The function $A : \mathbb{R} \to \mathbb{R}^{2\times 2}$ has the following properties:

(1) A is a 2π-periodic $C^2(\mathbb{R})$ function;

(2) we have

$$A(t)\xi \cdot \xi \;\geq\; \alpha|\xi|^2,\ \forall \xi \in \mathbb{R}^2,\ \forall t \in \mathbb{R}, \tag{4.3.8}$$

where α is given by Assumption 4.2.4;

(3) there exists a constant $C_1(A) > 0$, depending only on the $L^\infty(\mathbb{R})$ norms of a, b and c, such that

$$|A(t)\xi| \leq C_1(A)|\xi|,\ \forall \xi \in \mathbb{R}^2,\ \forall t \in \mathbb{R}; \tag{4.3.9}$$

(4) there exists a constant $C_2(A') > 0$, depending only on the $L^\infty(\mathbb{R})$ norms of a', b' and c', such that

$$|A'(t)\xi| \leq C_2(A')|\xi|,\ \forall \xi \in \mathbb{R}^2,\ \forall t \in \mathbb{R} \tag{4.3.10}$$

and

$$\big|(A(t) - A(s))\xi\big| \leq C_2(A') \cdot |t-s| \cdot |\xi|,\ \forall \xi \in \mathbb{R}^2,\ \forall s,t \in \mathbb{R}; \tag{4.3.11}$$

(5) there exists a constant $C_3(A'') > 0$, depending only on the $L^\infty(\mathbb{R})$ norms of a'', b'' and c'', such that

$$\big|(A'(t) - A'(s))\xi\big| \leq C_3(A'') \cdot |t-s| \cdot |\xi|,\ \forall \xi \in \mathbb{R}^2,\ \forall s,t \in \mathbb{R} \tag{4.3.12}$$

and

$$\left|\left(\frac{A(t) - A(s)}{t-s} - A'(t)\right)\xi\right| \leq C_3(A'') \cdot |t-s| \cdot |\xi|,\ \forall \xi \in \mathbb{R}^2,\ \forall s,t \in \mathbb{R},\ s \neq t. \tag{4.3.13}$$

Proposition 4.3.3. The function $\mathscr{B} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, defined in Definition 4.3.4, has the following properties:

(1) there exists a continuous increasing function $d_1 : [0,+\infty) \to [0,+\infty)$ such that $d_1(0) \geq 0$ and

$$|\mathscr{B}(t,\xi) - \mathscr{B}(t,\eta)| \leq d_1(|t|)\big(1 + |\xi| + |\eta|\big)|\xi - \eta|,\ \forall \xi, \eta \in \mathbb{R}^2,\ \forall t \in \mathbb{R};$$

(2) there exists a continuous increasing function $d_2 : [0,+\infty) \to [0,+\infty)$ such that $d_2(0) = 0$ and

$$|\mathscr{B}(t,\xi) - \mathscr{B}(s,\xi)| \leq d_2(|t-s|)(1 + |\xi|^2),\ \forall \xi \in \mathbb{R}^2,\ \forall s,t \in \mathbb{R}.$$

Remark 4.3.6. We are now under the hypothesis from [34]. In [33, Theorem 9.1], the authors prove that there exists an ε-independent constant $\mathcal{C} > 0$ such that:

$$\|\nabla \varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \mathcal{C} \ \text{ and } \ \|\varphi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \mathcal{C}.$$

We now move our attention to the cell problem and the homogenised matrix A_0 for the scalar problem. We proceed in the same fashion as in [34]:

Definition 4.3.5. For any fixed $t \in \mathbb{R}$, we define the homogenised matrix $A_0(t)$ as:

$$A_0(t)\xi = \int_{Y\setminus T} A(t)\big(\xi - \nabla\chi_\xi(\mathbf{x})\big)\,\mathrm{d}\mathbf{x},$$

where $\xi \in \mathbb{R}^2$ and χ_ξ is the unique solution of the following:

$$\begin{cases} -\mathrm{div}(A(t)\nabla\chi_\xi) = 0 & \text{in } Y \setminus T \\ A(t)\big(\xi - \nabla\chi_\xi\big)\cdot\nu = 0 & \text{on } \partial T \\ \chi_\xi \text{ is } Y\text{-periodic} \\ \displaystyle\int_{Y\setminus T}\chi_\xi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0. \end{cases} \tag{4.3.14}$$

Remark 4.3.7. The existence of solutions $\chi_\xi(\cdot,t) \in H^1(Y \setminus T)$ which are Y-periodic and with zero average over $Y \setminus T$ is studied in Subsection 4.5.3. We also prove there that the operator $t \to \chi_\xi(\cdot,t)$ is continuous and Fréchet differentiable on $\mathbb{R}$. The differentiability of this operator will become very useful for the proof of Proposition 4.3.6.

Definition 4.3.6. Let $a_0, b_0, c_0, d_0 : \mathbb{R} \to \mathbb{R}$ be the components of the homogenised matrix A_0, that is, for any $t \in \mathbb{R}$, we have:

$$A_0(t) = \begin{pmatrix} a_0(t) & b_0(t) \\ c_0(t) & d_0(t) \end{pmatrix}.$$

Remark 4.3.8. For $\xi = \mathbf{e}_1 = (1,0)$ and $\xi = \mathbf{e}_2 = (0,1)$, we denote the respective χ_ξ functions as χ_1 and χ_2.

Proposition 4.3.4. For any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, one has that $\chi_\xi = \xi_1\chi_1 + \xi_2\chi_2$.

Proof. The proof is immediate, due to the linearity of the cell problem. □

Definition 4.3.7. For any $t \in \mathbb{R}$, we introduce the corrector $C^\varepsilon(\cdot, t) \in \mathbb{R}^{2\times 2}$ as:

$$C^\varepsilon(\mathbf{x}, t) = C\left(\frac{1}{\varepsilon}\mathbf{x}, t\right), \text{ where } C(\mathbf{x}, t) = \begin{pmatrix} 1 - \dfrac{\partial \chi_1}{\partial x}(\mathbf{x}, t) & -\dfrac{\partial \chi_2}{\partial x}(\mathbf{x}, t) \\ \\ -\dfrac{\partial \chi_1}{\partial y}(\mathbf{x}, t) & 1 - \dfrac{\partial \chi_2}{\partial y}(\mathbf{x}, t) \end{pmatrix}, \ \forall \mathbf{x} \in Y \setminus T.$$

Proposition 4.3.5. For any $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$, one has that $C(\cdot, t)\xi = \xi - \nabla \chi_\xi(\cdot, t)$ in $Y \setminus T$.

Proof. The proof is immediate due to Proposition 4.3.4. □

We have now presented all the notations and requirements such that we are able to apply [34, Theorem 2.1] to our case and obtain that:

Theorem 4.3.1. There exists a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$, still denoted with subscript ε, a function $\varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and a Carathéodory function $\mathscr{B}_0 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ such that:

$$\begin{cases} (i)\ E_0\varphi_\varepsilon \rightharpoonup \theta_0\varphi_0 \text{ weakly in } L^2(\Omega) \text{ and weakly* in } L^\infty(\Omega), \\ (ii)\ E_0\Big(\mathscr{B}(\varphi_\varepsilon, \nabla\varphi_\varepsilon)\Big) \to \mathscr{B}_0(\varphi_0, \nabla\varphi_0) \text{ in } \mathcal{D}'(\Omega) \end{cases}$$

where θ_0 is defined in Definition 4.2.1 and E_0 in Definition 4.2.3. The function φ_0 is a solution of the following problem:

$$\begin{cases} -\text{div}(A_0(\varphi_0)\nabla\varphi_0) = \mathscr{B}_0(\varphi_0, \nabla\varphi_0) & \text{in } \Omega \\ \varphi_0 = 0 & \text{on } \partial\Omega \end{cases}$$

which is exactly (4.1.3), where A_0 is introduced in Definition 4.3.5 and the function $\mathscr{B}_0$ is given by:

$$\mathscr{B}_0(t, \xi) = \int_{Y\setminus T} \mathscr{B}\big(t, C(\mathbf{x}, t)\xi\big)\ \mathrm{d}\mathbf{x}$$

for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^2$, where the corrector $C(\cdot, t)$ is introduced in Definition 4.3.7. Moreover

$$-\text{div}\Big(A(E_0\varphi_\varepsilon)E_0\nabla\varphi_\varepsilon\Big) \to -\text{div}\big(A_0(\varphi_0)\nabla\varphi_0\big) \text{ strongly in } H^{-1}(\Omega).$$

In Subsection 4.5.4, the reader can find the proof of the following proposition:

Proposition 4.3.6. For any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^2$, we have that:

$$\mathscr{B}_0(t, \xi) = -\frac{1}{2}A_0'(t)\xi \cdot \xi + \frac{\theta_0\mu}{2}\sin(2t - 2\overline{\varphi}),$$

where θ_0 denotes the volume fraction and it is described in Definition 4.2.1.

Let us now introduce the following definition, based on Definition 4.3.6:

Definition 4.3.8. We denote by $L_0 : H^1(\Omega) \to H^{-1}(\Omega)$ the following differential operator:

$$L_0\varphi = a_0(\varphi)\frac{\partial^2\varphi}{\partial x^2} + \big(b_0(\varphi) + c_0(\varphi)\big)\frac{\partial^2\varphi}{\partial x\partial y} + d_0(\varphi)\frac{\partial^2\varphi}{\partial y^2},\ \forall\varphi \in H^1(\Omega).$$

We recall now at the beginning we started with φ_ε being a critical point of F_ε. Then, when deriving the PDE that φ_ε solves, we have made Remark 4.3.5, which gave us a connection between B and L, introduced in Definition 4.3.4 and Definition 4.3.3. Proposition 4.3.6 gives us the same equation:

$$\operatorname{div}(A_0(\varphi_0)\nabla\varphi_0) = L_0\varphi_0 - \Big(2\mathscr{B}_0(t,\xi) - \theta_0\mu\sin(2(\varphi_0 - \overline{\varphi}))\Big),$$

which is the key for the next corollary.

Corollary 4.3.1. The function φ_0, given by Theorem 4.3.1, is a critical point of the following energy functional:

$$F_0(\varphi) = \int_\Omega a_0(\varphi)\left(\frac{\partial\varphi}{\partial x}\right)^2 + \big(b_0(\varphi) + c_0(\varphi)\big)\cdot\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\varphi}{\partial y} + d_0(\varphi)\left(\frac{\partial\varphi}{\partial y}\right)^2 + \theta_0\mu\cos^2\left(\varphi - \overline{\varphi}\right)\,\mathrm{dx}. \tag{4.3.15}$$

We would like, before proving the main result of this chapter, to describe the technique to generate the energy functional for the $\mathbb{S}^1$-valued problem. First of all, let us recall (4.3.1):

$$A(t) = R(t)K(t)R(-t),\ \forall t \in \mathbb{R}.$$

In order to obtain the homogenised matrix K_0, we should use the same relation as before.

Definition 4.3.9. We define the homogenised matrix $K_0 : \mathbb{R} \to \mathbb{R}^{2\times 2}$ as

$$K_0(t) := R(-t)A_0(t)R(t),\ \forall t \in \mathbb{R},$$

where R is introduced in Definition 4.3.2 and A_0 in Definition 4.3.5. We denote the components of K_0 as $k_1^0, k_{21}^0, k_{22}^0, k_3^0 : \mathbb{R} \to \mathbb{R}$, that is:

$$K_0(t) = \begin{pmatrix} k_1^0(t) & k_{21}^0(t) \\ k_{22}^0(t) & k_3^0(t) \end{pmatrix},\ \forall t \in \mathbb{R}.$$

Let $\kappa_1^*, \kappa_2^*, \kappa_3^* : \mathbb{S}^1 \to \mathbb{R}$ be such that, for any $t \in \mathbb{R}$, we have:

$$\kappa_1^*(\cos t, \sin t) = k_1^0(t),\ \kappa_2^*(\cos t, \sin t) = k_{21}^0(t) + k_{22}^0(t) \text{ and } \kappa_3^*(\cos t, \sin t) = k_3^0(t).$$

The proof of the following proposition that describes the connection between the initial matrix of elastic coefficients, K, and the homogenised one, K_0, can be found in Subsection 4.5.5.

Proposition 4.3.7. One has that $K_0(t) = K(t) \cdot C_0(t)$, where

$$C_0(t) = R(-t) \cdot \left(\int_{Y \setminus T} C(\mathbf{x}, t) \, d\mathbf{x} \right) \cdot R(t), \ \forall t \in \mathbb{R}.$$

4.4 PROOF OF THE MAIN RESULT

We now prove the main result of this chapter.

Proof of Theorem 4.2.1. Let $\mathbf{u}_0 = (\cos \varphi_0, \sin \varphi_0)$, where φ_0 is given by Theorem 4.3.1. Since $\varphi_0 \in H_0^1(\Omega)$, then $\mathbf{u}_0 \in \mathbf{V}_0$.

Since φ_0 is a critical point of (4.3.15), then, using Definition 4.3.9, one could prove, by a similar argument as in Subsection 4.5.6, that $\mathbf{u}_0 \in \mathbf{V}_0$ is a critical point of:

$$\mathbf{F}_0(\mathbf{u}_0) = \int_\Omega \kappa_1^*(\mathbf{u}_0)(\text{curl } \mathbf{u}_0)^2 + \kappa_2^*(\mathbf{u}_0)(\text{curl } \mathbf{u}_0)(\text{div } \mathbf{u}_0) + \kappa_3^*(\mathbf{u}_0)(\text{div } \mathbf{u}_0)^2 d\mathbf{x} + \\ + \int_\Omega \theta_0 \mu (\mathbf{u}_0 \cdot \overline{\mathbf{u}})^2 \, d\mathbf{x},$$

where κ_1^*, κ_2^* and κ_3^* are introduced in Definition 4.3.9.

We prove now (4.2.2). By [33, Theorem 9.1], we have that $(\varphi_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega_\varepsilon)$, with the constant independent of ε (as previously mentioned in Remark 4.3.6). By Theorem 4.3.1, there exists a subsequence, with subscript still denoted by ε, such that $E_0\varphi_\varepsilon \rightharpoonup \theta_0\varphi_0$ weakly in $L^2(\Omega)$. Then this subsequence is uniformly bounded in $H^1(\Omega_\varepsilon)$ with the same ε-independent constant, hence, by [5, Lemma 2.3], there exists a sub-subsequence, with subscript still denoted by ε, and a function $\widetilde{\varphi_0} \in L^2(\Omega)$ such that $E_0\varphi_\varepsilon \rightharpoonup \theta_0\widetilde{\varphi_0}$ weakly in $L^2(\Omega)$. By the uniqueness of weak limits, we have $\varphi_0 = \widetilde{\varphi_0}$. Moreover, by the same lemma, $(E_0\varphi_\varepsilon)_{\varepsilon>0}$ is "compact" in the following sense: for any sequence $\psi_\varepsilon \in L^2(\Omega_\varepsilon)$ such that $E_0\psi_\varepsilon \rightharpoonup \theta_0\psi_0$ weakly in $L^2(\Omega)$ and for any function $\phi \in \mathcal{D}(\Omega)$, we have:

$$\int_{\Omega_\varepsilon} \phi\varphi_\varepsilon\psi_\varepsilon \, d\mathbf{x} \to \int_\Omega \theta_0\phi\varphi_0\psi_0 \, d\mathbf{x}.$$

By [5, Remark 2.4], we also have that:

$$\|\varphi_\varepsilon - \varphi_0\|_{L^2(\Omega_\varepsilon \cap \omega)} \to 0$$

as $\varepsilon \to 0$ for any open set ω satisfying $\overline{\omega} \subset \Omega$.

Since cosine is a Lipschitz function with constant 1, we have:

$$0 \le \|\cos\varphi_\varepsilon - \cos\varphi_0\|^2_{L^2(\Omega_\varepsilon \cap \omega)} = \int_{\Omega_\varepsilon \cap \omega} |\cos\varphi_\varepsilon - \cos\varphi_0|^2 \, d\mathbf{x} \le \int_{\Omega_\varepsilon \cap \omega} |\varphi_\varepsilon - \varphi_0|^2 \, d\mathbf{x} \to 0,$$

hence

$$\|\cos\varphi_\varepsilon - \cos\varphi_0\|_{L^2(\Omega_\varepsilon \cap \omega)} \to 0 \text{ as } \varepsilon \to 0$$

and, in a similar fashion:

$$\|\sin\varphi_\varepsilon - \sin\varphi_0\|_{L^2(\Omega_\varepsilon\cap\omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Since $\mathbf{u}_\varepsilon = \big(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon\big)$ and $\mathbf{u}_0 = \big(\cos\varphi_0, \sin\varphi_0\big)$, we obtain:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L^2(\Omega_\varepsilon\cap\omega;\mathbb{S}^1)} \to 0 \text{ as } \varepsilon \to 0,$$

for any open set ω such that $\overline{\omega} \subset \Omega$. □

4.5 APPENDIX

4.5.1 PROOFS FOR THE GENERAL ASSUMPTIONS SECTION

In this subsection, we use the following notations. Let $r, r_1, r_2 > 0$ be three real positive numbers such that $r_1 < r_2$. Then:

- B_r represents the ball of radius r in $\mathbb{R}^2$, that is, the set $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < r\}$;
- ∂B_r represents the circle of radius r in $\mathbb{R}^2$, that is, the set $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = r\}$;
- A_{r_1,r_2} represents the two dimensional annulus with radii r and R given by $A_{r_1,r_2} = B_{r_2} \setminus \overline{B_{r_1}}$.

Proof of Proposition 4.2.3. Let us remark first that since $T_\varepsilon^{i,j}$ is a compact, smooth and simply connected set, for any $i \in \{1, 2, \ldots, N_T\}$ and for any $j \in \{1, 2, \ldots, N_\varepsilon\}$, then by the smooth Riemann mapping theorem, there exists a diffeomorphism that transforms $T_\varepsilon^{i,j}$ into $\overline{B_1}$. Therefore, we can assume w.l.o.g. that $\partial T_\varepsilon^{i,j}$ is $\mathbb{S}^1$.

Let us prove first that $\mathscr{C}_\varepsilon \subset \widetilde{\mathscr{C}}_\varepsilon$. Let $\mathbf{u} \in \mathscr{C}_\varepsilon$. Since $\mathbf{u} \in \mathbf{V}_\varepsilon$, then $\mathbf{u}\big|_{\partial T_\varepsilon^{i,j}} \in H^{1/2}(\partial T_\varepsilon^{i,j}; \mathbb{S}^1)$. Applying [21, Proposition 12.2], we have that:

$$\deg(\mathbf{u}, \partial T_\varepsilon^{i,j}) = 0 \;\Leftrightarrow\; \exists\, \mathbf{E}_\varepsilon^{i,j}\mathbf{u} \in H^1(T_\varepsilon^{i,j}; \mathbb{S}^1) \text{ such that } \mathbf{E}_\varepsilon^{i,j} u = u \text{ on } \partial T_\varepsilon^{i,j}.$$

Therefore, we can extend the function $\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$ to a function $\mathbf{E}_\varepsilon\mathbf{u} \in H^1(\Omega; \mathbb{S}^1)$, where

$$\mathbf{E}_\varepsilon\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in \Omega_\varepsilon \\ \mathbf{E}_\varepsilon^{i,j}\mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in T_\varepsilon^{i,j},\ \forall i, j. \end{cases}$$

Since Ω is a bounded, smooth and simply connected domain from $\mathbb{R}^2$, using [21, Corollary 5.1], there exists a function $\varphi \in H^1(\Omega)$ such that:

$$\mathbf{E}_\varepsilon\mathbf{u}(\mathbf{x}) = \big(\cos(\varphi(\mathbf{x})), \sin(\varphi(\mathbf{x}))\big),\ \forall \mathbf{x} \in \Omega.$$

Let us denote by $\mathrm{Tr}_{\partial\Omega} : H^1(\Omega;\mathbb{S}^1) \to H^{1/2}(\partial\Omega;\mathbb{S}^1)$ the trace operator. Since $\mathbf{u} \in \mathbf{V}_\varepsilon$, then $\mathrm{Tr}_{\partial\Omega}\mathbf{u} = (1,0)$, hence $\mathrm{Tr}_{\partial\Omega}\mathbf{E}_\varepsilon\mathbf{u} = (1,0)$. But this implies that:

$$(1,0) = \mathrm{Tr}_{\partial\Omega}\big(\cos\varphi, \sin\varphi\big).$$

This implies that $\varphi\big|_{\partial\Omega} \in H^{1/2}(\partial\Omega;2\pi\mathbb{Z})$. We recall here that since Ω is a bounded, smooth and simply connected open set in $\mathbb{R}^2$, then, by the smooth Riemann mapping theorem, $\partial\Omega$ can be identified with $\mathbb{S}^1$ by a diffeomorphism. In this case, we apply [21, Corollary 6.2] and we obtain that the only functions from $H^{1/2}(\partial\Omega;2\pi\mathbb{Z})$ are constants. Hence, there exists a constant $m \in 2\pi\mathbb{Z}$ such that $\varphi\big|_{\partial\Omega} \equiv m$ on $\partial\Omega$. Therefore, there exists $\tilde{\varphi} := \varphi - m$ which is in V_ε (we have $\tilde{\varphi}\big|_{\partial\Omega} = \varphi_{\partial\Omega} - m = 0$) such that $\mathbf{u} = (\cos\tilde{\varphi}, \sin\tilde{\varphi})$ in Ω_ε. Therefore, $\mathbf{u} \in \widetilde{\mathscr{C}}_\varepsilon$.

Let us prove now that $\widetilde{\mathscr{C}}_\varepsilon \subset \mathscr{C}_\varepsilon$. Let $\mathbf{u} \in \widetilde{\mathscr{C}}_\varepsilon$ and $\varphi \in V_\varepsilon$ such that $\mathbf{u} = e^{i\varphi}$ a.e. in Ω_ε. Since φ is a real-valued function, we can extend φ to a function $E_\varepsilon\varphi \in H^1_0(\Omega)$ by using, for example, the harmonic extension operator in each of the holes. Therefore, there exists an extension $\mathbf{E}_\varepsilon : H^1(\Omega_\varepsilon;\mathbb{S}^1) \to H^1(\Omega;\mathbb{S}^1)$ such that $\mathbf{E}_\varepsilon\mathbf{u} \equiv \mathbf{u}$ in Ω_ε. This implies that we can extend $\mathbf{u}\big|_{\partial T_\varepsilon^{i,j}} \in H^{1/2}(\partial T_\varepsilon^{i,j};\mathbb{S}^1)$ in each of the holes $T_\varepsilon^{i,j}$. Using once again [21, Proposition 12.2], we obtain that $\deg(\mathbf{u}, \partial T_\varepsilon^{i,j}) = 0$, for any i, j, hence $\mathbf{u} \in \mathscr{C}_\varepsilon$. □

We present now the following auxiliary result:

Lemma 4.5.1. Let $0 < r_1 < r_2$ and $\mathbf{u} \in H^1(A_{r_1,r_2};\mathbb{S}^1)$. Then the following inequality holds:

$$2\pi \cdot |\deg(\mathbf{u}, \partial B_{r_1})|^2 \cdot \ln\left(\frac{r_2}{r_1}\right) \leq \int_{A_{r_1,r_2}} |\nabla\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}.$$

Proof. Let $\mathbf{v} : (r_1, r_2) \times [0, 2\pi] \to \mathbb{S}^1$ such that

$$\mathbf{v}(\rho,\theta) = \mathbf{u}(\rho e^{i\theta}).$$

Let us prove first that there exists a lifting for $\mathbf{v}$. Since $\mathbf{u} \in H^1(A_{r_1,r_2};\mathbb{S}^1)$, then $\mathbf{v} \in H^1((r_1,r_2) \times (0,2\pi);\mathbb{S}^1)$. We know that $(r_1,r_2) \times (0,2\pi)$ is a bounded, open and simply connected set in $\mathbb{R}^2$, but it is not smooth, hence we can not apply, for example, [21, Corollary 5.1]. However, by [52, Remark 2], there exists $\Lambda_2 : \mathbb{R}^2 \to \mathbb{R}^2$ a bi-Lipschitz transformation in that maps the unit ball into $(0,1)^2$. We can transform $(r_1,r_2) \times (0,2\pi)$ in a smooth way into $(0,1)^2$ and we denote this diffeomorphism by $\Phi_\square$. Then $\mathbf{v} \circ \Phi_\square^{-1} : (0,1)^2 \to \mathbb{S}^1$ and, hence, $\widetilde{\mathbf{v}} := v \circ \Phi_\square^{-1} \circ \Lambda_2 : B_1 \to \mathbb{S}^1$. Since $\Phi_\square$ is a diffeomorphism and Λ_2 is a bi-Lipschitz transformation, then we have that $\widetilde{\mathbf{v}} \in H^1(B_1;\mathbb{S}^1)$. Now B_1 is a bounded, smooth and simply connected domain in $\mathbb{R}^2$ so, by [21, Corollary 5.1], there exists a lifting $\widetilde{\varphi} \in H^1(B_1;\mathbb{R})$ such that $\widetilde{\mathbf{v}} = e^{i\widetilde{\varphi}}$. We define now $\varphi : (r_1,r_2) \times (0,2\pi) \to \mathbb{R}$ as $\varphi = \widetilde{\varphi} \circ \Lambda_2^{-1} \circ \Phi_\square$ and we see that $\varphi \in H^1((r_1,r_2) \times (0,2\pi))$ such that $\mathbf{v} = e^{i\varphi}$. Hence, we can write:

$$\mathbf{v}(\rho,\theta) = (\cos\varphi(\rho,\theta), \sin\varphi(\rho,\theta)).$$

By the same arguments from Remark 4.2.3, we actually have:

$$\deg\left(\mathbf{u}, \partial B_{r_1}\right) = \frac{\varphi(\rho, 2\pi) - \varphi(\rho, 0)}{2\pi},$$

where the right hand side is a constant from $\mathbb{Z}$ (by using, for example, [21, Proposition 12.14]).

Using now the chain rule and the polar change of variables, we have:

$$\begin{aligned}
\int_{A_{r_1,r_2}} |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} &= \int_{r_1}^{r_2} \int_0^{2\pi} \left(\left| \frac{\partial \mathbf{v}}{\partial \rho} \right|^2 + \frac{1}{\rho^2} \left| \frac{\partial \mathbf{v}}{\partial \theta} \right|^2 \right) \rho \, \mathrm{d}\theta \, \mathrm{d}\rho \\
&\geq \int_{r_1}^{r_2} \int_0^{2\pi} \rho^{-1} \left| \frac{\partial \mathbf{v}}{\partial \theta} \right|^2 \, \mathrm{d}\theta \, \mathrm{d}\rho = \\
&= \int_{r_1}^{r_2} \int_0^{2\pi} \rho^{-1} \left| \frac{\partial \varphi}{\partial \theta} \right|^2 \, \mathrm{d}\theta \, \mathrm{d}\rho,
\end{aligned}$$

where we have used that $\mathbf{v}(\rho, \theta) = (\cos \varphi(\rho, \theta), \sin \varphi(\rho, \theta))$. By using Hölder inequality, we get:

$$\begin{aligned}
\int_{A_{r_1,r_2}} |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} &\geq \int_{r_1}^{r_2} \rho^{-1} \cdot \frac{1}{2\pi} \left(\int_0^{2\pi} \frac{\partial \varphi}{\partial \theta} \, \mathrm{d}\theta \right)^2 \mathrm{d}\rho = \\
&= \frac{1}{2\pi} \int_{r_1}^{r_2} \rho^{-1} \cdot |\varphi(\rho, 2\pi) - \varphi(\rho, 0)|^2 \, \mathrm{d}\rho = \\
&= 2\pi \cdot |\deg(\mathbf{u}, \partial B_{r_1})|^2 \cdot \int_{r_1}^{r_2} \rho^{-1} \, \mathrm{d}\rho = \\
&= 2\pi \cdot |\deg(\mathbf{u}, \partial B_{r_1})|^2 \cdot \ln\left(\frac{r_2}{r_1} \right).
\end{aligned}$$

□

In the following paragraphs, we want to construct a neighbourhood $P_\varepsilon^{i,j}$ of each hole $T_\varepsilon^{i,j}$ contained in Ω_ε and a diffeomorphism $\Psi^{i,j} : P_\varepsilon^{i,j} \to A_{\varepsilon,2\varepsilon}$ with the determinant of its Jacobian bounded by an ε-independent constant.

For this, let us recall first that T^i represents the i-th component of T and, by assumption Assumption 4.2.2, T^i is a compact, bounded, smooth and simply connected set in $\mathbb{R}^2$. By the tubular neighbourhood theorem (see, for example, [56, Theorem 6.17]), there exists a neighbourhood U^i of ∂T^i such that it is diffeomorphic with $(-1,1) \times \mathbb{S}^1$. If we denote by $\Phi^i : (-1,1) \times \mathbb{S}^1 \to U^i$ the diffeomorphism given by the tubular neighbourhood theorem, we also have $\Phi^i(0, \mathbb{S}^1) = \partial T^i$. At the same time, we can choose $r_i \geq 1$, for any $i \in \{1, 2, \ldots, N_T\}$, such that $\Phi^{i_1}\left((0, 1/r_{i_1}) \times \mathbb{S}^1\right) \cap \Phi^{i_2}\left((0, 1/r_{i_2}) \times \mathbb{S}^1\right) = \emptyset$, for any $i_1, i_2 \in \{1, 2, \ldots, N_T\}$ with $i_1 = i_2$. More simply said, we can choose restrictions of Φ^i such that the neighbourhoods of ∂T^i that lie in Y (the unit periodic cell, that is $(0,1)^2$) are mutually disjoint. Let $P^i = \Phi^i\left((0, 1/r_i) \times \mathbb{S}^1\right)$.

Let now $\Phi : A_{1,2} \to (0,1) \times \mathbb{S}^1$ be defined as $\Phi(\mathbf{x}) = \left(|\mathbf{x}| - 1, \mathbf{x}\right)$. We define $\Psi^i : A_{1,2} \to P^i$ as $\Psi^i := \Phi^i \circ \Phi$ and it is easy to see that Ψ^i is a diffeomorphism that transforms $A_{1,2}$ into P^i.

We consider now the following notation. For any $j \in \{1, 2, \ldots, N_\varepsilon\}$ (j is an index for the number of periodic cells constructed in Ω), we denote by $\tau_j : \mathbb{R}^2 \to \mathbb{R}^2$ the following smooth map: $\tau_j(\mathbf{x}) = \mathbf{x} + \mathbf{x}_\varepsilon^j$, where $\mathbf{x}_\varepsilon^j$ is the center of the j-th periodic cell. Moreover, let $\Phi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ the following smooth contraction: $\Phi_\varepsilon(\mathbf{x}) = \varepsilon\mathbf{x}$.

By the previously introduced notations, we construct $P_\varepsilon^{i,j} = \Phi_\varepsilon \circ \tau_j(P^i)$, for any $i \in \{1, 2, \ldots, N_T\}$ and any $j \in \{1, 2, \ldots, N_\varepsilon\}$. It is easy to see that $P_\varepsilon^{i,j}$ is a neighbourhood of $\partial T_\varepsilon^{i,j}$ that is contained in Ω_ε. We now prove the following proposition.

Proposition 4.5.1. Let $i \in \{1, 2, \ldots, N_T\}$ and $j \in \{1, 2, \ldots, N_\varepsilon\}$. We define $\Psi^{i,j} : A_{\varepsilon,2\varepsilon} \to P_\varepsilon^{i,j}$ as

$$\Psi^{i,j}(\mathbf{x}) = \left(\Phi_\varepsilon \circ \tau_j \circ \Psi^i \circ \Phi_\varepsilon^{-1}\right)(\mathbf{x}), \ \forall \mathbf{x} \in A_{\varepsilon,2\varepsilon}.$$

Then $\Psi^{i,j}$ is a diffeomorphism such that $\det J_{i,j}$ can be bounded from above by an ε-independent constant, where $J_{i,j}$ is the Jacobian of $\Psi^{i,j}$. Moreover, we have $\Psi^{i,j}(\partial B_\varepsilon) = \partial T_\varepsilon^{i,j}$.

Proof. Since Φ_ε is a smooth contraction over $\mathbb{R}^2$, τ_j is a translation with a constant over $\mathbb{R}^2$ and Ψ^i is a diffeomorphism, then $\Psi^{i,j}$ is also a diffeomorphism.

Let $\mathbf{x} \in \partial B_\varepsilon$. Then $\Phi_\varepsilon^{-1}(\mathbf{x}) \in \partial B_1$, which implies that $\Psi^i \circ \Phi_\varepsilon^{-1}(\mathbf{x}) \in \partial T^i$. Applying now τ_j and Φ_ε, we obtain that $\Psi^{i,j}(\mathbf{x}) \in \partial T_\varepsilon^{i,j}$.

At the same time, we have $\det J_{\Phi_\varepsilon} = \varepsilon^2$, $\det J_{\tau_j} = 1$ and $\det J_{\Phi_\varepsilon^{-1}} = \varepsilon^{-2}$. Since Ψ^i was defined ε-independent, the proof is complete. Moreover, the previous computations show us that $textdet J_{i,j}$ does not depend on j. □

Proof of Proposition 4.2.4. Let $\delta > 0$ and $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ be a critical point of $\mathbf{F}_\varepsilon$, given by Assumption 4.2.5, such that $\mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon) < \delta$.

Since $\mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon; \mathbb{S}^1)$, then $\mathbf{u}_\varepsilon\big|_{\partial T_\varepsilon^{i,j}} \in H^{1/2}(\partial T_\varepsilon^{i,j}; \mathbb{S}^1)$ and $\mathbf{u}_\varepsilon\big|_{P_\varepsilon^{i,j}} \in H^1(P_\varepsilon^{i,j}; \mathbb{S}^1)$.

Let us define $\mathbf{u}_\varepsilon^{i,j} := \mathbf{u}\big|_{P_\varepsilon^{i,j}} \circ \Psi^{i,j}$, for any $i \in \{1, 2, \ldots, N_T\}$ and any $j \in \{1, 2, \ldots, N_\varepsilon\}$. Using Proposition 4.5.1, we obtain that $\mathbf{u}_\varepsilon^{i,j} \in H^1(A_{\varepsilon,2\varepsilon}; \mathbb{S}^1)$ and that:

$$\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j}) = \deg(\mathbf{u}_\varepsilon^{i,j}, \partial B_\varepsilon). \tag{4.5.1}$$

We now apply Lemma 4.5.1 to $\mathbf{u}_\varepsilon^{i,j}$ on $A_{\varepsilon,2\varepsilon}$ and we obtain:

$$2\pi \cdot \left|\deg(\mathbf{u}_\varepsilon^{i,j}, \partial B_\varepsilon)\right|^2 \cdot \ln\frac{2\varepsilon}{\varepsilon} \le \int_{A_{\varepsilon,2\varepsilon}} \left|\nabla \mathbf{u}_\varepsilon^{i,j}\right|^2 \mathrm{d}\mathbf{x}.$$

Using now the change of variables $\Psi^{i,j}$ and the fact that the determinant of its Jacobian can be bounded from above by an ε-independent constant, let us say $\ell_i > 0$, we obtain that:

$$2\pi \cdot \left|\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j})\right|^2 \cdot \ln 2 \le \ell_i \cdot \int_{P_\varepsilon^{i,j}} \left|\nabla \mathbf{u}_\varepsilon\right|^2 \mathrm{d}\mathbf{x},$$

where we have also used (4.5.1).

Since the family any two P^{i_1} and P^{i_2} are mutually disjoint whenever $i_1 \neq i_2$, then, by our construction, also any two sets of the type $P_\varepsilon^{i,j}$ are mutually disjoint. Moreover, since $P_\varepsilon^{i,j}$ is contained in Ω_ε, we obtain that $\bigcup_{i=1}^{N_T}\bigcup_{j=1}^{N_\varepsilon} P_\varepsilon^{i,j} \subset \Omega_\varepsilon$. By denoting $\ell = \max\{\ell_1, \ell_2, \dots, \ell_{N_T}\}$, we obtain:

$$2\pi \cdot \ln 2 \cdot \sum_{i=1}^{N_T}\sum_{j=1}^{N_\varepsilon} \left|\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j})\right|^2 \le \ell \cdot \int_{\Omega_\varepsilon} \left|\nabla \mathbf{u}_\varepsilon\right|^2 \, \mathrm{d}\mathbf{x}.$$

Using Proposition 4.2.2, one has:

$$\alpha \|\nabla \mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon} \le \mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon) \le \delta$$

and therefore

$$\sum_{i=1}^{N_T}\sum_{j=1}^{N_\varepsilon} \left|\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j})\right|^2 \le \frac{\ell}{2\pi \cdot \ln 2 \cdot \alpha} \cdot \delta.$$

We recall now that $\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j}) \in \mathbb{Z}$, for any i, j. Hence, for $\delta > 0$ small enough, we have:

$$\deg(\mathbf{u}_\varepsilon, \partial T_\varepsilon^{i,j}) = 0, \ \forall i \in \{1, 2, \dots, N_T\} \text{ and } \forall j \in \{1, 2, \dots, N_\varepsilon\},$$

which implies that $\mathbf{u}_\varepsilon \in \widetilde{\mathscr{C}}_\varepsilon$, by Proposition 4.2.3. □

4.5.2 PROOFS FOR THE PROPERTIES OF A AND $\mathscr{B}$

This subsection contains the proofs of Proposition 4.3.2 and Proposition 4.3.3.

Proof of Proposition 4.3.2. (1) Since a, b and c are 2π-periodic functions of class $C^2(\mathbb{R})$, then A has the same properties.

(2) For this, we use Proposition 4.3.1 and we proceed in a similar fashion as in Proposition 4.2.1. Let $\xi \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Then:

$$\begin{aligned}
&A(t)\xi \cdot \xi = a(t)\xi_1^2 + b(t)\xi_1\xi_2 + c(t)\xi_2^2 \\
&= \alpha|\xi|^2 + \big(a(t) - \alpha\big)\xi_1^2 + b(t)\xi_1\xi_2 + \big(c(t) - \alpha\big)\xi_2^2 \\
&= \alpha|\xi|^2 + \left(\sqrt{a(t) - \alpha} \cdot \xi_1 + \frac{b(t)}{2\sqrt{a(t) - \alpha}} \cdot \xi_2\right)^2 + \frac{4\big(a(t) - \alpha\big)\big(c(t) - \alpha\big) - b^2(t)}{4\big(a(t) - \alpha\big)}\xi_2^2,
\end{aligned}$$

hence

$$A(t)\xi \cdot \xi \ \ge \ \alpha|\xi|^2, \ \forall \xi \in \mathbb{R}^2, \ \forall t \in \mathbb{R}.$$

(3) Let $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$. Then:

$$\begin{aligned}
|A(t)\xi|^2 &= \big(a(t)\xi_1 + b(t)\xi_2/2\big)^2 + \big(b(t)\xi_1/2 + c(t)\xi_2\big)^2 \\
&= \big(a^2(t) + b^2(t)/4\big)\xi_1^2 + b(t)\big(a(t) + c(t)\big)\xi_1\xi_2 + \big(c^2(t) + b^2(t)/4\big)\xi_2^2.
\end{aligned}$$

Since $2\xi_1\xi_2 \leq \xi_1^2 + \xi_2^2$, for any $\xi \in \mathbb{R}^2$, we can write:

$$|A(t)\xi|^2 \leq \left(a^2(t) + b^2(t)/4\right)\xi_1^2 + |b(t)|\left(a(t)+c(t)\right)\left(\xi_1^2+\xi_2^2\right)/2 + \left(c^2(t)+b^2(t)/4\right)\xi_2^2,$$

which implies

$$|A(t)\xi|^2 \leq \mathcal{C}_1^2(A)|\xi|^2,$$

where $\mathcal{C}_1(A)$ can be chosen such that:

$$\mathcal{C}_1^2(A) = \|a\|^2_{L^\infty(\mathbb{R})} + \|b\|^2_{L^\infty(\mathbb{R})}/2 + \|c\|^2_{L^\infty(\mathbb{R})} + \|b\|_{L^\infty(\mathbb{R})}\left(\|a\|_{L^\infty(\mathbb{R})} + \|c\|_{L^\infty(\mathbb{R})}\right).$$

(4) Let $t, s \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$. Then:

$$\begin{aligned}
&\Big|\big(A(t)-A(s)\big)\xi\Big|^2 = \\
&= \Big(\big(a(t)-a(s)\big)\xi_1 + \big(b(t)-b(s)\big)\xi_2/2\Big)^2 + \Big(\big(b(t)-b(s)\big)\xi_1/2 + \big(c(t)-c(s)\big)\xi_2\Big)^2 \\
&= \Big(\big(a(t)-a(s)\big)^2 + \big(b(t)-b(s)\big)^2/4\Big)\xi_1^2 + \Big(\big(c(t)-c(s)\big)^2 + \big(b(t)-b(s)\big)^2/4\Big)\xi_2^2 \\
&\quad + \big(b(t)-b(s)\big)\Big(\big(a(t)-a(s)\big) + \big(c(t)-c(s)\big)\Big)\xi_1\xi_2 + \\
&\leq \left(\|a'\|^2_{L^\infty(\mathbb{R})} + \|b'\|^2_{L^\infty(\mathbb{R})}/4\right)\cdot|t-s|^2\cdot\xi_1^2 + + \left(\|c'\|^2_{L^\infty(\mathbb{R})} + \|b'\|^2_{L^\infty(\mathbb{R})}/4\right)\cdot|t-s|^2\cdot\xi_2^2 + \\
&\quad + \|b'\|_{L^\infty(\mathbb{R})}\left(\|a'\|_{L^\infty(\mathbb{R})} + \|c'\|_{L^\infty(\mathbb{R})}\right)\cdot|t-s|^2\cdot\left(\xi_1^2+\xi_2^2\right)/2 \Rightarrow \\
&\Rightarrow \Big|\big(A(t)-A(s)\big)\xi\Big|^2 \leq \mathcal{C}_2^2(A')\cdot|t-s|^2\cdot|\xi|^2,
\end{aligned}$$

where $\mathcal{C}_2(A')$ can be chosen such that:

$$\mathcal{C}_2^2(A') = \|a'\|^2_{L^\infty(\mathbb{R})} + \|b'\|^2_{L^\infty(\mathbb{R})}/2 + \|c'\|^2_{L^\infty(\mathbb{R})} + \|b'\|_{L^\infty(\mathbb{R})}\left(\|a'\|_{L^\infty(\mathbb{R})} + \|c'\|_{L^\infty(\mathbb{R})}\right).$$

For (4.3.10), one could mimic the arguments from proving (4.3.9) in order to obtain the same constant $\mathcal{C}_2(A')$, since $\mathcal{C}_1(A)$ and $\mathcal{C}_2(A')$ both have the same form, but one is depending on a, b and c, while the other on a', b' and c'.

(5) By taking

$$\mathcal{C}_3^2(A'') = \|a''\|^2_{L^\infty(\mathbb{R})} + \|b''\|^2_{L^\infty(\mathbb{R})}/2 + \|c''\|^2_{L^\infty(\mathbb{R})} + \|b''\|_{L^\infty(\mathbb{R})}\left(\|a''\|_{L^\infty(\mathbb{R})} + \|c''\|_{L^\infty(\mathbb{R})}\right), \quad (4.5.2)$$

one can apply the same arguments from the proof of (4.3.11) in order to prove (4.3.12).

Since $a \in C^1(\mathbb{R})$, for any $t, s \in \mathbb{R}$, with $s \neq t$, there exists, by the mean value theorem, a point $p_{s,t}$ between s and t (either the interval (s,t) or the interval (t,s)) such that:

$$a(t) - a(s) = a'(p_{s,t})\cdot(t-s).$$

At the same time, $a' \in C^1(\mathbb{R})$, hence, by the mean value theorem, there exists $q_{s,t}$ between t and $p_{s,t}$ such that:

$$a'(p_{s,t}) - a'(t) = a''(q_{s,t}) \cdot (t - p_{s,t}),$$

which translates into

$$\frac{a(t)-a(s)}{t-s} - a'(t) = a''(q_{s,t}) \cdot (t - p_{s,t}).$$

Since $p_{s,t}$ is between s and t, then $|t - p_{s,t}| \leq |t-s|$. This implies that:

$$\left| \frac{a(t)-a(s)}{t-s} - a'(t) \right| \leq |a''(q_{s,t})| \cdot |t-s| \leq \|a''\|_{L^\infty(\mathbb{R})} \cdot |t-s|. \tag{4.5.3}$$

By deriving the same inequalities as in (4.5.3) for b and c, we then obtain:

$$\begin{aligned}
&\left| \left(\frac{A(t)-A(s)}{t-s} - A'(t) \right) \xi \right|^2 = \\
&= \left(\left(\frac{a(t)-a(s)}{t-s} + a'(t) \right) \xi_1 + \frac{1}{2} \left(\frac{b(t)-b(s)}{t-s} - b'(t) \right) \xi_2 \right)^2 + \\
&\quad + \left(\frac{1}{2} \left(\frac{b(t)-b(s)}{t-s} - b'(t) \right) \xi_1 + \left(\frac{c(t)-c(s)}{t-s} - c'(t) \right) \xi_2 \right)^2 \\
&= \left(\left(\frac{a(t)-a(s)}{t-s} - a'(t) \right)^2 + \frac{1}{4} \left(\frac{b(t)-b(s)}{t-s} - b'(t) \right)^2 \right) \xi_1^2 + \\
&\quad + \left(\frac{b(t)-b(s)}{t-s} - b'(t) \right) \left(\frac{a(t)-a(s)}{t-s} - a'(t) + \frac{c(t)-c(s)}{t-s} - c'(t) \right) \xi_1 \xi_2 + \\
&\quad + \left(\frac{1}{4} \left(\frac{b(t)-b(s)}{t-s} - b'(t) \right)^2 + \left(\frac{c(t)-c(s)}{t-s} - c'(t) \right)^2 \right) \xi_2^2 \\
&\leq \left(\|a''\|^2_{L^\infty(\mathbb{R})} + \frac{1}{4} \|b''\|^2_{L^\infty(\mathbb{R})} \right) \xi_1^2 \cdot |t-s|^2 + \\
&\quad + \|b''\|_{L^\infty(\mathbb{R})} \cdot \left(\|a''\|_{L^\infty(\mathbb{R})} + \|c''\|_{L^\infty(\mathbb{R})} \right) \cdot \frac{\xi_1^2 + \xi_2^2}{2} \cdot |t-s|^2 + \\
&\quad + \left(\frac{1}{4} \|b''\|^2_{L^\infty(\mathbb{R})} + \|c''\|^2_{L^\infty(\mathbb{R})} \right) \cdot \xi_2^2 \cdot |t-s|^2 \\
&\leq C_3(A'') \cdot |\xi|^2 \cdot |t-s|^2,
\end{aligned}$$

where $C_3^2(A'')$ is introduced in (4.5.2), which implies (4.3.13). □

Proof of Proposition 4.3.3. (1) Let $t \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^2$. Then:

$$\begin{aligned}
2 \cdot |\mathscr{B}(t,\xi) - \mathscr{B}(t,\eta)| &= |-A'(t)\xi \cdot \xi + A'(t)\eta \cdot \eta| \\
&= |-A'(t)\xi \cdot \xi + A'(t)\xi \cdot \eta - A'(t)\xi \cdot \eta + A'(t)\eta \cdot \eta|,
\end{aligned}$$

so

$$\begin{aligned}
2\cdot|\mathscr{B}(t,\xi)-\mathscr{B}(t,\eta)| &\le |A'(t)\xi\cdot(\xi-\eta)|+|A'(t)(\xi-\eta)\cdot\eta| \\
&\le |A'(t)\xi|\cdot|\xi-\eta|+|A'(t)(\xi-\eta)|\cdot|\eta| \\
&\le \mathcal{C}_2(A')\cdot|\xi|\cdot|\xi-\eta|+\mathcal{C}_2(A')\cdot|\xi-\eta|\cdot|\eta| \\
&\le \mathcal{C}_2(A')\cdot\big(|\xi|+|\eta|\big)\cdot|\xi-\eta|\le \mathcal{C}_2(A')\cdot\big(1+|\xi|+|\eta|\big)\cdot|\xi-\eta|.
\end{aligned}$$

We choose $d_1(\cdot)=\dfrac{1}{2}\mathcal{C}_2(A')$ and conclude the proof for the first part.

(2) Let $s,t\in\mathbb{R}$ and $\xi\in\mathbb{R}^2$. We have:

$$\begin{aligned}
&2\cdot|\mathscr{B}(t,\xi)-\mathscr{B}(s,\xi)| = \big|\big(A'(s)-A'(t)\big)\xi\cdot\xi-\mu\big(\sin(2t-2\overline{\varphi})-\sin(2s-2\overline{\varphi})\big)\big| \\
&\le \big|\big(A'(s)-A'(t)\big)\xi\big|\cdot|\xi|+\mu\big|\sin(2t-2\overline{\varphi})-\sin(2s-2\overline{\varphi})\big| \\
&\le \mathcal{C}_3(A'')\cdot|t-s|\cdot|\xi|^2+2\mu\cdot|t-s| \\
&\le \big(\mathcal{C}_3(A'')+2\mu\big)\cdot|t-s|\cdot\big(1+|\xi|^2\big),
\end{aligned}$$

where we have used (4.3.12).

We choose $d_2(|t-s|)=\dfrac{1}{2}\big(\mathcal{C}_3(A'')+2\mu\big)\cdot|t-s|$ and we conclude the proof. □

4.5.3 PROPERTIES OF THE CELL PROBLEM

Let us fix $\xi\in\mathbb{R}^2$. In the following paragraphs, we are interested in studying the dependency between the parameters $A(t)$, given by Definition 4.3.2, and the solutions $\chi_\xi(\cdot,t)$ given by the cell problem:

$$\begin{cases}
a(t)\dfrac{\partial^2\chi_\xi}{\partial x^2}(\mathbf{x},t)+b(t)\dfrac{\partial^2\chi_\xi}{\partial x\partial y}(\mathbf{x},t)+c(t)\dfrac{\partial^2\chi_\xi}{\partial y^2}(\mathbf{x},t)=0 & \text{in } Y\setminus T\\
A(t)\big(\xi-\nabla\chi_\xi(\mathbf{x},t)\big)\cdot\nu=0 & \text{on } \partial T\\
\chi_\xi \text{ is } Y\text{-periodic} & \\
0=\displaystyle\int_{Y\setminus T}\chi_\xi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}:=\big(\chi_\xi(\cdot,t)\big)_{Y\setminus T}. &
\end{cases}\tag{4.5.4}$$

Remark 4.5.1. On ∂T we impose a nonhomogeneous Neumann boundary condition that should be understood in the same way as in [35, Section 4.5]. To be more specific, since $t\in\mathbb{R}$ and $\xi\in\mathbb{R}^2$ are fixed, we have $\operatorname{div}(A(t)\nabla\chi_\xi)=0$ in $Y\setminus T$. At the same time, we look for solutions χ_ξ in a subspace of $H^1(Y\setminus T)$, to be described later on, so we have $A(t)\nabla\chi_\xi\in\big(L^2(Y\setminus T)\big)^2$. This implies that

$$A(t)\nabla\chi_\xi\in H(Y\setminus T;\,\operatorname{div}):=\{\mathbf{v}\in\big(L^2(Y\setminus T)\big)^2\;:\;\operatorname{div}\mathbf{v}\in L^2(Y\setminus T)\}.$$

By [35, Proposition 3.47], we obtain that $A(t)\nabla\chi_\xi \cdot \nu \in H^{-1/2}(\partial T \cup \partial Y)$. Then, by saying that $A(t)\nabla\chi_\xi \cdot \nu = A(t)\xi \cdot \nu$ on ∂T we mean:

$$< A(t)\nabla\chi_\xi \cdot \nu, \psi >_{H^{-1/2}(\partial T),H^{1/2}(\partial T)} = < A(t)\xi \cdot \nu, \psi >_{H^{-1/2}(\partial T),H^{1/2}(\partial T)}, \ \forall \psi \in H^{1/2}(\partial T),$$

that is,

$$< A(t)\nabla\chi_\xi \cdot \nu, \psi >_{H^{-1/2}(\partial T),H^{1/2}(\partial T)} = \int_{\partial T} (A(t)\xi \cdot \nu)\psi \, \mathrm{d}\sigma, \ \forall \psi \in H^{1/2}(\partial T),$$

since for any $t \in \mathbb{R}$ and any $\xi \in \mathbb{R}$, we have $A(t)\xi \cdot \nu \in L^2(\partial T)$.

Remark 4.5.2. On ∂Y we impose a periodicity condition, that should be understood in the following sense. First, let $C^\infty_\#(Y)$ the subset of $C^\infty(\mathbb{R}^2)$ of Y-periodic functions. We then consider $C^\infty_\#(Y \setminus T)$ the subset of $C^\infty_\#(Y)$ obtained by a restriction on $Y \setminus T$. We then define $H^1_{\text{per}}(Y \setminus T)$ the closure of $C^\infty_\#(Y \setminus T)$ for the H^1 norm.

Definition 4.5.1. We define:

$$H^1_\#(Y \setminus T) = \left\{ \chi \in H^1_{\text{per}}(Y \setminus T) \ : \ (\chi)_{Y\setminus T} := \int_{Y\setminus T} \chi(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \right\}.$$

We also recall the Poincaré-Wirtinger inequality for the bounded connected open set $Y \setminus T$ with Lipschitz boundary and $H^1_\#(Y \setminus T)$ functions:

Proposition 4.5.2. There exists a constant $C_P(Y) > 0$ such that:

$$\|\chi\|_{L^2(Y\setminus T)} \leq C_P(Y) \|\nabla\chi\|_{L^2(Y\setminus T)},$$

for all $\chi \in H^1_\#(Y \setminus T)$.

The following proposition is a consequence of [53, Theorem 1.5.1.10], since ∂Y and ∂T are Lipschitz:

Proposition 4.5.3. Let $\mathrm{Tr} : H^1(Y \setminus T) \to H^{1/2}(\partial Y \cup \partial T)$ be the Sobolev trace operator. Then there exists a constant $C_{Tr}(Y \setminus T)$ such that

$$\|\mathrm{Tr}(\chi)\|_{L^2(\partial(Y\setminus T))} \leq C_{Tr}(Y \setminus T) \|\chi\|_{H^1(Y\setminus T)}, \ \forall \chi \in H^1(Y \setminus T).$$

We now prove the following auxiliary result.

Lemma 4.5.2. Let $t \in \mathbb{R}$ be fixed.

i) For any $\chi, \psi \in H^1_\#(Y \setminus T)$ such that $\operatorname{div}(A(t)\nabla\chi) \in L^2(Y \setminus T)$[1], we have:

$$< A(t)\nabla\chi \cdot \nu, \psi >_{H^{-1/2}(\partial Y),H^{1/2}(\partial Y)} = 0. \tag{4.5.5}$$

[1] We assume this condition such that (4.5.5) has the sense from Remark 4.5.1

ii) For any $\chi, \overline{\chi}, \psi \in H^1_\#(Y \setminus T)$ such that

$$\operatorname{div}\big(A(t)\nabla\overline{\chi} + A'(t)\nabla\chi\big) \in L^2(Y \setminus T),$$

we have

$$< \big(A(t)\nabla\overline{\chi} + A'(t)\nabla\chi\big) \cdot \nu, \psi >_{H^{-1/2}(\partial Y), H^{1/2}(\partial Y)} = 0. \tag{4.5.6}$$

Proof. Let us fix $t \in \mathbb{R}$.

i) We prove first (4.5.5) for $C^\infty_\#(Y \setminus T)$ functions. Hence, we assume $\chi, \psi \in C^\infty_\#(Y \setminus T)$. In this case, we have $\big(A(t)\nabla\chi \cdot \nu\big) \in L^2(\partial Y)$, hence:

$$\begin{aligned}
< A(t)\nabla\chi \cdot \nu, \psi >_{H^{-1/2}(\partial Y), H^{1/2}(\partial Y)} &= \int_{\partial Y} \big(A(t)\nabla\chi(\mathbf{x}) \cdot \nu\big)\psi(\mathbf{x}) \, \mathrm{d}\sigma \\
&= \int_0^1 a(t)\left(\frac{\partial\chi}{\partial x}(1,y)\psi(1,y) - \frac{\partial\chi}{\partial x}(0,y)\psi(0,y)\right) \mathrm{d}y + \\
&+ \int_0^1 \frac{b(t)}{2}\left(\frac{\partial\chi}{\partial y}(1,y)\psi(1,y) - \frac{\partial\chi}{\partial y}(0,y)\psi(0,y)\right) \mathrm{d}y + \\
&+ \int_0^1 \frac{b(t)}{2}\left(\frac{\partial\chi}{\partial x}(x,1)\psi(x,1) - \frac{\partial\chi}{\partial x}(x,0)\psi(x,0)\right) \mathrm{d}x + \\
&+ \int_0^1 c(t)\left(\frac{\partial\chi}{\partial y}(x,1)\psi(x,1) - \frac{\partial\chi}{\partial y}(x,0)\psi(x,0)\right) \mathrm{d}x.
\end{aligned}$$

Since χ is Y-periodic and smooth, then $\dfrac{\partial\chi}{\partial x}$ and $\dfrac{\partial\chi}{\partial y}$ are smooth and 1-periodic. Since ψ is also assumed to be Y-periodic, (4.5.5) follows for this case.

Let us leave $\chi \in C^\infty_\#(Y \setminus T)$ and let us assume now that $\psi \in H^1_\#(Y \setminus T)$. Then, there exists a sequence $(\psi_n)_{n\geq 1} \subset C^\infty_\#(Y \setminus T)$ such that

$$\psi_n \to \psi \text{ strongly in } H^1_\#(Y \setminus T) \text{ as } n \to +\infty.$$

Since the trace operator defined in Proposition 4.5.3 is continuous, we have:

$$\lim_{n\to+\infty} \|\psi - \psi_n\|_{H^{1/2}(\partial Y)} = 0. \tag{4.5.7}$$

At the same time, we still have $\big(A(t)\nabla\chi \cdot \nu\big) \in L^2(\partial Y)$, hence:

$$\begin{aligned}
\Big| < A(t)\nabla\chi \cdot \nu, \psi >_{H^{-1/2}(\partial Y), H^{1/2}(\partial Y)} \Big| &= \left| \int_{\partial Y} \big(A(t)\nabla\chi(\mathbf{x}) \cdot \nu\big)\psi(\mathbf{x}) \, \mathrm{d}\sigma \right| \\
= \left| \int_{\partial Y} \big(A(t)\nabla\chi(\mathbf{x}) \cdot \nu\big)\big(\psi(\mathbf{x}) - \psi_n(\mathbf{x}) + \psi_n(\mathbf{x})\big) \, \mathrm{d}\sigma \right| \\
\leq \left| \int_{\partial Y} \big(A(t)\nabla\chi \cdot \nu\big)\psi_n \, \mathrm{d}\sigma \right| + \left| \int_{\partial Y} \big(A(t)\nabla\chi \cdot \nu\big)\big(\psi - \psi_n\big) \, \mathrm{d}\sigma \right|.
\end{aligned}$$

Using (4.5.5) for χ and ψ_n, we obtain:

$$\begin{aligned}\left| < A(t)\nabla\chi\cdot\nu, \psi >_{H^{-1/2}(\partial Y), H^{1/2}(\partial Y)} \right| &\leq \left| \int_{\partial Y} \left(A(t)\nabla\chi\cdot\nu\right)\left(\psi-\psi_n\right)\,\mathrm{d}\sigma \right| \\ &\leq \left(\int_{\partial Y} \left|A(t)\nabla\chi\cdot\nu\right|^2 \mathrm{d}\sigma \right)^2 \left(\int_{\partial Y} \left|\psi_n-\psi\right|^2 \mathrm{d}\sigma \right)^{1/2} \\ &\leq \left(\int_{\partial Y} \left|A(t)\nabla\chi\cdot\nu\right|^2 \mathrm{d}\sigma \right)^2 \cdot \left\|\psi_n-\psi\right\|_{H^{1/2}(\partial Y)}.\end{aligned}$$

Since t and χ are fixed, using (4.5.7), we obtain (4.5.5) also for the pair $\chi \in C^\infty_\#(Y\setminus T)$ and $\psi \in H^1_\#(Y\setminus T)$.

We assume now that $\chi \in H^1_\#(Y\setminus T)$ such that $\operatorname{div}(A(t)\nabla\chi) \in L^2(Y\setminus T)$ and $\psi \in H^1_\#(Y\setminus T)$. By the construction of $H^1_\#(Y\setminus T)$, there exists a sequence $(\chi_n) \subset C^\infty_\#(Y\setminus T)$ such that $\chi_n \to \chi$ strongly in $H^1_\#(Y\setminus T)$, which implies:

$$\nabla\chi_n \to \nabla\chi \text{ strongly in } L^2(Y\setminus T).$$

We recall that by our choice of χ, we have, in the same fashion as in Remark 4.5.1, $A(t)\nabla\chi \in H(Y\setminus T;\operatorname{div})$. At the same time, we also have $A(t)\nabla\chi_n \in H(Y\setminus T;\operatorname{div})$. Using [35, Proposition 3.47], the map

$$\mathbf{v} \in H(Y\setminus T;\operatorname{div}) \to \mathbf{v}\cdot\nu \in H^{-1/2}(\partial Y\cup\partial T)$$

is linear and continuous. Thus:

$$\lim_{n\to\infty} \left\|A(t)\left(\nabla\chi_n-\nabla\chi\right)\cdot\nu\right\|_{H^{-1/2}(\partial Y)} = 0. \tag{4.5.8}$$

We recall here that:

$$\|\mathbf{v}\cdot\nu\|_{H^{-1/2}(\partial Y)} = \sup_{H^{1/2}(\partial Y)\setminus\{0\}} \|\psi\|^{-1}_{H^{1/2}(\partial Y)} \left| \int_{\partial Y} (\mathbf{v}\cdot\nu)\psi\,\mathrm{d}\sigma \right|.$$

Since for any $\psi \in H^1_\#(Y\setminus T)$, we have $\psi\big|_{\partial Y} \in H^{1/2}(\partial Y)$, then, by the previously proved equalities:

$$\left\|A(t)\nabla\chi_n\cdot\nu\right\|_{H^{-1/2}(\partial Y)} = 0. \tag{4.5.9}$$

Combining now (4.5.8) and (4.5.9), we conclude the proof.

ii) We proceed in a similar fashion as for i). It is easy to see that if we assume that $\chi, \overline{\chi}$ and ψ are in $C^\infty_\#(Y\setminus T)$, then (4.5.6) holds, by a similar argument as in the proof of i) for smooth functions.

For the case $\chi, \overline{\chi} \in C^\infty_\#(Y\setminus T)$ and $\psi \in H^1_\#(Y\setminus T)$, the same argument as in the case of i) applies.

Let now $\chi, \overline{\chi}, \psi \in H^1_\#(Y \setminus T)$ and $\left(\chi_n\right)_{n\geq 1}$, $\left(\overline{\chi_n}\right)_{n\geq 1}$ such that:

$$\chi_n \to \chi \text{ and } \overline{\chi}_n \to \overline{\chi} \text{ strongly in } H^1_\#(Y \setminus T),$$

which implies that

$$A(t)\nabla\overline{\chi}_n + A'(t)\nabla\chi_n \to A(t)\nabla\overline{\chi} + A'(t)\nabla\chi \text{ strongly in } \left(L^2(Y \setminus T)\right)^2.$$

Once again, due to our choices of χ, χ_n, $\overline{\chi}$, $\overline{\chi}_n$, we have:

$$\left(A(t)\overline{\chi} + A'(t)\chi\right) \cdot \nu,\ \left(A(t)\overline{\chi}_n + A'(t)\chi_n\right) \cdot \nu \in H^{-1/2}(\partial Y).$$

Using [35, Proposition 3.47], we obtain:

$$\lim_{n\to\infty} \left\|\left(A(t)\overline{\chi}_n + A'(t)\chi_n\right) \cdot \nu\right\|_{H^{-1/2}(\partial Y)} = \left\|\left(A(t)\overline{\chi} + A'(t)\chi\right) \cdot \nu\right\|_{H^{-1/2}(\partial Y)},$$

where we observe that the sequence under the limit is constant 0, since (4.5.6) holds for χ_n, $\overline{\chi_n}$ and any $\psi \in H^1_\#(Y \setminus T)$. □

The variational formulation of (4.1.2) is the following:

$$\int_{Y\setminus T} A(t)\nabla\chi_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} = < A(t)\nabla\chi_\xi \cdot \nu, \psi >_{H^{-1/2}(\partial Y \cup \partial T), H^{1/2}(\partial Y \cup \partial T)},$$

for any $\psi \in H^1_\#(Y \setminus T)$ and we recall that the right hand side is well defined, by the trace operator defined in Proposition 4.5.3. By Lemma 4.5.2 and Remark 4.5.1, we obtain that:

$$\int_{Y\setminus T} A(t)\nabla\chi_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} = \int_{\partial T} \left(A(t)\xi \cdot \nu\right)\psi \, \mathrm{d}\sigma,\ \forall \psi \in H^1_\#(Y \setminus T). \tag{4.5.10}$$

Definition 4.5.2. Let $\xi \in \mathbb{R}^2$ be fixed. We define $\mathscr{A} : \mathbb{R} \times H^1_\#(Y \setminus T) \times H^1_\#(Y \setminus T) \to \mathbb{R}$ and $f : \mathbb{R} \times H^1_\#(Y \setminus T) \to \mathbb{R}$ as follows:

$$\mathscr{A}(t, \chi, \psi) = \int_{Y\setminus T} A(t)\nabla\chi \cdot \nabla\psi \, \mathrm{d}\mathbf{x}$$

and

$$f(t, \psi) = \int_{\partial T} \left(A(t)\xi \cdot \nu\right)\psi \, \mathrm{d}\sigma. \tag{4.5.11}$$

We recall that, for any $t \in \mathbb{R}$ and any $\xi \in \mathbb{R}^2$, we have $\left(A(t)\xi \cdot \nu\right) \in L^2(\partial T)$ and, for any $\psi \in H^1_\#(Y \setminus T)$, $\psi\big|_{\partial T} \in H^{1/2}(\partial T)$, so (4.5.11) is well defined.

Proposition 4.5.4. For any $t \in \mathbb{R}$, there exists a unique solution $\chi^t_\xi \in H^1_\#(Y \setminus T)$ of (4.5.4).

Proof. We first prove that, for any $t \in \mathbb{R}$, $\mathscr{A}(t,\cdot,\cdot)$ is a continuous coercive bilinear form on $H^1_\#(Y \setminus T)$ and that $f(t,\cdot)$ is a bounded linear functional on $H^1_\#(Y \setminus T)$. The continuity of $\mathscr{A}(t,\cdot,\cdot)$ is given by:

$$\begin{aligned}\left|\mathscr{A}(t,\chi,\psi)\right| &\le \int_{Y\setminus T} |A(t)\nabla\chi|\cdot|\nabla\psi|\,\mathrm{d}\mathbf{x} \le \left(\int_{Y\setminus T} |A(t)\nabla\chi|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\left(\int_{Y\setminus T} |\nabla\psi|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\\ &\le C_1(A)\cdot|Y\setminus T|^{1/2}\cdot\|\nabla\chi\|_{L^2(Y\setminus T)}\cdot\|\nabla\psi\|_{L^2(Y\setminus T)}\\ &\le C_1(A)\cdot|Y\setminus T|^{1/2}\cdot\|\chi\|_{H^1(Y\setminus T)}\cdot\|\psi\|_{H^1(Y\setminus T)},\end{aligned}$$

for all $\chi,\psi \in H^1_\#(Y\setminus T)$, where we have used (4.3.9). Moreover, we have that:

$$\alpha\|\nabla\chi\|^2_{L^2(Y\setminus T)} \le \int_{Y\setminus T} A(t)\nabla\chi\cdot\nabla\chi\,\mathrm{d}\mathbf{x} = \mathscr{A}(t,\chi,\chi),\ \forall\chi\in H^1(Y\setminus T),\ \forall t\in\mathbb{R},$$

due to (4.3.8). Using now Proposition 4.5.2, we obtain that:

$$\|\chi\|_{H^1(Y\setminus T)} \le \big(1+C_P(Y)\big)\|\nabla\chi\|_{L^2(Y\setminus T)}.$$

Combining the last two inequalities, we obtain:

$$\frac{\alpha}{\big(1+C_P(Y)\big)^2}\|\chi\|^2_{H^1(Y\setminus T)} \le \mathscr{A}(t,\chi,\chi),$$

for all $\chi \in H^1_\#(Y\setminus T)$.

The continuity of $f(t,\cdot)$ is given by the following sequence of inequalities:

$$\begin{aligned}|f(t,\psi)| &\le \int_{\partial T}\left|\big(A(t)\xi\cdot\nu\big)\psi(\mathbf{x})\right|\mathrm{d}\sigma \le \left(\int_{\partial T}|A(t)\xi\cdot\nu|^2\,\mathrm{d}\sigma\right)^{1/2}\left(\int_{\partial T}|\psi(\mathbf{x})|^2\,\mathrm{d}\sigma\right)^{1/2}\\ &\le C_1(A)\cdot|\xi|\cdot|\partial T|^{1/2}\cdot\|\psi\|_{L^2(\partial T)} \le C_1(A)\cdot|\xi|\cdot|\partial T|^{1/2}\|\psi\|_{L^2(\partial T\cup\partial Y)}\\ &\le C_1(A)\cdot|\xi|\cdot|\partial T|^{1/2}\cdot C_{Tr}(Y\setminus T)\cdot\|\psi\|_{H^1(Y\setminus T)},\end{aligned}$$

where $C_{Tr}(Y\setminus T)$ is the constant given by Proposition 4.5.3 and we have used once again (4.3.9).

We are now under the hypothesis of the Lax-Milgram theorem, therefore, for any $t\in\mathbb{R}$, the equation:

$$\mathscr{A}(t,\chi,\psi) = f(t,\psi),\ \forall\psi\in H^1_\#(Y\setminus T)$$

admits a unique solution $\chi^t_\xi \in H^1_\#(Y\setminus T)$. □

Definition 4.5.3. Let $\mathscr{S} : \mathbb{R} \to H^1_\#(Y\setminus T)$ be defined as $\mathscr{S}(t) = \chi^t_\xi$, where χ^t_ξ is given by Proposition 4.5.4.

Proposition 4.5.5. The operator $\mathscr{S}$ is continuous.

Proof. Let us fix $t \in \mathbb{R}$ and let $s \in \mathbb{R}$. Then, for any $\psi \in H^1_\#(Y \setminus T)$, by (4.5.10), χ^t_ξ and χ^s_ξ satisfy:

$$\int_{Y\setminus T} A(t)\nabla\chi^t_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} = \int_{\partial T} (A(t)\xi\cdot\nu)\psi \, \mathrm{d}\sigma, \tag{4.5.12}$$

and

$$\int_{Y\setminus T} A(s)\nabla\chi^s_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} = \int_{\partial T} (A(s)\xi\cdot\nu)\psi \, \mathrm{d}\sigma. \tag{4.5.13}$$

Then, using (4.3.8) at the point $s \in \mathbb{R}$, we have that:

$$\begin{aligned}
&\alpha\|\nabla\chi^t_\xi - \nabla\chi^s_\xi\|^2_{L^2(Y\setminus T)} \le \int_{Y\setminus T} A(s)\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} = \\
&= \int_{Y\setminus T} A(s)\nabla\chi^t_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} - \int_{Y\setminus T} A(s)\nabla\chi^s_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} \\
&= \int_{Y\setminus T} A(t)\nabla\chi^t_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} + \int_{Y\setminus T} \big(A(s)-A(t)\big)\nabla\chi^t_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} - \\
&\qquad - \int_{\partial T} \big(A(s)\xi\cdot\nu\big)\big(\chi^t_\xi - \chi^s_\xi\big)\,\mathrm{d}\sigma \\
&= \int_{Y\setminus T} \big(A(s)-A(t)\big)\nabla\chi^t_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} + \int_{\partial T} \big((A(t)-A(s))\xi\cdot\nu\big)\big(\chi^t_\xi - \chi^s_\xi\big)\,\mathrm{d}\sigma,
\end{aligned}$$

where the terms on ∂T come from (4.5.12) and (4.5.13) for $\psi = \chi^t_\xi - \chi^s_\xi$. We estimate the last two terms separately. First:

$$\begin{aligned}
&\int_{Y\setminus T} \big(A(s)-A(t)\big)\nabla\chi^t_\xi\cdot\big(\nabla\chi^t_\xi - \nabla\chi^s_\xi\big)\,\mathrm{d}\mathbf{x} \le \\
&\quad \le \left(\int_{Y\setminus T} \left|\big(A(s)-A(t)\big)\nabla\chi^t_\xi\right|^2 \mathrm{d}\mathbf{x}\right)^{1/2} \cdot \left(\int_{Y\setminus T} \left|\nabla\chi^t_\xi - \nabla\chi^s_\xi\right|^2 \mathrm{d}\mathbf{x}\right)^{1/2} \\
&\quad \le \mathcal{C}_2(A')\cdot|t-s|\cdot\|\nabla\chi^t_\xi\|_{L^2(Y\setminus T)}\cdot\|\nabla\chi^t_\xi - \nabla\chi^s_\xi\|_{L^2(Y\setminus T)},
\end{aligned}$$

where we have used (4.3.11).

Now, we move our attention to the term containing ∂T:

$$\begin{aligned}
&\int_{\partial T} \Big(\big(A(t)-A(s)\big)\xi\cdot\nu\Big)\big(\chi^t_\xi - \chi^s_\xi\big)\,\mathrm{d}\sigma \le \\
&\le \left(\int_{\partial T} \left|\big(A(t)-A(s)\big)\xi\cdot\nu\right|^2 \mathrm{d}\sigma\right)^{1/2} \cdot \left(\int_{\partial T} |\chi^t_\xi - \chi^s_\xi|^2\,\mathrm{d}\sigma\right)^{1/2} \\
&\le \mathcal{C}_2(A')\cdot|t-s|\cdot|\xi|\cdot|\partial T|^{1/2}\cdot\left(\int_{\partial T\cup\partial Y} |\chi^t_\xi - \chi^s_\xi|^2\,\mathrm{d}\sigma\right)^{1/2} \\
&\le \left(\mathcal{C}_2(A')\cdot|\xi|\cdot|\partial T|^{1/2}\right)\cdot|t-s|\cdot\|\chi^t_\xi - \chi^s_\xi\|_{L^2(\partial(Y\setminus T))} \\
&\le \left(C_{Tr}(Y\setminus T)\cdot\mathcal{C}_2(A')\cdot|\xi|\cdot|\partial T|^{1/2}\right)\cdot|t-s|\cdot\|\chi^t_\xi - \chi^s_\xi\|_{H^1(Y\setminus T)} \\
&\le \left(\sqrt{1+C_P^2(Y)}\cdot C_{Tr}(Y\setminus T)\cdot\mathcal{C}_2(A')\cdot|\xi|\cdot|\partial T|^{1/2}\right)\cdot|t-s|\cdot\|\nabla\chi^t_\xi - \nabla\chi^s_\xi\|_{L^2(Y\setminus T)},
\end{aligned}$$

where we have used (4.3.11), Proposition 4.5.2 and Proposition 4.5.3.

Let

$$M(A', \xi, Y, T) = \max\left\{\left(\sqrt{1 + C_P^2(Y)} \cdot C_{Tr}(Y \setminus T) \cdot \mathcal{C}_2(A') \cdot |\xi| \cdot \left|\partial T\right|^{1/2}\right), \mathcal{C}_2(A')\right\}.$$

We have obtained that:

$$\alpha\|\nabla\chi_\xi^t - \nabla\chi_\xi^s\|^2_{L^2(Y\setminus T)} \leq M(A', \xi, Y, T) \cdot \left(1 + \|\nabla\chi_\xi^t\|_{L^2(Y\setminus T)}\right) \cdot |t-s| \cdot \|\nabla\chi_\xi^t - \nabla\chi_\xi^s\|_{L^2(Y\setminus T)}. \tag{4.5.14}$$

Since $\alpha > 0$, then treating (4.5.14) as a quadratic function in $\|\nabla\chi_\xi^t - \nabla\chi_\xi^s\|_{L^2(Y\setminus T)}$, we obtain:

$$0 \leq \|\nabla\chi_\xi^t - \nabla\chi_\xi^s\|_{L^2(Y\setminus T)} \leq \left(\alpha^{-1} \cdot M(A', \xi, Y, T) \cdot \left(1 + \|\nabla\chi_\xi^t\|_{L^2(Y\setminus T)}\right)\right) \cdot |t-s|.$$

If we fix $t \in \mathbb{R}$ and let $s \to t$, then $\nabla\chi_\xi^s \to \nabla\chi_\xi^t$ strongly in $L^2(Y \setminus T)$ by the previous inequality. Using the Poincaré inequality from Proposition 4.5.2, we finally obtain that $\chi_\xi^s \to \chi_\xi^t$ strongly in $H^1(Y \setminus T)$ and, therefore, in $H^1_\#(Y \setminus T)$. □

We now move our focus to proving that the operator $\mathscr{S}$ is differentiable. For this, we analyse the following PDE:

$$\begin{cases} \operatorname{div}(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t) = 0 & \text{in } Y \setminus T \\ \left(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t\right) \cdot \nu = A'(t)\xi \cdot \nu & \text{on } \partial T \\ \psi_\xi^t \in H^1_\#(Y \setminus T). \end{cases} \tag{4.5.15}$$

Remark 4.5.3. In the same fashion as in Remark 4.5.1, since

$$\operatorname{div}(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t) = 0 \in L^2(Y \setminus T),$$

then $A(t)\nabla\psi_\xi^t + A'(t)\chi_\xi^t \in H(Y \setminus T; \operatorname{div})$, which implies that

$$\left(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t\right) \cdot \nu \in H^{-1/2}(\partial T).$$

The boundary condition on ∂T should then be interpreted, variationally, as the pairing between $H^{-1/2}(\partial T)$ and $H^{1/2}(\partial T)$ functions.

At the same time, for any $t \in \mathbb{R}$ and any $\xi \in \mathbb{R}$, we have $A'(t)\xi \cdot \nu \in L^2(\partial T)$. Then, the variational formulation of (4.5.15) is the following:

$$\begin{aligned} \int_{Y\setminus T} &\left(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t\right) \cdot \nabla\psi \,\mathrm{d}\mathbf{x} = \\ &= < \left(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t\right) \cdot \nu, \psi >_{H^{-1/2}(\partial Y \cup \partial T), H^{1/2}(\partial Y \cup \partial T)} \\ &= < \left(A(t)\nabla\psi_\xi^t + A'(t)\nabla\chi_\xi^t\right) \cdot \nu, \psi >_{H^{-1/2}(\partial T), H^{1/2}(\partial T)} \\ &= < A'(t)\xi \cdot \nu, \psi >_{H^{-1/2}(\partial T), H^{1/2}(\partial T)} \\ &= \int_{\partial T} \left(A'(t)\xi \cdot \nu\right)\psi \,\mathrm{d}\sigma, \end{aligned}$$

for any $\psi \in H^1_\#(Y \setminus T)$, where we have used (4.5.6) for ψ^t_ξ and χ^t_ξ. Therefore:

$$\int_{Y\setminus T} A(t)\nabla\psi^t_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} = \int_{Y\setminus T} -A'(t)\nabla\chi^t_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} + \int_{\partial T} (A'(t)\xi \cdot \nu)\psi \, \mathrm{d}\sigma, \tag{4.5.16}$$

which can be written as:

$$\mathscr{A}(t, \psi^t_\xi, \psi) = \overline{f}(t, \psi),$$

where

$$\overline{f}(t, \psi) = \int_{\partial T} (A'(t)\xi \cdot \nu)\psi \, \mathrm{d}\sigma + \int_{Y\setminus T} -A'(t)\nabla\chi^t_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x}.$$

Proposition 4.5.6. There exists a unique $\psi^t_\xi \in H^1_\#(Y \setminus T)$ that solves (4.5.15).

Proof. We mimic the arguments from the proof of Proposition 4.5.4, that is, for any $t \in \mathbb{R}$, we apply Lax-Milgram theorem to get a unique $\psi^t_\xi \in H^1_\#(Y \setminus T)$ that solves:

$$\mathscr{A}(t, \psi^t_\xi, \psi) = \overline{f}(t, \psi), \ \forall \psi \in H^1_\#(Y \setminus T).$$

Since all the properties of $\mathscr{A}$ have already been proved in Proposition 4.5.4, it is sufficient to prove that $\overline{f} : H^1_\#(Y \setminus T) \to \mathbb{R}$ is a bounded linear functional. By Hölder inequality, we have:

$$\left| \int_{\partial T} (A'(t)\xi \cdot \nu)\psi \, \mathrm{d}\sigma \right| \leq \left(\int_{\partial T} |A'(t)\xi \cdot \nu|^2 \, \mathrm{d}\sigma \right)^{1/2} \cdot \left(\int_{\partial T} |\psi|^2 \, \mathrm{d}\sigma \right)^{1/2},$$

which implies

$$\left| \int_{\partial T} (A'(t)\xi \cdot \nu)\psi \, \mathrm{d}\sigma \right| \leq C_2(A') \cdot |\xi| \cdot |\partial T|^{1/2} \cdot C_{Tr}(Y \setminus T) \cdot \|\psi\|_{H^1(Y\setminus T)},$$

where we have used (4.3.10) and the trace inequality from Proposition 4.5.3. Using once again (4.3.10), we have:

$$\begin{aligned} \left| \int_{Y\setminus T} -A'(t)\nabla\chi^t_\xi \cdot \nabla\psi \, \mathrm{d}\mathbf{x} \right| &\leq \left(\int_{Y\setminus T} |A'(t)\nabla\chi^t_\xi|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} \left(\int_{Y\setminus T} |\nabla\psi|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} \\ &\leq C_2(A') \cdot |Y \setminus T|^{1/2} \cdot \|\nabla\chi^t_\xi\|_{L^2(Y\setminus T)} \cdot \|\psi\|_{H^1(Y\setminus T)}, \end{aligned}$$

from which we conclude. ☐

We recall that an operator $\mathscr{T} : \mathbb{R} \to H^1_\#(Y \setminus T)$ is Fréchet differentiable at a point $t_0 \in \mathbb{R}$ if there exists a bounded linear operator $\mathscr{T}'[t_0] : \mathbb{R} \to H^1_\#(Y \setminus T)$ such that:

$$\lim_{0<|h|\to 0} \frac{\|\mathscr{T}(t_0 + h) - \mathscr{T}(t_0) - \mathscr{T}'[t_0](h)\|_{H^1_\#(Y\setminus T)}}{|h|} = 0.$$

Proposition 4.5.7. The operator $\mathscr{S} : \mathbb{R} \to H^1_\#(Y \setminus T)$ is Fréchet differentiable on $\mathbb{R}$. Moreover, for any $t_0 \in \mathbb{R}$, we have $\mathscr{S}'[t_0](h) = h \cdot \psi^{t_0}_\xi$, for any $h \in \mathbb{R}$, where $\psi^{t_0}_\xi$ is given by Proposition 4.5.6.

Proof. Let us fix $t \in \mathbb{R}$ and let $s \in \mathbb{R}$ with $s \neq t$. Moreover, let χ_ξ^t and χ_ξ^s be the $H^1_\#(Y \setminus T)$ functions given by Proposition 4.5.4 and let ψ_ξ^t given by Proposition 4.5.6.

We prove first that:

$$\chi_\xi^{t,s} := (t-s)^{-1}(\chi_\xi^t - \chi_\xi^s) - \psi_\xi^t \to 0 \text{ strongly in } H^1_\#(Y \setminus T) \text{ as } s \to t.$$

Using the coercivity property of $\mathscr{A}(s,\cdot,\cdot)$ given by (4.3.8), one could write that:

$$\begin{aligned}
\alpha\|\nabla\chi_\xi^{t,s}\|^2_{L^2(Y\setminus T)} \leq \int_{Y\setminus T} A(s)\Big((t-s)^{-1}(\nabla\chi_\xi^t - \nabla\chi_\xi^s) - \nabla\psi_\xi^t\Big)\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} = \\
= \int_{Y\setminus T}(t-s)^{-1}A(s)(\nabla\chi_\xi^t - \nabla\chi_\xi^s)\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{Y\setminus T} A(s)\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} \\
= \int_{Y\setminus T}(t-s)^{-1}A(s)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{Y\setminus T}(t-s)^{-1}A(s)\nabla\chi_\xi^s\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \\
+ \int_{Y\setminus T}(A(t)-A(s))\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{Y\setminus T} A(t)\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx}
\end{aligned}$$

Using (4.5.13), we obtain:

$$\begin{aligned}
&\alpha\|\nabla\chi_\xi^{t,s}\|^2_{L^2(Y\setminus T)} \leq \\
&\leq \int_{Y\setminus T}(t-s)^{-1}A(s)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{\partial T}(t-s)^{-1}(A(s)\xi\cdot\nu)\chi_\xi^{t,s}\,\mathrm{d}\sigma + \\
&\quad + \int_{Y\setminus T}(A(t)-A(s))\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} + \int_{Y\setminus T} A'(t)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{\partial T}(A(t)\xi\cdot\nu)\chi_\xi^{t,s}\,\mathrm{d}\sigma
\end{aligned}$$

and then, by (4.5.16), we have:

$$\begin{aligned}
&\alpha\|\nabla\chi_\xi^{t,s}\|^2_{L^2(Y\setminus T)} \leq \\
&\quad\leq \int_{Y\setminus T}(t-s)^{-1}A(s)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} + \int_{\partial T}(t-s)^{-1}\big((A(t)-A(s))\xi\cdot\nu\big)\chi_\xi^{t,s}\,\mathrm{d}\sigma - \\
&\quad - \int_{\partial T}(t-s)^{-1}(A(t)\xi\cdot\nu)\chi_\xi^{t,s}\,\mathrm{d}\sigma + \int_{Y\setminus T}(A(t)-A(s))\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} + \\
&\quad + \int_{Y\setminus T} A'(t)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} - \int_{\partial T}(A'(t)\xi\cdot\nu)\chi_\xi^{t,s}\,\mathrm{d}\sigma.
\end{aligned}$$

Using (4.5.12), we obtain that:

$$\begin{aligned}
\alpha\|\nabla\chi_\xi^{t,s}\|^2_{L^2(Y\setminus T)} \leq \int_{Y\setminus T}\left(A'(t) - \frac{A(t)-A(s)}{t-s}\right)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx} + \\
+ \int_{\partial T}\left(\left(\frac{A(t)-A(s)}{t-s} - A'(t)\right)\xi\cdot\nu\right)\chi_\xi^{t,s}\,\mathrm{d}\sigma + \\
+ \int_{Y\setminus T}(A(t)-A(s))\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathbf{dx}
\end{aligned}$$

Now, using (4.3.13), we obtain:

$$
\begin{aligned}
&\left|\int_{Y\setminus T}\left(A'(t)-\frac{A(t)-A(s)}{t-s}\right)\nabla\chi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathrm{d}\mathbf{x}\right|\le\\
&\le\left(\int_{Y\setminus T}\left|\left(A'(t)-\frac{A(t)-A(s)}{t-s}\right)\nabla\chi_\xi^t\right|^2\mathrm{d}\mathbf{x}\right)^{1/2}\cdot\left(\int_{Y\setminus T}|\nabla\chi_\xi^{t,s}|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\le\mathcal{C}_3(A'')\cdot|t-s|\cdot\|\nabla\chi_\xi^t\|_{L^2(Y\setminus T)}\cdot\|\nabla\chi_\xi^{t,s}\|_{L^2(Y\setminus T)}
\end{aligned}
$$

and, by using the trace inequality from Proposition 4.5.3 and the Poincaré inequality from Proposition 4.5.2, we also obtain:

$$
\begin{aligned}
&\left|\int_{\partial T}\left(\left(\frac{A(t)-A(s)}{t-s}-A'(t)\right)\xi\cdot\nu\right)\chi_\xi^{t,s}\,\mathrm{d}\sigma\right|\le\\
&\le\left(\int_{\partial T}\left|\left(\frac{A(t)-A(s)}{t-s}-A'(t)\right)\xi\cdot\nu\right|^2\mathrm{d}\sigma\right)^{1/2}\cdot\left(\int_{\partial T}|\chi_\xi^{t,s}|^2\,\mathrm{d}\sigma\right)^{1/2}\\
&\le\mathcal{C}_3(A'')\cdot|t-s|\cdot|\partial T|^{1/2}\cdot|\xi|\cdot C_{Tr}(Y\setminus T)\cdot\|\chi_\xi^{t,s}\|_{H^1(Y\setminus T)}\\
&\le\left(\sqrt{1+C_P^2(Y)}\cdot C_{Tr}(Y\setminus T)\cdot\mathcal{C}_3(A'')\cdot|\xi|\cdot|\partial T|^{1/2}\right)\cdot|t-s|\cdot\|\nabla\chi_\xi^{t,s}\|_{L^2(Y\setminus T)}.
\end{aligned}
$$

We also have, by (4.3.11):

$$
\begin{aligned}
&\left|\int_{Y\setminus T}(A(t)-A(s))\nabla\psi_\xi^t\cdot\nabla\chi_\xi^{t,s}\,\mathrm{d}\mathbf{x}\right|\le\\
&\le\left(\int_{Y\setminus T}|(A(t)-A(s))\nabla\psi_\xi^t|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\cdot\left(\int_{Y\setminus T}|\nabla\chi_\xi^{t,s}|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\le C_2(A')\cdot|t-s|\cdot\|\nabla\psi_\xi^t\|_{L^2(Y\setminus T)}\cdot\|\nabla\chi_\xi^{t,s}\|_{L^2(Y\setminus T)}.
\end{aligned}
$$

Let us consider

$$
\begin{aligned}
M(A',A'',\xi,Y,T,\chi_\xi^t,\psi_\xi^t)=\max\Big\{&\mathcal{C}_3(A'')\cdot\|\nabla\chi_\xi^t\|_{L^2(Y\setminus T)},\ C_2(A')\cdot\|\nabla\psi_\xi^t\|_{L^2(Y\setminus T)},\\
&\sqrt{1+C_P^2(Y)}\cdot C_{Tr}(Y\setminus T)\cdot\mathcal{C}_3(A'')\cdot|\xi|\cdot|\partial T|^{1/2}\Big\}.
\end{aligned}
$$

Therefore, we have:

$$
\alpha\|\nabla\chi_\xi^{t,s}\|^2_{L^2(Y\setminus T)}\le M(A',A'',\xi,Y,T,\chi_\xi^t,\psi_\xi^t)\cdot|t-s|\cdot\|\nabla\chi_\xi^{t,s}\|_{L^2(Y\setminus T)},
$$

which implies, since $\alpha>0$, that:

$$
0\le\|\nabla\chi_\xi^{t,s}\|_{L^2(Y\setminus T)}\le M(A',A'',\xi,Y,T,\chi_\xi^t,\psi_\xi^t)\cdot|t-s|,\ \forall s,t\in\mathbb{R},\ s\ne t.
$$

By using the Poincaré inequality from Proposition 4.5.2, we obtain that

$$
\chi_\xi^{t,s}\to 0\text{ strongly in }H^1_\#(Y\setminus T)\text{ for }s\to t,\ s\ne t. \tag{4.5.17}
$$

Let $t_0 \in \mathbb{R}$ fixed and $\mathscr{S}'[t_0] : \mathbb{R} \to H^1_\#(Y \setminus T)$ defined as $\mathscr{S}'[t_0](h) = h \cdot \psi_\xi^{t_0}$, for all $h \in \mathbb{R}$. Then:

$$\lim_{0<|h|\to 0} \frac{\|\mathscr{T}(t_0+h) - \mathscr{T}(t_0) - \mathscr{T}'[t_0](h)\|_{H^1_\#(Y\setminus T)}}{|h|} = \lim_{0<|h|\to 0} \frac{\|\chi_\xi^{t_0+h} - \chi_\xi^{t_0} - h\cdot\psi_\xi^{t_0}\|_{H^1_\#(Y\setminus T}}{|h|} =$$

$$= \lim_{0<|h|\to 0} \left\| \frac{\chi_\xi^{t_0+h} - \chi_\xi^{t_0}}{|h|} - \psi_\xi^{t_0} \right\|_{H^1_\#(Y\setminus T)} = \lim_{0<|h|\to 0} \|\chi_\xi^{t_0,t_0+h}\|_{H^1_\#(Y\setminus T)} = 0,$$

by (4.5.17). □

4.5.4 PROOF OF THE PROPERTY OF B_0

Throughout this subsection, we are going to use the following notation:

$$w_\xi(\mathbf{x}, t) = \xi \cdot \mathbf{x} - \chi_\xi(\mathbf{x}, t),\ \forall \mathbf{x} \in Y \setminus T,\ \forall t \in \mathbb{R}.$$

Since $\chi_\xi(\cdot, t)$ solves (4.3.14), we see that $w_\xi(\cdot, t) \in H^1(Y \setminus T)$ and that it solves:

$$\begin{cases} -\operatorname{div}\big(A(t)\nabla w_\xi(\mathbf{x}, t)\big) = 0, & \text{in } Y \setminus T \\ A(t)\nabla w_\xi(\mathbf{x}, t) \cdot \nu = 0, & \text{on } \partial T \end{cases}$$

which generates, after using Lemma 4.5.2, the following equality:

$$\int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x}, t) \cdot \nabla\psi(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0,\ \forall \psi \in H^1_\#(Y \setminus T). \tag{4.5.18}$$

Proof of Proposition 4.3.6. Let $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$. We recall first that:

$$\begin{aligned} \mathscr{B}_0(t, \xi) &= \int_{Y\setminus T} \mathscr{B}(t, C(\mathbf{x}, t)\xi)\ \mathrm{d}\mathbf{x} \\ &= \int_{Y\setminus T} -\frac{1}{2}A'(t)C(\mathbf{x}, t)\xi \cdot C(\mathbf{x}, t)\xi + \frac{\mu}{2}\sin(2t - 2\overline{\varphi})\ \mathrm{d}\mathbf{x} \\ &= \int_{Y\setminus T} -\frac{1}{2}A'(t)\big(\xi - \nabla\chi_\xi(\mathbf{x}, t)\big) \cdot \big(\xi - \nabla\chi_\xi(\mathbf{x}, t)\big)\ \mathrm{d}\mathbf{x} + \frac{\mu}{2} \cdot |Y \setminus T| \cdot \sin(2t - 2\overline{\varphi}) \end{aligned}$$

where we have used Proposition 4.3.5. Moreover, due to Definition 4.2.1, we have $\theta_0 = |Y \setminus T|$, hence, in order to conclude the proof, we would like to prove that:

$$A_0'(t)\xi \cdot \xi = \int_{Y\setminus T} A'(t)\nabla w_\xi(\mathbf{x}, t) \cdot \nabla w_\xi(\mathbf{x}, t)\ \mathrm{d}\mathbf{x}.$$

For this, we start from Definition 4.3.5:

$$A_0(t)\xi = \int_{Y\setminus T} A(t)\big(\xi - \nabla\chi_\xi(\mathbf{x}, t)\big)\ \mathrm{d}\mathbf{x}.$$

Multiplying by ξ, we obtain:

$$\begin{aligned}
A_0(t)\xi\cdot\xi &= \int_{Y\setminus T} A(t)\big(\xi-\nabla\chi_\xi(\mathbf{x},t)\big)\cdot\xi\;\mathrm{d}\mathbf{x} = \\
&= \int_{Y\setminus T} A(t)\big(\xi-\nabla\chi_\xi(\mathbf{x},t)\big)\cdot\big(\xi-\nabla\chi_\xi(\mathbf{x},t)\big)\;\mathrm{d}\mathbf{x} + \int_{Y\setminus T} A(t)\big(\xi-\nabla\chi_\xi(\mathbf{x},t)\big)\cdot\nabla\chi_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} \\
&= \int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} + \int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla\chi_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} \\
&= \int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x},
\end{aligned}$$

due to (4.5.18), since $\chi_\xi(\cdot,t)\in H^1_\#(Y\setminus T)$. Let $J:\mathbb{R}^*\to\mathbb{R}$ be defined as:

$$\begin{aligned}
J(h) = &\frac{1}{h}\left(\int_{Y\setminus T} A(t+h)\nabla w_\xi(\mathbf{x},t+h)\cdot\nabla w_\xi(\mathbf{x},t+h)\;\mathrm{d}\mathbf{x}\right) - \\
&-\frac{1}{h}\left(\int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x}\right) - \\
&-\int_{Y\setminus T} A'(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} - \\
&-2\int_{Y\setminus T} A(t)\frac{\partial}{\partial t}\big(\nabla w_\xi(\mathbf{x},t)\big)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x},
\end{aligned}$$

where we recall that

$$\frac{\partial w_\xi}{\partial t}(\mathbf{x},t) = -\frac{\partial\chi_\xi}{\partial t}(\mathbf{x},t) = -\psi^t_\xi(\mathbf{x}) \text{ and } \frac{\partial}{\partial t}\nabla w_\xi(\mathbf{x},t) = -\nabla\psi^t_\xi(\mathbf{x}),$$

a.e. in $Y\setminus T$, due to Proposition 4.5.7.

Since $A(t)$ is symmetric, we can rewrite $J(h)$ as:

$$\begin{aligned}
J(h) = &\int_{Y\setminus T}\left(\frac{1}{h}\big(A(t+h)-A(t)\big)-A'(t)\right)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} + \\
&+\int_{Y\setminus T} A(t+h)\nabla w_\xi(\mathbf{x},t+h)\cdot\left(\frac{1}{h}\big(\nabla w_\xi(\mathbf{x},t+h)-\nabla w_\xi(\mathbf{x},t)\big)+\nabla\psi^t_\xi(\mathbf{x})\right)\;\mathrm{d}\mathbf{x} + \\
&+\int_{Y\setminus T} A(t+h)\left(\frac{1}{h}\big(\nabla w_\xi(\mathbf{x},t+h)-\nabla w_\xi(\mathbf{x},t)\big)+\nabla\psi^t_\xi(\mathbf{x})\right)\cdot\nabla w_\xi(\mathbf{x},t)\;\mathrm{d}\mathbf{x} - \\
&-\int_{Y\setminus T}\big(A(t+h)-A(t)\big)\nabla w_\xi(\mathbf{x},t+h)\cdot\nabla\psi^t_\xi(\mathbf{x})\;\mathrm{d}\mathbf{x} - \\
&-\int_{Y\setminus T} A(t)\big(\nabla w_\xi(\mathbf{x},t+h)-\nabla w_\xi(\mathbf{x},t)\big)\cdot\nabla\psi^t_\xi(\mathbf{x})\;\mathrm{d}\mathbf{x} - \\
&-\int_{Y\setminus T}\big(A(t+h)-A(t)\big)\nabla w_\xi(\mathbf{x},t)\cdot\nabla\psi^t_\xi(\mathbf{x})\;\mathrm{d}\mathbf{x} \\
:= &J_1(h)+J_2(h)+J_3(h)+J_4(h)+J_5(h)+J_6(h).
\end{aligned}$$

We have:

$$
\begin{aligned}
|J_1(h)| &\leq \int_{Y\setminus T} \left| \left(\frac{1}{h}(A(t+h) - A(t)) - A'(t) \right) \nabla w_\xi(\mathbf{x}, t+h) \right| \cdot \left| \nabla w_\xi(\mathbf{x}, t+h) \right| \, \mathrm{d}\mathbf{x} \\
&\leq \int_{Y\setminus T} \mathcal{C}_3(A'') \cdot |h| \cdot \left| \nabla w_\xi(\mathbf{x}, t+h) \right|^2 \mathrm{d}\mathbf{x} = \mathcal{C}_3(A'') \cdot |h| \cdot \|\nabla w_\xi(\cdot, t+h)\|^2_{L^2(Y\setminus T)} \\
&\leq \frac{1}{2} \cdot \mathcal{C}_3(A'') \cdot |h| \cdot \left(|\xi|^2 \cdot |Y \setminus T| + \|\nabla \chi_\xi(\cdot, t+h)\|^2_{L^2(Y\setminus T)} \right),
\end{aligned}
$$

where we have used (4.3.13). Due to Proposition 4.5.5, as $h \to 0$, we have that $\chi_\xi(\cdot, t+h) \to \chi_\xi(\cdot, t)$ strongly in $H^1_\#(Y \setminus T)$. Therefore, as $h \to 0$, we have $J_1(h) \to 0$.

For the rest of the terms, we first remark that

$$
\nabla w_\xi(\mathbf{x}, t+h) - \nabla w_\xi(\mathbf{x}, t) = \nabla \chi_\xi(\mathbf{x}, t+h) - \nabla \chi_\xi(\mathbf{x}, t).
$$

For $J_2(h)$, we apply Hölder inequality and obtain:

$$
\begin{aligned}
|J_2(h)| \leq &\left(\int_{Y\setminus T} \left| A(t+h) \nabla w_\xi(\mathbf{x}, t+h) \right|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \cdot \\
&\cdot \left(\int_{Y\setminus T} \left| \frac{1}{h}(\nabla w_\xi(\mathbf{x}, t+h) - \nabla w_\xi(\mathbf{x}, t)) + \nabla \psi^t_\xi(\mathbf{x}) \right|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \\
\leq &\,\mathcal{C}_1(A) \cdot \|\nabla \chi_\xi(\cdot, t+h)\|_{L^2(Y\setminus T)} \cdot \left\| \frac{1}{h}(\nabla \chi_\xi(\cdot, t+h) - \nabla \chi_\xi(\cdot, t)) - \nabla \psi^t_\xi \right\|_{L^2(Y\setminus T)},
\end{aligned}
$$

where we have used (4.3.9). As $h \to 0$, we have that $\chi_\xi(\cdot, t+h) \to \chi_\xi(\cdot, t)$ strongly in $H^1_\#(Y \setminus T)$, due to Proposition 4.5.5. At the same time, from Proposition 4.5.7 we also get that, as $h \to 0$:

$$
\left(\frac{1}{h}(\nabla \chi_\xi(\cdot, t+h) - \nabla \chi_\xi(\cdot, t)) - \nabla \psi^t_\xi \right) \to 0 \text{ strongly in } \left(L^2(Y \setminus T)\right)^2.
$$

So, as $h \to 0$, we obtain $J_2(h) \to 0$.

In a similar fashion we obtain that $J_3(h) \to 0$ as $h \to 0$.

For $J_4(h)$ and $J_6(h)$, for $s \in \{t, t+h\}$, we have, by applying Hölder inequality:

$$
\begin{aligned}
&\left| \int_{Y\setminus T} (A(t+h) - A(t)) \nabla w_\xi(\mathbf{x}, s) \cdot \nabla \psi^t_\xi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right| \leq \\
&\quad \leq \left(\int_{Y\setminus T} \left| (A(t+h) - A(t)) \nabla w_\xi(\mathbf{x}, s) \right|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \cdot \left(\int_{Y\setminus T} \left| \nabla \psi^t_\xi(\mathbf{x}) \right|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \\
&\quad \leq \mathcal{C}_2(A') \cdot |h| \cdot \|\nabla w_\xi(\cdot, s)\|_{L^2(Y\setminus T)} \cdot \|\nabla \psi^t_\xi\|_{L^2(Y\setminus T)} \\
&\quad \leq \mathcal{C}_2(A') \cdot |h| \cdot \|\nabla \psi^t_\xi\|_{L^2(Y\setminus T)} \cdot \left(|\xi| \cdot |Y \setminus T|^{1/2} + \|\nabla \chi_\xi(\cdot, s)\|_{L^2(Y\setminus T)} \right),
\end{aligned}
$$

where we have used (4.3.11). If $s = t + h$, then as $h \to 0$, we know from Proposition 4.5.5 that $\chi_\xi(\cdot, t+h) \to \chi_\xi(\cdot, t)$ strongly in $H^1_\#(Y \setminus T)$. Therefore, if $h \to 0$, we also obtain that $J_4(h) \to 0$ and $J_6(h) \to 0$.

For $J_5(h)$, we apply once again Hölder inequality and (4.3.9) and we obtain:

$$|J_5(h)| \le C_1(A)\cdot \|\nabla w_\xi(\cdot,t+h)-\nabla w_\xi(\cdot,t)\|_{L^2(Y\setminus T)}\cdot \|\nabla \psi_\xi^t\|_{L^2(Y\setminus T)} \Rightarrow$$
$$\Rightarrow |J_5(h)| \le C_1(A)\cdot \|\nabla \chi_\xi(\cdot,t+h)-\nabla \chi_\xi(\cdot,t)\|_{L^2(Y\setminus T)}\cdot \|\nabla \psi_\xi^t\|_{L^2(Y\setminus T)}.$$

Due to Proposition 4.5.5, as $h\to 0$, $\big(\nabla\chi_\xi(\cdot,t+h)-\nabla\chi_\xi(\cdot,t)\big)\to \mathbf{o}$ strongly in $\big(L^2(Y\setminus T)\big)^2$, from which we also obtain that $J_5(h)\to 0$ as $h\to 0$.

In this way, we have shown that $J(h)\to 0$ as $h\to 0$. At the same time, we have:

$$J(h)=\frac{1}{h}\big(A_0(t+h)-A_0(t)\big)\xi\cdot\xi-$$
$$-\int_{Y\setminus T} A'(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\,d\mathbf{x}-2\int_{Y\setminus T} A(t)\frac{\partial}{\partial t}\big(\nabla w_\xi(\mathbf{x},t)\big)\cdot\nabla w_\xi(\mathbf{x},t)\,d\mathbf{x},$$

which implies that, as $h\to 0$, we have:

$$J(h)\to A_0'(t)\xi\cdot\xi-\int_{Y\setminus T} A'(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\,d\mathbf{x}+2\int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla\psi_\xi^t(\mathbf{x})\,d\mathbf{x},$$

where we have just replaced the term $\frac{\partial}{\partial t}\nabla w_\xi(\mathbf{x},t)$ with $\big(-\nabla\psi_\xi^t(\mathbf{x})\big)$ and we have used that $A_0\in C^2(\mathbb{R})$. Moreover, we recall that $\psi_\xi^t\in H^1_\#(Y\setminus T)$, hence, by using (4.5.18), we have:

$$\int_{Y\setminus T} A(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla\psi_\xi^t(\mathbf{x})\,d\mathbf{x}=0.$$

Therefore,

$$\lim_{h\to 0} J(h)=0=A_0'(t)\xi\cdot\xi-\int_{Y\setminus T} A'(t)\nabla w_\xi(\mathbf{x},t)\cdot\nabla w_\xi(\mathbf{x},t)\,d\mathbf{x},$$

from which we conclude. □

4.5.5 THE DEPENDENCY BETWEEN K_0 AND K

Proof of Proposition 4.3.7. In order to express the relationship between K_0 and K, we start by obtaining a relationship between A_0 and A. We recall that:

$$A_0(t)\xi=\int_{Y\setminus T} A(t)\big(\xi-\nabla\chi_\xi\big)\,d\mathbf{x},\ \forall\xi\in\mathbb{R}^2.$$

Let $\tau=(\cos t,\sin t)$ and $\tau^\perp=(-\sin t,\cos t)$. Computing $A_0(t)\tau$ and $A_0(t)\tau^\perp$, one gets that:

$$A_0(t)\cdot\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}=|Y\setminus T|A(t)\cdot\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}-A(t)\cdot\int_{Y\setminus T}\begin{pmatrix}\frac{\partial\chi_\tau}{\partial x} & \frac{\partial\chi_{\tau^\perp}}{\partial x}\\[4pt] \frac{\partial\chi_\tau}{\partial y} & \frac{\partial\chi_{\tau^\perp}}{\partial y}\end{pmatrix}\,d\mathbf{x},$$

which, by Proposition 4.3.4 and Definition 4.3.2, becomes:

$$A_0(t)R(t) = |Y \setminus T| A(t)R(t) - A(t) \int_{Y\setminus T} \begin{pmatrix} \dfrac{\partial \chi_1}{\partial x} & \dfrac{\partial \chi_2}{\partial x} \\ \\ \dfrac{\partial \chi_1}{\partial y} & \dfrac{\partial \chi_2}{\partial y} \end{pmatrix} \cdot R(t) \, \mathrm{d}\mathbf{x}$$

which, by Definition 4.3.7, becomes:

$$A_0(t)R(t) = |Y \setminus T| A(t)R(t) + A(t) \int_{Y\setminus T} \Big(C(\mathbf{x}, t) - \mathbb{I}_2 \Big) \cdot R(t) \, \mathrm{d}\mathbf{x} \Rightarrow$$
$$\Rightarrow A_0(t)R(t) = A(t) \cdot \left(\int_{Y\setminus T} C(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} \right) \cdot R(t)$$

and, by multiplying with $R(-t)$ on the left and writing $\mathbb{I}_2 = R(t) \cdot R(-t)$, we obtain:

$$R(-t)A_0(t)R(t) = \Big(R(-t) \cdot A(t)R(t) \Big) \cdot R(-t) \cdot \left(\int_{Y\setminus T} C(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} \right) \cdot R(t).$$

This concludes the proof. □

4.5.6 PROVING THAT φ_ε IS INDEED A CRITICAL POINT OF F_ε

In this subsection, we prove the following used implication: if $\mathbf{u}_\varepsilon \in \mathbf{V}_\varepsilon$ is a critical point of $\mathbf{F}_\varepsilon$, then $\varphi_\varepsilon \in V_\varepsilon$ is a critical point of F_ε, where we recall that $\mathbf{V}_\varepsilon$ and V_ε are introduced in Definition 4.2.3, $\mathbf{F}_\varepsilon$ in (4.1.1) and F_ε in (4.3.2).

We proceed in a similar fashion as in [44, Theorem 5, p. 496].

Let us consider $\mathbf{v} \in H_0^1(\Omega_\varepsilon; \mathbb{R}^2) \cap L^\infty(\Omega_\varepsilon; \mathbb{R}^2)$. For τ small enough, one has $|\mathbf{u}_\varepsilon + \tau \mathbf{v}| \neq 0$ in Ω_ε, hence, we can define:

$$\mathbf{w}(\tau) = \frac{\mathbf{u}_\varepsilon + \tau \mathbf{v}}{|\mathbf{u}_\varepsilon + \tau \mathbf{v}|}.$$

Since $\mathbf{v} \equiv \mathbf{o}$ on $\partial\Omega$, then we have that $\mathbf{w}(\tau) \in \mathbf{V}_\varepsilon$ for τ small enough. Moreover, we have $\mathbf{w}(0) = \mathbf{u}_\varepsilon$ and $\mathbf{w}'(0) = \mathbf{v} - \big(\mathbf{u}_\varepsilon \cdot \mathbf{v}\big)\mathbf{u}_\varepsilon$.

Let

$$i_1(\tau) = \int_{\Omega_\varepsilon} \kappa_1(\mathbf{w}(\tau)) \big(\operatorname{curl} \mathbf{w}(\tau)\big)^2 \, \mathrm{d}\mathbf{x}.$$

Then

$$\frac{1}{\tau}\big(i_1(\tau) - i_1(0)\big) = \int_{\Omega_\varepsilon} \left(\frac{\kappa_1(\mathbf{w}(\tau)) - \kappa_1(\mathbf{w}(0))}{\tau} \right) \big(\operatorname{curl} \mathbf{w}(\tau)\big)^2 \, \mathrm{d}\mathbf{x} +$$
$$+ \int_{\Omega_\varepsilon} \kappa_1(\mathbf{w}(0)) \frac{\big(\operatorname{curl} \mathbf{w}(\tau)\big)^2 - \big(\operatorname{curl} \mathbf{w}(0)\big)^2}{\tau} \, \mathrm{d}\mathbf{x}.$$

Locally, we can write $\mathbf{w}(\tau) = \mathbf{w}(0) + \tau\mathbf{w}'(0) + o(\tau^2)$, hence, as $\tau \to 0$, we have that

$$\frac{\big(\text{curl } \mathbf{w}(\tau)\big)^2 - \big(\text{curl } \mathbf{w}(0)\big)^2}{\tau} \to \big(\text{curl } \mathbf{w}(0)\big)\big(\text{curl } \mathbf{w}'(0)\big).$$

At the same time, since κ_1 is of class C^2, we also have, as $\tau \to 0$, that:

$$\frac{\kappa_1(\mathbf{w}(\tau)) - \kappa_1(\mathbf{w}(0))}{\tau} \to \nabla\kappa_1(\mathbf{w}(0)) \cdot \mathbf{w}'(0)$$

and

$$\big(\text{curl } \mathbf{w}(\tau)\big)^2 \to \big(\text{curl } \mathbf{w}(0)\big)^2.$$

Therefore, we obtain that:

$$i_1'(0) = \int_{\Omega_\varepsilon} \big(\nabla\kappa_1(\mathbf{w}(0)) \cdot \mathbf{w}'(0)\big)\big(\text{curl } \mathbf{w}(0)\big)^2 + \kappa_1(\mathbf{w}(0))\big(\text{curl } \mathbf{w}(0)\big)\big(\text{curl } \mathbf{w}'(0)\big)\ \mathrm{d}\mathbf{x}.$$

Proceeding in the same fashion for all of the other components of $\mathbf{F}_\varepsilon$, we obtain that $\mathbf{u}_\varepsilon$ solves the following equation:

$$\begin{aligned}
0 = &\int_{\Omega_\varepsilon} \big(\nabla\kappa_1(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\big)\big(\text{curl } \mathbf{u}_\varepsilon\big)^2\ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} \big(\nabla\kappa_2(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\big)\big(\text{curl } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{u}_\varepsilon\big)\ \mathrm{d}\mathbf{x} + \\
+ &\int_{\Omega_\varepsilon} \big(\nabla\kappa_3(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\big)\big(\text{div } \mathbf{u}_\varepsilon\big)^2\ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} 2 \cdot \kappa_1(\mathbf{u}_\varepsilon)\big(\text{curl } \mathbf{u}_\varepsilon\big)\big(\text{curl } \mathbf{w}'(0)\big)\ \mathrm{d}\mathbf{x} + \\
+ &\int_{\Omega_\varepsilon} \kappa_2(\mathbf{u}_\varepsilon)\Big(\big(\text{curl } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{w}'(0)\big) + \big(\text{curl } \mathbf{w}'(0)\big)\big(\text{div } \mathbf{u}_\varepsilon\big)\Big)\ \mathrm{d}\mathbf{x} + \\
+ &\int_{\Omega_\varepsilon} 2 \cdot \kappa_3(\mathbf{u}_\varepsilon)\big(\text{div } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{w}'(0)\big)\ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} 2 \cdot \mu \cdot \big(\mathbf{u}_\varepsilon \cdot \overline{\mathbf{u}}\big)\big(\mathbf{w}'(0) \cdot \overline{\mathbf{u}}\big)\ \mathrm{d}\mathbf{x}.
\end{aligned} \tag{4.5.19}$$

At the same time, for τ sufficiently small, we also have that $\mathbf{w}(\tau) \in \mathscr{C}_\varepsilon$, that is, $\mathbf{w}(\tau)$ can be lifted, using $\mathbf{F}_\varepsilon(\mathbf{u}_\varepsilon) \leq \delta$ and Proposition 4.2.4. We know that $\mathbf{u}_\varepsilon = \big(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon\big)$, with $\varphi_\varepsilon \in V_\varepsilon$. Let $\Phi(\tau, \mathbf{v}) \in V_\varepsilon$ such that $\mathbf{w}(\tau) = \big(\cos\Phi(\tau,\mathbf{v}), \sin\Phi(\tau,\mathbf{v})\big)$. For $\tau = 0$, we have $\Phi(0,\mathbf{v}) = \varphi_\varepsilon$. Let

$$\psi_\mathbf{v} := \frac{1}{\tau}\big(\Phi(\tau,\mathbf{v}) - \varphi_\varepsilon\big).$$

Since $\varphi_\varepsilon, \Phi(\tau,\mathbf{v}) \in V_\varepsilon$, then we also have $\psi_\mathbf{v} \in V_\varepsilon$. This implies that we can write

$$\mathbf{w}(\tau) = \big(\cos(\varphi_\varepsilon + \tau\psi_\mathbf{v}), \sin(\varphi_\varepsilon + \tau\psi_\mathbf{v})\big)$$

which implies that

$$\mathbf{w}'(0) = \psi_\mathbf{v}\big(-\sin\varphi_\varepsilon, \cos\varphi_\varepsilon\big) = \psi_\mathbf{v}\mathbf{u}_\varepsilon^\perp.$$

At the same time, we have

$$\mathbf{w}'(0) = \mathbf{v} - (\mathbf{u}_\varepsilon \cdot \mathbf{v})\mathbf{u}_\varepsilon = (\mathbf{v} \cdot \mathbf{u}_\varepsilon^\perp)\mathbf{u}_\varepsilon^\perp.$$

The last two relations show us that actually $\psi_\mathbf{v} = \mathbf{v} \cdot \mathbf{u}_\varepsilon^\perp$.

Let us now deduce the equation for φ_ε, based on (4.5.19). We recall that:

$$\begin{cases} \text{curl } \mathbf{u}_\varepsilon = \cos \varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial x} + \sin \varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial y}, \\ \text{div } \mathbf{u}_\varepsilon = - \sin \varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial x} + \cos \varphi_\varepsilon \dfrac{\partial \varphi_\varepsilon}{\partial y}. \end{cases}$$

The first integral of (4.5.19) becomes:

$$\int_{\Omega_\varepsilon} \left(\nabla\kappa_1(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\right)\left(\text{curl } \mathbf{u}_\varepsilon\right)^2 \, d\mathbf{x} = \int_{\Omega_\varepsilon} \left(\left(\nabla\kappa_1(\mathbf{u}_\varepsilon) \cdot \mathbf{u}_\varepsilon^\perp\right)\left(\text{curl } \mathbf{u}_\varepsilon\right)^2 \right) \psi_\mathbf{v} \, d\mathbf{x}$$
$$= \int_{\Omega_\varepsilon} \left(\left(\nabla\kappa_1(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon) \cdot (-\sin\varphi_\varepsilon, \cos\varphi_\varepsilon)\right)\left(\text{curl}(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon)\right)^2 \right) \psi_\mathbf{v} \, d\mathbf{x}.$$

But since $k_1(t) = \kappa_1(\cos t, \sin t)$, for any $t \in \mathbb{R}$, one obtains that

$$k_1'(t) = \nabla\kappa_1(\cos t, \sin t) \cdot (-\sin t, \cos t),$$

hence:

$$\int_{\Omega_\varepsilon} \left(\nabla\kappa_1(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\right)\left(\text{curl } \mathbf{u}_\varepsilon\right)^2 \, d\mathbf{x} = \int_{\Omega_\varepsilon} \left(k_1'(\varphi_\varepsilon)\left(\text{curl}(\cos\varphi_\varepsilon, \sin\varphi_\varepsilon)\right)^2 \right) \psi_\mathbf{v} \, d\mathbf{x}$$
$$= \int_{\Omega_\varepsilon} k_1'(\varphi_\varepsilon) \left(\cos^2\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2 + 2\cos\varphi_\varepsilon \sin\varphi_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y} + \sin^2\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2 \right) \psi_\mathbf{v} \, d\mathbf{x}. \quad (4.5.20)$$

For the second and third integrals from (4.5.19), we proceed in a similar fashion and obtain:

$$\int_{\Omega_\varepsilon} \left(\nabla\kappa_2(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\right)\left(\text{curl } \mathbf{u}_\varepsilon\right)\left(\text{div } \mathbf{u}_\varepsilon\right) \, d\mathbf{x} =$$
$$= \int_{\Omega_\varepsilon} k_2'(\varphi_\varepsilon) \left(-\cos\varphi_\varepsilon \sin\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2 + \left(\cos^2\varphi_\varepsilon - \sin^2\varphi_\varepsilon\right) \frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y} \right) \psi_\mathbf{v} \, d\mathbf{x} +$$
$$+ \int_{\Omega_\varepsilon} k_2'(\varphi_\varepsilon) \cdot \cos\varphi_\varepsilon \sin\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2 \cdot \psi_\mathbf{v} \, d\mathbf{x} \quad (4.5.21)$$

and

$$\int_{\Omega_\varepsilon} \left(\nabla\kappa_3(\mathbf{u}_\varepsilon) \cdot \mathbf{w}'(0)\right)\left(\text{div } \mathbf{u}_\varepsilon\right)^2 \, d\mathbf{x} =$$
$$= \int_{\Omega_\varepsilon} k_3'(\varphi_\varepsilon) \left(\sin^2\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2 - 2\cos\varphi_\varepsilon \sin\varphi_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y} + \cos^2\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2 \right) \psi_\mathbf{v} \, d\mathbf{x}. \quad (4.5.22)$$

For the next three integrals, we need to compute $\mathrm{curl}(\mathbf{w}'(0))$ and $\mathrm{div}(\mathbf{w}'(0))$. For a generic scalar function f and a vector valued function $\mathbf{u}$, we have:

$$\begin{aligned}\mathrm{curl}(f\mathbf{u}) &= \mathrm{curl}(fu_1, fu_2) = \frac{\partial(fu_2)}{\partial x} - \frac{\partial(fu_1)}{\partial y} = \\ &= f\frac{\partial u_2}{\partial x} + u_2\frac{\partial f}{\partial x} - f\frac{\partial u_1}{\partial y} - u_1\frac{\partial f}{\partial y} \\ &= f\mathrm{curl}\ \mathbf{u} - \nabla f \cdot \mathbf{u}^{\perp},\end{aligned}$$

where $\mathbf{u}^{\perp} = (-u_2, u_1)$, and

$$\begin{aligned}\mathrm{div}(f\mathbf{u}) &= \mathrm{div}(fu_1, fu_2) = \frac{\partial(fu_1)}{\partial x} + \frac{\partial(fu_2)}{\partial y} = \\ &= f\frac{\partial u_1}{\partial x} + u_1\frac{\partial f}{\partial x} + f\frac{\partial u_2}{\partial y} + u_2\frac{\partial f}{\partial y} \\ &= f\mathrm{div}\ \mathbf{u} + \nabla f \cdot \mathbf{u}.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{curl}\ \mathbf{w}'(0) &= \mathrm{curl}(\psi_{\mathbf{v}}\mathbf{u}_{\varepsilon}^{\perp}) = \psi_{\mathbf{v}}\mathrm{curl}\ \mathbf{u}_{\varepsilon}^{\perp} - \nabla\psi_{\mathbf{v}} \cdot \left(\mathbf{u}_{\varepsilon}^{\perp}\right)^{\perp} \\ &= \psi_{\mathbf{v}}\mathrm{div}\ \mathbf{u}_{\varepsilon} + \nabla\psi_{\mathbf{v}} \cdot \mathbf{u}_{\varepsilon},\end{aligned}$$

where we have also used that $\mathrm{curl}\ \mathbf{u}^{\perp} = \mathrm{div}\ \mathbf{u}$ and $\left(\mathbf{u}^{\perp}\right)^{\perp} = -\mathbf{u}$.

Also

$$\begin{aligned}\mathrm{div}\ \mathbf{w}'(0) &= \mathrm{div}(\psi_{\mathbf{v}}\mathbf{u}_{\varepsilon}^{\perp}) = \psi_{\mathbf{v}}\mathrm{div}\ \mathbf{u}_{\varepsilon}^{\perp} + \nabla\psi_{\mathbf{v}} \cdot \mathbf{u}_{\varepsilon}^{\perp} \\ &= -\psi_{\mathbf{v}}\mathrm{curl}\ \mathbf{u}_{\varepsilon} + \nabla\psi_{\mathbf{v}} \cdot \mathbf{u}_{\varepsilon}^{\perp},\end{aligned}$$

where we have used that $\mathrm{div}\ \mathbf{u}^{\perp} = -\mathrm{curl}\ \mathbf{u}$.

This implies that:

$$\begin{aligned}&\int_{\Omega_\varepsilon} 2 \cdot \kappa_1(\mathbf{u}_\varepsilon)\big(\mathrm{curl}\ \mathbf{u}_\varepsilon\big)\big(\mathrm{curl}\ \mathbf{w}'(0)\big)\ \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} 2 \cdot k_1(\varphi_\varepsilon)\left(-\cos\varphi_\varepsilon \sin\varphi_\varepsilon \left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2\right)\psi_{\mathbf{v}}\ \mathrm{d}\mathbf{x} + \\ &\quad + \int_{\Omega_\varepsilon} 2 \cdot k_1(\varphi_\varepsilon)\left(\big(\cos^2\varphi_\varepsilon - \sin^2\varphi_\varepsilon\big)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y} + \cos\varphi_\varepsilon \sin\varphi_\varepsilon\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\psi_{\mathbf{v}}\ \mathrm{d}\mathbf{x} + \\ &\quad + \int_{\Omega_\varepsilon} 2 \cdot k_1(\varphi_\varepsilon)\left(\cos^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial x} + \cos\varphi_\varepsilon \sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x} + \\ &\quad + \int_{\Omega_\varepsilon} 2 \cdot k_1(\varphi_\varepsilon)\left(-\cos\varphi_\varepsilon \sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial x} + \sin^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}. \qquad (4.5.23)\end{aligned}$$

We split the next integral from (4.5.19) into two parts:

$$\int_{\Omega_\varepsilon} \kappa_2(\mathbf{u}_\varepsilon)\Big(\big(\text{curl } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{w}'(0)\big)\Big)\,\mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\big(-\cos^2\varphi_\varepsilon\big)\left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2\cdot\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-2\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y}-\sin^2\varphi_\varepsilon\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\cdot\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+\cos^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-\sin^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}$$

and

$$\int_{\Omega_\varepsilon} \kappa_2(\mathbf{u}_\varepsilon)\Big(\big(\text{curl } \mathbf{w}'(0)\big)\big(\text{div } \mathbf{u}_\varepsilon\big)\Big)\,\mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\cdot\sin^2\varphi_\varepsilon\cdot\left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2\cdot\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-2\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y}+\cos^2\varphi_\varepsilon\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\cdot\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial x}-\sin^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(\cos^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}$$

and by adding the two parts together we get:

$$\int_{\Omega_\varepsilon} \kappa_2(\mathbf{u}_\varepsilon)\Big(\big(\text{curl } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{w}'(0)\big)+\big(\text{curl } \mathbf{w}'(0)\big)\big(\text{div } \mathbf{u}_\varepsilon\big)\Big)\,\mathrm{d}\mathbf{x} =$$
$$=\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(\big(\sin^2\varphi_\varepsilon-\cos^2\varphi_\varepsilon\big)\left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2+\big(\cos^2\varphi_\varepsilon-\sin^2\varphi_\varepsilon\big)\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-4\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y}\right)\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(-2\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+\big(\cos^2\varphi_\varepsilon-\sin^2\varphi_\varepsilon\big)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} k_2(\varphi_\varepsilon)\left(\big(\cos^2\varphi_\varepsilon-\sin^2\varphi_\varepsilon\big)\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+2\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}. \tag{4.5.24}$$

For the term containing κ_3, we have:

$$\int_{\Omega_\varepsilon} 2\cdot\kappa_3(\mathbf{u}_\varepsilon)\big(\text{div } \mathbf{u}_\varepsilon\big)\big(\text{div } \mathbf{w}'(0)\big)\,\mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} 2\cdot k_3(\varphi_\varepsilon)\left(\cos\varphi_\varepsilon\sin\varphi_\varepsilon\left(\frac{\partial\varphi_\varepsilon}{\partial x}\right)^2\right)\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} 2\cdot k_3(\varphi_\varepsilon)\left(-\big(\cos^2\varphi_\varepsilon-\sin^2\varphi_\varepsilon\big)\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\varphi_\varepsilon}{\partial y}-\cos\varphi_\varepsilon\sin\varphi_\varepsilon\left(\frac{\partial\varphi_\varepsilon}{\partial y}\right)^2\right)\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} 2\cdot k_3(\varphi_\varepsilon)\left(\sin^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial x}-\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}+$$
$$+\int_{\Omega_\varepsilon} 2\cdot k_3(\varphi_\varepsilon)\left(-\cos\varphi_\varepsilon\sin\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial x}+\cos^2\varphi_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\frac{\partial\psi_{\mathbf{v}}}{\partial y}\right)\mathrm{d}\mathbf{x}. \tag{4.5.25}$$

For the last integral from (4.5.19), we recall that $\overline{\mathbf{u}} = (\cos\overline{\varphi}, \sin\overline{\varphi})$, hence:

$$\begin{aligned}\int_{\Omega_\varepsilon} 2\cdot\mu\cdot\left(\mathbf{u}_\varepsilon\cdot\overline{\mathbf{u}}\right)\left(\mathbf{w}'(0)\cdot\overline{\mathbf{u}}\right)\,\mathrm{d}\mathbf{x} = \\ &= \int_{\Omega_\varepsilon} 2\cdot\mu\cdot\left(\mathbf{u}_\varepsilon\cdot\overline{\mathbf{u}}\right)\left(\mathbf{u}_\varepsilon^\perp\cdot\overline{\mathbf{u}}\right)\psi_{\mathbf{v}};\mathrm{d}\mathbf{x} \\ &= \int_{\Omega_\varepsilon} -2\cdot\mu\cdot\cos(\varphi_\varepsilon-\overline{\varphi})\sin(\varphi_\varepsilon-\overline{\varphi})\cdot\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x} \\ &= \int_{\Omega_\varepsilon} -\mu\sin\left(2(\varphi_\varepsilon-\overline{\varphi})\right)\psi_{\mathbf{v}}\,\mathrm{d}\mathbf{x},\end{aligned}$$

which coincides with the last term from (4.3.3).

We want now to prove that Equations (4.5.20) to (4.5.25) generate (4.3.3). We only identify the coefficients of $\left(\dfrac{\partial\varphi_\varepsilon}{\partial x}\right)^2$ and $\dfrac{\partial\varphi_\varepsilon}{\partial x}\dfrac{\partial\psi_{\mathbf{v}}}{\partial x}$, since for all the others we can proceed in the same way. For $\left(\dfrac{\partial\varphi_\varepsilon}{\partial x}\right)^2$, the coefficient is generated by Equations (4.5.20) to (4.5.25) and it is:

$$\begin{aligned}&k_1'(\varphi_\varepsilon)\cos^2\varphi_\varepsilon - k_2'(\varphi_\varepsilon)\cos\varphi_\varepsilon\sin\varphi_\varepsilon + k_3'(\varphi_\varepsilon)\sin^2\varphi_\varepsilon - \\ &-2k_1(\varphi_\varepsilon)\cos\varphi_\varepsilon\sin\varphi_\varepsilon + k_2(\varphi_\varepsilon)\left(-\cos^2\varphi_\varepsilon+\sin^2\varphi_\varepsilon\right) + 2k_3(\varphi_\varepsilon)\cos\varphi_\varepsilon\sin\varphi_\varepsilon,\end{aligned}$$

which is exactly $a'(\varphi_\varepsilon)$, by Remark 4.3.3. The coefficient of $\dfrac{\partial\varphi_\varepsilon}{\partial x}\dfrac{\partial\psi_{\mathbf{v}}}{\partial x}$ is generated only by Equations (4.5.23) to (4.5.25) and it is:

$$2k_1(\varphi_\varepsilon)\cos^2\varphi_\varepsilon - 2k_2(\varphi_\varepsilon)\cos\varphi_\varepsilon\sin\varphi_\varepsilon + 2k_3(\varphi_\varepsilon)\sin^2\varphi_\varepsilon,$$

which is exactly $2a(\varphi_\varepsilon)$, by Remark 4.3.3. All the other terms can be identified in a similar fashion, hence (4.3.3) is recovered, from which we conclude that if $\mathbf{u}_\varepsilon$ is a critical point of $\mathbf{F}_\varepsilon$, then φ_ε is a critical point of F_ε.

5

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