MOTION OF A RIGID BODY IN A COMPRESSIBLE FLUID WITH NAVIER-SLIP BOUNDARY CONDITION

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ABSTRACT. In this work, we study the motion of a rigid body in a bounded domain which is filled with a compressible isentropic fluid. We consider the Navier-slip boundary condition at the interface as well as at the boundary of the domain. This is the first mathematical analysis of a compressible fluid-rigid body system where Navier-slip boundary conditions are considered. We prove existence of a weak solution of the fluid-structure system up to collision.

CONTENTS

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain occupied by a fluid and a rigid body. Let the rigid body $\mathcal{S}(t)$ be a regular, bounded domain and moving inside Ω. The motion of the rigid body is governed by the balance equations for linear and angular momentum. We assume that the fluid domain $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$ is filled with a viscous isentropic compressible fluid. We also assume the Navier-slip boundary conditions at the interface of the interaction of the fluid and the rigid body as well as at $\partial\Omega$. The fluid occupies, at $t = 0$, the domain $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$, where the initial position of the rigid body is $\mathcal{S}_0$. In equations (1.4)–(1.9), $\nu(t, x)$ is the unit normal to $\partial\mathcal{S}(t)$ at the point $x \in \partial\mathcal{S}(t)$, directed to the interior of the body. In (1.3) and (1.4)–(1.5), $g_\mathcal{F}$ and $g_\mathcal{S}$ are the specific body forces. Moreover, $\alpha > 0$ is a coefficient of friction. Here, the notation $u \otimes v$ is the tensor product of two vectors $u, v \in \mathbb{R}^3$ and it is defined as $u \otimes v = (u_i v_j)_{1 \leqslant i, j \leqslant 3}$. In the equations, $\rho_\mathcal{F}$ and $u_\mathcal{F}$ represent respectively the mass density and the velocity of the fluid, and the pressure of the fluid is denoted by $p_\mathcal{F}$.

We assume that the flow is in the barotropic regime and we focus on the isentropic case where the relation between $p_\mathcal{F}$ and $\rho_\mathcal{F}$ is given by the constitutive law:

$$p_\mathcal{F} = a_\mathcal{F} \rho_\mathcal{F}^\gamma, \tag{1.1}$$

with $a_\mathcal{F} > 0$ and the adiabatic constant $\gamma > \frac{3}{2}$, which is a necessary assumption for the existence of a weak solution of compressible fluids (see for example [9]).

As it is common, we set

$$\mathbb{T}(u_\mathcal{F}) = 2\mu_\mathcal{F}\mathbb{D}(u_\mathcal{F}) + \lambda_\mathcal{F}\operatorname{div} u_\mathcal{F}\mathbb{I},$$

where $\mathbb{D}(u_\mathcal{F}) = \frac{1}{2}\left(\nabla u_\mathcal{F} + \nabla u_\mathcal{F}^\top\right)$ denotes the symmetric part of the velocity gradient, $\nabla u_\mathcal{F}^\top$ is the transpose of $\nabla u_\mathcal{F}$, $\lambda_\mathcal{F}$ and $\mu_\mathcal{F}$ are the viscosity coefficients satisfying

$$\mu_\mathcal{F} > 0, \quad 3\lambda_\mathcal{F} + 2\mu_\mathcal{F} \geqslant 0.$$

The evolution of this fluid-structure system can be described by the following equations

$$\frac{\partial\rho_\mathcal{F}}{\partial t} + \operatorname{div}(\rho_\mathcal{F} u_\mathcal{F}) = 0, \quad t \in (0,T),\ x \in \mathcal{F}(t), \qquad (1.2)$$

$$\frac{\partial(\rho_\mathcal{F} u_\mathcal{F})}{\partial t} + \operatorname{div}(\rho_\mathcal{F} u_\mathcal{F} \otimes u_\mathcal{F}) - \operatorname{div}\mathbb{T}(u_\mathcal{F}) + \nabla p_\mathcal{F} = \rho_\mathcal{F} g_\mathcal{F}, \quad t \in (0,T),\ x \in \mathcal{F}(t), \qquad (1.3)$$

$$mh''(t) = -\int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_\mathcal{F}) - p_\mathcal{F}\mathbb{I})\nu\, d\Gamma + \int_{\mathcal{S}(t)} \rho_\mathcal{S} g_\mathcal{S}\, dx, \quad \text{in } (0,T), \qquad (1.4)$$

$$(J\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \times (\mathbb{T}(u_\mathcal{F}) - p_\mathcal{F}\mathbb{I})\nu\, d\Gamma + \int_{\mathcal{S}(t)} (x - h(t)) \times \rho_\mathcal{S} g_\mathcal{S}\, dx, \quad \text{in } (0,T), \qquad (1.5)$$

the boundary conditions

$$u_\mathcal{F} \cdot \nu = u_\mathcal{S} \cdot \nu, \quad \text{for } t \in (0,T),\ x \in \partial\mathcal{S}(t), \qquad (1.6)$$

$$(\mathbb{T}(u_\mathcal{F})\nu) \times \nu = -\alpha(u_\mathcal{F} - u_\mathcal{S}) \times \nu, \quad \text{for } t \in (0,T),\ x \in \partial\mathcal{S}(t), \qquad (1.7)$$

$$u_\mathcal{F} \cdot \nu = 0, \quad \text{on } (t,x) \in (0,T) \times \partial\Omega, \qquad (1.8)$$

$$(\mathbb{T}(u_\mathcal{F})\nu) \times \nu = -\alpha(u_\mathcal{F} \times \nu), \quad \text{on } (t,x) \in (0,T) \times \partial\Omega, \qquad (1.9)$$

and the initial conditions

$$\rho_\mathcal{F}(0,x) = \rho_{\mathcal{F}_0}(x), \quad (\rho_\mathcal{F} u_\mathcal{F})(0,x) = q_{\mathcal{F}_0}(x), \quad \forall\, x \in \mathcal{F}_0, \qquad (1.10)$$

$$h(0) = 0, \quad h'(0) = \ell_0, \quad \omega(0) = \omega_0. \qquad (1.11)$$

The Eulerian velocity $u_\mathcal{S}(t,x)$ at each point $x \in \mathcal{S}(t)$ of the rigid body is given by

$$u_\mathcal{S}(t,x) = h'(t) + \omega(t) \times (x - h(t)), \qquad (1.12)$$

where $h(t)$ is the position of the centre of mass and $h'(t)$, $\omega(t)$ are the translational and angular velocities of the rigid body.

The solid domain at time t is given by

$$\mathcal{S}(t) = \{h(t) + \mathbb{O}(t)x \mid x \in \mathcal{S}_0\},$$

where $\mathbb{O}(t) \in SO(3)$ is associated to the rotation of the rigid body:

$$\mathbb{O}'(t)\mathbb{O}^\top(t)x = \omega(t) \times x\ \forall\, x \in \mathbb{R}^3, \quad \mathbb{O}(0) = \mathbb{I}.$$

Observe that $\mathbb{O}'\mathbb{O}^\top$ is skew-symmetric as $\mathbb{O}\mathbb{O}^\top = \mathbb{I}$. The initial velocity of the rigid body is given by

$$u_\mathcal{S}(0,x) = u_{\mathcal{S}_0} := \ell_0 + \omega_0 \times x, \quad x \in \mathcal{S}_0. \qquad (1.13)$$

Here the mass density $\rho_\mathcal{S}$ of the body satisfies the following transport equation

$$\frac{\partial\rho_\mathcal{S}}{\partial t} + u_\mathcal{S} \cdot \nabla\rho_\mathcal{S} = 0, \quad t \in (0,T),\ x \in \mathcal{S}(t), \quad \rho_\mathcal{S}(0,x) = \rho_{\mathcal{S}_0}(x), \quad \forall\, x \in \mathcal{S}_0. \qquad (1.14)$$

Moreover, m is the mass of the solid and $J(t)$ is the moment of inertia tensor of the solid calculated with respect to $h(t)$. We express $h(t)$, m and $J(t)$ in the following way:

$$m = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \, dx, \tag{1.15}$$

$$h(t) = \frac{1}{m} \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \, x \, dx, \tag{1.16}$$

$$J(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \left[|x - h(t)|^2 \mathbb{I} - (x - h(t)) \otimes (x - h(t)) \right] \, dx. \tag{1.17}$$

In the remainder of this introduction, we present the weak formulation of the system, discuss our main result regarding the existence of weak solutions and put it in a larger perspective.

1.1. **Weak formulation.** We derive a weak formulation with the help of multiplication by appropriate test functions and integration by parts along with the application of the boundary conditions. Due to the presence of the Navier-slip boundary condition, the test functions will be discontinuous across the fluid-solid interface. We introduce the set of rigid velocity fields:

$$\mathcal{R}(\Omega) = \left\{ \zeta : \Omega \to \mathbb{R}^3 \mid \text{There exist } V, r, a \in \mathbb{R}^3 \text{ such that } \zeta(x) = V + r \times (x - a) \text{ for any } x \in \Omega \right\}. \tag{1.18}$$

For any $T > 0$, we define the test function space V_T as follows:

$$V_T = \left\{ \begin{array}{l} \phi \in C([0,T]; L^2(\Omega)) \text{ such that there exist } \phi_{\mathcal{F}} \in \mathcal{D}([0,T); \mathcal{D}(\overline{\Omega})), \; \phi_{\mathcal{S}} \in \mathcal{D}([0,T); \mathcal{R}(\Omega)) \\ \text{satisfying } \phi(t, \cdot) = \phi_{\mathcal{F}}(t, \cdot) \text{ on } \mathcal{F}(t), \quad \phi(t, \cdot) = \phi_{\mathcal{S}}(t, \cdot) \text{ on } \mathcal{S}(t) \text{ with} \\ \phi_{\mathcal{F}}(t, \cdot) \cdot \nu = \phi_{\mathcal{S}}(t, \cdot) \cdot \nu \text{ on } \partial\mathcal{S}(t), \; \phi_{\mathcal{F}}(t, \cdot) \cdot \nu = 0 \text{ on } \partial\Omega \text{ for all } t \in [0,T] \end{array} \right\}, \tag{1.19}$$

where $\mathcal{D}$ denotes the set of all infinitely differentiable functions that have compact support. We multiply equation (1.3) by a test function $\phi \in V_T$ and integrate over $\mathcal{F}(t)$ to obtain

$$\frac{d}{dt} \int_{\mathcal{F}(t)} \rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \phi_{\mathcal{F}} - \int_{\mathcal{F}(t)} \rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \frac{\partial}{\partial t} \phi_{\mathcal{F}} - \int_{\mathcal{F}(t)} (\rho_{\mathcal{F}} u_{\mathcal{F}} \otimes u_{\mathcal{F}}) : \nabla \phi_{\mathcal{F}} + \int_{\mathcal{F}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) : \mathbb{D}(\phi_{\mathcal{F}})$$

$$= \int_{\partial\Omega} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{F}} + \int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{F}} + \int_{\mathcal{F}(t)} \rho_{\mathcal{F}} g_{\mathcal{F}} \cdot \phi_{\mathcal{F}}. \tag{1.20}$$

We use the identity $(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (B \cdot C)(A \cdot D)$ to have

$$\mathbb{T}(u_{\mathcal{F}}) \nu \cdot \phi_{\mathcal{F}} = [\mathbb{T}(u_{\mathcal{F}}) \nu \cdot \nu](\phi_{\mathcal{F}} \cdot \nu) + [\mathbb{T}(u_{\mathcal{F}}) \nu \times \nu] \cdot (\phi_{\mathcal{F}} \times \nu),$$

$$\mathbb{T}(u_{\mathcal{F}}) \nu \cdot \phi_{\mathcal{S}} = [\mathbb{T}(u_{\mathcal{F}}) \nu \cdot \nu](\phi_{\mathcal{S}} \cdot \nu) + [\mathbb{T}(u_{\mathcal{F}}) \nu \times \nu] \cdot (\phi_{\mathcal{S}} \times \nu).$$

Now by using the definition of V_T and the boundary conditions (1.6)–(1.9), we get

$$\int_{\partial\Omega} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{F}} = -\alpha \int_{\partial\Omega} (u_{\mathcal{F}} \times \nu) \cdot (\phi_{\mathcal{F}} \times \nu), \tag{1.21}$$

$$\int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{F}} = -\alpha \int_{\partial\mathcal{S}(t)} [(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu] \cdot [(\phi_{\mathcal{F}} - \phi_{\mathcal{S}}) \times \nu] + \int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{S}}. \tag{1.22}$$

Using the rigid body equations (1.4)–(1.5), equation (1.14) and Reynolds' transport theorem, we obtain

$$\int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) \nu \cdot \phi_{\mathcal{S}} = -\frac{d}{dt} \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \phi_{\mathcal{S}} + \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \frac{\partial}{\partial t} \phi_{\mathcal{S}} + \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} g_{\mathcal{S}} \cdot \phi_{\mathcal{S}}. \tag{1.23}$$

Thus by combining the above relations (1.20)–(1.23) and then integrating from 0 to T, we have

$$
-\int\limits_0^T \int\limits_{\mathcal{F}(t)} \rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \frac{\partial}{\partial t} \phi_{\mathcal{F}} - \int\limits_0^T \int\limits_{\mathcal{S}(t)} \rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \frac{\partial}{\partial t} \phi_{\mathcal{S}} - \int\limits_0^T \int\limits_{\mathcal{F}(t)} (\rho_{\mathcal{F}} u_{\mathcal{F}} \otimes u_{\mathcal{F}}) : \nabla \phi_{\mathcal{F}} + \int\limits_0^T \int\limits_{\mathcal{F}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I}) : \mathbb{D}(\phi_{\mathcal{F}})
$$

$$
+ \alpha \int\limits_0^T \int\limits_{\partial \Omega} (u_{\mathcal{F}} \times \nu) \cdot (\phi_{\mathcal{F}} \times \nu) + \alpha \int\limits_0^T \int\limits_{\partial \mathcal{S}(t)} [(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu] \cdot [(\phi_{\mathcal{F}} - \phi_{\mathcal{S}}) \times \nu]
$$

$$
= \int\limits_0^T \int\limits_{\mathcal{F}(t)} \rho_{\mathcal{F}} g_{\mathcal{F}} \cdot \phi_{\mathcal{F}} + \int\limits_0^T \int\limits_{\mathcal{S}(t)} \rho_{\mathcal{S}} g_{\mathcal{S}} \cdot \phi_{\mathcal{S}} + \int\limits_{\mathcal{F}(0)} (\rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \phi_{\mathcal{F}})(0) + \int\limits_{\mathcal{S}(0)} (\rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \phi_{\mathcal{S}})(0). \quad (1.24)
$$

Definition 1.1. *Let $T > 0$, and let Ω and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. A triplet $(\mathcal{S}, \rho, u)$ is a bounded energy weak solution to system (1.2)–(1.11) if the following holds:*

- *$\mathcal{S}(t) \Subset \Omega$ is a bounded domain of $\mathbb{R}^3$ for all $t \in [0, T)$ such that*

$$
\chi_{\mathcal{S}}(t, x) := \mathbb{1}_{\mathcal{S}(t)}(x) \in L^\infty((0, T) \times \Omega). \quad (1.25)
$$

- *u belongs to the following space*

$$
U_T = \left\{ \begin{array}{l} u \in L^2(0, T; L^2(\Omega)) \text{ such that there exist } u_{\mathcal{F}} \in L^2(0, T; H^1(\Omega)), \ u_{\mathcal{S}} \in L^2(0, T; \mathcal{R}) \\ \text{satisfying } u(t, \cdot) = u_{\mathcal{F}}(t, \cdot) \text{ on } \mathcal{F}(t), \quad u(t, \cdot) = u_{\mathcal{S}}(t, \cdot) \text{ on } \mathcal{S}(t) \text{ with} \\ u_{\mathcal{F}}(t, \cdot) \cdot \nu = u_{\mathcal{S}}(t, \cdot) \cdot \nu \text{ on } \partial \mathcal{S}(t), \ u_{\mathcal{F}} \cdot \nu = 0 \text{ on } \partial \Omega \text{ for a.e } t \in [0, T] \end{array} \right\}.
$$

- *$\rho \geqslant 0$, $\rho \in L^\infty(0, T; L^\gamma(\Omega))$ with $\gamma > 3/2$, $\rho |u|^2 \in L^\infty(0, T; L^1(\Omega))$, where*

$$
\rho = (1 - \mathbb{1}_{\mathcal{S}}) \rho_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}} \rho_{\mathcal{S}}, \quad u = (1 - \mathbb{1}_{\mathcal{S}}) u_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}} u_{\mathcal{S}}.
$$

- *The continuity equation is satisfied in the weak sense, i.e.*

$$
\frac{\partial \rho_{\mathcal{F}}}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}}) = 0 \text{ in } \mathcal{D}'([0, T) \times \Omega), \quad \rho_{\mathcal{F}}(0, x) = \rho_{\mathcal{F}_0}(x), \ x \in \Omega. \quad (1.26)
$$

  Also, a renormalized continuity equation holds in a weak sense, i.e.

$$
\partial_t b(\rho_{\mathcal{F}}) + \operatorname{div}(b(\rho_{\mathcal{F}}) u_{\mathcal{F}}) + (b'(\rho_{\mathcal{F}}) - b(\rho_{\mathcal{F}})) \operatorname{div} u_{\mathcal{F}} = 0 \text{ in } \mathcal{D}'([0, T) \times \Omega), \quad (1.27)
$$

  for any $b \in C([0, \infty)) \cap C^1((0, \infty))$ satisfying

$$
|b'(z)| \leqslant c z^{-\kappa_0}, \ z \in (0, 1], \ \kappa_0 < 1, \qquad |b'(z)| \leqslant c z^{\kappa_1}, \ z \geqslant 1, \ -1 < \kappa_1 < \infty. \quad (1.28)
$$

- *The transport of $\mathcal{S}$ by the rigid vector field $u_{\mathcal{S}}$ holds (in the weak sense)*

$$
\frac{\partial \chi_{\mathcal{S}}}{\partial t} + \operatorname{div}(u_{\mathcal{S}} \chi_{\mathcal{S}}) = 0 \text{ in } (0, T) \times \Omega, \quad \chi_{\mathcal{S}}(0, x) = \mathbb{1}_{\mathcal{S}_0}(x), \ x \in \Omega. \quad (1.29)
$$

- *The density $\rho_{\mathcal{S}}$ of the rigid body $\mathcal{S}$ satisfies (in the weak sense)*

$$
\frac{\partial \rho_{\mathcal{S}}}{\partial t} + \operatorname{div}(u_{\mathcal{S}} \rho_{\mathcal{S}}) = 0 \text{ in } (0, T) \times \Omega, \quad \rho_{\mathcal{S}}(0, x) = \rho_{\mathcal{S}_0}(x), \ x \in \Omega. \quad (1.30)
$$

- *Balance of linear momentum holds in a weak sense, i.e. for all $\phi \in V_T$ the relation (1.24) holds.*
- *The following energy inequality holds for almost every $t \in (0, T)$:*

$$
E(t) + \int\limits_0^t \int\limits_{\mathcal{F}(\tau)} \left(2\mu_{\mathcal{F}} |\mathbb{D}(u_{\mathcal{F}})|^2 + \lambda_{\mathcal{F}} |\operatorname{div} u_{\mathcal{F}}|^2 \right) + \alpha \int\limits_0^t \int\limits_{\partial \Omega} |u_{\mathcal{F}} \times \nu|^2
$$

$$
+ \alpha \int\limits_0^t \int\limits_{\partial \mathcal{S}(\tau)} |(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu|^2 \leqslant \int\limits_0^t \int\limits_{\mathcal{F}(\tau)} \rho_{\mathcal{F}} g_{\mathcal{F}} \cdot u_{\mathcal{F}} + \int\limits_0^t \int\limits_{\mathcal{S}(\tau)} \rho_{\mathcal{S}} g_{\mathcal{S}} \cdot u_{\mathcal{S}} + E_0. \quad (1.31)
$$

where $E(t)$ and E_0 are given by

$$E(t) = \int\limits_{\mathcal{F}(t)} \frac{1}{2}\rho_{\mathcal{F}}|u_{\mathcal{F}}|^2 + \int\limits_{\mathcal{S}(t)} \frac{1}{2}\rho_{\mathcal{S}}|u_{\mathcal{S}}|^2 + \int\limits_{\mathcal{F}(t)} \frac{a_{\mathcal{F}}}{\gamma-1}\rho_{\mathcal{F}}^{\gamma}, \quad E_0 = \int\limits_{\mathcal{F}_0} \frac{1}{2}\frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}} + \int\limits_{\mathcal{S}_0} \frac{1}{2}\rho_{\mathcal{S}_0}|u_{\mathcal{S}_0}|^2 + \int\limits_{\mathcal{F}_0} \frac{a_{\mathcal{F}}}{\gamma-1}\rho_{\mathcal{F}_0}^{\gamma} \qquad (1.32)$$

Remark 1.2. *We stress that in the definition of the set U_T (in Definition 1.1) the function $u_{\mathcal{F}}$ on Ω is a regular extension of the velocity field $u_{\mathcal{F}}$ from $\mathcal{F}(t)$ to Ω, see (5.10)–(5.11). Correspondingly, $u_{\mathcal{S}} \in \mathcal{R}$ denotes a rigid extension from $\mathcal{S}(t)$ to Ω as in (1.12). Moreover, by the density $\rho_{\mathcal{F}}$ in (1.26), we mean an extended fluid density $\rho_{\mathcal{F}}$ from $\mathcal{F}(t)$ to Ω by zero, see (5.16)–(5.17). Correspondingly, $\rho_{\mathcal{S}}$ refers to an extended solid density from $\mathcal{S}(t)$ to Ω by zero.*

Remark 1.3. *In (1.26), the initial fluid density $\rho_{\mathcal{F}_0}$ on Ω represents a zero extension of $\rho_{\mathcal{F}_0}$ (defined in (1.10)) from $\mathcal{F}_0$ to Ω. Correspondingly, $\rho_{\mathcal{S}_0}$ in equation (1.30) stands for an extended initial solid density (defined in (1.14)) from $\mathcal{S}_0$ to Ω by zero. Obviously, $q_{\mathcal{F}_0}$ refers to an extended initial momentum from $\mathcal{F}_0$ to Ω by zero and $u_{\mathcal{S}_0} \in \mathcal{R}$ denotes a rigid extension from $\mathcal{S}_0$ to Ω as in (1.13).*

Remark 1.4. *We note that our continuity equation (1.26) is different from the corresponding one in [7]. We have to work with $u_{\mathcal{F}}$ instead of u because of the Navier boundary condition. The reason is that we need the $H^1(\Omega)$ regularity of the velocity in order to achieve the validity of the continuity equation in Ω. Observe that $u \in L^2(0,T;L^2(\Omega))$ but the extended fluid velocity has better regularity, in particular, $u_{\mathcal{F}} \in L^2(0,T;H^1(\Omega))$, see (5.10)–(5.11).*

Remark 1.5. *In the weak formulation (1.24), we need to distinguish between the fluid velocity $u_{\mathcal{F}}$ and the solid velocity $u_{\mathcal{S}}$. Due to the presence of the discontinuities in the tangential components of u and ϕ, neither $\partial_t\phi$ nor $\mathbb{D}(u)$, $\mathbb{D}(\phi)$ belong to $L^2(\Omega)$. That's why it is not possible to write (1.24) in a global and condensed form (i.e. integrals over Ω).*

Remark 1.6. *Let us mention that in the whole paper we assume the regularity of domains Ω and $\mathcal{S}_0$ as $C^{2+\kappa}$, $\kappa > 0$. However, we expect that our assumption on the regularity of the domain can be relaxed to a less regular domain like in the work of Kukučka [26].*

1.2. **Discussion and main result.** The mathematical analysis of systems describing the motion of a rigid body in a viscous *incompressible* fluid is nowadays well developed. The proof of existence of weak solutions until a first collision can be found in several papers, see [3, 4, 18, 24, 33]. Later, the possibility of collisions in the case of a weak solution was included, see [8, 32]. Moreover, it was shown that under Dirichlet boundary conditions collisions cannot occur, which is paradoxical with respect to real situations; for details see [20, 22, 23]. Neustupa and Penel showed that under a prescribed motion of the rigid body and under Navier-type of boundary conditions collision can occur [29]. After that Gérard-Varet and Hillairet showed that to construct collisions one needs to assume less regularity of the domain or different boundary conditions, see e.g. [15, 16, 17]. In the case of very high viscosity, under the assumption that rigid bodies are not touching each other or not touching the boundary at the initial time, it was shown that collisions cannot occur in finite time, see [10]. For an introduction we refer to the problem of a fluid coupled with a rigid body in the work by Galdi, see [13]. Let us also mention results on strong solutions, see e.g. [14, 34, 35].

A few results are available on the motion of a rigid structure in a *compressible* fluid with Dirichlet boundary conditions. The existence of strong solutions in the L^2-framework for small data up to a collision was shown in [1, 31]. The existence of strong solutions in the L^p setting based on $\mathcal{R}$-bounded operators was applied in the barotropic case [21] and in the full system [19].

The existence of a weak solution, also up to a collision but without smallness assumptions, was shown in [5]. Generalization of this result allowing collisions was given in [7]. The weak-strong uniqueness of a compressible fluid with a rigid body can be found in [25]. Existence of weak solutions in the case of Navier boundary conditions is not available yet; it is the topic of this article.

For many years, the *no-slip boundary condition* has been the most widely used given its success in reproducing the standard velocity profiles for incompressible/compressible viscous fluids. Although the no-slip hypothesis seems to be in good agreement with experiments, it leads to certain rather surprising conclusions. As we have already mentioned, the most striking one being the absence of collisions of rigid objects immersed in a linearly viscous fluid [20, 22].

The so-called *Navier boundary conditions*, which allow for slip, offer more freedom and are likely to provide a physically acceptable solution at least to some of the paradoxical phenomena resulting from the no-slip boundary condition, see, e.g. Moffat [28]. Mathematically, the behavior of the tangential component $[u]_{tan}$ is a delicate issue.

The main result of our paper (Theorem 1.7) asserts the local-in-time existence of a weak solution for the system involving the motion of a rigid body in a compressible fluid in the case of Navier boundary conditions at the interface with the solid and at the outer boundary. It is the first result in the context of rigid body-compressible fluid interaction in the case of Navier type of boundary conditions. Let us mention that the main difficulty which arises in our problem is the jump in the velocity through the interface boundary between the rigid body and the compressible fluid. This difficulty cannot be resolved by the approach introduced in the work of Desjardins, Esteban [5], or Feireisl [7] since they consider the velocity field continuous through the interface. Moreover, the techniques in the works by Gérard-Varet, Hillairet [16] and Chemetov, Nečasová [2] cannot be used directly as they are in the incompressible framework. Our weak solutions have to satisfy the jump of the velocity field through the interface boundary.

Our idea is to introduce a novel approximate scheme which combines the theory of compressible fluids introduced by P. L. Lions [27] and then developed by Feireisl [9] to get the strong convergence of the density (renormalized continuity equations, effective viscous flux, artificial pressure) together with ideas from Gérard-Varet, Hillairet [16] and Chemetov, Nečasová [2] concerning a penalization of the jump. We remark that such type of difficulties do not arise for the existence of weak solutions of compressible fluids without a rigid body neither for Dirichlet nor for Navier type of boundary conditions.

We emphasize the main issues that arise in the analysis of our system and the novel methodology that we adapt to deal with it:

- It is not possible to define a uniform velocity field as in [5, 7] due to the presence of a discontinuity through the interface of interaction. This is the reason why we introduce the regularized fluid velocity $u_{\mathcal{F}}$ and the solid velocity $u_{\mathcal{S}}$ and why we treat them separately.
- We introduce approximate problems and recover the original problem as a limit of the approximate ones. In fact, we consider several levels of approximations; in each level we ensure that our solution and the test function do not show a jump across the interface so that we can use several available techniques of compressible fluids (without body). In the limit, however, the discontinuity at the interface is recovered. The particular construction of the test functions is a delicate and crucial issue in our proof of Proposition 5.1.
- Recovering the velocity fields for the solid and fluid parts separately is also a challenging issue. We introduce a penalization in such a way that, in the last stage of the limiting process, this term allows us to recover the rigid velocity of the solid, see (5.8)–(5.9). The introduction of an appropriate extension operator helps us to recover the fluid velocity, see (5.12)–(5.13).
- Since we consider the compressible case, our penalization with parameter $\delta > 0$, see (2.3), is different from the penalization for the incompressible fluid in [16].
- Due to the Navier-slip boundary condition, no H^1 bound on the velocity on the whole domain is possible. We can only obtain the H^1 regularity of the extended velocities of the fluid and solid parts separately. We have introduced an artificial viscosity that vanishes asymptotically on the solid part so that we can capture the H^1 regularity for the fluid part (see step 1 of the proof of Theorem 1.7 in Section 5).
- We have already mentioned that the main difference with [16] is that we consider compressible fluid whereas they have considered an incompressible fluid. We have encountered several issues that are present due to the compressible nature of the fluid (vanishing viscosity in the continuity equation, recovering the renormalized continuity equation, identification of the pressure). One important point is to see that passing to the limit as δ tends to zero in the transport for the rigid body is not obvious because our velocity field does not have regularity $L^\infty(0, T, L^2(\Omega))$ as in the incompressible case see e.g. [16] but $L^2(0, T, L^2(\Omega))$ (here we have $\sqrt{\rho}u \in L^\infty(0, T, L^2(\Omega))$ only). To handle this problem, we apply Proposition 3.5 in the δ-level, see Section 5.

Next we present the main result of our paper.

Theorem 1.7. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$*

$$\mathrm{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma.$$

Let $g_{\mathcal{F}}$, $g_{\mathcal{S}} \in L^{\infty}((0,T) \times \Omega)$ and the pressure $p_{\mathcal{F}}$ be determined by (1.1) *with $\gamma > 3/2$. Assume that the initial data (defined in the sense of Remark 1.3) satisfy*

$$\rho_{\mathcal{F}_0} \in L^{\gamma}(\Omega), \quad \rho_{\mathcal{F}_0} \geqslant 0 \ a.e. \ in \ \Omega, \quad \rho_{\mathcal{S}_0} \in L^{\infty}(\Omega), \quad \rho_{\mathcal{S}_0} > 0 \ a.e. \ in \ \mathcal{S}_0, \tag{1.33}$$

$$q_{\mathcal{F}_0} \in L^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad q_{\mathcal{F}_0} \mathbb{1}_{\{\rho_{\mathcal{F}_0}=0\}} = 0 \ a.e. \ in \ \Omega, \quad \frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}} \mathbb{1}_{\{\rho_{\mathcal{F}_0}>0\}} \in L^1(\Omega), \tag{1.34}$$

$$u_{\mathcal{S}_0} = \ell_0 + \omega_0 \times x \quad \forall \ x \in \Omega \ with \ \ell_0, \ \omega_0 \in \mathbb{R}^3. \tag{1.35}$$

Then there exists $T > 0$ (depending only on $\rho_{\mathcal{F}_0}$, $\rho_{\mathcal{S}_0}$, $q_{\mathcal{F}_0}$, $u_{\mathcal{S}_0}$, $g_{\mathcal{F}}$, $g_{\mathcal{S}}$, dist$(\mathcal{S}_0, \partial\Omega)$) such that a finite energy weak solution to (1.2)–(1.11) *exists on $[0, T]$. Moreover,*

$$\mathcal{S}(t) \Subset \Omega, \quad \text{dist}(\mathcal{S}(t), \partial\Omega) \geqslant \frac{3\sigma}{2}, \quad \forall \ t \in [0, T].$$

Remark 1.8. *We want to mention that in the absence of rigid body, the existence of at least one bounded energy weak solution for compressible fluid with Navier-slip on the outer boundary has been stated in* [30, Theorem 7.69]. *This is the same class of regularity as the fluid part in our main result.*

Remark 1.9. *We can establish the existence result Theorem 1.7 in the case when the frictional coefficients for the outer boundary and the moving solid are not the same. Precisely, we can replace* (1.7) *and* (1.9) *by the following boundary conditions:*

$$(\mathbb{T}(u_{\mathcal{F}})\nu) \times \nu = -\alpha_1(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu, \quad for \ t \in (0,T), \ x \in \partial\mathcal{S}(t),$$
$$(\mathbb{T}(u_{\mathcal{F}})\nu) \times \nu = -\alpha_2(u_{\mathcal{F}} \times \nu), \quad on \ (t,x) \in (0,T) \times \partial\Omega,$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are the frictional coefficients for the rigid body and the outer boundary respectively.

The outline of the paper is as follows. We introduce three levels of approximation schemes in Section 2. In Section 3, we describe some results on the transport equation, which are needed in all the levels of approximation. The existence results of approximate solutions have been proved in Section 4. Section 4.1 and Section 4.2 are dedicated to the construction and convergence analysis of the Faedo-Galerkin scheme associated to the finite dimensional approximation level. We discuss the limiting system associated to the vanishing viscosity in Section 4.3. Section 5 is devoted to the main part: we derive the limit as the parameter δ tends to zero.

2. Approximate solutions

In this section, we present the approximate problems by combining the penalization method, introduced in [16], and the approximation scheme developed in [12] along with a careful treatment of the boundary terms of the rigid body to solve the original problem (1.2)–(1.11). There are three levels of approximations with the parameters N, ε, δ. Let us briefly explain these approximations:

- The parameter N is connected with solving the momentum equation using the Faedo-Galerkin approximation.
- The parameter $\varepsilon > 0$ is connected with a new diffusion term $\varepsilon\Delta\rho$ in the continuity equation together with a term $\varepsilon\nabla\rho\nabla u$ in the momentum equation.
- The parameter $\delta > 0$ is connected with the approximation in the viscosities (see (2.8)) together with a penalization of the boundary of the rigid body to get smoothness through the interface (see (2.3)) and together with the artificial pressure containing the approximate coefficient, see (2.7).

At first, we state the existence results for the different levels of approximation schemes and then we will prove these later on. We start with the δ-level of approximation via an artificial pressure. We are going to consider the following approximate problem: Let $\delta > 0$. Find a triplet $(S^{\delta}, \rho^{\delta}, u^{\delta})$ such that

- $\mathcal{S}^{\delta}(t) \Subset \Omega$ is a bounded, regular domain for all $t \in [0, T]$ with

$$\chi_{\mathcal{S}}^{\delta}(t,x) := \mathbb{1}_{\mathcal{S}^{\delta}(t)}(x) \in L^{\infty}((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \forall \, 1 \leqslant p < \infty. \tag{2.1}$$

- The velocity field $u^{\delta} \in L^2(0,T; H^1(\Omega))$, and the density function $\rho^{\delta} \in L^{\infty}(0,T; L^{\beta}(\Omega))$, $\rho^{\delta} \geqslant 0$ satisfy

$$\frac{\partial\rho^{\delta}}{\partial t} + \text{div}(\rho^{\delta}u^{\delta}) = 0 \text{ in } \mathcal{D}'([0,T) \times \Omega). \tag{2.2}$$

- For all $\phi \in H^1(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}(\Omega))$, where $r = \max\left\{\beta+1, \frac{\beta+\theta}{\theta}\right\}$, $\beta \geqslant \max\{8,\gamma\}$ and $\theta = \frac{2}{3}\gamma - 1$ with $\phi \cdot \nu = 0$ on $\partial\Omega$ and $\phi|_{t=T} = 0$, the following holds:

$$-\int_0^T \int_\Omega \rho^\delta \left(u^\delta \cdot \frac{\partial}{\partial t}\phi + u^\delta \otimes u^\delta : \nabla\phi\right) + \int_0^T \int_\Omega \left(2\mu^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi) + \lambda^\delta \operatorname{div} u^\delta \mathbb{I} : \mathbb{D}(\phi) - p^\delta(\rho^\delta)\mathbb{I} : \mathbb{D}(\phi)\right)$$

$$+ \alpha \int_0^T \int_{\partial\Omega} (u^\delta \times \nu) \cdot (\phi \times \nu) + \alpha \int_0^T \int_{\partial\mathcal{S}^\delta(t)} [(u^\delta - P_\mathcal{S}^\delta u^\delta) \times \nu] \cdot [(\phi - P_\mathcal{S}^\delta \phi) \times \nu] + \frac{1}{\delta} \int_0^T \int_\Omega \chi_\mathcal{S}^\delta (u^\delta - P_\mathcal{S}^\delta u^\delta) \cdot (\phi - P_\mathcal{S}^\delta \phi)$$

$$= \int_0^T \int_\Omega \rho^\delta g^\delta \cdot \phi + \int_\Omega (\rho^\delta u^\delta \cdot \phi)(0), \quad (2.3)$$

  where $\mathcal{P}_\mathcal{S}^\delta$ is defined in (2.9) below.
- $\chi_\mathcal{S}^\delta(t,x)$ satisfies (in the weak sense)

$$\frac{\partial \chi_\mathcal{S}^\delta}{\partial t} + P_\mathcal{S}^\delta u^\delta \cdot \nabla \chi_\mathcal{S}^\delta = 0 \ \text{ in } (0,T) \times \Omega, \quad \chi_\mathcal{S}^\delta|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \ \text{ in } \Omega. \quad (2.4)$$

- $\rho^\delta \chi_\mathcal{S}^\delta(t,x)$ satisfies (in the weak sense)

$$\frac{\partial}{\partial t}(\rho^\delta \chi_\mathcal{S}^\delta) + P_\mathcal{S}^\delta u^\delta \cdot \nabla(\rho^\delta \chi_\mathcal{S}^\delta) = 0 \ \text{ in } (0,T) \times \Omega, \quad (\rho^\delta \chi_\mathcal{S}^\delta)|_{t=0} = \rho_0^\delta \mathbb{1}_{\mathcal{S}_0} \ \text{ in } \Omega. \quad (2.5)$$

- Initial data are given by

$$\rho^\delta(0,x) = \rho_0^\delta(x), \quad \rho^\delta u^\delta(0,x) = q_0^\delta(x), \quad x \in \Omega. \quad (2.6)$$

Above we have used the following quantities:

- The specific body force is defined as

$$g^\delta = (1 - \chi_\mathcal{S}^\delta)g_\mathcal{F} + \chi_\mathcal{S}^\delta g_\mathcal{S}.$$

- The artificial pressure is given by

$$p^\delta(\rho) = a^\delta \rho^\gamma + \delta\rho^\beta, \quad \text{with} \quad a^\delta = a_\mathcal{F}(1 - \chi_\mathcal{S}^\delta), \quad (2.7)$$

  where $a_\mathcal{F} > 0$ and γ and β are exponents (by abuse of notation) and they satisfy $\gamma > 3/2$, $\beta \geqslant \max\{8,\gamma\}$.
- The viscosity coefficients are given by

$$\mu^\delta = (1 - \chi_\mathcal{S}^\delta)\mu_\mathcal{F} + \delta^2 \chi_\mathcal{S}^\delta, \quad \lambda^\delta = (1 - \chi_\mathcal{S}^\delta)\lambda_\mathcal{F} + \delta^2 \chi_\mathcal{S}^\delta \quad \text{so that} \quad \mu^\delta > 0, \ 2\mu^\delta + 3\lambda^\delta \geqslant 0. \quad (2.8)$$

- The orthogonal projection onto rigid fields, $P_\mathcal{S}^\delta : L^2(\Omega) \to L^2(\mathcal{S}^\delta(t)) \cap \mathcal{R}(\mathcal{S}^\delta(t))$, is such that, for all $t \in [0,T]$ and $u \in L^2(\Omega)$, we have $P_\mathcal{S}^\delta u \in \mathcal{R}$ and it is given by

$$P_\mathcal{S}^\delta u(t,x) = \frac{1}{m^\delta} \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta u + \left((J^\delta)^{-1} \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta ((y - h^\delta(t)) \times u) \ dy\right) \times (x - h^\delta(t)), \quad \forall x \in \Omega, \quad (2.9)$$

  where h^δ, m^δ and J^δ are defined as

$$h^\delta(t) = \frac{1}{m^\delta} \int_{\mathbb{R}^3} \rho^\delta \chi_\mathcal{S}^\delta x \ dx, \quad m^\delta = \int_{\mathbb{R}^3} \rho^\delta \chi_\mathcal{S}^\delta \ dx,$$

$$J^\delta(t) = \int_{\mathbb{R}^3} \rho^\delta \chi_\mathcal{S}^\delta \left[|x - h^\delta(t)|^2 \mathbb{I} - (x - h^\delta(t)) \otimes (x - h^\delta(t))\right] \ dx.$$

Remark 2.1. *The penalization which we apply in our case is different from that in [7]. We do not use the high viscosity limit but our penalization contains an L^2 penalization (see (2.3)), which is necessary because of the discontinuity of the velocity field through the fluid-structure interface. Moreover, we consider a penalization of the viscosity coefficients (2.8) together with the additional regularity of the pressure, see (2.7). This approach is completely new.*

A weak solution of problem (1.2)–(1.11) in the sense of Definition 1.1 will be obtained as a weak limit of the solution $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ of system (2.1)–(2.6) as $\delta \to 0$. The existence result of the approximate system reads:

Proposition 2.2. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$*

$$\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma.$$

Let $g^\delta = (1 - \chi_{\mathcal{S}}^\delta) g_{\mathcal{F}} + \chi_{\mathcal{S}}^\delta g_{\mathcal{S}} \in L^\infty((0, T) \times \Omega)$ and

$$\delta > 0, \ \gamma > 3/2, \ \beta \geqslant \max\{8, \gamma\}. \tag{2.10}$$

Further, let the pressure p^δ be determined by (2.7) and the viscosity coefficients μ^δ, λ^δ be given by (2.8). Assume that the initial conditions satisfy

$$\rho_0^\delta \in L^\beta(\Omega), \quad \rho_0^\delta \geqslant 0 \ a.e. \ in \ \Omega, \quad \rho_0^\delta \mathbb{1}_{\mathcal{S}_0} \in L^\infty(\Omega), \quad \rho_0^\delta \mathbb{1}_{\mathcal{S}_0} > 0 \ a.e. \ in \ \mathcal{S}_0, \tag{2.11}$$

$$q_0^\delta \in L^{\frac{2\beta}{\beta+1}}(\Omega), \quad q_0^\delta \mathbb{1}_{\{\rho_0^\delta = 0\}} = 0 \ a.e. \ in \ \Omega, \quad \frac{|q_0^\delta|^2}{\rho_0^\delta} \mathbb{1}_{\{\rho_0 > 0\}} \in L^1(\Omega). \tag{2.12}$$

Let the initial energy

$$E^\delta[\rho_0^\delta, q_0^\delta] = \int_\Omega \left(\frac{1}{2} \frac{|q_0^\delta|^2}{\rho_0^\delta} \mathbb{1}_{\{\rho_0^\delta > 0\}} + \frac{a^\delta(0)}{\gamma - 1} (\rho_0^\delta)^\gamma + \frac{\delta}{\beta - 1} (\rho_0^\delta)^\beta \right) := E_0^\delta$$

be uniformly bounded with respect to δ. Then there exists $T > 0$ (depending only on E_0^δ, $g_{\mathcal{F}}$, $g_{\mathcal{S}}$, $\operatorname{dist}(\mathcal{S}_0, \partial\Omega)$) such that system (2.1)–(2.6) admits a finite energy weak solution $(S^\delta, \rho^\delta, u^\delta)$, which satisfies the following energy inequality:

$$E^\delta[\rho^\delta, q^\delta] + \int_0^T \int_\Omega \left(2\mu^\delta |\mathbb{D}(u^\delta)|^2 + \lambda^\delta |\operatorname{div} u^\delta|^2 \right) + \alpha \int_0^T \int_{\partial\Omega} |u^\delta \times \nu|^2 + \alpha \int_0^T \int_{\partial\mathcal{S}^\delta(t)} |(u^\delta - P_{\mathcal{S}}^\delta u^\delta) \times \nu|^2$$

$$+ \frac{1}{\delta} \int_0^T \int_\Omega \chi_{\mathcal{S}}^\delta |u^\delta - P_{\mathcal{S}}^\delta u^\delta|^2 \leqslant \int_0^T \int_\Omega \rho^\delta g^\delta \cdot u^\delta + E_0^\delta. \tag{2.13}$$

Moreover,

$$\operatorname{dist}(\mathcal{S}^\delta(t), \partial\Omega) \geqslant 2\sigma, \quad \forall \ t \in [0, T],$$

and the solution satisfies the following properties:

(1) *For $\theta = \frac{2}{3}\gamma - 1$, $s = \gamma + \theta$,*

$$\|(a^\delta)^{1/s} \rho^\delta\|_{L^s((0,T)\times\Omega)} + \delta^{\frac{1}{\beta+\theta}} \|\rho^\delta\|_{L^{\beta+\theta}((0,T)\times\Omega)} \leqslant c. \tag{2.14}$$

(2) *The couple (ρ^δ, u^δ) satisfies the identity*

$$\partial_t b(\rho^\delta) + \operatorname{div}(b(\rho^\delta) u^\delta) + [b'(\rho^\delta) \rho^\delta - b(\rho^\delta)] \operatorname{div} u^\delta = 0, \tag{2.15}$$

a.e. in $(0, T) \times \Omega$ for any $b \in C([0, \infty)) \cap C^1((0, \infty))$ satisfying (1.28).

In order to prove Proposition 2.2, we consider a problem with another level of approximation: the ε-level approximation is obtained via the continuity equation with dissipation accompanied by the artificial pressure in the momentum equation. We want to find a triplet $(S^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ such that we can obtain a weak solution $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ of the system (2.1)–(2.6) as a weak limit of the sequence $(\mathcal{S}^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$. For $\varepsilon > 0$, the triplet is supposed to satisfy:

- $\mathcal{S}^\varepsilon(t) \Subset \Omega$ is a bounded, regular domain for all $t \in [0, T]$ with

$$\chi_{\mathcal{S}}^\varepsilon(t, x) := \mathbb{1}_{\mathcal{S}^\varepsilon(t)}(x) \in L^\infty((0, T) \times \Omega) \cap C([0, T]; L^p(\Omega)), \forall 1 \leqslant p < \infty. \tag{2.16}$$

- The velocity field $u^\varepsilon \in L^2(0, T; H^1(\Omega))$ and the density function $\rho^\varepsilon \in L^\infty(0, T; L^\beta(\Omega)) \cap L^2(0, T; H^1(\Omega))$, $\rho^\varepsilon \geqslant 0$ satisfy

$$\frac{\partial \rho^\varepsilon}{\partial t} + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \varepsilon \Delta \rho^\varepsilon \ in \ (0, T) \times \Omega, \quad \frac{\partial \rho^\varepsilon}{\partial \nu} = 0 \ on \ \partial\Omega. \tag{2.17}$$

- For all $\phi \in H^1(0, T; L^2(\Omega)) \cap L^{\beta+1}(0, T; W^{1,\beta+1}(\Omega))$ with $\phi \cdot \nu = 0$ on $\partial\Omega$, $\phi|_{t=T} = 0$, where $\beta \geqslant \max\{8, \gamma\}$, the following holds:

$$
-\int_0^T \int_\Omega \rho^\varepsilon \left(u^\varepsilon \cdot \frac{\partial}{\partial t}\phi + u^\varepsilon \otimes u^\varepsilon : \nabla\phi \right) + \int_0^T \int_\Omega \left(2\mu^\varepsilon \mathbb{D}(u^\varepsilon) : \mathbb{D}(\phi) + \lambda^\varepsilon \operatorname{div} u^\varepsilon \mathbb{I} : \mathbb{D}(\phi) - p^\varepsilon(\rho^\varepsilon)\mathbb{I} : \mathbb{D}(\phi) \right)
$$

$$
+\int_0^T \int_\Omega \varepsilon \nabla u^\varepsilon \nabla\rho^\varepsilon \cdot \phi + \alpha \int_0^T \int_{\partial\Omega} (u^\varepsilon \times \nu) \cdot (\phi \times \nu) + \alpha \int_0^T \int_{\partial\mathcal{S}^\varepsilon(t)} [(u^\varepsilon - P_{\mathcal{S}}^\varepsilon u^\varepsilon) \times \nu] \cdot [(\phi - P_{\mathcal{S}}^\varepsilon \phi) \times \nu]
$$

$$
+\frac{1}{\delta}\int_0^T \int_\Omega \chi_{\mathcal{S}}^\varepsilon(u^\varepsilon - P_{\mathcal{S}}^\varepsilon u^\varepsilon) \cdot (\phi - P_{\mathcal{S}}^\varepsilon \phi) = \int_0^T \int_\Omega \rho^\varepsilon g^\varepsilon \cdot \phi + \int_\Omega (\rho^\varepsilon u^\varepsilon \cdot \phi)(0). \quad (2.18)
$$

- $\chi_{\mathcal{S}}^\varepsilon(t, x)$ satisfies (in the weak sense)

$$
\frac{\partial \chi_{\mathcal{S}}^\varepsilon}{\partial t} + P_{\mathcal{S}}^\varepsilon u^\varepsilon \cdot \nabla \chi_{\mathcal{S}}^\varepsilon = 0 \text{ in } (0, T) \times \Omega, \quad \chi_{\mathcal{S}}^\varepsilon|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \Omega. \quad (2.19)
$$

- $\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon(t, x)$ satisfies (in the weak sense)

$$
\frac{\partial}{\partial t}(\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon) + P_{\mathcal{S}}^\varepsilon u^\varepsilon \cdot \nabla(\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon) = 0 \text{ in } (0, T) \times \Omega, \quad (\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon)|_{t=0} = \rho_0^\varepsilon \mathbb{1}_{\mathcal{S}_0} \text{ in } \Omega. \quad (2.20)
$$

- The initial data are given by

$$
\rho^\varepsilon(0, x) = \rho_0^\varepsilon(x), \quad \rho^\varepsilon u^\varepsilon(0, x) = q_0^\varepsilon(x) \quad \text{in } \Omega, \quad \frac{\partial \rho_0^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0. \quad (2.21)
$$

Above we have used the following quantities:

- The specific body force is defined as

$$
g^\varepsilon = (1 - \chi_{\mathcal{S}}^\varepsilon)g_{\mathcal{F}} + \chi_{\mathcal{S}}^\varepsilon g_{\mathcal{S}}. \quad (2.22)
$$

- The artificial pressure is given by

$$
p^\varepsilon(\rho) = a^\varepsilon \rho^\gamma + \delta\rho^\beta, \quad \text{with} \quad a^\varepsilon = a_{\mathcal{F}}(1 - \chi_{\mathcal{S}}^\varepsilon), \quad (2.23)
$$

  where $a_{\mathcal{F}}, \delta > 0$, and the exponents γ and β satisfy $\gamma > 3/2$, $\beta \geqslant \max\{8, \gamma\}$.
- The viscosity coefficients are given by

$$
\mu^\varepsilon = (1 - \chi_{\mathcal{S}}^\varepsilon)\mu_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}}^\varepsilon, \quad \lambda^\varepsilon = (1 - \chi_{\mathcal{S}}^\varepsilon)\lambda_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}}^\varepsilon \quad \text{so that} \quad \mu^\varepsilon > 0, \; 2\mu^\varepsilon + 3\lambda^\varepsilon \geqslant 0. \quad (2.24)
$$

- $P_{\mathcal{S}}^\varepsilon : L^2(\Omega) \to L^2(\mathcal{S}^\varepsilon(t)) \cap \mathcal{R}(\mathcal{S}^\varepsilon(t))$ is the orthogonal projection onto rigid fields; it is defined as in (2.9) with $\chi_{\mathcal{S}}^\delta$ replaced by $\chi_{\mathcal{S}}^\varepsilon$.

Remark 2.3. *Above, the triplet $(\mathcal{S}^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ should actually be denoted by $(\mathcal{S}^{\delta,\varepsilon}, \rho^{\delta,\varepsilon}, u^{\delta,\varepsilon})$. The dependence on δ is due to the penalization term $\left(\frac{1}{\delta}\int_0^T \int_\Omega \chi_{\mathcal{S}}^\varepsilon(u^\varepsilon - P_{\mathcal{S}}^\varepsilon u^\varepsilon) \cdot (\phi - P_{\mathcal{S}}^\varepsilon \phi) \right)$ in (2.18) and in the viscosity coefficients μ^ε, λ^ε in (2.24). To simplify the notation, we omit δ here.*

In Section 4.2 we will prove the following existence result of the approximate system (2.16)–(2.21):

Proposition 2.4. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$,*

$$
\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma.
$$

Let $g^\varepsilon = (1 - \chi_{\mathcal{S}}^\varepsilon)g_{\mathcal{F}} + \chi_{\mathcal{S}}^\varepsilon g_{\mathcal{S}} \in L^\infty((0, T) \times \Omega)$ and β, δ, γ be given as in (2.10). Further, let the pressure p^ε be determined by (2.23) and the viscosity coefficients μ^ε, λ^ε be given by (2.24). The initial conditions satisfy, for some $\underline{\rho}, \overline{\rho} > 0$,

$$
0 < \underline{\rho} \leqslant \rho_0^\varepsilon \leqslant \overline{\rho} \text{ in } \Omega, \quad \rho_0^\varepsilon \in W^{1,\infty}(\Omega), \quad q_0^\varepsilon \in L^2(\Omega). \quad (2.25)
$$

Let the initial energy

$$E^\varepsilon[\rho_0^\varepsilon, q_0^\varepsilon] = \int\limits_\Omega \left(\frac{1}{2} \frac{|q_0^\varepsilon|^2}{\rho_0^\varepsilon} \mathbb{1}_{\{\rho_0^\varepsilon > 0\}} + \frac{a^\varepsilon(0)}{\gamma - 1} (\rho_0^\varepsilon)^\gamma + \frac{\delta}{\beta - 1} (\rho_0^\varepsilon)^\beta \right) := E_0^\varepsilon$$

be uniformly bounded with respect to δ and ε. Then there exists $T > 0$ (depending only on E_0^ε, $g_{\mathcal{F}}$, $g_{\mathcal{S}}$, $\mathrm{dist}(\mathcal{S}_0, \partial\Omega)$) such that system (2.16)–(2.21) admits a weak solution $(S^\varepsilon, \rho^\varepsilon, u^\varepsilon)$, which satisfies the following energy inequality:

$$E^\varepsilon[\rho^\varepsilon, q^\varepsilon] + \int\limits_0^T \int\limits_\Omega \left(2\mu^\varepsilon |\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon |\operatorname{div} u^\varepsilon|^2 \right) + \delta\varepsilon\beta \int\limits_0^T \int\limits_\Omega (\rho^\varepsilon)^{\beta-2} |\nabla\rho^\varepsilon|^2 + \alpha \int\limits_0^T \int\limits_{\partial\Omega} |u^\varepsilon \times \nu|^2$$

$$+ \alpha \int\limits_0^T \int\limits_{\partial\mathcal{S}^\varepsilon(t)} |(u^\varepsilon - P_{\mathcal{S}}^\varepsilon u^\varepsilon) \times \nu|^2 + \frac{1}{\delta} \int\limits_0^T \int\limits_\Omega \chi_{\mathcal{S}}^\varepsilon |u^\varepsilon - P_{\mathcal{S}}^\delta u^\varepsilon|^2 \leqslant \int\limits_0^T \int\limits_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon + E_0^\varepsilon. \quad (2.26)$$

Moreover,

$$\mathrm{dist}(\mathcal{S}^\varepsilon(t), \partial\Omega) \geqslant 2\sigma, \quad \forall\, t \in [0, T],$$

and the solution satisfies

$$\partial_t \rho^\varepsilon, \ \Delta\rho^\varepsilon \in L^{\frac{5\beta - 3}{4\beta}}((0, T) \times \Omega),$$

$$\sqrt{\varepsilon} \|\nabla\rho^\varepsilon\|_{L^2((0,T)\times\Omega)} + \|\rho^\varepsilon\|_{L^{\beta+1}((0,T)\times\Omega)} + \|(a^\varepsilon)^{\frac{1}{\gamma+1}} \rho^\varepsilon\|_{L^{\gamma+1}((0,T)\times\Omega)} \leqslant c, \quad (2.27)$$

where c is a positive constant depending on δ but independent of ε.

To solve the problem (2.16)–(2.21), we need yet another level of approximation. The N-level approximation is obtained via a Faedo-Galerkin approximation scheme.

Suppose that $\{e_k\}_{k \geqslant 1} \subset \mathcal{D}(\overline{\Omega})$ with $e_k \cdot \nu = 0$ on $\partial\Omega$ is a basis of $L^2(\Omega)$. We set

$$X_N = \operatorname{span}(e_1, \ldots, e_N).$$

X_N is a finite dimensional space with scalar product induced by the scalar product in $L^2(\Omega)$. As X_N is finite dimensional, norms on X_N induced by $W^{k,p}$ norms, $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ are equivalent. We also assume that

$$\bigcup_N X_N \text{ is dense in } \left\{ v \in W^{1,p}(\Omega) \mid v \cdot \nu = 0 \text{ on } \partial\Omega \right\}, \text{ for any } 1 \leqslant p < \infty.$$

Such a family X_N has been constructed in [11, Theorem 11.19, page 460].

The task is to find a triplet (S^N, ρ^N, u^N) satisfying:

- $\mathcal{S}^N(t) \Subset \Omega$ is a bounded, regular domain for all $t \in [0, T]$ with

$$\chi_{\mathcal{S}}^N(t, x) := \mathbb{1}_{\mathcal{S}^N(t)}(x) \in L^\infty((0, T) \times \Omega) \cap C([0, T]; L^p(\Omega)), \ \forall\, 1 \leqslant p < \infty. \quad (2.28)$$

- The velocity field $u^N(t, \cdot) = \sum\limits_{k=1}^N \alpha_k(t) e_k$ with $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in C([0, T])^N$ and the density function $\rho^N \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$, $\rho^N > 0$ satisfies

$$\frac{\partial\rho^N}{\partial t} + \operatorname{div}(\rho^N u^N) = \varepsilon\Delta\rho^N \text{ in } (0, T) \times \Omega, \quad \frac{\partial\rho^N}{\partial\nu} = 0 \text{ on } \partial\Omega. \quad (2.29)$$

- For all $\phi \in \mathcal{D}([0,T); X_N)$ with $\phi \cdot \nu = 0$ on $\partial\Omega$, the following holds:

$$-\int_0^T \int_\Omega \rho^N \left(u^N \cdot \frac{\partial}{\partial t}\phi + u^N \otimes u^N : \nabla\phi \right) + \int_0^T \int_\Omega \left(2\mu^N \mathbb{D}(u^N) : \mathbb{D}(\phi) + \lambda^N \operatorname{div} u^N \mathbb{I} : \mathbb{D}(\phi) - p^N(\rho^N)\mathbb{I} : \mathbb{D}(\phi) \right)$$

$$\int_0^T \int_\Omega \varepsilon \nabla u^N \nabla \rho^N \cdot \phi + \alpha \int_0^T \int_{\partial\Omega} (u^N \times \nu) \cdot (\phi \times \nu) + \alpha \int_0^T \int_{\partial\mathcal{S}^N(t)} [(u^N - P_\mathcal{S}^N u^N) \times \nu] \cdot [(\phi - P_\mathcal{S}^N \phi) \times \nu]$$

$$+ \frac{1}{\delta} \int_0^T \int_\Omega \chi_\mathcal{S}^N (u^N - P_\mathcal{S}^N u^N) \cdot (\phi - P_\mathcal{S}^N \phi) = \int_0^T \int_\Omega \rho^N g^N \cdot \phi + \int_\Omega (\rho^N u^N \cdot \phi)(0). \quad (2.30)$$

- $\chi_\mathcal{S}^N(t,x)$ satisfies (in the weak sense)

$$\frac{\partial \chi_\mathcal{S}^N}{\partial t} + P_\mathcal{S}^N u^N \cdot \nabla \chi_\mathcal{S}^N = 0 \text{ in } (0,T) \times \Omega, \quad \chi_\mathcal{S}^N|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \Omega. \quad (2.31)$$

- $\rho^N \chi_\mathcal{S}^N(t,x)$ satisfies (in the weak sense)

$$\frac{\partial}{\partial t}(\rho^N \chi_\mathcal{S}^N) + P_\mathcal{S}^N u^N \cdot \nabla(\rho^N \chi_\mathcal{S}^N) = 0 \text{ in } (0,T) \times \Omega, \quad (\rho^N \chi_\mathcal{S}^N)|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_0} \text{ in } \Omega. \quad (2.32)$$

- The initial data are given by

$$\rho^N(0) = \rho_0^N, \quad u^N(0) = u_0^N \quad \text{in } \Omega, \quad \frac{\partial \rho_0^N}{\partial \nu}\Big|_{\partial\Omega} = 0. \quad (2.33)$$

Above we have used the following quantities:

- The specific body force is defined as

$$g^N = (1 - \chi_\mathcal{S}^N)g_\mathcal{F} + \chi_\mathcal{S}^N g_\mathcal{S}. \quad (2.34)$$

- The artificial pressure is given by

$$p^N(\rho) = a^N \rho^\gamma + \delta \rho^\beta, \quad \text{with} \quad a^N = a_\mathcal{F}(1 - \chi_\mathcal{S}^N), \quad (2.35)$$

where $a_\mathcal{F}, \delta > 0$ and the exponents γ and β satisfy $\gamma > 3/2, \ \beta \geqslant \max\{8, \gamma\}$.
- The viscosity coefficients are given by

$$\mu^N = (1 - \chi_\mathcal{S}^N)\mu_\mathcal{F} + \delta^2 \chi_\mathcal{S}^N, \quad \lambda^N = (1 - \chi_\mathcal{S}^N)\lambda_\mathcal{F} + \delta^2 \chi_\mathcal{S}^N \quad \text{so that} \quad \mu^N > 0, \ 2\mu^N + 3\lambda^N \geqslant 0. \quad (2.36)$$

- $P_\mathcal{S}^N : L^2(\Omega) \to L^2(\mathcal{S}^N(t)) \cap \mathcal{R}(\mathcal{S}^N(t))$ is the orthogonal projection onto rigid fields; it is defined as in (2.9) with $\chi_\mathcal{S}^\delta$ replaced by $\chi_\mathcal{S}^N$.

Remark 2.5. *Actually the triplet $\left(\mathcal{S}^N, \rho^N, u^N\right)$ above should be denoted by $\left(\mathcal{S}^{\delta,\varepsilon,N}, \rho^{\delta,\varepsilon,N}, u^{\delta,\varepsilon,N}\right)$. The dependence on δ and ε is due to the penalization term $\left(\frac{1}{\delta}\int_0^T \int_\Omega \chi_\mathcal{S}^N(u^N - P_\mathcal{S}^N u^N) \cdot (\phi - P_\mathcal{S}^N \phi)\right)$, the viscosity coefficients μ^N, λ^N and the artificial dissipative term $(\varepsilon\Delta\rho)$. To simplify the notation, we omit δ and ε here.*

A weak solution $(\mathcal{S}^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ to the system (2.16)–(2.21) is obtained through the limit of $(\mathcal{S}^N, \rho^N, u^N)$ as $N \to \infty$. The existence result of the approximate solution of the Faedo-Galerkin scheme reads:

Proposition 2.6. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$,*

$$\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma.$$

Let $g^N = (1 - \chi_\mathcal{S}^N)g_\mathcal{F} + \chi_\mathcal{S}^N g_\mathcal{S} \in L^\infty((0,T) \times \Omega)$ and β, δ, γ be given by (2.10). Further, let the pressure p^N be determined by (2.35) and the viscosity coefficients μ^N, λ^N be given by (2.36). The initial conditions are assumed to satisfy

$$0 < \underline{\rho} \leqslant \rho_0^N \leqslant \overline{\rho} \quad \text{in } \Omega, \quad \rho_0^N \in W^{1,\infty}(\Omega), \quad u_0^N \in X_N. \quad (2.37)$$

Let the initial energy

$$E^N(\rho_0^N, q_0^N) = \int_\Omega \left(\frac{1}{\rho_0^N} |q_0^N|^2 \mathbb{1}_{\{\rho_0 > 0\}} + \frac{a^N(0)}{\gamma - 1} (\rho_0^N)^\gamma + \frac{\delta}{\beta - 1} (\rho_0^N)^\beta \right) := E_0^N$$

be uniformly bounded with respect to N, ε, δ. Then there exists $T > 0$ (depending only on E_0^N, $g_\mathcal{F}$, $g_\mathcal{S}$, $\overline{\rho}$, $\underline{\rho}$, $\mathrm{dist}(\mathcal{S}_0, \partial\Omega)$) such that the problem (2.28)–(2.33) admits a solution $(\mathcal{S}^N, \rho^N, u^N)$ and it satisfies the energy inequality:

$$E^N[\rho^N, q^N] + \int_0^T \int_\Omega \left(2\mu^N |\mathbb{D}(u^N)|^2 + \lambda^N |\mathrm{div}\, u^N|^2 \right) + \delta\varepsilon\beta \int_0^T \int_\Omega (\rho^N)^{\beta-2} |\nabla \rho^N|^2 + \alpha \int_0^T \int_{\partial\Omega} |u^N \times \nu|^2$$

$$+ \alpha \int_0^T \int_{\partial\mathcal{S}^N(t)} |(u^N - P_\mathcal{S}^N u^N) \times \nu|^2 + \frac{1}{\delta} \int_0^T \int_\Omega \chi_\mathcal{S}^N |u^N - P_\mathcal{S}^N u^N|^2 \leqslant \int_0^T \int_\Omega \rho^N g^N \cdot u^N + E_0^N.$$

Moreover,

$$\mathrm{dist}(\mathcal{S}^N(t), \partial\Omega) \geqslant 2\sigma, \quad \forall\, t \in [0, T].$$

We prove the above proposition in Section 4.1.

3. Isometric propagators and the motion of the body

In this section, we state and prove some results regarding the transport equation that we use in our analysis. We mainly concentrate on the following equation:

$$\frac{\partial \chi_\mathcal{S}}{\partial t} + \mathrm{div}(P_\mathcal{S} u \chi_\mathcal{S}) = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad \chi_\mathcal{S}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3, \tag{3.1}$$

where $P_\mathcal{S} u \in \mathcal{R}(\Omega)$ is given by

$$P_\mathcal{S} u(t, x) = \frac{1}{m} \int_\Omega \rho \chi_\mathcal{S} u + \left(J^{-1} \int_\Omega \rho \chi_\mathcal{S} ((y - h(t)) \times u)\, dy \right) \times (x - h(t)), \quad \forall\, (t, x) \in (0, T) \times \mathbb{R}^3. \tag{3.2}$$

In [16, Proposition 3.1], the existence of a solution to (3.1) and the characterization of the transport of the rigid body have been established with constant ρ in the expression (3.2) of $P_\mathcal{S} u$. Here we deal with the case when ρ is evolving. We start with some existence results when the velocity field and the density satisfy certain regularity assumptions.

Proposition 3.1. *Let $u \in C([0, T]; \mathcal{D}(\overline{\Omega}))$ and $\rho \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega))$. Then the following holds true:*

(1) *There is a unique solution $\chi_\mathcal{S} \in L^\infty((0, T) \times \mathbb{R}^3) \cap C([0, T]; L^p(\mathbb{R}^3))\ \forall\, 1 \leqslant p < \infty$ to (3.1). More precisely,*

$$\chi_\mathcal{S}(t, x) = \mathbb{1}_{\mathcal{S}(t)}(x), \quad \forall\, t \geqslant 0,\ \forall\, x \in \mathbb{R}^3.$$

If the isometric propagator $\eta_{t,s}$, associated to $P_\mathcal{S} u$ is defined by

$$\frac{\partial \eta_{t,s}}{\partial t}(y) = P_\mathcal{S} u(t, \eta_{t,s}(y)), \quad \forall\, (t, s, y) \in (0, T)^2 \times \mathbb{R}^3, \quad \eta_{s,s}(y) = y, \quad \forall\, y \in \mathbb{R}^3, \tag{3.3}$$

then

$$(t, s) \mapsto \eta_{t,s} \in C^1([0, T]^2; C^\infty_{loc}(\mathbb{R}^3)),$$

Moreover, we also have $\mathcal{S}(t) = \eta_{t,0}(\mathcal{S}_0)$.

(2) *Let $\rho_0 \mathbb{1}_{\mathcal{S}_0} \in L^\infty(\mathbb{R}^3)$. Then there is a unique solution $\rho \chi_\mathcal{S} \in L^\infty((0, T) \times \mathbb{R}^3) \cap C([0, T]; L^p(\mathbb{R}^3)),\ \forall\, 1 \leqslant p < \infty$ to the following equation:*

$$\frac{\partial(\rho\chi_\mathcal{S})}{\partial t} + \mathrm{div}((\rho\chi_\mathcal{S}) P_\mathcal{S} u) = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad \rho\chi_\mathcal{S}|_{t=0} = \rho_0 \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3. \tag{3.4}$$

Proof. Following [16, Proposition 3.1], we observe that proving existence of solution to (3.1) is equivalent to establishing the well-posedness of the ordinary differential equation

$$\frac{d}{dt}\eta_{t,0} = U_{\mathcal{S}}(t, \eta_{t,0}), \quad \eta_{0,0} = \mathbb{I}, \tag{3.5}$$

where $U_{\mathcal{S}} \in \mathcal{R}$ is given by

$$U_{\mathcal{S}}(t, x) = \frac{1}{m}\int_{\mathcal{S}_0} \rho(t, \eta_{t,0}(y))\mathbb{1}_{\Omega}u(t, \eta_{t,0}(y))\, dy + \left(J^{-1}\int_{\mathcal{S}_0} \rho(t, \eta_{t,0}(y))\mathbb{1}_{\Omega}((\eta_{t,0}(y) - h(t)) \times u(t, \eta_{t,0}(y)))\, dy\right) \times (x - h(t)).$$

According to the Cauchy-Lipschitz theorem, equation (3.5) admits the unique C^1 solution if $U_{\mathcal{S}}$ is continuous in (t, η) and uniformly Lipschitz in η. Thus, it is enough to establish the following result analogous to [16, Lemma 3.2]: Let $u \in C([0,T]; \mathcal{D}(\overline{\Omega}))$ and $\rho \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega))$. Then the function

$$\mathcal{M} : [0,T] \times \mathrm{Isom}(\mathbb{R}^3) \mapsto \mathbb{R}, \quad \mathcal{M}(t, \eta) = \int_{\mathcal{S}_0} \rho(t, \eta(y))\mathbb{1}_{\Omega}(\eta(y))u(t, \eta(y))$$

is continuous in (t, η) and uniformly Lipschitz in η over $[0,T]$. Observe that the continuity in the t-variable is obvious. Moreover, for two isometries η_1 and η_2, we have

$$\mathcal{M}(t, \eta_1) - \mathcal{M}(t, \eta_2)$$
$$= \int_{\mathcal{S}_0} \rho(t, \eta_1(y))\mathbb{1}_{\Omega}(\eta_1(y))(u(t, \eta_1(y)) - u(t, \eta_2(y))) + \int_{\mathcal{S}_0} \rho(t, \eta_1(y))(\mathbb{1}_{\Omega}(\eta_1(y)) - \mathbb{1}_{\Omega}(\eta_2(y)))u(t, \eta_2(y))$$
$$+ \int_{\mathcal{S}_0} (\rho(t, \eta_1(y)) - \rho(t, \eta_2(y)))\mathbb{1}_{\Omega}(\eta_2(y))u(t, \eta_2(y)) := M_1 + M_2 + M_3.$$

As $\rho \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega))$, the estimates of the terms M_1 and M_2 are similar to [16, Lemma 3.2]. The term M_3 can be estimated in the following way:

$$|M_3| \leqslant C\|\rho\|_{L^{\infty}(0,T; H^1(\Omega))}\|u\|_{L^{\infty}(0,T; L^2(\Omega))}\|\eta_1 - \eta_2\|_{\infty}.$$

This finishes the proof of the first part of Proposition 3.1. The second part of this Proposition is similar and we skip it here. $\square$

Next we prove the analogous result of [16, Proposition 3.3, Proposition 3.4] on strong and weak sequential continuity which are essential to establish the existence result of the Galerkin approximation scheme in Section 4.1. The result obtained in the next proposition is used to establish the continuity of the fixed point map in the proof of the existence of Galerkin approximation.

Proposition 3.2. *Let $\rho_0^N \in W^{1,\infty}(\Omega)$, let $\rho^k \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ be the solution to*

$$\frac{\partial \rho^k}{\partial t} + \mathrm{div}(\rho^k u^k) = \Delta \rho^k \ \text{in} \ (0,T) \times \Omega, \quad \frac{\partial \rho^k}{\partial \nu} = 0 \ \text{on} \ \partial\Omega, \quad \rho^k(0, x) = \rho_0^N(x) \quad \text{in} \ \Omega, \quad \frac{\partial \rho_0^k}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{3.6}$$

$$u^k \to u \ \text{strongly in} \ C([0,T]; \mathcal{D}(\overline{\Omega})), \quad \chi_{\mathcal{S}}^k \ \text{is bounded in} \ L^{\infty}((0,T) \times \mathbb{R}^3) \ \text{satisfying}$$

$$\frac{\partial \chi_{\mathcal{S}}^k}{\partial t} + \mathrm{div}(P_{\mathcal{S}}^k u^k \chi_{\mathcal{S}}^k) = 0 \ \text{in} \ (0,T) \times \mathbb{R}^3, \quad \chi_{\mathcal{S}}^k|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \ \text{in} \ \mathbb{R}^3, \tag{3.7}$$

and let $\{\rho^k \chi_{\mathcal{S}}^k\}$ be a bounded sequence in $L^{\infty}((0,T) \times \mathbb{R}^3)$ satisfying

$$\frac{\partial}{\partial t}(\rho^k \chi_{\mathcal{S}}^k) + \mathrm{div}(P_{\mathcal{S}}^k u^k (\rho^k \chi_{\mathcal{S}}^k)) = 0 \ \text{in} \ (0,T) \times \mathbb{R}^3, \quad \rho^k \chi_{\mathcal{S}}^k|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_0} \ \text{in} \ \mathbb{R}^3, \tag{3.8}$$

where $P_{\mathcal{S}}^k : L^2(\Omega) \to L^2(\mathcal{S}^k(t)) \cap \mathcal{R}(\mathcal{S}^k(t))$ is the orthogonal projection onto rigid fields with $\mathcal{S}^k(t) \Subset \Omega$ being a bounded, regular domain for all $t \in [0, T]$. Then

$$\chi_{\mathcal{S}}^k \to \chi_{\mathcal{S}} \text{ weakly-}* \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)), \ \forall \ 1 \leqslant p < \infty,$$

$$\rho^k \chi_{\mathcal{S}}^k \to \rho \chi_{\mathcal{S}} \text{ weakly-}* \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)), \ \forall \ 1 \leqslant p < \infty,$$

where $\chi_{\mathcal{S}}$ and $\rho \chi_{\mathcal{S}}$ are satisfying (3.1) and (3.4) with initial data $\mathbb{1}_{\mathcal{S}_0}$ and $\rho_0^N \mathbb{1}_{\mathcal{S}_0}$, respectively. Moreover,

$$P_{\mathcal{S}}^k u^k \to P_{\mathcal{S}} u \text{ strongly in } C([0,T]; C_{loc}^\infty(\mathbb{R}^3)),$$

$$\eta_{t,s}^k \to \eta_{t,s} \text{ strongly in } C^1([0,T]^2; C_{loc}^\infty(\mathbb{R}^3)).$$

Proof. As $\{u^k\}$ converges strongly in $C([0,T]; \mathcal{D}(\overline{\Omega}))$ and $\{\rho^k \chi_{\mathcal{S}}^k\}$ is bounded in $L^\infty((0,T) \times \mathbb{R}^3)$, we obtain that $P_{\mathcal{S}}^k u^k$ is bounded in $L^2(0,T; \mathcal{R})$. Thus, up to a subsequence,

$$P_{\mathcal{S}}^k u^k \to \overline{u_{\mathcal{S}}} \text{ weakly in } L^2(0,T; \mathcal{R}). \tag{3.9}$$

Here, obviously $P_{\mathcal{S}}^k u^k \in L^1(0,T; L_{loc}^\infty(\mathbb{R}^3))$, $\mathrm{div}(P_{\mathcal{S}}^k u^k) = 0$ and $\overline{u_{\mathcal{S}}} \in L^1(0,T; W_{loc}^{1,1}(\mathbb{R}^3))$ satisfies

$$\frac{\overline{u_{\mathcal{S}}}}{1 + |x|} \in L^1(0,T; L^1(\mathbb{R}^3)).$$

Moreover, $\{\chi_{\mathcal{S}}^k\}$ is bounded in $L^\infty((0,T) \times \mathbb{R}^3)$, $\chi_{\mathcal{S}}^k$ satisfies (3.7) and $\{\rho^N \chi_{\mathcal{S}}^k\}$ is bounded in $L^\infty((0,T) \times \mathbb{R}^3)$, $\rho^k \chi_{\mathcal{S}}^k$ satisfies (3.8). As we have verified all the required conditions, we can apply [6, Theorem II.4, Page 521] to obtain

$$\chi_{\mathcal{S}}^k \text{ converges weakly-}* \text{ in } L^\infty((0,T) \times \mathbb{R}^3), \text{ strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

$$\rho^k \chi_{\mathcal{S}}^k \text{ converges weakly-}* \text{ in } L^\infty((0,T) \times \mathbb{R}^3), \text{ strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty).$$

Let the limit of $\chi_{\mathcal{S}}^k$ be denoted by $\chi_{\mathcal{S}}$, it satisfies

$$\frac{\partial \chi_{\mathcal{S}}}{\partial t} + \mathrm{div}(\overline{u_{\mathcal{S}}} \ \chi_{\mathcal{S}}) = 0 \text{ in } (0,T) \times \mathbb{R}^3, \quad \chi_{\mathcal{S}}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3.$$

Let the weak limit of $\rho^k \chi_{\mathcal{S}}^k$ be denoted by $\overline{\rho \chi_{\mathcal{S}}}$; it satisfies

$$\frac{\partial (\overline{\rho \chi_{\mathcal{S}}})}{\partial t} + \mathrm{div}(\overline{u_{\mathcal{S}}} \ \overline{\rho \chi_{\mathcal{S}}}) = 0 \text{ in } (0,T) \times \mathbb{R}^3, \quad \overline{\rho \chi_{\mathcal{S}}}|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3.$$

We follow the similar analysis as for the fluid case explained in [30, Section 7.8.1, Page 362] to conclude that

$$\rho^k \to \rho \text{ strongly in } L^p((0,T) \times \Omega), \quad \forall \ 1 \leqslant p < \frac{4}{3}\beta \text{ with } \ \beta \geqslant \max\{8, \gamma\}, \ \gamma > 3/2.$$

The strong convergences of ρ^k and $\chi_{\mathcal{S}}^k$ help us to identify the limit:

$$\overline{\rho \chi_{\mathcal{S}}} = \rho \chi_{\mathcal{S}}.$$

Using the convergences of $\rho^k \chi_{\mathcal{S}}^k$ and u^k in the equation

$$P_{\mathcal{S}}^k u^k(t,x) = \frac{1}{m^k} \int_\Omega \rho^k \chi_{\mathcal{S}}^k u^k + \left((J^k)^{-1} \int_\Omega \rho^k \chi_{\mathcal{S}}^k ((y - h^k(t)) \times u^k) \ dy \right) \times (x - h^k(t)),$$

and the convergence in (3.9), we conclude that

$$\overline{u_{\mathcal{S}}} = P_{\mathcal{S}} u.$$

The convergence of the isometric propagator $\eta_{t,s}^k$ follows from the convergence of $P_{\mathcal{S}}^k u^k$ and equation (3.3). $\qquad \square$

We need the next result on weak sequential continuity to analyze the limiting system of Faedo-Galerkin as $N \to \infty$ in Section 4.2. The proof is similar to that of Proposition 3.2 and we skip it here.

Proposition 3.3. *Let us assume that $\rho_0^N \in W^{1,\infty}(\Omega)$ with $\rho_0^N \to \rho_0$ in $W^{1,\infty}(\Omega)$, ρ^N satisfies (3.6) and*

$$\rho^N \to \rho \text{ strongly in } L^p((0,T) \times \Omega), \ 1 \leqslant p < \frac{4}{3}\beta \text{ with } \beta \geqslant \max\{8, \gamma\}, \ \gamma > 3/2.$$

Let $\{u^N, \chi_{\mathcal{S}}^N\}$ be a bounded sequence in $L^\infty(0,T; L^2(\Omega)) \times L^\infty((0,T) \times \mathbb{R}^3)$ satisfying (3.7). Let $\{\rho^N \chi_{\mathcal{S}}^N\}$ be a bounded sequence in $L^\infty((0,T) \times \mathbb{R}^3)$ satisfying (3.8). Then, up to a subsequence, we have

$$u^N \to u \text{ weakly-} * \text{ in } L^\infty(0,T; L^2(\Omega)),$$

$$\chi_{\mathcal{S}}^N \to \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)), \ \forall \ 1 \leqslant p < \infty,$$

$$\rho^N \chi_{\mathcal{S}}^N \to \rho \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)), \ \forall \ 1 \leqslant p < \infty,$$

where $\chi_{\mathcal{S}}$ and $\rho \chi_{\mathcal{S}}$ satisfying (3.1) and (3.4), respectively. Moreover,

$$P_{\mathcal{S}}^N u^N \to P_{\mathcal{S}} u \text{ weakly-} * \text{ in } L^\infty(0,T; C_{loc}^\infty(\mathbb{R}^3)),$$

$$\eta_{t,s}^N \to \eta_{t,s} \text{ weakly-} * \text{ in } W^{1,\infty}((0,T)^2; C_{loc}^\infty(\mathbb{R}^3)).$$

At the level of the Galerkin approximation, we have boundedness of $\sqrt{\rho^N} u^N$ in $L^\infty(0,T; L^2(\Omega))$ and ρ^N is strictly positive, which means that we get the boundedness of u^N in $L^\infty(0,T; L^2(\Omega))$. So, we can use Proposition 3.3 in the convergence analysis of the Galerkin scheme. In the case of the ε-level for the compressible fluid, we have boundedness of $\sqrt{\rho^\varepsilon} u^\varepsilon$ in $L^\infty(0,T; L^2(\Omega))$ but ρ^ε is only non-negative. On the other hand, we establish boundedness of u^ε in $L^2(0,T; H^1(\Omega))$. We need the following result for the convergence analysis of the vanishing viscosity limit in Section 4.3.

Proposition 3.4. *Let $\rho_0^\varepsilon \in W^{1,\infty}(\Omega)$ with $\rho_0^\varepsilon \to \rho_0$ in $L^\beta(\Omega)$, ρ^ε satisfies*

$$\frac{\partial \rho^\varepsilon}{\partial t} + \text{div}(\rho^\varepsilon u^\varepsilon) = \Delta \rho^\varepsilon \text{ in } (0,T) \times \Omega, \quad \frac{\partial \rho^\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad \rho^\varepsilon(0,x) = \rho_0^\varepsilon(x) \text{ in } \Omega, \quad \frac{\partial \rho_0^\varepsilon}{\partial \nu}\big|_{\partial\Omega} = 0.,$$

and

$$\rho^\varepsilon \to \rho \text{ weakly in } L^{\beta+1}((0,T) \times \Omega), \text{ with } \beta \geqslant \max\{8, \gamma\}, \ \gamma > 3/2. \tag{3.10}$$

Let $\{u^\varepsilon, \chi_{\mathcal{S}}^\varepsilon\}$ be a bounded sequence in $L^2(0,T; H^1(\Omega)) \times L^\infty((0,T) \times \mathbb{R}^3)$ satisfying

$$\frac{\partial \chi_{\mathcal{S}}^\varepsilon}{\partial t} + \text{div}(P_{\mathcal{S}}^\varepsilon u^\varepsilon \chi_{\mathcal{S}}^\varepsilon) = 0 \text{ in } (0,T) \times \mathbb{R}^3, \quad \chi_{\mathcal{S}}^\varepsilon|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3, \tag{3.11}$$

and let $\{\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon\}$ be a bounded sequence in $L^\infty((0,T) \times \mathbb{R}^3)$ satisfying

$$\frac{\partial}{\partial t}(\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon) + \text{div}(P_{\mathcal{S}}^\varepsilon u^\varepsilon (\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon)) = 0 \text{ in } (0,T) \times \mathbb{R}^3, \quad \rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon|_{t=0} = \rho_0^\varepsilon \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3, \tag{3.12}$$

where $P_{\mathcal{S}}^\varepsilon : L^2(\Omega) \to L^2(\mathcal{S}^\varepsilon(t))$ is the orthogonal projection onto rigid fields with $\mathcal{S}^\varepsilon(t) \Subset \Omega$ being a bounded, regular domain for all $t \in [0,T]$. Then up to a subsequence, we have

$$u^\varepsilon \to u \text{ weakly in } L^2(0,T; H^1(\Omega)),$$

$$\chi_{\mathcal{S}}^\varepsilon \to \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

$$\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon \to \rho \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

with $\chi_{\mathcal{S}}$ and $\rho \chi_{\mathcal{S}}$ satisfying (3.1) and (3.4) respectively. Moreover,

$$P_{\mathcal{S}}^\varepsilon u^\varepsilon \to P_{\mathcal{S}} u \text{ weakly in } L^2(0,T; C_{loc}^\infty(\mathbb{R}^3)),$$

$$\eta_{t,s}^\varepsilon \to \eta_{t,s} \text{ weakly in } H^1((0,T)^2; C_{loc}^\infty(\mathbb{R}^3)).$$

Proof. As $\{u^\varepsilon\}$ is a bounded sequence in $L^2(0,T; H^1(\Omega))$ and $\{\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon\}$ is bounded in $L^\infty((0,T) \times \mathbb{R}^3)$, we obtain that $\{P_{\mathcal{S}}^\varepsilon u^\varepsilon\}$ is bounded in $L^2(0,T; \mathcal{R})$. Thus, up to a subsequence,

$$P_{\mathcal{S}}^\varepsilon u^\varepsilon \to \overline{u_{\mathcal{S}}} \text{ weakly in } L^2(0,T; \mathcal{R}). \tag{3.13}$$

Here, obviously $P_{\mathcal{S}}^\varepsilon u^\varepsilon \in L^1(0,T; L_{loc}^\infty(\mathbb{R}^3))$, $\text{div}(P_{\mathcal{S}}^\varepsilon u^\varepsilon) = 0$ and $\overline{u_{\mathcal{S}}} \in L^1(0,T; W_{loc}^{1,1}(\mathbb{R}^3))$ satisfies

$$\frac{\overline{u_{\mathcal{S}}}}{1 + |x|} \in L^1(0,T; L^1(\mathbb{R}^3)).$$

Moreover, $\{\chi_{\mathcal{S}}^{\varepsilon}\}$ is bounded in $L^{\infty}((0,T) \times \mathbb{R}^3)$, $\chi_{\mathcal{S}}^{\varepsilon}$ satisfies (3.11) and $\{\rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon}\}$ is bounded in $L^{\infty}((0,T) \times \mathbb{R}^3)$, $\rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon}$ satisfies (3.12). As we have verified all the required conditions, we can apply [6, Theorem II.4, Page 521] to obtain

$$\chi_{\mathcal{S}}^{\varepsilon} \text{ converges weakly-} * \text{ in } L^{\infty}((0,T) \times \mathbb{R}^3), \text{ strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

$$\rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon} \text{ converges weakly-} * \text{ in } L^{\infty}((0,T) \times \mathbb{R}^3), \text{ strongly in } C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty).$$

Let the limit of $\chi_{\mathcal{S}}^{\varepsilon}$ be denoted by $\chi_{\mathcal{S}}$; it satisfies

$$\frac{\partial \chi_{\mathcal{S}}}{\partial t} + \operatorname{div}(\overline{u_{\mathcal{S}}} \ \chi_{\mathcal{S}}) = 0 \text{ in } \mathbb{R}^3, \quad \chi_{\mathcal{S}}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3.$$

Let the limit of $\rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon}$ be denoted by $\overline{\rho\chi_{\mathcal{S}}}$; it satisfies

$$\frac{\partial (\overline{\rho\chi_{\mathcal{S}}})}{\partial t} + \operatorname{div}(\overline{u_{\mathcal{S}}} \ \overline{\rho\chi_{\mathcal{S}}}) = 0 \text{ in } (0,T) \times \mathbb{R}^3, \quad \overline{\rho\chi_{\mathcal{S}}}|_{t=0} = \rho_0 \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3.$$

The weak convergence of ρ^{ε} and strong convergence of $\chi_{\mathcal{S}}^{\varepsilon}$ help us to identify the limit:

$$\overline{\rho\chi_{\mathcal{S}}} = \rho\chi_{\mathcal{S}}.$$

Using the convergences of $\rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon}$ and u^{ε} in the equation

$$P_{\mathcal{S}}^{\varepsilon} u^{\varepsilon}(t,x) = \frac{1}{m^{\varepsilon}} \int_{\Omega} \rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon} u^{\varepsilon} + \left((J^{\varepsilon})^{-1} \int_{\Omega} \rho^{\varepsilon}\chi_{\mathcal{S}}^{\varepsilon}((y-h^{\varepsilon}(t)) \times u^{\varepsilon}) \ dy \right) \times (x - h^{\varepsilon}(t)),$$

and the convergence in (3.13), we conclude that

$$\overline{u_{\mathcal{S}}} = P_{\mathcal{S}} u.$$

The convergence of the isometric propagators $\eta_{t,s}^{\varepsilon}$ follows from the convergence of $P_{\mathcal{S}}^{\varepsilon} u^{\varepsilon}$ and equation (3.3). $\qquad\square$

In the limit of u^{δ}, we can expect the boundedness of the limit only in $L^2(0,T; L^2(\Omega))$ but not in $L^2(0,T; H^1(\Omega))$. That is why we need a different sequential continuity result, which we use in Section 5.

Proposition 3.5. *Let $\rho_0^{\delta} \in L^{\beta}(\Omega)$ with $\rho_0^{\delta} \to \rho_0$ in $L^{\gamma}(\Omega)$, let ρ^{δ} satisfy*

$$\frac{\partial \rho^{\delta}}{\partial t} + \operatorname{div}(\rho^{\delta} u^{\delta}) = 0 \ in \ (0,T) \times \Omega, \quad \rho^{\delta}(0,x) = \rho_0^{\delta}(x) \ in \ \Omega,$$

and

$$\rho^{\delta} \to \rho \ weakly \ in \ L^{\gamma+\theta}((0,T) \times \Omega), \ with \ \gamma > 3/2, \ \theta = \frac{2}{3}\gamma - 1. \tag{3.14}$$

Let $\{u^{\delta}, \chi_{\mathcal{S}}^{\delta}\}$ be a bounded sequence in $L^2(0,T; L^2(\Omega)) \times L^{\infty}((0,T) \times \mathbb{R}^3)$ satisfying

$$\frac{\partial \chi_{\mathcal{S}}^{\delta}}{\partial t} + \operatorname{div}(P_{\mathcal{S}}^{\delta} u^{\delta} \chi_{\mathcal{S}}^{\delta}) = 0 \ in \ (0,T) \times \mathbb{R}^3, \quad \chi_{\mathcal{S}}^{\delta}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \ in \ \mathbb{R}^3, \tag{3.15}$$

and let $\{\rho^{\delta}\chi_{\mathcal{S}}^{\delta}\}$ be a bounded sequence in $L^{\infty}((0,T) \times \mathbb{R}^3)$ satisfying

$$\frac{\partial}{\partial t}(\rho^{\delta}\chi_{\mathcal{S}}^{\delta}) + \operatorname{div}(P_{\mathcal{S}}^{\delta} u^{\delta}(\rho^{\delta}\chi_{\mathcal{S}}^{\delta})) = 0 \ in \ (0,T) \times \mathbb{R}^3, \quad \rho^{\delta}\chi_{\mathcal{S}}^{\delta}|_{t=0} = \rho_0^{\delta}\mathbb{1}_{\mathcal{S}_0} \ in \ \mathbb{R}^3, \tag{3.16}$$

where $P_{\mathcal{S}}^{\delta} : L^2(\Omega) \to L^2(\mathcal{S}^{\delta}(t))$ is the orthogonal projection onto rigid fields with $\mathcal{S}^{\delta}(t) \Subset \Omega$ being a bounded, regular domain for all $t \in [0,T]$. Then, up to a subsequence, we have

$$u^{\delta} \to u \ weakly \ \ in \ L^2(0,T; L^2(\Omega)),$$

$$\chi_{\mathcal{S}}^{\delta} \to \chi_{\mathcal{S}} \ weakly\text{-} * \ in \ L^{\infty}((0,T) \times \mathbb{R}^3) \ and \ \ strongly \ \ in \ C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

$$\rho^{\delta}\chi_{\mathcal{S}}^{\delta} \to \rho\chi_{\mathcal{S}} \ weakly\text{-} * \ in \ L^{\infty}((0,T) \times \mathbb{R}^3) \ and \ \ strongly \ \ in \ C([0,T]; L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

with $\chi_{\mathcal{S}}$ and $\rho\chi_{\mathcal{S}}$ satisfying (3.1) and (3.4) respectively. Moreover,

$$P_{\mathcal{S}}^{\delta} u^{\delta} \to P_{\mathcal{S}} u \ weakly \ \ in \ L^2(0,T; C_{loc}^{\infty}(\mathbb{R}^3)),$$

$$\eta_{t,s}^{\delta} \to \eta_{t,s} \ weakly \ \ in \ H^1((0,T)^2; C_{loc}^{\infty}(\mathbb{R}^3)).$$

Proof. As $\{u^\delta\}$ is a bounded sequence in $L^2(0, T; L^2(\Omega))$ and $\{\rho^\delta \chi_{\mathcal{S}}^\delta\}$ is bounded in $L^\infty((0, T) \times \mathbb{R}^3)$, we obtain that $\{P_{\mathcal{S}}^\delta u^\delta\}$ is bounded in $L^2(0, T; \mathcal{R})$. Thus, up to a subsequence,

$$P_{\mathcal{S}}^\delta u^\delta \to \overline{u_{\mathcal{S}}} \text{ weakly in } L^2(0, T; \mathcal{R}). \tag{3.17}$$

Here, obviously $P_{\mathcal{S}}^\delta u^\delta \in L^1(0, T; L^\infty_{loc}(\mathbb{R}^3))$, $\operatorname{div}(P_{\mathcal{S}}^\delta u^\delta) = 0$ and $\overline{u_{\mathcal{S}}} \in L^1(0, T; W^{1,1}_{loc}(\mathbb{R}^3))$ satisfies

$$\frac{\overline{u_{\mathcal{S}}}}{1 + |x|} \in L^1(0, T; L^1(\mathbb{R}^3)).$$

Moreover, $\{\chi_{\mathcal{S}}^\delta\}$ is bounded in $L^\infty((0, T) \times \mathbb{R}^3)$, $\chi_{\mathcal{S}}^\delta$ satisfies (3.15) and $\{\rho^\delta \chi_{\mathcal{S}}^\delta\}$ is bounded in $L^\infty((0, T) \times \mathbb{R}^3)$, $\rho^\delta \chi_{\mathcal{S}}^\delta$ satisfies (3.16). Now we can apply [6, Theorem II.4, Page 521] to obtain

$$\chi_{\mathcal{S}}^\delta \text{ converges weakly-} * \text{ in } L^\infty((0, T) \times \mathbb{R}^3), \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)) \ (1 \leqslant p < \infty),$$

$$\rho^\delta \chi_{\mathcal{S}}^\delta \text{ converges weakly-} * \text{ in } L^\infty((0, T) \times \mathbb{R}^3) \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)) \ (1 \leqslant p < \infty).$$

Let the weak limit of $\chi_{\mathcal{S}}^\delta$ be denoted by $\chi_{\mathcal{S}}$. Then it satisfies

$$\frac{\partial \chi_{\mathcal{S}}}{\partial t} + \operatorname{div}(\overline{u_{\mathcal{S}}} \ \chi_{\mathcal{S}}) = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad \chi_{\mathcal{S}}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3,$$

Let the limit of $\rho^\delta \chi_{\mathcal{S}}^\delta$ be denoted by $\overline{\rho \chi_{\mathcal{S}}}$; it satisfies

$$\frac{\partial (\overline{\rho \chi_{\mathcal{S}}})}{\partial t} + \operatorname{div}(\overline{u_{\mathcal{S}}} \ \overline{\rho \chi_{\mathcal{S}}}) = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad \overline{\rho \chi_{\mathcal{S}}}|_{t=0} = \rho_0 \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3.$$

From (3.14), we know that

$$\rho^\delta \to \rho \text{ weakly in } L^{\gamma+\theta}((0, T) \times \Omega), \text{ with } \gamma > 3/2, \theta = \frac{2}{3}\gamma - 1.$$

The weak convergence of ρ^δ to ρ and strong convergence of $\chi_{\mathcal{S}}^\delta$ to $\chi_{\mathcal{S}}$ help us to identify the limit:

$$\overline{\rho \chi_{\mathcal{S}}} = \rho \chi_{\mathcal{S}},$$

Using the convergences of $\rho^\delta \chi_{\mathcal{S}}^\delta$ and u^δ in the equation

$$P_{\mathcal{S}}^\delta u^\delta(t, x) = \frac{1}{m^\delta} \int_\Omega \rho^\delta \chi_{\mathcal{S}}^\delta u^\delta + \left((J^\delta)^{-1} \int_\Omega \rho^\delta \chi_{\mathcal{S}}^\delta ((y - h^\delta(t)) \times u^\delta) \ dy \right) \times (x - h^\delta(t)),$$

and the convergence in (3.17), we conclude that

$$\overline{u_{\mathcal{S}}} = P_{\mathcal{S}} u.$$

The convergence of the isometric propagator $\eta_{t,s}^\delta$ follows from the convergence of $P_{\mathcal{S}}^\delta u^\delta$ and equation (3.3). $\qquad \square$

4. EXISTENCE PROOFS OF APPROXIMATE SOLUTIONS

In this section, we present the proofs of the existence results of the three approximation levels. We start with the N-level approximation in Section 4.1 and the limit as $N \to \infty$ in Section 4.2, which yields existence at the ε-level. The convergence of $\varepsilon \to 0$, considered in Section 4.3, then shows existence of solutions at the δ-level. The final limit problem as $\delta \to 0$ is the topic of Section 5.

4.1. **Existence of the Faedo-Galerkin approximation.**

In this subsection, we construct a solution $(\mathcal{S}^N, \rho^N, u^N)$ to the problem (2.28)–(2.33). First we recall a known maximal regularity result for the parabolic problem (2.29):

Proposition 4.1. [30, Proposition 7.39, Page 345] *Suppose that Ω is a regular bounded domain and assume $\rho_0 \in W^{1,\infty}(\Omega)$, $\underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho}$, $u \in L^\infty(0, T; W^{1,\infty}(\Omega))$. Then the parabolic problem (2.29) admits a unique solution in the solution space*

$$\rho \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

and it satisfies

$$\underline{\rho} \exp\left(-\int_0^\tau \| \operatorname{div} u(s) \|_{L^\infty(\Omega)} \ ds \right) \leqslant \rho(\tau, x) \leqslant \overline{\rho} \exp\left(\int_0^\tau \| \operatorname{div} u(s) \|_{L^\infty(\Omega)} \ ds \right) \tag{4.1}$$

for any $\tau \in [0, T]$.

Proof of Proposition 2.6. The idea is to view our Galerkin approximation as a fixed point problem and then apply Schauder's fixed point theorem to it. We set

$$B_{R,T} = \{u \in C([0,T]; X_N), \ \|u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant R\},$$

for R and T positive which will be fixed in Step 3.

Step 1: Continuity equation and transport of the body. Given $u \in B_{R,T}$, let ρ be the solution to

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = \varepsilon \Delta \rho \text{ in } (0,T) \times \Omega, \quad \frac{\partial \rho}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad \rho(0) = \rho_0^N, \quad 0 < \underline{\rho} \leqslant \rho_0^N \leqslant \overline{\rho}, \tag{4.2}$$

and let $\chi_{\mathcal{S}}$ satisfy

$$\frac{\partial \chi_{\mathcal{S}}}{\partial t} + P_{\mathcal{S}} u \cdot \nabla \chi_{\mathcal{S}} = 0, \quad \chi_{\mathcal{S}}|_{t=0} = \mathbb{1}_{\mathcal{S}_0}, \tag{4.3}$$

and

$$\frac{\partial}{\partial t}(\rho \chi_{\mathcal{S}}) + P_{\mathcal{S}} u \cdot \nabla(\rho \chi_{\mathcal{S}}) = 0, \quad (\rho \chi_{\mathcal{S}})|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_0}, \tag{4.4}$$

where $P_{\mathcal{S}} u \in \mathcal{R}$ and it is given by (3.2).

Since $\rho_0^N \in W^{1,\infty}(\Omega)$, $u \in B_{R,T}$ in (4.2), we can apply Proposition 4.1 to conclude that $\rho > 0$ and

$$\rho \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

Moreover, by Proposition 3.1 we obtain

$$\chi_{\mathcal{S}} \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \ \forall 1 \leqslant p < \infty,$$
$$\rho \chi_{\mathcal{S}} \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \ \forall 1 \leqslant p < \infty.$$

Consequently, we define

$$\mu = (1 - \chi_{\mathcal{S}})\mu_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}}, \quad \lambda = (1 - \chi_{\mathcal{S}})\lambda_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}} \text{ so that } \mu > 0, \ 2\mu + 3\lambda \geqslant 0,$$
$$g = (1 - \chi_{\mathcal{S}})g_{\mathcal{F}} + \chi_{\mathcal{S}} g_{\mathcal{S}}, \quad p(\rho) = a\rho^\gamma + \delta\rho^\beta \quad \text{with} \quad a = a_{\mathcal{F}}(1 - \chi_{\mathcal{S}}).$$

Step 2: Momentum equation. Given $u \in B_{R,T}$, let us consider the following equation satisfied by $\widetilde{u} : [0,T] \mapsto X_N$:

$$-\int_0^T \int_\Omega \rho\Big(\frac{\partial \widetilde{u}}{\partial t} \cdot e_j + (u \cdot \nabla e_j) \cdot \widetilde{u}\Big) + \int_0^T \int_\Omega \Big(2\mu \mathbb{D}(\widetilde{u}) : \mathbb{D}(e_j) + \lambda \operatorname{div} \widetilde{u} \mathbb{I} : \mathbb{D}(e_j) - p(\rho)\mathbb{I} : \mathbb{D}(e_j)\Big)$$

$$+ \int_0^T \int_\Omega \varepsilon \nabla \widetilde{u} \nabla \rho \cdot e_j + \alpha \int_0^T \int_{\partial\Omega} (\widetilde{u} \times \nu) \cdot (e_j \times \nu) + \alpha \int_0^T \int_{\partial \mathcal{S}^N(t)} [(\widetilde{u} - P_{\mathcal{S}}\widetilde{u}) \times \nu] \cdot [(e_j - P_{\mathcal{S}} e_j) \times \nu]$$

$$+ \frac{1}{\delta} \int_0^T \int_\Omega \chi_{\mathcal{S}}(\widetilde{u} - P_{\mathcal{S}}\widetilde{u}) \cdot (e_j - P_{\mathcal{S}} e_j) = \int_0^T \int_\Omega \rho g \cdot e_j, \tag{4.5}$$

where $\rho, \chi_{\mathcal{S}}$ are defined as in Step 1. We can write

$$\widetilde{u}(t, \cdot) = \sum_{i=1}^N g_i(t)e_i, \quad \widetilde{u}(0) = u_0^N = \sum_{i=1}^N \left(\int_\Omega u_0 \cdot e_i\right) e_i.$$

The coefficients $\{g_i\}$ of $\widetilde{u}$ satisfy the ordinary differential equation,

$$\sum_{i=1}^N a_{i,j} g_i'(t) + \sum_{i=1}^N b_{i,j} g_i(t) = f_j(t), \quad g_i(0) = \int_\Omega u_0^N \cdot e_i, \tag{4.6}$$

where $a_{i,j}$, $b_{i,j}$ and f_j are given by

$$a_{i,j} = \int\limits_0^T \int\limits_\Omega \rho e_i e_j,$$

$$b_{i,j} = \int\limits_0^T \int\limits_\Omega \rho(u \cdot \nabla e_j) \cdot e_i + \int\limits_0^T \int\limits_\Omega \Big(2\mu \mathbb{D}(e_i) : \mathbb{D}(e_j) + \lambda \operatorname{div} e_i \mathbb{I} : \mathbb{D}(e_j) \Big) + \int\limits_0^T \int\limits_\Omega \varepsilon \nabla e_i \nabla \rho \cdot e_j$$

$$+ \alpha \int\limits_0^T \int\limits_{\partial\Omega} (e_i \times \nu) \cdot (e_j \times \nu) + \alpha \int\limits_0^T \int\limits_{\partial\mathcal{S}(t)} [(e_i - P_\mathcal{S}e_i) \times \nu] \cdot [(e_j - P_\mathcal{S}e_j) \times \nu] + \frac{1}{\delta} \int\limits_0^T \int\limits_\Omega \chi_\mathcal{S}(e_i - P_\mathcal{S}e_i) \cdot (e_j - P_\mathcal{S}e_j),$$

$$f_j = \int\limits_0^T \int\limits_\Omega \rho g \cdot e_j + \int\limits_0^T \int\limits_\Omega p(\rho)\mathbb{I} : \mathbb{D}(e_j).$$

Observe that the positive lower bound of ρ in Proposition 4.1 guarantees the invertibility of the matrix $(a_{i,j}(t))_{1\leqslant i,j\leqslant N}$. We use the regularity of ρ (Proposition 4.1), of $\chi_\mathcal{S}$ and of the propagator associated to $P_\mathcal{S}u$ (Proposition 3.1) to conclude the continuity of $(a_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $(b_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $(f_i(t))_{1\leqslant i\leqslant N}$. The existence and uniqueness theorem for ordinary differential equations gives that system (4.6) has a unique solution defined on $[0,T]$ and therefore equation (4.5) has a unique solution

$$\widetilde{u} \in C([0,T]; X_N).$$

Step 3: Well-definedness of $\mathcal{N}$. Let us define a map

$$\mathcal{N} : B_{R,T} \to C([0,T], X_N)$$
$$u \mapsto \widetilde{u},$$

where $\widetilde{u}$ satisfies (4.5). Since we know the existence of $\widetilde{u} \in C([0,T]; X_N)$ to the problem (4.5), we have that $\mathcal{N}$ is well-defined from $B_{R,T}$ to $C([0,T]; X_N)$. Now we establish the fact that $\mathcal{N}$ maps $B_{R,T}$ to itself for suitable R and T.

We fix

$$0 < \sigma < \frac{1}{2} \operatorname{dist}(\mathcal{S}_0, \partial\Omega).$$

Given $u \in B_{R,T}$, we want to estimate $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))}$. We have the following identities via a simple integration by parts:

$$\int\limits_0^t \int\limits_\Omega \rho \widetilde{u}' \cdot \widetilde{u} = -\frac{1}{2} \int\limits_0^t \int\limits_\Omega \frac{\partial\rho}{\partial t} |\widetilde{u}|^2 + \frac{1}{2}(\rho|\widetilde{u}|^2)(t) - \frac{1}{2}\rho_0|u_0|^2, \tag{4.7}$$

$$\int\limits_0^T \int\limits_\Omega \rho(u \cdot \nabla)\widetilde{u} \cdot \widetilde{u} = -\frac{1}{2} \int\limits_0^T \int\limits_\Omega \operatorname{div}(\rho u)|\widetilde{u}|^2, \tag{4.8}$$

$$\int\limits_\Omega \nabla(\rho^\gamma) \cdot \widetilde{u} = \frac{\gamma}{\gamma-1} \int\limits_\Omega \nabla(\rho^{\gamma-1}) \cdot \rho\widetilde{u} = -\frac{\gamma}{\gamma-1} \int\limits_\Omega \rho^{\gamma-1} \operatorname{div}(\rho\widetilde{u}) = \frac{1}{\gamma-1}\frac{d}{dt} \int\limits_\Omega \rho^\gamma - \frac{\varepsilon\gamma}{\gamma-1} \int\limits_\Omega \rho^{\gamma-1}\Delta\rho$$

$$= \frac{1}{\gamma-1}\frac{d}{dt} \int\limits_\Omega \rho^\gamma + \varepsilon\gamma \int\limits_\Omega \rho^{\gamma-2}|\nabla\rho|^2 \geqslant \frac{1}{\gamma-1}\frac{d}{dt} \int\limits_\Omega \rho^\gamma. \tag{4.9}$$

Similarly,

$$\int\limits_\Omega \nabla(\rho^\beta) \cdot \widetilde{u} = \frac{1}{\beta-1}\frac{d}{dt} \int\limits_\Omega \rho^\beta + \varepsilon\beta \int\limits_\Omega \rho^{\beta-2}|\nabla\rho|^2. \tag{4.10}$$

We multiply equation (4.5) by g_j, add these equations for $j = 1, 2, ..., N$, use the relations (4.7)–(4.10) and the continuity equation (4.2) to obtain the following energy estimate:

$$\int_\Omega \left(\frac{1}{2}\rho|\widetilde{u}|^2 + \frac{a}{\gamma - 1}\rho^\gamma + \frac{\delta}{\beta - 1}\rho^\beta\right) + \int_0^T \int_\Omega \left(2\mu|\mathbb{D}(\widetilde{u})|^2 + \lambda|\operatorname{div}\widetilde{u}|^2\right) + \delta\varepsilon\beta\int_0^T\int_\Omega \rho^{\beta-2}|\nabla\rho|^2 + \alpha\int_0^T\int_{\partial\Omega}|\widetilde{u}\times\nu|^2$$

$$+ \alpha\int_0^T\int_{\partial\mathcal{S}(t)}|(\widetilde{u} - P_\mathcal{S}\widetilde{u})\times\nu|^2 + \frac{1}{\delta}\int_0^T\int_\Omega \chi_\mathcal{S}|\widetilde{u} - P_\mathcal{S}\widetilde{u}|^2 \leqslant \int_0^T\int_\Omega \rho g\cdot\widetilde{u} + \int_\Omega\left(\frac{1}{2}\frac{\rho_0^N}{|q_0^N|^2}\mathbb{1}_{\{\rho_0>0\}} + \frac{a}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta\right)$$

$$\leqslant \sqrt{\overline{\rho}}T\left(\frac{1}{2\widetilde{\varepsilon}}\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\widetilde{\varepsilon}}{2}\|\sqrt{\rho}\widetilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\right) + \int_\Omega\left(\frac{1}{2}\frac{\rho_0^N}{|q_0^N|^2}\mathbb{1}_{\{\rho_0>0\}} + \frac{a}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta\right). \quad (4.11)$$

An appropriate choice of $\widetilde{\varepsilon}$ in (4.11) gives us

$$\|\widetilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant \frac{4\overline{\rho}}{\underline{\rho}}T^2\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}}E_0^N,$$

where $\overline{\rho}$ and $\underline{\rho}$ are the upper and lower bounds of ρ. In order to get $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant R$, we need

$$R^2 \geqslant \frac{4\overline{\rho}}{\underline{\rho}}T^2\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}}E_0^N. \quad (4.12)$$

We also need to verify that for T small enough and for any $u \in B_{R,T}$,

$$\inf_{u\in B_{R,T}}\operatorname{dist}(\mathcal{S}(t),\partial\Omega) \geqslant 2\sigma > 0 \quad (4.13)$$

holds. We can write $\mathcal{S}(t) = \eta_{t,0}(\mathcal{S}_0)$ with the isometric propagator $\eta_{t,s}$ (see equation (3.3) for the precise definition) associated to the rigid field $P_\mathcal{S}u = h'(t) + \omega(t)\times(y - h(t))$. From [16, Proposition 4.6, Step 2], we conclude: proving (4.13) is equivalent to establishing the following bound:

$$\sup_{t\in[0,T]}|\partial_t\eta_{t,0}(t,y)| < \frac{\operatorname{dist}(\mathcal{S}_0,\partial\Omega) - 2\sigma}{T}, \quad t\in[0,T], y\in\mathcal{S}_0. \quad (4.14)$$

Using equation (3.3), we have

$$|\partial_t\eta_{t,0}(t,y)| = |P_\mathcal{S}u(t,\eta_{t,0}(t,y))|.$$

The expressions

$$P_\mathcal{S}u(t,x) = h'(t) + \omega_i(t)\times(x - h(t)), \quad x\in\mathbb{R}^3$$

and

$$\eta_{t,0}(t,y) = h(t) + \mathbb{O}(t)(y) \text{ where } \mathbb{O}(t)\in SO(3), \ y\in\mathcal{S}_0,$$

along with the isometric property of the propagator $\eta_{t,0}$ yield the following:

$$|P_\mathcal{S}u(t,\eta_{t,0}(t,y))| = |h'(t) + \omega(t)\times(\eta_{t,0}(y) - \eta_{t,0}(0))| \leqslant |h'(t)| + |\omega(t)||\eta_{t,0}(y) - \eta_{t,0}(0)| = |h'(t)| + |\omega(t)||y|.$$

Furthermore, if $\overline{\rho}$ is the upper bound of ρ, then for $u \in B_{R,T}$

$$|h'(t)|^2 + J(t)\omega(t)\cdot\omega(t) = \int_{\mathcal{S}(t)}\rho|P_\mathcal{S}u(t,\cdot)|^2 \leqslant \int_\Omega\rho|u(t,\cdot)|^2 \leqslant \overline{\rho}R^2 \quad (4.15)$$

for any R and $t\in(0,T)$. As $J(t)$ is congruent to $J(0)$, they have the same eigenvalues and we have

$$\lambda_0|\omega(t)|^2 \leqslant J(t)\omega(t)\cdot\omega(t),$$

where λ_0 is the smallest eigenvalue of $J(0)$. Observe that for $t\in[0,T]$, $y\in\mathcal{S}_0$,

$$|h'(t)| + |\omega(t)||y| \leqslant \sqrt{2}(|h'(t)|^2 + |\omega(t)|^2|y|^2)^{1/2} \leqslant \sqrt{2}\max\{1,|y|\}(|h'(t)|^2 + |\omega(t)|^2)^{1/2}$$
$$\leqslant C_0\left(|h'(t)|^2 + J(t)\omega(t)\cdot\omega(t)\right)^{1/2}, \quad (4.16)$$

where $C_0 = \sqrt{2}\frac{\max\{1,|y|\}}{\min\{1,\lambda_0\}^{1/2}}$. Thus, with the help of (4.15)–(4.16) and the relation of R in (4.12), we can conclude that any

$$T < \frac{\text{dist}(\mathcal{S}_0, \partial\Omega) - 2\sigma}{C_0|\overline{\rho}|^{1/2}[\frac{4\overline{\rho}}{\underline{\rho}}T^2\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}}E_0^N]^{1/2}}, \tag{4.17}$$

satisfies the relation (4.13). Thus, we choose T satisfying (4.17) and fix it. Then we choose R as in (4.12) to conclude that $\mathcal{N}$ maps $B_{R,T}$ to itself.

Step 4: Continuity of $\mathcal{N}$. We show that if a sequence $\{u^k\} \subset B_{R,T}$ is such that $u^k \to u$ in $B_{R,T}$, then $\mathcal{N}(u^k) \to \mathcal{N}(u)$ in $B_{R,T}$. As $\text{span}(e_1, e_2, ..., e_N)$ is a finite dimensional subspace of $\mathcal{D}(\overline{\Omega})$, we have $u^k \to u$ in $C([0,T]; \mathcal{D}(\overline{\Omega}))$. Given $\{u^k\} \subset B_{R,T}$, we have that $\rho^k \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ is the solution to (4.2), $\chi_{\mathcal{S}}^k$ is bounded in $L^\infty((0,T) \times \mathbb{R}^3)$ satisfying (4.3) and $\{\rho^k\chi_{\mathcal{S}}^k\}$ is a bounded sequence in $L^\infty((0,T) \times \mathbb{R}^3)$ satisfying (4.4). We apply Proposition 3.2 to obtain

$$\chi_{\mathcal{S}}^k \to \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)), \forall 1 \leqslant p < \infty,$$
$$P_{\mathcal{S}}^k u^k \to P_{\mathcal{S}} u \text{ strongly in } C([0,T]; C^\infty_{loc}(\mathbb{R}^3)),$$
$$\eta_{t,s}^k \to \eta_{t,s} \text{ strongly in } C^1([0,T]^2; C^\infty_{loc}(\mathbb{R}^3)).$$

We use the continuity argument as in Step 2 to conclude

$$a_{i,j}^k \to a_{i,j}, \quad b_{i,j}^k \to b_{i,j}, \quad f_j^k \to f_j \text{ strongly in } C([0,T]),$$

and so we obtain

$$\mathcal{N}(u^k) = \widetilde{u}^k \to \widetilde{u} = \mathcal{N}(u) \text{ strongly in } C([0,T]; X_N).$$

Step 5: Compactness of $\mathcal{N}$. If $\widetilde{u}(t) = \sum_{i=1}^N g_i(t)e_i$, we can view (4.6) as

$$A(t)G'(t) + B(t)G(t) = F(t),$$

where $A(t) = (a_{i,j}(t))_{1 \leqslant i,j \leqslant N}$, $B(t) = (b_{i,j}(t))_{1 \leqslant i,j \leqslant N}$, $F(t) = (f_i(t))_{1 \leqslant i \leqslant N}$, $G(t) = (g_i(t))_{1 \leqslant i \leqslant N}$. We deduce

$$|g_i'(t)| \leqslant R|A^{-1}(t)||B(t)| + |A^{-1}(t)||F(t)|.$$

Thus, we have

$$\sup_{t \in [0,T]} \left(|g_i(t)| + |g_i'(t)|\right) \leqslant C.$$

This also implies

$$\sup_{u \in B_{R,T}} \|\mathcal{N}(u)\|_{C^1([0,T]; X_N)} \leqslant C.$$

The $C^1([0,T]; X_N)$-boundedness of $\mathcal{N}(u)$ allows us to apply the Arzelà-Ascoli theorem to obtain compactness of $\mathcal{N}$ in $B_{R,T}$.

Now we are in a position to apply Schauder's fixed point theorem to $\mathcal{N}$ to conclude the existence of a fixed point $u^N \in B_{R,T}$. Then we define ρ^N satisfying the continuity equation (2.29) on $(0,T) \times \Omega$, and $\chi_{\mathcal{S}}^N = \mathbb{1}_{\mathcal{S}^N}$ is the corresponding solution to the transport equation (2.31) on $(0,T) \times \mathbb{R}^3$. It only remains to justify the momentum

equation (2.30). We multiply equation (4.5) by $\psi \in \mathcal{D}([0, T))$ to obtain:

$$
-\int_0^T \int_\Omega \rho^N \Big(\frac{\partial u^N}{\partial t} \cdot \psi(t)e_j + (u^N \cdot \nabla(\psi(t)e_j)) \cdot u^N \Big) + \int_0^T \int_\Omega \varepsilon \nabla u^N \nabla \rho^N \cdot (\psi(t)e_j) + \alpha \int_0^T \int_{\partial\Omega} (u^N \times \nu) \cdot (\psi(t)e_j \times \nu)
$$

$$
+ \int_0^T \int_\Omega \Big(2\mu^N \mathbb{D}(u^N) : \mathbb{D}(\psi(t)e_j) + \lambda^N \operatorname{div} u^N \mathbb{I} : \mathbb{D}(\psi(t)e_j) - p^N(\rho^N)\mathbb{I} : \mathbb{D}(\psi(t)e_j) \Big)
$$

$$
+ \alpha \int_0^T \int_{\partial\mathcal{S}^N(t)} [(u^N - P_\mathcal{S}^N u^N) \times \nu] \cdot [(\psi(t)e_j - P_\mathcal{S}^N \psi(t)e_j) \times \nu] + \frac{1}{\delta} \int_0^T \int_\Omega \chi_\mathcal{S}(u^N - P_\mathcal{S}^N u^N) \cdot (\psi(t)e_j - P_\mathcal{S}^N \psi(t)e_j)
$$

$$
= \int_0^T \int_\Omega \rho^N g^N \cdot \psi(t)e_j, \quad (4.18)
$$

We have the following identities via integration by parts:

$$
\int_0^T \rho^N \frac{\partial u^N}{\partial t} \cdot \psi(t)e_j = -\int_0^T \frac{\partial \rho^N}{\partial t} u^N \cdot \psi(t)e_j - \int_0^T \rho^N u^N \cdot \psi'(t)e_j - (\rho^N u^N \cdot \psi e_j)(0), \quad (4.19)
$$

$$
\int_\Omega \rho^N(u^N \cdot \nabla(\psi(t)e_j)) \cdot u^N = -\int_\Omega \operatorname{div}(\rho^N u^N)(\psi(t)e_j \cdot u^N) - \int_\Omega \rho^N(u^N \cdot \nabla)u^N \cdot \psi(t)e_j. \quad (4.20)
$$

Thus we can use the relations (4.19)–(4.20) and continuity equation (2.29) in the identity (4.18) to obtain equation (2.30) for all $\phi \in \mathcal{D}([0, T); X_N)$. $\qquad\square$

4.2. **Convergence of the Faedo-Galerkin scheme and the limiting system.** In Proposition 2.6, we have already constructed a solution $(\mathcal{S}^N, \rho^N, u^N)$ to the problem (2.28)–(2.33). In this section, we establish Proposition 2.4 by passing to the limit in (2.28)–(2.33) as $N \to \infty$ to recover the solution of (2.16)–(2.21), i.e. of the ε-level approximation.

Proof of Proposition 2.4. If we replace ϕ by u^N in (2.30), then as in (4.11), we derive

$$
E^N[\rho^N, q^N] + \int_0^T \int_\Omega \Big(2\mu^N |\mathbb{D}(u^N)|^2 + \lambda^N |\operatorname{div} u^N|^2 \Big) + \delta\varepsilon\beta \int_0^T \int_\Omega (\rho^N)^{\beta-2} |\nabla\rho^N|^2 + \alpha \int_0^T \int_{\partial\Omega} |u^N \times \nu|^2
$$

$$
+ \alpha \int_0^T \int_{\partial\mathcal{S}^N(t)} |(u^N - P_\mathcal{S}^N u^N) \times \nu|^2 + \frac{1}{\delta} \int_0^T \int_\Omega \chi_\mathcal{S}^N |u^N - P_\mathcal{S}^N u^N|^2 \leqslant \int_0^T \int_\Omega \rho^N g^N \cdot u^N + E_0^N, \quad (4.21)
$$

where

$$
E^N[\rho^N, q^N] = \int_\Omega \Big(\frac{1}{2}\rho^N |u^N|^2 + \frac{a^N}{\gamma-1}(\rho^N)^\gamma + \frac{\delta}{\beta-1}(\rho^N)^\beta \Big).
$$

Following the idea of the footnote in [30, Page 368], the initial data (ρ_0^N, u_0^N) is constructed in such a way that

$$
\rho_0^N \to \rho_0^\varepsilon \text{ in } W^{1,\infty}(\Omega), \quad \rho_0^N u_0^N \to q_0^\varepsilon \text{ in } L^2(\Omega)
$$

and

$$
\int_\Omega \Big(\frac{1}{2}\rho_0^N |u_0^N|^2 \mathbb{1}_{\{\rho_0^N > 0\}} + \frac{a^N}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta \Big) \to \int_\Omega \Big(\frac{1}{2}\frac{|q_0^\varepsilon|^2}{\rho_0^\varepsilon}\mathbb{1}_{\{\rho_0^\varepsilon > 0\}} + \frac{a^\varepsilon}{\gamma-1}(\rho_0^\varepsilon)^\gamma + \frac{\delta}{\beta-1}(\rho_0^\varepsilon)^\beta \Big) \text{ as } N \to \infty.
$$
$$(4.22)$$

Precisely, we approximate q_0^ε by a sequence q_0^N satisfying (2.37) and such that (4.22) is valid. It is sufficient to take $u_0^N = P_N(\frac{q_0^\varepsilon}{\rho_0^\varepsilon})$, where by P_N we denote the orthogonal projection of $L^2(\Omega)$ onto X_N. Proposition 2.6 is valid with

these new initial data and we have existence of a solution $(\mathcal{S}^N, \rho^N, u^N)$. The construction of ρ^N and (4.1) imply that $\rho^N > 0$. Thus the energy estimate (4.21) yields that up to a subsequence

(1) $u^N \to u^\varepsilon$ weakly-$*$ in $L^\infty(0, T; L^2(\Omega))$ and weakly in $L^2(0, T; H^1(\Omega))$,
(2) $\rho^N \to \rho^\varepsilon$ weakly-$*$ in $L^\infty(0, T; L^\beta(\Omega))$,
(3) $\nabla \rho^N \to \nabla \rho^\varepsilon$ weakly in $L^2((0, T) \times \Omega)$.

We follow the similar analysis as for the fluid case explained in [30, Section 7.8.1, Page 362] to conclude that

- $\rho^N \to \rho^\varepsilon$ in $C([0, T]; L^\beta_{weak}(\Omega))$ and $\rho^N \to \rho^\varepsilon$ strongly in $L^p((0, T) \times \Omega)$, $\forall\, 1 \leqslant p < \frac{4}{3}\beta$,
- $\rho^N u^N \to \rho^\varepsilon u^\varepsilon$ weakly in $L^2(0, T; L^{\frac{6\beta}{\beta+6}})$ and weakly-$*$ in $L^\infty(0, T; L^{\frac{2\beta}{\beta+1}})$.

We also know that $\chi_\mathcal{S}^N$ is a bounded sequence in $L^\infty((0, T) \times \mathbb{R}^3)$ satisfying (2.31) and $\{\rho^N \chi_\mathcal{S}^N\}$ is a bounded sequence in $L^\infty((0, T) \times \mathbb{R}^3)$ satisfying (2.32). We use Proposition 3.3 to conclude

$$\chi_\mathcal{S}^N \to \chi_\mathcal{S}^\varepsilon \text{ weakly-}* \text{ in } L^\infty((0, T) \times \mathbb{R}^3) \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)), \ \forall\, 1 \leqslant p < \infty, \qquad (4.23)$$

with $\chi_\mathcal{S}^\varepsilon$ satisfying (2.19) along with (2.16). Thus, we have recovered the transport equation for the body (2.19). From (4.23) and the definitions of g^N and g^ε in (2.34) and (2.22), it follows that

$$g^N \to g^\varepsilon \text{ weakly-}* \text{ in } L^\infty((0, T) \times \mathbb{R}^3) \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)) \ \forall\, 1 \leqslant p < \infty. \qquad (4.24)$$

These convergence results make it possible to pass to the limit $N \to \infty$ in (2.29) to achieve (2.17). Now we concentrate on the limit of the momentum equation (2.30). The four most difficult terms are:

$$A^N(t, e_k) = \int_{\partial \mathcal{S}^N(t)} [(u^N - P_\mathcal{S}^N u^N) \times \nu] \cdot [(e_k - P_\mathcal{S}^N e_k) \times \nu], \quad B^N(t, e_k) = \int_\Omega \rho^N u^N \otimes u^N : \nabla e_k,$$

$$C^N(t, e_k) = \int_\Omega \varepsilon \nabla u^N \nabla \rho^N \cdot e_k, \quad D^N(t, e_k) = \int_\Omega (\rho^N)^\beta \mathbb{I} : \mathbb{D}(e_k), \quad 1 \leqslant k \leqslant N.$$

To analyze the term $A^N(t, e_k)$, we do a change of variables to rewrite it in a fixed domain and use the convergence results from Proposition 3.2 for the projection and the isometric propagator:

$$P_\mathcal{S}^N u^N \to P_\mathcal{S}^\varepsilon u^\varepsilon \text{ weakly-}* \text{ in } L^\infty(0, T; C^\infty_{loc}(\mathbb{R}^3)),$$

$$\eta_{t,s}^N \to \eta_{t,s}^\varepsilon \text{ weakly-}* \text{ in } W^{1,\infty}((0, T)^2; C^\infty_{loc}(\mathbb{R}^3)).$$

We follow a similar analysis as in [16, Page 2047–2048] to conclude that A^N converges weakly in $L^1(0, T)$ to

$$A(t, e_k) = \int_{\partial \mathcal{S}^\varepsilon(t)} [(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \times \nu] \cdot [(e_k - P_\mathcal{S}^\varepsilon e_k) \times \nu].$$

We proceed as explained in the fluid case [30, Section 7.8.2, Page 363–365] to analyze the limiting process for the other terms $B^N(t, e_k)$, $C^N(t, e_k)$, $D^N(t, e_k)$. The limit of $B^N(t, e_k)$ follows from the fact [30, Equation (7.8.22), Page 364] that

$$\rho^N u^N \otimes u^N \to \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \text{ weakly in } L^2(0, T; L^{\frac{6\beta}{4\beta+3}}(\Omega)). \qquad (4.25)$$

To get the limit of $C^N(t, e_k)$, we use [30, Equation (7.8.26), Page 365]:

$$\varepsilon \nabla u^N \nabla \rho^N \to \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon \text{ weakly in } L^2(0, T; L^{\frac{5\beta-3}{4\beta}}(\Omega)),$$

and the limit of $D^N(t, e_k)$ is obtained by using [30, Equation (7.8.8), Page 362]:

$$\rho^N \to \rho^\varepsilon \text{ strongly in } L^p(0, T; \Omega), \quad 1 \leqslant p < \frac{4}{3}\beta. \qquad (4.26)$$

Thus, using the above convergence results for B^N, C^N, D^N and the fact that

$$\bigcup_N X_N \text{ is dense in } \{v \in W^{1,p}(\Omega) \mid v \cdot \nu = 0 \text{ on } \partial\Omega\} \text{ for any } p \in [1, \infty),$$

we conclude the following weak convergences in $L^1(0,T)$:

$$B^N(t,\phi^N) \to B(t,\phi^\varepsilon) = \int_\Omega \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon : \nabla\phi^\varepsilon,$$

$$C^N(t,\phi^N) \to C(t,\phi^\varepsilon) = \int_\Omega \varepsilon \nabla u^\varepsilon \nabla\rho^\varepsilon \cdot \phi^\varepsilon,$$

$$D^N(t,\phi^N) \to D(t,\phi^\varepsilon) = \int_\Omega (\rho^\varepsilon)^\beta \mathbb{I} : \mathbb{D}(\phi^\varepsilon).$$

Thus we have achieved (2.18) as a limit of equation (2.30) as $N \to \infty$. Hence, we have established the existence of a solution $(\mathcal{S}^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ to system (2.16)–(2.21). Now we establish energy inequality (2.26) and estimates independent of ε:

- The weak convergence of $\rho^N |u^N|^2$ in (4.25) and strong convergence of ρ^N in (4.26) ensures that, up to a subsequence,

$$\int_\Omega \left(\frac{1}{2}\rho^N |u^N|^2 + \frac{a^N}{\gamma-1}(\rho^N)^\gamma + \frac{\delta}{\beta-1}(\rho^N)^\beta\right) \to \int_\Omega \left(\frac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 + \frac{a^\varepsilon}{\gamma-1}(\rho^\varepsilon)^\gamma + \frac{\delta}{\beta-1}(\rho^\varepsilon)^\beta\right) \text{ in } \mathcal{D}'((0,T)). \quad (4.27)$$

- Due to the weak lower semicontinuity of convex functionals, the weak convergence of u^N in $L^2(0,T;H^1(\Omega))$, the strong convergence of $\chi_\mathcal{S}^N$ in $C([0,T];L^p(\Omega))$ and the strong convergence of $P_\mathcal{S}^N$ in $C([0,T];C_{loc}^\infty(\mathbb{R}^3))$, we obtain

$$\int_0^T \psi \int_\Omega \left(2\mu^\varepsilon |\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon |\operatorname{div} u^\varepsilon|^2\right) \leqslant \liminf_{N\to\infty} \int_0^T \psi \int_\Omega \left(2\mu^N |\mathbb{D}(u^N)|^2 + \lambda^N |\operatorname{div} u^N|^2\right), \quad (4.28)$$

$$\int_0^T \psi \int_\Omega \chi_\mathcal{S}^\varepsilon |u^\varepsilon - P_\mathcal{S}^\delta u^\varepsilon|^2 \leqslant \liminf_{N\to\infty} \int_0^T \psi \int_\Omega \chi_\mathcal{S}^N |u^N - P_\mathcal{S} u^N|^2, \quad (4.29)$$

  where ψ is a smooth non-negative function on $(0,T)$.

- Using the fact that $\nabla\rho^N \to \nabla\rho$ strongly in $L^2((0,T)\times\Omega)$ (by [30, Equation (7.8.25), Page 365]), strong convergence of ρ^N in (4.26) and Fatou's lemma, we have

$$\int_0^T \psi \int_\Omega (\rho^\varepsilon)^{\beta-2} |\nabla\rho^\varepsilon|^2 \leqslant \liminf_{N\to\infty} \int_0^T \psi \int_\Omega (\rho^N)^{\beta-2} |\nabla\rho^N|^2. \quad (4.30)$$

- For passing to the limit in the boundary terms, we follow the idea of [16]. We introduce the extended velocities U^N, $U_\mathcal{S}^N$ to whole $\mathbb{R}^3$ associated to $\mathcal{E}u^N$, $P_\mathcal{S}^N u^N$ respectively. They are defined by:

$$\mathcal{E}u^N(t,\eta_{t,0}^N(y)) = J_{\eta_{t,0}^N}\Big|_y U^N(t,y), \quad P_\mathcal{S}^N u^N(t,\eta_{t,0}^N(y)) = J_{\eta_{t,0}^N}\Big|_y U_\mathcal{S}^N(t,y)$$

  where $\mathcal{E}: H^1(\Omega) \to H^1(\mathbb{R}^3)$ is the extension operator and $J_{\eta_{t,0}^N}$ is the Jacobian matrix of $\eta_{t,0}^N$. According to [16, Lemma A.2], we have the weak convergences of U^N, $U_\mathcal{S}^N$ to U^ε, $U_\mathcal{S}^\varepsilon$ in $L^2(0,T;H_{loc}^1(\mathbb{R}^3))$. These facts along with the lower semicontinuity of the L^2-norm yield

$$\int_0^T \psi \int_{\partial\mathcal{S}^\varepsilon(t)} |(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon)\times\nu|^2 = \int_0^T \psi \int_{\partial\mathcal{S}_0} |(U^\varepsilon - U_\mathcal{S}^\varepsilon)\times\nu|^2$$

$$\leqslant \liminf_{N\to\infty} \int_0^T \psi \int_{\partial\mathcal{S}_0} |(U^N - U_\mathcal{S}^N)\times\nu|^2 = \liminf_{N\to\infty} \int_0^T \psi \int_{\partial\mathcal{S}^N(t)} |(u^N - P_\mathcal{S} u^N)\times\nu|^2. \quad (4.31)$$

In the above, the first and the last equality in (4.31) follows from the change of variables formula. Similar arguments also help us to obtain

$$\int\limits_0^T \psi \int\limits_{\partial\Omega} |u^\varepsilon \times \nu|^2 \leqslant \liminf_{N\to\infty} \int\limits_0^T \psi \int\limits_{\partial\Omega} |u^N \times \nu|^2. \tag{4.32}$$

- Regarding the term on the right hand side of (4.21), the weak convergence of u^N in $L^2(0,T;H^1(\Omega))$, the strong convergence of ρ^N in (4.26) and the strong convergence of g^N in (4.24) yield

$$\int\limits_0^T \psi \int\limits_\Omega \rho^N g^N \cdot u^N \to \int\limits_0^T \psi \int\limits_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon, \ \text{ as } N \to \infty. \tag{4.33}$$

We can obtain the following differential form of energy inequality by using the above results (4.27)–(4.33):

$$\frac{d}{dt} E^\varepsilon[\rho^\varepsilon, q^\varepsilon] + \int\limits_\Omega \left(2\mu^\varepsilon |\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon |\operatorname{div} u^\varepsilon|^2 \right) + \delta\varepsilon\beta \int\limits_\Omega (\rho^\varepsilon)^{\beta-2}|\nabla\rho^\varepsilon|^2$$
$$+ \alpha \int\limits_{\partial\Omega} |u^\varepsilon \times \nu|^2 + \alpha \int\limits_{\partial\mathcal{S}^\varepsilon(t)} |(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \times \nu|^2 + \frac{1}{\delta} \int\limits_\Omega \chi_\mathcal{S}^\varepsilon |u^\varepsilon - P_\mathcal{S}^\delta u^\varepsilon|^2 \leqslant \int\limits_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon \ \text{ in } \mathcal{D}'((0,T)). \tag{4.34}$$

Since, $E^\varepsilon[\rho^\varepsilon, q^\varepsilon] \in L^\infty((0,T))$, we can apply the if and only if relation between differential and integral form of energy inequality as stated in [30, Equation (7.1.27)-(7.1.28), Page 317]. Hence, we have established energy inequality (2.26):

$$E^\varepsilon[\rho^\varepsilon, q^\varepsilon] + \int\limits_0^T \int\limits_\Omega \left(2\mu^\varepsilon |\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon |\operatorname{div} u^\varepsilon|^2 \right) + \delta\varepsilon\beta \int\limits_0^T \int\limits_\Omega (\rho^\varepsilon)^{\beta-2}|\nabla\rho^\varepsilon|^2$$
$$+ \alpha \int\limits_0^T \int\limits_{\partial\Omega} |u^\varepsilon \times \nu|^2 + \alpha \int\limits_0^T \int\limits_{\partial\mathcal{S}^\varepsilon(t)} |(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \times \nu|^2 + \frac{1}{\delta} \int\limits_0^T \int\limits_\Omega \chi_\mathcal{S}^\varepsilon |u^\varepsilon - P_\mathcal{S}^\delta u^\varepsilon|^2 \leqslant \int\limits_0^T \int\limits_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon + E_0^\varepsilon, \tag{4.35}$$

where

$$E^\varepsilon[\rho^\varepsilon, q^\varepsilon] = \int\limits_\Omega \left(\frac{1}{2} \frac{|q^\varepsilon|^2}{\rho^\varepsilon} + \frac{a^\varepsilon}{\gamma-1} (\rho^\varepsilon)^\gamma + \frac{\delta}{\beta-1} (\rho^\varepsilon)^\beta \right).$$

We obtain as in [30, Equation (7.8.14), Page 363]:

$$\partial_t \rho^\varepsilon, \ \Delta\rho^\varepsilon \in L^{\frac{5\beta-3}{4\beta}}((0,T) \times \Omega).$$

Regarding the $\sqrt{\varepsilon}\|\nabla\rho^\varepsilon\|_{L^2((0,T)\times\Omega)}$ estimate in (2.27), we have to multiply (2.17) by ρ^ε and integrate by parts to obtain

$$\frac{1}{2} \int\limits_\Omega |\rho^\varepsilon(t)|^2 + \varepsilon \int\limits_0^T \int\limits_\Omega |\nabla\rho^\varepsilon(t)|^2 = \frac{1}{2} \int\limits_\Omega |\rho_0^\varepsilon|^2 - \frac{1}{2} \int\limits_0^T \int\limits_\Omega |\rho^\varepsilon|^2 \operatorname{div} u^\varepsilon \leqslant \frac{1}{2} \int\limits_\Omega |\rho_0^\varepsilon|^2 + \sqrt{T} \|\rho^\varepsilon\|_{L^\infty(0,T;L^4(\Omega))}^2 \|\operatorname{div} u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}.$$

Now, the pressure estimates $\|\rho^\varepsilon\|_{L^{\beta+1}((0,T)\times\Omega)}$ and $\|\rho^\varepsilon\|_{L^{\gamma+1}((0,T)\times\Omega)}$ in (2.27) can be derived by means of the test function $\phi(t,x) = \psi(t)\Phi(t,x)$ with $\Phi(t,x) = \mathcal{B}[\rho^\varepsilon - \overline{m}]$ in (2.18), where

$$\psi \in \mathcal{D}(0,T), \quad \overline{m} = |\Omega|^{-1} \int\limits_\Omega \rho^\varepsilon,$$

and $\mathcal{B}$ is the Bogovskii operator related to Ω (for details about $\mathcal{B}$, see [30, Section 3.3, Page 165]). After taking this special test function and integrating by parts, we obtain

$$
\int_0^T \psi \int_\Omega \left(a^\varepsilon(\rho^\varepsilon)^\gamma + \delta(\rho^\varepsilon)^\beta \right)\rho^\varepsilon = \int_0^T \psi \int_\Omega \left(a^\varepsilon(\rho^\varepsilon)^\gamma + \delta(\rho^\varepsilon)^\beta \right)\overline{m} + \int_0^T 2\psi \int_\Omega \mu^\varepsilon \mathbb{D}(u^\varepsilon) : \mathbb{D}(\Phi) + \int_0^T \psi \int_\Omega \lambda^\varepsilon \rho^\varepsilon \operatorname{div} u^\varepsilon
$$

$$
- \overline{m} \int_0^T \psi \int_\Omega \lambda^\varepsilon \operatorname{div} u^\varepsilon + \int_0^T \psi \int_\Omega \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon \cdot \Phi + \alpha \int_0^T \psi \int_{\partial \mathcal{S}^\varepsilon(t)} [(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \times \nu] \cdot [(\Phi - P_\mathcal{S}^\varepsilon \Phi) \times \nu]
$$

$$
+ \frac{1}{\delta} \int_0^T \psi \int_\Omega \chi_\mathcal{S}^\varepsilon (u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \cdot (\Phi - P_\mathcal{S}^\varepsilon \Phi) + \int_0^T \psi \int_\Omega \rho^\varepsilon g^\varepsilon \cdot \Phi. \quad (4.36)
$$

We see that all the terms can be estimated as in [30, Section 7.8.4, Pages 366–368] except the penalization term. Using Hölder's inequality and bounds from energy estimate (2.26), the penalization term can be handled in the following way

$$
\int_0^T \psi \int_\Omega \chi_\mathcal{S}^\varepsilon (u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon) \cdot (\Phi - P_\mathcal{S}^\varepsilon \Phi) \leqslant |\psi|_{C[0,T]} \left(\int_0^T \int_\Omega \chi_\mathcal{S}^\varepsilon |(u^\varepsilon - P_\mathcal{S}^\varepsilon u^\varepsilon)|^2 \right)^{1/2} \|\Phi\|_{L^2((0,T)\times\Omega)} \leqslant C|\psi|_{C[0,T]}, \quad (4.37)
$$

where in the last inequality we have used $\|\Phi\|_{L^2(\Omega)} \leqslant c\|\rho^\varepsilon\|_{L^2(\Omega)}$ and the energy inequality (2.26). Thus, we have an improved regularity of the density and we have established the required estimates of (2.27).

The only remaining thing is to check the following fact: there exists T small enough such that if $\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma$, then

$$
\operatorname{dist}(\mathcal{S}^\varepsilon(t), \partial\Omega) \geqslant 2\sigma > 0 \quad \forall\, t \in [0, T]. \quad (4.38)
$$

It is equivalent to establishing the following bound:

$$
\sup_{t\in[0,T]} |\partial_t \eta_{t,0}(t, y)| < \frac{\operatorname{dist}(\mathcal{S}_0, \partial\Omega) - 2\sigma}{T}, \quad y \in \mathcal{S}_0. \quad (4.39)
$$

We show as in Step 3 of the proof of Proposition 2.6 that (see (4.13)–(4.16)):

$$
|\partial_t \eta_{t,0}^\varepsilon(t, y)| \leqslant |(h^\varepsilon)'(t)| + |\omega^\varepsilon(t)||y - h^\varepsilon(t)| \leqslant C_0 \left(\int_\Omega \rho^\varepsilon |u^\varepsilon(t)|^2 \right)^{1/2}, \quad (4.40)
$$

where $C_0 = \sqrt{2}\frac{\max\{1,|y|\}}{\min\{1,\lambda_0\}^{1/2}}$. Moreover, the energy estimate (4.35) yields

$$
\frac{d}{dt}E^\varepsilon[\rho^\varepsilon, q^\varepsilon] + \int_\Omega \left(2\mu^\varepsilon |\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon |\operatorname{div} u^\varepsilon|^2 \right) \leqslant \int_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon \leqslant E^\varepsilon[\rho^\varepsilon, q^\varepsilon] + \frac{1}{2\gamma_1}\left(\frac{\gamma-1}{2\gamma} \right)^{\gamma_1/\gamma} \|g^\varepsilon\|^{2\gamma_1}_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)},
$$

with $\gamma_1 = 1 - \frac{1}{\gamma}$, which implies

$$
E^\varepsilon[\rho^\varepsilon, q^\varepsilon] \leqslant e^T E_0^\varepsilon + CT\|g^\varepsilon\|^{2\gamma_1}_{L^\infty((0,T)\times\Omega)}. \quad (4.41)
$$

Thus, with the help of (4.39) and (4.40)–(4.41), we can conclude that for any T satisfying

$$
T < \frac{\operatorname{dist}(\mathcal{S}_0, \partial\Omega) - 2\sigma}{C_0 \left[e^T E_0^\varepsilon + CT\|g^\varepsilon\|^{2\gamma_1}_{L^\infty((0,T)\times\Omega)} \right]^{1/2}},
$$

the relation (4.38) holds. This completes the proof of Proposition 2.4. $\qquad \square$

4.3. **Vanishing dissipation in the continuity equation and the limiting system.** In this section, we prove Proposition 2.2 by taking $\varepsilon \to 0$ in the system (2.16)–(2.21). In order to do so, we have to deal with the problem of identifying the pressure corresponding to the limiting density. First of all, following the idea of the footnote in [30, Page 381], the initial data $(\rho_0^\varepsilon, q_0^\varepsilon)$ is constructed in such a way that

$$\rho_0^\varepsilon > 0, \quad \rho_0^\varepsilon \in W^{1,\infty}(\Omega), \quad \rho_0^\varepsilon \to \rho_0^\delta \text{ in } L^\beta(\Omega), \quad q_0^\varepsilon \to q_0^\delta \text{ in } L^{\frac{2\beta}{\beta+1}}(\Omega)$$

and

$$\int_\Omega \left(\frac{|q_0^\varepsilon|^2}{\rho_0^\varepsilon} \mathbb{1}_{\{\rho_0^\varepsilon > 0\}} + \frac{a}{\gamma-1}(\rho_0^\varepsilon)^\gamma + \frac{\delta}{\beta-1}(\rho_0^\varepsilon)^\beta \right) \to \int_\Omega \left(\frac{|q_0^\delta|^2}{\rho_0^\delta} \mathbb{1}_{\{\rho_0^\delta > 0\}} + \frac{a}{\gamma-1}(\rho_0^\delta)^\gamma + \frac{\delta}{\beta-1}(\rho_0^\delta)^\beta \right) \text{ as } \varepsilon \to 0.$$

More precisely, let $(\rho_0^\delta, q_0^\delta)$ satisfy (2.11)–(2.12); then, following [30, Section 7.10.7, Page 392], we can find $\rho_0^\varepsilon \in W^{1,\infty}(\Omega)$, $\rho_0^\varepsilon > 0$ by defining

$$\rho_0^\varepsilon = \mathcal{K}_\varepsilon(\rho_0^\delta) + \varepsilon,$$

where $\mathcal{K}_\varepsilon$ is the standard regularizing operator in the space variable. Then our initial density satisfies

$$\rho_0^\varepsilon \to \rho_0^\delta \text{ strongly in } L^\beta(\Omega).$$

We define

$$\overline{q_0^\varepsilon} = \begin{cases} q_0^\delta \sqrt{\dfrac{\rho_0^\varepsilon}{\rho_0^\delta}} & \text{if } \rho_0^\delta > 0, \\ 0 & \text{if } \rho_0^\delta = 0. \end{cases}$$

From (2.12), we know that

$$\frac{|\overline{q_0^\varepsilon}|}{\sqrt{\rho_0^\varepsilon}} \in L^2(\Omega).$$

Due to a density argument, there exists $h^\varepsilon \in W^{1,\infty}(\Omega)$ such that

$$\left\| \frac{q_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}} - h^\varepsilon \right\|_{L^2(\Omega)} < \varepsilon.$$

Now, we set $q_0^\varepsilon = h^\varepsilon \sqrt{\rho_0^\varepsilon}$, which implies that

$$q_0^\varepsilon \to q_0^\delta \text{ in } L^{\frac{2\beta}{\beta+1}}(\Omega),$$

and

$$E_0^\varepsilon \to E_0^\delta.$$

Proof of Proposition 2.2. The estimates (2.26) and (2.27) help us to conclude that, up to an extraction of a subsequence, we have

$$u^\varepsilon \to u^\delta \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.42}$$

$$\rho^\varepsilon \to \rho^\delta \text{ weakly in } L^{\beta+1}((0,T) \times \Omega), \text{ weakly-} * \text{ in } L^\infty(0,T;L^\beta(\Omega)), \tag{4.43}$$

$$(\rho^\varepsilon)^\gamma \to \overline{(\rho^\delta)^\gamma} \text{ weakly in } L^{\frac{\beta+1}{\gamma}}((0,T) \times \Omega), \tag{4.44}$$

$$(\rho^\varepsilon)^\beta \to \overline{(\rho^\delta)^\beta} \text{ weakly in } L^{\frac{\beta+1}{\beta}}((0,T) \times \Omega), \tag{4.45}$$

$$\varepsilon \nabla \rho^\varepsilon \to 0 \text{ strongly in } L^2((0,T) \times \Omega) \tag{4.46}$$

as $\varepsilon \to 0$. Below, we denote by $\left(\rho^\delta, u^\delta, \overline{(\rho^\delta)^\gamma}, \overline{(\rho^\delta)^\beta} \right)$ also the extended version of the corresponding quantities in $(0,T) \times \mathbb{R}^3$.

Step 1: Limit of the transport equation. We obtain from Proposition 2.4 that ρ^ε satisfies (2.17), $\{u^\varepsilon, \chi_{\mathcal{S}}^\varepsilon\}$ is a bounded sequence in $L^2(0, T; H^1(\Omega)) \times L^\infty((0, T) \times \mathbb{R}^3)$ satisfying (2.19) and $\{\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon\}$ is a bounded sequence in $L^\infty((0, T) \times \mathbb{R}^3)$ satisfying (2.20). Thus, we can use Proposition 3.4 to conclude that up to a subsequence:

$$\chi_{\mathcal{S}}^\varepsilon \to \chi_{\mathcal{S}}^\delta \text{ weakly-} * \text{ in } L^\infty((0, T) \times \mathbb{R}^3) \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)) \ (1 \leqslant p < \infty), \tag{4.47}$$

$$\rho^\varepsilon \chi_{\mathcal{S}}^\varepsilon \to \rho^\delta \chi_{\mathcal{S}}^\delta \text{ weakly-} * \text{ in } L^\infty((0, T) \times \mathbb{R}^3) \text{ and strongly in } C([0, T]; L^p_{loc}(\mathbb{R}^3)) \ (1 \leqslant p < \infty), \tag{4.48}$$

with $\chi_{\mathcal{S}}^\delta$ and $\rho^\delta \chi_{\mathcal{S}}^\delta$ satisfying (2.4) and (2.5) respectively. Moreover,

$$P_{\mathcal{S}}^\varepsilon u^\varepsilon \to P_{\mathcal{S}}^\delta u^\delta \text{ weakly in } L^2(0, T; C^\infty_{loc}(\mathbb{R}^3)). \tag{4.49}$$

Hence, we have recovered the regularity of $\chi_{\mathcal{S}}^\delta$ in (2.1) and the transport equations (2.4) and (2.5) as $\varepsilon \to 0$.

Step 2: Limit of the continuity and the momentum equation. We follow the ideas of [30, Auxiliary lemma 7.49] to conclude: if $\rho^\delta, u^\delta, \overline{(\rho^\delta)^\gamma}, \overline{(\rho^\delta)^\beta}$ are defined by (4.42)–(4.45), we have

- (ρ^δ, u^δ) satisfies:

$$\frac{\partial \rho^\delta}{\partial t} + \operatorname{div}(\rho^\delta u^\delta) = 0 \text{ in } \mathcal{D}'([0, T) \times \mathbb{R}^3). \tag{4.50}$$

- For all $\phi \in H^1(0, T; L^2(\Omega)) \cap L^r(0, T; W^{1,r}(\Omega))$, where $r = \max\left\{\beta + 1, \frac{\beta + \theta}{\theta}\right\}$, $\beta \geqslant \max\{8, \gamma\}$ and $\theta = \frac{2}{3}\gamma - 1$ with $\phi \cdot \nu = 0$ on $\partial\Omega$ and $\phi|_{t=T} = 0$, the following holds:

$$-\int_0^T \int_\Omega \rho^\delta \left(u^\delta \cdot \frac{\partial}{\partial t}\phi + u^\delta \otimes u^\delta : \nabla\phi\right) + \int_0^T \int_\Omega \left(2\mu^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi) + \lambda^\delta \operatorname{div} u^\delta \mathbb{I} : \mathbb{D}(\phi) - \left(a^\delta \overline{(\rho^\delta)^\gamma} + \delta \overline{(\rho^\delta)^\beta}\right) \mathbb{I} : \mathbb{D}(\phi)\right)$$

$$+ \alpha \int_0^T \int_{\partial\Omega} (u^\delta \times \nu) \cdot (\phi \times \nu) + \alpha \int_0^T \int_{\partial\mathcal{S}^\delta(t)} \left[(u^\delta - P_{\mathcal{S}}^\delta u^\delta) \times \nu\right] \cdot \left[(\phi - P_{\mathcal{S}}^\delta \phi) \times \nu\right]$$

$$+ \frac{1}{\delta} \int_0^T \int_\Omega \chi_{\mathcal{S}}^\delta (u^\delta - P_{\mathcal{S}}^\delta u^\delta) \cdot (\phi - P_{\mathcal{S}}^\delta \phi) = \int_0^T \int_\Omega \rho^\delta g^\delta \cdot \phi + \int_\Omega (\rho^\delta u^\delta \cdot \phi)(0). \tag{4.51}$$

- The couple (ρ^δ, u^δ) satisfies the identity

$$\partial_t b(\rho^\delta) + \operatorname{div}(b(\rho^\delta)u^\delta) + [b'(\rho^\delta)\rho^\delta - b(\rho^\delta)] \operatorname{div} u^\delta = 0 \text{ in } \mathcal{D}'([0, T) \times \mathbb{R}^3), \tag{4.52}$$

with any $b \in C([0, \infty)) \cap C^1((0, \infty))$ satisfying (1.28).
- $\rho^\delta \in C([0, T]; L^p(\Omega))$, $1 \leqslant p < \beta$.

We outline the main lines of the proof of the above mentioned result. We prove (4.50) by passing to the limit $\varepsilon \to 0$ in equation (2.17) with the help of the convergence of the density in (4.43), (4.46) and the convergence of the momentum [30, Section 7.9.1, page 370]

$$\rho^\varepsilon u^\varepsilon \to \rho^\delta u^\delta \text{ weakly-} * \text{ in } L^\infty(0, T; L^{\frac{2\beta}{\beta+1}}(\Omega)), \text{ weakly in } L^2(0, T; L^{\frac{6\beta}{\beta+6}}(\Omega)). \tag{4.53}$$

We obtain identity (4.51), corresponding to the momentum equation, by passing to the limit in (2.18). To pass to the limit, we use the convergences of the density and the velocity (4.42)–(4.45) and of the transport part (4.47)–(4.49) along with the convergence of the product of the density and the velocity (4.53) and the convergence of the following terms [30, Section 7.9.1, page 371]:

$$\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon \to \rho^\delta u_i^\delta u_j^\delta \text{ weakly in } L^2(0, T; L^{\frac{6\beta}{4\beta+3}}(\Omega)), \quad i, j = 1, 2, 3,$$

$$\varepsilon(\nabla\rho^\varepsilon \cdot \nabla)u^\varepsilon \to 0 \text{ weakly in } L^{\frac{5\beta-3}{4\beta}}((0, T) \times \Omega).$$

Since, we have already established the continuity equation (4.50) and the function $b \in C([0, \infty)) \cap C^1((0, \infty))$ satisfies (1.28), the renormalized continuity equation (4.52) follows from the application of [30, Lemma 6.9, page 307]. Moreover, the regularity of the density $\rho^\delta \in C([0, T]; L^p(\Omega))$, $1 \leqslant p < \beta$ follows from [30, Lemma 6.15, page 310] via the appropriate choice of the renormalization function b in (4.52) and with the help of the regularities of

$\rho^\delta \in L^\infty(0, T; L^\beta_{loc}(\mathbb{R}^3)) \cap C([0, T]; L^\beta_{loc}(\Omega))$, $u^\delta \in L^2(0, T; H^1_{loc}(\mathbb{R}^3))$. Hence we have established the continuity equation (2.2) and the renormalized one (2.15).

Step 3: Limit of the pressure term. In this step, our aim is to identify the term $\left(\overline{(\rho^\delta)^\gamma} + \delta\overline{(\rho^\delta)^\beta}\right)$ by showing that $\overline{(\rho^\delta)^\gamma} = (\rho^\delta)^\gamma$ and $\overline{(\rho^\delta)^\beta} = (\rho^\delta)^\beta$. To prove this, we need some compactness of ρ^ε, which is not available. However, the quantity $(\rho^\varepsilon)^\gamma + \delta(\rho^\varepsilon)^\beta - (2\mu + \lambda)\rho^\varepsilon \mathrm{div} u^\varepsilon$, called "effective viscous flux", possesses a convergence property that helps us to identify the limit of our required quantity. We have the following weak and weak-∗ convergences from the boundedness of their corresponding norms [30, Section 7.9.2, page 373]:

$$\rho^\varepsilon \mathrm{div} u^\varepsilon \to \overline{\rho^\delta \mathrm{div} u^\delta} \text{ weakly in } L^2(0, T; L^{\frac{2\beta}{2+\beta}}(\Omega)), \tag{4.54}$$

$$(\rho^\varepsilon)^{\gamma+1} \to \overline{(\rho^\delta)^{\gamma+1}} \text{ weakly-} \ast \text{ in } [C((0, T) \times \Omega)]', \tag{4.55}$$

$$(\rho^\varepsilon)^{\beta+1} \to \overline{(\rho^\delta)^{\beta+1}} \text{ weakly-} \ast \text{ in } [C((0, T) \times \Omega)]'. \tag{4.56}$$

We apply the following result regarding the "effective viscous flux" from [30, Lemma 7.50, page 373]: Let u^δ, ρ^δ, $\overline{(\rho^\delta)^\gamma}$, $\overline{(\rho^\delta)^\beta}$, $\overline{(\rho^\delta)^{\gamma+1}}$, $\overline{(\rho^\delta)^{\beta+1}}$, $\overline{\rho^\delta \mathrm{div} u^\delta}$ be defined in (4.42)–(4.45), (4.54)–(4.56). Then we have

$$\overline{(\rho^\delta)^{\gamma+1}} \in L^{\frac{\beta+1}{\gamma+1}}((0, T) \times \Omega), \quad \overline{(\rho^\delta)^{\beta+1}} \in L^1((0, T) \times \Omega), \tag{4.57}$$

$$\overline{(\rho^\delta)^{\gamma+1}} + \delta\overline{(\rho^\delta)^{\beta+1}} - (2\mu + \lambda)\overline{\rho^\delta \mathrm{div} u^\delta} = \overline{(\rho^\delta)^\gamma}\rho^\delta + \delta\overline{(\rho^\delta)^\beta}\rho^\delta - (2\mu + \lambda)\rho^\delta \mathrm{div} u^\delta \text{ a.e. in } (0, T) \times \Omega. \tag{4.58}$$

Using the above relation (4.57) and an appropriate choice of the renormalization function in (4.52), we deduce the strong convergence of the density as in [30, Lemma 7.51, page 375]: Let ρ^δ, $\overline{(\rho^\delta)^\gamma}$, $\overline{(\rho^\delta)^\beta}$, $\overline{(\rho^\delta)^{\gamma+1}}$, $\overline{(\rho^\delta)^{\beta+1}}$ be defined in (4.43)–(4.45), (4.54)–(4.55). Then we have

$$\overline{(\rho^\delta)^\gamma} = (\rho^\delta)^\gamma, \quad \overline{(\rho^\delta)^\beta} = (\rho^\delta)^\beta \text{ a.e. in } (0, T) \times \Omega.$$

In particular,

$$\rho^\varepsilon \to \rho^\delta \text{ strongly in } L^p((0, T) \times \Omega), \ 1 \leqslant p < \beta + 1. \tag{4.59}$$

Thus, we have identified the pressure term in equation (4.51). Hence, we have recovered the momentum equation (2.3) and we have proved the existence of a weak solution $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ to system (2.1)–(2.6). It remains to prove the energy inequality (2.13) and the improved regularity for the density (2.14).

Step 4: Energy inequality and improved regularity of the density. Due to the convergences

$$\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon \to \rho^\delta u_i^\delta u_j^\delta \text{ weakly in } L^2(0, T; L^{\frac{6\beta}{4\beta+3}}(\Omega)), \quad i, j = 1, 2, 3,$$

$$\rho^\varepsilon \to \rho^\delta \text{ strongly in } L^p((0, T) \times \Omega), \ 1 \leqslant p < \beta + 1,$$

we have

$$\int_\Omega \rho^\varepsilon |u^\varepsilon|^2 \to \int_\Omega \rho^\delta |u^\delta|^2 \text{ weakly in } L^2(0, T),$$

$$\int_\Omega \left((\rho^\varepsilon)^\gamma + \delta(\rho^\varepsilon)^\beta\right) \to \int_\Omega \left((\rho^\delta)^\gamma + \delta(\rho^\delta)^\beta\right) \text{ weakly in } L^{\frac{\beta+1}{\beta}}(0, T).$$

In particular,

$$\int_\Omega \left(\rho^\varepsilon |u^\varepsilon|^2 + \frac{a^\varepsilon}{\gamma-1}(\rho^\varepsilon)^\gamma + \frac{\delta}{\beta-1}(\rho^\varepsilon)^\beta\right) \to \int_\Omega \left(\rho^\delta |u^\delta|^2 + \frac{a^\delta}{\gamma-1}(\rho^\delta)^\gamma + \frac{\delta}{\beta-1}(\rho^\delta)^\beta\right) \text{ weakly in } L^{\frac{\beta+1}{\beta}}(0, T).$$

Due to the weak lower semicontinuity of the corresponding L^2 norms, the weak convergence of u^ε in $L^2(0, T; H^1(\Omega))$, the strong convergence of ρ^ε in $L^p((0, T) \times \Omega)$, $1 \leqslant p < \beta + 1$, the strong convergence of $\chi_\mathcal{S}^\varepsilon$ in $C([0, T]; L^p(\Omega))$ and the strong convergence of $P_\mathcal{S}^\varepsilon$ in $C([0, T]; C^\infty_{loc}(\mathbb{R}^3))$, we follow the idea explained in (4.28)–(4.33) to pass to the limit

as $\varepsilon \to 0$ in the other terms of inequality (4.34) to obtain

$$\frac{d}{dt} E^\delta[\rho^\delta, q^\delta] + \int_\Omega \left(2\mu^\delta |\mathbb{D}(u^\delta)|^2 + \lambda^\delta |\operatorname{div} u^\delta|^2 \right) + \alpha \int_{\partial\Omega} |u^\delta \times \nu|^2 + \alpha \int_{\partial\mathcal{S}^\delta(t)} |(u^\delta - P_{\mathcal{S}}^\delta u^\delta) \times \nu|^2$$

$$+ \frac{1}{\delta} \int_\Omega \chi_{\mathcal{S}}^\delta |u^\delta - P_{\mathcal{S}}^\delta u^\delta|^2 \leqslant \int_\Omega \rho^\delta g^\delta \cdot u^\delta \quad \text{in } \mathcal{D}'((0,T)). \quad (4.60)$$

Since, $E^\delta[\rho^\delta, q^\delta] \in L^\infty((0,T))$, we can apply the relation between differential and integral form of energy inequality as stated in [30, Equation (7.1.27)-(7.1.28), Page 317] to establish the energy inequality (2.13).

To establish the regularity (2.14), we use an appropriate test function of the type

$$\mathcal{B} \left((\rho^\delta)^\theta - |\Omega|^{-1} \int_\Omega (\rho^\delta)^\theta \right)$$

in the momentum equation (2.3), where $\mathcal{B}$ is the Bogovskii operator. The detailed proof is in the lines of [30, Section 7.9.5, pages 376-381] and the extra terms can be treated as we have already explained in (4.36)–(4.37). Moreover, we follow the same idea as in the proof of Proposition 2.4 (precisely, the calculations in (4.38)–(4.41)) to conclude that there exists T small enough such that if $\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma$, then

$$\operatorname{dist}(\mathcal{S}^\delta(t), \partial\Omega) \geqslant 2\sigma > 0 \quad \forall \, t \in [0,T]. \quad (4.61)$$

This settles the proof of Proposition 2.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

5. PROOF OF THE MAIN RESULT

We have already established the existence of a weak solution $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ to system (2.1)–(2.6) in Proposition 2.2. In this section, we study the convergence analysis and the limiting behaviour of the solution as $\delta \to 0$ and recover a weak solution to system (1.2)–(1.11), i.e., we show Theorem 1.7.

Proof of Theorem 1.7. Step 0: Initial data. We consider initial data $\rho_{\mathcal{F}_0}, q_{\mathcal{F}_0}, \rho_{\mathcal{S}_0}, q_{\mathcal{S}_0}$ satisfying the conditions (1.33)–(1.35). In this step we present the construction of the approximate initial data $(\rho_0^\delta, q_0^\delta)$ satisfying (2.11)–(2.12) so that, in the limit $\delta \to 0$, we can recover the initial data $\rho_{\mathcal{F}_0}$ and $q_{\mathcal{F}_0}$ on $\mathcal{F}_0$. We set

$$\rho_0 = \rho_{\mathcal{F}_0}(1 - \mathbb{1}_{\mathcal{S}_0}) + \rho_{\mathcal{S}_0}\mathbb{1}_{\mathcal{S}_0},$$

$$q_0 = q_{\mathcal{F}_0}(1 - \mathbb{1}_{\mathcal{S}_0}) + \rho_{\mathcal{S}_0} u_{\mathcal{S}_0} \mathbb{1}_{\mathcal{S}_0}.$$

Similarly as in [30, Section 7.10.7, Page 392], we can find $\rho_0^\delta \in L^\beta(\Omega)$ by defining

$$\rho_0^\delta = \mathcal{K}_\delta(\rho_0) + \delta, \quad (5.1)$$

where $\mathcal{K}_\delta$ is the standard regularizing operator in the space variable. Then our initial density satisfies

$$\rho_0^\delta \to \rho_0 \text{ strongly in } L^\gamma(\Omega). \quad (5.2)$$

We define

$$\overline{q_0^\delta} = \begin{cases} q_0 \sqrt{\dfrac{\rho_0^\delta}{\rho_0}} & \text{if } \rho_0 > 0, \\ 0 & \text{if } \rho_0 = 0. \end{cases} \quad (5.3)$$

From (1.34), we know that

$$\frac{|\overline{q_0^\delta}|}{\sqrt{\rho_0^\delta}} \in L^2(\Omega).$$

Due to a density argument, there exists $h^\delta \in W^{1,\infty}(\Omega)$ such that

$$\left\| \frac{q_0^\delta}{\sqrt{\rho_0^\delta}} - h^\delta \right\|_{L^2(\Omega)} < \delta.$$

Now, we set $q_0^\delta = h^\delta \sqrt{\rho_0^\delta}$, which implies that

$$q_0^\delta \to q_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\Omega)$$

and

$$E^\delta[\rho_0^\delta, q_0^\delta] \to E[\rho_0, q_0].$$

Next we start with the sequence of approximate solutions ρ^δ, u^δ of the system (2.1)–(2.6) (Proposition 2.2). Since the energy $E^\delta[\rho_0^\delta, q_0^\delta]$ is uniformly bounded with respect to δ, we have from inequality (2.13) that

$$\|\sqrt{\rho^\delta} u^\delta\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\rho^\delta\|_{L^\infty(0,T;L^\gamma(\Omega))}^2 + \|\sqrt{2\mu^\delta}\mathbb{D}(u^\delta)\|_{L^2((0,T)\times\Omega)}^2 + \|\sqrt{\lambda^\delta}\operatorname{div} u^\delta\|_{L^2((0,T)\times\Omega)}^2$$
$$+ \frac{1}{\delta}\|\sqrt{\chi_{\mathcal{S}}^\delta}\left(u^\delta - P_{\mathcal{S}}^\delta u^\delta\right)\|_{L^2((0,T)\times\Omega)}^2 \leqslant C, \quad (5.4)$$

with C independent of δ.

<u>Step 1: Recovery of the transport equation for body.</u> Since $\{u^\delta, \chi_{\mathcal{S}}^\delta\}$ is a bounded sequence in $L^2(0,T;L^2(\Omega)) \times L^\infty((0,T)\times\mathbb{R}^3)$ satisfying (2.4), we can apply Proposition 3.5 to conclude that: up to a subsequence, we have

$$u^\delta \to u \text{ weakly in } L^2(0,T;L^2(\Omega)),$$

$$\chi_{\mathcal{S}}^\delta \to \chi_{\mathcal{S}} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L_{loc}^p(\mathbb{R}^3)) \ (1 \leqslant p < \infty), \quad (5.5)$$

with

$$\chi_{\mathcal{S}}(t,x) = \mathbb{1}_{\mathcal{S}(t)}(x), \quad \mathcal{S}(t) = \eta_{t,0}(\mathcal{S}_0),$$

where $\eta_{t,s} \in H^1((0,T)^2; C_{loc}^\infty(\mathbb{R}^3))$ is the isometric propagator. Moreover,

$$P_{\mathcal{S}}^\delta u^\delta \to P_{\mathcal{S}} u \text{ weakly in } L^2(0,T;C_{loc}^\infty(\mathbb{R}^3)), \quad (5.6)$$
$$\eta_{t,s}^\delta \to \eta_{t,s} \text{ weakly in } H^1((0,T)^2; C_{loc}^\infty(\mathbb{R}^3)).$$

Also, we obtain that $\chi_{\mathcal{S}}$ satisfies

$$\frac{\partial \chi_{\mathcal{S}}}{\partial t} + \operatorname{div}(P_{\mathcal{S}} u \chi_{\mathcal{S}}) = 0 \text{ in } \Omega, \quad \chi_{\mathcal{S}}(t,x) = \mathbb{1}_{\mathcal{S}(t)}(x).$$

Now we set

$$u_{\mathcal{S}} = P_{\mathcal{S}} u \quad (5.7)$$

to recover the transport equation (1.29). Note that we have already recovered the regularity of $\chi_{\mathcal{S}}$ in (1.25).

Observe that the fifth term of inequality (5.4) yields

$$\sqrt{\chi_{\mathcal{S}}^\delta}\left(u^\delta - P_{\mathcal{S}}^\delta u^\delta\right) \to 0 \text{ strongly in } L^2((0,T)\times\Omega). \quad (5.8)$$

The strong convergence of $\chi_{\mathcal{S}}^\delta$ and the weak convergence of u^δ and $P_{\mathcal{S}}^\delta u^\delta$ imply that

$$\chi_{\mathcal{S}}\left(u - u_{\mathcal{S}}\right) = 0. \quad (5.9)$$

To analyze the behaviour of the velocity field in the fluid part, we introduce the following continuous extension operator:

$$\mathcal{E}_u^\delta(t): \left\{u \in H^1(\mathcal{F}^\delta(t)), \ u \cdot \nu = 0 \text{ on } \partial\Omega\right\} \to H^1(\Omega). \quad (5.10)$$

Let us set

$$u_{\mathcal{F}}^\delta(t,\cdot) = \mathcal{E}_u^\delta(t)\left[u^\delta(t,\cdot)|_{\mathcal{F}^\delta}\right]. \quad (5.11)$$

We have

$$\{u_{\mathcal{F}}^\delta\} \text{ is bounded in } L^2(0,T;H^1(\Omega)), \quad u_{\mathcal{F}}^\delta = u^\delta \text{ on } \mathcal{F}^\delta, \text{ i.e. } (1-\chi_{\mathcal{S}}^\delta)(u^\delta - u_{\mathcal{F}}^\delta) = 0. \quad (5.12)$$

Thus, the strong convergence of $\chi_{\mathcal{S}}^\delta$ and the weak convergence of $u_{\mathcal{F}}^\delta \to u_{\mathcal{F}}$ in $L^2(0,T;H^1(\Omega))$ yield that

$$(1-\chi_{\mathcal{S}})\left(u - u_{\mathcal{F}}\right) = 0. \quad (5.13)$$

By combining the relations (5.9)–(5.13), we conclude that the limit u of u^δ satisfies $u \in L^2(0,T;L^2(\Omega))$ and there exists $u_{\mathcal{F}} \in L^2(0,T;H^1(\Omega))$, $u_{\mathcal{S}} \in L^2(0,T;\mathcal{R})$ such that $u(t,\cdot) = u_{\mathcal{F}}(t,\cdot)$ on $\mathcal{F}(t)$ and $u(t,\cdot) = u_{\mathcal{S}}(t,\cdot)$ on $\mathcal{S}(t)$.

<u>Step 2: Recovery of the continuity equations.</u> We recall that $\rho^\delta \chi_{\mathcal{S}}^\delta(t,x)$ satisfies (2.5), i.e.

$$\frac{\partial}{\partial t}(\rho^\delta \chi_{\mathcal{S}}^\delta) + P_{\mathcal{S}}^\delta u^\delta \cdot \nabla(\rho^\delta \chi_{\mathcal{S}}^\delta) = 0, \quad (\rho^\delta \chi_{\mathcal{S}}^\delta)|_{t=0} = \rho_0^\delta \mathbb{1}_{\mathcal{S}_0}.$$

We proceed as in Proposition 3.5 to conclude that

$$\rho^\delta \chi_{\mathcal{S}}^\delta \to \rho \chi_{\mathcal{S}} \text{ weakly-} * \text{ in } L^\infty((0,T) \times \mathbb{R}^3) \text{ and strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)) \ (1 \leqslant p < \infty), \tag{5.14}$$

and $\rho \chi_{\mathcal{S}}$ satisfies

$$\frac{\partial}{\partial t}(\rho \chi_{\mathcal{S}}) + P_{\mathcal{S}} u \cdot \nabla(\rho \chi_{\mathcal{S}}) = 0, \quad (\rho \chi_{\mathcal{S}})|_{t=0} = \rho_{\mathcal{S}_0} \mathbb{1}_{\mathcal{S}_0}.$$

We set

$$\rho_{\mathcal{S}} = \rho \chi_{\mathcal{S}} \tag{5.15}$$

and use the definition of $u_{\mathcal{S}}$ in (5.7) to conclude that $\rho_{\mathcal{S}}$ satisfies:

$$\frac{\partial \rho_{\mathcal{S}}}{\partial t} + \operatorname{div}(u_{\mathcal{S}} \rho_{\mathcal{S}}) = 0 \text{ in } (0,T) \times \Omega, \quad \rho_{\mathcal{S}}(0,x) = \rho_{\mathcal{S}_0}(x) \mathbb{1}_{\mathcal{S}_0} \text{ in } \Omega.$$

Thus, we recover the equation of continuity (1.30) for the density of the rigid body.

We introduce the following extension operator:

$$\mathcal{E}_\rho^\delta(t) : \left\{ \rho \in L^{\gamma+\theta}(\mathcal{F}^\delta(t)) \right\} \to L^{\gamma+\theta}(\Omega),$$

given by

$$\mathcal{E}_\rho^\delta(t) \left[\rho^\delta(t,\cdot)|_{\mathcal{F}^\delta} \right] = \begin{cases} \rho^\delta(t,\cdot)|_{\mathcal{F}^\delta} & \text{in } \mathcal{F}^\delta(t), \\ 0 & \text{in } \Omega \setminus \mathcal{F}^\delta(t). \end{cases} \tag{5.16}$$

Let us set

$$\rho_{\mathcal{F}}^\delta(t,\cdot) = \mathcal{E}_\rho^\delta(t) \left[\rho^\delta(t,\cdot)|_{\mathcal{F}^\delta} \right]. \tag{5.17}$$

From estimates (2.13), (2.14), (5.12) and the definition of $\rho_{\mathcal{F}}^\delta$ in (5.17), we obtain that

$$u_{\mathcal{F}}^\delta \to u_{\mathcal{F}} \text{ weakly in } L^2(0,T; H^1(\Omega)), \tag{5.18}$$

$$\rho_{\mathcal{F}}^\delta \to \rho_{\mathcal{F}} \text{ weakly in } L^{\gamma+\theta}((0,T) \times \Omega), \ \theta = \frac{2}{3}\gamma - 1 \text{ and weakly-} * \text{ in } L^\infty(0,T; L^\beta(\Omega)), \tag{5.19}$$

$$(\rho_{\mathcal{F}}^\delta)^\gamma \to \overline{\rho_{\mathcal{F}}^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T) \times \Omega), \tag{5.20}$$

$$\delta(\rho_{\mathcal{F}}^\delta)^\beta \to 0 \text{ weakly in } L^{\frac{\beta+\theta}{\beta}}((0,T) \times \Omega). \tag{5.21}$$

Next, we follow the ideas of [30, Auxiliary lemma 7.53, Page 384] to assert: if $u_{\mathcal{F}}, \rho_{\mathcal{F}}, \overline{\rho_{\mathcal{F}}^\gamma}$ are defined by (5.18)–(5.20), we have

- $(\rho_{\mathcal{F}}, u_{\mathcal{F}})$ satisfies:

$$\frac{\partial \rho_{\mathcal{F}}}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}}) = 0 \text{ in } \mathcal{D}'([0,T) \times \mathbb{R}^3). \tag{5.22}$$

- The couple $(\rho_{\mathcal{F}}, u_{\mathcal{F}})$ satisfies the identity

$$\partial_t \overline{b(\rho_{\mathcal{F}})} + \operatorname{div}(\overline{b(\rho_{\mathcal{F}})} u_{\mathcal{F}}) + \overline{[b'(\rho_{\mathcal{F}})\rho_{\mathcal{F}} - b(\rho_{\mathcal{F}})] \operatorname{div} u_{\mathcal{F}}} = 0 \text{ in } \mathcal{D}'([0,T) \times \mathbb{R}^3), \tag{5.23}$$

for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfying (1.28) and the weak limits $\overline{b(\rho_{\mathcal{F}})}$ and $\overline{[b'(\rho_{\mathcal{F}})\rho_{\mathcal{F}} - b(\rho_{\mathcal{F}})] \operatorname{div} u_{\mathcal{F}}}$ being defined in the following sense:

$$b(\rho_{\mathcal{F}}^\delta) \to \overline{b(\rho_{\mathcal{F}})} \text{ weakly-} * \text{ in } L^\infty(0,T; L^{\frac{\gamma}{1+\kappa_1}}(\mathbb{R}^3)),$$

$$[b'(\rho_{\mathcal{F}}^\delta)\rho_{\mathcal{F}}^\delta - b(\rho_{\mathcal{F}}^\delta)] \operatorname{div} u_{\mathcal{F}}^\delta \to \overline{[b'(\rho_{\mathcal{F}})\rho_{\mathcal{F}} - b(\rho_{\mathcal{F}})] \operatorname{div} u_{\mathcal{F}}} \text{ weakly in } L^2(0,T; L^{\frac{2\gamma}{2+2\kappa_1+\gamma}}(\mathbb{R}^3)).$$

We outline the main idea of the proof of the asserted result. We use the strong convergence of density in weaker space and weak convergence of velocity to obtain the convergence of the momentum [30, Section 7.10.1, equation (7.10.7), page 383]

$$\rho_{\mathcal{F}}^\delta u_{\mathcal{F}}^\delta \to \rho_{\mathcal{F}} u_{\mathcal{F}} \text{ weakly-} * \text{ in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)), \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\mathbb{R}^3)). \tag{5.24}$$

We derive (5.22) by letting $\delta \to 0$ in equation (2.2) with the help of the convergence of the density in (5.19) and the convergence of momentum in (5.24).

Recall that when we pass to the limit $\varepsilon \to 0$, we do have $\rho_{\mathcal{F}}^\delta \in L^2((0,T) \times \Omega)$. But in this step, we do not have $\rho_{\mathcal{F}} \in L^2((0,T) \times \Omega)$. So, it is not straightforward to obtain the renormalized continuity equation. Observe that

this difficulty is not present in the case of $\gamma \geqslant \frac{9}{5}$ as in that case, $\rho_{\mathcal{F}} \in L^{\gamma+\theta}((0,T) \times \Omega) \subset L^2((0,T) \times \Omega)$ since $\gamma + \theta = \frac{5}{3}\gamma - 1 \geqslant 2$ for $\gamma \geqslant \frac{9}{5}$.

We use equation (2.15) and [30, Lemma 1.7] to establish that $\{b(\rho_{\mathcal{F}}^{\delta})\}$ is uniformly continuous in $W^{-1,s}(\Omega)$ with $s = \min\left\{\frac{6\gamma}{6\kappa_1+6+\gamma}, 2\right\}$, where the function $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfies (1.28). We apply [30, Lemma 6.2, Lemma 6.4] to get

$$b(\rho_{\mathcal{F}}^{\delta}) \to \overline{b(\rho_{\mathcal{F}})} \text{ in } C([0,T]; L^{\frac{\gamma}{1+\kappa_1}}(\Omega)),$$
$$b(\rho_{\mathcal{F}}^{\delta}) \to \overline{b(\rho_{\mathcal{F}})} \text{ strongly in } L^p(0,T; W^{-1,2}(\Omega)), \quad 1 \leqslant p < \infty.$$

The above mentioned limits together with (5.18) help us to conclude

$$b(\rho_{\mathcal{F}}^{\delta})u_{\mathcal{F}}^{\delta} \to \overline{b(\rho_{\mathcal{F}})}u_{\mathcal{F}} \text{ weakly in } L^2\left(0,T; L^{\frac{6\gamma}{6\kappa_1+6+\gamma}}(\Omega)\right).$$

Eventually, we obtain (5.23) by taking the limit $\delta \to 0$ in (2.15).

Step 3: Recovery of the renormalized continuity equation. By the method of an effective viscous flux with an appropriate choice of functions [30, Lemma 7.55, page 386], we establish boundedness of oscillations of the density sequence and we have an estimate for the amplitude of oscillations [30, Lemma 7.56, page 386]:

$$\limsup_{\delta \to 0} \int_0^T \int_{\Omega} [T_k(\rho_{\mathcal{F}}^{\delta}) - T_k(\rho_{\mathcal{F}})]^{\gamma+1} \leqslant \int_0^T \int_{\Omega} \left[\overline{\rho_{\mathcal{F}}^{\gamma}T_k(\rho_{\mathcal{F}})} - \overline{\rho_{\mathcal{F}}^{\gamma}}\,\overline{T_k(\rho_{\mathcal{F}})}\right],$$

where $T_k(\rho_{\mathcal{F}}) = \min\{\rho_{\mathcal{F}}, k\}$, $k > 0$, are cut-off operators and $\overline{T_k(\rho_{\mathcal{F}})}$, $\overline{\rho_{\mathcal{F}}^{\gamma}T_k(\rho_{\mathcal{F}})}$ stand for the weak limits of $T_k(\rho_{\mathcal{F}}^{\delta})$, $(\rho_{\mathcal{F}}^{\delta})^{\gamma}T_k(\rho_{\mathcal{F}}^{\delta})$. This result allows us to estimate the quantities

$$\sup_{\delta > 0} \|\rho_{\mathcal{F}}^{\delta}\mathbb{1}_{\{\rho_{\mathcal{F}}^{\delta} \geqslant k\}}\|_{L^p((0,T)\times\Omega)}, \quad \sup_{\delta > 0} \|T_k(\rho_{\mathcal{F}}^{\delta}) - \rho_{\mathcal{F}}^{\delta}\|_{L^p((0,T)\times\Omega)},$$
$$\|\overline{T_k(\rho_{\mathcal{F}})} - \rho_{\mathcal{F}}\|_{L^p((0,T)\times\Omega)}, \quad \|T_k(\rho_{\mathcal{F}}) - \rho_{\mathcal{F}}\|_{L^p((0,T)\times\Omega)} \text{ with } k > 0, \ 1 \leqslant p < \gamma + \theta.$$

Using the above estimate and taking the renormalized function $b = T_k$ in (5.23), after several computations we obtain, the following statement [30, Lemma 7.57, page 388]: Let $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfy (1.28) with $\kappa_1 + 1 \leqslant \frac{\gamma+\theta}{2}$ and let $u_{\mathcal{F}}, \rho_{\mathcal{F}}$ be defined by (5.18)–(5.19). Then the renormalized continuity equation (1.27) reads

$$\partial_t b(\rho_{\mathcal{F}}) + \operatorname{div}(b(\rho_{\mathcal{F}})u_{\mathcal{F}}) + (b'(\rho_{\mathcal{F}}) - b(\rho_{\mathcal{F}}))\operatorname{div} u_{\mathcal{F}} = 0 \text{ in } \mathcal{D}'([0,T) \times \Omega).$$

So far, we have recovered the transport equation of the body (1.29), the continuity equation (1.26) and the renormalized one (1.27). It remains to prove the momentum equation (1.24) and establish the energy inequality (1.31).

Step 4: Recovery of the momentum equation. Notice that the test functions in the weak formulation of momentum equation (1.24) belong to the space V_T (the space is defined in (1.19)), which is a space of discontinuous functions. Precisely,

$$\phi = (1 - \chi_{\mathcal{S}})\phi_{\mathcal{F}} + \chi_{\mathcal{S}}\phi_{\mathcal{S}} \text{ with } \phi_{\mathcal{F}} \in \mathcal{D}([0,T); \mathcal{D}(\overline{\Omega})), \quad \phi_{\mathcal{S}} \in \mathcal{D}([0,T); \mathcal{R}),$$

satisfying

$$\phi_{\mathcal{F}} \cdot \nu = 0 \text{ on } \partial\Omega, \quad \phi_{\mathcal{F}} \cdot \nu = \phi_{\mathcal{S}} \cdot \nu \text{ on } \partial\mathcal{S}(t).$$

Whereas, if we look at the test functions in momentum equation (2.3) in the δ-approximation, we see that it involves an $L^p(0,T; W^{1,p}(\Omega))$-regularity. Hence we approximate this discontinuous test function by a sequence of test functions that belong to $L^p(0,T; W^{1,p}(\Omega))$. The idea is to construct an approximation $\phi_{\mathcal{S}}^{\delta}$ of ϕ without jumps at the interface such that

$$\phi_{\mathcal{S}}^{\delta}(t,x) = \phi_{\mathcal{F}}(t,x) \quad \forall \ t \in (0,T), \ x \in \partial\mathcal{S}^{\delta}(t), \tag{5.25}$$

and

$$\phi_{\mathcal{S}}^{\delta}(t,\cdot) \approx \phi_{\mathcal{S}}(t,\cdot) \text{ in } \mathcal{S}^{\delta}(t) \text{ away from a } \delta^{\vartheta} \text{ neighborhood of } \partial\mathcal{S}^{\delta}(t) \text{ with } \vartheta > 0. \tag{5.26}$$

In the spirit of [16, Proposition 5.1], at first, we give the precise result regarding this construction and then we will continue the proof of Theorem 1.7.

Proposition 5.1. *Let $\phi \in V_T$ and $\vartheta > 0$. Then there exists a sequence*

$$\phi^\delta \in H^1(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}(\Omega)), \ \text{where } r = \max\left\{\beta+1, \frac{\beta+\theta}{\theta}\right\}, \ \beta \geqslant \max\{8,\gamma\} \ \text{and} \ \theta = \frac{2}{3}\gamma - 1$$

of the form

$$\phi^\delta = (1 - \chi_\mathcal{S}^\delta)\phi_\mathcal{F} + \chi_\mathcal{S}^\delta \phi_\mathcal{S}^\delta \tag{5.27}$$

that satisfies for all $p \in [1,\infty)$:

(1) $\|\chi_\mathcal{S}^\delta(\phi_\mathcal{S}^\delta - \phi_\mathcal{S})\|_{L^p((0,T)\times\Omega)} = \mathcal{O}(\delta^{\vartheta/p})$,
(2) $\phi^\delta \to \phi$ strongly in $L^p((0,T)\times\Omega)$,
(3) $\|\phi^\delta\|_{L^p(0,T;W^{1,p}(\Omega))} = \mathcal{O}(\delta^{-\vartheta(1-1/p)})$,
(4) $\|\chi_\mathcal{S}^\delta(\partial_t + P_\mathcal{S}^\delta u^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S})\|_{L^2(0,T;L^p(\Omega))} = \mathcal{O}(\delta^{\vartheta/p})$,
(5) $(\partial_t + P_\mathcal{S}^\delta u^\delta \cdot \nabla)\phi^\delta \to (\partial_t + P_\mathcal{S} u \cdot \nabla)\phi$ weakly in $L^2(0,T;L^p(\Omega))$.

We give the proof of Proposition 5.1 at the end of this section. Next we continue the proof of Theorem 1.7.

Step 4.1: Linear terms of the momentum equation. We use ϕ^δ (constructed in Proposition 5.1) as the test function in (2.3). Then we take the limit $\delta \to 0$ in (2.3) to recover equation (1.24). Let us analyze the passage to the limit in the linear terms. To begin with, we recall the following convergences of the velocities of the fluid part and the solid part, cf. (5.18) and (5.6):

$$(1 - \chi_\mathcal{S}^\delta)u_\mathcal{F}^\delta = (1 - \chi_\mathcal{S}^\delta)u^\delta, \quad u_\mathcal{F}^\delta \to u_\mathcal{F} \text{ weakly in } L^2(0,T;H^1(\Omega)),$$

$$u_\mathcal{S}^\delta = P_\mathcal{S}^\delta u^\delta, \ u_\mathcal{S} = P_\mathcal{S}u, \quad u_\mathcal{S}^\delta \to u_\mathcal{S} \text{ weakly in } L^2(0,T;C_{loc}^\infty(\mathbb{R}^3)).$$

Let us start with the diffusion term $2\mu^\delta\mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta) + \lambda^\delta \operatorname{div} u^\delta \mathbb{I} : \mathbb{D}(\phi^\delta)$ in (2.3). We write

$$\int_0^T \int_\Omega 2\mu^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta) = \int_0^T \int_\Omega \left(2\mu_\mathcal{F}(1 - \chi_\mathcal{S}^\delta)\mathbb{D}(u_\mathcal{F}^\delta) + \delta^2 \chi_\mathcal{S}^\delta \mathbb{D}(u^\delta)\right) : \mathbb{D}(\phi^\delta)$$

$$= \int_0^T \int_\Omega 2\mu_\mathcal{F}(1 - \chi_\mathcal{S}^\delta)\mathbb{D}(u_\mathcal{F}^\delta) : \mathbb{D}(\phi_\mathcal{F}) + \delta^2 \int_0^T \int_\Omega \chi_\mathcal{S}^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta).$$

The strong convergence of $\chi_\mathcal{S}^\delta$ to $\chi_\mathcal{S}$ and the weak convergence of $u_\mathcal{F}^\delta$ to $u_\mathcal{F}$ imply that

$$\int_0^T \int_\Omega 2\mu_\mathcal{F}(1 - \chi_\mathcal{S}^\delta)\mathbb{D}(u_\mathcal{F}^\delta) : \mathbb{D}(\phi_\mathcal{F}) \to \int_0^T \int_\Omega 2\mu_\mathcal{F}(1 - \chi_\mathcal{S})\mathbb{D}(u_\mathcal{F}) : \mathbb{D}(\phi_\mathcal{F}).$$

We know from (5.4), definition of μ^δ in (2.8) and Proposition 5.1 (with $p = 2$ case) that

$$\|\delta \chi_\mathcal{S}^\delta \mathbb{D}(u^\delta)\|_{L^2((0,T)\times\Omega)} \leqslant C, \quad \|\phi^\delta\|_{L^2(0,T;H^1(\Omega))} = \mathcal{O}(\delta^{-\vartheta/2}).$$

These estimates yield that

$$\left|\delta^2 \int_0^T \int_\Omega \chi_\mathcal{S}^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta)\right| \leqslant \delta\|\delta\chi_\mathcal{S}^\delta \mathbb{D}(u^\delta)\|_{L^2((0,T)\times\Omega)}\|\mathbb{D}(\phi^\delta)\|_{L^2(0,T;L^2(\Omega))} \leqslant C\delta^{1-\vartheta/2}. \tag{5.28}$$

If we consider $\vartheta < 2$ and $\delta \to 0$, we have

$$\delta^2 \int_0^T \int_\Omega \chi_\mathcal{S}^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta) \to 0.$$

Hence,

$$\int_0^T \int_\Omega \left(2\mu^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi^\delta) + \lambda_\mathcal{F} \operatorname{div} u_\mathcal{F} \mathbb{I} : \mathbb{D}(\phi^\delta)\right) \to \int_0^T \int_{\mathcal{F}(t)} \left(2\mu_\mathcal{F}\mathbb{D}(u_\mathcal{F}) + \lambda_\mathcal{F} \operatorname{div} u_\mathcal{F}\mathbb{I}\right) : \mathbb{D}(\phi_\mathcal{F})$$

as $\delta \to 0$. Next we consider the boundary term on $\partial\Omega$ in (2.3). The weak convergence of $u_{\mathcal{F}}^{\delta}$ to $u_{\mathcal{F}}$ in $L^2(0, T; H^1(\Omega))$ yields

$$\int\limits_0^T \int\limits_{\partial\Omega} (u^{\delta} \times \nu) \cdot (\phi^{\delta} \times \nu) = \int\limits_0^T \int\limits_{\partial\Omega} (u_{\mathcal{F}}^{\delta} \times \nu) \cdot (\phi_{\mathcal{F}} \times \nu) \to \int\limits_0^T \int\limits_{\partial\Omega} (u_{\mathcal{F}} \times \nu) \cdot (\phi_{\mathcal{F}} \times \nu) \text{ as } \delta \to 0.$$

To deal with the boundary term on $\partial\mathcal{S}^{\delta}(t)$ we do a change of variables such that this term becomes an integral on the fixed boundary $\partial\mathcal{S}_0$. Then we pass to the limit as $\delta \to 0$ and afterwards transform back to the moving domain. Next, we introduce the notation $r_{\mathcal{S}}^{\delta} = P_{\mathcal{S}}^{\delta}\phi^{\delta}$ to write the following:

$$\int\limits_0^T \int\limits_{\partial\mathcal{S}^{\delta}(t)} [(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta}) \times \nu] \cdot [(\phi^{\delta} - P_{\mathcal{S}}^{\delta}\phi^{\delta}) \times \nu] = \int\limits_0^T \int\limits_{\partial\mathcal{S}^{\delta}(t)} [(u_{\mathcal{F}}^{\delta} - u_{\mathcal{S}}^{\delta}) \times \nu] \cdot [(\phi_{\mathcal{F}}^{\delta} - r_{\mathcal{S}}^{\delta}) \times \nu]$$

$$= \int\limits_0^T \int\limits_{\partial\mathcal{S}_0} [(U_{\mathcal{F}}^{\delta} - U_{\mathcal{S}}^{\delta}) \times \nu] \cdot [(\Phi_{\mathcal{F}}^{\delta} - R_{\mathcal{S}}^{\delta}) \times \nu],$$

where we denote by capital letters the corresponding velocity fields and test functions in the fixed domain. By Proposition 5.1 we have that $\phi^{\delta} \to \phi$ strongly in $L^2(0, T; L^6(\Omega))$. Hence we obtain, as in Proposition 3.5, that

$$r_{\mathcal{S}}^{\delta} \to r_{\mathcal{S}} = P_{\mathcal{S}}\phi \text{ strongly in } L^2(0, T; C_{loc}^{\infty}(\mathbb{R}^3)).$$

Now using [16, Lemma A.2], we obtain the convergence in the fixed domain

$$R_{\mathcal{S}}^{\delta} \to R_{\mathcal{S}} \text{ strongly in } L^2(0, T; H^{1/2}(\partial\mathcal{S}_0)).$$

Similarly, the convergences of $u_{\mathcal{F}}^{\delta}$ and $u_{\mathcal{S}}^{\delta}$ with [16, Lemma A.2] imply

$$U_{\mathcal{F}}^{\delta} \to U_{\mathcal{F}}, \ U_{\mathcal{S}}^{\delta} \to U_{\mathcal{S}} \quad \text{weakly in } L^2(0, T; H^1(\Omega)).$$

These convergence results and going back to the moving domain gives

$$\int\limits_0^T \int\limits_{\partial\mathcal{S}^{\delta}(t)} [(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta}) \times \nu] \cdot [(\phi^{\delta} - P_{\mathcal{S}}^{\delta}\phi^{\delta}) \times \nu] = \int\limits_0^T \int\limits_{\partial\mathcal{S}_0} [(U_{\mathcal{F}}^{\delta} - U_{\mathcal{S}}^{\delta}) \times \nu] \cdot [(\Phi_{\mathcal{F}}^{\delta} - R_{\mathcal{S}}^{\delta}) \times \nu]$$

$$\to \int\limits_0^T \int\limits_{\partial\mathcal{S}_0} [(U_{\mathcal{F}} - U_{\mathcal{S}}) \times \nu] \cdot [(\Phi_{\mathcal{F}} - R_{\mathcal{S}}) \times \nu] = \int\limits_0^T \int\limits_{\partial\mathcal{S}(t)} [(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu] \cdot [(\phi_{\mathcal{F}} - \phi_{\mathcal{S}}) \times \nu].$$

The penalization term can be estimated in the following way:

$$\left| \frac{1}{\delta} \int\limits_0^T \int\limits_{\Omega} \chi_{\mathcal{S}}^{\delta}(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta}) \cdot (\phi^{\delta} - P_{\mathcal{S}}^{\delta}\phi^{\delta}) \right| = \left| \frac{1}{\delta} \int\limits_0^T \int\limits_{\Omega} \chi_{\mathcal{S}}^{\delta}(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta}) \cdot ((\phi^{\delta} - \phi_{\mathcal{S}}) - P_{\mathcal{S}}^{\delta}(\phi^{\delta} - \phi_{\mathcal{S}})) \right|$$

$$= \left| \frac{1}{\delta} \int\limits_0^T \int\limits_{\Omega} \chi_{\mathcal{S}}^{\delta}(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta}) \cdot (\phi_{\mathcal{S}}^{\delta} - \phi_{\mathcal{S}}) \right| \leqslant \delta^{-1/2} \frac{1}{\delta^{1/2}} \left\| \sqrt{\chi_{\mathcal{S}}^{\delta}} \left(u^{\delta} - P_{\mathcal{S}}^{\delta}u^{\delta} \right) \right\|_{L^2((0,T)\times\Omega)} \left\| \sqrt{\chi_{\mathcal{S}}^{\delta}}(\phi_{\mathcal{S}}^{\delta} - \phi_{\mathcal{S}}) \right\|_{L^2(0,T;L^2(\Omega))}$$

$$\leqslant C\delta^{-1/2+\vartheta/2}, \tag{5.29}$$

where we have used the estimates obtained from (5.4) and Proposition 5.1. By choosing $\vartheta > 1$ and taking $\delta \to 0$, the penalization term vanishes. Note that we also need $\vartheta < 2$ in view of (5.28).

Step 4.2: Nonlinear terms of the momentum equation. In this step, we analyze the following terms:

$$\int_0^T \int_\Omega \rho^\delta \left(u^\delta \cdot \frac{\partial}{\partial t}\phi + u^\delta \otimes u^\delta : \nabla\phi^\delta \right) = \int_0^T \int_\Omega \rho_\mathcal{F}^\delta (1-\chi_\mathcal{S}^\delta) u_\mathcal{F}^\delta \cdot \frac{\partial}{\partial t}\phi_\mathcal{F} + \int_0^T \int_\Omega \rho_\mathcal{F}^\delta (1-\chi_\mathcal{S}^\delta) u_\mathcal{F}^\delta \otimes u_\mathcal{F}^\delta : \nabla\phi_\mathcal{F}$$

$$+ \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)\phi^\delta \cdot u^\delta \quad (5.30)$$

The strong convergence of $\chi_\mathcal{S}^\delta$ to $\chi_\mathcal{S}$ and the weak convergence of $\rho^\delta u_\mathcal{F}^\delta$ to $\rho_\mathcal{F} u_\mathcal{F}$ (see (5.24)) imply

$$\int_0^T \int_\Omega \rho_\mathcal{F}^\delta (1-\chi_\mathcal{S}^\delta) u_\mathcal{F}^\delta \cdot \frac{\partial}{\partial t}\phi_\mathcal{F} \to \int_0^T \int_\Omega \rho_\mathcal{F} (1-\chi_\mathcal{S}) u_\mathcal{F} \cdot \frac{\partial}{\partial t}\phi_\mathcal{F} \quad \text{as } \delta \to 0. \quad (5.31)$$

We use the convergence result for the convective term from [30, Section 7.10.1, page 384]

$$\rho_\mathcal{F}^\delta (u_\mathcal{F}^\delta)_i (u_\mathcal{F}^\delta)_j \to \rho_\mathcal{F} (u_\mathcal{F})_i (u_\mathcal{F})_j \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{4\gamma+3}}(\Omega)), \quad \forall i,j \in \{1,2,3\},$$

to pass to the limit in the second term of the right-hand side of (5.30):

$$\int_0^T \int_\Omega \rho_\mathcal{F}^\delta (1-\chi_\mathcal{S}^\delta) u_\mathcal{F}^\delta \otimes u_\mathcal{F}^\delta : \nabla\phi_\mathcal{F} \to \int_0^T \int_\Omega \rho_\mathcal{F} (1-\chi_\mathcal{S}) u_\mathcal{F} \otimes u_\mathcal{F} : \nabla\phi_\mathcal{F}. \quad (5.32)$$

Next we consider the third term on the right-hand side of (5.30):

$$\int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)\phi^\delta \cdot u^\delta = \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S}) \cdot u^\delta + \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta \partial_t \phi_\mathcal{S} \cdot u^\delta$$

$$+ \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (u_\mathcal{S}^\delta \cdot \nabla)\phi_\mathcal{S} \cdot u^\delta =: T_1^\delta + T_2^\delta + T_3^\delta.$$

We write

$$T_1^\delta = \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S}) \cdot (u^\delta - P_\mathcal{S}^\delta u^\delta) + \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S}) \cdot P_\mathcal{S}^\delta u^\delta.$$

We estimate these terms in the following way:

$$\left| \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S}) \cdot (u^\delta - P_\mathcal{S}^\delta u^\delta) \right|$$

$$\leqslant \|\rho^\delta \chi_\mathcal{S}^\delta\|_{L^\infty((0,T)\times\Omega)} \|\chi_\mathcal{S}^\delta (\partial_t + P_\mathcal{S}^\delta u^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S})\|_{L^2(0,T;L^6(\Omega))} \frac{1}{\delta^{1/2}} \left\| \sqrt{\chi_\mathcal{S}^\delta} \left(u^\delta - P_\mathcal{S}^\delta u^\delta \right) \right\|_{L^2((0,T)\times\Omega)} \delta^{1/2},$$

$$\left| \int_0^T \int_\Omega \rho^\delta \chi_\mathcal{S}^\delta (\partial_t + u_\mathcal{S}^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S}) \cdot P_\mathcal{S}^\delta u^\delta \right|$$

$$\leqslant \|\rho^\delta \chi_\mathcal{S}^\delta\|_{L^\infty((0,T)\times\Omega)} \|\chi_\mathcal{S}^\delta (\partial_t + P_\mathcal{S}^\delta u^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S})\|_{L^2(0,T;L^6(\Omega))} \|P_\mathcal{S}^\delta u^\delta\|_{L^2((0,T)\times\Omega)},$$

where we have used $\rho^\delta \chi_\mathcal{S}^\delta \in L^\infty((0,T)\times\Omega)$ as it is a solution to (2.5). Moreover, by Proposition 5.1 (with the case $p=6$), we know that for $\vartheta > 0$

$$\|\chi_\mathcal{S}^\delta (\partial_t + P_\mathcal{S}^\delta u^\delta \cdot \nabla)(\phi^\delta - \phi_\mathcal{S})\|_{L^2(0,T;L^6(\Omega))} = \mathcal{O}(\delta^{\vartheta/6}).$$

Hence,
$$T_1^\delta \to 0 \text{ as } \delta \to 0. \tag{5.33}$$

Observe that
$$T_2^\delta = \int\limits_0^T \int\limits_\Omega \rho^\delta \chi_S^\delta \partial_t \phi_S \cdot (u^\delta - P_S^\delta u^\delta) + \int\limits_0^T \int\limits_\Omega \rho^\delta \chi_S^\delta \partial_t \phi_S \cdot P_S^\delta u^\delta.$$

Now we use the following convergences:

- the strong convergence of $\sqrt{\chi_S^\delta}\left(u^\delta - P_S^\delta u^\delta\right) \to 0$ in $L^2((0,T) \times \Omega)$ (see the fifth term of inequality (5.4)),
- the strong convergence of χ_S^δ to χ_S (see the convergence in (5.5)),
- the weak convergence of $\rho^\delta \chi_S^\delta P_S^\delta u^\delta$ to $\rho \chi_S P_S u$ (see the convergences in (5.14) and (5.6)),

to deduce
$$T_2^\delta \to \int\limits_0^T \int\limits_{S(t)} \rho \chi_S \partial_t \phi_S \cdot P_S u \quad \text{as} \quad \delta \to 0.$$

Recall the definition of u_S in (5.7) and the definition of ρ_S in (5.15) to conclude
$$T_2^\delta \to \int\limits_0^T \int\limits_{S(t)} \rho_S \partial_t \phi_S \cdot u_S \quad \text{as} \quad \delta \to 0. \tag{5.34}$$

Notice that
$$T_3^\delta = \int\limits_0^T \int\limits_\Omega \rho^\delta \chi_S^\delta (u_S^\delta \cdot \nabla)\phi_S \cdot u^\delta = \int\limits_0^T \int\limits_\Omega \rho^\delta \chi_S^\delta (u_S^\delta \otimes u_S^\delta) : \nabla\phi_S = \int\limits_0^T \int\limits_\Omega \rho^\delta \chi_S^\delta (u_S^\delta \otimes u_S^\delta) : \mathbb{D}(\phi_S) = 0. \tag{5.35}$$

Eventually, combining the results (5.31)–(5.35), we obtain
$$\int\limits_0^T \int\limits_\Omega \rho^\delta \left(u^\delta \cdot \frac{\partial}{\partial t}\phi + u^\delta \otimes u^\delta : \nabla\phi\right) \to \int\limits_0^T \int\limits_{F(t)} \rho_F u_F \cdot \frac{\partial}{\partial t}\phi_F + \int\limits_0^T \int\limits_{S(t)} \rho_S u_S \cdot \frac{\partial}{\partial t}\phi_S + \int\limits_0^T \int\limits_{F(t)} (\rho_F u_F \otimes u_F) : \nabla\phi_F.$$

Step 4.3: Pressure term of the momentum equation. We use the definition of ϕ^δ
$$\phi^\delta = (1 - \chi_S^\delta)\phi_F + \chi_S^\delta \phi_S^\delta,$$

to write
$$\int\limits_0^T \int\limits_\Omega \left(a^\delta(\rho^\delta)^\gamma + \delta(\rho^\delta)^\beta\right) \mathbb{I} : \mathbb{D}(\phi^\delta) = \int\limits_0^T \int\limits_\Omega \left[a_F(1-\chi_S^\delta)(\rho_F^\delta)^\gamma + \delta(1-\chi_S^\delta)(\rho_F^\delta)^\beta\right] \mathbb{I} : \mathbb{D}(\phi_F),$$

where we have used the fact that $\operatorname{div}\phi_S^\delta = 0$. Due to the strong convergence of χ_S^δ to χ_S and the weak convergence of $(\rho_F^\delta)^\gamma$, $(\rho_F^\delta)^\beta$ in (5.20), (5.21) we obtain
$$\int\limits_0^T \int\limits_\Omega a_F(1-\chi_S^\delta)(\rho_F^\delta)^\gamma \mathbb{I} : \mathbb{D}(\phi_F) \to \int\limits_0^T \int\limits_\Omega a_F(1-\chi_S)\overline{(\rho_F)^\gamma}\mathbb{I} : \mathbb{D}(\phi_F) \text{ as } \delta \to 0,$$

and
$$\int\limits_0^T \int\limits_\Omega \delta(1-\chi_S^\delta)(\rho_F^\delta)^\beta \mathbb{I} : \mathbb{D}(\phi_F) \to 0 \text{ as } \delta \to 0.$$

In order to establish (1.24), it only remains to show that $\overline{\rho_F^\gamma} = \rho_F^\gamma$. This is equivalent to establishing some strong convergence result of the sequence ρ_F^δ. Let $\{\rho_F^\delta\}$ be the sequence and ρ_F be its weak limit from (5.19). We have the following strong convergence of density [30, Lemma 7.60, page 391]:
$$\rho_F^\delta \to \rho_F \text{ in } L^p((0,T) \times \Omega), \quad 1 \leqslant p < \gamma + \theta.$$

This immediately yields $\overline{\rho_{\mathcal{F}}^{\gamma}} = \rho_{\mathcal{F}}^{\gamma}$. Thus, we have recovered the weak form of the momentum equation.

Step 5: Recovery of the energy estimate. We derive from (2.13) that

$$\int_{\Omega} \left(\rho^{\delta} |u^{\delta}|^2 + \frac{a_{\mathcal{F}}}{\gamma - 1}(1 - \chi_{\mathcal{S}}^{\delta})(\rho_{\mathcal{F}}^{\delta})^{\gamma} + \frac{\delta}{\beta - 1}(\rho_{\mathcal{F}}^{\delta})^{\beta} \right) + \int_0^T \int_{\Omega} \left(2\mu_{\mathcal{F}}(1 - \chi_{\mathcal{S}}^{\delta})|\mathbb{D}(u_{\mathcal{F}}^{\delta})|^2 + \lambda_{\mathcal{F}}(1 - \chi_{\mathcal{S}}^{\delta})|\operatorname{div} u_{\mathcal{F}}^{\delta}|^2 \right) + \alpha \int_0^T \int_{\partial\Omega} |u^{\delta} \times \nu|^2$$

$$+ \alpha \int_0^T \int_{\partial\mathcal{S}^{\delta}(t)} |(u^{\delta} - P_{\mathcal{S}}^{\delta} u^{\delta}) \times \nu|^2 \leqslant \int_0^T \int_{\Omega} \rho^{\delta} g^{\delta} \cdot u^{\delta} + \int_{\Omega} \left(\frac{|q_0^{\delta}|^2}{\rho_0^{\delta}} \mathbb{1}_{\{\rho_0^{\delta} > 0\}} + \frac{a}{\gamma - 1}(\rho_0^{\delta})^{\gamma} + \frac{\delta}{\beta - 1}(\rho_0^{\delta})^{\beta} \right).$$

To see the limiting behaviour of the above inequality as δ tends to zero, we observe that the limit as $\delta \to 0$ is similar to the limit $\varepsilon \to 0$. Hence we obtain the energy inequality (1.31).

Step 6: Rigid body is away from boundary. It remains to check that there exists T small enough such that if $\operatorname{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma$, then

$$\operatorname{dist}(\mathcal{S}(t), \partial\Omega) \geqslant \frac{3\sigma}{2} > 0 \quad \forall\, t \in [0, T]. \tag{5.36}$$

Let us introduce the following notation:

$$(\mathcal{U})_{\sigma} = \left\{ x \in \mathbb{R}^3 \mid \operatorname{dist}(x, \mathcal{U}) < \sigma \right\},$$

for an open set $\mathcal{U}$ and $\sigma > 0$. We recall the following result [16, Lemma 5.4]: Let $\sigma > 0$. There exists $\delta_0 > 0$ such that for all $0 < \delta \leqslant \delta_0$,

$$\mathcal{S}^{\delta}(t) \subset (\mathcal{S}(t))_{\sigma/4} \subset (\mathcal{S}^{\delta}(t))_{\sigma/2}, \quad \forall\, t \in [0, T]. \tag{5.37}$$

Note that condition (5.37) and the relation (4.61), i.e., $\operatorname{dist}(\mathcal{S}^{\delta}(t), \partial\Omega) \geqslant 2\sigma > 0$ for all $t \in [0, T]$ imply our required estimate (5.36). Thus, we conclude the proof of Theorem 1.7. $\qquad\square$

It remains to prove Proposition 5.1. The main difference between Proposition 5.1 and [16, Proposition 5.1] is the time regularity of the approximate test functions. Since here we only have weak convergence of u^{δ} in $L^2(0, T; L^2(\Omega))$, according to Proposition 3.5 we have convergence of $\eta_{t,s}^{\delta}$ in $H^1((0, T)^2; C_{loc}^{\infty}(\mathbb{R}^3))$. In [16, Proposition 5.1], they have weak convergence of u^{δ} in $L^{\infty}(0, T; L^2(\Omega))$, which yields higher time regularity of the propagator $\eta_{t,s}^{\delta}$ in $W^{1,\infty}((0, T)^2; C_{loc}^{\infty}(\mathbb{R}^3))$.

Proof of Proposition 5.1. The proof relies on the construction of the approximation $\phi_{\mathcal{S}}^{\delta}$ of $\phi_{\mathcal{S}}$ so that we can avoid the jumps at the interface for the test functions such that (5.25)–(5.26) holds.

The idea is to write the test functions in Lagrangian coordinates through the isometric propagator $\eta_{t,s}^{\delta}$ so that we can work on the fixed domain. Let $\Phi_{\mathcal{F}}$, $\Phi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}^{\delta}$ be the transformed quantities in the fixed domain related to $\phi_{\mathcal{F}}$, $\phi_{\mathcal{S}}$ and $\phi_{\mathcal{S}}^{\delta}$ respectively:

$$\phi_{\mathcal{S}}(t, \eta_{t,0}^{\delta}(y)) = J_{\eta_{t,0}^{\delta}}\Big|_y (\Phi_{\mathcal{S}}(t, y)), \quad \phi_{\mathcal{F}}(t, \eta_{t,0}^{\delta}(y)) = J_{\eta_{t,0}^{\delta}}\Big|_y \Phi_{\mathcal{F}}(t, y) \quad \text{and} \quad \phi_{\mathcal{S}}^{\delta}(t, \eta_{t,0}^{\delta}(y)) = J_{\eta_{t,0}^{\delta}}\Big|_y \Phi_{\mathcal{S}}^{\delta}(t, y), \tag{5.38}$$

where $J_{\eta_{t,0}^{\delta}}$ is the Jacobian matrix of $\eta_{t,0}^{\delta}$. Note that if we define

$$\Phi^{\delta}(t, y) = (1 - \chi_{\mathcal{S}}^{\delta})\Phi_{\mathcal{F}} + \chi_{\mathcal{S}}^{\delta}\Phi_{\mathcal{S}}^{\delta},$$

then the definition of ϕ^{δ} in (5.27) gives

$$\phi^{\delta}(t, \eta_{t,0}^{\delta}(y)) = J_{\eta_{t,0}^{\delta}}\Big|_y (\Phi^{\delta}(t, y)). \tag{5.39}$$

Thus, the construction of the approximation $\phi_{\mathcal{S}}^{\delta}$ satisfying (5.25)–(5.26) is equivalent to building the approximation $\Phi_{\mathcal{S}}^{\delta}$ so that there is no jump for the function Φ^{δ} at the interface and the following holds:

$$\Phi_{\mathcal{S}}^{\delta}(t, x) = \Phi_{\mathcal{F}}(t, x) \quad \forall\, t \in (0, T),\ x \in \partial\mathcal{S}_0,$$

and

$$\Phi_{\mathcal{S}}^{\delta}(t, \cdot) \approx \Phi_{\mathcal{S}}(t, \cdot) \text{ in } \mathcal{S}_0 \text{ away from a } \delta^{\vartheta} \text{ neighborhood of } \partial\mathcal{S}_0 \text{ with } \vartheta > 0.$$

Explicitly, we set (inspired by [16, Pages 2055-2058]):

$$\Phi_{\mathcal{S}}^{\delta} = \Phi_{\mathcal{S},1}^{\delta} + \Phi_{\mathcal{S},2}^{\delta}, \tag{5.40}$$

with

$$\Phi_{\mathcal{S},1}^{\delta} = \Phi_{\mathcal{S}} + \chi(\delta^{-\vartheta}z)\left[(\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) - ((\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) \cdot e_z)e_z\right], \tag{5.41}$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth truncation function which is equal to 1 in a neighborhood of 0 and z is a coordinate transverse to the boundary $\partial\mathcal{S}_0 = \{z = 0\}$. Moreover, to make $\Phi_{\mathcal{S}}^{\delta}$ divergence-free in $\mathcal{S}_0$, we need to take $\Phi_{\mathcal{S},2}^{\delta}$ such that

$$\operatorname{div} \Phi_{\mathcal{S},2}^{\delta} = -\operatorname{div} \Phi_{\mathcal{S},1}^{\delta} \quad \text{in } \mathcal{S}_0, \quad \Phi_{\mathcal{S},2}^{\delta} = 0 \quad \text{on } \partial\mathcal{S}_0.$$

Observe that, the explicit form (5.41) of $\Phi_{\mathcal{S},1}^{\delta}$ yields

$$\operatorname{div} \Phi_{\mathcal{S},2}^{\delta} = -\operatorname{div} \Phi_{\mathcal{S},1}^{\delta} = -\chi(\delta^{-\vartheta}z)\operatorname{div}\left[(\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) - ((\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) \cdot e_z)e_z\right]. \tag{5.42}$$

Thus, the expressions (5.41)–(5.42) give us: for all $p < \infty$,

$$\|\Phi_{\mathcal{S},1}^{\delta} - \Phi_{\mathcal{S}}\|_{H^1(0,T;L^p(\mathcal{S}_0))} \leqslant C\delta^{\vartheta/p}, \tag{5.43}$$

$$\|\Phi_{\mathcal{S},1}^{\delta} - \Phi_{\mathcal{S}}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_0))} \leqslant C\delta^{-\vartheta(1-1/p)}, \tag{5.44}$$

and

$$\|\Phi_{\mathcal{S},2}^{\delta}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_0))} \leqslant C\|\chi(\delta^{-\vartheta}z)\operatorname{div}\left[(\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) - ((\Phi_{\mathcal{F}} - \Phi_{\mathcal{S}}) \cdot e_z)e_z\right]\|_{H^1(0,T;L^p(\mathcal{S}_0))} \leqslant C\delta^{\vartheta/p}. \tag{5.45}$$

Using the decomposition (5.40) of $\Phi_{\mathcal{S}}^{\delta}$ and the estimates (5.43)–(5.44), (5.45), we obtain

$$\|\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}}\|_{H^1(0,T;L^p(\mathcal{S}_0))} \leqslant C\delta^{\vartheta/p},$$

$$\|\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_0))} \leqslant C\delta^{-\vartheta(1-1/p)}.$$

Furthermore, we combine the above estimates with the uniform bound of the propagator $\eta_{t,0}^{\delta}$ in $H^1(0,T;C^{\infty}(\Omega))$ to obtain

$$\left\|J_{\eta_{t,0}^{\delta}}|_y(\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}})\right\|_{H^1(0,T;L^p(\mathcal{S}_0))} \leqslant C\delta^{\vartheta/p}, \tag{5.46}$$

$$\left\|J_{\eta_{t,0}^{\delta}}|_y(\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}})\right\|_{H^1(0,T;W^{1,p}(\mathcal{S}_0))} \leqslant C\delta^{-\vartheta(1-1/p)}. \tag{5.47}$$

Observe that due to the change of variables (5.38) and estimate (5.46):

$$\|\chi_{\mathcal{S}}^{\delta}(\phi_{\mathcal{S}}^{\delta} - \phi_{\mathcal{S}})\|_{L^p((0,T)\times\Omega)} \leqslant C\|J_{\eta_{t,0}^{\delta}}|_y(\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}})\|_{L^p((0,T)\times\mathcal{S}_0)} \leqslant C\delta^{\vartheta/p}. \tag{5.48}$$

Since

$$\|\phi^{\delta} - \phi\|_{L^p((0,T)\times\Omega)} \leqslant \|(\chi_{\mathcal{S}}^{\delta} - \chi_{\mathcal{S}})\phi_{\mathcal{F}}\|_{L^p((0,T)\times\Omega)} + \|\chi_{\mathcal{S}}^{\delta}(\phi_{\mathcal{S}}^{\delta} - \phi_{\mathcal{S}})\|_{L^p((0,T)\times\Omega)} + \|(\chi_{\mathcal{S}}^{\delta} - \chi_{\mathcal{S}})\phi_{\mathcal{S}}\|_{L^p((0,T)\times\Omega)},$$

using the strong convergence of $\chi_{\mathcal{S}}^{\delta}$ and the estimate (5.48), we conclude that

$$\phi^{\delta} \to \phi \quad \text{strongly in } L^p((0,T) \times \Omega).$$

We use estimate (5.44) and the relation (5.39) to obtain

$$\|\phi^{\delta}\|_{L^p(0,T;W^{1,p}(\Omega))} \leqslant \delta^{-\vartheta(1-1/p)}.$$

Moreover, the change of variables (5.38) and estimate (5.46) give

$$\|\chi_{\mathcal{S}}^{\delta}(\partial_t + P_{\mathcal{S}}^{\delta}u^{\delta} \cdot \nabla)(\phi^{\delta} - \phi_{\mathcal{S}})\|_{L^2(0,T;L^p(\Omega))} \leqslant C\left\|\frac{d}{dt}\left(J_{\eta_{t,0}^{\delta}}\Big|_y(\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}})\right)\right\|_{L^2(0,T;L^p(\mathcal{S}_0))}$$
$$\leqslant C\left\|J_{\eta_{t,0}^{\delta}}|_y(\Phi_{\mathcal{S}}^{\delta} - \Phi_{\mathcal{S}})\right\|_{H^1(0,T;L^p(\mathcal{S}_0))} \leqslant C\delta^{\vartheta/p}. \tag{5.49}$$

The above estimate (5.49), strong convergence of $\chi_{\mathcal{S}}^{\delta}$ to $\chi_{\mathcal{S}}$ in $C(0,T;L^p(\Omega))$ and weak convergence of $P_{\mathcal{S}}^{\delta}u^{\delta}$ to $P_{\mathcal{S}}u$ weakly in $L^2(0,T;C_{loc}^{\infty}(\mathbb{R}^3))$, give us

$$(\partial_t + P_{\mathcal{S}}^{\delta}u^{\delta} \cdot \nabla)\phi^{\delta} \to (\partial_t + P_{\mathcal{S}}u \cdot \nabla)\phi \quad \text{weakly in } L^2(0,T;L^p(\Omega)),$$

where

$$\phi^{\delta} = (1 - \chi_{\mathcal{S}}^{\delta})\phi_{\mathcal{F}} + \chi_{\mathcal{S}}^{\delta}\phi_{\mathcal{S}}^{\delta} \quad \text{and} \quad \phi = (1 - \chi_{\mathcal{S}})\phi_{\mathcal{F}} + \chi_{\mathcal{S}}\phi_{\mathcal{S}}.$$

$\square$

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Compliance with Ethical Standards

Conflict of interest

The authors declare that there are no conflicts of interest.

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