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Homogenization of a 2D tidal dynamics equation

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Abstract: In this paper we study the asymptotic behavior of the solutions of the two dimensional tidal equations by using the sigma-convergence method. We prove that the sequence of solutions of the original problem converges in suitable topologies to the solution of a homogenized problem of the same type.

Keywords: homogenization; tiidal equation; sigma convergence

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0. Introduction

Ocean tides have been investigated by many authors starting from [13,18]. In the last few decades rapid progress in theoretical and experimental studies of ocean tides has been achieved and they are being used to study important problems not only in oceanography but also in atmospheric sciences, geophysics as well as in electronics and telecommunications. Laplace [14] was the first author to give the first major theoretical formulation for water tides on a rotating globe: he formulated a system of partial differential equations relating the horizontal flow to the surface height of the ocean. The existence and uniqueness of the deterministic tide equation by using the classical compactness method have been proved in [9,18]. In this work we consider a deterministic analogue of a tidal dynamics model studied by Manna et al. [17] and originally proposed by Marchuk and Kagan [18] where they considered the tidal dynamics model which can be obtained from taking the shallow water model on a rotating sphere which is a slight generalization of the Laplace model.

Our objective is to carry out the homogenization of the problem (1.2)-(1.5) under a suitable structural assumption on the coefficients of the operator involved in (1.2). These assumptions cover a wide range of concrete behavious such as the classical periodicity assumption, the almost periodicity hypothesis and much more. In order to achieve our goal, we shall use the notion of sigma-convergence [22] which is roughly a formulation of the well-known two-scale convergence method [5] in the context of algebras with mean value [22,24–26]. This is the so called deterministic homogenization theory which includes the periodic homogenization theory as a special case.

The work is organized as follows. In Section 2, we state the ε-problem and derive some useful a priori estimates. Section 3 deals with the fundamentals of the sigma-convergence method. The homogenization process is performed in Section4 while in Section 5 we provide some applications of the main homogenization result.

1. Setting of the problem and uniform estimates

1.1. Statement of the problem

The tidal dynamics system developed by Manna et al. [17] for suitably normalized velocity $\mathbf{u}$ and tide height z reads as

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) + g\nabla z = f \text{ in } Q = \Omega \times (0,T) \\ \dfrac{\partial z}{\partial t} + \operatorname{div}(h\mathbf{u}) = 0 \text{ in } Q \\ \mathbf{u} = 0 \text{ on } \partial\Omega \times (0,T) \\ \mathbf{u}(x,0) = \mathbf{u}^0(x) \text{ and } z(x,0) = z^0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset in, where $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{pmatrix} -\alpha \,\triangle & \eta \\ \eta & -\alpha\,\triangle \end{pmatrix} \text{ and } \mathbf{B}(u) = \gamma \left|\mathbf{u} + \omega^0\right| \left(\mathbf{u} + \omega^0\right),$$

α and η (the Coriolis parameter) being positive constants, ω^0 a given function, $\gamma(x) = r/h(x)$ with h a given positive function.

In this work, we neglect the Coriolis parameter ($\eta = 0$), so that $\mathbf{A}(\mathbf{u}) = -\alpha\Delta\mathbf{u}$. However, instead of the Laplace operator, we rather consider a general linear elliptic operator of order 2 in divergence form, leading to the investigation of the asymptotic behaviour, as $0 < \varepsilon \to 0$ of the sequence of solutions $(\mathbf{u}_\varepsilon, z_\varepsilon)$ of the system (1.2)-(1.5) below

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial t} - \operatorname{div}\left(A_0\left(x, \frac{x}{\varepsilon}\right)\nabla \mathbf{u}_\varepsilon\right) + \mathbf{B}(\mathbf{u}_\varepsilon) + g\nabla z_\varepsilon = \mathbf{f} \text{ in } Q \tag{1.2}$$

$$\frac{\partial z_\varepsilon}{\partial t} + \operatorname{div}(h\mathbf{u}_\varepsilon) = 0 \text{ in } Q \tag{1.3}$$

$$\mathbf{u}_\varepsilon = 0 \text{ on } \partial\Omega \times (0,T) \tag{1.4}$$

$$\mathbf{u}_\varepsilon(x,0) = \mathbf{u}^0(x) \text{ and } z_\varepsilon(x,0) = z^0(x) \text{ in } \Omega, \tag{1.5}$$

where Ω is a Lipschitz bounded domain of $\mathbb{R}^2$, T a positive real number. Here $\mathbf{u}_\varepsilon$ and z_ε represent the total transport 2-D vector (the vertical integral of the velocity) and the deviation of the free surface with respect to the ocean bottom, respectively. In (1.2)-(1.5) ∇ (resp. div) is the gradient (resp. divergence) operator in Ω and the functions A_0, h, $\mathbf{u}^0$, z^0 and $\mathbf{B}$ are constrained as follows:

(A1) $A_0 \in \mathcal{C}(\overline{\Omega}, L^\infty(\mathbb{R}^2_y))^{2\times 2}$ is a symmetric matrix with

$$A_0(x,y)\xi\cdot\xi \geq \alpha\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2,\ x \in \overline{\Omega} \text{ and a.e. } y \in \mathbb{R}^2, \tag{1.6}$$

where $\alpha > 0$ is a given constant not depending on x, y and ξ. In the following we will also denote $A_0^\varepsilon(x) = A_0\left(x, \frac{x}{\varepsilon}\right)$ $(x \in \Omega)$.

(A2) The operator $\mathbf{B}$ is defined on $L^4(\Omega)^2$ by $\mathbf{B}(\mathbf{v}) = \gamma\left|\mathbf{v} + \omega^0\right|\left(\mathbf{v} + \omega^0\right)$ $\left(\mathbf{v} \in L^4(\Omega)^2\right)$ where $\omega^0 \in L^2(0,T;H^1_0(\Omega)^2)$ is a given function and $\gamma(x) = r/h(x)$ (for a fixed real number $r > 0$), h being a continuously differentiable function satisfying

$$\min_{x\in\Omega} h(x) = \beta > 0,\ \max_{x\in\Omega} h(x) = \mu \text{ and } \max_{x\in\Omega}|\nabla h(x)| \leq M,$$

where M is some positive constant which equals to zero at a constant ocean depth. The functions $\mathbf{u}^0$, z^0 and $\mathbf{f}$ are such that $\mathbf{u}^0 \in L^2(\Omega)^2$, $z^0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^2)$, and g is the gravitational constant.

(**A3**) We assume further that for all $x \in \overline{\Omega}$, the matrix-function $A_0(x,\cdot)$ has its entries in $B^2_{\mathcal{A}}(\mathbb{R}^2)$ where $\mathcal{A}$ is an algebra with mean value on in $\mathbb{R}^2$, while $B^2_{\mathcal{A}}(\mathbb{R}^2)$ stands for the generalized Besicovitch space associated to $\mathcal{A}$.

Remark 1.1. The operator $\mathbf{B}$ sends continuously $L^4(\Omega)^2$ into $L^2(\Omega)^2$ with the following properties (see [17, Lemma 3.3]): for $\mathbf{u},\mathbf{v} \in L^4(\Omega)^2$, we have

$$(\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}) \geq 0; \tag{1.7}$$

$$\|\mathbf{B}(\mathbf{u})\|_{L^2(\Omega)^2} \leq \|\gamma\|_\infty \|\mathbf{u}\|_{L^4(\Omega)^2}; \tag{1.8}$$

$$\|\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v})\|_{L^2(\Omega)^2} \leq \|\gamma\|_\infty \left(\|\mathbf{u}\|_{L^4(\Omega)^2} + \|\mathbf{v}\|_{L^4(\Omega)^2}\right) \|\mathbf{u}-\mathbf{v}\|_{L^4(\Omega)^2}. \tag{1.9}$$

The Assumption (**A3**), which depends on the algebra with mean value $\mathcal{A}$, is crucial in the homogenization process. It shows how the microstructures are distributed in the medium Ω, and therefore allows us to pass to the limit.

Before dealing with the well-posedness of (1.2)-(1.5), we first need to define the concept of solutions we will deal with.

Definition 1.1. Let $\mathbf{u}^0 \in L^2(\Omega)^2$, $z^0 \in L^2(\Omega)$, $\mathbf{f} \in L^2\left(0,T;H^{-1}(\Omega)^2\right)$, $\omega^0 \in L^2\left(0,T;H^1_0(\Omega)^2\right)$ and $0 < T < \infty$. The couple $(\mathbf{u}_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ is a weak solution to the problem (1.2)-(1.5) if

$$\begin{aligned} \mathbf{u}_\varepsilon &\in \mathcal{C}\left(0,T;L^2(\Omega)^2\right) \cap L^2\left(0,T;H^1_0(\Omega)^2\right); \\ \frac{\partial \mathbf{u}_\varepsilon}{\partial t} &\in L^2\left(0,T;H^{-1}(\Omega)^2\right); \\ z_\varepsilon &\in L^\infty\left(0,T;L^2(\Omega)\right), \ \frac{\partial z_\varepsilon}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right); \end{aligned}$$

and for all $\varphi \in L^2\left(0,T;H^1_0(\Omega)^2\right)$ and $\psi \in L^2\left(0,T;L^2(\Omega)\right)$, we have

$$\begin{aligned} &\int_0^T \left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t}, \varphi\right) dt + \int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla\varphi dxdt + \int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi dxdt + \int_Q g\nabla z_\varepsilon \varphi dxdt \\ &= \int_0^T (\mathbf{f}(t), \varphi(t))dt \end{aligned} \tag{1.10}$$

and

$$\int_0^T \left(\frac{\partial z_\varepsilon}{\partial t}, \psi\right) dt + \int_0^T (\mathrm{div}(h\mathbf{u}_\varepsilon), \psi)\, dt = 0. \tag{1.11}$$

In the above definition, $(\cdot,\cdot)$ stands for the duality pairings between any Hilbert space X and its topological dual X'. We also recall that the operator div $\left(A_0\left(x,\frac{x}{\varepsilon}\right)\nabla\mathbf{u}_\varepsilon\right)$ acts on a diagonal way, that is, for $\mathbf{v} = (v_1, v_2) \in H^1_0(\Omega)^2$, we have

$$\begin{aligned} \left(\mathrm{div}\left(A_0\left(x,\frac{x}{\varepsilon}\right)\nabla\mathbf{u}_\varepsilon\right), \mathbf{v}\right) &= -\int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{v} dxdt \\ &\equiv -\sum_{i=1}^2 \int_Q A_0^\varepsilon \nabla u_\varepsilon^i \cdot \nabla v_i dxdt \end{aligned}$$

where $\mathbf{u}_\varepsilon = (u^i_\varepsilon)_{1\leq i \leq 2}$. This being so, the following existence and uniqueness result holds.

Theorem 1.1. *Under assumptions* (**A1**)-(**A2**)*, there exists (for each $\varepsilon > 0$) a unique weak solution* $(\mathbf{u}_\varepsilon, z_\varepsilon)$ *to the problem* (1.2)-(1.5) *in the sense of Definition* 1.1.

Proof. We note that in the problem stated in [17], if we replace the Laplace operator by $-\operatorname{div}(A_0^\varepsilon \nabla \mathbf{u}_\varepsilon)$ and we neglect therein the Coriolis parameter, then the proof follows exactly the lines of that of [17, Propositions 3.6 and 3.7]. □

1.2. A priori estimates

The following result will be useful in deriving the uniform estimates for $(\mathbf{u}_\varepsilon, z_\varepsilon)$

Lemma 1.1 ([17, Lemma 3.1])**.** *For any real-valued smooth functions φ and ψ with compact support in $\mathbb{R}^2$, we have*

$$\|\varphi\psi\|_{L^2(\Omega)} \leq \left\|\varphi \frac{\partial \varphi}{\partial x_1}\right\|_{L^1(\Omega)} \left\|\psi \frac{\partial \psi}{\partial x_2}\right\|_{L^1(\Omega)} \tag{1.12}$$

$$\|\varphi\|_{L^4(\Omega)}^4 \leq 2\,\|\varphi\|_{L^2(\Omega)}^2 \|\nabla\varphi\|_{L^2(\Omega)}^2 . \tag{1.13}$$

The following lemma provides us with the a priori estimates.

Lemma 1.2. *Under assumptions* **(A1)**-**(A2)** *the weak solution $(\mathbf{u}_\varepsilon, z_\varepsilon)$ of problem* (1.2)-(1.5) *in the sense of Definition* 1.1 *satisfies the following estimates*

$$\sup_{0\leq t\leq T} \|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2} \leq C; \tag{1.14}$$

$$\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{H_0^1(\Omega)^2}^2 \, dt \leq C; \tag{1.15}$$

$$\left\|\frac{\partial \mathbf{u}_\varepsilon}{\partial t}\right\|_{L^2\left(0,T;H^{-1}(\Omega)^2\right)} \leq C; \tag{1.16}$$

$$\sup_{0\leq t\leq T} \|z_\varepsilon(t)\|_{L^2(\Omega)} \leq C; \tag{1.17}$$

$$\left\|\frac{\partial z_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \leq C \tag{1.18}$$

where the positive constant C is independent of ε.

Proof. We first deal with equation (1.2). By taking the scalar product in $L^2(\Omega)^2$ of equation (1.2) with $\mathbf{u}_\varepsilon$ and using (1.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}^2 + (A_0^\varepsilon \nabla \mathbf{u}_\varepsilon(t), \nabla \mathbf{u}_\varepsilon(t)) + (B(\mathbf{u}_\varepsilon(t)), \mathbf{u}_\varepsilon(t)) \\ + (g\nabla z_\varepsilon(t), \mathbf{u}_\varepsilon(t)) = (\mathbf{f}(t), \mathbf{u}_\varepsilon(t)) . \tag{1.19}$$

By the divergence theorem we have

$$(g\nabla z_\varepsilon(t), \mathbf{u}_\varepsilon(t)) = -(g z_\varepsilon(t), \operatorname{div}(\mathbf{u}_\varepsilon(t))) . \tag{1.20}$$

Applying Young's inequality in the form

$$ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2 \tag{1.21}$$

to (1.20) (with $\delta = \frac{2g}{\alpha}$), we obtain

$$\begin{aligned}|g\left(\nabla z_{\varepsilon}\left(t\right),\mathbf{u}_{\varepsilon}\left(t\right)\right)| &= \left|-g\left(z_{\varepsilon}\left(t\right),\operatorname{div}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)\right)\right| \\ &\leq \frac{g}{2}\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\frac{\alpha}{2g}\left\|\operatorname{div}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)\right\|^{2}_{L^{2}(\Omega)}\right) \\ &\leq \frac{g}{2}\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\frac{\alpha}{2g}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}\right). \end{aligned} \tag{1.22}$$

In (1.7) if we take $\mathbf{u}=\mathbf{u}_{\varepsilon}$ and $\mathbf{v}=0$ to get $\left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)-\gamma\left|\omega^{0}\right|^{2},\mathbf{u}_{\varepsilon}\left(t\right)\right)\geq 0$, which yields

$$\begin{aligned}\left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right),\mathbf{u}_{\varepsilon}\left(t\right)\right) &= \left(\mathbf{B}\left(\mathbf{u}_{\varepsilon}\left(t\right)\right)-\gamma\left|\omega^{0}\right|^{2},\mathbf{u}_{\varepsilon}\left(t\right)\right)+\left(\gamma\left|\omega^{0}\left(t\right)\right|^{2},\mathbf{u}_{\varepsilon}\left(t\right)\right) \\ &\geq \left(\gamma\left|\omega^{0}\left(t\right)\right|^{2},\mathbf{u}_{\varepsilon}\left(t\right)\right) \\ &\geq -\frac{r}{\beta}\left\|\omega^{0}\left(t\right)\right\|^{2}_{L^{4}(\Omega)^{2}}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|_{L^{2}(\Omega)^{2}} \\ &\geq -\frac{r}{2\beta}\left(\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}}+\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right). \end{aligned} \tag{1.23}$$

Using again (1.21) but this time with $\delta = 1$, we get

$$\left(\mathbf{f}(t),\mathbf{u}_{\varepsilon}\left(t\right)\right)\leq\frac{1}{2}\left(\left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}+\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right). \tag{1.24}$$

Putting together (1.6), (1.22), (1.23) and (1.24), we derive from (1.19) the following

$$\begin{aligned}&\frac{d}{dt}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}+2\alpha\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}} \\ &\leq \left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\frac{r}{\beta}\left(\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}}+\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}\right) \\ &+g\left(\frac{2g}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\frac{\alpha}{2g}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}\right)+\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}} \\ &= \left(1+\frac{r}{\beta}\right)\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}+\frac{2g^{2}}{\alpha}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\frac{r}{\beta}\left\|\omega^{0}\left(t\right)\right\|^{4}_{L^{4}(\Omega)^{2}} \\ &+\frac{\alpha}{2}\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}+\left\|\mathbf{f}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}. \end{aligned} \tag{1.25}$$

Integrating (1.25) with respect to t, we obtain

$$\begin{aligned}&\left\|\mathbf{u}_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)^{2}}+2\alpha\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}ds \\ &\leq \left(1+\frac{r}{\beta}\right)\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{L^{2}(\Omega)^{2}}ds+\frac{2g^{2}}{\alpha}\int_{0}^{t}\left\|z_{\varepsilon}\left(s\right)\right\|^{2}_{L^{2}(\Omega)}ds+\frac{r}{\beta}\int_{0}^{t}\left\|\omega^{0}\left(s\right)\right\|^{2}_{L^{4}(\Omega)^{2}}ds \\ &+\frac{\alpha}{2}\int_{0}^{t}\left\|\mathbf{u}_{\varepsilon}\left(s\right)\right\|^{2}_{H^{1}_{0}(\Omega)^{2}}ds+\int_{0}^{t}\left\|\mathbf{f}\left(s\right)\right\|^{2}_{L^{2}(\Omega)^{2}}ds+\left\|\mathbf{u}^{0}\left(t\right)\right\|_{L^{2}(\Omega)^{2}}. \end{aligned} \tag{1.26}$$

Next dealing with (1.3) which we multiply by $z_{\varepsilon}(t)$ and then integrate the resulting equality over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\left\|z_{\varepsilon}\left(t\right)\right\|^{2}_{L^{2}(\Omega)}+\left(\operatorname{div}\left(h\mathbf{u}_{\varepsilon}\left(t\right)\right),z_{\varepsilon}\left(t\right)\right)=0. \tag{1.27}$$

But

$$\begin{aligned}
|(\operatorname{div}(h\mathbf{u}_\varepsilon(t)), z_\varepsilon(t))| &= |(h \operatorname{div} \mathbf{u}_\varepsilon(t), z_\varepsilon(t)) + (\mathbf{u}_\varepsilon(t)\cdot\nabla h, z_\varepsilon(t))| \\
&\le |(h \operatorname{div} \mathbf{u}_\varepsilon(t), z_\varepsilon(t))| + |(\mathbf{u}_\varepsilon(t)\cdot\nabla h, z_\varepsilon(t))| \\
&\le \|h\|_\infty \|\mathbf{u}_\varepsilon(t)\|_{H_0^1(\Omega)^2} \|z_\varepsilon(t)\|_{L^2(\Omega)} + M \|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2} \|z_\varepsilon(t)\|_{L^2(\Omega)} \\
&\le \frac{\mu}{2}\left(\frac{\alpha}{2\mu}\|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(\Omega)^2} + \frac{2\mu}{\alpha}\|z_\varepsilon(t)\|^2_{L^2(\Omega)}\right) \\
&+ \frac{M}{2}\left(\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)}\right). \qquad (1.28)
\end{aligned}$$

Taking into account (1.28) and integrating (1.27) in t gives

$$\begin{aligned}
\|z_\varepsilon(t)\|^2_{L^2(\Omega)} &\le M\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2} ds + \left(\frac{2\mu^2}{\alpha} + M\right)\int_0^t \|z_\varepsilon(s)\|^2_{L^2(\Omega)} ds \\
&+ \frac{\alpha}{2}\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2} ds + \left\|z^0(t)\right\|_{L^2(\Omega)}. \qquad (1.29)
\end{aligned}$$

Summing up inequalities (1.26) and (1.29) gives readily

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2} ds \\
&\le \lambda_1 \int_0^t \left(\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(s)\|^2_{L^2(\Omega)}\right) ds + \frac{r}{\beta}\int_0^t \left\|\omega^0(s)\right\|^4_{L^4(\Omega)^2} ds + \lambda_2,
\end{aligned}$$

where

$$\lambda_1 = \max\left(1 + M + \frac{r}{\beta}, \frac{2\mu^2}{\alpha} + M + \frac{2g^2}{\alpha}\right)$$

and

$$\lambda_2 = \int_0^T \|\mathbf{f}(s)\|^2_{L^2(\Omega)^2} ds + \left\|\mathbf{u}^0\right\|^2_{L^2(\Omega)^2} + \left\|z^0\right\|^2_{L^2(\Omega)}.$$

Now, appealing to inequality (1.13) (in Lemma 1.1) and owing to the fact that $\omega^0 \in L^2(0, T; H_0^1(\Omega)^2)$, we have

$$\begin{aligned}
\left\|\omega^0(s)\right\|^4_{L^4(\Omega)} &\le C\left\|\omega^0(s)\right\|^2_{L^4(\Omega)} \left\|\nabla\omega^0(s)\right\|^2_{L^2(\Omega)} \\
&\le C\left\|\omega^0(s)\right\|^4_{H_0^1(\Omega)} \quad \text{for a.e. } s \in (0, T),
\end{aligned}$$

so that

$$\left\|\omega^0(s)\right\|_{L^2(0,T;L^4(\Omega)^2)} \le C\left\|\omega^0(s)\right\|_{L^2(0,T;H_0^1(\Omega)^2)} \le C.$$

We are therefore led to

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t \|\mathbf{u}_\varepsilon(s)\|^2_{H_0^1(\Omega)^2} ds \\
&\le C + \lambda_1\int_0^t (\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\Omega)^2} + \|z_\varepsilon(s)\|^2_{L^2(\Omega)}) ds.
\end{aligned}$$

Applying the Gronwall inequality leads to

$$\sup_{0\le t\le T}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^2} \le C, \ \sup_{0\le t\le T}\|z_\varepsilon(t)\|_{L^2(\Omega)} \le C, \ \int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{H_0^1(\Omega)^2} dt \le C. \qquad (1.30)$$

From (1.10) we obtain for all $\varphi \in L^2(0,T;H_0^1(\Omega)^2)$,

$$\left|\left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t},\varphi\right)\right| \leq C\left\|\mathbf{u}_\varepsilon\right\|_{L^2(0,T;H_0^1(\Omega)^2)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)} + \left\|\mathbf{B}\left(\mathbf{u}_\varepsilon\right)\right\|_{L^2(Q)^2}\left\|\varphi\right\|_{L^2(Q)^2} + C\left\|z_\varepsilon\right\|_{L^2(Q)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)} + \left\|\mathbf{f}\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\left\|\varphi\right\|_{L^2(0,T;H_0^1(\Omega)^2)}.$$

Next, using the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\|\mathbf{B}(\mathbf{u}_\varepsilon)\|_{L^2(Q)} \leq C\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^4(\Omega)^2)} \leq C\|\mathbf{u}_\varepsilon\|_{L^2(0,T;H_0^1(\Omega)^2)}.$$

We therefore infer from (1.30) that

$$\left|\left(\frac{\partial \mathbf{u}_\varepsilon}{\partial t},\varphi\right)\right| \leq C\|\varphi\|_{L^2(0,T;H_0^1(\Omega)^2)},$$

from which

$$\left\|\frac{\partial \mathbf{u}_\varepsilon}{\partial t}\right\|_{L^2\left(0,T;H^{-1}(\Omega)^2\right)} \leq C.$$

We follow the same way of reasoning to see that

$$\left\|\frac{\partial z_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

This concludes the proof. □

2. Fundamentals of the sigma-convergence method

In this section we recall the main properties and some basic facts about the concept of sigma-convergence. We refer the reader to [24,25] for the details about most of the results of this section.

Let $\mathcal{A}$ be an algebra with mean value on $\mathbb{R}^d$ (integer $d \geq 1$), that is, a closed subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous real-valued functions on $\mathbb{R}^d$, BUC($\mathbb{R}^d$), which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for every $u \in \mathcal{A}$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ $(u^\varepsilon(x) = u(x/\varepsilon))$ weakly$*$-converges in $L^\infty(\mathbb{R}^d)$ to some real number $M(u)$ (called the mean value of u) as $\varepsilon \to 0$. The mean value expresses as

$$M(u) = \lim_{R\to\infty} ⨍_{B_R} u(y)dy \text{ for } u \in \mathcal{A} \tag{2.1}$$

where we have set $⨍_{B_R} = |B_R|^{-1}\int_{B_R}$.

For $1 \leq p < \infty$, we define the Marcinkiewicz space $\mathfrak{M}^p(\mathbb{R}^d)$ to be the set of functions $u \in L^p_{loc}(\mathbb{R}^d)$ such that

$$\limsup_{R\to\infty} ⨍_{B_R} |u(y)|^p\, dy < \infty.$$

Endowed with the seminorm

$$\|u\|_p = \left(\limsup_{R\to\infty} ⨍_{B_R} |u(y)|^p\, dy\right)^{1/p},$$

$\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space. Next, we define the *generalized Besicovitch space* $B^p_{\mathcal{A}}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ as the closure of the algebra with mean value $\mathcal{A}$ in $\mathfrak{M}^p(\mathbb{R}^d)$. Then for any $u \in B^p_{\mathcal{A}}(\mathbb{R}^d)$ we have that

$$\|u\|_p = \left(\lim_{R\to\infty} \fint_{B_R} |u(y)|^p \, dy\right)^{\frac{1}{p}} = (M(|u|^p))^{\frac{1}{p}}. \tag{2.2}$$

In this regard, we consider the space

$$B^{1,p}_{\mathcal{A}}(\mathbb{R}^d) = \{u \in B^p_{\mathcal{A}}(\mathbb{R}^d) : \nabla_y u \in (B^p_{\mathcal{A}}(\mathbb{R}^d))^d\}$$

endowed with the seminorm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla_y u\|_p^p\right)^{\frac{1}{p}},$$

which is a complete seminormed space. The Banach counterpart of the previous spaces are defined as follows. We set $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d) = B^p_{\mathcal{A}}(\mathbb{R}^d)/\mathcal{N}$ where $\mathcal{N} = \{u \in B^p_{\mathcal{A}}(\mathbb{R}^d) : \|u\|_p = 0\}$. We define $\mathcal{B}^{1,p}_{\mathcal{A}}(\mathbb{R}^d)$ mutatis mutandis: replace $B^p_{\mathcal{A}}(\mathbb{R}^d)$ by $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$ and $\partial/\partial y_i$ by $\overline{\partial}/\partial y_i$, where $\overline{\partial}/\partial y_i$ is defined by

$$\frac{\overline{\partial}}{\partial y_i}(u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \text{ for } u \in B^{1,p}_{\mathcal{A}}(\mathbb{R}^d). \tag{2.3}$$

It is important to note that $\overline{\partial}/\partial y_i$ is also defined as the infinitesimal generator in the ith direction coordinate of the strongly continuous group $\mathcal{T}(y) : \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d) \to \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$; $\mathcal{T}(y)(u+\mathcal{N}) = u(\cdot + y) + \mathcal{N}$. Let us denote by $\varrho : B^p_{\mathcal{A}}(\mathbb{R}^d) \to \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d) = B^p_{\mathcal{A}}(\mathbb{R}^d)/\mathcal{N}$, $\varrho(u) = u + \mathcal{N}$, the canonical surjection. We remark that if $u \in B^{1,p}_{\mathcal{A}}(\mathbb{R}^d)$ then $\varrho(u) \in \mathcal{B}^{1,p}_{\mathcal{A}}(\mathbb{R}^d)$ with further

$$\frac{\overline{\partial}\varrho(u)}{\partial y_i} = \varrho\left(\frac{\partial u}{\partial y_i}\right),$$

as seen above in (2.3).

We assume in the sequel that the algebra $\mathcal{A}$ is ergodic, that is, any $u \in \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$ which is invariant under $(\mathcal{T}(y))_{y\in\mathbb{R}^d}$ is constant in $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$, i.e., if $\mathcal{T}(y)u = u$ for every $y \in \mathbb{R}^d$, then $\|u - c\|_p = 0$ where c is a constant function in $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d)$. Let us also recall the following property [19,22]:

- The mean value M viewed as defined on $\mathcal{A}$, extends by continuity to a positive continuous linear form (still denoted by M) on $B^p_{\mathcal{A}}(\mathbb{R}^d)$. For each $u \in B^p_{\mathcal{A}}(\mathbb{R}^d)$ and all $a \in \mathbb{R}^d$, we have $M(u(\cdot + a)) = M(u)$, and $\|u\|_p = \left[M(|u|^p)\right]^{1/p}$.

To the space $B^p_{\mathcal{A}}(\mathbb{R}^d)$ we also attach the following *corrector* space

$$B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d) = \{u \in W^{1,p}_{loc}(\mathbb{R}^d) : \nabla_y u \in B^p_{\mathcal{A}}(\mathbb{R}^d)^d \text{ and } M(\nabla_y u) = 0\}.$$

In $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ we identify two elements by their gradients: $u = v$ in $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ iff $\nabla_y(u - v) = 0$, i.e. $\|\nabla_y(u - v)\|_p = 0$. We may therefore equip $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d)$ with the gradient norm $\|u\|_{\#,p} = \|\nabla_y u\|_p$. This defines a Banach space [6, Theorem 3.12] containing $B^{1,p}_{\mathcal{A}}(\mathbb{R}^d)$ as a subspace.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q))$ $(1 \leq p < \infty)$ is said to:

(i) *weakly* Σ*-converge* in $L^p(Q)$ to $u_0 \in L^p(Q; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$ if as $\varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x,t) f\left(x,t,\frac{x}{\varepsilon}\right) dxdt \to \int_Q M(u_0(x,t,\cdot) f(x,t,\cdot)) dxdt \tag{2.4}$$

for every $f \in L^{p'}(Q; \mathcal{A}))$, $\frac{1}{p} + \frac{1}{p'} = 1$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ;

(ii) strongly Σ*-converge* in $L^p(Q)$ to $u_0 \in L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$ if (2.4) holds and further

$$\|u_\varepsilon\|_{L^p(Q)} \to \|u_0\|_{L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))}\,. \tag{2.5}$$

We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-strong Σ.

Remark 2.1. 1) We can prove that the weak Σ-convergence in $L^p\,(Q)$ implies the weak convergence in $L^p\,(Q)$. 2) The convergence (2.4) still holds true for $f \in \mathcal{C}(\overline{Q}; B^{p',\infty}_{\mathcal{A}}\left(\mathbb{R}^d\right))$, where $B^{p',\infty}_{\mathcal{A}}\left(\mathbb{R}^d\right) = B^{p'}_{\mathcal{A}}\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$, $\frac{1}{p}+\frac{1}{p'}=1$.

The following results are the main properties of sigma-convergence and they can be found in [19,22,24]. Before we can state them, we need to define what we call a fundamental sequence. By a *fundamental sequence* we term any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ (denoted here below by E) of real numbers satisfying $0<\varepsilon_n\leq 1$ and $\varepsilon_n \to 0$ when $n\to\infty$.

(SC)$_1$ For $1<p<\infty$, any sequence which is bounded in $L^p\,(Q)$ possesses a weakly Σ-convergent subsequence.

(SC)$_2$ Let $1<p<\infty$. Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(0,T;W^{1,p}_0(\Omega))$ be a sequence which satisfies the following estimate

$$\sup_{\varepsilon\in E} \|u_\varepsilon\|_{L^p\left(0,T;W^{1,p}_0(\Omega)\right)} < \infty.$$

Then there exist a subsequence E' from E and a couple (u_0,u_1) with $u_0 \in L^p(0,T;W^{1,p}_0(\Omega))$ and $u_1 \in L^p(Q;B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^d))$ such that as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p\,(Q)\text{-weak }\Sigma$$

and

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^p(Q)\text{-weak }\Sigma,\ 1\leq i\leq d. \tag{2.6}$$

(SC)$_3$ Let $1<p,q<\infty$ and $r\geq 1$ be such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\leq 1$. Assume that $(u_\varepsilon)_{\varepsilon>0} \subset L^q(Q)$ is weakly Σ-convergent in $L^q(Q)$ to some $u_0 \in L^q(Q;\mathcal{B}^q_{\mathcal{A}}(\mathbb{R}^d))$ and $(v_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ is strongly Σ-convergent in $L^p(Q)$ to some $v_0 \in L^p(Q;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^d))$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon>0}$ is weakly Σ-convergent in $L^r(Q)$ to $u_0 v_0$.

3. Homogenization result

3.1. Passage to the limit

First we set

$$\mathbb{V} = \left\{ \mathbf{u} \in L^2\left(0,T;H^1_0(\Omega)^2\right) : \mathbf{u}' = \frac{\partial \mathbf{u}}{\partial t} \in L^2\left(0,T;H^{-1}(\Omega)^2\right) \right\};$$

$$\begin{aligned}\mathbb{H} &= \left\{ z \in L^2\left(0,T;L^2(\Omega)\right) : z' = \frac{\partial z}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right) \right\}\\ &= H^1(0,T;L^2(\Omega))\end{aligned}$$

The space $\mathbb{V}$ and $\mathbb{H}$ are Hilbert spaces with obvious norms. Moreover the imbeddings $\mathbb{V} \hookrightarrow L^2\left(0,T;L^2(\Omega)^2\right)$ and $\mathbb{H} \hookrightarrow L^2\left(0,T;L^2(\Omega)\right)$ are compact.

Now in view of a priori estimates in Lemma 1.2, the sequences $(\mathbf{u}_\varepsilon)_\varepsilon$ and $(z_\varepsilon)_\varepsilon$ are bounded in $\mathbb{V}$ and in $\mathbb{H}$ respectively. Hence given a fundamental sequence E, there exist a subsequence E' of E and a couple $(\mathbf{u}_0, z_0) \in \mathbb{V} \times \mathbb{H}$ such that, as $E' \ni \varepsilon \to 0$,

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \text{ in } \mathbb{V}\text{-weak}; \tag{3.1}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \text{ in } L^2(0,T;L^2(\Omega)^2)\text{-strong}; \tag{3.2}$$

$$z_\varepsilon \to z_0 \text{ in } \mathbb{H}\text{-weak}; \tag{3.3}$$

$$z_\varepsilon \to z_0 \text{ in } L^2\left(0,T;L^2(\Omega)\right)\text{-strong}. \tag{3.4}$$

Taking into account the estimates (1.14) to (1.18), we derive (by a diagonal process) the existence of a subsequence of E' (still denoted by E') and of a function $\mathbf{u}_1 \in L^2(Q; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$ such that, as $E' \ni \varepsilon \to 0$,

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial x_i} \to \frac{\partial \mathbf{u}_0}{\partial x_i} + \frac{\partial \mathbf{u}_1}{\partial y_i} \text{ in } L^2(Q)^2\text{-weak } \Sigma,\ i = 1,2. \tag{3.5}$$

It follows that $(\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}^1_0 = \mathbb{V} \times L^2(Q; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$.

Now, for an element $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1) \in \mathbb{F}^1_0$, we set

$$\mathbb{D}\mathbf{v} = \nabla \mathbf{v}_0 + \nabla_y \mathbf{v}_1 = (\mathbb{D}_i \mathbf{v})_{1\le i\le 2} \text{ where } \mathbb{D}_i\mathbf{v} = \frac{\partial \mathbf{v}_0}{\partial x_i} + \frac{\partial \mathbf{v}_1}{\partial y_i},\ i = 1,2$$

with $\frac{\partial \mathbf{v}_0}{\partial x_i} + \frac{\partial \mathbf{v}_1}{\partial y_i} = \left(\frac{\partial \mathbf{v}_0^j}{\partial x_i} + \frac{\partial \mathbf{v}_1^j}{\partial y_i}\right)_{1\le j\le 2}$. The smooth counterpart of $\mathbb{F}^1_0$ is defined by $\mathcal{F}^\infty_0 = \mathcal{C}^\infty_0(Q)^2 \otimes \mathcal{C}^\infty_0\left(Q; (\mathcal{A}^\infty/\mathbb{R})^2\right)$.

With this in mind, the following result holds true.

Proposition 3.1. *Let $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{F}^1_0$ and $z_0 \in \mathbb{H}$. Then $\mathbf{u}$ and z_0 solve the following variational problem:*

$$\begin{aligned} &-\int_Q \mathbf{u}_0 \frac{\partial \varphi_0}{\partial t} dxdt + \int_Q M\left(A_0 \mathbb{D}\mathbf{u} \cdot \mathbb{D}\varphi\right) dxdt + \int_Q \mathbf{B}(\mathbf{u}_0)\, \varphi_0 dxdt + \int_Q g \nabla z_0 \varphi_0 dxdt \\ &= \int_0^T (\mathbf{f}(t), \varphi_0(t))\, dt \end{aligned} \tag{3.6}$$

$$-\int_Q z_0 \frac{\partial \psi_0}{\partial t} dxdt - \int_Q h\mathbf{u}_0 \cdot \nabla \psi_0 dxdt = 0 \tag{3.7}$$

for all $\varphi = (\varphi_0, \varphi_1) \in \mathcal{F}^\infty_0$ and $\psi_0 \in \mathcal{C}^\infty_0(Q)$.

Proof. Let $\varphi = (\varphi_0, \varphi_1)$ and ψ_0 be as above, and define

$$\varphi_\varepsilon = \varphi_0(x,t) + \varphi_1\left(x, t, \frac{x}{\varepsilon}\right) \text{ for } (x,t) \in Q.$$

Taking $(\varphi_\varepsilon, \psi_0)$ as a test function in the variational form of (1.2)-(1.5), we obtain

$$\begin{aligned} &-\int_Q \mathbf{u}_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} dxdt + \int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi_\varepsilon dxdt + \int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi_\varepsilon dxdt + \int_Q g \nabla z_\varepsilon \varphi_\varepsilon dxdt \\ &= \int_0^T (\mathbf{f}(t), \varphi_\varepsilon(t))\, dt \end{aligned} \tag{3.8}$$

and

$$-\int_Q z_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt - \int_Q h\mathbf{u}_\varepsilon \cdot \nabla \psi_0 dxdt = 0. \tag{3.9}$$

Using the identities

$$\frac{\partial \varphi_\varepsilon}{\partial t} = \frac{\partial \varphi_0}{\partial t} + \varepsilon \left(\frac{\partial \varphi_1}{\partial t} \right)^\varepsilon \text{ and } \nabla \varphi_\varepsilon = \nabla \varphi_0 + (\nabla_y \varphi_1)^\varepsilon + \varepsilon (\nabla \varphi_1)^\varepsilon,$$

we infer that, as $\varepsilon \to 0$,

$$\frac{\partial \varphi_\varepsilon}{\partial t} \to \frac{\partial \varphi_0}{\partial t} \text{ in } L^2 \left(0, T; H^{-1}(\Omega)^2\right) \text{-weak} \tag{3.10}$$

$$\nabla \varphi_\varepsilon \to \nabla \varphi_0 + \nabla_y \varphi_1 \text{ in } L^2(Q)^{2\times 2}\text{-strong } \Sigma \tag{3.11}$$

$$\varphi_\varepsilon \to \varphi_0 \text{ in } L^2(Q)^2\text{-strong.} \tag{3.12}$$

Let us consider each of the equations (3.8) and (3.9) separately. We first consider (3.8) and using the convergence results (3.2) and (3.10), we obtain

$$\int_Q \mathbf{u}_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} dxdt \to \int_Q \mathbf{u}_0 \frac{\partial \varphi_0}{\partial t} dxdt. \tag{3.13}$$

Considering the second term of the left hand-side of (3.8) and owing the fact that $A_0 \in \mathcal{C}(\overline{Q}; B^{2,\infty}_{\mathcal{A}}(\mathbb{R}^2)^{2\times 2})$, we use A_0 as a test function and property $(\text{SC})_3$ (recall that we have (3.5) and (3.11)) to get

$$\int_Q A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \varphi_\varepsilon dxdt \to \int_Q M\left(A_0 \mathbb{D}\mathbf{u} \cdot \mathbb{D}\varphi\right) dxdt. \tag{3.14}$$

Let us show that

$$\int_Q \mathbf{B}(\mathbf{u}_\varepsilon)\varphi_\varepsilon dxdt \to \int_Q \mathbf{B}(\mathbf{u}_0)\varphi_0 dxdt. \tag{3.15}$$

First, we have from (3.2) that, up to a subsequence of E' not relabeled, $\mathbf{u}_\varepsilon \to \mathbf{u}_0$ a.e. in Q. Hence from the continuity of $\mathbf{B}$, we entail

$$\mathbf{B}(\mathbf{u}_\varepsilon) \to \mathbf{B}(\mathbf{u}_0) \text{ a.e. in } Q.$$

we infer from the boundedness of the sequence $(\mathbf{B}(\mathbf{u}_\varepsilon))_{\varepsilon>0}$ that $\mathbf{B}(\mathbf{u}_\varepsilon) \to \mathbf{B}(\mathbf{u}_0)$ in $L^2(Q)^2$-weak. Putting this together with (3.12), we obtain (3.15). We also easily obtain

$$\int_0^T (\mathbf{f}(t), \varphi_\varepsilon(t))\, dt \to \int_0^T (\mathbf{f}(t), \varphi_0(t)) dt. \tag{3.16}$$

Next, the convergence results (3.3) and (3.12) yield

$$\int_Q g \nabla z_\varepsilon \varphi_\varepsilon dxdt \to \int_Q g \nabla z_0 \varphi_0 dxdt. \tag{3.17}$$

As for equation (3.9), we use the strong convergence (3.4) associated to (3.12) to get

$$\int_Q z_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt \to \int_Q z_0 \frac{\partial \psi_0}{\partial t} dxdt.$$

Concerning the second term in (3.9), we infer from (3.2) that

$$\int_Q h \mathbf{u}_\varepsilon \cdot \nabla \psi_0 dxdt \to \int_Q h \mathbf{u}_0 \cdot \nabla \psi_0 dxdt,$$

thereby completing the proof of the proposition. □

108 *3.2. Homogenized problem*

We intend here to derive the problem whose the couple $(\mathbf{u}_0, z_0)$ is solution. To achieve this, we first uncouple equation (3.6), which is equivalent to the system consisting of (3.18) and (3.19) below:

$$-\int_Q \mathbf{u}_0 \frac{\partial \varphi_0}{\partial t} dxdt + \int_Q M\left(A_0 \mathbb{D}\mathbf{u} \cdot \nabla'_0\right) dxdt + \int_Q \mathbf{B}\left(\mathbf{u}_0\right) \varphi_0 dxdt + \int_Q g \nabla z_0 \cdot \varphi_0 dxdt$$
$$= \int_0^T \left(\mathbf{f}(t), \varphi_0(t)\right) dt; \tag{3.18}$$

$$\int_Q M\left(A_0 \mathbb{D}\mathbf{u} \cdot \nabla_y \varphi_1\right) dxdt = 0. \tag{3.19}$$

Choosing in (3.19)

$$\varphi_1(x,t,y) = \theta(x,t)\mathbf{v}(y) \text{ where } \theta \in C_0^\infty(Q), \mathbf{v} \in (\mathcal{A}^\infty)^2, \tag{3.20}$$

we obtain

$$M\left(A_0 \mathbb{D}\mathbf{u} \cdot \nabla \mathbf{v}\right) = 0 \text{ for all } \mathbf{v} \in (\mathcal{A}^\infty)^2. \tag{3.21}$$

Let us deal with (3.21). To this end, fix $\xi \in \mathbb{R}^{2\times 2}$ and consider the corrector problem:

$$\begin{cases} \text{Find } \mathfrak{B}(\xi) \in \mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2) \text{ such that :} \\ -\operatorname{div}_y[A_0(x,\cdot)(\xi + \nabla_y \mathfrak{B}(\xi))] = 0 \text{ in } \mathbb{R}^2. \end{cases} \tag{3.22}$$

Then in view of the properties of the matrix $A_0(x,\cdot)$, we infer from [10,23] that (3.22) possesses a unique solution in $\mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2)$. Coming back to (3.22) and taking there $\xi = \nabla \mathbf{u}_0(x,t)$, testing the resulting equation with φ_1 as in (3.20), we get by the uniqueness of the solution of (3.22) that $\mathbf{u}_1(x,t,y) = \mathfrak{B}(\nabla \mathbf{u}_0(x,t)(x,y)$. This shows that $\mathfrak{B}(\nabla \mathbf{u}_0)$ belongs to $L^2(0,T;\mathcal{C}(\overline{\Omega}; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2)^2))$. Clearly, if χ_j^ℓ is the solution of (3.22) corresponding to $\xi = \xi_j^\ell = (\delta_{ij}\delta_{k\ell})_{1\le i,k\le 2}$ (that is all the entries of ξ are zero except the entry occupying the jth row and the ℓth column which is equal to 1), then

$$\mathbf{u}_1 = \sum_{j,\ell=1}^{2} \frac{\partial u_0^\ell}{\partial x_j} \chi_j^\ell \text{ where } \mathbf{u}_0 = (u_0^\ell)_{1\le \ell \le 2}. \tag{3.23}$$

We recall again that χ_j^ℓ depends on x as it is the case for A_0. In the variational form of (3.18), we insert the value of $\mathbf{u}_1$ obtained in (3.23) to get the equation

$$\frac{\partial \mathbf{u}_0}{\partial t} - \operatorname{div}(\widehat{A}_0(x)\nabla \mathbf{u}_0) + \mathbf{B}\left(\mathbf{u}_0\right) + g\nabla z_0 = \mathbf{f} \text{ in } Q. \tag{3.24}$$

where $\widehat{A}_0(x) = (\widehat{a}_{ij}^{k\ell}(x))_{1\le i,j,k,\ell\le 2}$, $\widehat{a}_{ij}^{k\ell}(x) = a_{\text{hom}}(\chi_j^\ell + P_j^\ell, \chi_i^k + P_i^k)$ with $P_j^\ell = y_j e^\ell$ (e^ℓ the ℓth vector of the canonical basis of $\mathbb{R}^2$) and

$$a_{\text{hom}}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i,j,k=1}^{2} M\left(a_{ij} \frac{\partial u^k}{\partial y_j} \frac{\partial v^k}{\partial y_i}\right) \text{ where } A_0 = (a_{ij})_{1\le i,j\le 2}.$$

Also the equation (3.7) is equivalent to

$$\frac{\partial z_0}{\partial t} + \operatorname{div}(h\mathbf{u}_0) = 0 \text{ in } Q. \tag{3.25}$$

Finally putting together the equations (3.24)-(3.25) associated to the boundary and initial conditions, we are led to the homogenized problem, viz.

$$\begin{cases} \dfrac{\partial \mathbf{u}_0}{\partial t} - \operatorname{div}(\widehat{A}_0(x)\nabla \mathbf{u}_0) + \mathbf{B}\left(\mathbf{u}_0\right) + g\nabla z_0 = \mathbf{f} \text{ in } Q \\ \dfrac{\partial z_0}{\partial t} + \operatorname{div}(h\mathbf{u}_0) = 0 \text{ in } Q \\ \mathbf{u}_0 = 0 \text{ on } \partial\Omega \times (0,T) \\ \mathbf{u}_0(x,0) = \mathbf{u}^0(x), z_0(x,0) = z^0(x) \text{ in } \Omega. \end{cases} \tag{3.26}$$

It can be easily shown that the matrix $\widehat{A}_0$ of homogenized coefficients has entries in $\mathcal{C}(\overline{\Omega})$, and is uniformly elliptic, so that under the conditions (**A1**)-(**A2**), the problem (3.26) possesses a unique solution $(\mathbf{u}_0, z_0)$ with $\mathbf{u}_0 \in L^2\left(0,T;H_0^1(\Omega)^2\right)$ and $z_0 \in L^2\left(0,T;L^2(\Omega)\right)$. Since the solution of (3.26) is unique, we infer that the whole sequence $(\mathbf{u}_\varepsilon, z_\varepsilon)$ converges in a suitable space towards $(\mathbf{u}_0, z_0)$ as stated in the following result, which is the main result of the work.

Theorem 3.1. *Assume that* (**A1**) *to* (**A3**) *hold. For any $\varepsilon > 0$ let $(\mathbf{u}_\varepsilon, z_\varepsilon)$ be the unique solution of problem* (1.2) *to* (1.5)*. Then the sequence $(\mathbf{u}_\varepsilon, z_\varepsilon)$ converges strongly in $L^2(Q)^2 \times L^2(Q)$ to the solution of problem* (3.26).

Proof. The proof is a consequence of the previous steps. □

4. Some concrete applications of Theorem 3.1

The homogenization of problem has been made possible under the fundamental assumption (**A3**). Some physical situations that lead to (**A3**) are listed below.

Problem 1 (**Periodic Homogenization**)**.** The homogenization of (1.2)-(1.5) holds under the periodicity assumption that the matrix-function $A_0(x,\cdot)$ is periodic with period 1 in each coordinate, for any $x \in \overline{\Omega}$. In that case, we have $\mathcal{A} = \mathcal{C}_{per}(Y)$, where $Y = (0,1)^2$ and $\mathcal{C}_{per}(Y)$ is the algebra of continuous Y-periodic functions defined in $\mathbb{R}^2$. It is easy to see that $B^2_{\mathcal{A}}(\mathbb{R}^2) = L^2_{per}(Y) \equiv \{u \in L^2_{loc}(\mathbb{R}^2) : u$ is Y-periodic$\}$, and the mean value expresses as $M(u) = \int_Y u(y)dy$. The homogenized matrix is hence defined by $\widehat{A}_0(x) = (\widehat{a}^{k\ell}_{ij}(x))_{1\le i,j,k,\ell\le 2}$, $\widehat{a}^{k\ell}_{ij}(x) = a_{\text{hom}}(\chi^\ell_j + P^\ell_j, \chi^k_i + P^k_i)$ with $P^\ell_j = y_j e^\ell$ (e^ℓ the ℓth vector of the canonical basis of $\mathbb{R}^2$) and

$$a_{\text{hom}}(\boldsymbol{u},\boldsymbol{v}) = \sum_{i,j,k=1}^{2} \int_Y a_{ij}\frac{\partial u^k}{\partial y_j}\frac{\partial v^k}{\partial y_i}dy \text{ where } A_0 = (a_{ij})_{1\le i,j\le 2}.$$

where here χ^ℓ_j is the solution of the cell problem

$$\chi^\ell_j(x,\cdot) \in H^1_\#(Y)^2 : -\operatorname{div}_y(A_0(x,\cdot)(\xi^\ell_j + \nabla_y \chi^\ell_j(x,\cdot))) = 0 \text{ in } Y$$

with $H^1_\#(Y) = \{v \in H^1_{per}(Y) : \int_Y v dy = 0\}$, $H^1_{per}(Y) = \{v \in L^2_{per}(Y) : \nabla_y v \in L^2_{per}(Y)^2\}$ and $\xi^\ell_j = (\delta_{ij}\delta_{k\ell})_{1\le i,k\le 2}$.

Problem 2 (**Almost periodic Homogenization**)**.** We may consider the homogenization problem for (1.2)-(1.5) under the assumption that the coefficients of the matrix $A_0(x,\cdot)$ are Besicovitch almost periodic functions [1]. In that case, hypothesis (**A3**) holds true with $\mathcal{A} = \text{AP}(\mathbb{R}^2)$, where $\text{AP}\left(\mathbb{R}^2\right)$ is the algebra of Bohr almost periodic functions on $\mathbb{R}^2$ [3]. The mean value of a function $u \in \text{AP}\left(\mathbb{R}^2\right)$ is the unique constant belonging to the close convex hull of the family of the translates $(u(\cdot + a))_{a\in\mathbb{R}^2}$.

Problem 3 (**Weakly almost periodic Homogenization**)**.** We may solve the homogenization problem for (1.2)-(1.5) under the assumption: the function $A_0(x,\cdot)$ is weakly almost periodic, that is, the matrix $A_0(x,\cdot)$ has its entries in the algebra with mean value $\mathcal{A} = \text{WAP}\left(\mathbb{R}^2\right)$ (where $\text{WAP}\left(\mathbb{R}^2\right)$ is the algebra of continuous weakly almost periodic functions on $\mathbb{R}^2$; see e.g., [7]).

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Sample Availability: Samples of the compounds ...... are available from the authors.