axioms

Article

New Relation-Theoretic Fixed Point Theorems in Fuzzy Metric Spaces with an Application to Fractional Differential Equations

Samera M. Saleh [1], Waleed M. Alfaqih [2,*], Salvatore Sessa [3,*] and Ferdinando Di Martino [3]

1 Department of Mathematics, Faculty of Science, Taiz University, Taiz 6803, Yemen; samirasaleh2007@yahoo.com
2 Department of Mathematics, Hajjah University, Hajjah 1729, Yemen
3 Dipartimento di Architettura, Università degli Studi di Napoli Federico II, Via Toledo 402, 80134 Napoli, Italy; fdimarti@unina.it
* Correspondence: waleedmohd2016@gmail.com (W.M.A.); sessa@unina.it (S.S.)

Abstract: In this paper, we introduce the notion of fuzzy $\mathcal{R}-\psi-$contractive mappings and prove some relevant results on the existence and uniqueness of fixed points for this type of mappings in the setting of non-Archimedean fuzzy metric spaces. Several illustrative examples are also given to support our newly proven results. Furthermore, we apply our main results to prove the existence and uniqueness of a solution for Caputo fractional differential equations.

Keywords: fuzzy metric space; fixed point; binary relation; $\mathcal{R}-\psi-$contractive mappings; Caputo fractional differential equation

MSC: 47H10; 54H25

check for updates

Citation: Saleh, S.M.; Alfaqih, W.M.; Sessa, S.; Di Martino, F. New Relation-Theoretic Fixed Point Theorems in Fuzzy Metric Spaces with an Application to Fractional Differential Equations. *Axioms* **2022**, *11*, 117. https://doi.org/10.3390/axioms11030117

Academic Editor: Hsien-Chung Wu

Received: 15 January 2022
Accepted: 26 February 2022
Published: 7 March 2022

1. Introduction and Preliminaries

The concept of fuzzy sets was initially presented by Zadeh [1] in 1965, wherein he defined a fuzzy set as: a fuzzy set $\mathcal{M}$ on a non-empty set $\mathcal{X}$ is a function from $\mathcal{X}$ to $[0,1]$. This concept plays a very important role in several scientific and engineering applications. Thereafter, Kramosil and Michalek [2] introduced the notion of fuzzy metric spaces which has been modified later on by George and Veeramani [3] holding the Hausdorffness property for such modified spaces.

The fuzzy fixed point theory was started by Grabiec [4] in 1988, wherein he presented the concepts of G-Cauchy sequences and G-complete fuzzy metric spaces and provided a fuzzy metric version of Banach's contraction principle. To date, many fixed point results have been provided on such spaces. In fact, the above mentioned concept of G-completeness is not a very natural notion, as even $\mathbb{R}$ (the set of real numbers) is not complete in this sense. In this quest, in 1994 George and Veeramani [3] slightly modified the concepts of fuzzy metric spaces and M-Cauchy sequences wherein they found a Hausdorff topology in their new defined notion of fuzzy metric spaces. Later, in 2002, Gregori and Sapena [5] defined fuzzy contractive mappings and proved a very natural extension of the well-known Banach contraction principle for such mappings in G-complete as well as M-complete fuzzy metric spaces. Mihet [6] in 2008 extended the class of Gregori and Sapena's fuzzy contractive mappings [5] and proved a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michalek.

On the other hand, relation-theoretic fixed point theory is a relatively new direction of fixed point theory. This direction was initiated by Turinici [7] and it becomes a very active area after the appearance of the great results due to Ran and Reurings [8] and Nieto and Lopez [9,10] wherein they provided a new version for Banach contraction principle equipping the contractive condition with an ordered binary relation. The authors in [8–10] provided several interested applications to boundary value problems and matrix equations which supported their fixed point results strongly. Thereafter, a lot of fixed point theorems have been provided in which various definitions of binary relations were equipped (e.g., [11–15] and several others).

Now, let us recall some basic definitions, notions, and results which will be needed in the following.

Definition 1 ([2]). *A continuous t-norm $*$ is a continuous binary operation $* : [0,1] \times [0,1] \to [0,1]$ which is commutative and associative and satisfies:*

(i) $t * 1 = t \; \forall \; t \in [0,1]$;

(ii) $t * s \le u * v$ *whenever* $t \le u$ *and* $s \le v \; \forall \; t,s,u,v \in [0,1]$.

The following are some well-known examples of continuous t-norm: $t * s = \min\{t,s\}$, $t * s = ts$, and $t * s = \max\{t + s - 1, 0\}$, $\forall t,s \in [0,1]$.

Kramosil and Michalek [2] defined fuzzy metric spaces as under.

Definition 2 ([2]). *Let $\mathcal{M}$ be a fuzzy set on $\mathcal{X}^2 \times [0,\infty)$ and $*$ a continuous t-norm. Assume that ($\forall \; x,y,z \in \mathcal{X}$ and $t,s > 0$):*

(KM-i) $\mathcal{M}(x,y,0) = 0$;

(KM-ii) $\mathcal{M}(x,y,t) = 1$ *iff* $x = y$;

(KM-iii) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;

(KM-iv) $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \le \mathcal{M}(x,z,t+s)$;

(KM-v) $\mathcal{M}(x,y,.) : [0,\infty) \to [0,1]$ *is left continuous.*

Then $(\mathcal{X},\mathcal{M},*)$ is called a fuzzy metric space (Kramosil and Michalek's sense).

Definition 3. *If we replace the axiom (KM-iv) by:*

(KM-iv)$'$ $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \le \mathcal{M}(x,z,\max\{t,s\}) \; \forall \; x,y,z \in \mathcal{X}$ *and* $t,s > 0$, *then $(\mathcal{X},\mathcal{M},*)$ is known as a non-Archimedean fuzzy metric space. It is easy to check that the triangular inequality (KM-iv)$'$ implies (KM-iv), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.*

The topology of a fuzzy metric space (Kramosil and Michalek's sense) is not Hausdorff in general. In order to have Hausdorffness property, George and Veeramani [3,16] slightly modified the definition of fuzzy metric spaces such that the topology of the newly defined fuzzy metric space becomes Hausdorff.

Definition 4 ([3,16]). *Let $\mathcal{M}$ be a fuzzy set on $\mathcal{X}^2 \times (0,\infty)$ and $*$ a continuous t-norm. Assume that ($\forall \; x,y,z \in \mathcal{X}$ and $t,s > 0$):*

(GV-i) $\mathcal{M}(x,y,t) > 0$;

(GV-ii) $\mathcal{M}(x,y,t) = 1$ *iff* $x = y$;

(GV-iii) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;

(GV-iv) $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \le \mathcal{M}(x,z,t+s)$;

(GV-v) $\mathcal{M}(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

Then $(\mathcal{X},\mathcal{M},*)$ is called a fuzzy metric space (George and Veeramani's sense).

Remark 1 ([3]). *The topology of a fuzzy metric space in the sense of Definition 4 is Hausdorff.*

Remark 2 ([3]). *Every fuzzy metric space in the sense of Definition 4 is a fuzzy metric space in the sense of Definition 2, the converse is not true in general.*

Example 1. *Let $(\mathcal{X},d)$ be an ordinary metric space and let ϕ be a nondecreasing and continuous function from $(0,\infty) \to (0,1)$ such that $\lim_{t\to\infty} \phi(t) = 1$ (Some examples of these functions are $\phi(t) = \frac{t}{1+t}$, $\phi(t) = 1 - e^{-t}$, and $\phi(t) = e^{-\frac{1}{t}}$). Let $a * b \le ab$ for all $a,b \in [0,1]$. For each $t \in (0,\infty)$, define*

$$\mathcal{M}(x,y,t) = [\phi(t)]^{d(x,y)}, \quad \forall x,y \in X.$$

It is easy to see that $(X, M, *)$ is a non-Archimedean fuzzy metric space.

Remark 3 ([2]). *For all $x, y \in \mathcal{X}$, $\mathcal{M}(x, y, .)$ is a non-decreasing mapping.*

Definition 5 ([3,4,16]). *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. A sequence $\{x_n\} \subseteq \mathcal{X}$ is said to be*
(i) convergent to $x \in \mathcal{X}$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = 1 \, \forall \, t > 0,$$

in this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) Cauchy if $\forall \, \epsilon > 0$ and $t > 0$, $\exists N \in \mathbb{N}$ satisfying

$$\mathcal{M}(x_n, x_{n+p}, t) > 1 - \epsilon \, \forall \, n \geq N \text{ and } p \in \mathbb{N}_0.$$

Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. If every Cauchy sequence in $\mathcal{X}$ is convergent in $\mathcal{X}$, then $\mathcal{X}$ is said to be complete.

Lemma 1 ([17]). *If $(\mathcal{X}, \mathcal{M}, *)$ is a fuzzy metric space, then $\mathcal{M}$ is a continuous function on $\mathcal{X}^2 \times (0, \infty)$.*

Definition 6 ([18]). *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. Then the mapping M is said to be continuous on $\mathcal{X}^2 \times (0, \infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $\mathcal{X}^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in \mathcal{X}^2 \times (0, \infty)$, i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Roldán-López-de-Hierro [19] defined a comparison function $\psi : [0, 1] \to [0, 1]$ which satisfies:

(A) ψ is non-decreasing and left continuous;
(B) $\psi(t) < t$ for all $t \in (0, 1)$;
(C) $\psi(0) = 0$.

Let Ψ denotes the family of all such functions ψ.

For example, $\psi(t) = t^2$ for all $t \in [0, 1]$. Notice that, using the previous definition, the condition $\psi(1) = 1$ is not necessarily true.

Remark 4 ([19]). *Let $\psi \in \Psi$.*

(i) $\psi(t) \leq t$ *for all* $t \in [0, 1]$;
(ii) *if* $\psi(t_0) = t_0$ *for some* $t_0 \in (0, 1]$, *then* $t_0 = 1$.
(iii) *If* $\{t_n\} \subset [0, 1]$ *and* $\psi(t_n) \to 1$, *then* $t_n \to 1$.

Now, we recall some relation-theoretic notions as follows.

Definition 7 ([20]). *A subset $\mathcal{R}$ of $\mathcal{X}^2$ is called a binary relation on $\mathcal{X}$. If $(x, y) \in \mathcal{R}$ (we may write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$), then we say that "x is related to y under $\mathcal{R}$". If either $x\mathcal{R}y$ or $y\mathcal{R}x$, then we write $[x, y] \in \mathcal{R}$.*

Observe that $\mathcal{X}^2$ is a binary relation on $\mathcal{X}$ called the universal relation. In this presentation, $\mathcal{X}$ is to a non-empty set and $\mathcal{R}$ refers for a non-empty binary relation on $\mathcal{X}$.

Definition 8 ([21,22]). *A binary relation $\mathcal{R}$ on a non-empty set $\mathcal{X}$ is said to be:*
(i) reflexive if $x\mathcal{R}x \; \forall \; x \in \mathcal{X}$;
(ii) transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z \; \forall \; x, y, z \in \mathcal{X}$;
(iii) antisymmetric if $x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y \; \forall \; x, y \in \mathcal{X}$;
(iv) partial order if it is reflexive, antisymmetric and transitive;
(v) complete if $[x, y] \in \mathcal{R} \; \forall \; x, y \in \mathcal{X}$;
(vi) f-closed if $(x, y) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R} \;\; \forall x, y \in \mathcal{X}$ where $f : \mathcal{X} \to \mathcal{X}$ is a mapping.

Definition 9 ([23]). *Let $\mathcal{X}$ be a non-empty set and $\mathcal{R}$ be a binary relation on $\mathcal{X}$. A sequence $\{x_n\} \subseteq \mathcal{X}$ is said to be an $\mathcal{R}$-preserving sequence if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.*

Recently, Alfaqih et al. [24] presented a relation-theoretic version for the fuzzy version of Banach contractive principle wherein the authors introduced relation-theoretic versions of several fuzzy metrical notions as follows.

Definition 10 ([24]). *A binary relation $\mathcal{R}$ on $\mathcal{X}$ is said to be an $\mathcal{M}$-self-closed if given any convergent $\mathcal{R}$-preserving sequence $\{x_n\} \subseteq \mathcal{X}$ which converges (in fuzzy sense) to some $x \in \mathcal{X}$, $\exists \{x_{n_k}\} \subseteq \{x_n\}$ with $(x_{n_k}, x) \in \mathcal{R}$.*

Example 2 ([24]). *Let $\mathcal{X} = (0, 4]$ and $*$ be the product t-norm given by $t * s = ts \; \forall t, s \in [0, 1]$. Define $\mathcal{M}$ by ($\forall x, y \in \mathcal{X}$ and $t > 0$)*

$$\mathcal{M}(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \\ \frac{2t}{2t+|x-y|}, & \text{if } t \neq 0. \end{cases}$$

Define $\mathcal{R}$ on $\mathcal{X}$ by

$$\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (1, 4), (2, 4)\}.$$

Observe that if $\{x_n\}$ is an $\mathcal{R}$-preserving sequence which converges to some $x \in \mathcal{X}$, then $\exists N \in \mathbb{N}$ such that either $x_n = 1 \; \forall \; n \geq N$ or $x_n = 2 \; \forall \; n \geq N$. Therefore, $\{x_{N+i}\}_{i \in \mathbb{N}}$ is a subsequence of $\{x_n\}$ such that $x_{N+i}\mathcal{R}x$ for each $i \in \mathbb{N}$. Hence, $\mathcal{R}$ is $\mathcal{M}$-self closed.

Definition 11. *A sequence $\{x_n\} \subseteq \mathcal{X}$ is called $\mathcal{R}$-Cauchy if $x_n\mathcal{R}x_{n+1} \; \forall \; n \in \mathbb{N}_0$ and $\forall \; \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ satisfying (for all $t > 0$)*

$$\mathcal{M}(x_n, x_{n+p}, t) > 1 - \epsilon \; \forall \; n \geq N \text{ and } p \in \mathbb{N}_0.$$

Remark 5. *Every Cauchy sequence is an $\mathcal{R}$-Cauchy sequence, for any arbitrary binary relation $\mathcal{R}$. $\mathcal{R}$-Cauchyness coincides with Cauchyness if $\mathcal{R}$ is taken to be the universal relation.*

Definition 12. *A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ which is endowed with a binary relation $\mathcal{R}$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-Cauchy sequence is convergent in $\mathcal{X}$.*

Remark 6. *Every complete fuzzy metric space is $\mathcal{R}$-complete fuzzy metric space, for any arbitrary binary relation $\mathcal{R}$. $\mathcal{R}$-completeness coincides with completeness if $\mathcal{R}$ is taken to be the universal relation.*

The present paper aims to introduce the concept of fuzzy $\mathcal{R} - \psi -$contractive mappings and prove some relevant results on the existence and uniqueness of fixed points for such mappings in the setting of non-Archimedean fuzzy metric spaces (in Kramosil and Michalek's sense as well as George and Veeramani's sense) which extended and generalized the results in [6,19]. We also provide some illustrative examples which support our work. In the last section, we apply our newly fixed point results to prove the existence and uniqueness of solutions for Caputo fractional differential equations.

2. Main Results

We start our main section with the following lemma which will be useful in the proof of our main results.

Lemma 2. *Let $f : \mathcal{X} \to \mathcal{X}$ and $\mathcal{R}$ a transitive binary relation which is $f-$closed. Assume that there exists $x_0 \in \mathcal{X}$ such that $x_0 \mathcal{R} f x_0$ and define $\{x_n\}$ in $\mathcal{X}$ by $x_n = f x_{n-1}$, for all $n \in \mathbf{N}_0$. Then*

$$x_m \mathcal{R} x_n \quad \text{for all } m, n \in \mathbf{N}_0 \text{ with } m < n. \tag{1}$$

Proof. As there exists $x_0 \in \mathcal{X}$ such that $x_0 \mathcal{R} f x_0$ and $x_n = f x_{n-1}$, then $x_0 \mathcal{R} x_1$. As $\mathcal{R}$ is $f-$closed and $x_0 \mathcal{R} x_1$, we deduce that $x_1 \mathcal{R} x_2$. By continuing this process, we find $x_n \mathcal{R} x_{n+1}$ for all $n \in \mathbf{N}_0$. Suppose that $m < n$, so $x_m \mathcal{R} x_{m+1}$ and $x_{m+1} \mathcal{R} x_{m+2}$ Due to the transitivity of $\mathcal{R}$, we find $x_m \mathcal{R} x_{m+2}$. Similarly, as $x_m \mathcal{R} x_{m+2}$ and $x_{m+2} \mathcal{R} x_{m+3}$, we find $x_m \mathcal{R} x_{m+3}$. By continuing this process, we obtain $x_m \mathcal{R} x_n$ for all $m, n \in \mathbf{N}_0$ with $m < n$. As required. □

Next, we introduce the notion of KM-fuzzy $\mathcal{R} - \psi-$contractive mapping as follows:

Definition 13. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a non-Archimedean fuzzy metric space (in the sense of Kramosil and Michalek), $\mathcal{R}$ a binary relation on $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{X}$. We say that f is a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping if there exists $\psi \in \Psi$ such that (for all $x, y \in \mathcal{X}$ and all $t > 0$ with $x\mathcal{R}y$)*

$$\mathcal{M}(x,y,t) > 0 \Rightarrow \min\{\mathcal{M}(x,y,t), \max\{\mathcal{M}(fx,x,t), \mathcal{M}(y,fy,t)\}\} \leq \psi(\mathcal{M}(fx,fy,t)). \tag{2}$$

The following is an example of a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping.

Example 3. *Let $\mathcal{X} = [0,\infty)$ and let $*$ be the product $t-$norm given by $t * s = ts\ \forall t, s \in [0,1]$. Define $\mathcal{M} : \mathcal{X}^2 \times [0,\infty) \to [0,1]$ for all $x, y \in \mathcal{X}$ by*

$$\mathcal{M}(x,y,t) = \begin{cases} 0, & \text{if } t = 0, \\ (\frac{t}{1+t})^{|x-y|}, & \text{if } t \neq 0. \end{cases}$$

Define $f : \mathcal{X} \to \mathcal{X}$, $\psi : [0,1] \to [0,1]$, and $\mathcal{R}$ on $\mathcal{X}$ by

$$fx = \begin{cases} e^{-x} & \text{if } x \in [0,2) \\ \frac{x+3}{2} & \text{if } x \in [2,5] \\ e^{-x} + 6 & \text{if } x \in (5,\infty) \end{cases}, \quad \psi(t) = t^3, \quad x\mathcal{R}y \Leftrightarrow x, y \in [2,5],\ x \leq y.$$

Then f is a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping as we will prove later on.

Now, we are equipped to state and prove our first main result as under.

Theorem 1. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a non-Archimedean fuzzy metric space (in the sense of Kramosil and Michalek) equipped with a binary relation $\mathcal{R}$ and $f : \mathcal{X} \to \mathcal{X}$. Assume that $\mathcal{X}$ is an $\mathcal{R}-$complete and f is a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping such that:*

(i) there exists x_0 in $\mathcal{X}$ such that $x_0 \mathcal{R} f x_0$ and $\mathcal{M}(x_0, f x_0, t) > 0$ for all $t > 0$;
(ii) $\mathcal{R}$ is transitive and $f-$closed;
(iii) one of the following holds:
  (a) f is continuous or
  (b) $\mathcal{R}$ is $\mathcal{M}-$self-closed.

Then f has a fixed point in $\mathcal{X}$.

Proof. From (i), there exists $x_0 \in \mathcal{X}$ such that $x_0 \mathcal{R} f x_0$ and $\mathcal{M}(x_0, fx_0, t) > 0$ for all $t > 0$. Define a sequence $\{x_n\}$ in $\mathcal{X}$ by $fx_n = x_{n+1}$, for all $n \in \mathbf{N}_0$. If $x_n = x_{n+1}$, for some $n \in \mathbf{N}_0$, then x_n is a fixed point of f. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbf{N}_0$.

As $\mathcal{M}(x_0, fx_0, t) = \mathcal{M}(x_0, x_1, t) > 0$ for all $t > 0$, and in view of Lemma 2 and (2), we obtain

$$\begin{aligned} & \min\{\mathcal{M}(x_0, x_1, t), \max\{\mathcal{M}(fx_0, x_0, t), \mathcal{M}(x_1, fx_1, t)\}\} \leq \psi(\mathcal{M}(fx_0, fx_1, t)) \\ \Rightarrow\ & \min\{\mathcal{M}(x_0, x_1, t), \max\{\mathcal{M}(x_1, x_0, t), \mathcal{M}(x_1, x_2, t)\}\} \leq \psi(\mathcal{M}(x_1, x_2, t)) \\ \Rightarrow\ & 0 < \mathcal{M}(x_0, x_1, t) \leq \psi(\mathcal{M}(x_1, x_2, t)) \leq \mathcal{M}(x_1, x_2, t). \end{aligned} \tag{3}$$

If there is some $t_0 > 0$ such that $\mathcal{M}(x_1, x_2, t_0) = 0$, then $\psi(\mathcal{M}(x_1, x_2, t_0)) = 0$. This implies that $\mathcal{M}(x_1, x_2, t_0) = 0$, (due to condition (C) of the definition of ψ) which contradicts (3). Therefore, $\mathcal{M}(x_1, x_2, t) > 0$ for all $t > 0$. Continuing with the same scenario, we deduce that for all $n \in \mathbf{N}_0$ and all $t > 0$,

$$0 < \mathcal{M}(x_{n-1}, x_n, t) \leq \psi(\mathcal{M}(x_n, x_{n+1}, t)) \leq \mathcal{M}(x_n, x_{n+1}, t) < 1,$$

for all $n \in \mathbf{N}_0$ and all $t > 0$, which implies that the sequence $\{\mathcal{M}(x_n, x_{n+1}, t)\}$ is non-decreasing sequence and bounded above. Hence, there exists $0 < \delta(t) \leq 1$ for all $t > 0$ such that $\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, t) = \delta(t)$.

Now, we show that $\delta(t) = 1$ for all $t > 0$. If there is $t_0 > 0$ such that $\delta(t_0) < 1$ then

$$0 < \mathcal{M}(x_{n-1}, x_n, t_0) \leq \psi(\mathcal{M}(x_n, x_{n+1}, t_0)) \leq \mathcal{M}(x_n, x_{n+1}, t_0) \leq \delta(t_0) < 1, \tag{4}$$

hence, $0 < \delta(t_0) < 1$. As ψ is left-continuous and $\{\mathcal{M}(x_n, x_{n+1}, t)\}$ is non-decreasing sequence of positive numbers, letting $n \to \infty$ in (4) we obtain $\psi(\delta(t_0)) = \delta(t_0)$, a contradiction $(\delta(t_0) \in (0,1))$. Therefore, $\delta(t) = 1$ for all $t > 0$. That is,

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, t) = 1. \tag{5}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, \mathcal{M}, *)$. If on the contrary, $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon \in (0,1)$ and some $t_0 > 0$ such that, for all $k \in \mathbf{N}_0$, there exist $m(k), n(k) \in \mathbf{N}_0$ such that $k \leq n(k) \leq m(k)$ satisfies

$$\begin{aligned} \mathcal{M}(x_{m(k)}, x_{n(k)}, t_0) &\leq 1 - \varepsilon, \\ \mathcal{M}(x_{m(k)-1}, x_{n(k)}, t_0) &> 1 - \varepsilon, \quad \text{for all } k \in \mathbf{N}_0. \end{aligned}$$

As $(\mathcal{X}, \mathcal{M}, *)$ is non-Archimedean, we have for all $k \in \mathbf{N}_0$,

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{M}(x_{m(k)}, x_{n(k)}, t_0) \\ &\geq \mathcal{M}(x_{m(k)}, x_{m(k)-1}, t_0) * \mathcal{M}(x_{m(k)-1}, x_{n(k)}, t_0) \\ &> \mathcal{M}(x_{m(k)}, x_{m(k)-1}, t_0) * (1 - \varepsilon). \end{aligned}$$

Letting $k \to \infty$, and using that $*$ is continuous, and (5) we can conclude that

$$\lim_{k\to\infty} \mathcal{M}(x_{m(k)}, x_{n(k)}, t_0) = 1 - \varepsilon. \tag{6}$$

Additionally, as $(\mathcal{X}, \mathcal{M}, *)$ is non-Archimedean, we have (for all $k \in \mathbf{N}_0$)

$$\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0) \geq \mathcal{M}(x_{m(k)-1}, x_{n(k)}, t_0) * \mathcal{M}(x_{n(k)}, x_{n(k)-1}, t_0) > (1 - \varepsilon) * \mathcal{M}(x_{n(k)}, x_{n(k)-1}, t_0),$$

and

$$\mathcal{M}(x_{m(k)}, x_{n(k)}, t_0) \geq \mathcal{M}(x_{m(k)}, x_{m(k)-1}, t_0) * \mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0) * \mathcal{M}(x_{n(k)-1}, x_{n(k)}, t_0).$$

Taking $k \to \infty$, in the above inequalities and using (5), (6), we find

$$\lim_{k\to\infty} \mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1 - \varepsilon. \tag{7}$$

That is, $\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0) > 0$ whenever k is large enough. Now, using (2) and Lemma 2, we have, (for all k)

$$\begin{aligned}&\min\{\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0), \max\{\mathcal{M}(fx_{m(k)-1}, x_{m(k)-1}, t_0), \mathcal{M}(x_{n(k)-1}, fx_{n(k)-1}, t_0)\}\} \\ &\leq \psi(\mathcal{M}(fx_{m(k)-1}, fx_{n(k)-1}, t_0)).\end{aligned}$$

Hence,

$$\begin{aligned}&\min\{\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0), \max\{\mathcal{M}(x_{m(k)}, x_{m(k)-1}, t_0), \mathcal{M}(x_{n(k)-1}, x_{n(k)}, t_0)\}\} \\ &\leq \psi(\mathcal{M}(x_{m(k)}, x_{n(k)}, t_0)).\end{aligned}$$

Letting $k \to \infty$, and using (5)–(7) and the fact that ψ is left-continuous we deduce that

$$1 - \varepsilon < \min\{1 - \varepsilon, \max\{1, 1\}\} \leq \psi(1 - \varepsilon) \Rightarrow 1 - \varepsilon \leq \psi(1 - \varepsilon) < 1 - \varepsilon,$$

a contradiction. Hence, $\{x_n\}$ must be a Cauchy sequence in $(\mathcal{X}, \mathcal{M}, *)$. Now, we have $\{x_n\}$, an $\mathcal{R}-$Cauchy sequence, and $(\mathcal{X}, \mathcal{M}, *)$, an $\mathcal{R}-$complete, so there exists $x \in \mathcal{X}$ such that $x_n \to x$.

Now, if f is continuous, then taking the limit as $n \to \infty$ on the both sides of $x_{n+1} = fx_n$, $n \in \mathbf{N}_0$, we obtain $x = fx$.

Otherwise, if $\mathcal{R}$ is $M-$self-closed, then there exists a subsequence $\{x_{n(k)}\} \subseteq \{x_n\}$ such that $x_{n(k)}\mathcal{R}x$ for all $k \in \mathbf{N}_0$. We claim that $x = f(x)$. As $\lim_{k\to\infty} x_{n(k)} = x$ we have $\lim_{k\to\infty} \mathcal{M}(x_{n(k)}, x, t) = 1$ for all $t > 0$. Then $\mathcal{M}(x_{n(k)}, x, t) > 0$ when k is large enough for all $t > 0$ and as $x_{n(k)}\mathcal{R}x$, from condition (2) we find

$$\begin{aligned}&\min\{\mathcal{M}(x_{n(k)}, x, t), \max\{\mathcal{M}(fx_{n(k)}, x_{n(k)}, t), \mathcal{M}(x, fx, t)\}\} \\ \leq\ &\psi(\mathcal{M}(fx_{n(k)}, fx, t)).\end{aligned}$$

Thus,

$$\begin{aligned}&\min\{\mathcal{M}(x_{n(k)}, x, t), \max\{\mathcal{M}(x_{n(k)+1}, x_{n(k)}, t), \mathcal{M}(x, fx, t)\}\} \\ \leq\ &\psi(\mathcal{M}(x_{n(k)+1}, fx, t)).\end{aligned}$$

Letting $k \to \infty$, and using (5), $\lim_{k\to\infty} \mathcal{M}(x_{n(k)}, x, t) = 1$, we find

$$1 = \min\{1, \max\{1, \mathcal{M}(x, fx, t)\}\} \leq \lim_{k\to\infty} \psi(\mathcal{M}(x_{n(k)+1}, fx, t)).$$

This means that

$$\lim_{k\to\infty} \psi(\mathcal{M}(x_{n(k)+1}, fx, t)) = 1.$$

Hence, from Remark 4 (iii) and the continuity of $\mathcal{M}$, we obtain

$$\lim_{k\to\infty} \mathcal{M}(x_{n(k)+1}, fx, t) = 1.$$

Thus, $\lim_{k\to\infty} x_{n(k)+1} = fx$. The uniqueness of the limit gives that $fx = x$. This ends the proof. □

Next, we provide the following uniqueness theorem.

Theorem 2. *In addition to the hypotheses of Theorem 1, if the following condition holds:*

(iv) *for all $x, y \in Fix(f)$, there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$ and $y\mathcal{R}z$, $\mathcal{M}(x,z,t) > 0$ and $\mathcal{M}(y,z,t) > 0$ for all $t > 0$.*

Then the fixed point of f is unique.

Proof. In view of Theorem 1, $Fix(f) \neq \phi$. Let $x, y \in Fix(f)$, by condition (iv) there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$, $y\mathcal{R}z$, $\mathcal{M}(x,z,t) > 0$ and $\mathcal{M}(y,z,t) > 0$ for all $t > 0$. Define $z_0 = z$ and $z_{n+1} = fz_n$ for all $n \geq 0$. We claim that $x = y$. As $x\mathcal{R}z_0$, $\mathcal{M}(x,z_0,t) > 0$ for all $t > 0$, then from (2) we have

$$\begin{aligned} & \min\{\mathcal{M}(x,z_0,t), \max\{\mathcal{M}(fx,x,t), \mathcal{M}(z_0,fz_0,t)\}\} \leq \psi(\mathcal{M}(fx,fz_0,t)) \\ \Rightarrow \quad & \min\{\mathcal{M}(x,z_0,t), \max\{\mathcal{M}(x,x,t), \mathcal{M}(z_0,z_1,t)\}\} \leq \psi(\mathcal{M}(x,z_1,t)) \\ \Rightarrow \quad & \min\{\mathcal{M}(x,z_0,t), \max\{1, \mathcal{M}(z_0,z_1,t)\}\} \leq \psi(\mathcal{M}(x,z_1,t)) \\ \Rightarrow \quad & \min\{\mathcal{M}(x,z_0,t), 1\} \leq \psi(\mathcal{M}(x,z_1,t)) \\ \Rightarrow \quad & 0 < \mathcal{M}(x,z_0,t) \leq \psi(\mathcal{M}(x,z_1,t)) \leq \mathcal{M}(x,z_1,t). \end{aligned}$$

By induction, we find $\mathcal{M}(x,z_n,t) > 0$ for all $n \in \mathbf{N}_0$ and $t > 0$ and as $\mathcal{R}$ is f-closed, we conclude that (by induction), $x\mathcal{R}z_n$ for all $n \in \mathbf{N}_0$. Hence

$$\begin{aligned} & \min\{\mathcal{M}(x,z_n,t), \max\{\mathcal{M}(fx,x,t), \mathcal{M}(z_n,fz_n,t)\}\} \leq \psi(\mathcal{M}(fx,fz_n,t)) \\ & \Rightarrow \min\{\mathcal{M}(x,z_n,t), \max\{\mathcal{M}(x,x,t), \mathcal{M}(z_n,z_{n+1},t)\}\} \leq \psi(\mathcal{M}(x,z_{n+1},t)) \\ & \Rightarrow 0 < \mathcal{M}(x,z_n,t) \leq \psi(\mathcal{M}(x,z_{n+1},t)) \leq \mathcal{M}(x,z_{n+1},t). \end{aligned} \tag{8}$$

Thus, $\{\mathcal{M}(x,z_n,t)\}$ is non-decreasing and bounded above. Hence, there exists $0 < \gamma(t) \leq 1$ for all $t > 0$ such that $\lim_{n\to\infty} \mathcal{M}(x,z_n,t) = \gamma(t)$. Letting $n \to \infty$ in (8), and as ψ is left-continuous, we find $\psi(\gamma(t)) = \gamma(t)$. Therefore, in view of Remark 4, we deduce that $\gamma(t) = 1$ for all $t > 0$. Thus,

$$\lim_{n\to\infty} \mathcal{M}(x,z_n,t) = 1 \;\text{ for all }\; t > 0.$$

Similarly, we can show that

$$\lim_{n\to\infty} \mathcal{M}(y,z_n,t) = 1 \;\text{ for all }\; t > 0.$$

As $(\mathcal{X}, \mathcal{M}, *)$ is non-Archimedean, we find (for all $n \in \mathbf{N}_0$)

$$\mathcal{M}(x,y,t) \geq \mathcal{M}(x,z_n,t) * \mathcal{M}(z_n,y,t).$$

Letting $n \to \infty$, and using the continuity of $*$, we can conclude that

$$\mathcal{M}(x,y,t) \geq 1 * 1 = 1 \Longrightarrow \mathcal{M}(x,y,t) = 1.$$

Hence, $x = y$. As required. □

Now, we present the following example which exhibits the utility of Theorems 1 and 2.

Example 4. *Consider the mapping f given in Example 3. We are going to show that all the hypotheses of Theorem 1 are satisfied.*

Proof. It is obvious that $(\mathcal{X}, \mathcal{M}, *)$ is $\mathcal{R}-$complete non- Archimedean fuzzy metric space (see [25], Example 1.3).

Note that

- $\mathcal{R}$ is transitive on $[2,5]$;
- $2 \in [2,5]$, $f(2) = 2.5 \in [2,5]$ and $2 \leq f(2)$ hence $2\mathcal{R}f(2)$;

- for all $x, y \in [2,5]$ where $x \leq y$, we see that $\frac{x+3}{2}, \frac{y+3}{2} \in [2.5,4] \subset [2,5]$ and $\frac{x+3}{2} \leq \frac{y+3}{2}$, so when $x\mathcal{R}y$ we have $f(x)\mathcal{R}f(y)$, that means $\mathcal{R}$ is $f-$closed;
- if $\{x_n\} \subseteq \mathcal{X}$ is $\mathcal{R}$-preserving sequence, that is $x_n\mathcal{R}x_{n+1}$ then $x_n \leq x_{n+1}$, $x_n, x_{n+1} \in [2,5]$ for all $n \geq n_0$. Hence, $\{x_n\}$ is non-decreasing sequence and bounded above, that is
$$\lim_{n\to\infty} x_n = \sup_{n \geq n_0} x_n = x \in [2,5].$$
Therefore, $x_n \leq x$, and $x_n, x \in [2,5]$ for all $n \geq n_0$. Thus, $x_n\mathcal{R}x$ and $\mathcal{R}$ is $\mathcal{M}-$self-closed.

Now, we show that f is a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping. For all $x, y \in \mathcal{X}$ we have
$$\psi(\mathcal{M}(fx, fy, t)) = (\frac{t}{t+1})^{3|fx-fy|} = (\frac{t}{t+1})^{\frac{3}{2}|x-y|}.$$

Hence, if $x\mathcal{R}y$ and $\mathcal{M}(x, y, t) > 0$ we find
$$\min\left\{(\frac{t}{t+1})^{|x-y|}, \max\left\{(\frac{t}{t+1})^{|fx-x|}, (\frac{t}{t+1})^{|fy-y|}\right\}\right\} \leq (\frac{t}{t+1})^{|x-y|} \leq (\frac{t}{t+1})^{\frac{3}{2}|x-y|}.$$

Therefore,
$$\min\{\mathcal{M}(x,y,t), \max\{\mathcal{M}(fx,x,t), \mathcal{M}(y,fy,t)\}\} \leq \psi(\mathcal{M}(fx,fy,t)) \quad \forall x,y \in \mathcal{X}.$$

Thus, f is a KM-fuzzy $\mathcal{R} - \psi-$contractive mapping. Then all the hypotheses of of Theorem 1 are satisfied and 3 is a fixed point of f. Observe that Theorem 2 is also satisfied on $[2,5]$, and 3 is the unique fixed point of f. $\square$

If we put $\psi(t) = kt$, where $k \in (0,1)$ in Theorems 1 and 2 we have the following corollary.

Corollary 1. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an $\mathcal{R}-$complete non-Archimedean fuzzy metric space (in the sense of Kramosil and Michalek) with a binary relation $\mathcal{R}$ and $f : \mathcal{X} \to \mathcal{X}$ be mapping such that there exists $k \in (0,1)$ and for all $x, y \in \mathcal{X}$, all $t > 0$ with $x\mathcal{R}y$,*
$$\mathcal{M}(x,y,t) > 0 \Rightarrow \min\{\mathcal{M}(x,y,t), \max\{\mathcal{M}(fx,x,t), \mathcal{M}(y,fy,t)\}\} \leq k\mathcal{M}(fx,fy,t).$$

Additionally,

(i) there exists x_0 in $\mathcal{X}$ such that $x_0\mathcal{R}fx_0$ and $\mathcal{M}(x_0, fx_0, t) > 0$ for all $t > 0$;
(ii) $\mathcal{R}$ is transitive and $f-$closed;
(iii) one of the following holds:
 (a) f is continuous or
 (b) $\mathcal{R}$ is $\mathcal{M}-$self-closed.

Then f has a fixed point in $\mathcal{X}$. In addition, if the following condition holds

(iv) for all $x, y \in Fix(f)$, there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$, $y\mathcal{R}z$, $\mathcal{M}(x,z,t) > 0$ and $\mathcal{M}(y,z,t) > 0$ for all $t > 0$.

Then the fixed point is unique.

In the rest of this, we show that Theorems 1 and 2 can be achieved in the setting of $\mathcal{R}-$complete non-Archimedean fuzzy metric spaces (in the sense of George and Veeramani).

Now, we define GV-fuzzy $\mathcal{R} - \psi-$contractive as under.

Definition 14. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a non-Archimedean fuzzy metric space (in the sense of George and Veeramani), $\mathcal{R}$ a binary relation and $f : \mathcal{X} \to \mathcal{X}$ a mapping. We say that f is a GV-fuzzy $\mathcal{R} - \psi-$contractive mapping if there exists $\psi \in \Psi$ such that, for all $x, y \in \mathcal{X}$ with $x\mathcal{R}y$,*

$$\min\{\mathcal{M}(x, y, t), \mathcal{M}(fx, x, t), \mathcal{M}(y, fy, t)\} \leq \psi(\mathcal{M}(fx, fy, t)). \tag{9}$$

Next, we provide the following Theorems in the sense of George and Veeramani fuzzy metric space.

Theorem 3. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a non-Archimedean fuzzy metric space (in the sense of George and Veeramani) with a binary relation $\mathcal{R}$ and $f : \mathcal{X} \to \mathcal{X}$. Assume that $\mathcal{X}$ is an $\mathcal{R}-$complete and f is a GV-fuzzy $\mathcal{R} - \psi-$contractive mapping such that:*

(i) *there exists x_0 in $\mathcal{X}$ with $x_0\mathcal{R}fx_0$;*
(ii) *$\mathcal{R}$ is transitive and $f-$closed;*
(iii) *one of the following holds:*
    (a) *f is continuous or*
    (b) *$\mathcal{R}$ is $\mathcal{M}-$self-closed.*

Then f has a fixed point in $\mathcal{X}$.

Proof. From (i) there exists $x_0 \in \mathcal{X}$ such that $x_0\mathcal{R}fx_0$. Define a sequence $\{x_n\}$ in $\mathcal{X}$ by $fx_n = x_{n+1}$, for all $n \in \mathbf{N}_0$. If $x_n = x_{n+1}$, for some $n \in \mathbf{N}_0$, then x_n is a fixed point of f. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbf{N}_0$. As $x_0\mathcal{R}x_1$ and in view of (9), we obtain

$$\begin{aligned} & \min\{\mathcal{M}(x_0, x_1, t), \mathcal{M}(fx_0, x_0, t), \mathcal{M}(x_1, fx_1, t)\} \leq \psi(\mathcal{M}(fx_0, fx_1, t)) \\ \Rightarrow\ & \min\{\mathcal{M}(x_0, x_1, t), \mathcal{M}(x_1, x_0, t), \mathcal{M}(x_1, x_2, t)\} \leq \psi(\mathcal{M}(x_1, x_2, t)). \end{aligned} \tag{10}$$

$$If \min\{\mathcal{M}(x_0, x_1, t), \mathcal{M}(x_1, x_2, t)\} = \mathcal{M}(x_1, x_2, t) \Rightarrow \psi(\mathcal{M}(x_1, x_2, t)) = \mathcal{M}(x_1, x_2, t),$$

by Definition 1 we find $\mathcal{M}(x_1, x_2, t) = 1$, which is a contradiction. Hence,

$$0 < \mathcal{M}(x_0, x_1, t) \leq \psi(\mathcal{M}(x_1, x_2, t)) \leq \mathcal{M}(x_1, x_2, t).$$

Continuing this process, we deduce that

$$0 < \mathcal{M}(x_{n-1}, x_n, t) \leq \psi(\mathcal{M}(x_n, x_{n+1}, t)) \leq \mathcal{M}(x_n, x_{n+1}, t),$$

for all $n \in \mathbf{N}_0$. As the proof of Theorem 1 we have

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, t) = 1. \tag{11}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, \mathcal{M}, *)$. If, on the contrary, $\{x_n\}$ is not a Cauchy sequence, then as the proof of Theorem 1 we find

$$\lim_{k\to\infty} \mathcal{M}(x_{m(k)}, x_{n(k)}, t_0) = 1 - \varepsilon, \tag{12}$$

$$\lim_{k\to\infty} \mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1 - \varepsilon. \tag{13}$$

Now, using the contractive condition (9) and Lemma 2, we have for all k,

$$\begin{aligned} & \min\{\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0), \mathcal{M}(fx_{m(k)-1}, x_{m(k)-1}, t_0), \mathcal{M}(x_{n(k)-1}, fx_{n(k)-1}, t_0)\} \\ & \leq \psi(\mathcal{M}(fx_{m(k)-1}, fx_{n(k)-1}, t_0)). \end{aligned}$$

Hence,

$$\min\{\mathcal{M}(x_{m(k)-1}, x_{n(k)-1}, t_0), \mathcal{M}(x_{m(k)}, x_{m(k)-1}, t_0), \mathcal{M}(x_{n(k)-1}, x_{n(k)}, t_0)\}$$
$$\leq \psi(\mathcal{M}(x_{m(k)}, x_{n(k)}, t_0)).$$

Letting $k \to \infty$, and using (11)–(13) and the left-continuity of ψ, we find that

$$1-\varepsilon < \min\{1-\varepsilon, 1, 1\} \leq \psi(1-\varepsilon) \Rightarrow 1-\varepsilon \leq \psi(1-\varepsilon) < 1-\varepsilon,$$

a contradiction. Hence, $\{x_n\}$ must be a Cauchy sequence in $(\mathcal{X}, \mathcal{M}, *)$. As $(\mathcal{X}, \mathcal{M}, *)$ is $\mathcal{R}-$complete, there exists $x \in \mathcal{X}$ such that $x_n \to x$. From condition (a), if f is continuous, as the proof of Theorem 1 we have $x = fx$.

From condition (b) if $\mathcal{R}$ is $M-$self-closed, then there exists a subsequence $\{x_{n(k)}\} \subseteq \{x_n\}$ such that $\lim_{k\to\infty} x_{n(k)} = x$ and $x_{n(k)}\mathcal{R}x$ for all $k \in \mathbf{N}_0$. Suppose that $x \neq f(x)$, and from condition (9) we find

$$\min\{\mathcal{M}(x_{n(k)}, x, t), \mathcal{M}(fx_{n(k)}, x_{n(k)}, t), \mathcal{M}(x, fx, t)\} \leq \psi(\mathcal{M}(fx_{n(k)}, fx, t)).$$

Thus,

$$\min\{\mathcal{M}(x_{n(k)}, x, t), \mathcal{M}(x_{n(k)+1}, x_{n(k)}, t), \mathcal{M}(x, fx, t)\} \leq \psi(\mathcal{M}(x_{n(k)+1}, fx, t)).$$

Letting $k \to \infty$, and using (11), $\lim_{k\to\infty} \mathcal{M}(x_{n(k)}, x, t) = 1$, we find

$$\mathcal{M}(x, fx, t) = \min\{1, 1, \mathcal{M}(x, fx, t)\} \leq \lim_{k\to\infty} \psi(\mathcal{M}(x_{n(k)+1}, fx, t)).$$

As ψ is left-continuous and $\mathcal{M}$ is continuous, we have

$$\mathcal{M}(x, fx, t) \leq \psi(\mathcal{M}(x, fx, t)) < \mathcal{M}(x, fx, t).$$

Hence, from Remark 4 (iii), we find $\mathcal{M}(x, fx, t) = 1$. As required. That is $fx = x$. □

Next, we provide the following uniqueness theorem.

Theorem 4. *In addition to the hypotheses of Theorem 3, if the following condition holds:*

(iv) for all $x, y \in Fix(f)$, there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$, $y\mathcal{R}z$ and $z\mathcal{R}fz$.

Then the fixed point of f is unique.

Proof. In view of Theorem 3, $Fix(f) \neq \phi$. Let $x, y \in Fix(f)$, by condition (iv) there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$, $y\mathcal{R}z$. Define $z_{n+1} = fz_n$ for all $n \geq 0$ and $z_0 = z$. As $z\mathcal{R}fz$ then as the proof of Theorem 3 we have

$$\lim_{n\to\infty} \mathcal{M}(z_n, z_{n+1}, t) = 1. \tag{14}$$

We claim that $x = y$. As $x\mathcal{R}z_0$, and $\mathcal{R}$ is f-closed, we find by induction $x\mathcal{R}z_n$ for all $n \in \mathbf{N}_0$. then from (9) we have

$$\begin{aligned} & \min\{\mathcal{M}(x, z_n, t), \mathcal{M}(fx, x, t), \mathcal{M}(z_n, fz_n, t)\} \leq \psi(\mathcal{M}(fx, fz_n, t)) \\ \Rightarrow\quad & \min\{\mathcal{M}(x, z_n, t), \mathcal{M}(x, x, t), \mathcal{M}(z_n, z_{n+1}, t)\} \leq \psi(\mathcal{M}(x, z_{n+1}, t)) \\ \Rightarrow\quad & \min\{\mathcal{M}(x, z_n, t), \mathcal{M}(z_n, z_{n+1}, t)\} \leq \psi(\mathcal{M}(x, z_{n+1}, t)). \end{aligned}$$

Case I: if $\min\{\mathcal{M}(x, z_n, t), \mathcal{M}(z_n, z_{n+1}, t)\} = \mathcal{M}(x, z_n, t)$ for all $n \geq n_0$ we have

$$\mathcal{M}(x, z_n, t) \leq \psi(\mathcal{M}(x, z_{n+1}, t)) \leq \mathcal{M}(x, z_{n+1}, t).$$

Thus, $\{\mathcal{M}(x, z_n, t)\}$ is non-decreasing and bounded above. So, as in Theorem 2

$$\lim_{n\to\infty} \mathcal{M}(x, z_n, t) = 1 \Rightarrow \lim_{n\to\infty} z_n = x.$$

Case II: if $\min\{\mathcal{M}(x, z_n, t), \mathcal{M}(z_n, z_{n+1}, t)\} = \mathcal{M}(z_n, z_{n+1}, t)$ for all $n \geq n_0$ we have

$$\mathcal{M}(z_n, z_{n+1}, t) \leq \psi(\mathcal{M}(x, z_{n+1}, t)).$$

By taking $n \to \infty$ and using (14) we find

$$\begin{aligned} & 1 \leq \lim_{n\to\infty} \psi(\mathcal{M}(x, z_{n+1}, t)) \\ \Rightarrow\ & 1 = \lim_{n\to\infty} \psi(\mathcal{M}(x, z_{n+1}, t)) \\ \Rightarrow\ & 1 = \lim_{n\to\infty} \mathcal{M}(x, z_{n+1}, t) \\ \Rightarrow\ & \lim_{n\to\infty} z_{n+1} = x. \end{aligned}$$

Therefore, from two cases we conclude that

$$\lim_{n\to\infty} z_n = x. \tag{15}$$

Similarly, we can show that

$$\lim_{n\to\infty} z_n = y. \tag{16}$$

As $(\mathcal{X}, \mathcal{M}, *)$ is Hausdorff then from (15) and (16), we obtain $x = y$. This ends the proof. □

If we put $\psi(t) = kt$ where $k \in (0,1)$ in Theorems 3 and 4 we have the following corollary.

Corollary 2. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an $\mathcal{R}-$complete non-Archimedean fuzzy metric space (in the sense of George and Veeramani) with a binary relation $\mathcal{R}$ and $f : \mathcal{X} \to \mathcal{X}$ be mapping such that there exists $k \in (0,1)$ and for all $x, y \in \mathcal{X}$, with $x\mathcal{R}y$,*

$$\min\{\mathcal{M}(x, y, t), \mathcal{M}(fx, x, t), \mathcal{M}(y, fy, t)\} \leq k\mathcal{M}(fx, fy, t).$$

Furthermore,

(i) there exists x_0 in $\mathcal{X}$ such that $x_0\mathcal{R}fx_0$;
(ii) $\mathcal{R}$ is transitive and $f-$closed;
(iii) one of the following holds:
  (a) f is continuous or
  (b) $\mathcal{R}$ is $\mathcal{M}-$self-closed.

Then f has a fixed point in $\mathcal{X}$. In addition if the following condition holds

(iv) for all $x, y \in Fix(f)$, there exists $z \in \mathcal{X}$ such that $x\mathcal{R}z$, $y\mathcal{R}z$, and $z\mathcal{R}fz$.

Then the fixed point is unique.

3. Application to Nonlinear Fractional Differential Equations

In this section, we apply our main results to study the existence of a solution of boundary value problems for fractional differential equations involving the Caputo fractional derivative.

Let $\mathcal{X} = C([0,1], \mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|.$$

Define $\mathcal{M} : \mathcal{X}^2 \times (0,\infty) \to [0,1]$ for all $x,y \in \mathcal{X}$, by

$$\mathcal{M}(x,y,\tau) = e^{\frac{-\|x-y\|_\infty}{\tau}}, \qquad \forall \tau \in (0,\infty).$$

It is well known that $(\mathcal{X},\mathcal{M},*)$ is a complete non-Archimedean fuzzy metric space with $a * b = a \cdot b$, $\forall\, a,b \in [0,1]$ (see [17,25]). Define a binary relation $\mathcal{R}$ on $\mathcal{X}$ by

$$x\mathcal{R}y \Leftrightarrow x(t) \le y(t) \text{ for all } x,y \in \mathcal{X}, t \in [0,1].$$

As $(\mathcal{X},\mathcal{M},*)$ is a complete non-Archimedean fuzzy metric space with $a * b = a \cdot b$, $\forall\, a,b \in [0,1]$, then $(\mathcal{X},\mathcal{M},*)$ is an $\mathcal{R}-$complete non-Archimedean fuzzy metric space with $a * b = a \cdot b$, $\forall\, a,b \in [0,1]$. In addition, it is easy to see that $\mathcal{R}$ is transitive.

Now, let us recall the following basic notions which will be needed subsequently.

Definition 15 ([26])**.** *For a function u given on the interval $[a,b]$ the Caputo fractional derivative of function u order $\beta > 0$ is defined by*

$$({}^c\mathcal{D}^{\beta}_{a+})u(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t (t-s)^{n-\beta-1}u^{(n)}(s)ds,\ (n-1 \le \beta < n,\ n = [\beta]+1), \tag{17}$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is a gamma function.

Consider the boundary value problem for fractional order differential equation given by:

$$\begin{cases} {}^c\mathcal{D}^{\beta}_{0+}(x(t)) = h(t,x(t)), & (t \in [0,1],\ 2 < \beta \le 3); \\ x(0) = c_0,\ x'(0) = c_0^*,\ x''(1) = c_1, & \end{cases} \tag{18}$$

where ${}^c\mathcal{D}^{\beta}_{0+}$ denotes the Caputo fractional derivative of order β, $h : [0,1] \to \mathbb{R}$ is a continuous function and c_0, c_0^*, c_1 are real constants.

Definition 16 ([27])**.** *A function $x \in C^3([0,1],\mathbb{R})$, with its β-derivative existing on $[0,1]$ is said to be a solution of (18) if x satisfies the equation ${}^c\mathcal{D}^{\beta}_{0+}(x(t)) = h(t,x(t))$ on $[0,1]$ and the conditions $x(0) = c_0$, $x'(0) = c_0^*$, $x''(1) = c_1$,*

The following lemma is required in what follows.

Lemma 3 ([27])**.** *Let $2 < \beta \le 3$ and let $u : [0,1] \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds - \frac{t^2}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}u(s)ds + c_0 + c_0^*t + \frac{c_1}{2}t^2$$

if and only if x is a solution of the fractional boundary value problems

$$\begin{aligned} &{}^c\mathcal{D}^{\beta}_{0+}(x(t)) = u(t), \\ &x(0) = c_0,\ x'(0) = c_0^*,\ x''(1) = c_1, \end{aligned}$$

where

$$x''(1) = 2c_2 + \frac{1}{\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}u(s)ds = c_1,\ \ c_i, c_0^* \in \mathbb{R},\ i = 0,1,2.$$

Now, we state and prove our main result in this section.

Theorem 5. *Suppose that*

(i) *for all* $x, y \in \mathcal{X}$, $x \leq y$, $t \in [0,1]$ *there exists* $\lambda > 0$ *such that*

$$|h(t,x(t)) - h(t,y(t))| \leq \lambda |x(t) - y(t)|, \text{ where}$$
$$0 < \frac{1}{k} = \lambda\Big(\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\Big) < 1; \tag{19}$$

(ii) *there exists* $x_0 \in \mathcal{X}$ *such that*

$$\begin{aligned} x_0(t) \quad \leq \quad & \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} h(s, x_0(s)) ds \\ & - \frac{t^2}{2\Gamma(\beta-2)} \int\limits_0^1 (1-s)^{\beta-3} h(s, x_0(s)) ds + c_0 + c_0^* t + \frac{c_1}{2} t^2; \end{aligned}$$

(iii) *h is nondecreasing in the second variable;*
Then, the Equation (18) has a unique solution in $\mathcal{X}$.

Proof. Define $H : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} Hx(t) \quad = \quad & \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} h(s, x(s)) ds \\ & - \frac{t^2}{2\Gamma(\beta-2)} \int\limits_0^1 (1-s)^{\beta-3} h(s, x(s)) ds + c_0 + c_0^* t + \frac{c_1}{2} t^2. \end{aligned}$$

where

$$c_1 = 2c_2 + \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\beta-3} h(s, x(s)) ds, \; c_i, c_0^* \in \mathbb{R}, \; (i = 0,1,2) \text{ are constant.}$$

First, we show that H is continuous. Let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = x$ in $\mathcal{X}$. Then for each $t \in [0,1]$

$$\begin{aligned} |Hx_n(t) - Hx(t)| \leq & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |h(s, x_n(s)) - h(s, x(s))| ds \\ & + \frac{1}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} |h(s, x_n(s)) - h(s, x(s))| ds. \end{aligned}$$

As h is a continuous function, we have

$$\begin{aligned} & \lim_{n\to\infty} \|h(s, x_n(s)) - h(s, x(s))\|_\infty = 0 \\ \Leftrightarrow \; & \lim_{n\to\infty} \|Hx_n - Hx\|_\infty = 0 \\ \Leftrightarrow \; & \lim_{n\to\infty} e^{\frac{-\|Hx_n - Hx\|_\infty}{\tau}} = 1 \\ \Leftrightarrow \; & \lim_{n\to\infty} \mathcal{M}(Hx_n, Hx, \tau) = 1 \\ \Leftrightarrow \; & \lim_{n\to\infty} Hx_n = Hx. \end{aligned}$$

Hence, H is continuous.

Clearly, the fixed points of the operator H are solutions of the Equation (18). We will use Theorem 3 to prove that H has a fixed point.

Therefore, we show that H is a GV-fuzzy $\mathcal{R}-\psi-$contractive mapping. Let $x,y\in\mathcal{X}$, $x\mathcal{R}y$ so $x(t)\le y(t)$, for all $t\in[0,1]$. Observe that

$$\begin{aligned}
\big|Hx(t)-Hy(t)\big| &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|h(s,x(s))-h(s,y(s))|ds \\
&\quad+\frac{t^2}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}|h(s,x(s))-h(s,y(s))|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|h(s,x(s))-h(s,y(s))|ds \\
&\quad+\frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}|h(s,x(s))-h(s,y(s))|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda|x(s)-y(s)|ds \\
&\quad+\frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\lambda|x(s)-y(s)|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda\big\|x-y\big\|_\infty ds+\frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\lambda\big\|x-y\big\|_\infty ds \\
&\le \frac{\lambda\big\|x-y\big\|_\infty}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds+\frac{\lambda\big\|x-y\big\|_\infty}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}ds \\
&\le \lambda\Big(\frac{1}{\Gamma(\beta+1)}+\frac{1}{2\Gamma(\beta-1)}\Big)\big\|x-y\big\|_\infty \\
&= \frac{1}{k}\big\|x-y\big\|_\infty.
\end{aligned}$$

Hence,

$$k\big\|Hx-Hy\big\|_\infty\le\big\|x-y\big\|_\infty.$$

This gives

$$e^{-\frac{k\|Hx-Hy\|_\infty}{\tau}}\ge e^{-\frac{\|x-y\|_\infty}{\tau}}.$$

Therefore,

$$\psi(\mathcal{M}(Hx,Hy,\tau))\ge\mathcal{M}(x,y,\tau)\ge\min\{\mathcal{M}(x,y,\tau),\mathcal{M}(Hx,x,\tau),\mathcal{M}(y,Hy,\tau)\},$$

with $\psi(t)=t^k$ and $k>1$. This shows that H is a GV-fuzzy $\mathcal{R}-\psi-$contractive mapping. From (ii), we conclude that $x_0(t)\mathcal{R}Hx_0(t)$, for all $t\in[0,1]$, then $x_0\mathcal{R}Hx_0$ that is, the condition (i) of Theorem 3 is satisfied. Let $x,y\in\mathcal{X}$, $x(t)\le y(t)$ for all $t\in[0,1]$, from (iii), as h is nondecreasing in the second variable, we have

$$\begin{aligned} Hx(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s,x(s))ds + c_0 + c_0^* t + c_2 t^2 \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s,y(s))ds + c_0 + c_0^* t + c_2 t^2 \\ &= Hy(t). \end{aligned}$$

we conclude that $Hx(t) \leq Hy(t)$ for all $t \in [0,1]$, then $Hx \leq Hy$ (i.e., $x\mathcal{R}y \Rightarrow Hx\mathcal{R}Hy$) that is, $\mathcal{R}$ is H-closed and the condition (iii) of Theorem 3 satisfies. Therefore, all the hypotheses of Theorem 3 are satisfied. Hence, H has a fixed point which is a solution for the Equation (18) in $\mathcal{X}$. Finally, observe that if $x, y \in \mathcal{X}$ are two fixed points of H in $\mathcal{X}$, then $x \leq \max\{x,y\}$, $y \leq \max\{x,y\}$, and $z = \max\{x,y\} \in \mathcal{X}$. Additionally, $\mathcal{M}(x,z,t) > 0$ and $\mathcal{M}(x,y,t) > 0$ for all $t > 0$ (due to Definition 4). Therefore, Theorem 4 is also satisfied. Hence, the fixed point of H is unique and thus the solution of (18) is also unique in $\mathcal{X}$. This ends the proof. □

Finally, we provide the following example which supports Theorem 5.

Example 5. *Consider the boundary value problem of fractional differential equation*

$$\begin{aligned} D_{0^+}^{\frac{5}{2}} x(t) = \frac{x(t)}{5(1+x(t))}, \quad t \in [0,1], \\ x(0) = 0, x'(0) = 0, x''(1) = 1. \end{aligned} \tag{20}$$

Take

$$f(t,x(t)) = \frac{x(t)}{5(1+x(t))}, \quad (t,x(t)) \in [0,1] \times [0,\infty)$$

Let $x(t), y(t) \in [0,\infty)$ and $t \in [0,1]$. Then

$$\begin{aligned} |f(t,x(t)) - f(t,y(t))| &= \frac{1}{5} \left| \frac{x(t)}{1+x(t)} - \frac{y(t)}{1+y(t)} \right| \\ &= \frac{1}{5} \left| \frac{x(t)-y(t)}{(1+x(t))(1+y(t))} \right| \\ &\leq \frac{1}{5} |x(t) - y(t)|. \end{aligned}$$

Hence, condition (i) of Theorem 5 is satisfied with $\lambda = \frac{1}{5}$. Now, we check that $\lambda\left[\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right] < 1$.

$$\frac{1}{5}\left[\frac{1}{\Gamma(\frac{7}{2})} + \frac{1}{2\Gamma(\frac{3}{2})}\right] = \frac{23}{15\sqrt{\pi}} < 1.$$

Hence, (19) holds. Taking $x_0 = 0$ then,

$$0 \leq \frac{1}{\Gamma(\frac{5}{2})} \int_0^t (t-s)^{\frac{3}{2}} h(s,0)ds - \frac{t^2}{2\Gamma(\frac{1}{2})} \int_0^1 (1-s)^{\frac{-1}{2}} h(s,0)ds + \frac{t^2}{2} = \frac{t^2}{2}, \quad t \in [0,1].$$

This shows that condition (ii) of Theorem 5 is also fulfilled. Additionally, if $x \leq y$ we conclude $fx \leq fy$. Therefore, condition (iii) of Theorem 5 holds. Therefore, Equation (20) has a unique solution on $[0,1]$.

4. Conclusions

We introduced the concept of fuzzy $\mathcal{R}-\psi-$contractive mappings and studied some relevant results on the existence and uniqueness of fixed points for such mappings in the setting of non-Archimedean fuzzy metric spaces (in Kramosil and Michalek's sense as well as George and Veeramani's sense). These results extended and generalized the results of [6,19]. We also provided some illustrative examples which supported our work. In the application section, we proved the existence and uniqueness of solutions for Caputo fractional differential equations.

Author Contributions: All the authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
3. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
4. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389. [CrossRef]
5. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
6. Mihet, D. Fuzzy φ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2008**, *159*, 739–744. [CrossRef]
7. Turinici, M. Abstract comparison principles and multivariable Gronwall-Bellman inequalities. *J. Math. Anal. Appl.* **1986**, *117*, 100–127. [CrossRef]
8. Ran, A.C.; Reurings, M.C. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
9. Nieto, J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
10. Nieto, J.J.; Rodríguez-López, R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Math. Sin. Engl. Ser.* **2007**, *23*, 2205–2212. [CrossRef]
11. Alfaqih, W.M.; Imdad, M.; Gubran, R.; Khan, I.A. Relation-theoretic coincidence and common fixed point results under $(F,\mathcal{R})_g$-contractions with an application. *Fixed Point Theory Appl.* **2019**, *2019*, 12. [CrossRef]
12. Ben-El-Mechaiekh, H. The Ran–Reurings fixed point theorem without partial order: A simple proof. *J. Fixed Point Theory Appl.* **2014**, *1*, 373–383. [CrossRef]
13. Imdad, M.; Khan, Q.; Alfaqih, W.M.; Gubran, R. A relation theoretic $(F,\mathcal{R})$-contraction principle with applications to matrix equations. *Bull. Math. Anal. Appl.* **2018**, *10*, 1–12.
14. Saleh, H.N.; Khan, I.A.; Imdad, M.; Alfaqih, W.M. New fuzzy ϕ-fixed point results employing a new class of fuzzy contractive mappings. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5391–5402. [CrossRef]
15. Saleh, S.M.; Sessa, S.; Alfaqih, W.M.; Shaddad, F. Common fixed point results for almost $\mathcal{R}_g$-Geraghty type contraction mappings in b_2-metric spaces with an application to integral equations. *Axioms* **2021**, *10*, 101. [CrossRef]
16. George, A.; Veeramani, P. On some results of analysis for fuzzy metric spaces. *Fuzzy Sets Syst.* **1997**, *90*, 365–368. [CrossRef]
17. Rodrigúez-López, J.; Romaguera, S. The Hausdorff fuzzy metric on compact sets. *Fuzzy Sets Syst.* **2004**, *147*, 273–283. [CrossRef]
18. Phiangsungnoen, S.; Cho, Y.J.; Kumam, P. *Fixed Point Results for Modified Various Contractions in Fuzzy Metric Spaces via α Admissible*; Faculty of Science and Mathematics, University of Nis: Nis, Serbia, 2016. [CrossRef]
19. Roldán-López-de-Hierro, A.-F.; Karapinar, E.; Manro, S.; Some new fixed point theorems in fuzzy metric spaces. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2257–2264. [CrossRef]
20. Lipschutz, S. *Schaum's Outline of Theory and Problems of Set Theory and Related Topics*; Tata McGraw-Hill Publishing: New York, NY, USA, 1976.
21. Lipschutz, S. *Theory and Problems of Set Theory and Related Topics*; Schaum Publishing: Mequon, WI, USA, 1964.
22. Maddux, R. *Boolean Algebras, Series: Studies in Logic and the Foundations of Mathematics*, 1st ed.; Elsevier Science: Amsterdam, The Netherlands, 2006; Volume 150.
23. Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [CrossRef]
24. Alfaqiha, W.M.; Ali, B.; Imdad, M.; Sessa, S. Fuzzy relation-theoretic contraction principle. *J. Intell. Fuzzy Syst.* **2021**, *40*, 4491–4501. [CrossRef]

25. Altun , I.; Mihet, D. Ordered non-Archimedean fuzzy metric spaces and some fixed point results. *Fixed Point Theory Appl.* **2010**, *11*, 782680. [CrossRef]
26. Kilbas, A.A.; Marzan, S.A. Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. *Russ. Differ. Uravn.* **2005**, *41*, 82–86, 142; English Translation *Differ. Equ.* **2005**, *41*, 84–89. [CrossRef]
27. Agarwal, R.P.; Benchohra, M.; Hamani, S. Boundary value problems for fractional differential equations. *Georgian Math. J.* **2009**, *16*, 401–411. [CrossRef]