# Some remarks about level sets of Cesaro averages of binary digits

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**Abstract**

The problem of averaging the binary digits of numbers in $[0,1]$ is considered. It is well known that Lebesgue a.e. in $[0,1]$ the usual Cesaro average is equal to $\frac{1}{2}$ and that the Hausdorff dimension of the set where the Cesaro average is equal to $\alpha$ is given by an entropy function $d(\alpha)$. We prove that if $\alpha \neq \frac{1}{2}$ then the Hausdorff measure $\mathcal{H}^{d(\alpha)}$ of such set is infinite. We moreover explicitly construct an infinite matrix $T$ (in a class $\mathcal{M}$ of Toeplitz matrices regular with respect to Cesaro averages) such that the Hausdorff dimension of the set of the points not having Cesaro average and where the $T$-generalized average is $\alpha$ is still given by $d(\alpha)$.

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## 1. Introduction

In this paper we consider the classic problem of averaging the binary digits of numbers in [0,1] and of studying the (Hausdorff) dimension and measure of some sets related to these averages.

Let us more precisely consider $t \in [0,1]$, the sequence $x(t) = (x_n(t))_n$ of its binary digits (cf. (2.4)) and the sequence of their averages $y(t) = (y_n(t))_n$ given by

$$y_n(t) = \frac{1}{n} \sum_{k=1}^{n} x_k(t), \forall n \in \mathbb{N} \tag{1.1}$$

We call the 'Cesaro average' of the binary digits of $t$ the quantity, when it exists:

$$\lim_{n \to +\infty} y_n(t). \tag{1.2}$$

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A classical result due to Borel is that, for almost every $t \in [0,1]$ (with respect to the Lebesgue measure) the Cesaro average is $\frac{1}{2}$ (see [S], example 1 page 369).

Let the $s$-dimensional Hausdorff measure and the Hausdorff dimension be respectively defined by (2.2) and (2.3) and let us set

$$F^{\alpha} = \left\{ t \in [0,1] : \lim_{n} y_n(t) = \alpha \right\} \tag{1.3}$$

A coincise expression for the Borel result quoted above is $\mathcal{H}^1\left(F^{\frac{1}{2}}\right) = 1$.

Another well known result (see Theorem 14 of [E] or Proposition 10.1 of [F]) states that the set $F^{\alpha}$ has Hausdorff dimension $d(\alpha)$, where the entropy function $d(t)$ is given by

$$d(t) = \begin{cases} -\left(t \log_2(t) + (1-t)\log_2(1-t)\right), & \forall t \in (0,1) \\ 0, & \text{if } t = 0, 1. \end{cases} \tag{1.4}$$

In the present paper we prove that if $\alpha \neq \frac{1}{2}$ then $\mathcal{H}^{d(\alpha)}(F^{\alpha}) = +\infty$ (Corollary 3.2).

It is moreover possible to generalize the definitions given by (1.1) $\div$ (1.3).

To be more precise, let $\omega = \{x : \mathbb{N} \to \mathbb{R}\}$ the set of the sequences of real numbers, then having in mind Toeplitz summation method (cf. [Ha], pag.41), we consider an infinite matrix $T = (a_{nk})_{n,k\in\mathbb{N}}$ of real numbers, lower triangular (i.e. $a_{nk} = 0$ if $k > n$), and define, for every $x = (x_n)_n \in \omega$, $T(x) = (T(x)_n)_n$ by

$$T(x)_n = \sum_{k=1}^{\infty} a_{nk} x_k. \tag{1.5}$$

Then we pose

$$T\text{-}F^{\alpha} = \left\{ t \in [0,1] : \lim_{n} T(x(t))_n = \alpha \right\}. \tag{1.6}$$

If $t \in T\text{-}F^{\alpha}$ we call $\alpha$ the $T$-generalized average of the binary digits of $t$.

Let the matrix $C_1$ be defined by

$$(C_1)_{h,k} = \begin{cases} \dfrac{1}{h}, & \text{if } k \leq h, \\ 0, & \text{otherwise;} \end{cases} . \tag{1.7}$$

it is called Cesaro matrix of order 1 and we obviously have $F^{\alpha} = C_1\text{-}F^{\alpha}$

If we consider the following class of matrices

$$\mathcal{M} = \left\{ T \text{ lower triangular matrix} : \begin{array}{l} \limsup\limits_{n} (T(x))_n \leq \limsup\limits_{n} (C_1(x))_n \\ \liminf\limits_{n} (T(x))_n \geq \liminf\limits_{n} (C_1(x))_n \end{array}, \forall x \in \omega \right\}, \tag{1.8}$$

it is easy to see (cf. Proposition 4.1) that if $T \in \mathcal{M}$ then $F^{\alpha} \subset T\text{-}F^{\alpha}$ and $\mathcal{H}^{d(\alpha)}\left((T\text{-}F^{\alpha}) \setminus F^{\alpha}\right) = 0$; in particular this implies $\dim_H\left((T_0\text{-}F^{\alpha}) \setminus F^{\alpha}\right) \leq d(\alpha)$.

We eventually prove by an explicit example (Theorem 4.4) that there exists a matrix $T_0 \in \mathcal{M}$ such that

$$\dim_H\left((T_0\text{-}F^{\alpha}) \setminus F^{\alpha}\right) = d(\alpha).$$

## 2. Notations and preliminary results

Let us denote by $\mathbb{N} = \{1, 2, 3, ...\}$, by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and by $\overline{\mathbb{R}}$=$\mathbb{R}\cup\{-\infty, +\infty\}$ Given a finite subset $M \subset \mathbb{N}$ we will denote by $\mathrm{card}\,(M)$ the number of its elements. Given a subset $E \subseteq \mathbb{R}$ we will denote by $\mathrm{diam}\,(E) = \sup\{|x-y| : x, y \in E\}$ its diameter and if in addition $E$ is a measurable, we define by $|E|$ its Lebesgue measure.

Let $\delta > 0$ and $s \geq 0$ real numbers and let us pose

$$\mathcal{H}^s_\delta(E) = \inf \sum_{n=1}^{\infty} \mathrm{diam}^s(F_n) \tag{2.1}$$

where the family $\{F_n\}_{n\in\mathbb{N}}$ is a countable covering of $E$ such that $\mathrm{diam}(F_n) < \delta$, $\forall n \in \mathbb{N}$ and the infimum is taken on this kind of families. The $s$-dimensional Hausdorff outer measure of $E$ is given as usual by

$$\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E), \tag{2.2}$$

while Hausdorff dimension of $E$ is given by

$$\dim_H(E) = \inf\{s \in \mathbb{R}{:}\mathcal{H}^s(E) = 0\}. \tag{2.3}$$

Let us denote by $\omega = \{x : \mathbb{N} \to \mathbb{R}\}$ the set of the sequences of real numbers and $c = \{x : \mathbb{N} \to \mathbb{R}{:}\ \lim_n x_n = l \in \mathbb{R}\}$ the set of converging ones.

Given $t \in \mathbb{R}$, we will denote by $[t]$ the integer part of $t$, i.e. $[t] = \max\{m \in \mathbf{Z} : m \leq t\}$ and by $I$ the interval $[0, 1]$.

Let us call

$$\mathcal{D} = \left\{t \in I : \exists p \in \mathbb{N}_0,\ q \in \mathbb{N} \text{ s.t. } t = \frac{p}{2^q}\right\}$$

the set of dyadic points. Let us observe that $\mathcal{D}$ is countable and therefore $\dim_H(\mathcal{D}) = 0$.

Let $t \in I$. We define the sequence $x(t) = \{x_n(t)\}_n$ in the following way

$$x_n(t) = [2^n t] - 2\left[2^{n-1}t\right] \qquad \forall n \in \mathbb{N}. \tag{2.4}$$

Such sequence is the one of the binary digits of $t$ (if $t \in \mathcal{D}$, it can be expressed in two ways as binary numbers: e.g. $\frac{1}{2} = 0,1_2$ and also $\frac{1}{2} = 0,0\bar{1}_2$; the sequence defined corresponds in this case to the representation with a finite number of digits equal to 1).

For a fixed $n \in \mathbb{N}$, $x_n(t)$ is a step function assuming only values 0 and 1

$$x_n(t) = \frac{1}{2}\left(\chi_{[0,1)} + \sum_{j=0}^{2^n-1}(-1)^{j+1}\chi_{[\frac{j}{2^n},\frac{j+1}{2^n})}(t)\right) \qquad \forall t \in I,$$

where for a set $A$, the function $\chi_A$ is the characteristic function of $A$.

Now let $y(t) = (y_n(t))_n$ the sequence defined by (1.1); $y_n(t)$ is a step function constant on every interval $[\frac{j}{2^n}, \frac{j+1}{2^n}), j = 0, 1, ..., 2^n - 1$, and takes only values $\frac{k}{n}, k = 0, 1, ..., n$.

Moreover

$$\left|\left\{t : y_n(t) = \frac{k}{n}\right\}\right| = \binom{n}{k} 2^{-n}, \tag{2.5}$$

where $\binom{n}{k}$ is the binomial coefficient of $n$ over $k$.

For every $\alpha, \beta \in I$ we define

$$\begin{array}{cc} \multicolumn{2}{c}{F^\alpha = \left\{t \in I : \lim_n y_n(t) = \alpha\right\},} \\ G^\alpha = \{t \in I : \limsup_n y_n(t) = \alpha\}, & G_\alpha = \{t \in I : \liminf_n y_n(t) = \alpha\}, \\ S^\alpha = \{t \in I : \limsup_n y_n(t) \geq \alpha\}, & S_\alpha = \{t \in I : \liminf_n y_n(t) \leq \alpha\}, \\ \multicolumn{2}{c}{G_\alpha^\beta = \{t \in I : \liminf_n y_n(t) = \alpha \text{ and } \limsup_n y_n(t) = \beta\}.} \end{array} \tag{2.6}$$

Obvious relations among the sets defined above are

$$F^\alpha = G_\alpha^\alpha, \qquad G^\alpha = \cup_{0\leq\beta\leq\alpha} G_\beta^\alpha, \qquad G_\alpha = \cup_{\alpha\leq\beta\leq 1} G_\alpha^\beta, \tag{2.7}$$

$$G_\alpha^\beta = G^\beta \cap G_\alpha, \qquad S^\alpha = \cup_{\alpha\leq\beta\leq 1} G^\beta, \qquad S_\alpha = \cup_{0\leq\beta\leq\alpha} G_\beta$$

for every $\alpha$ and $\beta$ in $I$.

Therefore obvious relations among the Hausdorff dimensions of such sets are

$$\dim_H(F^\alpha) \leq \dim_H(G^\alpha) \leq \sup_{\alpha\leq\beta\leq 1} \dim_H(G^\beta) \leq \dim_H(S^\alpha), \tag{2.8}$$

$$\dim_H(F^\alpha) \leq \dim_H(G_\alpha) \leq \sup_{0\leq\beta\leq\alpha} \dim_H(G_\beta) \leq \dim_H(S_\alpha), \tag{2.9}$$

$$\dim_H(G^\alpha) \geq \sup_{0\leq\beta\leq\alpha} \dim_H(G_\beta^\alpha), \qquad \dim_H(G_\alpha) \geq \sup_{\alpha\leq\beta\leq 1} \dim_H(G_\alpha^\beta), \tag{2.10}$$

$$\dim_H(G_\alpha^\beta) \leq \min\left\{\dim_H(G^\beta), \dim_H(G_\alpha)\right\} \tag{2.11}$$

for every $\alpha$ and $\beta$ in $I$.

We collect in the following theorem the known results about the dimension of set $F^\alpha$, $G_\alpha$, $G^\alpha$, $S_\alpha$, $S^\alpha$, $G_\alpha^\beta$.

**Theorem 2.1.** *Let $\alpha, \beta \in I$ and let $F^\alpha$, $G_\alpha$, $G^\alpha$, $S_\alpha$, $S^\alpha$, $G_\alpha^\beta$ be defined by (2.6). Then*

*i)* $\dim_H(G_\alpha^\beta) = \min\{d(\alpha), d(\beta)\}, \ \forall \alpha, \beta \in [0,1],$

*ii)* $\dim_H(F^\alpha) = d(\alpha),$

*iii)* $\dim_H(G_\alpha) = \dim_H(G^\alpha) = d(\alpha),$

*iv)* $\dim_H(S_\alpha) = \begin{cases} d(\alpha), & \text{if } \alpha \leq 1/2 \\ 1, & \text{if } \alpha \geq 1/2 \end{cases}, \dim_H(S^\alpha) = \begin{cases} 1, & \text{if } \alpha \leq 1/2 \\ d(\alpha), & \text{if } \alpha \geq 1/2 \end{cases} \quad \forall \alpha \in [0,1].$

**Proof.** The result i) is the theorem 6 proved in [C] .

We observe that ii) is a direct consequence of i). Statements iii) and iv) follow from theorem 14 and the related corollary at page 87 of [E]. ■

Let us observe that the sets defined by (2.6) can have dimension strictly between 0 and 1.

Now let $T$ be a infinite matrix lower triangular. Recall the definitions (2.6) and the definition of the set $T\text{-}F^{\alpha}$ given by (1.6). Then we can define in analogous way the sets $T\text{-}G^{\alpha}$, $T\text{-}G_{\alpha}$, $T\text{-}G^{\alpha}_{\beta}$, $T\text{-}S^{\alpha}$, $T\text{-}S_{\alpha}$.

Then it is easy to deduce the following proposition.

**Proposition 2.2.** *If $T \in \mathcal{M}$ (the class defined by 1.8), then*

$$\begin{aligned} \dim_H(T\text{-}F^{\alpha}) &= \dim_H(T\text{-}G_{\alpha}) = \dim_H(T\text{-}G^{\alpha}) = d(\alpha) \\ \dim_H(T\text{-}S_{\alpha}) &= \begin{cases} d(\alpha), & \text{if } \alpha \le 1/2 \\ 1, & \text{if } \alpha \ge 1/2 \end{cases}, \dim_H(T\text{-}S^{\alpha}) = \begin{cases} 1, & \text{if } \alpha \le 1/2 \\ d(\alpha), & \text{if } \alpha \ge 1/2 \end{cases} \quad \forall \alpha \in [0,1]. \end{aligned}$$

**Proof.** We have to recall the following inclusions which are consequence of definition of class $\mathcal{M}$

$$\begin{aligned} F^{\alpha} &\subseteq T\text{-}F^{\alpha} \subseteq T\text{-}G^{\alpha} \subseteq T\text{-}S^{\alpha} \subseteq S^{\alpha}, \\ F^{\alpha} &\subseteq T\text{-}F^{\alpha} \subseteq T\text{-}G_{\alpha} \subseteq T\text{-}S_{\alpha} \subseteq S_{\alpha}. \end{aligned}$$

Then we can apply theorem 2.1. ■

**Remark 1.** *It is nontrivial to evaluate the Hausdorff dimension of $T\text{-}G^{\beta}_{\alpha}$.*

*In the paper [C2] it is proved that if $T \in \mathcal{M}$ then* $\dim_H(T\text{-}G^{\beta}_{\alpha}) = \dim_H(G^{\beta}_{\alpha}) = \min\{d(\alpha), d(\beta)\}$.

We now give a generalization of Cantor like subsets of $I = [0,1]$ (slightly more general than the one given in [C], definition 3).

**Definition 2.3.** *Let us consider a sequence $\{k_h\}_h$ , $\{q_h\}_h \subseteq \mathbb{N}$ such that*

$$1 \le k_h < q_h \qquad \forall h \in \mathbb{N}.$$

*Furthermore, for every $h \in \mathbb{N}$ we consider a $k_h$-tuple of integers between 0 and $q_h - 1$*

$$0 \le p_h^1 < p_h^2 < ... < p_h^{k_h} < q_h.$$

*Let us denote*

$$P_h = \left(p_h^1, p_h^2, ..., p_h^{k_h}\right).$$

*Let us construct the following sequence of sets* $\{C_h\}_h$

$$C_1 = \cup_{i_1=1}^{k_1} \left[ \frac{p_1^{i_1}}{q_1} + \frac{1}{q_1} I \right], \quad C_2 = \cup_{i_1=1}^{k_1} \cup_{i_2=1}^{k_2} \left[ \frac{p_1^{i_1}}{q_1} + \frac{1}{q_1} \left[ \frac{p_2^{i_2}}{q_2} + \frac{1}{q_2} I \right] \right], \quad ...$$

$$C_h = \cup_{i_1=1}^{k_1} \cup_{i_2=1}^{k_2} ... \cup_{i_h=1}^{k_h} \left[ \frac{p_1^{i_1}}{q_1} + \frac{1}{q_1} \left[ \frac{p_2^{i_2}}{q_2} + \frac{1}{q_2} \left[ ... \left[ \frac{p_h^{i_h}}{q_h} + \frac{1}{q_h} I \right] ... \right] \right] \right], \quad ... \tag{2.12}$$

*and define*

$$C = \cap_{h=1}^{+\infty} C_h. \tag{2.13}$$

In other words $C$ is a set obtained in a way similar to the Cantor set.

Every $C_h$ is an essential disjoint union of $k_1 k_2 \cdots k_h$ intervals of length $(q_1 q_2 \cdots q_h)^{-1}$; you obtain $C_{h+1}$ from $C_h$ performing the following steps:

a) divide $I$ in $q_{h+1}$ intervals;

b) choose $k_{h+1}$ intervals among them according to (order) numbers $p_{h+1}^1, ..., p_{h+1}^{k_{h+1}}$;

c) scale down the set obtained in b) to the length of the intervals of $C_h$;

d) replace every interval of $C_h$ with the set obtained in c), translated by the left endpoint of the interval.

Let us first recall the following inequality proved in [C] (see lemma 2).

**Lemma 2.4.** *Let $m, n$ be natural numbers such that $n \geq 1$, $0 \leq m \leq n$;*

let $d$ the function defined by (1.4). Then

$$n\, d(\frac{m}{n}) - \frac{1}{2} \log_2(n) - 1 \leq \log_2 \binom{n}{m} \leq n\, d(\frac{m}{n}).$$

The following lemma holds.

**Lemma 2.5.** *Let $C$ be a set constructed like in definition 2.3. Let moreover $C'$ be the set obtained by the same construction where $I = [0,1]$ is replaced by $I' = [0,1[$.*

*Let $\gamma > 0$ and assume that there exists $\lambda > 0$ and $h_0 \in \mathbb{N}$ such that*

$$k_1 k_2 \cdots k_{h-1} \geq \lambda \left( q_1 q_2 \cdots q_h \right)^{\gamma} \qquad \forall h \geq h_0. \tag{2.14}$$

*Then*

$$\mathcal{H}^{\gamma}\left(C'\right) = \mathcal{H}^{\gamma}\left(C\right) > 0. \tag{2.15}$$

**Proof.** The equality in (2.15) easily follows from the following inclusions

$$C' \subseteq C \subseteq C' \cup \mathcal{D}$$

where $\mathcal{D}$ is the set of dyadic points.

Let us now prove the inequality in (2.15).

Let $\{B_j\}_j$ a countable covering of $C$ with open balls such that $\text{diam}\,(B_j) < \dfrac{1}{q_1 q_2 \cdots q_{h_0}}$ for every $j \in \mathbb{N}$. By the compatcness of $C$ we can assume that exists $\nu \in \mathbb{N}$ such that $\{B_j\}_{1 \le j \le \nu}$ is still a covering of $C$. For every $1 \le j \le \nu$ there exists $h_j \ge h_0$ such that

$$\frac{1}{q_1 q_2 \cdots q_{h_j}} \le \text{diam}\,(B_j) < \frac{1}{q_1 q_2 \cdots q_{h_{j-1}}}; \tag{2.16}$$

Let $m = \max\{h_j : 1 \le j \le \nu\}$ and observe that $C$ is contained in $C_m$ that in turn is the essential disjoint union of $k_1 k_2 \cdots k_m$ intervals of length $(q_1 q_2 \cdots q_m)^{-1}$, $C_m = C_m^1 \cup C_m^2 \cup ... \cup C_m^{k_1 k_2 \cdots k_m}$.

Let us define

$$\mu_j \doteq \frac{\text{card}\,\{i = 1, ..., k_1 k_2 \cdots k_m : B_j \cap C_m^i \ne \emptyset\}}{k_1 k_2 \cdots k_m}. \tag{2.17}$$

Since for every $i = 1, ..., k_1 k_2 \cdots k_m$ the interval $C_m^i$ contains points of $C$ and $\{B_j\}_{1 \le j \le \nu}$ is a covering of $C$ we have

$$\sum_{j=1}^{\nu} \mu_j \ge 1. \tag{2.18}$$

If we divide $[0,1]$ in $q_1 q_2 \cdots q_{h_{j-1}}$ intervals, $B_j$ can have nonempty intersection with at most two such intervals, and each of these intervals contains $k_{h_j} k_{h_{j+1}} \cdots k_m$ intervals of $C_m$.

By (2.17), (2.14) and (2.16) we have

$$\mu_j \le \frac{2 k_{h_j} k_{h_{j+1}} \cdots k_m}{k_1 k_2 \cdots k_m} = \frac{2}{k_1 k_2 \cdots k_{h_j - 1}} \le \frac{2}{\lambda} \left( \frac{1}{q_1 q_2 \cdots q_{h_j}} \right)^{\gamma} \le \frac{2}{\lambda} \left(\text{diam}(B_j)\right)^{\gamma} \tag{2.19}$$

Then (2.19) and (2.18) give

$$\sum_{j=1}^{\nu} \text{diam}\,(B_j)^{\gamma} \ge \frac{\lambda}{2} \sum_{j=1}^{\nu} \mu_j = \frac{\lambda}{2} > 0$$

whence, taking into acoount definitions (2.1) and (2.2) the thesis follows. ■

The following result (see also lemma 12 in [C2]) is a particular case of lemma 2.5.

**Lemma 2.6.** *Let $q \in \mathbb{N}$, $(z_h)_h \subseteq \mathbb{N}$ a sequence such that $z_h \le q$, $\forall h \in \mathbb{N}$,*

$$E_h = \left\{ t \in [0,1) : \sum_{i=1}^{q} x_{(k-1)q+i}\,(t) = z_k, \forall 1 \le k \le h \right\}$$

*and*

$$E = \cap_{h=1}^{\infty} E_h.$$

*Then*

i) $E_h$ *can be obtained as in definition 2.3, with* $I$ *replaced by* $[0,1)$, $q_h = 2^q$ *and* $k_h = \binom{q}{z_h}$;

ii)

$$\dim_H(E) \geq \liminf_h \frac{1}{hq} \sum_{j=1}^{h} \log_2 \binom{q}{z_j}.$$

## 3. The computation of the measure of level sets of Cesaro averages.

**Proposition 3.1.** *Let* $\alpha \in \left(\frac{1}{2}, 1\right)$. *Let* $C'$ *be the set defined by*

$$C' = \left\{ t \in [0,1] : \left[ k\left(\alpha - \frac{6}{\sqrt{k}}\right) \right] < \sum_{j=\frac{(k-1)k}{2}+1}^{\frac{k(k+1)}{2}} x_j(t) \leq [k\alpha], \ \forall k \in \mathbb{N} \right\}. \tag{3.1}$$

*Then*

$$C' \subset F^\alpha \text{ and } \mathcal{H}^{d(\alpha)}(C') > 0; \tag{3.2}$$

*(3.2) can be obtained in a similar way if* $\alpha \in \left(0, \frac{1}{2}\right)$.

**Proof.** Let us first verify that $C' \subset F^\alpha$.

If $t \in C'$ and $n \in \mathbb{N}$ then we have

$$y_{\frac{n(n+1)}{2}}(t) = \frac{2}{n(n+1)} \sum_{j=1}^{\frac{n(n+1)}{2}} x_j(t) \leq \frac{2\alpha}{n(n+1)} \sum_{k=1}^{n} k = \alpha \tag{3.3}$$

and

$$y_{\frac{n(n+1)}{2}}(t) = \frac{2}{n(n+1)} \sum_{j=1}^{\frac{n(n+1)}{2}} x_j(t) \geq \frac{2\alpha}{n(n+1)} \sum_{k=1}^{n} \left(k - 6\sqrt{k}\right) \geq \alpha - \frac{12\alpha\sqrt{n}}{n(n+1)} \geq \alpha\left(1 - \frac{12}{\sqrt{n}}\right). \tag{3.4}$$

Let now $k, n \in \mathbb{N}$ such that $\frac{n(n+1)}{2} < k \leq \frac{(n+1)(n+2)}{2}$. Then if $t \in C'$ we have

$$y_k(t) \leq \frac{2}{(n+1)(n+2)} \left( \frac{n(n+1)}{2} y_{\frac{n(n+1)}{2}}(t) + (n+1) \right) \leq \frac{n\alpha + 2}{(n+2)} \tag{3.5}$$

and

$$y_k(t) \geq \frac{n}{(n+2)} y_{\frac{n(n+1)}{2}}(t) \geq \frac{\alpha n}{(n+2)} \left(1 - \frac{12}{\sqrt{n}}\right). \tag{3.6}$$

By (3.3), (3.4) (3.5) and (3.6) we easily get

$$\lim_k y_k(t) = \alpha. \tag{3.7}$$

In order to complete the proof we only have to prove that $\mathcal{H}^{d(\alpha)}(C') > 0$.

If we consider the construction given by definition 2.3 where $I = [0,1]$ is replaced by $I' = [0,1[$ and, for every $h \in \mathbb{N}$, $P_h$ given by

$$P_h = \left\{ 0 \leq m \leq 2^{h-1} : \left(\alpha - \frac{6}{\sqrt{h}}\right) \leq \sum_{j=1}^{h} x_j\left(\frac{m}{2^h}\right) \leq \alpha \right\} \tag{3.8}$$

then it is easy to verify that

$$q_h = 2^h, \ k_h = \sum_{m=\left[h\left(\alpha - \frac{6}{\sqrt{h}}\right)\right]+1}^{[h\alpha]} \binom{h}{m} \tag{3.9}$$

$$C'_h = \left\{ t \in [0,1] : k\left(\alpha - \frac{6}{\sqrt{k}}\right) \leq \sum_{j=\frac{(k-1)k}{2}+1}^{\frac{k(k+1)}{2}} x_j(t) \leq k\alpha, \ \forall k \leq h \right\}. \tag{3.10}$$

and that $C' = \cap_{h=1}^{+\infty} C'_h$ .

Let us now recall that by lemma 2.4 we have

$$\frac{2^{hd\left(\frac{m}{h}\right)}}{2\sqrt{h}} \leq \binom{h}{m} \leq 2^{hd\left(\frac{m}{h}\right)}. \tag{3.11}$$

Let $h_0$ such that $[h\alpha] - \left[h\left(\alpha - \frac{6}{\sqrt{h}}\right)\right] > 4\sqrt{h}$ and $\left[h\left(\alpha - \frac{6}{\sqrt{h}}\right)\right] > \frac{1}{2}h$ for every $h \geq h_0$.

Then for every $m$ in the sum in (3.9) we have $\frac{1}{2} < \frac{m}{h} < \alpha$ and $d\left(\frac{m}{h}\right) > d(\alpha)$. Then we get

$$k_h \geq 4\sqrt{h}\frac{2^{hd\left(\frac{m}{h}\right)}}{2\sqrt{h}} \geq 2^{hd(\alpha)+1} > 2^{(h+1)d(\alpha)}.$$

Then

$$\begin{aligned} k_1 k_2 \cdots k_{h-1} &\geq k_{h_0+1} k_{h_0+2} \cdots k_{h-1} \geq 2^{(h_0+2)d(\alpha)} 2^{(h_0+3)d(\alpha)} \cdots 2^{hd(\alpha)} = \\ &= \left(q_{h_0+2} q_{h_0+3} \cdots q_h\right)^{d(\alpha)} \geq \left(\frac{1}{q_1 q_2 \cdots q_{h_0+1}}\right)^{d(\alpha)} \left(q_1 q_2 \cdots q_h\right)^{d(\alpha)}. \end{aligned}$$

Then assumption (2.14) of Lemma 2.5 is satisfied and by this Lemma we obtain $\mathcal{H}^{d(\alpha)}(C') > 0$ and the thesis in the case $\alpha \in \left(\frac{1}{2}, 1\right)$.

If $\alpha \in \left(0, \frac{1}{2}\right)$ we can perform a similar proof giving an analogous definition of $C'$. ■

**Corollary 3.2.** *Let $\alpha \in [0,1]$, $\alpha \neq \frac{1}{2}$. Then $\mathcal{H}^{d(\alpha)}(F^\alpha) = +\infty$.*

**Proof.** If $\alpha = 0$ or $\alpha = 1$ then $d(\alpha) = 0$ and $\mathcal{H}^{d(\alpha)} \equiv \mathcal{H}^0$ is the counting measure. Since $\mathrm{card}(F^0) = \mathrm{card}(F^1) = +\infty$ the thesis follows in this case.

If $\alpha \in (0,1)$ by the equalities

$$F^\alpha = \left(F^\alpha \cap \left[0, \frac{1}{2}\right)\right) \cup \left(F^\alpha \cap \left[\frac{1}{2}, 1\right)\right) = \left(\frac{1}{2}F^\alpha\right) \cup \left(\frac{1}{2} + \frac{1}{2}F^\alpha\right)$$

and the properties of Hausdorff measure we deduce

$$\mathcal{H}^{d(\alpha)}(F^\alpha) = 2^{1-d(\alpha)}\mathcal{H}^{d(\alpha)}(F^\alpha); \tag{3.12}$$

if $\alpha \neq \frac{1}{2}$ then $1 - d(\alpha) > 0$ and (3.12) gives $\mathcal{H}^{d(\alpha)}(F^\alpha) = 0$ or $\mathcal{H}^{d(\alpha)}(F^\alpha) = +\infty$; by Proposition 3.1 $\mathcal{H}^{d(\alpha)}(F^\alpha) > 0$ and the thesis follows. ■

## 4. Some remarks about level sets of generalized averages.

In this section we consider a matrix $T$ in the class $\mathcal{M}$ and the generalized averages level sets $T$-$F^\alpha$ defined by (1.6).

We have the simple following proposition.

**Proposition 4.1.** $\mathcal{H}^{d(\alpha)}(T\text{-}F^\alpha) = +\infty$; $\mathcal{H}^{d(\alpha)}((T\text{-}F^\alpha)\setminus F^\alpha) = 0$.

**Proof.** The thesis follows from the inclusions

$$\begin{aligned} F^\alpha &\subset T\text{-}F^\alpha \\ (T\text{-}F^\alpha)\setminus F^\alpha &\subset \left(\cup\left\{S^\lambda : \lambda > \alpha,\ \lambda \in \mathbb{Q}\right\}\right) \cap \left(\cup\left\{S_\mu : \mu > \alpha,\ \mu \in \mathbb{Q}\right\}\right) \end{aligned} \tag{4.1}$$

and the observation that at least one of the sets in the intersection in (4.1) has $\mathcal{H}^{d(\alpha)}$ measure equal to zero. ■

The equality $\mathcal{H}^{d(\alpha)}((T\text{-}F^\alpha)\setminus F^\alpha) = 0$ obviously implies $\dim_H((T\text{-}F^\alpha)\setminus F^\alpha) \leq d(\alpha)$. Anyway this Hausdorff dimension can be equal to $d(\alpha)$, i.e. there exists a matrix $T_0 \in \mathcal{M}$ such that

$$\dim_H((T_0\text{-}F^\alpha) \setminus (F^\alpha)) = d(\alpha).$$

Let us first state the following lemmas that are useful in proof of theorem 4.4.

**Lemma 4.2.** *Let $0 < \alpha < 1$ and $p_1$, $p_2$, $q \in \mathbb{N}$ such that $\frac{p_1}{q} < \alpha < \frac{p_2}{q}$. Then there exists a sequence $(s_h)_h \subseteq \mathbb{N}$ such that:*

$$\begin{aligned} &i)\ s_h \in \{p_1, p_2\},\ \forall h \in \mathbb{N}, \\ &ii)\ C = \left\{t \in [0,1] : \sum_{j=(h-1)q+1}^{hq} x_j(t) = s_h,\ \forall h \in \mathbb{N}\right\} \subseteq F^\alpha. \end{aligned}$$

**Proof.** Let us define the sequence $(s_h)_h$ used in lemma 2.6 as follows

$$s_1 = p_1,\ s_2 = p_2$$

$$s_{h+1} = \begin{cases} p_1, & \text{if } \frac{\sum_{j=1}^{h} s_j}{hq} \geq \alpha, \\ p_2, & \text{if } \frac{\sum_{j=1}^{h} s_j}{hq} < \alpha, \end{cases} \quad \text{if } h \geq 2. \tag{4.2}$$

Let us take $t \in C$ and $n \geq 2q+1$. If $s_{\left[\frac{n-1}{q}\right]+1} = p_2$ we have $t \in C$

$$y_n(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(t) = \frac{1}{n}\left(\sum_{j=1}^{\left[\frac{n-1}{q}\right]q} x_j(t) + \sum_{j=\left[\frac{n-1}{q}\right]q+1}^{n} x_j(t)\right) < \frac{1}{n}\left(\alpha\left[\frac{n-1}{q}\right]q + p_2\right). \tag{4.3}$$

If $s_{\left[\frac{n-1}{q}\right]+1} = p_1$, let $\overline{h} = \max\left\{h < \left[\frac{n-1}{q}\right] : s_{h+1} = p_2\right\}$. Then if, $t \in C$, we have

$$\begin{aligned} y_n(t) &= \frac{1}{n}\left(\sum_{j=1}^{\overline{h}q} x_j(t) + \sum_{j=\overline{h}q+1}^{(\overline{h}+1)q} x_j(t) + \sum_{j=(\overline{h}+1)q+1}^{n} x_j(t)\right) < \\ &< \frac{1}{n}\left(\alpha\overline{h}q + p_2 + \left(\left[\frac{n-1}{q}\right] - \overline{h}\right)p_1\right) < \\ &< \frac{1}{n}\left(\alpha\overline{h}q + \left(\left[\frac{n-1}{q}\right] - \overline{h}\right)\alpha q + p_2\right) = \frac{1}{n}\left(\left[\frac{n-1}{q}\right]\alpha q + p_2\right). \end{aligned} \tag{4.4}$$

By (4.3) and (4.4) we get

$$\limsup_n y_n(t) \leq \alpha. \tag{4.5}$$

In a similar way can easily be proved that

$$\liminf_n y_n(t) \geq \alpha. \tag{4.6}$$

By (4.2), (4.5) and (4.6) we obtain the thesis. ■

**Lemma 4.3.** *Let us define the functions*

$$\begin{aligned} \Phi(j) &= 2^{[\log_2 j]} + j - 1, && j \in \mathbb{N}, \\ \Psi(j) &= \left(\Phi\left(\left[\frac{j-1}{q}\right] + 1\right) - \left[\frac{j-1}{q}\right] - 1\right)q + j, && j \in \mathbb{N}. \end{aligned}$$

*Then* $\Phi$ *and* $\Psi$ *are strictly increasing; moreover*

$$M = \Phi(\mathbb{N}) = \left\{m \in \mathbb{N} : \exists k \in \mathbb{N} \text{ such that } 2^k - 1 \leq m \leq 2^{k-1}3 - 2\right\}$$

$$\begin{aligned} S &= \Psi(\mathbb{N}) = \{j \in \mathbb{N} : \exists m \in M \textit{ such that } (m-1)q + 1 \le j \le mq\} = \\ &= \left\{j \in \mathbb{N} : \exists k \in \mathbb{N} \textit{ such that } \left(2^{k+1} - 2\right) q + 1 \le j \le \left(2^k 3 - 2\right) q\right\}. \end{aligned}$$

*and the inverse functions are given by*

$$\begin{aligned} \Phi^{-1}(h) &= -2^{[\log_2(h+1)]-1} + h + 1, & h \in M, \\ \Psi^{-1}(h) &= \left(\Phi^{-1}\left(\left[\frac{h-1}{q}\right] + 1\right) - \left[\frac{h-1}{q}\right] - 1\right) q + h, & h \in S. \end{aligned}$$

*We eventually have that if $h \in S$ and $h \to +\infty$, then*

$$\frac{\Psi^{-1}(h)}{h} \to \frac{1}{2}. \tag{4.7}$$

**Proof.** The proof is elementary. In order to get (4.7), we just observe that $\lim_{j\to+\infty} \frac{\Phi(j)}{j} = \lim_{j\to+\infty} \frac{\Psi(j)}{j} = 2$. ■

**Theorem 4.4.** *There exists $T_0 \in \mathcal{M}$ such that*

$$\dim_H \left( (T_0\text{-}F^\alpha) \setminus F^\alpha \right) = d(\alpha) \qquad \forall \alpha \in [0,1]. \tag{4.8}$$

**Proof.** Let $M$ and $S$ the sets introduced in lemma 4.3 and let us pose

$$k_n = |S \cap \{1, ..., n\}|.$$

Let us pose

$$a_{nk} = \frac{1}{k_n} \chi_{S \cap \{1,...,n\}}.$$

If we pose $\widetilde{T_0} = (a_{nk})_{n,k}$, the matrix $T_0 = \widetilde{T_0} \circ C_1$ defines a matrix in $\mathcal{M}$ because it satisfies condition 2 of theorem 1.8. We claim that $T_0$ satisfies (4.8).

If $\alpha = 0$ or $\alpha = 1$, the thesis is obvious. Let $0 < \alpha < 1$. We observe that for every $\varepsilon > 0$ there exist $p_1, p_2, q \in \mathbb{N}$ such that

$$0 < \alpha - \varepsilon < \frac{p_1}{q} < \alpha < \frac{p_2}{q} < \alpha + \varepsilon < 1. \tag{4.9}$$

and that $\frac{1}{2} \frac{\log_2 q}{q} + \frac{1}{q} < \varepsilon$.

We can write

$$\alpha = \lambda \frac{p_1}{q} + (1 - \lambda) \frac{p_2}{q} \tag{4.10}$$

and assume, without loss of generality, that $\left|\alpha - \frac{p_1}{q}\right| < \left|\alpha - \frac{p_2}{q}\right|$ (and therefore $\lambda > \frac{1}{2}$).

Let us prove that it is possible to construct a set $E$ as in lemma 2.6 such that

$$\begin{array}{ll} i) & z_k \in \{p_1, p_2\}, \ \forall k \in \mathbb{N}; \\ ii) & E \subseteq (T_0\text{-}F^{\alpha}) \setminus F^{\alpha}. \end{array} \tag{4.11}$$

By lemma 2.6 and lemma 2.4, we have

$$\begin{aligned} \dim_H(E) &\geq \min\left\{\frac{1}{q}\log_2\binom{q}{p_1}, \frac{1}{q}\log_2\binom{q}{p_2}\right\} > \\ &> \min\{d(\alpha-\varepsilon), d(\alpha+\varepsilon)\} - \varepsilon. \end{aligned} \tag{4.12}$$

Since $E \subseteq (T_0\text{-}F^{\alpha}) \setminus F^{\alpha}$, by continuity of $d$ and the arbitrarness of $\varepsilon$, we obtain

$$\dim_H((T_0\text{-}F^{\alpha}) \setminus F^{\alpha}) \geq d(\alpha).$$

Let $C$ and $(s_h)_h$ given by lemma 4.2, let us define the sequence $(m_h)_h$ by

$$m_h \overset{def.}{=} \left|\left\{j \in \mathbb{N} : 2^{h-1} \leq j \leq 2^h - 1 \text{ and } s_j = p_2\right\}\right|, \ h \in \mathbb{N}, \tag{4.13}$$

and the sequence $(s_h)_h$ as follows

$$z_m = \begin{cases} s_{\Phi^{-1}(m)}, & \text{if } m \in M, \\ p_1, & \text{if } \exists h \in \mathbb{N}\text{:}2^{h-1}3 - 1 \leq m \leq 2^{h-1}3 - 2 + m_h \\ p_2, & \text{if } \exists h \in \mathbb{N}\text{:}2^{h-1}3 - 1 + m_h \leq m \leq 2^{h+1} - 2 \end{cases}. \tag{4.14}$$

(since $m_h \leq 2^{h-1}$, $\forall h \in \mathbb{N}$, $2^{h-1}3 - 1 \leq 2^{h-1}3 - 2 + m_h \leq 2^{h+1} - 2$).

Let $E$ the set constructed in lemma 2.6 using the sequence $(z_h)_h$. Then the i) of (4.11) is obviously satisfied and we have just to prove ii) of (4.11), that is

$$E \subseteq (T_0\text{-}F^{\alpha}) \setminus F^{\alpha}. \tag{4.15}$$

Let $t \in E$ and observe that

$$(T_0 x)_n(t) = \sum_{k=1}^{n} a_{nk} y_k(t) = \frac{1}{k_n} \sum_{k \in S \cap \{1,\ldots,n\}} y_k(t). \tag{4.16}$$

By (4.16), to obtain (4.15) is sufficient to prove that if $t \in E$ the sequence $(y_k(t))_k$ does not converge, while the subsequence $(y_k(t))_{k \in S}$ converges to $\alpha$.

Let $v \in E$ and let us define $t$ by

$$x_j(t) = x_{\Psi(j)}(v), \quad \forall j \in \mathbb{N}, \tag{4.17}$$

where $\Psi$ is given in lemma 4.3.

By (4.17) it follows

$$\sum_{j=j_1}^{j_2} x_j(t) = \sum_{j \in S,\, j=\Psi(j_1)}^{\Psi(j_2)} x_j(v) = \sum_{j \in S,\, j=\Psi(j_1)}^{\Psi(j_2+1)-1} x_j(v), \ \forall j_1, j_2 \in \mathbb{N}. \tag{4.18}$$

In particular, since $\Psi((h-1)q+1) = (\Phi(h)-1)q+1$, $\Psi(hq) = \Phi(h)q$ and $\Phi(h) \in M$

$$\sum_{j=(h-1)q+1}^{hq} x_j(t) = \sum_{j=(\Phi(h)-1)q+1}^{\Phi(h)q} x_j(v) = z_{\Phi(h)} = s_h, \quad \forall h \in \mathbb{N} \tag{4.19}$$

therefore $t \in C$.

Moreover, since $\Psi\left(\left(2^k-1\right)q+1\right) = \left(2^{k+1}-2\right)q+1$

$$\sum_{j=1}^{\left(2^k-1\right)q} x_j(t) = \sum_{j\in S,\, j=1}^{\left(2^{k+1}-2\right)q} x_j(v), \quad \forall k \in \mathbb{N} \tag{4.20}$$

By (4.14) we also have

$$\sum_{j\in S,\, j=1}^{\left(2^k-2\right)q} x_j(v) = \sum_{j\notin S,\, j=1}^{\left(2^k-2\right)q} x_j(v), \quad \forall k \in \mathbb{N} \tag{4.21}$$

Let now $h \in S$, set

$$k_h = \max\left\{k \in \mathbb{N} : \left(2^k-2\right)q+1 \le h\right\} = \left[\log_2\left(2+\frac{h-1}{q}\right)\right]$$

and observe that

$$h - \Psi^{-1}(h) = \left(2^{k_h-1}-1\right)q \quad \text{and} \quad \Psi\left(\left(2^{k_h-1}-1\right)q+1\right) = \left(2^{k_h}-2\right)q+1; \tag{4.22}$$

then by (4.19) $\div$ (4.22) we have

$$\begin{aligned}
\sum_{j=1}^{h} x_j(v) &= \sum_{j=1}^{\left(2^{k_h}-2\right)q} x_j(v) + \sum_{j=\left(2^{k_h}-2\right)q+1}^{h} x_j(v) = \\
&= \sum_{j\in S,\, j=1}^{\left(2^{k_h}-2\right)q} x_j(v) + \sum_{j\notin S,\, j=1}^{\left(2^{k_h}-2\right)q} x_j(v) + \sum_{j=\left(2^{k_h}-2\right)q+1}^{h} x_j(v) = \\
&= 2\sum_{j=1}^{\left(2^{k_h-1}-1\right)q} x_j(t) + \sum_{j=\left(2^{k_h-1}-1\right)q+1}^{\Psi^{-1}(h)} x_j(t) = \sum_{j=1}^{\Psi^{-1}(h)} x_j(t) + \sum_{j=1}^{h-\Psi^{-1}(h)} x_j(t).
\end{aligned}$$

Therefore

$$\begin{aligned}
y_h(v) &= \frac{1}{h}\left(\sum_{j=1}^{\Psi^{-1}(h)} x_j(t) + \sum_{j=1}^{h-\Psi^{-1}(h)} x_j(t)\right) = \\
&= \frac{1}{h}\left(\Psi^{-1}(h)\, y_{\Psi^{-1}(h)}(t) + \left(h-\Psi^{-1}(h)\right) y_{h-\Psi^{-1}(h)}(t)\right), \quad h \in S.
\end{aligned}$$

then, since $t \in C$, by lemma 4.2 and (4.7) $y_h(v)$ tends to $\alpha$, as $h \to +\infty$ in $S$.

Let $v \in E$ and $t \in C$ be defined by (4.17). By (4.20) we have

$$y_{\left(2^{k+1}-2\right)q}(v) = y_{\left(2^{k}-1\right)q}(t), \ \forall k \in \mathbb{N}. \tag{4.23}$$

Let us observe that, since $y_h(t) \xrightarrow{h} \alpha$

$$\frac{\left(2^k-1\right)q}{2^{k-1}q} y_{\left(2^k-1\right)q}(t) - \frac{\left(2^{k-1}-1\right)q}{2^{k-1}q} y_{\left(2^{k-1}-1\right)q}(t) \to \alpha, \text{ as } k \to +\infty. \tag{4.24}$$

But we have

$$= \frac{y_{\left(2^k-1\right)q}(t)\left(2^k-1\right)q - y_{\left(2^{k-1}-1\right)q}(t)\left(2^{k-1}-1\right)q}{2^{k-1}q} = \tag{4.25}$$

$$= \frac{m_k p_1 + \left(2^{k-1}-m_k\right)p_2}{2^{k-1}q} = \frac{m_k}{2^{k-1}}\frac{p_1}{q} + \left(1 - \frac{m_k}{2^{k-1}}\right)\frac{p_2}{q},$$

where $m_k$ is defined by (4.13).

Then by (4.10), (4.24) and (4.25) we have that

$$\frac{m_k}{2^{k-1}} \to \lambda > \frac{1}{2}. \tag{4.26}$$

Therefore if $k$ is large $m_k > 2^{k-2}$ and by (4.7)

$$\text{if } \left(2^{k-1}3-1\right) \leq m \leq 2^{k-1}3 - 2 + 2^{k-2}, \qquad \text{then } s_m = p_1.$$

Let us take $n_k = \left(2^{k-1}3 + 2^{k-2} - 2\right)q$, we have by (4.23) and (4.26)

$$\begin{aligned}
y_{n_k}(v) &= \frac{1}{n_k}\sum_{j=1}^{n_k} x_j(v) = \\
&= \frac{1}{n_k}\left(q\left(2^k-2\right)y_{\left(2^k-2\right)q}(v) + \sum_{j=\left(2^k-2\right)q+1}^{\left(2^{k-1}3-2\right)q} x_j(v) + \sum_{j=\left(2^{k-1}3-2\right)q+1}^{n_k} x_j(v)\right) = \\
&= \frac{1}{n_k}\left(2q\left(2^{k-1}-1\right)y_{\left(2^{k-1}-1\right)q}(t) + \left(m_k p_1 + \left(2^{k-1}-m_k\right)p_2\right) + 2^{k-2}p_1\right)
\end{aligned}$$

that, as $k \to +\infty$, tends to

$$\frac{4}{7}\alpha + \frac{2}{7}\left(\lambda\frac{p_1}{q} + (1-\lambda)\frac{p_2}{q}\right) + \frac{1}{7}\frac{p_1}{q} = \frac{6}{7}\alpha + \frac{1}{7}\frac{p_1}{q} \neq \alpha.$$

So (4.11) is fully satisfied by $E$. Therefore by (4.11), (4.12) and the arbitrarness of $\varepsilon > 0$, the thesis follows. ■

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