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Shape programming of a magnetic elastica

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We consider a cantilever beam which possesses a possibly non-uniform permanent magnetization, and whose shape is controlled by an applied magnetic field. We model the beam as a plane elastic curve and we suppose that the magnetic field acts upon the beam by means of a distributed couple that pulls the magnetization towards its direction. Given a list of target shapes, we look for a design of the magnetization profile and for a list of controls such that the shapes assumed by the beam when acted upon by the controls are as close as possible to the targets, in an averaged sense. To this effect, we formulate and solve an optimal design and control problem leading to the minimization of a functional which we study by both direct and indirect methods. In particular, we prove that minimizers exist, solve the associated Lagrange-multiplier formulation (besides non-generic cases), and are unique at least for sufficiently low intensities of the controlling magnetic fields. To achieve the latter result, we use two nested fixed-point arguments relying on the Lagrange-multiplier formulation of the problem, a method which also suggests a numerical scheme. Various relevant open question are also discussed.

Keywords: Rods; nonlinear elasticity; optimal design; optimal control.

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1. Introduction and Main Results

1.1. *Motivation*

Recent technological developments have made it possible to assemble, with pinpoint accuracy of composition and texture, elastic materials which can convert into deformation, and hence motion, a diversity of energetic inputs in the form of heat, light, chemical agents, electric and magnetic fields. These advances make it possible to craft devices which can change their shape through distributed actuation, mimicking biological examples such as elephant trunks and octopus arms, which are best suited for interacting with complex environments.[20] In particular, engineers find these materials appealing for applications at small scales, such as for instance microsurgery and drug delivery,[23] where the implementation of conventional technologies proves difficult. For these applications, a key requirement is the ability to attain a large variety of shapes: for example, the locomotion of miniature robots based on crawling involves negotiation of obstacles of all sorts in confined spaces,[15] thus requiring high adaptability; likewise, locomotion based on swimming requires *ad hoc* shape-control strategies such as a distinct power- and recovery-stroke.[21]

Shape control of devices with distributed actuation cannot be addressed with the conventional engineering practice of designing separately power, kinematics, and control: in order to achieve a desired motion strategy, the device morphology and the stimulus must be designed *at the same time*.[19,26] This state of matters has stimulated substantial theoretical work concerning *shape programming*, i.e. the design of textures and controls that produce desired shapes, a topic which is becoming increasingly relevant in theoretical elasticity (see e.g. Refs. 1 and 2).

Additional problems arise when miniature devices are required to operate untethered. In fact, since dissipative effects are dominant at small scales, self-powered devices require high-density energy storage mechanisms. In this respect, magnetic actuation offers several advantages: it can remotely provide both control and power, it offers fast response, and it does not affect the surrounding medium by polarization.[27]

Among the many available magneto-elastic materials, the so-called magnetorheological elastomers (MREs) are particularly suited for shape programming. Originally devised as viscoelastic solids whose mechanical response could be controlled by an applied magnetic field,[13] MREs are obtained by embedding magnetic particles in a soft elastomeric matrix. Thanks to their compliance, MREs find applications in circumstances when large displacements are in need.[34] Moreover, their magnetic properties can be finely controlled.[18] Furthermore, the theoretical modeling of MREs is well established (see for instance Refs. 10 and 17), their stability at both the macroscopic[24] and the microstructural level[28] has been investigated, and *ad-hoc* computational techniques[25] are available.

Proofs of concept exist[15,21,30] that MREs can be used to fabricate small-scale untethered microrobots, which can walk, crawl and swim. Indeed, when crafted in the form of thin bodies, such as rods or plates, magnetorheological elastomers display a very large range of motion.[29,34] In this respect, shape programming appears to be rather intriguing even for a simple mechanical model such Euler's Elastica, which is at the basis of the model we adopt in the present paper. This is not surprising, since the qualitative and quantitative properties of equilibrium solutions for elastic curves in a diversity of settings are still the objects of intense mathematical research (see e.g. Refs. 9, 11, 16 and 22).

This paper is meant as a contribution towards the development of a systematic mathematical framework for shape programming of magnetic materials, with an emphasis on obtaining rigorous results. With this aim in mind, we focus on a mechanical model featuring a planar *cantilever beam* with *permanent magnetization* having constant intensity but variable direction, under a *spatially-constant magnetic field*, as shown in Fig. 1 below.

For this model, we formulate an optimal design-and-control problem which may verbally be described as follows: given a list of pre-assigned target shapes, choose a magnetization profile (the *morphology*) and a list of applied fields (the *stimuli*) such that the shapes attained by the beam under the action of these fields best approximate the given shapes, in some averaged sense. Admittedly, the formulation we choose ignores a certain amount of the physics which comes into play in actual engineering applications. For example, the interaction force with the surrounding medium is being ignored. Likewise, this formulation ignores the dynamic effects of moving from one shape to another as the applied field varies (see the discussion in Sec. 1.6). On the other hand, experimental evidence from Ref. 21 shows that the results obtained in such simplified setting can still furnish valuable guidance to the design of actual devices.

1.2. *The mathematical model*

We model the cantilever beam as a magnetized *planar elastica*, and we describe its configuration through the parametric curve $\vec{r} : (0,1) \to \mathbb{R}^2$ defined by

$$\vec{r}(s) = \ell \int_0^s \vec{m}(\vartheta(\xi))\,\mathrm{d}\xi, \tag{1.1}$$

where ℓ is the length of the beam, $\vec{m} : \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\vec{m}(v) = (\cos(v), \sin(v)), \tag{1.2}$$

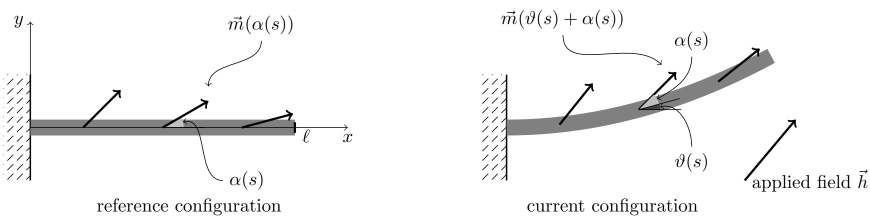

Fig. 1. A cantilever beam with a permanent magnetization of uniform intensity and angle $\alpha(s)$ with respect to the tangent.

and $\vartheta(s)$ is the *rotation at* s. With slight abuse of language, we shall refer to the function $\vartheta : (0,1) \to \mathbb{R}$ as the *shape of the beam*. Since the beam is clamped, the shape must satisfy the essential boundary condition:

$$\vartheta(0) = 0,$$

which holds irrespectively of the loading environment.

The beam has a permanent magnetization per unit length, whose intensity is a constant M_0 (its unit in the S.I. System is ampere·meter^{-2} [Am^{-2}]), and whose orientation with respect to the tangent line is given by a possibly non-uniform *relative angle* $\alpha(s) \in \mathbb{R}$, $s \in (0,1)$. We assume that the relative angle $\alpha(s)$ is not affected by the magnetic field and by the deformation process. Hence, the vector fields

$$\vec{m}(\alpha) = (\cos(\alpha), \sin(\alpha)), \quad \vec{m}(\vartheta + \alpha) = (\cos(\alpha + \vartheta), \sin(\alpha + \vartheta))$$

are the *orientation of the magnetization* in the undeformed, respectively, deformed, configurations.

Theories of magnetoelastic rods (see for instance Refs. 6, 7 and 12) predict that when a spatially constant magnetic field $\vec{H}$ [Am^{-1}] is applied to the beam, any stable equilibrium configuration must be a local minimizer of the renormalized *magnetoelastic energy*

$$\mathcal{E}(\vartheta) = \int_0^1 \left(\frac{1}{2} (\vartheta')^2 - \vec{h} \cdot \vec{m}(\vartheta + \alpha) \right) \mathrm{d}s, \tag{1.3}$$

where a dot denotes the scalar product, $\vec{h}$ is the renormalized magnetic field defined by $\vec{h} = \mu_0 \frac{M_0 \ell^2}{S} \vec{H}$, with μ_0 [Hm^{-1}] the magnetic permeability of vacuum and S [Nm^2] the *bending stiffness*. The vector $\vec{h}$ is dimensionless, since its modulus $|\vec{h}| = (\mu_0 M_0 H \ell)/(S\ell^{-1})$ can be written as the ratio between the *magnetic energy* $\frac{1}{2}\mu_0 M_0 H \ell$ that must be expended to immerse the beam in the magnetic field, and the *elastic energy* S/ℓ that must be stored in the system to impart the curvature ℓ^{-1} to the beam.

1.3. *The state equation*

For $\vec{h} = 0$ the magnetoelastic energy has the unique minimizer $\vartheta = 0$, which corresponds through (1.1) to the straight configuration. As detailed in Sec. 4 (see Corollary 4.1), for given, arbitrary $\vec{h}$ and α the magnetoelastic energy has at least one minimizer, which furthermore solves the *Euler–Lagrange system*

$$\begin{cases} -\vartheta'' - \vec{h} \cdot D\vec{m}(\alpha + \vartheta) = 0 & \text{in } I := (0,1), \\ \vartheta(0) = 0, \quad \vartheta'(1) = 0, \end{cases} \tag{P_ϑ}$$

where

$$D\vec{m}(v) = (-\sin v, \cos v), \quad \text{for all } v \in \mathbb{R} \tag{1.4}$$

is the derivative of the function $\vec{m}$ defined in (1.2); moreover, such minimizer is unique if

$$|\vec{h}| < c_p^{-2}, \tag{1.5}$$

where $c_p = 2/\pi$ is the best constant in the Poincaré-type inequality

$$\int_0^1 v^2 \le c_p^2 \int_0^1 (v')^2 \quad \text{for all } v \in C^1([0,1]) \text{ such that } v(0) = 0.$$

The *state equation* (P_ϑ) is a variant of the well-known *elastica equation*. Given α, (P_ϑ) defines a *solution operator*

$$\Theta_\alpha : B(0, c_p^{-2}) \ni \vec{h} \mapsto \vartheta = \Theta_\alpha(\vec{h}) \tag{1.6}$$

which maps the *control* $\vec{h}$ into the *state* $\vartheta = \Theta_\alpha(\vec{h})$. The manifold of attainable configurations parametrized by the chart Θ_α is two-dimensional. Thus, one may hope that complex motions, such as for instance those required for applications to microswimmers,[3,4] could be realized, at least with reasonable approximation, by a judicious choice of a fixed magnetization profile and a time varying magnetic field. References 15 and 21 offer experimental evidence of this possibility.

1.4. *The optimal design-control problem*

In this paper, we are concerned with the following situation. We are given a list of n prescribed *target shapes*,

$$\overline{\boldsymbol{\vartheta}} = (\bar{\vartheta}_1, \dots, \bar{\vartheta}_n) : [0,1] \to \mathbb{R}^n,$$

which the beam should ideally attain by applying n *controls*: these are the n magnetic fields

$$\vec{\boldsymbol{h}} = (\vec{h}_1, \dots, \vec{h}_n) \in \mathbb{R}^{2n},$$

with $\vec{h}_i = (h_{ix}, h_{iy}) \in \mathbb{R}^2$. At our disposal is also a *design*, the magnetization α of the beam. Thus, we look for a design α and a control $\vec{\boldsymbol{h}}$ such that the shapes $\vartheta_i = \Theta_\alpha(\vec{h}_i)$ attained by the beam when applying the magnetic fields $\vec{h}_i$, namely the solutions of

$$\begin{cases} -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0 \quad \text{in } (0,1), \\ \vartheta_i(0) = 0, \quad \vartheta_i'(1) = 0, \end{cases} \qquad i = 1, \dots, n, \tag{P_{ϑ_i}}$$

are "as close as possible" to the targets $\overline{\vartheta}_i$. The precise meaning of "closeness" depends on the choice of the *c*ost functional $\mathcal{C}$, which we define as follows:

$$\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = \mathcal{C}_{\varepsilon,\gamma}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = \frac{1}{2}\sum_{i=1}^n \int_0^1 |\vartheta_i - \bar{\vartheta}_i|^2 + \frac{\varepsilon}{2}\int_0^1 |\alpha'|^2 + \frac{\gamma}{2}\sum_{i=1}^n |\vec{h}_i|^2, \tag{1.7}$$

where $\varepsilon > 0$ and $\gamma > 0$ are positive parameters.

Remark 1.1. (The cost functional) The choice of the cost functional $\mathcal{C}$ in (1.7) deserves a discussion. The first integral has an obvious interpretation, since we aim at minimizing the distance between the n attained shapes ϑ_i and the n target shapes $\overline{\vartheta}_i$. The second and third term, which penalize inhomogeneities of the magnetization density, respectively, high intensities of the applied magnetic fields, are key technical ingredients, since they render the cost functional coercive with respect to a topology that guarantees compactness of minimizing sequences.

Our precise mathematical formulation of the problem thus involves three ingredients:

(i) the *admissible space*

$$\begin{aligned}\mathcal{H} &= \{(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) : \vec{\boldsymbol{h}} \in \mathbb{R}^{2n}, \alpha \in H^1_{0L}(I), \boldsymbol{\vartheta} \in H^1_{0L}(I)^n\} \\ &= \mathbb{R}^{2n} \times H^1_{0L}(I) \times H^1_{0L}(I)^n,\end{aligned} \tag{1.8}$$

where

$$H^1_{0L}(I) := \{v \in H^1(I) : v(0) = 0\}; \tag{1.9}$$

(ii) the *cost functional* $\mathcal{C} : \mathcal{H} \to \mathbb{R}$, defined by (1.7) for all $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$;

(iii) the *admissible set*

$$\mathcal{A} = \{(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H} : \vartheta_i \text{ solves } (P_{\vartheta_i}) \text{ for every } i = 1, \dots, n\}. \tag{1.10}$$

With these three ingredients, we may formulate the following *Optimal Control-Design Problem*:

$$\text{minimize } \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \text{ among all } (\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A}. \tag{1.11}$$

Simple calculations using angle sum identities show that (P_{ϑ_i}) and (1.7) are invariant under an equal rotation of the vectors $\vec{h}_i$ and $\vec{m}(\alpha)$ (see also the proof of part (iii) of Theorem 1.1): therefore, in (1.8) we have set $\alpha(0) = 0$ without losing generality.

1.5. *Our results*

Using the direct method of the Calculus of Variations, we prove in Sec. 3 the existence of a minimizer. In this respect, the penalization in the definition of $\mathcal{C}$ is crucial in guaranteeing coercivity for generic targets.

Theorem 1.1. (i) *For any $\varepsilon > 0$, $\gamma > 0$, and any $\overline{\boldsymbol{\vartheta}} \in L^2(I)^n$, the cost functional $\mathcal{C}_{\varepsilon,\gamma}$ has a minimizer in the admissible set $\mathcal{A}$. Furthermore, any minimizer is such that*

$$\Big(\max_{i=1,\dots,n}\{|\vec{h}_i|\}\Big)^2 \le \frac{\overline{\Theta}^2}{\gamma}, \quad \textit{where } \overline{\Theta}^2 = \sum_{i=1}^n \int_0^1 \overline{\vartheta}_i^2. \tag{1.12}$$

(ii) *For any* attainable *target* $\overline{\boldsymbol{\vartheta}}$, *i.e. any* $\overline{\boldsymbol{\vartheta}}$ *such that* $(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}}) \in \mathcal{A}$ *for some* $\bar{\vec{\boldsymbol{h}}} \in \mathbb{R}^{2n}$ *and some* $\bar{\alpha} \in H^1_{0L}(I)$, *minimizers of* $\mathcal{C}_{\varepsilon,\varepsilon}$ *converge to a minimizer of* $\mathcal{C}_{0,0}$ *as* ε *tends to* 0.

(iii) *For* $n = 1$, *any* $\bar{\vartheta} \in H^3(I)$ *with* $\bar{\vartheta}(0) = 0$ *and* $\bar{\vartheta}'(1) = 0$ *is attainable.*

Remark 1.2. For attainable targets, one has $\mathcal{C}_{0,0}(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}}) = 0$, hence existence of minimizers of $\mathcal{C}_{0,0}$ is trivial. However, the class of attainable targets is a non-dense subset of $H^1_{0L}(I)^n$, and existence of minimizers of $\mathcal{C}_{0,0}$ seems to be nontrivial for generic targets (it is not even clear if $\inf \mathcal{C}_{0,0}$ will be positive or not). This motivates introducing the penalization terms, and part (ii)–(iii) of Theorem 1.1 legitimates this choice. We note on passing that any attainable $\overline{\boldsymbol{\vartheta}}$ belongs to $H^3(I)^n$ (see Corollary 4.1), hence the assumption $\bar{\vartheta} \in H^3(I)$ in (iii) is not restrictive.

An important consequence of (1.12) is that, for $\overline{\Theta}$ sufficiently small and/or γ sufficiently large, each of the applied magnetic field $\vec{h}_i$ satisfies (1.5); thus, if $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer with $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$, then each state ϑ_i is the unique solution of its state system (P_{ϑ_i}): this means that the mechanical equilibria identified by the minimization of $\mathcal{C}$ are stable. In other words, for $\overline{\Theta}$ sufficiently small and/or γ sufficiently large each configuration ϑ_i corresponds to a stable minimizer of the magnetoelastic energy if the corresponding $\vec{\boldsymbol{h}}$ and α are taken as fixed.

The previous result neither implies uniqueness of the *triplet* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$, nor provides a constructive scheme for its numerical approximation. Focusing on these two aspects, we investigate the Lagrange-multiplier reformulation of (1.11). This reformulation amounts to finding a critical point of the *Lagrangian*

$$\mathcal{L}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) := \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) - \sum_{i=1}^{n} \int_0^1 \lambda_i \big(-\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i)\big),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is the *Lagrange multiplier*. Differentiation of $\mathcal{L}$ yields, formally, the following system:

$$\begin{cases} (P_{\vartheta_i})\colon -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0, & \vartheta_i(0) = \vartheta_i'(1) = 0, \\ (P_{\lambda_i})\colon -\lambda_i'' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = \vartheta_i - \bar{\vartheta}_i, & \lambda_i(0) = \lambda_i'(1) = 0, \\ (P_\alpha)\colon -\varepsilon\alpha'' + \sum_{i=1}^{n} \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = 0, & \varepsilon\alpha(0) = \varepsilon\alpha'(1) = 0, \\ (P_{\vec{h}_i})\colon \gamma\vec{h}_i = -\int_0^1 \lambda_i D\vec{m}(\alpha + \vartheta_i) & \end{cases} \tag{1.13}$$

for every $i = 1, \ldots, n$, where

$$D^2\vec{m}(v) = (-\cos v, -\sin v), \quad v \in \mathbb{R}, \tag{1.14}$$

is the second derivative of $\vec{m}$. According to the standard theory of constrained minimization through Lagrange multipliers in Banach spaces, whose main results we summarize in Appendix A, a minimizer $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ of the cost functional in the admissible set corresponds to a stationary point $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ of $\mathcal{L}$ for some Lagrange multiplier $\boldsymbol{\lambda}$ if that point is *regular*, in the sense that the *constraint mapping* $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$ (the dual of $H^1_{0L}(I)^n$), defined by

$$\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u}\rangle = \sum_{i=1}^{n} \left\{ \int_0^1 \vartheta_i' u_i' - \int_0^1 \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) u_i \right\}, \quad \forall\, \boldsymbol{u} \in H^1_{0L}(I)^n, \tag{1.15}$$

is *Fréchet differentiable* at $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ and its differential DG is *surjective*. We apply this theory in Sec. 5, where we study the Fréchet differentiability of the cost function $\mathcal{C}$ and of the constraint mapping G, as well as the surjectivity of the Fréchet differential of the latter. Let $r^+ = \max\{r, 0\}$, $r^- = \max\{-r, 0\} \geq 0$ (so that $r = r^+ - r^-$). We show the following.

Theorem 1.2. *Let* $\varepsilon \geq 0$, $\gamma \geq 0$, *and let* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *be a minimizer of* $\mathcal{C}$ *in* $\mathcal{A}$.

(i) *if, for all* $i = 1, \ldots, n$, $\mu = 1$ *is not an eigenvalue of the Sturm–Liouville operator*

$$\begin{cases} -u'' + (r_i^- + 1)u = \mu(r_i^+ + 1)u \quad \textit{in } (0,1), \quad r_i := \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i), \\ u(0) = 0, \quad u'(1) = 0, \end{cases} \tag{1.16}$$

then $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *is a regular point of* $\mathcal{A}$;

(ii) *in particular,* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *is a regular point of* $\mathcal{A}$ *if*

$$\max_{i=1,\ldots,n} \{|\vec{h}_i|\} < c_p^{-2}; \tag{1.17}$$

(iii) *if* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *is a regular point of* $\mathcal{A}$, *then there exists a Lagrange multiplier* $\boldsymbol{\lambda} \in H^1_{0L}(I)^n$ *such that* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ *is a solution of system* (1.13). *Furthermore,* $\alpha'(0) = 0$ *if* $\varepsilon > 0$.

Remark 1.3. The condition in (i) is equivalent to asking that the problem

$$-u'' - \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i)u = 0 \quad \text{in } (0,1), \quad u(0) = 0, \quad u'(1) = 0, \tag{1.18}$$

has only the null solution. This fact, however, does *not* directly imply surjectivity of (1.18), for which we need to invoke the theory of Sturm–Liouville operators. Such theory also guarantees that the eigenvalues of (1.16) are discrete (cf. Theorem A.1). Therefore, the current formulation of (i) highlights the fact that, besides non-generic "resonant" cases, any minimizer $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a solution to the Lagrangian system (1.13). Note also that in such generic cases a function α which does not satisfy $\alpha'(0) = 0$ is precluded from being the design of a minimizer if $\varepsilon > 0$. In addition, any minimizer is a solution to (1.13) if (1.17) holds.

Remark 1.4. As an immediate consequence of Theorem 1.1, the bound (1.12), and Theorem 1.2, we obtain that if $\varepsilon > 0$ and $\gamma > \overline{\Theta}^2 c_p^4$ then there exists a solution $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{H} \times H^1_{0L}(I)^n$ to system (1.13) such that $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer of $\mathcal{C}$ in $\mathcal{A}$.

The existence of a Lagrange multiplier justifies the approach proposed in Ref. 8 to numerically approximate the minimizer of $\mathcal{C}$, which is based on (1.13). In Sec. 6 we prove by a contraction argument that, at least for γ sufficiently large, System (1.13) has a unique solution (see Proposition 6.1). As a by-product, we have the following.

Theorem 1.3. *Let* $\overline{\boldsymbol{\vartheta}} \in C([0,1])^n$, $\varepsilon > 0$, *and let* $K > 0$ *such that*

$$K < c_p^{-2}. \tag{1.19}$$

Then exists $\gamma_* = \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for every* $\gamma > \gamma_*$ *there exists a unique solution of system* (1.13) *within the following set*:

$$(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{H} \times H^1_{0L}(I)^n \quad \textit{such that} \quad \max_{i=1,\dots,n} \{|\vec{h}_i|\} \le K < c_p^{-2}. \tag{1.20}$$

Moreover,

$$\|\vartheta_i\|_\infty \le |\vec{h}_i| \quad \textit{for all } i = 1, \dots, n. \tag{1.21}$$

Theorems 1.1–1.3 combine into the following.

Theorem 1.4. *Let* $\overline{\boldsymbol{\vartheta}} \in C([0,1])^n$, $\varepsilon > 0$, *and let* $K > 0$ *such that* (1.19) *holds. Then there exists* $\gamma_{**} = \gamma_{**}(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for any* $\gamma > \gamma_{**}$ *the minimizer in Theorem* 1.1 *is unique. Furthermore, it coincides with the unique solution to* (1.13), *whence it is smooth and such that* $\alpha'(0) = 0$, *and* (1.21) *holds.*

Proof. Let $(\vec{\boldsymbol{h}}^{(j)}, \alpha^{(j)}, \boldsymbol{\vartheta}^{(j)})$, $j = 1, 2$ be two minimizers. Let $\gamma > \overline{\Theta}^2 / K^2$. By (1.12) in Theorem 1.1, both minimizers satisfy

$$\left(\max_{i=1,\dots,n} |\vec{h}_i^{(j)}| \right)^2 \le \frac{\overline{\Theta}^2}{\gamma} < K^2 < c_p^{-4}, \quad j = 1, 2. \tag{1.22}$$

In particular, (1.17) holds for both. Hence, by Theorem 1.2, there exist $\boldsymbol{\lambda}^{(j)} \in H^1_{0L}(I)^n$ such that $(\vec{\boldsymbol{h}}^{(j)}, \alpha^{(j)}, \boldsymbol{\vartheta}^{(j)}, \boldsymbol{\lambda}^{(j)}) \in \mathcal{H} \times H^1_{0L}(I)^n$ are solutions to system (1.13). Assume in addition that $\gamma > \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$. Then, by (1.22) and Theorem 1.3, the two quadruplets, whence the two minimizers, coincide: therefore, the proof is complete by choosing $\gamma_{**} = \max\{\overline{\Theta}^2/K^2, \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)\}$. □

Theorem 1.4 states that for $\gamma > \gamma_{**}$ the minimum is unique and may be numerically approximated by solving the Euler–Lagrange system (1.13) (hence, not necessarily by a direct approach, although the latter is used to prove the existence of the minimum). In fact, it is through the uniqueness of the solution of the Euler–Lagrange system that we are able to assert the uniqueness of the minimum.

Remark 1.5. While Theorem 1.4 holds for any target $\overline{\boldsymbol{\vartheta}}$ (even very large ones), the minimizing state $\boldsymbol{\vartheta}$ will anyway be such that $\|\vartheta_i\|_\infty \leq |\vec{h}_i| < c_p^{-2}$ for all $i = 1, \dots, n$ (see (1.21)). Now, $c_p^{-2} > 3\pi/4$ is still a relevant value of the maximal rotation. However, one should bear in mind that large values of γ may turn into minimizers with $|\vec{h}_i|$ much smaller than c_p^{-2}, and thus insufficient to drive the attained shapes close to the targets. This means that, *under the uniqueness conditions of Theorem* 1.4, minimizing states may turn out to be far away from the targets when the latter ones are "large", a disappointing result from the point of view of engineering applications. In this respect, see also the comments to (a) and (b) below.

1.6. *Open problems*

We view the results in Sec. 1.5 as first steps in the mathematical analysis of the mechanical system under consideration. Indeed, quite a few relevant and interesting challenges are left open.

The first one is the existence of minimizers in the absence of penalization terms, i.e. with $\varepsilon = \gamma = 0$. While this is obvious in the non-generic case of attainable targets (see Remark 1.2), it otherwise appears to be a nontrivial problem. In fact, for generic targets, it might even be that the control-design minimization problem (1.11) is not well posed if $\varepsilon = \gamma = 0$ and $n > 1$ (see Remark 1.2).

The second one concerns *uniqueness* of minimizers, on which our results are admittedly limited by two conditions:

(a) a moderate maximal intensity of the applied field ($|\vec{h}| < c_p^{-2} = \pi^2/4$, see (1.19) and (1.20));
(b) a possibly large penalization constant ($\gamma > \gamma_{**}$).

We do not know whether (a) is optimal or not for the full design-control problem (1.11). However, as detailed in Remark 4.2, we know that c_p^{-2} is optimal for the state equation (P_ϑ). On the other hand, we believe that (b) is mainly technical (see Remark 6.3), and that uniqueness may hold even for values of γ which are much smaller than γ_{**}. Improving the current bound would require a refinement of the estimates of $\boldsymbol{\vartheta}$ in terms of $\overline{\boldsymbol{\vartheta}}$ in the Lagrangian formulation, a challenging but important goal for further developments (see also Remark 7.2).

Still related to uniqueness, Theorem 1.2 shows that, besides non-generic cases, minimizers are critical points of $\mathcal{L}$. We expect that, above a certain threshold (being it c_p^{-2} or larger), multiple critical points of $\mathcal{L}$ will emerge: it would be very interesting to develop selection criteria for identifying absolute minimizer(s) among multiple critical points.

The last major open question is of a different nature, and concerns the possibility of passing from a "static" to a dynamic framework, in which the rod moves in time following a prescribed path. In this framework, the n targets would represent discrete snapshots of such continuous movement.

Further remarks are presented in the concluding Sec. 7.

2. Notation and Preliminaries

In this section, we introduce some notation as a complement to that already defined in Sec. 1, and we collect preliminary results that will be needed in our subsequent developments. Other standard results are contained in Appendix A.

Given a vector $\vec{v} = (v_x, v_y) \in \mathbb{R}^2$, we let $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$ be its Euclidean norm and, for $\vec{w}$ another vector, we let $\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y$ be the scalar product between $\vec{v}$ and $\vec{w}$. Given a list of vectors $\vec{\boldsymbol{v}} = (\vec{v}_1, \dots, \vec{v}_n)$, with $\vec{v}_i \in \mathbb{R}^2$ for $i = 1, \dots, n$, we let $|\vec{\boldsymbol{v}}| := |\vec{v}_1| + \cdots + |\vec{v}_n|$. For $f : I \to \mathbb{R}$ a measurable function, we use the abbreviation $\|f\|_p \equiv \|f\|_{L^p(I)}$ for all exponents $p \geq 1$. We use similar abbreviations for measurable vector-valued functions. We recall that, by the Sobolev embedding theorem, $H^1_{0L}(I) \subset C([0,1])$, where $C([0,1])$ is the space of the continuous functions on $[0,1]$. We record for later use the inequality

$$\|v\|_\infty^2 \leq \int_0^1 (v')^2 \quad \text{for all } v \in H^1_{0L}(I), \tag{2.1}$$

which is sharp, as can be seen by taking $v(x) = x$.

We denote by $c_p = 2/\pi$ the *best constant* in the *Poincaré-type inequality*:

$$\int_0^1 v^2 \leq c_p^2 \int_0^1 (v')^2 \quad \text{for all } v \in H^1_{0L}(I). \tag{2.2}$$

It follows from (2.2) and from the definition (1.9) that

$$\|v\|^2 := \int_0^1 (v')^2 \tag{2.3}$$

is equivalent to the Sobolev norm on $H^1_{0L}(I)$. Accordingly, we henceforth shall use the norm (2.3) to endow $H^1_{0L}(I)$ with a Hilbert-space structure. For $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$ we define $\|(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})\|_{\mathcal{H}} = |\vec{\boldsymbol{h}}| + \|\alpha\| + \|\boldsymbol{\vartheta}\|$. It follows that $\mathcal{H}$ is an Hilbert space.

For $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ Banach spaces we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, and we let $\|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}$ be the operator norm. Moreover, we write $\langle \cdot, \cdot \rangle_{\mathcal{X}', \mathcal{X}}$ to denote the pairing between a Banach space $\mathcal{X}$ and its dual: in fact, we will omit the indexing whenever the space $\mathcal{X}$ is clear from the context.

If not otherwise specified, we will denote by C a generic constant whose value may possibly change within the same chain of inequalities, and by $C(\cdot)$ constants whose value only depend on the parameters and variables listed within parentheses.

Finally, we observe that the function $\vec{m} : \mathbb{R} \to \mathbb{R}^2$ defined in (1.2) is bounded, infinitely differentiable and its Nth derivative, defined consistently with (1.4), is

$$D^N \vec{m}(v) = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}^N \vec{m}(v), \quad \text{for all } N \in \mathbb{N}, \tag{2.4}$$

namely, $D^N\vec{m}(v)$ is the vector obtained by rotating $\vec{m}(v)$ in the counter-clockwise direction by the amount $N\pi/2$. Thus,

$$\left|D^N\vec{m}(v)\right| = 1, \quad \text{for all } v \in \mathbb{R} \quad \text{and} \quad N \in \mathbb{N}. \tag{2.5}$$

Hence,

$$|D^N\vec{m}(v_1) - D^N\vec{m}(v_2)| \le \int_{v_1}^{v_2} |D^{N+1}\vec{m}(v)|dv = |v_1 - v_2| \quad \text{for all } v_1, \quad v_2 \in \mathbb{R}.$$

As a consequence of this observation, we record three bounds which will be used several times.

Lemma 2.1. *Let* $(\vec{h},\alpha,\vartheta,\lambda),(\tilde{\vec{h}},\tilde{\alpha},\tilde{\vartheta},\tilde{\lambda}) \in \mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *and* $N\in\mathbb{N}$. *Then*

$$\left|\lambda D^N\vec{m}(\alpha+\vartheta) - \tilde{\lambda}D^N\vec{m}(\tilde{\alpha}+\tilde{\vartheta})\right| \le |\lambda-\tilde{\lambda}| + |\tilde{\lambda}|\big(|\alpha-\tilde{\alpha}| + |\vartheta-\tilde{\vartheta}|\big), \tag{2.6a}$$

$$\left|\vec{h}\cdot D^N\vec{m}(\alpha+\vartheta) - \tilde{\vec{h}}\cdot D^N\vec{m}(\tilde{\alpha}+\tilde{\vartheta})\right| \le |\vec{h}-\tilde{\vec{h}}| + |\tilde{\vec{h}}|\big(|\alpha-\tilde{\alpha}| + |\vartheta-\tilde{\vartheta}|\big), \tag{2.6b}$$

$$\begin{aligned}\left|\lambda\vec{h}\cdot D^N\vec{m}(\alpha+\vartheta) - \tilde{\lambda}\tilde{\vec{h}}\cdot D^N\vec{m}(\tilde{\alpha}+\tilde{\vartheta})\right| &\le |\vec{h}||\lambda-\tilde{\lambda}| + |\tilde{\lambda}||\vec{h}-\tilde{\vec{h}}| \\ &\quad + |\tilde{\vec{h}}||\tilde{\lambda}|\big(|\alpha-\tilde{\alpha}| + |\vartheta-\tilde{\vartheta}|\big).\end{aligned} \tag{2.6c}$$

3. Existence of a Minimizer

In this section, we address the existence of a minimizer to the optimal control-design problem (1.11).

Proof of Theorem 1.1. We recall that the admissible set is defined in (1.10). We begin by noting that $(\vec{\mathbf{0}},0,\mathbf{0})\in\mathcal{A}$, hence $\mathcal{A}$ is not empty. Next, we let

$$m = \inf_{\mathcal{A}} C(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})$$

and we consider a minimizing sequence, i.e. a sequence $\{(\vec{\boldsymbol{h}}_k,\alpha_k,\boldsymbol{\vartheta}_k)\}\subset\mathcal{A}$ with $\vec{\boldsymbol{h}}_k = (\vec{h}_{k1},\ldots,\vec{h}_{kn})$, $\vec{h}_{ki} = (h_{kix},h_{kiy})$ and $\boldsymbol{\vartheta}_k = (\vartheta_{k1},\ldots,\vartheta_{kn})$, such that $\mathcal{C}(\vec{\boldsymbol{h}}_k,\alpha_k,\boldsymbol{\vartheta}_k)\to m$ as $k\to+\infty$. In particular, by the definition of $\mathcal{C}$, a constant C exists such that

$$\sum_{i=1}^n\int_0^1|\vartheta_{ki}|^2 + \int_0^1|\alpha_k'|^2 + \sum_{i=1}^n|\vec{h}_{ki}|^2 \le C \tag{3.1}$$

for all $k\in\mathbb{N}$. Moreover ϑ_{ki} satisfies

$$\int_0^1\big(\vartheta_{ki}'v' - \vec{h}_{ki}\cdot D\vec{m}(\alpha_k+\vartheta_{ki})v\big) = 0, \quad \forall\, v\in H^1_{0L}(I) \tag{3.2}$$

for all $k\in\mathbb{N}$ and $i=1,\ldots,n$. Choosing ϑ_{ki} as test function in (3.2) and recalling (3.1), we obtain

$$\|\vartheta_{ki}\| \le 2C \quad \text{for all } k\in\mathbb{N} \text{ and every } i=1,\ldots,n.$$

Hence, by a standard compactness argument, $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$ exists such that, by passing to a subsequence (not relabeled),

$$\begin{aligned}
&\vec{h}_{ki} \to \vec{h}_i \text{ in } \mathbb{R}^2, \text{ for every } i = 1, \dots, n,\\
&\alpha_k \to \alpha \text{ weakly in } H^1_{0L}(I) \text{ and uniformly in } C([0,1]),\\
&\vartheta_{ki} \to \vartheta_i \text{ weakly in } H^1_{0L}(I) \text{ and uniformly in } C([0,1]), \text{ for every } i = 1, \dots, n.
\end{aligned} \tag{3.3}$$

Letting k tend to infinity in (3.2) and using the convergence statement (3.3), we conclude that ϑ_i is a weak solution of (P_{ϑ_i}) for every $i = 1, \dots, n$, so $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A}$. Moreover, by lower semi-continuity,

$$\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \leq \liminf_{k\to\infty} \mathcal{C}(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k) = m.$$

This implies that $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer of $\mathcal{C}$. In addition, since $(\mathbf{0}, 0, \vec{\mathbf{0}}) \in \mathcal{A}$, for any minimizer we have

$$\frac{\gamma}{2}\sum_{i=1}^n |\vec{h}_i|^2 \leq \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \leq \mathcal{C}(\vec{\mathbf{0}}, 0, \mathbf{0}) = \frac{1}{2}\sum_{i=1}^n \int_0^1 |\bar{\vartheta}_i|^2,$$

which implies (1.12). This proves part (i) of the theorem.

We now restrict attention to attainable targets: that is, we assume that $\overline{\boldsymbol{\vartheta}}$ is such that $(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}}) \in \mathcal{A}$ for some $\bar{\vec{\boldsymbol{h}}} \in \mathbb{R}^{2n}$ and some $\bar{\alpha} \in H^1_{0L}(I)$. Note that in this case $\mathcal{C}_{0,0}(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}}) = 0$, so $(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}})$ is a minimizer of $\mathcal{C}_{0,0}$. Let $(\vec{\boldsymbol{h}}_\varepsilon, \alpha_\varepsilon, \boldsymbol{\vartheta}_\varepsilon)$ be a minimizer of $\mathcal{C}_{\varepsilon,\varepsilon}$ in $\mathcal{A}$. We have $\mathcal{C}_{\varepsilon,\varepsilon}(\vec{\boldsymbol{h}}_\varepsilon, \alpha_\varepsilon, \boldsymbol{\vartheta}_\varepsilon) \leq \mathcal{C}_{\varepsilon,\varepsilon}(\bar{\vec{\boldsymbol{h}}}, \bar{\alpha}, \overline{\boldsymbol{\vartheta}})$, that is,

$$\sum_{i=1}^n \int_0^1 |\vartheta_{\varepsilon i} - \bar{\vartheta}_i|^2 + \varepsilon\int_0^1 |\alpha'_\varepsilon|^2 + \varepsilon\sum_{i=1}^n |\vec{h}_{\varepsilon i}|^2 \leq \varepsilon\int_0^1 |\bar{\alpha}'|^2 + \varepsilon\sum_{i=1}^n |\bar{\vec{h}}_i|^2.$$

This means that

$$\sum_{i=1}^n \int_0^1 |\vartheta_{\varepsilon i} - \bar{\vartheta}_i|^2 \overset{\varepsilon\to 0}{\to} 0 \quad \text{and} \quad \int_0^1 |\alpha'_\varepsilon|^2 + \sum_{i=1}^n |\vec{h}_{\varepsilon i}|^2 \leq \int_0^1 |\bar{\alpha}'|^2 + \sum_{i=1}^n |\bar{\vec{h}}_i|^2.$$

Therefore, arguing as above, we see that for a subsequence $\vec{\boldsymbol{h}}_\varepsilon \to \vec{\boldsymbol{h}}_0$ in $\mathbb{R}^{2n}$, $\alpha_\varepsilon \rightharpoonup \alpha_0$ in $H^1_{0L}(I)$, $\boldsymbol{\vartheta}_\varepsilon \rightharpoonup \boldsymbol{\vartheta}_0 = \overline{\boldsymbol{\vartheta}}$ in $H^1_{0L}(I)^n$ and $(\vec{\boldsymbol{h}}_0, \alpha_0, \overline{\boldsymbol{\vartheta}}) \in \mathcal{A}$. In addition, it is obvious that $\mathcal{C}_{0,0}(\vec{\boldsymbol{h}}_0, \alpha_0, \overline{\boldsymbol{\vartheta}}) = 0$. Therefore, $(\vec{\boldsymbol{h}}_0, \alpha_0, \overline{\boldsymbol{\vartheta}})$ is a minimizer of $\mathcal{C}_{0,0}$ in $\mathcal{A}$.

In order to prove (iii), fix $\bar{\vartheta} \in H^3(I) \subset C^2([0,1])$ with $\bar{\vartheta}(0) = 0$ and $\bar{\vartheta}'(1) = 0$. Let $H > \max_{s\in[0,1]} |\overline{\vartheta}''(s)|$ and set

$$\bar{\vec{h}} = (H\cos\psi, H\sin\psi), \quad \bar{\alpha}(s) = \arcsin\left(\frac{\overline{\vartheta}''(s)}{H}\right) - \overline{\vartheta}(s) + \psi_0.$$

Hence, choosing $\psi = -\arcsin\left(\frac{\bar{\vartheta}''(0)}{H}\right)$, we have $\alpha(0) = 0$. Moreover, since $\bar{\vartheta}'' \in H^1(I)$ and arcsin is a Lipschitz continuous function in any compact subset of $(-1,1)$, we deduce that $\bar{\alpha} \in H^1_{0L}(I)$. This implies that $(\bar{\vec{h}}, \bar{\alpha}, \bar{\vartheta}) \in \mathcal{H}$.

Finally, after straightforward computations using angle sum identities, one sees that

$$\begin{cases} -\bar{\vartheta}'' - \bar{\vec{h}} \cdot D\vec{m}(\bar{\alpha} + \bar{\vartheta}) = 0 \quad \text{in } I, \\ \bar{\vartheta}(0) = 0, \quad \bar{\vartheta}'(1) = 0. \end{cases}$$

This shows that $\bar{\vartheta}$ is always attainable. □

4. The Basic Equation

The next sections will be devoted to the analysis of the Lagrange-multiplier system (1.13). Its equations, (P_{ϑ_i}), (P_{λ_i}) and (P_α), share the following structure:

$$\begin{cases} -v'' + f(s,v) = 0 \qquad \text{in } (0,1), \\ v(0) = 0, \quad v'(1) = 0. \end{cases} \tag{4.1}$$

In particular, with reference to (1.13), we have that

$$\text{(4.1) is equivalent to: } \begin{cases} (P_{\vartheta_i}) \text{ for } f(s,v) = -\vec{h}_i \cdot D\vec{m}(\alpha(s) + v); \\ (P_{\lambda_i}) \text{ for } f(s,v) \\ \qquad = -v\vec{h}_i \cdot D^2\vec{m}(\alpha(s) + \vartheta_i(s)) + \overline{\vartheta}_i(s) - \vartheta_i(s); \\ (P_\alpha) \text{ for } f(s,v) = \dfrac{1}{\varepsilon}\displaystyle\sum_{i=1}^n \lambda_i(s)\vec{h}_i \cdot D^2\vec{m}(v + \vartheta_i(s)). \end{cases} \tag{4.2}$$

Definition 4.1. Let $f \in L^1(I \times \mathbb{R})$. A function v belonging to $H^1_{0L}(I)$ is a (weak) solution to problem (4.1) if

$$\int_0^1 v'w' + \int_0^1 f(s,v)w = 0 \quad \text{for all } w \in H^1_{0L}(I). \tag{4.3}$$

In the following lemma we provide (to the extent we need) uniqueness, existence, and boundedness results for solutions of (4.1).

Lemma 4.1. *Let $f \in L^\infty(I \times \mathbb{R})$, let $L \in (0, c_p^{-2})$ be such that*

$$|f(s,v_1) - f(s,v_2)| \le L|v_1 - v_2| \quad \textit{for a.e. } s \in I \textit{ and for all } v_1, v_2 \in \mathbb{R} \tag{4.4}$$

and let c_p be defined by (2.2)*. Then there exists a unique solution v to problem* (4.1) *in the sense of Definition 4.1, and this solution satisfies the bounds*

$$\|v\| \leq \frac{c_p}{1 - Lc_p^2} \|f(s,0)\|_\infty, \quad \|v\|_\infty \leq \|f\|_\infty. \tag{4.5}$$

Proof. Let $f_0(s) = f(s,0)$. For $g(s,v) = \int_0^v f(s,t)dt$, we let:

$$\mathscr{F}(w) = \frac{1}{2}\int_0^1 w'^2 + \int_0^1 g(s,w), \quad w \in H^1_{0L}(I).$$

This position defines a Gâteaux-differentiable, weakly-lower semicontinuous functional $\mathscr{F} : H^1_{0L}(I) \to \mathbb{R}$. Since

$$|g(s,v)| \leq \int_0^v |f(s,t)|dt \overset{(4.4)}{\leq} |f_0(s)||v| + \frac{1}{2}L|v|^2,$$

we have

$$\mathscr{F}(w) \geq \frac{1}{2}\int_0^1 w'^2 - \int_0^1 |f_0 w| - \frac{1}{2}L\int_0^1 w^2 \geq \frac{1}{2}(1 - Lc_p^2)\|w\|^2 - \|f_0\|_2\|w\|_2,$$

whence, by (2.2),

$$\mathscr{F}(w) \geq \frac{1}{2}(1 - Lc_p^2)\|w\|^2 - c_p\|f_0\|_\infty\|w\|.$$

This inequality implies that $\mathscr{F}$ is *coercive*, thanks to the hypothesis $L < c_p^{-2}$. The coercivity and the lower semicontinuity of $\mathscr{F}$ imply, by a standard argument, that $\mathscr{F}$ has a minimizer v in $H^1_{0L}(I)$ (see Ref. 14). Since $\mathscr{F}$ is Gâteaux differentiable, v is also a weak solution of problem (4.1).

In order to prove uniqueness, let v_1 and v_2 be two weak solutions of (4.1). According to Definition 4.1, $v_1 - v_2$ is a legal test function for the weak formulation of (4.1). We use this test in (4.3). On taking the difference between the resulting equations we obtain

$$\int_0^1 [(v_1 - v_2)']^2 + \int_0^1 (f(s,v_1) - f(s,v_2))(v_1 - v_2) = 0.$$

It follows from the assumption on f and from the Poincaré inequality (2.2) that

$$\int_0^1 (f(s,v_1) - f(s,v_2))(v_1 - v_2) \leq L\int_0^1 |v_1 - v_2|^2 \leq Lc_p^2\|v_1 - v_2\|^2,$$

whence $(1 - Lc_p^2)\|v_1 - v_2\|^2 \leq 0$ and thence $v_1 = v_2$, given that $Lc_p^2 < 1$.

Taking v as test function in (4.3) we obtain that

$$\|v\|^2 = \int_0^1 v'^2 = -\int_0^1 f(s,v)v \leq \int_0^1 |f_0(s)||v| + L\int_0^1 |v|^2$$
$$\leq \|f_0\|_\infty \left(\int_0^1 v^2\right)^{1/2} + L\int_0^1 |v|^2 \overset{(2.2)}{\leq} c_p\|f_0\|_\infty\|v\| + Lc_p^2\|v\|^2,$$

whence the first bound in (4.5). Finally, the second bound in (4.5) is immediate from the representation formula

$$v(s) = \int_0^s \int_{s'}^1 f(s'', v(s''))ds''\, ds'. \qquad \square$$

Remark 4.1. (Regularity and boundary values of the solution to problem (4.1)) Under the assumption $f \in L^\infty(I \times \mathbb{R})$ of Lemma 4.1, we note that if v is a weak solution to problem (4.1), then

$$v \in H^2(I) = \{w \in L^2(I) : w', w'' \in L^2(I)\}.$$

The Sobolev embedding theorem (see for instance Sec. 2.1 of Ref. 5) implies that $v \in C^1([0,1])$, and that the boundary conditions are satisfied pointwise. Indeed, since $v \in H^1_{0L}(I)$, we have that $v(0) = 0$. Moreover, multiplying by an arbitrary function $w \in H^1_{0L}(I)$ and integrating in I equation (4.1) we obtain

$$\int_0^1 -v''w + \int_0^1 f(s,v)w = 0 \quad \text{for all } w \in H^1_{0L}(I). \tag{4.6}$$

Integrating by parts the first term of the left-hand side of (4.6) we have

$$v'(1)w(1) = \int_0^1 (v'w)' = \int_0^1 v'w' + \int_0^1 f(s,v)w \overset{(4.3)}{=} 0 \quad \text{for all } w \in H^1_{0L}(I).$$

This implies that $v'(1) = 0$.

As a by-product of the previous discussion, we obtain the following.

Corollary 4.1. *Let $\vec{h} \in \mathbb{R}^2$ and $\alpha : I \to \mathbb{R}$ measurable. Then the energy functional $\mathcal{E}$ defined by* (1.3) *has a minimizer $\vartheta \in H^1_{0L}$. Furthermore, ϑ solves* (P_ϑ) *and it is unique if* (1.5) *holds. Finally, if $\alpha \in H^1(I)$, then $\vartheta \in H^3(I)$.*

Proof. Existence of a minimizer of $\mathcal{E}$ in $H^1_{0L}(I)$ can be proved with the same arguments used in Sec. 3. By standard variational considerations, ϑ is a solution to (4.3) with $f(s,\vartheta) = -\vec{h} \cdot D\vec{m}(\alpha(s) + \vartheta)$: hence, by Lemma 4.1, it is unique if $|\vec{h}| < c_p^{-2}$. Moreover, if $\alpha \in H^1(I)$, we deduce that $\vec{h} \cdot D\vec{m}(\alpha(s) + \vartheta) \in H^1(I)$. Hence, by (P_ϑ), $\vartheta'' \in H^1(I)$, i.e. $\vartheta \in H^3(I)$. $\square$

Remark 4.2. The condition $|\vec{h}| < c_p^{-2}$ in Corollary 4.1 is optimal in general, as the following counterexample shows. Taking $\alpha = 0$ and $\vec{h} = (-H, 0)$ in (P_ϑ) yields

$$\begin{cases} -\vartheta'' - H \sin\vartheta = 0 & \text{in } (0,1), \\ \vartheta(0) = 0, \quad \vartheta'(1) = 0. \end{cases} \tag{4.7}$$

Problem (4.7) is the same one which governs a cantilever under a compressive thrust applied at its free end. It admits the trivial solution $\vartheta = 0$ for every $H \in \mathbb{R}$. This solution is unique for $H < c_p^{-2} = \pi^2/4$. However, a nontrivial branch emanates from the singular point $(\vartheta, H) = (0, c_p^{-2})$, and uniqueness is lost for $H > c_p^{-2}$. Indeed, after integration, a strictly increasing solution $\vartheta(s)$ with $\vartheta(1) = \vartheta_1 > 0$ is implicitly given by

$$\int_0^{\vartheta(s)} \frac{dt}{\sqrt{\cos t - \cos\vartheta_1}} = \sqrt{2H}s, \quad \sqrt{2H} = \int_0^{\vartheta_1} \frac{dt}{\sqrt{\cos t - \cos\vartheta_1}} = \sqrt{2}K\left(\sin\frac{\vartheta_1}{2}\right), \tag{4.8}$$

where $K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ is the complete elliptic integral of the first kind (for the last equality in (4.8), one uses the change of variables $\sin\phi = \sin\frac{t}{2}/\sin\frac{\vartheta_1}{2}$). It is easily checked that $K(k)$ is increasing, $K(k) \to \frac{\pi}{2}$ as $k \to 0^+$ and $K(k) \to +\infty$ as $k \to 1^-$: hence for any $H > \pi^2/4 = c_p^{-2}$ the second equation in (4.8) has a solution $\theta_1(H)$, and inverting the first one we obtain $\vartheta(s)$. We aside mention that, when $\vec{h} = (0, H)$ is taken instead of $\vec{h} = (-H, 0)$, numerical evidence in Ref. 7 suggests uniqueness of solutions to (P_ϑ) for values of H substantially larger than c_p^{-2}.

Remark 4.3. One may wonder whether, under the condition $|\vec{h}| < c_p^{-2}$, the discrepancy between the full nonlinear theory and a simpler, linearized theory for (P_ϑ) would be negligible, so as to motivate an employment of the latter to simplify computations. In this respect, we remark that a simple numerical computation of the equilibrium shape for $\vec{h} = (0, H)$ and H close to c_p^{-2} yields a discrepancy between nonlinear and linear theory of nearly 40%.

5. The Lagrange Multiplier Formulation

We recall the definition (1.15) of the constraint mapping:

$$\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u}\rangle = \sum_{i=1}^{n} \left\{ \int_0^1 \vartheta_i' u_i' - \int_0^1 \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) u_i \right\}. \tag{5.1}$$

Since $|\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u}\rangle| \le C(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta})\|\boldsymbol{u}\|$ for every $\boldsymbol{u} \in H^1_{0L}(I)^n$, $G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a linear bounded functional. Thus, (5.1) defines a map $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$. Thanks to the equivalence

$$(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A} \Leftrightarrow G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = 0,$$

we can write

$$\mathcal{A} \overset{(1.10)}{=} \{(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H} : G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = 0\}. \tag{5.2}$$

Proposition A.2 in Appendix A of this paper provides sufficient conditions for the existence of the Lagrange multiplier $\boldsymbol{\lambda}$. We are going to use this proposition as a tool to characterize the minimizers of $\mathcal{C}$ in $\mathcal{A}$. To this aim, we need to assess the regularity of the functional $\mathcal{C}$ and of the operator G; the next statement concerns their Fréchet differentiability, which we shall obtain as a consequence of Proposition A.1 and the following lemma.

Lemma 5.1. *Let $\varepsilon, \gamma \geq 0$. The operators $\mathcal{C} : \mathcal{H} \to \mathbb{R}$ and $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$ are C^1, with $D\mathcal{C} : \mathcal{H} \to \mathcal{H}'$ and $DG : \mathcal{H} \to \mathcal{L}(\mathcal{H}, (H^1_{0L}(I)^n)')$ being represented by*

$$D\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota}) = \gamma \sum_{i=1}^{n} \vec{h}_i \cdot \vec{k}_i + \varepsilon \int_0^1 \alpha'\beta' + \sum_{i=1}^{n} \int_0^1 (\vartheta_i - \bar{\vartheta}_i)\iota_i, \tag{5.3}$$

respectively,

$$\begin{aligned} &\langle DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota}), \boldsymbol{u}\rangle \\ &\quad = \sum_{i=1}^{n} \int_0^1 -\vec{k}_i \cdot D\vec{m}(\alpha + \vartheta_i) u_i \\ &\qquad + \sum_{i=1}^{n} \int_0^1 \left\{ \iota_i' u_i' - \vec{h}_i \cdot \big(D^2\vec{m}(\alpha + \vartheta_i)\iota_i + D^2\vec{m}(\alpha + \vartheta_i)\beta\big) u_i \right\}, \end{aligned} \tag{5.4}$$

for every $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), (\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota}) \in \mathcal{H}$ and $\boldsymbol{u} \in H^1_{0L}(I)^n$.

Proof. Fix $\boldsymbol{\varphi} := (\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$. We consider a sequence $\{\boldsymbol{\varphi}_k\} := \{(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k)\}$, with $\vec{\boldsymbol{h}}_k = (\vec{h}_{k1}, \ldots, \vec{h}_{kn})$, $\vec{h}_{ki} = (h_{kix}, h_{kiy})$ and $\boldsymbol{\vartheta}_k = (\vartheta_{k1}, \ldots, \vartheta_{kn})$ such that $\|\boldsymbol{\varphi}_k - \boldsymbol{\varphi}\|_{\mathcal{H}} \to 0$ as $k \to +\infty$. In particular, C exists such that

$$|\vec{\boldsymbol{h}}_k| \leq C. \tag{5.5}$$

First we focus on $\mathcal{C}$. We trivially have $\mathcal{C}(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k) \to \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ in $\mathbb{R}$, hence $\mathcal{C}$ is continuous. The Gâteaux derivative $\mathcal{C}'(\boldsymbol{\varphi})$ can be computed explicitly via Definition A.1, and it coincides with the right-hand side of (5.3). In order to show that $\mathcal{C}$ is C^1, we write (using the Cauchy–Schwarz inequality)

$$\begin{aligned} &|\left(\mathcal{C}'(\boldsymbol{\varphi}) - \mathcal{C}'(\boldsymbol{\varphi}_k)\right)(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota})| \\ &\quad \overset{(5.3)}{\leq} \left\{ \gamma \sum_{i=1}^{n} |\vec{h}_i - \vec{h}_{ki}| + \varepsilon\|\alpha - \alpha_k\| + \sum_{i=1}^{n} \|\vartheta_i - \vartheta_{ki}\|_2 \right\} \|(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota})\|_{\mathcal{H}}, \end{aligned}$$

hence

$$\|\mathcal{C}'(\boldsymbol{\varphi}) - \mathcal{C}'(\boldsymbol{\varphi}_k)\|_{\mathcal{H}'} \leq C\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_k\|_{\mathcal{H}}.$$

Thus, the Gâteaux derivative $\mathcal{C}'$ of $\mathcal{C}$ is (Lipschitz) continuous with respect to the operator norm and, by applying Proposition A.1, we conclude that $\mathcal{C}$ is Fréchet differentiable, that its differential is $D\mathcal{C} = \mathcal{C}'$ and that therefore $\mathcal{C}$ is C^1.

Now we focus our attention on G, by first proving that G is continuous. To this aim, we fix $\boldsymbol{u} = (u_1, \ldots, u_n) \in H^1_{0L}(I)^n$ and we compute

$$\begin{aligned} &\langle G(\boldsymbol{\varphi}) - G(\boldsymbol{\varphi}_k), \boldsymbol{u}\rangle \\ &\quad \overset{(5.1)}{=} \sum_{i=1}^{n} \int_0^1 (\vartheta_i - \vartheta_{ki})' u_i' - \sum_{i=1}^{n} \int_0^1 \big(\vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) - \vec{h}_{ki} \cdot D\vec{m}(\alpha_k + \vartheta_{ki})\big) u_i \\ &\quad \overset{(2.6b)}{\leq} \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_k\|\|\boldsymbol{u}\| + \sum_{i=1}^{n} \int_0^1 \big(|\vec{h}_i - \vec{h}_{ki}| + |\vec{h}_{ki}|\big(|\alpha - \alpha_k| + |\vartheta_i - \vartheta_{ki}|\big)\big)|u_i|. \end{aligned}$$

Then, by making use of Hölder and Poincaré inequalities, we deduce the inequality

$$|\langle G(\boldsymbol{\varphi}) - G(\boldsymbol{\varphi}_k), \boldsymbol{u}\rangle| \le C\big((1+|\vec{\boldsymbol{h}}_k|)\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\| + |\vec{\boldsymbol{h}}-\vec{\boldsymbol{h}}_k| + |\vec{\boldsymbol{h}}_k|\|\alpha-\alpha_k\|\big)\|\boldsymbol{u}\|,$$

whence

$$\begin{aligned}\|G(\boldsymbol{\varphi}) - G(\boldsymbol{\varphi}_k)\|_{(H^1_{0L}(I)^n)'} &\le C\big((1+|\vec{\boldsymbol{h}}_k|)\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\| + |\vec{\boldsymbol{h}}-\vec{\boldsymbol{h}}_k| + |\vec{\boldsymbol{h}}_k|\|\alpha-\alpha_k\|\big)\\ &\overset{(5.5)}{\le} C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_k\|_{\mathcal{H}} \to 0,\end{aligned}$$

hence G is (Lipschitz) continuous. The Gâteaux derivative $G'(\boldsymbol{\varphi}) : \mathcal{H} \to (H^1_{0L}(I)^n)'$ can be computed explicitly via its definition, and it coincides with the right-hand side of (5.4). Hence we deduce that

$$\begin{aligned}&\big\langle\big(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k)\big)(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota}),\boldsymbol{u}\big\rangle\\ &\quad= -\sum_{i=1}^n\int_0^1 \vec{k}_i\cdot(D\vec{m}(\alpha+\vartheta_i) - D\vec{m}(\alpha_k+\vartheta_{ki}))u_i\\ &\qquad -\sum_{i=1}^n\int_0^1 \big(\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i) - \vec{h}_{ki}\cdot D^2\vec{m}(\alpha_k+\vartheta_{ki})\big)(\beta+\iota_i)u_i\\ &\quad\overset{(2.6b)}{\le} \sum_{i=1}^n\int_0^1 |\vec{k}_i|(|\alpha-\alpha_k|+|\vartheta_i-\vartheta_{ki}|)|u_i|\\ &\qquad +\sum_{i=1}^n\int_0^1 \big(|\vec{h}_i-\vec{h}_{ki}| + |\vec{h}_{ki}|(|\alpha-\alpha_k|+|\vartheta_i-\vartheta_{ki}|)\big)|\beta+\iota_i||u_i|\\ &\quad\overset{(5.5)}{\le} C\big(\big(|\vec{\boldsymbol{h}}-\vec{\boldsymbol{h}}_k| + \|\alpha-\alpha_k\| + \|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\|\big)(|\vec{\boldsymbol{k}}| + \|\beta\| + \|\boldsymbol{\iota}\|)\big)\|\boldsymbol{u}\|,\end{aligned}$$

where in the last inequality we have also used Hölder and Poincaré inequalities. It follows that

$$\begin{aligned}&\|(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{(H^1_{0L}(I)^n)'}\\ &\qquad= \sup_{\|\boldsymbol{u}\|_{H^1_{0L}(I)^n}=1} |\langle(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota}),\boldsymbol{u}\rangle|\\ &\qquad\le C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_k\|_{\mathcal{H}}\|(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{\mathcal{H}},\end{aligned}$$

hence that

$$\begin{aligned}&\|G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k)\|_{\mathcal{L}(\mathcal{H},(H^1_{0L}(I)^n)')}\\ &\qquad= \sup_{\|(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{\mathcal{H}}=1} \|(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{(H^1_{0L}(I)^n)'}\\ &\qquad\le C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_k\|_{\mathcal{H}}.\end{aligned}$$

This implies, applying Proposition A.1, that $DG = G'$ and that G is C^1. □

We may now prove Theorem 1.2.

Proof of Theorem 1.2. Let $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})$ be a minimizer of $\mathcal{C}$ in $\mathcal{A}$. We need to prove that $DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})$ is surjective, that is, for every $T\in(H^1_{0L}(I)^n)'$ there exists $\boldsymbol{\psi}\in\mathcal{H}$ such that

$$\langle DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\boldsymbol{\psi}),\boldsymbol{u}\rangle=\langle T,\boldsymbol{u}\rangle\quad\text{for all }\boldsymbol{u}=(u_1,\dots,u_n)\in H^1_{0L}(I)^n. \tag{5.6}$$

It suffices to show that (5.6) has a solution $\boldsymbol{\psi}$ of the form $\boldsymbol{\psi}=(\vec{\boldsymbol{0}},0,\boldsymbol{v})$ for some $\boldsymbol{v}\in H^1_{0L}(I)^n$. In this case, the left-hand side of (5.6) defines a bilinear form, $a: H^1_{0L}(I)^n\times H^1_{0L}(I)^n\to\mathbb{R}$:

$$a(\boldsymbol{v},\boldsymbol{u}):=\langle DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\vec{\boldsymbol{0}},0,\boldsymbol{v}),\boldsymbol{u}\rangle\overset{(5.4)}{=}\sum_{i=1}^n\int_0^1 v_i'u_i'-\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)v_iu_i. \tag{5.7}$$

Applying Theorem A.2 to each component of DG (with $\delta=1$, $\tilde{L}=DG_i$ and $\tilde{r}=\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)$) we obtain (i) and (ii) of Theorem 1.2.

In order to prove (iii), assume that $DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})$ is surjective. In view of Lemma 5.1 and Proposition A.2, there exists a Lagrange multiplier $\boldsymbol{\lambda}\in H^1_{0L}(I)^n$ such that $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta},\boldsymbol{\lambda})$ satisfies

$$D\mathcal{C}(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\cdot)=\langle DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\cdot),\boldsymbol{\lambda}\rangle\quad\text{in }\mathcal{H}'. \tag{5.8}$$

It follows from (5.3) and (5.4) that (5.8) evaluated in $(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\in\mathcal{H}$ is equivalent to

$$\begin{aligned}\gamma\sum_{i=1}^n\vec{h}_i\cdot\vec{k}_i+\varepsilon\int_0^1\alpha'\beta'+\sum_{i=1}^n\int_0^1(\vartheta_i-\bar{\vartheta}_i)\iota_i&=\sum_{i=1}^n\int_0^1-\vec{k}_i\cdot\lambda_iD\vec{m}(\alpha+\vartheta_i)\\&+\sum_{i=1}^n\int_0^1-\lambda_i\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)\beta+\sum_{i=1}^n\int_0^1\left\{\lambda_i'\iota_i'-\lambda_i\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)\iota_i\right\}.\end{aligned} \tag{5.9}$$

Since $D\mathcal{C}$ and DG are linear with respect to $(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})$, (5.9) is equivalent to

$$\begin{cases}\displaystyle\int_0^1\left\{\lambda_i'\iota_i'-\lambda_i\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)\iota_i\right\}=\int_0^1(\vartheta_i-\bar{\vartheta}_i)\iota_i,\\ \displaystyle\varepsilon\int_0^1\alpha'\beta'+\sum_{i=1}^n\int_0^1\lambda_i\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)\beta=0, & \forall\, i=1,\dots,n.\\ \displaystyle\gamma\vec{k}_i\cdot\vec{h}_i=-\vec{k}_i\cdot\int_0^1\lambda_iD\vec{m}(\alpha+\vartheta_i),\end{cases} \tag{5.10}$$

Recalling Remark 4.1 and adding to (5.10) the constraint $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})\in\mathcal{A}$, we conclude that $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta},\boldsymbol{\lambda})$ is a solution to (1.13).

In order to deduce $\alpha'(0)=0$ if $\varepsilon>0$, we recall that $\vartheta_i,\alpha,\lambda_i\in C^1([0,1])$ (see Remark 4.1). Therefore, using (P_α), we obtain $\alpha\in C^2(I)$ and

$$\alpha'(1)-\alpha'(0)\;=\;\int_0^1\alpha''\overset{(P_\alpha)}{=}\frac{1}{\varepsilon}\sum_{i=1}^n\int_0^1\lambda_i\vec{h}_i\cdot D^2\vec{m}(\alpha+\vartheta_i)$$

$$\stackrel{(2.4)}{=} \frac{1}{\varepsilon}\sum_{i=1}^{n}\int_0^1 \vec{h}_i \cdot \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \lambda_i D\vec{m}(\alpha+\vartheta_i)$$

$$\stackrel{(P_{\vec{h}_i})}{=} -\frac{\gamma}{\varepsilon}\sum_{i=1}^{n}\vec{h}_i \cdot \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \vec{h}_i = 0.$$

Hence $\alpha'(1) = \alpha'(0) \stackrel{(P_\alpha)}{=} 0$. □

6. A Constructive Scheme; Uniqueness of Solutions to the Lagrange Multiplier Formulation

In this section, where we assume $\varepsilon, \gamma > 0$, we introduce a constructive scheme to obtain solutions of the Euler–Lagrange formulation (1.13). We will prove its contractivity and, as a by-product, uniqueness of solutions to (1.13) (Theorem 1.3). The scheme consists of two steps and works as follows.

Step 1. In the first step, we fix $\alpha \in C([0,1])$. We introduce the set

$$D := \left\{\vec{\boldsymbol{h}} \in \mathbb{R}^{2n} : \max_{1\le i\le n} |\vec{h}_i| \le K\right\}, \quad \text{with } K < c_p^{-2} \text{ (cf. (1.19))}. \tag{6.1}$$

We will show that the chain

$$\vec{\boldsymbol{h}} \stackrel{(P_{\vartheta_i})}{\mapsto} \boldsymbol{\vartheta} = (\vartheta_1,\dots,\vartheta_n) \stackrel{(P_{\lambda_i})}{\mapsto} \boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \stackrel{(P_{\vec{h}_i})}{\mapsto} \vec{\boldsymbol{T}}^{(\alpha)}(\vec{\boldsymbol{h}}) \in \mathbb{R}^{2n} \tag{6.2}$$

defines a map $\vec{\boldsymbol{T}}^{(\alpha)} : D \to D$. We then show that $\vec{\boldsymbol{T}}^{(\alpha)}$ is a contraction for γ sufficiently large. Then, by Proposition A.3, there exists a unique fixed point of $\vec{\boldsymbol{T}}^{(\alpha)}$ in D, $\vec{\boldsymbol{h}}(\alpha)$:

$$\vec{\boldsymbol{h}}(\alpha) = \vec{\boldsymbol{T}}^{(\alpha)}(\vec{\boldsymbol{h}}(\alpha)).$$

Step 2. In view of Step 1, we define $A : C([0,1]) \to C([0,1])$ as the unique function such that

$$\begin{cases} -A(\alpha)'' = -\dfrac{1}{\varepsilon}\displaystyle\sum_{i=1}^{n}\lambda_i(\alpha)\vec{h}_i(\alpha)\cdot D^2(\alpha+\vartheta_i(\alpha)) \quad \text{in } (0,1), \\ A(\alpha)|_0 = A(\alpha)'|_1 = 0, \end{cases} \tag{6.3}$$

where $\vartheta_i(\alpha), \lambda_i(\alpha)$ are the unique solutions to (P_{ϑ_i}), respectively, (P_{λ_i}), with $\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}(\alpha)$. We will prove that A is a contraction, hence it has a unique fixed point, for γ sufficiently large.

Thanks to (6.3) and to (6.2), a quadruplet $(\vec{\boldsymbol{h}}(\alpha), \alpha, \boldsymbol{\vartheta}(\alpha), \boldsymbol{\lambda}(\alpha))$ is a solution to System (1.13) if and only if α is a fixed point of A. In particular, this implies the uniqueness result in Theorem 1.3.

Remark 6.1. There are three key features of System (1.13) that allow us to show that the maps $\vec{\boldsymbol{T}}^{(\alpha)}$ and A are contractions. Namely:

- the boundary-value problems in (1.13) share the same structure, that of (4.1);
- all lowest-order terms on the left-hand sides of the differential equations in (1.13) are *proportional* to the norm $|\vec{h}_i|$ of the applied fields, which in turn are controlled (for a minimizer) by the target shapes $\overline{\boldsymbol{\vartheta}}$ and by the regularization constant γ through the bound (1.12);
- the equation for $\vec{h}_i$ in (1.13) contain on the right-hand side the pre-factor γ^{-1}. Accordingly, as long as γ is large, we can control the applied fields and hence also the solutions of (1.13).

We now prove the assertions formulated above.

Proposition 6.1. *Let D as in* (6.1)*. Then*:

(i) *For any $\alpha \in C([0,1])$ there exists γ_1 (depending on $\bar{\vartheta}$ and K) such that for any $\gamma > \gamma_1$ the map $\vec{\boldsymbol{T}}^{(\alpha)} : D \to \mathbb{R}^{2n}$ defined in* (6.2) *has a unique fixed point in D, $\vec{\boldsymbol{h}}(\alpha)$; in particular,*

$$\max_{1\le i\le n} |\vec{h}_i(\alpha)| \le K < c_p^{-2};$$

(ii) *there exists $\gamma_* > \gamma_1$ (depending on $\bar{\vartheta}$, K and ε) such that the map $A : C([0,1]) \to C([0,1])$ defined in* (6.3) *has a unique fixed point, $\alpha = A(\alpha)$. Furthermore, $\alpha \in C^2([0,1])$ and $\alpha(0) = \alpha'(1) = 0$;*

(iii) *consequently, Theorem* 1.3 *holds.*

Proof. We divide the proof into steps.

(A) There exists γ_0 such that $\vec{\boldsymbol{T}}^{(\alpha)}$ maps D in itself. Let $\vec{\boldsymbol{h}} \in D$. Thanks to the first equivalence in (4.2), (2.6b) and (1.19), we can apply Lemma 4.1 with $L = K$: for every $i = 1, \dots, n$ there exists a unique solution $\vartheta_i \in H^1_{0L}(I)$ of (P_{ϑ_i}), which satisfies

$$\|\vartheta_i\|_\infty \overset{(4.5)_2}{\le} |\vec{h}_i| \le K, \quad i = 1, \dots, n. \tag{6.4}$$

By the same argument, the second equivalence in (4.2) and (2.6c) allow to apply Lemma 4.1 with $L = K$ and $f_0 = \overline{\vartheta}_i - \vartheta_i$: for every $i = 1, \dots, n$ there exists a unique solution $\lambda_i \in H^1_{0L}(I)$ of (P_{λ_i}), such that

$$\begin{aligned} \|\lambda_i\|_\infty &\overset{(4.5)_1}{\le} \frac{c_p}{1 - Kc_p^2}\|\vartheta_i - \overline{\vartheta}_i\|_\infty \le \frac{c_p}{1 - Kc_p^2}\big(\|\vartheta_i\|_\infty + \|\overline{\vartheta}_i\|_\infty\big) \\ &\overset{(6.4)}{\le} \frac{c_p}{1 - Kc_p^2}\big(K + \|\overline{\vartheta}_i\|_\infty\big) = C, \end{aligned} \tag{6.5}$$

where from now on C denotes a generic constant depending on $\overline{\boldsymbol{\vartheta}}$ and K, but independent of γ and ε. Therefore,

$$|\vec{T}_i^{(\alpha)}(\vec{\boldsymbol{h}})| \overset{(6.2),(1.13)_4}{=} \frac{1}{\gamma}\left|\int_0^1 \lambda_i D\vec{m}(\alpha+\vartheta_i)\right| \overset{(2.5)}{\leq} \frac{2}{\gamma}\|\lambda_i\|_\infty \overset{(6.5)}{\leq} \frac{C}{\gamma}, \quad i=1,\dots,n. \tag{6.6}$$

This implies that, for $\gamma > \gamma_0$ large enough, the operator $\vec{\boldsymbol{T}}^{(\alpha)}$ maps D in itself.

For reasons which will be clarified later, we postpone the proof of (i), and for the moment we assume it to be true.

(B) Proof of (ii) assuming (i). Assume $\gamma > \gamma_1$, with γ_1 as given in (i). For α and $\tilde{\alpha}$ in $C([0,1])$, let $\vartheta_i = \vartheta_i(\alpha)$ and $\tilde{\vartheta}_i = \tilde{\vartheta}_i(\alpha)$, respectively, $\lambda_i = \lambda_i(\alpha)$ and $\tilde{\lambda}_i = \tilde{\lambda}_i(\alpha)$, be the unique solutions to (P_{ϑ_i}), respectively, (P_{λ_i}), with $\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}(\alpha)$ and $\vec{\tilde{\boldsymbol{h}}} = \vec{\tilde{\boldsymbol{h}}}(\tilde{\alpha})$ as defined in (i). It follows from (6.4) and (6.5) that

$$\|\vartheta_i\|_\infty + \|\lambda_i\|_\infty \leq C \quad \text{and} \quad \|\tilde{\vartheta}_i\|_\infty + \|\tilde{\lambda}_i\|_\infty \leq C, \quad i=1,\dots,n. \tag{6.7}$$

Let $A(\alpha)$ and $A(\tilde{\alpha})$ be defined by (6.3), and note that (6.3) is equivalent to

$$A(\alpha)(s) := -\frac{1}{\varepsilon}\sum_{i=1}^n \int_0^s \int_{s'}^1 \lambda_i(s'')\vec{h}_i \cdot D^2(\alpha(s'') + \vartheta_i(s''))ds''ds', \quad \forall\, s \in [0,1]; \tag{6.8}$$

in particular, $A(\alpha) \in C^2([0,1])$. Therefore,

$$\begin{aligned}
&\|A(\alpha) - A(\tilde{\alpha})\|_\infty \\
&\quad \overset{(6.8)}{\leq} \frac{1}{\varepsilon}\sum_{i=1}^n \|\lambda_i\vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i\vec{\tilde{h}}_i \cdot D^2\vec{m}(\tilde{\alpha}+\tilde{\vartheta}_i)\|_\infty \\
&\quad \overset{(2.6c)}{\leq} \frac{1}{\varepsilon}\sum_{i=1}^n \big(|\vec{h}_i|\|\lambda_i - \tilde{\lambda}_i\|_\infty + \|\tilde{\lambda}_i\|_\infty|\vec{h}_i - \vec{\tilde{h}}_i| \\
&\qquad + |\vec{\tilde{h}}_i|\|\tilde{\lambda}_i\|_\infty\big(\|\alpha-\tilde{\alpha}\|_\infty + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty\big)\big) \\
&\quad \overset{(6.1)}{\underset{(6.7)}{\leq}} \frac{C}{\varepsilon}\sum_{i=1}^n \big(|\vec{h}_i - \vec{\tilde{h}}_i| + \|\alpha-\tilde{\alpha}\|_\infty + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty + \|\lambda_i - \tilde{\lambda}_i\|_\infty\big).
\end{aligned} \tag{6.9}$$

Now we will estimate the right-hand side of (6.9). Taking $\lambda_i - \tilde{\lambda}_i$ as test function in the weak formulations for λ_i and $\tilde{\lambda}_i$ (cf. (4.2) and (4.3)) and subtracting the resulting equations, we obtain

$$\begin{aligned}
\int_0^1 |(\lambda_i - \tilde{\lambda}_i)'|^2 \;=\; &\int_0^1 \big(\lambda_i\vec{h} \cdot D^2\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i\vec{\tilde{h}}_i \cdot D^2\vec{m}(\tilde{\alpha}+\tilde{\vartheta}_i)\big)(\lambda_i - \tilde{\lambda}_i) \\
&+ \int_0^1 (\vartheta_i - \tilde{\vartheta}_i)(\lambda_i - \tilde{\lambda}_i)
\end{aligned}$$

$$\overset{(2.6c)}{\leq} \big(\|\tilde\lambda_i\|_\infty|\tilde{\vec h}_i|+1\big)\int_0^1 |\vartheta_i-\tilde\vartheta_i||\lambda_i-\tilde\lambda_i| + \|\tilde\lambda_i\|_\infty|\vec h_i-\tilde{\vec h}_i|\int_0^1|\lambda_i-\tilde\lambda_i| + \|\tilde\lambda_i\|_\infty|\tilde{\vec h}_i|\int_0^1|\alpha-\tilde\alpha||\lambda_i-\tilde\lambda_i| + |\vec h_i|\int_0^1|\lambda_i-\tilde\lambda_i|^2. \tag{6.10}$$

We estimate the last summand on the right-hand side of (6.10) using the Poincaré inequality and the definition of D:

$$|\vec h_i|\int_0^1|\lambda_i-\tilde\lambda_i|^2 \overset{(2.2),(6.1)}{\leq} Kc_p^2\int_0^1|(\lambda_i-\tilde\lambda_i)'|^2.$$

Absorbing this summand on the left-hand side of (6.10), we obtain:

$$\begin{aligned}\underbrace{(1-Kc_p^2)}_{>0 \text{ by } (1.19)}\int_0^1|(\lambda_i-\tilde\lambda_i)'|^2 &\leq \left(\int_0^1|\lambda_i-\tilde\lambda_i|\right)\Big(\big(\|\tilde\lambda_i\|_\infty|\tilde{\vec h}_i|+1\big)\|\vartheta_i-\tilde\vartheta_i\|_\infty \\ &\quad + \|\tilde\lambda_i\|_\infty|\vec h_i-\tilde{\vec h}_i| + \|\tilde\lambda_i\|_\infty|\tilde{\vec h}_i|\|\alpha-\tilde\alpha\|_\infty\Big) \\ &\overset{(6.1),(6.7)}{\leq} C\left(\int_0^1|\lambda_i-\tilde\lambda_i|\right)\big(|\vec h_i-\tilde{\vec h}_i|+\|\vartheta_i-\tilde\vartheta_i\|_\infty+\|\alpha-\tilde\alpha\|_\infty\big).\end{aligned}$$

Using Hölder and Poincaré inequalities, we deduce

$$\|\lambda_i-\tilde\lambda_i\|_\infty \overset{(2.1)}{\leq} \|\lambda_i-\tilde\lambda_i\| \leq C\big(|\vec h_i-\tilde{\vec h}_i|+\|\vartheta_i-\tilde\vartheta_i\|_\infty+\|\alpha-\tilde\alpha\|_\infty\big). \tag{6.11}$$

In order to estimate $\|\vartheta_i-\tilde\vartheta_i\|_\infty$, we follow the same line of argument. We choose $\vartheta_i-\tilde\vartheta_i$ as test function in the weak formulations for ϑ_i and for $\tilde\vartheta_i$ (cf. (4.3)): subtracting the resulting equations, we have

$$\begin{aligned}\int_0^1|(\vartheta_i-\tilde\vartheta_i)'|^2 &= \int_0^1\big(\vec h_i\cdot D\vec m(\alpha+\vartheta_i)-\tilde{\vec h}_i\cdot D\vec m(\tilde\alpha+\tilde\vartheta_i)\big)(\vartheta_i-\tilde\vartheta_i) \\ &\overset{(2.6b)}{\leq} \big(|\vec h_i-\tilde{\vec h}_i|+|\vec h_i|\|\alpha-\tilde\alpha\|_\infty\big)\int_0^1|\vartheta_i-\tilde\vartheta_i| + |\vec h_i|\int_0^1|\vartheta_i-\tilde\vartheta_i|^2.\end{aligned}$$

As above, the second summand may be absorbed on the left-hand side via Poincaré inequality and the assumption that $K<c_p^{-2}$, whereas the first one can be treated by Hölder and Poincaré inequality (the specific constant being irrelevant in this case). Altogether, we obtain

$$\|\vartheta_i-\tilde\vartheta_i\|_\infty \overset{(2.1)}{\leq} \|\vartheta_i-\tilde\vartheta_i\| \leq C\big(|\vec h_i-\tilde{\vec h}_i|+\|\alpha-\tilde\alpha\|_\infty\big). \tag{6.12}$$

Now we estimate $|\vec{h}_i - \vec{\tilde{h}}_i|$. By the definition of $T_i^{(\alpha)}$ we deduce that

$$\begin{aligned} |\vec{h}_i - \vec{\tilde{h}}_i| &\leq \frac{1}{\gamma}\|\lambda_i D\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i D\vec{m}(\tilde{\alpha}+\tilde{\vartheta}_i)\|_\infty \\ &\overset{(2.6a),(6.7)}{\leq} \frac{C}{\gamma}\big(\|\alpha-\tilde{\alpha}\|_\infty + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty + \|\lambda_i - \tilde{\lambda}_i\|_\infty\big). \end{aligned} \tag{6.13}$$

Inserting (6.11) and (6.12) in (6.13), we deduce that there exists C_2 such that

$$|\vec{h}_i - \vec{\tilde{h}}_i| \leq \frac{C_2}{\gamma}\big(\|\alpha-\tilde{\alpha}\|_\infty + |\vec{h}_i - \vec{\tilde{h}}_i|\big). \tag{6.14}$$

Taking $\gamma_2 = C_2$, we have $C_2/\gamma < 1$ for $\gamma > \gamma_2$, so that

$$|\vec{h}_i - \vec{\tilde{h}}_i| \leq \frac{C_2}{\gamma - C_2}\|\alpha-\tilde{\alpha}\|_\infty \tag{6.15}$$

for $\gamma > \gamma_2$. Using (6.15) in (6.12), we obtain

$$\|\vartheta_i - \tilde{\vartheta}_i\|_\infty \leq \frac{C}{\gamma - C_2}\|\alpha-\tilde{\alpha}\|_\infty. \tag{6.16}$$

In turn, using (6.15) and (6.16) in (6.11), we obtain

$$\|\lambda_i - \tilde{\lambda}_i\|_\infty \leq \frac{C}{\gamma - C_2}\|\alpha-\tilde{\alpha}\|_\infty. \tag{6.17}$$

Finally, inserting (6.15)–(6.17) into (6.9), we deduce that there exist C_3 such that

$$\|A(\alpha) - A(\tilde{\alpha})\|_\infty \leq \frac{1}{\varepsilon}\frac{C_3}{\gamma - C_2}\|\alpha-\tilde{\alpha}\|_\infty. \tag{6.18}$$

We now set $\gamma_3 = C_2 + C_3/\varepsilon > \gamma_2$, so that the prefactor in (6.18) is smaller than 1 for every $\gamma > \max\{\gamma_1, \gamma_3\} =: \gamma_*$. By Proposition A.3, for $\gamma > \gamma_*$ there exists a unique fixed point $\alpha \in C([0,1])$ of A.

We finally return to the proof of (i), which we postponed since its proof is simpler than that of (ii), in that we may use the same estimates as in (B) with $\alpha = \tilde{\alpha}$.

(C) Proof of (i). We prove that $\vec{\boldsymbol{T}}^{(\alpha)}$ is a contraction. Let $\gamma > \gamma_0$, as given in (A). Given $\vec{\boldsymbol{h}}, \vec{\tilde{\boldsymbol{h}}} \in D$, we define ϑ_i and $\tilde{\vartheta}_i$, respectively, λ_i and $\tilde{\lambda}_i$, as the corresponding unique solutions of (P_{ϑ_i}), respectively, (P_{λ_i}). Then, the same arguments of (B) may be applied with $\alpha = \tilde{\alpha}$, yielding

$$\|\lambda_i - \tilde{\lambda}_i\|_\infty \leq C\big(|\vec{h}_i - \vec{\tilde{h}}_i| + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty\big) \tag{6.19}$$

(cf. (6.11)) and

$$\|\vartheta_i - \tilde{\vartheta}_i\|_\infty \overset{(2.1)}{\leq} \|\vartheta_i - \tilde{\vartheta}_i\| \leq C|\vec{h}_i - \vec{\tilde{h}}_i| \tag{6.20}$$

(cf. (6.12)). Therefore,

$$\begin{aligned}&|\vec{T}_i^{(\alpha)}(\vec{\boldsymbol{h}}) - \vec{T}_i^{(\alpha)}(\vec{\tilde{\boldsymbol{h}}})|\\ &\overset{(6.2),(1.13)_4}{=} \frac{1}{\gamma}\left|\int_0^1 \left(\lambda_i D\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i D\vec{m}(\alpha+\tilde{\vartheta}_i)\right)\right|\\ &\overset{(2.6a),(6.5)}{\leq} \frac{C}{\gamma}\left\{\|\lambda_i - \tilde{\lambda}_i\|_\infty + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty\right\} \overset{(6.19),(6.20)}{\leq} \frac{C_1}{\gamma}|\vec{h}_i - \vec{\tilde{h}}_i|.\end{aligned}$$

Choosing $\gamma_1 = C_1$, we conclude that $\vec{\boldsymbol{T}}^{(\alpha)}$ is a contraction for every $\gamma > \gamma_1$.

(D) Proof of (iii). Theorem 1.3 is an immediate consequence of (i) and (ii). Indeed, let $\alpha = A(\alpha)$ be the fixed point of A identified in (ii), and let $\vec{\boldsymbol{h}}(\alpha) = \vec{\boldsymbol{T}}^{(\alpha)}(\vec{\boldsymbol{h}}(\alpha))$ be the fixed point identified in (i). Then, by construction, the quadruplet $(\vec{\boldsymbol{h}}(\alpha), \alpha, \boldsymbol{\vartheta}(\alpha), \boldsymbol{\lambda}(\alpha))$ is a solution to system (1.13) in the class (1.20). Viceversa, if two solutions of (1.13) exist in that class, then they are both fixed points of A, hence they coincide. □

Remark 6.2. Under the provision that $K < 1$, a bound similar to (6.19) might be obtained from the representation formula

$$\lambda_i - \tilde{\lambda}_i = \int_0^s \int_{s'}^1 \left(\lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i \vec{\tilde{h}}_i \cdot D^2\vec{m}(\alpha+\tilde{\vartheta}_i)\right) ds'' ds'.$$

Indeed, from

$$\|\lambda_i - \tilde{\lambda}_i\|_\infty \overset{(2.6c)}{\leq} \|\tilde{\lambda}_i\|_\infty |\vec{\tilde{h}}_i| \|\vartheta_i - \tilde{\vartheta}_i\|_\infty + \|\tilde{\lambda}_i\|_\infty |\vec{h}_i - \vec{\tilde{h}}_i| + |\vec{h}_i| \|\lambda_i - \tilde{\lambda}_i\|_\infty,$$

we obtain, for $K < 1$,

$$\begin{aligned}\|\lambda_i - \tilde{\lambda}_i\|_\infty &\overset{(6.5),(6.1)}{\leq} \frac{1}{1-K}\left(KC(\overline{\boldsymbol{\vartheta}}, K)\|\vartheta_i - \tilde{\vartheta}_i\|_\infty + C(\overline{\boldsymbol{\vartheta}}, K)|\vec{h}_i - \vec{\tilde{h}}_i|\right)\\ &\leq C(\overline{\boldsymbol{\vartheta}}, K)\left(\|\vartheta_i - \tilde{\vartheta}_i\|_\infty + |\vec{h}_i - \vec{\tilde{h}}_i|\right).\end{aligned}$$

However, the requirement $K < 1$ is stricter than our assumption (1.19) because $c_p < 1$.

Remark 6.3. Note that $\gamma_{**} = \max(\overline{\Theta}^2/K^2, \gamma_*)$ blows up both as K tends to 0 and as K tends to c_p^{-2}. The blow up for K small is obvious: as K tends to 0 the maximum allowed applied field tends to 0 in intensity, and to limit the applied field we need γ large. The blow up as $K \to c_p^{-2}$ is of a technical nature and follows from the blow up of γ_*. In turn, the blow up of γ_* follows from the estimates in the proof of Theorem 1.3, which become degenerate as K tends to c_p^{-2} (see e.g. (6.5) and (6.6)). Ultimately, this is because our estimates rely only on $(4.5)_1$ in Lemma 4.1, which becomes degenerate when the Lipschitz constant L (identified with the intensity of the magnetic field) approaches c_p^{-2}.

Let us conclude the section with a digression on the case in which α is fixed. Part (i) of Proposition 6.1 may be rephrased as follows.

Proposition 6.2. *Let $\overline{\boldsymbol{\vartheta}} \in C([0,1])^n$ and let $K > 0$ such that* (1.19) *holds. Then there exists $\gamma_* = \gamma_*(\overline{\boldsymbol{\vartheta}}, K)$ such that for every $\gamma > \gamma_*$ and any $\alpha \in C([0,1])$ there exists a unique solution of the system*

$$\begin{cases} -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0, & \vartheta_i(0) = \vartheta_i'(1) = 0, \\ -\lambda_i'' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = \vartheta_i - \bar{\vartheta}_i, & \lambda_i(0) = \lambda_i'(1) = 0, \qquad i = 1, \dots, n, \\ \vec{h}_i = -\dfrac{1}{\gamma} \displaystyle\int_0^1 \lambda_i D\vec{m}(\alpha + \vartheta_i), \end{cases} \tag{6.21}$$

within the following set: $(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in D \times H^1_{0L}(I)^n \times H^1_{0L}(I)^n$.

Remark 6.4. For fixed α, the solution in Proposition 6.2 is the unique stationary point of the functional

$$\tilde{\mathcal{C}}(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) = \frac{1}{2} \sum_{i=1}^n \int_0^1 |\vartheta_i - \bar{\vartheta}_i|^2 + \frac{\gamma}{2} \sum_{i=1}^n |\vec{h}_i|^2, \tag{6.22}$$

in the admissible set

$$(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) \in \tilde{\mathcal{A}} := \big\{ (\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) \in D \times H^1_{0L}(I)^n : \ \vartheta_i \text{ solves } (P_{\vartheta_i}) \text{ for every } i = 1, \dots, n \big\}.$$

Therefore, arguing as we did for the full problem, for γ sufficiently large it follows from Proposition 6.2 that:

- $\tilde{\mathcal{C}}$ has a unique minimizer;
- looking for the minimum $(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta})$ of $\tilde{\mathcal{C}}$ is equivalent to looking for the fixed point of $\vec{\boldsymbol{T}}^{(\alpha)}$.

7. Concluding Remarks

We have considered a beam clamped at one side, modeled as a planar elastica. The beam has a permanent magnetization (the design α), hence it deforms under the action of spatially-constant magnetic fields $\vec{h}_i$, $i = 1, \dots, n$ (the controls). Given a list of n prescribed target shapes $\overline{\vartheta}_i$ ($i = 1, \dots, n$), we have looked for optimal design and controls in order for the corresponding shapes ϑ_i ($i = 1, \dots, n$) of the beam to get as close as possible to the corresponding targets. Choosing the cost functional as in (1.7) has lead us to the formulation of an optimal design-control problem (cf. (1.11)), whose minimization has been studied by both direct and indirect methods. Loosely speaking, we have shown that:

- minimizers $(\vec{\boldsymbol{h}}, \alpha)$ exist (Theorem 1.1);
- provided the intensity of $\vec{\boldsymbol{h}}$ is sufficiently small (cf. (1.17)), and otherwise in "generic" cases (see Theorem 1.2 and the comments below it), minimizers solve the Lagrange multiplier formulation (1.13);

- if the parameter γ penalizing the cost of the fields' intensity is sufficiently large, the minimizer is unique, satisfies (1.17), and is the unique solution to the Lagrange multiplier formulation (1.13) (Theorems 1.3 and 1.4).

In what follows, we briefly discuss a numerical scheme which naturally emerges from the proof of Theorem 1.3, as well as a different choice of the cost functional, using residuals. We also point out two possible generalizations of our choice of the cost.

Remark 7.1. (The numerical scheme) The proof of Theorem 1.3 suggests an alternative to the numerical scheme proposed in Ref. 8. The new scheme is based on two nested loops. In the inner loop, α is fixed and $\vec{\boldsymbol{h}}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\vartheta}$ are computed by a fixed point iteration scheme which uses, in the order, equations (P_{ϑ_i}), (P_{λ_i}) and $(P_{\vec{h}_i})$; in the outer loop, α is updated by using the equation (P_α) with $\vec{\boldsymbol{h}}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\vartheta}$ obtained from the inner loop. Each loop terminates when the update of each variable results in an increment below a certain tolerance *tol*. The algorithm is described in the pseudocode below. Note that, in this algorithm, steps to be performed for $i = 1, \ldots, n$ do not need to be carried out sequentially, but can also be done in parallel, since they are independent on each other.

Initialization:
$\alpha \leftarrow$ initial guess $\alpha^{(0)}$;
$\vec{h}_i \leftarrow$ initial guess $\vec{h}_i^{(0)}$, $i = 1, \ldots, n$;
$\lambda_i \leftarrow$ initial guess $\lambda_i^{(0)}$, $i = 1, \ldots, n$;
tol $\leftarrow$ tolerance;

repeat
  repeat
    $\vartheta_i \leftarrow$ solve (P_{ϑ_i}), $i = 1, \ldots, n$;
    $\lambda_i \leftarrow$ solve (P_{λ_i}), $i = 1, \ldots, n$;
    $\vec{h}_i^{\text{old}} \leftarrow \vec{h}_i$;
    $\vec{h}_i \leftarrow$ solve $(P_{\vec{h}_i})$, $i = 1, \ldots, n$;
  until $\max_{i=1}^n |\vec{h}_i^{\text{old}} - \vec{h}_i| \leq$ tol;
  $\alpha^{\text{old}} \leftarrow \alpha$;
  $\alpha \leftarrow$ solve (P_α);
until $\|\alpha^{\text{old}} - \alpha\|_\infty \leq$ tol;

Remark 7.2. (Using residuals to assess shape attainment) Shape programming has been addressed in Ref. 21 under slightly more general conditions than those considered in this paper. In particular, Ref. 21 allows the magnetization intensity to be non-constant and the magnetic field to be non-uniform, and assigns a different weight to each shape. Within our framework (constant magnetic intensity, uniform applied field, and same weight for all shapes), the approach proposed in Ref. 21 would lead to the minimization of the following functional:

$$\widetilde{E}(\vec{\boldsymbol{h}}, \alpha) = \sum_{i=1}^n \int_0^1 \left| -\overline{\vartheta}_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \overline{\vartheta}_i) \right|^2. \tag{7.1}$$

The integrands in (7.1) represent *residuals*, in the sense that they vanish on attainable targets. Such minimization would be carried out in the space of designs α

whose first k Fourier coefficients are in a bounded set and control fields $\vec{\boldsymbol{h}}$ whose magnitude does not exceed a constant K. It would be useful to have estimates of the *attainment error*:

$$E(\vec{\boldsymbol{h}}, \alpha) = \frac{1}{2} \sum_{i=1}^{n} \int_0^1 |\overline{\vartheta}_i - \Theta_\alpha(\vec{h}_i)|^2$$

(cf. (1.6)) for solutions of both the optimization problem considered in Ref. 21 and the problem considered in this paper. In this respect, a first problem to be solved would be obtaining a bound of $E(\vec{\boldsymbol{h}}, \alpha)$ in terms of $\widetilde{E}(\vec{\boldsymbol{h}}, \alpha)$, where $(\vec{\boldsymbol{h}}, \alpha)$ is a minimizer of (7.1).

Remark 7.3. (Variable intensity of the magnetization) Further developments of the present work may include a variable intensity of the magnetization. In this case, if we let $\mu(s)M_0$ be the magnetization density in the undeformed configuration, then the energy functional (1.3) would be replaced by

$$\widetilde{\mathcal{E}}(\vartheta) = \int_0^1 \frac{1}{2}(\vartheta')^2 - \mu\vec{h} \cdot \vec{m}(\vartheta + \alpha).$$

Such modification would also require a regularization to limit the oscillations of μ, as well as a penalization of negative values. Instead of choosing μ and α as design variables for the magnetization, one might choose the vector $\vec{\mu} = \mu\vec{m}(\alpha)$. In terms of this vector, the energy would take the form

$$\widetilde{\mathcal{E}}(\vartheta) = \int_0^1 \frac{1}{2}(\vartheta')^2 - \vec{h} \cdot \mathbf{R}(\vartheta)\vec{\mu},$$

where $\mathbf{R}(v) = \left(\begin{smallmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{smallmatrix}\right)$ is the counterclockwise rotation of the angle v. Such extension should be accompanied by a penalization of the oscillation of the vector field $\vec{\mu}$.

Remark 7.4. (Non-quadratic costs) A nontrivial generalization of the present work consists in considering more general costs, of the form

$$\sum_{i=1}^{n} \|\overline{\vartheta}_i - \vartheta\|_{L^2}^2 + e(\alpha, \alpha') + g(\vec{\boldsymbol{h}}). \tag{7.2}$$

Of particular interest might be obstacle-type penalization. Such more general situation would likely require different techniques, with respect to those used in this paper.

Appendix A

Review of calculus in Banach spaces

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces, $\mathcal{U}$ an open subset of $\mathcal{X}$. We shall also consider a generic map $F : \mathcal{U} \to \mathcal{Y}$.

Definition A.1. F has Gâteaux derivative $F'(x_0)$ at the point $x_0 \in \mathcal{U}$ if there exist

$$F'(x_0)(v) := \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}, \quad \forall\, v \in \mathcal{X}.$$

Definition A.2. F is called Fréchet differentiable at $x_0 \in \mathcal{U}$ if there exists $DF(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that

$$\lim_{\|h\|_{\mathcal{X}} \to 0} \frac{\|F(x_0 + h) - F(x) - DF(x_0)(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

Moreover we give the notion of continuous differentiable operator. Let T belong to $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. We recall that the operator norm is defined by

$$\|T\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup_{\{0 \neq x \in \mathcal{X}\}} \frac{\|T(x)\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = \sup_{\|x\|_{\mathcal{X}} = 1} \|T(x)\|_{\mathcal{Y}}.$$

Definition A.3. We say that F is C^1 if $DF(x)$ exists for every $x \in \mathcal{U}$ and $DF : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a continuous operator.

We recall the following proposition linking Gâteaux derivability and Fréchet differentiability.

Proposition A.1. *If F admits Gâteaux derivative $F'(x)$ in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of x_0 and $F' : \mathcal{V} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is continuous at x_0, then F is Fréchet differentiable at x_0 and $DF(x_0) = F'(x_0)$. Moreover if $F' : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a continuous operator, then $DF = F'$ and F is C^1.*

Proof. See Ref. 32, p. 274. □

We denote by $\mathcal{X}'$ the dual space of $\mathcal{X}$ and by $\langle \cdot, \cdot \rangle : \mathcal{X}' \times \mathcal{X} \to \mathbb{R}$ the duality pairing defined as $\langle S, t \rangle = S(t)$, for every $t \in \mathcal{X}, S \in \mathcal{X}'$.

Proposition A.2. (*Existence of a Lagrange multiplier*: *Ref.* 32, *p.* 270) *Let $f : \mathcal{U} \subset \mathcal{X} \to \mathbb{R}$ and $G : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y}$ be C^1 on an open neighborhood $\mathcal{U}$ of $\tilde{x}$. Suppose that $\tilde{x}$ is an extremum of f on the set $\{x \in \mathcal{U} : G(x) = 0\}$ and that*

$$DG(\tilde{x}) : \mathcal{X} \to \mathcal{Y} \quad \textit{is a surjective linear operator.}$$

Then there exists $\lambda \in \mathcal{Y}'$, a Lagrange multiplier, *such that $Df(\tilde{x}) - \langle \lambda, DG(\tilde{x}) \rangle = 0$.*

Proposition A.3. (*Contraction Theorem*, *Ref.* 31) *Let $T : \mathcal{X} \to \mathcal{X}$. If $L \in (0, 1)$ exists such that*

$$\|T(x) - T(y)\|_{\mathcal{X}} \leq L\|x - y\|_{\mathcal{X}} \quad \textit{for all } x, y \in \mathcal{X},$$

then T admits a unique fixed point $x^* \in \mathcal{X}$ (i.e. $T(x^*) = x^*$).

Surjectivity of Sturm–Liouville operators

We prove a result on surjectivity of Sturm–Liouville operators onto dual spaces, which is crucial in the proof of Theorem 1.2 and for which we could not find a reference. For $\mu \in \mathbb{R}$, let $a_\mu : H^1_{0L}(I) \times H^1_{0L}(I)$ be the bilinear symmetric form defined by

$$a_\mu(u,v) = \int_0^1 u'v' + \int_0^1 quv - \mu \int_0^1 ruv, \tag{A.1}$$

where $q, r : I \to \mathbb{R}$ with $r > 0$ in I and $q, r \in L^1(I)$. Let L_μ be the associated linear operator:

$$L_\mu : H^1_{0L}(I) \to H^1_{0L}(I)', \quad L_\mu(u)(v) = a_\mu(u,v), \quad \forall\, v \in H^1_{0L}(I). \tag{A.2}$$

We note that $L_\mu(u) = 0$ in $H^1_{0L}(I)'$ if and only if u is a weak solution to

$$\begin{cases} -u'' + qu = \mu r u & \text{in } (0,1), \\ u(0) = 0, \quad u'(1) = 0. \end{cases} \tag{A.3}$$

We introduce the weighted scalar product $(u,v)_r = \int_0^1 ruv$, with corresponding norm $\|u\|_r^2 = (u,u)_r$, and define $L^2(I,r) := \{u \in L^1(I) : \|u\|_r < +\infty\}$. Following the standard nomenclature in Sturm–Liouville theory, we say the following.

Definition A.4. $\mu \in \mathbb{R}$ is an *eigenvalue* of L_μ if there exists a nonzero function $\varphi \in H^1_{0L}(I)$, called *eigenfunction*, such that φ is a solution of $L_\mu(\varphi) = 0$.

The eigenvalues of L_μ are characterized as follows.

Theorem A.1. (*Theorem* 4.6.2 *of Ref.* 33) *Let L_μ be as in* (A.1) *and* (A.2). *Then*

(i) *all eigenvalues of L_μ are real and simple*;
(ii) *the eigenvalues of L_μ are an infinite and countable set, $\{\mu_k : k \in \mathbb{N}\}$, bounded from below*: $-\infty < \mu_1 < \mu_2 < \mu_3 < \cdots$ *and $\mu_k \to \infty$ as $k \to \infty$*;
(iii) *the sequence of eigenfunction $\{\varphi_k\}$ is complete in $L^2(I,r)$ and can be normalized to be an orthonormal sequence in $L^2(I,r)$.*

Remark A.5. It is readily checked that:

(i) If $r \in L^\infty(I)$ and $\inf r > 0$, then $L^2(I,r) = L^2(I)$;
(ii) If $q \geq 0$, then $\mu_k > 0$ for every $k \in \mathbb{N}$.

Lemma A.1. *Let L_μ be as in* (A.1) *and* (A.2) *and let $q \geq 0$ and $r \in L^\infty(I)$ be such that* $\inf r > 0$. *Then for every $k \in \mathbb{N}$*

$$\mu_k = \min\left\{ J(v) := \int_0^1 ((v')^2 + qv^2) : v \in E^\perp_{k-1},\ \|v\|_r = 1 \right\}, \tag{A.4}$$

where $E^\perp_0 = H^1_{0L}(I)$ and $E^\perp_k = \{v \in H^1_{0L}(I) : (\varphi_i, v)_r = 0, \quad \forall\, i = 1, \ldots, k\}$ for $k \geq 1$.

Proof. Let $m_k = \inf\{J(v) : \ v \in E_{k-1}^{\perp}, \ \|v\|_r = 1\}$. Since J is weakly lower semicontinuous and coercive and $\{v \in E_{k-1}^{\perp} : \|v\|_r = 1\}$ is weakly closed in $H^1_{0L}(I)$, there exists a minimizer $u_k \in E_{k-1}^{\perp}$, $m_k = J(u_k)$ and $\|u_k\|_r = 1$. Furthermore, u_k solves

$$\int_0^1 u_k' v' + \int_0^1 q u_k v = m_k \int_0^1 r u_k v, \quad \forall\, v \in E_{k-1}^{\perp}. \tag{A.5}$$

We now prove that $m_k = \mu_k$. In view of Remark A.5(i), we can decompose any $v \in H^1_{0L}(I)$ as $v = \sum_{i=1}^{k-1} v_i \varphi_i + w$ with $w \in E_{k-1}^{\perp}$. Noting that

$$\int_0^1 u_k' \varphi_i' + \int_0^1 q u_k \varphi_i = \mu_i \int_0^1 r u_k \varphi_i \overset{u_k \in E_{k-1}^{\perp}}{=} 0 \quad \text{for } i = 1, \ldots, k-1,$$

we deduce that (A.5) holds for any $v \in H^1_{0L}(I)$. This implies that m_k is an eigenvalue, and, since $u_k \in E_{k-1}^{\perp}$, $m_k \geq \mu_k$. On the other hand $m_k = J(u_k) \leq J(\varphi_k) = \mu_k$, hence we conclude that $m_k = \mu_k$. □

Proposition A.4. *Let L_μ be as in* (A.1) *and* (A.2). *Let $q \geq 0$ and $r \in L^\infty(I)$ be such that* $\inf r > 0$. *If μ is not an eigenvalue of L_μ, then L_μ is surjective.*

Proof. Our goal is to prove that for every $T \in H^1_{0L}(I)'$ there exists $u \in H^1_{0L}(I)$ such that $L_\mu(u)(v) = \langle T, v\rangle$ for all $v \in H^1_{0L}(I)$. Since $\mu \notin \{\mu_k\}$, we may distinguish two cases: (a) $\mu < \mu_1$; (b) $\mu_k < \mu < \mu_{k+1}$ for some $k \geq 1$. We first analyze case (b).

Case (b). Thanks to Theorem A.1(iii) and Remark A.5(i), if $u, v \in H^1_{0L}(I)$ then there exist coefficients $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ such that

$$u = \sum_{i=1}^k u_i \varphi_i + \overline{u} \quad \text{and} \quad v = \sum_{i=1}^k v_i \varphi_i + \overline{v}, \tag{A.6}$$

where $\overline{u}, \overline{v} \in E_k^{\perp}$. Using the bilinearity of a, we have

$$\begin{aligned} L_\mu(u)(v) \overset{(A.6)}{=} & \sum_{i,j=1}^k u_i v_j L_\mu(\varphi_i)(\varphi_j) + \sum_{i=1}^k u_i L_\mu(\varphi_i)(\overline{v}) \\ & + \sum_{j=1}^k v_j L_\mu(\overline{u})(\varphi_j) + L_\mu(\overline{u})(\overline{v}). \end{aligned} \tag{A.7}$$

Since φ_i is an eigenfunction, we deduce that

$$L_\mu(\varphi_i)(\overline{v}) \overset{(A.2)}{=} \int_0^1 \varphi_i' \overline{v}' + \int_0^1 q \varphi_i \overline{v} - \mu \int_0^1 r \varphi_i \overline{v} = (\mu_i - \mu) \int_0^1 r \varphi_i \overline{v} \overset{\overline{v} \in E_k^{\perp}}{=} 0$$

for $i = 1, \ldots, k$. Analogously, $L_\mu(\overline{u})(\varphi_j) = 0$ for $j = 1, \ldots, k$. Hence (A.7) turns into

$$L_\mu(u)(v) = \sum_{i,j=1}^k u_i v_j L_\mu(\varphi_i)(\varphi_j) + L_\mu(\overline{u})(\overline{v}). \tag{A.8}$$

By definition of L_μ and using first that φ_i is an eigenfunction with eigenvalue μ_i and then Theorem A.1(iii), we have

$$\begin{aligned} L_\mu(\varphi_i)(\varphi_j) &= \int_0^1 \varphi_i'\varphi_j' + \int_0^1 q\varphi_i\varphi_j - \mu\int_0^1 r\varphi_i\varphi_j \\ &= (\mu_i - \mu)\int_0^1 r\varphi_i\varphi_j \\ &= \begin{cases} (\mu_i - \mu) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{A.9}$$

It follows from (A.8) and (A.9) that L_μ is surjective if and only for every $T \in H^1_{0L}(I)'$ there exist coefficients $u_1, \ldots, u_k$ and $\overline{u} \in E_k^\perp$ such that

$$\sum_{i=1}^k u_i v_i(\mu_i - \mu) + L_\mu(\overline{u})(\overline{v}) = \langle T, v\rangle, \quad \forall\, v \in H^1_{0L}(I). \tag{A.10}$$

The coefficients $u_1, \ldots, u_k$ are readily identified: choosing $v = \varphi_i$ in (A.10), since $L_\mu(\overline{u})(\varphi_i) = 0$ and $\mu \notin \{\mu_i\}$ we deduce

$$u_i = \frac{1}{\mu_i - \mu}\langle T, \varphi_i\rangle, \quad \forall\, i = 1, \ldots, k. \tag{A.11}$$

Assume for a moment that

$$\exists\, \overline{u} \in E_k^\perp : L_\mu(\overline{u})(\overline{v}) \overset{(A.2)}{=} a_\mu(\overline{u}, \overline{v}) = \langle T, \overline{v}\rangle, \quad \forall\, \overline{v} \in E_k^\perp. \tag{A.12}$$

Then u defined as in (A.6) satisfies (A.10). Indeed, plugging (A.11) and (A.12) into the left-hand side of (A.10) we obtain

$$\begin{aligned} L_\mu(u)(v) &= \sum_{i=1}^k v_i\langle T, \varphi_i\rangle + \langle T, \overline{v}\rangle \\ &= \left\langle T, \sum_{i=1}^k v_i\varphi_i + \overline{v} \right\rangle = \langle T, v\rangle. \end{aligned}$$

Therefore, it remains to prove (A.12).

Proof of (A.12)**.** We note that $E_k^\perp \subset H^1_{0L}(I)$ is a Hilbert space. Therefore, $T \in H^1_{0L}(I)' \subset (E_k^\perp)'$. Hence (A.12) follows from Lax–Milgram theorem, once we show that the bilinear form a, restricted to $E_k^\perp \times E_k^\perp$, is bounded and coercive. Applying Hölder inequality, (2.1) and (2.2), we have

$$\begin{aligned} |a(\overline{u}, \overline{v})| &= \left|\int_0^1 \overline{u}'\overline{v}' + \int_0^1 q\overline{u}\overline{v} - \mu\int_0^1 r\overline{u}\overline{v}\right| \\ &\le \|\overline{u}\|\|\overline{v}\| + \|q\|_{L^1(I)}\|\overline{u}\|_\infty\|\overline{v}\|_\infty + \mu c_p^2\|r\|_\infty\|\overline{u}\|\|\overline{v}\| \le C\|\overline{u}\|\|\overline{v}\|, \end{aligned}$$

hence a is bounded. In order to prove that a is coercive, we recall that by (ii) of Remark A.5, $0 < \mu_k < \mu < \mu_{k+1}$, and we estimate

$$a(\overline{u},\overline{u}) = \int_0^1 \left((\overline{u}')^2 + q\overline{u}^2 - \mu r\overline{u}^2\right)$$
$$= \left(1 - \frac{\mu}{\mu_{k+1}}\right)\int_0^1 \left((\overline{u}')^2 + q\overline{u}^2\right) + \frac{\mu}{\mu_{k+1}}\int_0^1 \left((\overline{u}')^2 + q\overline{u}^2 - \mu_{k+1} r\overline{u}^2\right). \tag{A.13}$$

Since $\overline{u} \in E_k^{\perp}$, (A.4) implies that the second summand on the right-hand side of (A.13) is nonnegative; hence

$$a(\overline{u},\overline{u}) \geq \left(1 - \frac{\mu}{\mu_{k+1}}\right)\left(\int_0^1 (\overline{u}')^2 + \int_0^1 q\overline{u}^2\right) \overset{q\geq 0}{\geq} \left(1 - \frac{\mu}{\mu_{k+1}}\right)\|\overline{u}\|^2_{E_k^{\perp}}.$$

Since $\mu < \mu_{k+1}$, a is coercive.

Case (a). The result follows by arguing in the same way used in the proof of (A.12) in case (b), choosing $k = 0$ and recalling that $E_0^{\perp} = H^1_{0L}(I)$. □

In the body of this manuscript, we deal with linear operators $\tilde{L} : H^1_{0L}(I) \to H^1_{0L}(I)'$ defined as

$$\tilde{L}(u)(v) = \int_0^1 u'v' - \int_0^1 \tilde{r}uv, \quad \forall\, v \in H^1_{0L}(I), \tag{A.14}$$

with $\tilde{r} \in L^\infty(I)$ which may not be positive. However, it turns out that Proposition A.4 may be applied. Indeed, we have $\tilde{L} = \tilde{L}_1$, where

$$\tilde{L}_\mu(u)(v) = \int_0^1 u'v' + \int_0^1 (\tilde{r}^- + \delta)uv - \mu\int_0^1 (\tilde{r}^+ + \delta)uv, \quad \forall\, v \in H^1_{0L}(I), \tag{A.15}$$

with $r^- = \max(-r, 0)$, $r^+ = \max(r, 0)$ and $\delta > 0$ an arbitrary constant.

Theorem A.2. *Let $\tilde{L}$ as in* (A.14) *and let $\tilde{r} \in L^\infty(I)$.*

(i) *If* 1 *is not an eigenvalue of $\tilde{L}_\mu$ in the sense of Definition* A.4, *then $\tilde{L}$ is surjective.*
(ii) *If $\|\tilde{r}^+\|_\infty < c_p^{-2}$, then $\tilde{L}$ is surjective.*

Proof. The proof of (i) is an immediate consequence of Proposition A.4, since $\tilde{L}_1 = \tilde{L}$. In order to prove (ii), we choose $\delta < c_p^{-2} - \|\tilde{r}^+\|_\infty$ in the definition (A.15) of $\tilde{L}_\mu$. Let φ_1 be an eigenfunction of μ_1, normalized so that $\int_0^1 (\tilde{r}^+ + \delta)\varphi_1^2 = 1$. Then

$$\mu_1 = \frac{\displaystyle\int_0^1 \left((\varphi_1')^2 + (\tilde{r}^- + \delta)(\varphi_1^2)\right)}{\displaystyle\int_0^1 (\tilde{r}^+ + \delta)\varphi_1^2} \geq \frac{\displaystyle\int_0^1 (\varphi_1')^2}{(\|\tilde{r}^+\|_\infty + \delta)\displaystyle\int_0^1 \varphi_1^2} \geq \frac{c_p^{-2}}{(\|\tilde{r}^+\|_\infty + \delta)} > 1,$$

hence 1 is not an eigenvalue. □

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