Environmental degradation and indeterminacy of equilibrium selection

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June 16, 2021

Abstract

This paper analyzes an intertemporal optimization problem in which agents derive utility from three goods: leisure, a public environmental good and the consumption of a produced good. The global analysis of the dynamic system generated by the optimization problem shows that global indeterminacy may arise: given the initial values of the state variables, the economy may converge to different steady states, by choosing different initial values of the control variable.

Keywords: economic growth model; environmental depletion; global analysis; global indeterminacy; poverty trap.

1 Introduction

In this paper we analyze global dynamics of the system proposed in Antoci et al. (2020), where only local stability analysis of steady states was developed, while other dynamic features were suggested via numerical simulations.

Antoci et al. (2020) consider an economy constituted by a continuum of identical economic agents, which have to solve an intertemporal optimization problem where the state variables are the stock of physical capital K accumulated by each agent and the stock E of a free access renewable environmental resource. The control variables are agents' labour input L and consumption C of a produced good. At each instant of time $t \in [0, \infty)$, each agent produces the output Y by the following production function:

$$Y = K^{\alpha} L^{1-\alpha} \tag{1}$$

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where $1 > \alpha > 0$. In each instant of time, economic agents' utility is measured by the function:

$$U(C, L, E) = \frac{\left[CE^{\beta}(1-L)^{\gamma}\right]^{1-\delta} - 1}{1-\delta} \tag{2}$$

where $1 - L(t)$ represents leisure and parameters satisfy the conditions: β, γ, $\delta > 0$, $\delta \neq 1$. This function is jointly concave in C and $1 - L$ if $\delta > \frac{\gamma}{1+\gamma}$. Each agent solves the following maximization problem:

$$\underset{C,\, L}{MAX} \int_0^{\infty} \frac{\left[CE^{\beta}(1-L)^{\gamma}\right]^{1-\delta} - 1}{1-\delta} e^{-\rho t} dt \tag{3}$$

subject to the constraints:

$$\dot{K} = K^{\alpha} L^{1-\alpha} - C \tag{4}$$

$$\dot{E} = E(\overline{E} - E) - \varepsilon \overline{Y} \tag{5}$$

with $K(0) > 0$ and $E(0) > 0$ given, $K(t)$, $E(t)$, $C(t) \geq 0$ and $1 \geq L(t) \geq 0$ for every $t \in [0, +\infty)$. The parameter $\rho > 0$ measures the subjective discount rate. The symbols $\dot{K}$ and $\dot{E}$ represent, respectively, the time derivatives of K and E.

Equation (4) models the accumulation process of productive capital; according to it, the (net) investment in new capital is equal to the difference between the produced output Y and consumption C. Equation (5) gives the time evolution of the stock of the environmental resource E; $\overline{Y}$ is the economy-wide average output and the parameter $\varepsilon > 0$ measures the negative impact of $\overline{Y}$ on E. The parameter $\overline{E} > 0$ represents the value of E to which the stock of the environmental resource converges starting from an initial value $E(0) > 0$, in absence of the negative impact due to the production process of output Y ($\overline{E}$ is the "carrying capacity" of the natural resource).

In solving problem (3)-(5), each agent considers $\overline{Y}$ as exogenously determined. Indeed, as there exists a continuum of economic agents, each of them considers her own impact on $\overline{Y}$ as negligible. However, since agents are identical, ex post $\overline{Y} = Y$ holds. This implies that the trajectories resulting from our model are not optimal (i.e. they do not describe the social optimum). However, they represent Nash equilibria in the sense that, along them, no agent has an incentive to modify her choices if the others don't modify theirs.

Antoci et al. (2020) analyzed the dynamic system obtained by applying the *Maximum Principle* to problem (3)-(5), focusing on local analysis. They showed that there exist at most two steady states $P_1 = (K^*, E_1, L^*)$ and $P_2 = (K^*, E_2, L^*)$, with $E_1 < E_2$, and analyzed their local stability properties. In the present paper, we complete the analysis by studying global dynamics. Our analysis highlights the dynamic regimes that may be observed, and proves that global indeterminacy may occur. That is, given the initial conditions $K(0)$ and

$E(0)$, both steady states can be reached by choosing different initial values of the control variable L. Antoci et al. (2020) only touched upon questions of global indeterminacy and used numerical simulations to illustrate such a phenomenon.

Indeterminacy scenarios may be observed because economic agents are unable to coordinate their choices, since each of them takes the economy-wide average output $\overline{Y}$ (and, consequently, the time evolution of E) as exogenously given. This result implies that one cannot predict a priori where the economy will eventually converge to.

Our work derives global indeterminacy results through an analytical characterization of the invariant surfaces separating different regimes of the trajectories (the same approach is followed in Antoci et al. 2011). Our approach differs from previous contributions to the global indeterminacy literature, which are based only on bifurcation techniques (e.g., Bella et al. 2017, Mattana et al. 2009). For a review of the literature on global indeterminacy see, among the others, Mino (2017).

Differently from previous studies in the literature on global indeterminacy (see, for a review, Bella et al. 2017, Mino 2017), where indeterminacy occurs in contexts in which economic agents do not take into account the positive effects (positive externalities) of their decisions on the production activity or on the accumulation process of human capital, in our model global indeterminacy results from the assumption that agents do not internalize the negative effects (negative externalities) of their decisions on the environmental resource. Other growth models with environmental assets exhibit global indeterminacy scenarios; see, among the others, Antoci et al. (2011, 2019), Yanase (2011), Fernández et al. (2012), Carboni and Russu (2013), Bella and Mattana (2018), Russu (2020). However, they analyze global indeterminacy in very different theoretical contexts (see Caravaggio and Sodini, 2018, for a review of the literature).

Finally, it is worth to stress that in the present work we are not dealing with another important problem of dynamic optimization, that is, the existence of indifference points. Starting from these points, two or more optimal solutions exist, giving rise to the same value of the objective function (see the seminal contributions of Sethi, 1977, and Skiba, 1978). Vice versa, in our optimization problem, the trajectories that an economy may follow, in a global indeterminacy scenario, are Nash equilibria but do not represent optimal solutions, in that each economic agent does not take into account the negative impact that her choices have on the dynamics of E. Therefore, when multiple equilibrium trajectories exist, starting from the same initial values of the state variables, economic agents may select one along which the value of the objective function is lower than in others, due to a coordination failure.

The paper will be structured as follows. In Section 2 the dynamic system generated by the maximization problem is derived. Sections 3 and 4 deal, respectively, with local and global analysis of the system.

2 The dynamic system

The current value Hamiltonian function associated to problem (3)-(5) is:

$$H = \frac{\left[CE^{\beta}(1-L)^{\gamma}\right]^{1-\delta} - 1}{1-\delta} + \lambda\left(K^{\alpha}L^{1-\alpha} - C\right)$$

where λ is the co-state variable associated to K. By applying the *Maximum Principle*, we obtain:

$$\dot{K} = \frac{\partial H}{\partial \lambda} = K^{\alpha}L^{1-\alpha} - C \tag{6}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial K} = \lambda\left(\rho - \alpha K^{\alpha-1}L^{1-\alpha}\right) \tag{7}$$

where C and L satisfy the following conditions[1]:

$$\frac{\partial H}{\partial C} = C^{-\delta}E^{\beta(1-\delta)}\left(1-L\right)^{\gamma(1-\delta)} - \lambda = 0 \tag{8}$$

$$\frac{\partial H}{\partial L} = 0 \text{ i.e. } -\gamma C^{1-\delta}E^{\beta(1-\delta)}\left(1-L\right)^{\gamma(1-\delta)} + (1-\alpha)\lambda(1-L)K^{\alpha}L^{-\alpha} = 0 \tag{9}$$

Each agent considers the time evolution of E, given by equation (5), as exogenously determined. The context is that considered by Wirl (1997).

In Antoci et al. (2020) it is showed that equations (6)-(7) and conditions (8)-(9) give rise to the following dynamic system:

$$\dot{K} = \frac{1}{\gamma}\frac{K^{\alpha}}{L^{\alpha}}\left[L(1-\alpha+\gamma) - (1-\alpha)\right] \tag{10}$$

$$\dot{E} = E(\overline{E} - E) - \varepsilon K^{\alpha}L^{1-\alpha} \tag{11}$$

$$\dot{L} = f(L)\left[\rho - \alpha\frac{L^{1-\alpha}}{K^{1-\alpha}} + \frac{\alpha\delta}{K}\dot{K} - \frac{\beta(1-\delta)}{E}\dot{E}\right] \tag{12}$$

where:

$$f(L) = \frac{L(1-L)}{\left[(\gamma+1)\delta - \gamma - \alpha\delta\right]L + \alpha\delta}$$

and $f(L) > 0$, for $1 > L > 0$, recalling $\delta > \gamma/(\gamma+1)$.

We will analyze the dynamics of the system (10)-(12) **in the set**:[2]

[1] It is easy to check that, according to the utility and production functions we adopted, the representative agent always chooses $C > 0$ and $0 < L < 1$.

[2] Notice that equation (12) is not defined for $E = 0$ and $K = 0$, and remember that the parameter $\overline{E}$ represents the value to which the variable E tends (for $t \to +\infty$) in absence of the negative impact due to the production process of output Y.

$$S = \{K > 0, \quad 0 < E < \overline{E}, \quad 0 < L < 1\} \tag{13}$$

The variables K and E are state variables, while the variable L is a control variable. So, the initial values $K(0)$ and $E(0)$ are determined by the "history" of the economy (according to the terminology used by Krugman, 1991), while the initial value $L(0)$ is determined by the "expectations" of economic agents.

3 Steady states in the set S

Antoci et al. (2020) showed that there exist at most two steady states of system (10)-(12), $P_1 = (K^*, E_1, L^*)$ and $P_2 = (K^*, E_2, L^*)$ with $0 < E_1 \leq E_2$, where:

a) $L^* = (1-\alpha)/(1-\alpha+\gamma)$;

b) $K^* = (\alpha/\rho)^{1/(1-\alpha)} L^*$;

c) E_1 and E_2 are the real solutions of the equation $E(\overline{E}-E) = \varepsilon(K^*)^\alpha(L^*)^{1-\alpha}$, which exist if and only if $\varepsilon \leq \frac{\overline{E}^2}{4(K^*)^\alpha(L^*)^{1-\alpha}} = \frac{(1-\alpha+\gamma)\overline{E}^2}{4(1-\alpha)}\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}$.

The previous paper also investigated the local stability properties of P_1 and P_2 and showed that P_1 is, generically, either a local attractor or a saddle with one-dimensional stable manifold, while P_2 is, generically, either a repeller or a saddle with two-dimensional stable manifold. In particular, P_1 may be a local attractor only if $\delta > 1$.

According to such results, the steady state P_2 can be (generically) reached by the economy only when it is a saddle with a two-dimensional stable manifold. In such a case, given initial conditions $K(0)$ and $E(0)$ close enough to the values of K and E in P_2 (K^* and E_2, respectively), then generically there exists a unique initial value $L(0)$ of the control variable L such that the trajectory starting from $(K(0), E(0), L(0))$ converges to P_2.

The steady state P_1 can be (generically) reached only when it is attractive. Remember that the steady state P_1 is characterized by a lower level of the stock of environmental resource, $E_1 < E_2$. So, when it is attractive, it is a poverty trap. Furthermore, a *local indeterminacy* scenario occurs: given initial conditions $K(0)$ and $E(0)$ close enough to the values of K and E in P_1 (that is, K^* and E_1), there exists a continuum of initial values $L(0)$ of the control variable L such that the trajectory starting from $(K(0), E(0), L(0))$ converges to P_1 (see Benhabib and Farmer, 1999). On the contrary, when P_1 is a saddle with a one-dimensional manifold, then it cannot be, generically, reached.

We remark that P_1 can be attractive only if $\delta > 1$. When such condition holds, we have that consumption C and environmental resource E are "Edgeworth substitutes": ceteris paribus, the increase in utility $U(C, L, E)$ (see (2)) deriving from an increase in C depends negatively on the stock E of the environmental good:

$$\frac{\partial U(C,L,E)}{\partial C\partial E}=(1-\delta)\,\beta E^{\beta-1}\frac{(1-L)^{\gamma}}{\left[CE^{\beta}\,(1-L)^{\gamma}\right]^{\delta}}<0 \text{ if } \delta>1 \tag{14}$$

So, if $\delta > 1$, environmental degradation induces economic agents to increase consumption C. That is, they tend to "substitute" the benefits deriving from the "consumption" of the free access natural resource by the benefits deriving from the consumption C. To implement such a substitution process, economic agents have to work more to increase the production of output Y and, consequently, their consumption level C. Such a substitution process has a self-reinforcing nature: environmental degradation induces agents to produce more output Y; an increase in Y generates further environmental degradation, and so on. This process may end up leading the economy on a welfare-reducing trajectory converging to P_1.

If, on the contrary, $\delta \in (0,1)$, then the economy is unlikely to converge to P_1, as such steady state cannot generically be reached. In this context, P_2 is the unique steady state that can be reached. According to (14), if $\delta < 1$, we have that the increase in utility $U(C,L,E)$ deriving from an increase in C depends positively on the stock E of the environmental good. In such a case, it is said that C and E are "Edgeworth complements". To set ideas, consider playing sports in a green area. If the quality and the extent of it is high (that is, the value of E is high), then the utility of buying a bike to go for a ride in the area is high. In such a case, C and E go hand in hand (they are Edgeworth complements). The opposite holds if we consider the utility deriving from purchasing a gym membership, where individuals can use a bike or walk on a treadmill, rather than play sports en plein air. In such a case, the expensive private consumption C "replaces" the use of the free access environmental resource E, and the utility generated by such a consumption choice is negatively related to the value of E (C and E are Edgeworth substitutes). Similar examples apply to many other private consumer goods (see, among the others, Antoci and Borghesi, 2012).

4 Global analysis

It is easily observed that there exist, for any value of δ, two open regions whose trajectories converge, respectively, to $K = 0$ and $E = 0$ in a finite time. Then, such trajectories cannot come back to S. In the following we investigate the case where the two steady states P_1 and P_2 exist and are, respectively, a sink and a saddle (with two-dimensional stable manifold). In fact, in Figure 1 we consider a set of parameter values for which such a case occurs. More generally, we can observe that the steady states coordinates do not depend on the parameters β and δ, which, however, influence their stability. Specifically, given any $\delta > 1$, P_1 is a sink and P_2 a saddle if β is sufficiently high. Then, letting δ decrease, sooner or later (for some $\overline{\delta} > 1$) a supercritical Hopf bifurcation generically occurs: P_1 becomes a saddle with one-dimensional stable manifold and an attracting limit cycle arises around it. On the other hand, a Hopf bifurcation may also concern P_2, which may become a source surrounded by a limit cycle with a two-dimensional stable manifold.

Actually, the conclusion section is devoted to considerations and conjectures about the global dynamics evolution.

As a first step, we describe the geometric shape of the surfaces $\dot{K} = 0$, $\dot{E} = 0$ and $\dot{L} = 0$.

- $\dot{K} = 0$ is clearly given by the plane:

$$L = L^* = \frac{1-\alpha}{1-\alpha+\gamma}$$

- $\dot{E} = 0$ is, for any $L = L_0$, $0 < L_0 < 1$, a "parabolic" curve:

$$K^\alpha = \frac{1}{\varepsilon L_0^{1-\alpha}} E(E - \overline{E})$$

  The maximum value of K is is reached when $E = \overline{E}/2$. Calling it $\widetilde{K}(L_0)$, we observe that $\widetilde{K}(L_0)$ decreases when L_0 increases, and $\lim\limits_{L_0 \to 0} \widetilde{K}(L_0) = +\infty$.

- In order to describe $\dot{L} = 0$, we multiply by E the expression in square brackets in formula (12). Hence, we obtain a second degree equation in E which, for any given $L_0 \in (0,1)$, has two, one or zero solutions. In fact, it can be checked that there exist two values, $0 \le L^{'} < L^{''} < 1$, such that when $L_0 \in (L^{'}, L^{''})$, $\dot{L} = 0$ is given, in the positive quadrant of the plane $L = L_0$, by the union of two curves, $E = E^{'}(K, L_0)$ and $E = E^{''}(K, L_0)$, $E^{'} \le E^{''}$, such that there exist two values $\overline{K}$ and $\overline{\overline{K}}$, $\overline{K} < \overline{\overline{K}}$, for which:

$$E^{'}(\overline{K}, L_0) = E^{''}(\overline{K}, L_0)$$
$$E^{'}(\overline{\overline{K}}, L_0) = E^{''}(\overline{\overline{K}}, L_0)$$

  Moreover, as $L_0 \in (L^{'}, L^{''})$, $\left\{\dot{L} = 0\right\} \cap \{L = L_0\}$ intersects $\left\{\dot{E} = 0\right\} \cap \{L = L_0\}$ in two points $\left(\widehat{K}, \widehat{E}_1\right)$ and $\left(\widehat{K}, \widehat{E}_2\right)$, with $\overline{K} < \widehat{K} < \overline{\overline{K}}$. In other words, as $L_0 \in (L^{'}, L^{''})$, $\left\{\dot{L} = 0\right\} \cap \{L = L_0\}$ has an oval shape, which may or may not intersect the line $E = \overline{E}$.

  Moreover, one can check that $L^{'} > 0$ if $\alpha > 1/2$, while $L^{'} = 0$ if $\alpha < 1/2$. In the former case, all the trajectories starting from points where $L < L^{'}$ tend to $K = 0$. In both cases, anyway, $\widehat{K}$ increases as $L \to L^{'}$, and, if $L^{'} = 0$, $\lim\limits_{L_0 \to 0} \widehat{K} = +\infty$.

We can also observe that, posed $\widehat{E} = E^{'}(\widehat{K}, L_0) = E^{''}(\widehat{K}, L_0)$, $\widehat{E} < \overline{E}/2$. In fact, having written $\dot{L} = 0$ as a second degree equation in E, $G(K, E, L_0) = 0$, the pair $(\widehat{K}, \widehat{E})$ satisfies $\partial G/\partial E = 0$. Since at that point $\dot{E} > 0$, the claim easily follows.
Finally, when $L_0 \in (L^{''}, 1)$, it can be checked that, in the plane $L = L_0$, $\dot{L} > 0$ when $\dot{E} \geq 0$ and the two curves $E = E^{'}(K, L_0)$ and $E = E^{''}(K, L_0)$ intersect at one point, which, however, may not belong to S.

Now we can prove the following result.

Theorem 1 *Assume that the steady states P_1 and P_2 exist and are, respectively, a sink and a saddle with two-dimensional stable manifold. Then P_2 belongs to the boundary of the attraction basin of P_1. Moreover, the stable manifold of P_2 separates trajectories tending to P_1 from trajectories tending to $K = 0$.*

Proof. The strategy of the proof consists in following the backward trajectory from a point Q_0 belonging to a small disc D on the plane $L = L^*$, contained in the attraction basin of P_1, where $\dot{L} > 0$ and $\dot{E} < 0$. It is shown, through some technical steps, that such a trajectory reaches generically a maximum of E at a point $Q_1 = \left(K_1, \widetilde{E}_1, L_1\right)$, with $\widetilde{E}_1 > \frac{\overline{E}}{2}$, $L_1 < L_0$ and $\dot{K} < 0$, $\dot{E} = 0$, $\dot{L} > 0$. Next we see that on the plane $L = L_1$ there exists an arc $K = K(E, L_1)$, defined for $\frac{\overline{E}}{2} \leq E \leq \widetilde{E}_1$, whose forward trajectories reach $K = 0$. Therefore a limit case is represented by a point Q_0, possibly coinciding with P_1, whose backward trajectory, after a number (possibly infinite) of rotations, tends to a point where $\dot{E} = 0$ with $E > \frac{\overline{E}}{2}$, but does not cross again $\dot{E} = 0$. In fact, that implies such a trajectory to tend to the saddle P_2, while, on the basis of what has been seen above, it is proved that there exist trajectories tending to P_2 as limit cases of trajectories reaching $K = 0$. This proves the Theorem's statement.

Coming to the details, take, in the plane $L = L^*$, a disc D centered in P_1 contained in the basin of attraction of P_1. Take a point of D where $\dot{L} > 0$ and $\dot{E} < 0$, say $Q_0 = (K_0, E_0, L^*)$, and follow the backward (i.e. when $t < 0$) trajectory of Q_0. Clearly, if $\alpha > 1/2$, one must meet a minimum of $L(t)$, since all the trajectories starting from some $L \in (0, L^{'})$ tend to $(0, \overline{E}, 0)$. Vice versa, in the case $\alpha < 1/2$, we expect that the backward trajectory of Q_0 might tend, as $t \to -\infty$, to $L = 0$ and consequently, recalling $\lim_{L \to 0} \widehat{K} = +\infty$, to $K = +\infty$ and $E = \overline{E}$. To this end, we can study the local stability of $(K, E, L) = (+\infty, \overline{E}, 0)$, for example by the change of variables $H = K^{\alpha - 1}$, $E = E$, $M = F(L)$, such that $F'(L) = 1/f(L)$ and $F(0) = 0$. So the characteristic polynomial of the Jacobian matrix at $(H, E, M) = (0, \overline{E}, 0)$ is computed to be:

$$-\lambda(\lambda^2 + b\lambda + c) = 0$$

where the sign of b is ambiguous, but $c<0$. Hence such a steady state is a saddle point and, by an arbitrarily small change of Q_0, one can avoid that along the backward trajectory $L \to 0$.

Hence, following the backward trajectory of Q_0, we will find, sooner or later, a minimum of L and a maximum of E, while K remains bounded. Precisely, recalling that a maximum of K corresponds, in the forward trajectory, to $\dot{L}<0$, the maximum of E will be reached at a point of the forward trajectory where $\dot{K}<0$ and $\dot{L}>0$. Assume, then, it occurs at $L=L_1<L^*$. Thus the backward trajectory of Q_0 intersects $L=L_1$ at a point $Q_1=(K_1,\widetilde{E}_1,L_1)$, with $\dot{K}(Q_1)<0$, $\dot{E}(Q_1)=0$, $\dot{L}(Q_1)>0$, $\ddot{E}(Q_1)<0$. Therefore, it is easily observed that $\widetilde{E}_1>\overline{E}/2$.

Moreover, consider the arc of $\dot{L}=0$ on the plane $L=L_1$, defined by $K=K(E,L_1)$, with $E\in\left[\overline{E}/2,\widehat{E}_2\right)$, where $\dot{E}>0$. Then along such an arc K is increasing and a trajectory crossing it, at a point where $\dot{L}=0$ and $\dot{K}<0$, cannot cross again $\dot{L}=0$, since, for $L_2<L_1$, $K(E,L_2)>K(E,L_1)$ when $E\geq\overline{E}/2$ (by $K(E,\cdot)$ we mean the lower edge of the oval corresponding to $\dot{L}=0$). Hence, on the plane $L=L_1$ we can consider a curve, from $E=\overline{E}/2$ to $(\widehat{K},\widehat{E}_2)$, whose forward trajectories reach $K=0$.

Now, we are looking for trajectories separating the basin of P_1 from the region whose trajectories reach $K=0$. In fact, as backwards trajectories from points belonging to the basin of P_1 spiral away from P_1, we can imagine that the backward trajectory of Q_1 intersects again $\dot{E}=0$ when $E>\overline{E}/2$ at a point $Q_2=(K_2,\widetilde{E}_2,L_2)$, where $E_2>\overline{E}/2$ and $K_1>K_2>K^*$.

Actually, we can find, as a limit case, a point $Q_0=(K_0,E_0,L^*)$, possibly coinciding with P_1, belonging to the basin of P_1, $E_1\leq E_0<\overline{E}$, where $\dot{E}(Q_0)\leq 0$ and $\dot{L}(Q_0)\geq 0$, such that its backward trajectory, after a number n, $0\leq n\leq +\infty$, of rotations, tends to a point of $\dot{E}=0$ with $E>\overline{E}/2$, but does not cross again $\dot{E}=0$. Therefore such a point is the saddle P_2.

On the other hand, take some $L_0\in(L^{'},L^*)$. Consider the segment of $E=\overline{E}/2$ on the plane where $K\in(\overline{K},\widetilde{K})$, defined by $\dot{L}\left(\overline{K},\overline{E}/2,L_0\right)=0$ and $\dot{E}\left(\widetilde{K},\overline{E}/2,L_0\right)=0$. Then the trajectory from $\left(\overline{K},\overline{E}/2,L_0\right)$, as we have observed, reaches $K=0$, and so do trajectories starting from points $\left(K,\overline{E}/2,L_0\right)$ where $\overline{K}<K<\widehat{K}_0$, $\widehat{K}_0$ being a suitable point of the interval $(\overline{K},\widetilde{K})$. Thus the trajectory from $\left(K_0,\overline{E}/2,L_0\right)$ will tend to $\dot{L}=0$ but cannot cross it; hence, it will converge to P_2.

In conclusion, we have proved that there exists a trajectory joining P_1 and P_2 and that the stable manifold of P_2 separates trajectories tending to P_1 from trajectories reaching $K=0$. ■

The following global indeterminacy result derives from Theorem 1, recalling the description of the surface $\dot{L} = 0$.

Corollary 2 *Given the conditions of Theorem 1, there exists a surface* Σ *, whose equation is* $K = K(E, L)$*,* $L' \leq \widetilde{L} < L < L^*$*,* $\overline{E}/2 < E < \widetilde{E}(L) \leq \widehat{E}_2(L)$*, with the following properties:*

- *the trajectory through a point* $Q_0 = (K_0, E_0, L_0) \in \Sigma$ *converges to* P_2*;*
- *for each* $Q_0 \in \Sigma$ *there exists* $\mu > 0$ *such that: the trajectory from a point* (K_0, E_0, L)*,* $L_0 - \mu < L < L_0$*, reaches* $K = 0$ *in finite time, while the trajectory from a point* (K_0, E_0, L)*,* $L_0 < L < L_0 + \mu$*, converges to the sink* P_1*.*

Proof. We have shown, in the proof of Theorem 1, that there exist points in the attraction basin of P_1, with an initial $E < \overline{E}/2$, whose backward trajectories cross $\dot{E} = 0$ at points, say, $\widetilde{Q}_0 = (\widetilde{K}_0, \widetilde{E}_0, \widetilde{L}_0)$, with $\widetilde{L}_0 < L^*$, $\widetilde{E}_0 > \overline{E}/2, \widetilde{K}_0 > \widehat{K}\left(\widetilde{L}_0\right)$. Moreover, as a limit case, one such backward trajectory tends to the saddle P_2. On the other hand, for each $\widetilde{L}_0 \in (L', L^*)$, there exists a non-decreasing[3] arc $K = K\left(E, \widetilde{L}_0\right)$, $\overline{E}/2 < E < \widehat{E}_2\left(\widetilde{L}_0\right)$, on the plane $L = \widetilde{L}_0$, along which $\dot{K} < 0$, $\dot{E}$, $\dot{L} > 0$, whose trajectories converge to P_2; while the trajectories starting from points with the same values of E and L, but a lower value of K, reach $K = 0$. Hence, the above arc separates the basin of P_1 from the trajectories reaching $K = 0$, when $\widetilde{L}_0$ belongs to a suitable interval $\left(\widetilde{L}, L^*\right)$, $\widetilde{L} \geq L'$. Now, omitting the decoration, consider a point $Q_0 = (K_0, E_0, L_0)$ of the arc and a point $Q_0' = (K_0, E_0, L_0 - \mu)$, $\mu > 0$ being sufficiently small. Hence, since $\dot{K} < 0$, $\dot{E}$, $\dot{L} > 0$ in Q_0', the trajectory of Q_0' will meet $L = L_0$ below the arc and, thus, will tend to $K = 0$. Conversely, the trajectory of the point $Q_0'' = (K_0, E_0, L_0 + \mu)$ will necessarily converge to P_1, if $\mu > 0$ is small enough. ■

According to Theorem 1 and Corollary 2, when $\delta > 1$, global indeterminacy can be observed: given the initial values of the state variables K and E, different steady states can be reached by choosing different initial values of the control variable L. More specifically, under the assumptions of Theorem 1, for each point $Q_0 = (K_0, E_0, L_0)$, $L_0 < L^*$, belonging to the surface Σ (and so converging

[3] In fact, consider a point $Q_0 = (K_0, E_0, \widetilde{L}_0)$ lying *just* below the arc, where $\dot{K} < 0$, $\dot{E}, \dot{L} > 0$. Hence, the trajectory of Q_0 will cross, first, $\dot{L} = 0$ and then will converge to $(0, \overline{E}, 0)$. Consider a point $Q_0' = (K_0, E_0', \widetilde{L}_0)$, with $E_0' - E_0 > 0$ sufficiently small. Then it is easily checked that, being $\dot{E} > 0$, $\frac{\partial \dot{L}}{\partial E} < 0$, while, at parity of E, $\dot{E}$ is higher if K and L are lower. As a consequence, it is not difficult to show that the trajectory from Q_0' reaches $\dot{L} = 0$ before the one from Q_0, converging as well to P_0. This proves that the function $K = K\left(E, \widetilde{L}_0\right)$, describing the arc, is not decreasing.

to the saddle point P_2), there exists $\mu > 0$ such that the trajectory starting from a point (K_0, E_0, L), with $L_0 < L < L_0 + \mu$, converges to the sink P_1, which is a poverty trap (see Fig.1).

That is, they choose to work more with the objective to increase the production and consumption of output. This allows them to replace the deteriorated environmental resource E by the consumption C of the produced good, and therefore to protect themselves from environmental degradation. However, their choices drive the economy towards the poverty trap P_1. The convergence to P_1 can be interpreted, in this context, as the consequence of a coordination failure of economic agents.

In the real world, some consumption goods are Edgeworth substitutes for environmental resources, others Edgeworth complements. In the aggregate model like the one proposed in this paper, a value of the parameter δ greater than 1 represents a context in which, at the aggregate level, substitutability prevails over complementarity; vice versa, if $\delta < 1$. At the best of our knowledge, there does not exist an empirical study giving an estimate of the degree of substitutability between environmental resources and private consumption goods. The contribution of our analysis is to show that substitutability between E and C, which is associated to the well-known notion of weak sustainability (Solow, 1974, 1986, 1993; Hartwick, 1977, 1978), may have a counterintuitive effect, in that it may favour the emergence of global indeterminacy scenarios and the consequent existence of trajectories approaching a welfare-reducing outcome.

These findings hold only if the production process of Y has a negative impact on the environment, that is, if $\varepsilon > 0$. If green technologies are used (i.e. $\varepsilon = 0$), the model admits only one saddle point with two-dimensional stable manifold, therefore no indeterminacy occurs in that case (see Antoci et al., 2020). The same holds if, as showed in Antoci et al. (2020), output Y is taxed at a constant rate $\tau \in (0, 1)$, the revenues τY are used for environmental protection (environmental defensive expenditures), and the positive effect of defensive expenditures on E is high enough with respect to the negative impact of Y (measured by the parameter ε).

5 Concluding remarks

As we have shown in the previous section, when P_1 and P_2 are, respectively, a sink and a saddle (implying, in particular, $\delta > 1$), there exist three open regions (and generically, we expect, no other one), say R_1, R_2 and R_3, filled, respectively, by trajectories reaching (in a finite time) the plane $E = 0$, converging to the sink P_1 and reaching (in a finite time) the plane $K = 0$. We have also proven that R_2 and R_3 are separated by the stable manifold of P_2.

In fact, we can conjecture, on the basis of the previous arguments, a likely evolution of the system dynamics. Let us start from Theorem 1 hypotheses. Now, assume δ, initially greater than 1, decreases. Then at some value of δ, say, δ_0, P_1 loses its stability and may be "replaced" by an attracting cycle Γ (Hopf bifurcation, see Fig.2) while P_2 continues to be a saddle. So, when δ

furtherly decreases, we expect Γ to expand (see Fig.3) until, when $\delta = 1$, the cycle explodes reaching the plane $E = 0$.

From such a point on, i.e. when $\delta < 1$, there should exist generically two open regions, $R_1$and R_3, whose trajectories reach, respectively, in a finite time the planes $E = 0$ and $K = 0$. Hence, we expect that the separatrix between the two regions would be, precisely, the stable manifold of P_2 (if P_2 has remained a saddle). In particular, the trajectories converging to P_2 would be the only ones staying in S when $t \to +\infty$. This occurrence is shown in Figure 4.

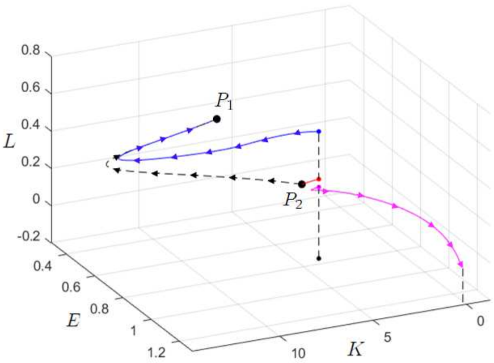

Figure 1. Global indeterminacy scenario. Parameter values: $\overline{E} = 1.3$; $\alpha = 0.3$, $\beta = 24.84$, $\gamma = 1.3$, $\delta = 1.06$, $\varepsilon = 0.4$, $\rho = 0.04$. Initial values of the state variables $E(0) = 1.04025820702807$; $K(0) = 5.99124478008392$. For $L(0) = 0.425697415099600$ the trajectory evolves on the stable manifold of P_2 converging to the saddle P_2. For a larger value of $L(0)$ (in figure $L(0) = 0.681115864159360$) trajectories converge to the sink P_1. For lower values of $L(0)$ (in figure $L(0) = 0.383127673589640$) trajectories reach the plane $K = 0$ in finite time. The dashed curve shows the branch of the unstable manifold of P_2 converging to the sink P_1.

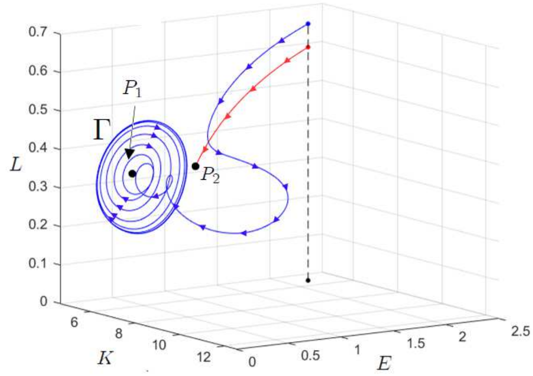

Figure 2. Hopf bifurcation and global indeterminacy. The parameter values are the same as in the previous figure except for $\delta = 1.035$. Trajectories converging to the limit cycle Γ surrounding P_1 coexist with trajectories on the stable manifold of the saddle P_2. Initial values of the state variables for the trajectories in figure are $E(0) = 2.20617886672602$, $K(0) = 5.35230152562421$. The red trajectory converging to the saddle P_2 starts with $L(0) = 0.605972067919661$, the one converging to Γ with $L(0) = 0.666569274711627$.

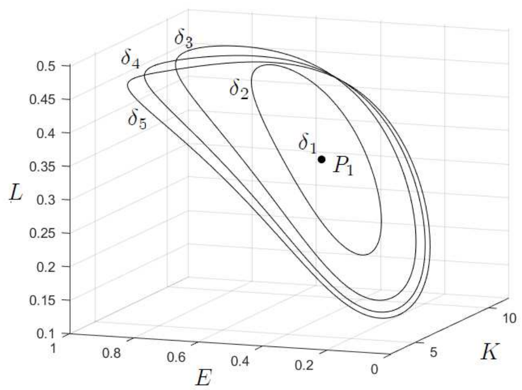

Figure 3. Evolution of the attractor of the system. The parameter values are the same as in the previous figures except for δ. Starting from $\delta = \delta_1 > 1.0374$ for which $P_1 = (6.225232325, 0.3491884150, 0.35)$ is a sink (coordinates of the stationary states are independent of δ), if we let δ decrease, P_1 undergoes a supercritical Hopf bifurcation ($\delta \simeq 1.0374$). Letting δ decrease further we observe an expansion of the attractive limit cycle surrounding P_1. The labels near the closed curves indicate the values of δ in the simulation: $\delta_2 = 1.035$, $\delta_3 = 1.03$, $\delta_4 = 1.028$, $\delta_5 = 1.027$.

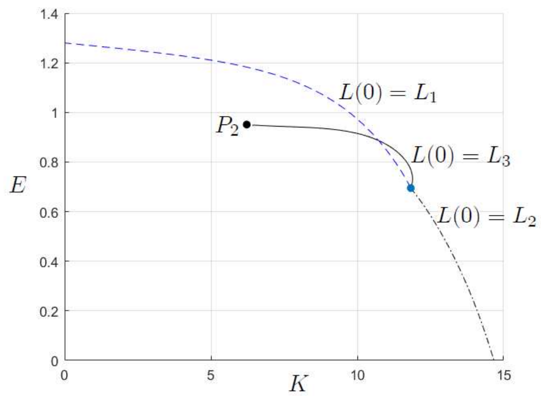

Figure 4. Case $\delta < 1$. The parameter values are the same as in the previous figures except for $\delta = 0.95$**.** No attractor exists. The phasespace is projected on the plane K, E. Three trajectories are depicted starting from the same initial values of the state variables K, E ($K(0) = 11.8279$, $E(0) = 0.6949$), but with different initial values of the jumping variable L. The dashed one reaches (in finite time) the plane $K = 0$ ($L(0) = L_1 = 0.2800$); the dash-dot one reaches (in finite time) the plane $E = 0$ ($L(0) = L_2 = 0.4550$); the solid one tends (in infinite time) to the saddle point P_2 ($L(0) = L_3 = 0.3585$).

Acknowledgements *The authors gratefully acknowledge two anonymous reviewers for valuable comments and suggestions. Mauro Sodini also acknowledges financial support by VSB-TUO project SP2021/15.*

References

[1] Antoci A., Galeotti M., Russu P., 2011. Poverty trap and global indeterminacy in a growth model with open-access natural resources. Journal of Economic Theory 146, 569-591.

[2] Antoci A., Borghesi S., 2012. Preserving or escaping? On the welfare effects of environmental self-protective choices. Journal of Socio-Economics 41, 248–254.

[3] Antoci A., Gori L., Sodini M., Ticci E., 2019. Maladaptation and global indeterminacy. Environment and Development Economics 24, 643-659.

[4] Antoci A., Borghesi S., Galeotti M., Russu P., 2020. Living in an uncertain world: Environment substitution, local and global indeterminacy. Journal of Economic Dynamics and Control, in press, https://doi.org/10.1016/j.jedc.2020.103929.

[5] Bella G., Mattana P., Venturi B., 2017. Shilnikov chaos in the Lucas model of endogenous growth. Journal of Economic Theory 172, 451–477.

[6] Bella G., Mattana P., 2018. Global indeterminacy and equilibrium selection in a model with depletion of non-renewable resources. Decisions Economics Finance 41,, 187–202.

[7] Benhabib J., Farmer R.E., 1999. Indeterminacy and sunspots in macroeconomics. In: J.B. Taylor, M. Woodford (eds.), Handbook of Macroeconomics, North-Holland, Amsterdam, 387–448.

[8] Caravaggio A., Sodini M., 2018. Nonlinear dynamics in coevolution of economic and environmental systems. Frontiers in Applied Mathematics and Statistics 4, 1–17.

[9] Carboni O.A., Russu P., 2013. Linear production function, externalities and indeterminacy in a capital-resource growth model. Journal of Mathematical Economics 49, 422-428.

[10] Fernández E., Pérez R., Ruiz J., 2012. The environmental Kuznets curve and equilibrium indeterminacy. Journal of Economic Dynamics and Control 36, 1700–1717.

[11] Hartwick J.M., 1977. Intergenerational equity and the investing of rents from exhaustible resources. The American Economic Review 67(5), 972–974.

[12] Hartwick J.M., 1978. Substitution among exhaustible resources and intergenerational equity. The Review of Economic Studies 45(2), 347–354.

[13] Krugman P., 1991. History versus expectations. Quarterly Journal of Economics 106, 651–667.

[14] Mattana P., Nishimura K., Shigoka T., 2009. Homoclinic bifurcation and global indeterminacy of equilibrium in a two-sector endogenous growth model. International Journal of Economic Theory 5, 25–47.

[15] Mino K., 2017. Growth and Business Cycles with Equilibrium Indeterminacy. Advances in Japanese Business and Economics 13, Springer, Tokyo Japan.

[16] Russu P., 2020. Mathematical analysis of an economic growth model with perfect-substitution technologies. Nonlinear Analysis: Modelling and Control 25(1), 84-107.

[17] Sethi S.P., 1977. Nearest feasible paths in optimal control problems: Theory, examples and counterexamples. Journal of Optimization Theory and Applications 23, 563–579.

[18] Skiba A., 1978. Optimal growth with a convex-concave production function. Econometrica 46, 527–540.

[19] Solow R.M., 1974. Intergenerational equity and exhaustible resources. Review of Economic Studies: Symposium of the Economics of Exhaustible Resources 29–46.

[20] Solow R.M., 1986. On the intergenerational allocation of natural resources. Scandinavian Journal of Economics 88(1), 141—9.

[21] Solow R.M., 1993. An almost practical step towards sustainability. Resources Policy 16, 162–72.

[22] Wirl F., 1997. Stability and limit cycles in one-dimensional dynamic optimizations of competitive agents with a market externality. Journal of Evolutionary Economics 7, 73–89.

[23] Yanase A., 2011. Impatience, pollution, and indeterminacy. Journal of Economic Dynamics and Control 35, 1789–1799.