Lieb-Robinson bounds for commutator-bounded operators

Isabeau Prémont-Schwarz,[1,2,*] Alioscia Hamma,[1] Israel Klich,[3] and Fotini Markopoulou-Kalamara[1,2]

[1]*Perimeter Institute for Theoretical Physics, 31 Caroline Street N, Waterloo, Ontario N2L 2Y5, Canada*
[2]*Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
[3]*Department of Physics, University of Virginia, 382 McCormick Road, Charlottesville, Virginia 22904, USA*

We generalize the Lieb-Robinson theorem to systems whose Hamiltonian is the sum of local operators whose commutators are bounded.

The principle of locality is at the heart of the foundations of all modern physics. In quantum field theory, the principle of locality is enforced by an exact light cone. Whenever two (bosonic) observables are spacelike separated, they have to commute, so that neither can have any causal influence on the other. In ordinary quantum mechanics, no explicit request for locality is imposed, and it is, in principle, possible to signal between arbitrarily far apart points in an arbitrarily short time. Nevertheless, a simple perturbation analysis shows that such an influence must decay exponentially with the distance between the observables. The seminal work by Lieb and Robinson [1] has made this statement rigorous for nonrelativistic spin systems. In essence, it states that any quantum system whose Hilbert space is composed of a tensor product of local, finite-dimensional Hilbert spaces and whose Hamiltonian is the sum of local operators will have an approximately maximum speed of signals. Here, local just means that every operator has as a support the tensor product of a few degrees of freedom. The approximation consists of the fact that outside the effective light cone there is an exponentially decaying tail.

Recently, Lieb-Robinson bounds (LRBs) have received renewed interest in both the fields of theoretical condensed matter and quantum information theory [2–15]. In particular, the LRB has been used to prove that a nonvanishing spectral gap implies an exponential clustering in the ground state [6,8,11]. Further developments can be found in Ref. [9], where the LRB is used also to argue about the existence of dynamics. The LRB has also been instrumental in the recent extension of the Lieb-Schultz-Mattis theorem to higher dimensions [12,16]. In Ref. [7,24], it has been shown how the LRBs can be exploited to find general scaling laws for entanglement. In Ref. [2], these techniques have been exploited to characterize the creation of topological order. The locality of dynamics has important consequences on the simulability of quantum spin systems. In Refs. [17,18], it has been shown that one-dimensional gapped spin systems can be efficiently simulated. A review of some of the most relevant aspects of the locality of dynamics for quantum spins systems can be found in Ref. [13]. Other developments of significant interest include Refs. [14,15] which show that it is possible to entangle macroscopically separated nanoelectromechanical oscillators of the oscillator chain and that the resulting entanglement is robust to decoherence. Such a system is of great interest for its possible application as a quantum channel and as a tool to investigate the boundary between the classical and quantum worlds.

The LRBs have found a more exotic use in the field of emergent gravity, where one wants to study locality, geometry, and Lorentz symmetry as emergent phenomena [19,20]. An example of the usefulness of the LRBs can be found in Ref. [22], where it was shown that, in spin systems with emergent electromagnetism [23], the speed of light is also the maximum speed of signals, without imposing, from the beginning, any Lorentz invariance. This raises the issue whether even Lorentz invariance could be emergent.

One problem with the LRBs is that it is difficult to obtain bounds for unbounded Hamiltonians. In the usual Lieb-Robinson settings, the Hamiltonian must be a sum of local *bounded* operators. If the unbounded terms in the Hamiltonian are completely local, that is, if they are on-site terms, it is possible to prove a Lieb-Robinson theorem using the usual technique [10]. In the specific case of coupled harmonic oscillators on a graph with local interactions, it was proven in Ref. [3] that the LRB is valid for canonical and Weyl operators, and a proof for a generalization to general operators is outlined. Algebraic suppression (instead of the usual exponential suppression of the LRB) is shown to result from nonlocal algebraic interactions. As an interesting corollary, Ref. [3] shows how the approximate locality implied by the LRB becomes exact in the continuous limit for the Klein-Gordon field.

In this paper, we will show how one can find a bound to the maximum speed of interactions in the case of a class of unbounded spin Hamiltonians. It is not true that for any unbounded Hamiltonian, a Lieb-Robinson bound exists [21]. Here, we want to show that one can derive an LRB if the Hamiltonian is the sum of local operators, *whose commutators are bounded*. Therefore, there is no necessity for even the nonlocal terms to be bounded, as long as their commutators are. More specifically, we show that for quantum systems whose Hilbert space is the tensor product of local Hilbert spaces associated with vertices and edges of a graph, if the Hamiltonian is the sum of local operators Φ_i, each with a support on a region of the graph with a diameter less than a fixed number R, if each of these local operators Φ_i is noncommuting with less than v other local operator terms of the Hamiltonian Φ_j, and if for any two of these operators we have $\|[\Phi_i]\Phi_j\| < K$ and for any three operators $\|[\Phi_i[\Phi_j\Phi_k]]\| < Q$ for two positive numbers K and Q, then we have that for any two local operators Φ_i and Φ_i which are terms in the Hamiltonian and whose support is separated by a graph distance d, that

$$\|[\Phi_i(t), \Phi_j(0)]\| \leqslant \tilde{M} \exp[\lambda(v_{LR}t - d)], \tag{1}$$

where $\tilde{M}$ is a constant and v_{LR}, the limit on the speed of propagation of information, depends only on local operators of

*ipremont-schwarz@perimeterinstitute.ca

the Hamiltonian as it is they who affect the propagation. This bound can be generalized to any local observables O_P and O_Q with supports P and Q, respectively, that satisfy the following local observable operator conditions: (i) The graph distance d separating P and Q is greater than R, (ii) the number of terms Φ_i of the Hamiltonian whose support has a nonempty intersection with P is $n_P < \infty$, and (iii) there exists F_P and F_Q such that, for all terms Φ_i and Φ_j of the Hamiltonian, the inequalities $\|[O_P, \Phi_i]\| < F_P K$, $\|[O_Q, \Phi_i]\| < F_Q K$, and $\|[O_Q, [\Phi_i, \Phi_j]]\| < F_Q Q$ are satisfied. The generalized bound is then

$$\|[O_P(t), O_Q(0)]\| \leqslant \tilde{\tilde{M}} F_P F_Q n_P (n_P + 1) \exp[\lambda(v_{LR} t - d)]. \tag{2}$$

To motivate our discussion, let us start by the most trivial example: Consider the case of a Hamiltonian $H = \sum h_i$ which is composed of a sum of local terms h_i which are commuting, such as the quantum Ising model without the transverse field. In such a case, there is simply no propagation of signals: Indeed, for any local operator O_A we have $O_A(t) = e^{itH} O_A e^{-iHt} = e^{itH_A} O_A e^{-iH_A t}$, where $H_A = \sum_{i:[h_i, O_A] \neq 0} h_i$, since there is a finite number of h_i in H_A, and they are of finite range; $O_A(t)$ is also strictly local for arbitrary long times t, irrespective of the norm of the h_i operators. This suggests that it is desirable to find LRBs in terms of the norm of the commutators rather than the norm of the local terms h_i.

Let us outline a simple example. Consider a system of parallel quantum wires. We place fermions on the wires, and these are usually described by one-dimensional Luttinger liquids and have approximately a linear dispersion relation. We place a density-density interaction between the wires. Labeling the wires by the index j, the system can be described by the following Hamiltonian:

$$H_{\text{wires}} = \sum_j [-i \partial/\partial x_j + V(x_j - x_{j+1})], \tag{3}$$

which is commutator bounded in the sense of this paper as long as both $|\partial_{x_j} V|$ and $|\partial_{x_i} \partial_{x_j} V|$ are bounded. Another example involves a generalized Dicke model, describing an array of spins interacting with a boson field via

$$H = \sum h_n; \quad h_n = \sigma_n^z (b_n^\dagger + b_n + i b_{n+1}^\dagger - i b_{n+1}), \tag{4}$$

where b_n are boson creation operators and σ_n^z is the nth spin. It is easy to check that, in this case, the commutator $[h_n, h_{n+1}] = -2i \sigma_n^z \sigma_{n+1}^z$ is bounded. [In fact, this particular Hamiltonian can also be written as a sum of commuting terms $\tilde{h}_n = b_n(\sigma_n^z - i\sigma_{n-1}^z) + \text{H.c.}$].

We consider Hamiltonians that are the sum of two different types of operators Φ_0 and Φ_1:

$$H \equiv \sum_{i \in S_0} h_0 \Phi_0^i + \sum_{j \in S_1} h_1 \Phi_1^j. \tag{5}$$

Here, S_0, S_1 are two sets of labels, h_0 and h_1 are two coupling constants, and $[\Phi_0^i, \Phi_0^j] = [\Phi_1^i, \Phi_1^j] = 0$ for every i, j. As an example, consider the Ising model. Then, $\Phi_0^i = \sigma_i^x \sigma_{i+1}^x$ and $\Phi_1^i = \sigma_i^z$. We call the subgraph which is the support of the operator Φ_q^m, $\Gamma(q, m)$ and for $(a, b) \in \{0, 1\}^2$, we define

$$K_{a \ b}^{i \ j}(t) \equiv [\Phi_a^i(t), \Phi_b^j(t)]. \tag{6}$$

We consider what we will refer to as commutator-bounded R-local quantum systems. For such systems, the commutators and the commutators of commutators of operators of the Hamiltonian are uniformly bounded while the operators themselves may be unbounded and the operators of the Hamiltonian have support on subgraphs of size less than R, for R an arbitrary natural number. Explicitly, the diameter of all $\Gamma(q, m)$ is less than R, and for any three operators Φ_a^i, Φ_b^j, and Φ_c^k appearing in the Hamiltonian with the coupling constants h_a, h_b, and h_c, we have that $h_a h_b \|[\Phi_a^i, \Phi_b^j]\| < K$ and $h_a h_b h_c \|[[\Phi_a^i, \Phi_b^j], \Phi_c^k]\| < Q$, where K and Q are positive real numbers. Note that a bounded system which is uniformly bounded by $\tilde{K}$, must satisfy $K \leqslant 2\tilde{K}^2$ as well as $Q \leqslant 4\tilde{K}^3$, and thus, boundedness implies commutator boundedness.

By taking the derivative of Eq. (6) with respect to t, we obtain $[K_{a \ b}^{i_1 \ j}(t)]' = [[-iH(t), \Phi_a^i(t)], \Phi_b^j]$, after keeping only the terms in $H(t)$ which do not commute with $\Phi_a^i(t)$, and after some algebra, we get [here and in the following, by $a + 1$ we mean $a + 1 \bmod (2)$]

$$\left[K_{a \ b}^{i_1 \ j}(t)\right]' = \left[K_{a \ b}^{i_1 \ j}(t), \left[-ih_{a+1} \sum_{i_2 \in Z_{i_1}} \Phi_{a+1}^{i_2}(t)\right]\right]$$
$$+ (-ih_{a+1}) \sum_{i_2 \in Z_{i_1}} \left[\Phi_a^{i_1}(t), \left[\Phi_{a+1}^{i_2}(t), \Phi_b^j\right]\right], \tag{7}$$

where, if $i \in S_0$, then Z_i is the finite subset of S_1 such that $j \in Z_i \Leftrightarrow \Gamma(0, i) \cap \Gamma(1, j) \neq \emptyset$ and vice versa for $i \in S_1$.

Taking the second derivative, and using the fact that $[\Phi_a^i(t), \Phi_a^j(t)] = 0$, we obtain, after some algebraic manipulation,

$$K_{a \ b}^{i_1 \ j \ ''}(t)$$
$$= -i \left[K_{a \ b}^{i_1 \ j \ '}(t), \sum_{j_2 \in Z_{i_1}} h_{a+1} \Phi_{a+1}^{j_2}(t) + \sum_{i_3 \in Z_{i_2}} h_a \Phi_a^{i_3}(t)\right]$$
$$- h_{a+1}^2 \sum_{j_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} \left[[\Phi_a^{i_1}(t), \Phi_{a+1}^{i_2}(t)], [\Phi_{a+1}^{j_2}(t), \Phi_b^j]\right]$$
$$- h_a h_{a+1} \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \left[[\Phi_a^{i_1}(t), \Phi_{a+1}^{i_2}(t)], [\Phi_a^{i_3}(t), \Phi_b^j]\right]. \tag{8}$$

Defining the unitary operator $U_{2+3}(t) \equiv \exp\{-it[\sum_{j_2 \in Z_{i_1}} h_{a+1} \Phi_{a+1}^{j_2}(t) + \sum_{i_3 \in Z_{i_2}} h_a \Phi_a^{i_3}(t)]\}$ and its associated unitary evolution $T_{2+3}(t) O \equiv U_{2+3}(t)^\dagger O U_{2+3}(t)$, integrating Eq. (8), and taking the norm, we obtain, after some manipulations,

$$\left\|K_{a \ b}^{i_1 \ j \ '}(t)\right\| \leqslant h_{a+1} \sum_{i_2 \in Z_{i_1}} \left\|[[\Phi_a^{i_1}, \Phi_{a+1}^{i_2}], \Phi_b^j]\right\|$$
$$+ \int_0^t ds \left(2h_{a+1}^2 K \sum_{j_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} \left\|[\Phi_{a+1}^{j_2}(s), \Phi_b^j]\right\|\right.$$
$$\left. + 2h_a h_{a+1} K \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \left\|[\Phi_a^{i_3}(s), \Phi_b^j]\right\|\right), \tag{9}$$

where we used the fact that $\|[\Phi_a^i(t), \Phi_{a+1}^j(t)]\| \leqslant K$. By integrating Eq. (9), we get

$$
\begin{aligned}
\big\| K_{ab}^{i_1 j}(t) \big\| \leqslant{}& \big\| \big[\Phi_a^{i_1}, \Phi_b^j \big] \big\| + h_{a+1} \sum_{i_2 \in Z_{i_1}} \big\| \big[\big[\Phi_a^{i_1}, \Phi_{a+1}^{i_2} \big] \Phi_b^j \big] \big\| t \\
&+ \int_0^t ds \int_0^s dl \left(2 h_{a+1}^2 K \sum_{i_2 \in Z_{i_1}} \sum_{j_2 \in Z_{i_1}} \big\| K_{a+1 b}^{j_2 j}(l) \big\| \right. \\
&\left. + 2 h_a h_{a+1} K \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \big\| K_{ab}^{i_3 j}(l) \big\| \right).
\end{aligned}
\tag{10}
$$

Since the commutators are bounded, we have $\|[\Phi_a^{i_1}, \Phi_b^j]\| \leqslant K$ and $\|[[\Phi_a^{i_1}, \Phi_{a+1}^{i_2}], \Phi_b^j]\| \leqslant Q$ for some $K, Q > 0$. Noting that $\|[\Phi_a^{i_1}, \Phi_b^j]\| = 0$ if $\Gamma(a,i_1)$ and $\Gamma(b,j)$ do not overlap and $\|[[\Phi_a^{i_1}, \Phi_{a+1}^{i_2}], \Phi_b^j]\| = 0$ if $\Gamma(b,j)$ does not overlap with either

$\Gamma(a,i_1)$ or $\Gamma(a+1,i_2)$, we see that Eq. (10) implies

$$
\begin{aligned}
\big\| K_{ab}^{i_1 j}(t) \big\| \leqslant{}& K \delta_{i_1}^j + \sum_{i_2 \in Z_{i_1}} h_{a+1} Q \delta_{i_1 \cup i_2}^j t \\
&+ \int_0^t ds \int_0^s dl \left(2 h_{a+1}^2 \sum_{j_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} \big\| K_{a+1 b}^{j_2 j}(l) \big\| \right. \\
&\left. + 2 h_a h_{a+1} \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \big\| K_{ab}^{i_3 j}(l) \big\| \right),
\end{aligned}
\tag{11}
$$

where we have used the following symbol:

$$
\delta_i^k := \begin{cases} 1 & \text{if } \Gamma(a_i,i) \cap \Gamma(a_k,k) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}
\tag{12}
$$

Solving for $\|K_{ab}^{i_1 j}(t)\|$, we find

$$
\begin{aligned}
\big\| K_{ab}^{i_1 j}(t) \big\| \leqslant{}& K \delta_{i_1}^j + \sum_{i_2 \in Z_{i_1}} Q h_{a+1} \delta_{i_1 \cup i_2}^j t + \sum_{j_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} 2 h_{a+1}^2 K \delta_{j_2}^j \frac{t^2}{2!} + \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} 2 h_a h_{a+1} K \delta_{i_3}^j \frac{t^2}{2!} + \sum_{j_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{j_2}} 2 h_{a+1}^2 h_a \\
&\times K Q \delta_{j_2 \cup i_3}^j \frac{t^3}{3!} + \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \sum_{i_4 \in Z_{i_3}} 2 h_{a+1} h_a^2 K Q \delta_{i_3 \cup i_4}^j \frac{t^3}{3!} + \int_0^t dv \int_0^v du \int_0^u ds \int_0^s dl \left((2 h_{a+1} h_a)^2 K^2 \right. \\
&\times \sum_{i_2 \in Z_{i_1}} \sum_{j_2 \in Z_{i_1}} \sum_{i_3 \in Z_{j_2}} \sum_{j_3 \in Z_{j_2}} \big\| K_{ab}^{j_3 j}(l) \big\| + (2 h_{a+1}^2)(2 h_{a+1} h_a) K^2 \sum_{i_2 \in Z_{i_1}} \sum_{j_2 \in Z_{i_1}} \sum_{i_3 \in Z_{j_2}} \sum_{i_4 \in Z_{i_4}} \big\| K_{a+1 b}^{j_4 j}(l) \big\| + (2 h_{a+1} h_a)^2 K^2 \\
&\times \sum_{i_2 \in Z_{i_1}} \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \sum_{i_4 \in Z_{i_3}} \big\| K_{a+1 b}^{i_4 j}(l) \big\| + (2 h_a^2)(2 h_{a+1} h_a) K^2 \sum_{i_2 \in Z_{i_1}} \sum_{i_3 \in Z_{i_2}} \sum_{i_3 \in Z_{1_2}} \sum_{j_3 \in Z_{i_2}} \big\| K_{ab}^{j_3 j}(l) \big\| \right).
\end{aligned}
\tag{13}
$$

Iterating this procedure, we obtain by induction

$$
\big\| K_{ab}^{i_1 j}(t) \big\| \leqslant M \sum_{n=0}^\infty \sqrt{2 h_0 h_1 K}^n \frac{|t|^n}{n!} c_n,
\tag{14}
$$

where $M = \sqrt{2} \max\{\frac{h_0}{h_1}, \frac{h_1}{h_0}\} \times \max\{\frac{1}{K}, 1\} \times \max\{\frac{\sqrt{K}}{Q}, 1\}$ and where c_n is a combinatorial factor counting the number of linking operator chains of n operators between $\Gamma(a,i_1)$ and $\Gamma(b,j)$. What we call an operator chain is heuristically a sequence of intersecting operators linking the initial and final operators.

The process of constructing the sequence of operators forming the chain is as follows: The $2j$th operator in the chain has to be noncommuting with the $(2j-1)$th one. This imposes that the two consecutive operators of a chain have to (1) be of a different interaction type and (2) have overlapping support. For the odd-numbered operators, there is an extra choice: The $(2k+1)$th operator can be an operator that does not commute with the $2k$th operator (as for the even case) *or* an operator that does not commute with the $(2k-1)$th operator. That is, even operators in the sequence must be noncommuting with the previous operator in the sequence, and odd operators in the sequence must be noncommuting with either of the two previous operators in the sequence. From the recursive Eq. (11), we see that if we start with an operator i_1 of type a, the next operator in the chain must be an operator $i_2 \in Z_{i_1}$ of the other type ($a+1$). The fact that it is in Z_{i_1} means that its

support overlaps with i_1's which is similar to what was found in the bounded case. However, if we look at the operator that comes after i_2, we see that we have two distinct possibilities. The first [second double sum under the integrals of Eq. (11)] is that it can be $i_3 \in Z_{i_2}$, an operator of type a (different from $a+1$) whose support overlaps with i_2's; if this were the only possibility, we would have exactly the same situation as we had for bounded systems, but the first double sum under the integrals of Eq. (11) adds another possibility. That second possibility (first double sum) is choosing an operator $j_2 \in Z_{i_1}$ after the operator i_2, j_2 is, like i_2, an operator of type $a+1$ which (by virtue of being in Z_{i_1}) has a support overlapping i_1's. To find the next operator after that we reiterate Eq. (11), and, thus, like the first operator after i_1, we need to choose an operator which is of a different type than the last one (be it j_2 or i_3) and has overlapping support with the last one; thus, at this point, we cannot change our mind. We can thus see the process of building the chain as, for every two choices, we must choose an operator that links with the previous one, but every other choice, we can also choose an operator that links to the penultimate one instead.

Because, for every two choices in building up the chain, we must choose an operator of a different type than the previous one, in the end, the chain contains the same number of operators of type 0 as of type 1 (plus or minus one). This means that there will be the same number of factors of h_1 as of h_0 in every term; hence, we can pull them out of the sums

over chains and simply write an overall factor of $\sqrt{h_0 h_1}$ in front while passing from Eq. (13) to Eq. (14).

Furthermore, we can always find a bound of the following type for c_n:

$$c_n \leqslant \tilde{M} \gamma^n e^{\lambda[(n/\xi)-d]}, \qquad (15)$$

where λ is an arbitrary positive real number. This is because the $\Gamma(a,i)$'s have a diameter of R or less. Hence, if the distance d between the initial and final points is greater than R^n, then there are no possible linking operator chains of n local operators between the initial and final points. Furthermore, since at every odd step along the chain there is a choice of at most ν local operators to choose from for the next operator in the chain and at every even step there is, at most, 2ν operators to choose from, there is a maximum of $(\sqrt{2}\nu)^n$ possible local operator chains of n operators starting from any given position. Thus, we certainly have that

$$c_n \leqslant \sqrt{2}^n \nu^n e^{\lambda(Rn-d)}, \qquad (16)$$

where λ is arbitrary. Using Eq. (15) with Eq. (14), we obtain the LRB of Eq. (1),

$$\left\| \left[\Phi_a^i(t), \Phi_b^j(0) \right] \right\| \leqslant \tilde{M} \exp \lambda \left(2\sqrt{h_0 h_1 K} \frac{\gamma}{\lambda} e^{\lambda/\xi} t - d \right), \quad (17)$$

where $\tilde{\tilde{M}} = \tilde{M} M$. To obtain the generalization to local operators O_P and O_Q satisfying the local observable operator conditions enounced earlier, we introduce $\tilde{K}_a^{i_1}(t) \equiv [\Phi_a^i(t), O_Q(0)]$. Using exactly the same procedure used to obtain Eq. (9), we get

$$\begin{aligned}
&\| [O_P(t), O_Q(0)'(t)] \| \\
&\leqslant \int_0^t ds \left(2 h_b h_a K F_P \sum_{j_2 \in Z_P} \sum_{i_2 \in Z_P} \left\| \tilde{K}_b^{j_2}(s) \right\| \right. \\
&\quad \left. + 2 h_a h_{a+1} K F_P \sum_{i_2 \in Z_P} \sum_{i_3 \in Z_{i_2}} \left\| \tilde{K}_a^{i_3}(s) \right\| \right) \\
&\leqslant 2 \max \left\{ h_0^2, h_1^2 \right\} n_P (n_P + 1) \int_0^t ds \left\| \tilde{K}_a^k(s) \right\|, \quad (18)
\end{aligned}$$

where Z_P is the set of of labels of the terms of the Hamiltonian which do not commute with O_P ($\|Z_P\| = n_P$), where k is such that $\int_0^t ds \, \| \tilde{K}_a^k(s) \| = \max_{i \in Z_P} \int_0^t ds \, \| \tilde{K}_a^i(s) \|$ and where, unlike in Eq. (9), the terms containing no integrals do not appear here because of the condition that $d > R$. $\| \tilde{K}_a^k(s) \|$ can then be treated in exactly the same way as $\| K_{ab}^{i,j}(s) \|$ was, with the only exception that while bounding the final commutators [i.e., when we place the δ of Eq. (12)], we will need an extra factor of F_Q. Thus, we obtain Eq. (2):

$$\begin{aligned}
\| [O_P(t), O_Q(0)] \| &\leqslant F_P F_Q n_P (n_P + 1) \tilde{\tilde{M}} \\
&\times \exp \lambda \left(2 \sqrt{h_0 h_1 K} \frac{\gamma}{\lambda} e^{\lambda/\xi} t - d \right).
\end{aligned} \qquad (19)$$

Optimizing for λ, we have that the Lieb-Robinson speed is thus

$$v_{LR} = 2 \frac{\gamma}{\xi} e \sqrt{h_0 h_1 K}. \qquad (20)$$

We can compare Eq. (19) with the bound obtained for Hamiltonians composed of bounded local operators and for bounded local observables O_P and O_Q which is [22]

$$\begin{aligned}
&\| [O_P(t), O_Q(0)] \| \\
&\leqslant \| O_P \| \| O_Q \| n_P \tilde{M} \exp \lambda \left(2 \sqrt{h_0 h_1} \frac{\gamma}{\lambda} e^{\lambda/\xi} t - d \right). \quad (21)
\end{aligned}$$

To summarize, in this paper, we have shown that an LRB exists for those Hamiltonians that are the sum of local operators whose commutator is bounded. This allows for treating a class of systems with unbounded operators.

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