Pinning Control of Higher Order Nonlinear Network Systems

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***Abstract*— In this letter, we study the problem of controlling via pinning the motion of nonlinear network systems of any order whose dynamics are in controllable canonical form. Different from existing works that either focus on spontaneous synchronization, assume linear dynamics or rely on dynamics cancellation, here we provide a constructive method to prove pinning controllability towards the desired trajectory selected by the pinner. We introduce an algorithmic procedure that associates to any connected topology a suitable Lyapunov function for the network system. The approach is demonstrated on an illustrative example.**

***Index Terms*— Complex networks, companion form, higher-order systems, nonlinear network systems, pinning control.**

I. Introduction

COORDINATING the behavior of coupled dynamical systems attracted an intense research effort in the last decades due to the wide range of applications [1], including truck platooning [2], formation control [3], [4], and swarm robotics [5]. Among the distributed approaches to the control of network systems, pinning control emerged as a viable strategy when only a handful of nodes can directly receive the control input. The strategy prescribes that an additional node (virtual or physical) sharing the same nodes' dynamics, which is called *pinner* or *leader*, injects a control signal only to a subset of so-called *pinned* nodes. First introduced in the field of partial differential equations [6], the approach was later extended to control network of systems described by ordinary differential equations, see e.g. [7]–[10].

In this paper, we use pinning control to steer a network of nonlinear systems in companion form towards a desired solution described by the pinner. Due to the relevance for applications, the problem of controlling networks of systems in companion form has been the focus of extensive literature, which nevertheless mainly dealt with linear dynamics. The research community first offered solutions to the linear second-order consensus problem [11]–[14], to then extend the analysis to specific classes of nonlinear systems, see e.g. [15]–[17]. More recently, the focus shifted towards higher-order systems [18]–[20] but, to the best of our knowledge, no paper has dealt with the pinning control problem of nonlinear network systems in companion form, and the only result for this kind of networks is on spontaneous (uncontrolled) synchronization [21]. An alternative control approach has been explored in [22], where a neural-network based adaptive control algorithm estimates and cancels out the individual dynamics, thus guaranteeing a bounded tracking error given that the vector field describing the individual is nought at the origin.

Different from the existing literature, here we do not cancel the individual dynamics of the nodes and employ proportional-derivative (PD) distrubuted coupling layers to drive the higher order nodes of the controlled network towards the pinner's trajectory. PI and PID coupling protocols have been employed for spontaneous synchronization of both linear and nonlinear network systems, see e.g. [21], [23], [24]. Here, we broaden the approach proposed in [21] to deal with the presence of the pinner, which is essential to achieve the control goal. Specifically, we prove that a network system in companion form of any order n can be pinning controlled by using a PD controller where the derivative action is of order $n-1$. Interestingly, our method is constructive, in the sense that we provide an algorithmic procedure that associates to each individual dynamics and network topology a suitable Lyapunov function that proves the pinning controllability of the higher order nonlinear network system and simultaneously provides a set of suitable control gains. The viability of the proposed approach is demonstrated on a testbed example.

II. Mathematical background

In this section, we report some fundamental properties of graphs and matrices that will be exploited in derivation of the paper's results. For more details, we refer the reader to [25].

Matrix properties. Given a symmetric matrix $\Xi \in \mathbb{R}^{\rho\times\rho}$, i.e. $\Xi = \Xi^{\mathrm{T}}$, the following result holds:

Lemma 1: [25, Theorem 4.2.2] Let $\Xi \in \mathbb{R}^{\rho\times\rho}$ be a symmetric matrix. Then, for all $y \in \mathbb{R}^{\rho}$ we have $\xi^{\min} y^{\mathrm{T}} y \leq y^{\mathrm{T}} \Xi y \leq \xi^{\max} y^{\mathrm{T}} y$, where $\xi^{\min}$ and $\xi^{\max}$ are the smallest and largest eigenvalue of Ξ.

For any index $h \in \{1, \ldots, \rho\}$, the $h \times h$ top left submatrix obtained from Ξ is denoted as a *leading principal submatrix* and its determinant is called a *leading principal minor*. Analogously, the $h\times h$ bottom right submatrix is denoted as a *trailing principal submatrix* and its determinant a *trailing principal minor*. Here, we report the Sylvester's criterion [25], which, given a symmetric matrix, provides a necessary and sufficient

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on its principal minors to assess positive definiteness of the whole matrix.

Lemma 2: [25, Theorem 7.2.5] Let $\xi \in \mathbb{R}^{\rho\times\rho}$ be a symmetric matrix. Then, Ξ is positively defined iff all its leading (or, equivalently, trailing) principal minors, i.e., for $h = 1, \dots, \rho$, are positive definite.

Graphs. A weighted graph of order N is a triple $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = \{1, 2, \dots, N\}$ is the set of *nodes*, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of *edges*, and the set $\mathcal{A} \subset \mathbb{R}^{|\mathcal{E}|}$ of the nonnegative weights associated to the edges. A graph $\mathcal{G}$ can be represented by the *adjacency matrix* $A = A(\mathcal{G}) \in \mathbb{R}^{N\times N}$. If $(i, j) \in \mathcal{E}$, the ij-th element a_{ij} of A is equal to the positive weight associated to (i, j), while it is zero otherwise. For all $i \in \mathcal{N}$, the set of out-neighbors of node i is $\mathcal{N}_i = \{j : (i, j) \in \mathcal{E}\}$. The *Laplacian matrix* $L = L(\mathcal{G}) \in \mathbb{R}^{N\times N}$ is defined as $L = \Delta - A$, where $\Delta = \mathrm{diag}\{\kappa_1, \dots, \kappa_N\}$, with $\kappa_i = \sum_{j=1}^{N} a_{ij}$ being the weighted out-degree of node i. If the graph is undirected, then $L = L^{\mathrm{T}}$, and its real eigenvalues can be sorted in ascending order as $0 = \lambda^{(1)} \le \lambda^{(2)} \le \dots \le \lambda^{(N)}$. Additionally, if $\mathcal{G}$ is connected, $\lambda^{(2)} > 0$. Denoting $\lambda^{\min}(i, \epsilon)$ the smallest eigenvalue of the symmetric matrix $L + \epsilon\mathfrak{e}_i\mathfrak{e}_i^{\mathrm{T}}$, with $\mathfrak{e}_i$ being the ith versor in $\mathbb{R}^N$, we can now report the following useful lemma:

Lemma 3: [8, Equation (26)] Given a weighted undirected connected graph G with associated Laplacian L, for any $i = 1, \dots, N$, $\epsilon > 0$, we have

$$\lambda^{\min}(i, \epsilon) \ge \epsilon\lambda^{(2)}/N\big(\lambda^{(2)} + \epsilon\big) > 0. \tag{1}$$

Vector fields. Given two scalars $n, d > 0$, we consider a vector field $f(t, x) : \mathbb{R}_0^+ \times \mathbb{R}^\rho \to \mathbb{R}^d$, where $t \in \mathbb{R}_0^+$ and $x \in \mathbb{R}^\rho$, with $\rho = nd$. The following definitions and lemma hold:

Definition 1: The vector field $f(t, x)$ is *globally Lipschitz* with respect to x if, for all $y, z \in \mathbb{R}^\rho$, $t \ge 0$, there exists a constant $w > 0$ s.t. $\|f(t, y) - f(t, z)\| \le w\|y - z\|$. The scalar w is the *Lipschitz constant* of f.

Definition 2: Consider that any vector $v \in \mathbb{R}^\rho$ can be written as the stack $\big[v_1^{\mathrm{T}}, \dots, v_n^{\mathrm{T}}\big]^{\mathrm{T}}$, with $v_i \in \mathbb{R}^d$. The vector field $f(t, x)$ is *globally* Γ *weak-Lipschitz* with respect to x if there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{d\times d}$ and a positive scalar w s.t., for all $x, y \in \mathbb{R}^\rho$, $t \ge 0$, and $i \in \{1, \dots, n\}$, $(x_i - y_i)^{\mathrm{T}}\Gamma[f(t, x) - f(t, y)] \le w\|x - y\|^2$. The scalar w is the *weak-Lipschitz constant* of f.

Lemma 4: If the vector field $f(t, x)$ is Lipschitz with constant w', then it is also weak-Lipschitz for any positive definite matrix $\Gamma \in \mathbb{R}^{d\times d}$ with weak-Lipschitz constant $w = w'\|\Gamma\|^n$.

Proof: Let us consider the stack vector field $F_i(t, x) \in \mathbb{R}^{nd}$ defined as $F_i(t, x) = \big[\varphi_1^{\mathrm{T}}, \dots, \varphi_n^{\mathrm{T}}\big]^{\mathrm{T}}$, where $\varphi_j = 0_d$ if $j \ne i$ and $\varphi_i = f(t, x)$, with $0_d \in \mathbb{R}^d$ being a zero vector. For all $i \in \{1, \dots, n\}$, noting that $\|F_i(t, x) - F_i(t, y)\| = \|f(t, x) - f(t, y)\|$, we can write

$$\begin{aligned}&(x_i - y_i)^{\mathrm{T}}\Gamma[f(t, x) - f(t, y)]\\ &= (x - y)^{\mathrm{T}}(I_n \otimes \Gamma)[F_i(t, x) - F_i(t, y)] \le w'\|\Gamma\|^n\|x - y\|^2.\end{aligned}$$

■

III. Problem formulation

In this paper, we study the pinning synchronization problem for a complex network of N higher-order systems in canonical control form coupled through an undirected weighted graph $\mathcal{G} = (\mathcal{N}, \mathcal{V}, \mathcal{A})$. Specifically, each node $i \in \mathcal{N}$ corresponds to a *follower* system, whose state $x^{(i)} \in \mathbb{R}^{nd}$ is defined as $x^{(i)} = \Big[x_1^{(i)\,\mathrm{T}}, \dots, x_n^{(i)\,\mathrm{T}}\Big]^{\mathrm{T}}$, where $x_h^{(i)} = \Big[x_h^{(i),[1]}, \dots, x_h^{(i),[d]}\Big]^{\mathrm{T}} \in \mathbb{R}^d$ is the hth *state component* of node i. For all $h = 1, \dots, n$, $j = 1, \dots, d$, the *scalar component* $x_h^{(i),[j]} \in \mathbb{R}$ of $x_h^{(i)} \in \mathbb{R}^d$ represents the $(h-1)$th order derivative of the component $x_1^{(i),[j]}$ of $x_1^{(i)}$. The dynamics of the ith node can be written in the following form:

$$\begin{aligned}\dot{x}_l^{(i)} &= x_{l+1}^{(i)}, \qquad l = 1, \dots, n-1,\\ \dot{x}_n^{(i)} &= f(t, x^{(i)}) + u^{(i)},\end{aligned} \tag{2}$$

where $u^{(i)} = \big[u^{(i),[1]}, \dots, u^{(i),[d]}\big]^{\mathrm{T}} \in \mathbb{R}^d$ is the distributed control action at node i, and $f \in \mathbb{R}_0^+ \times \mathbb{R}^{nd} \to \mathbb{R}^d$.

Dynamical systems in form (2) are familiar to the control community since they encompass the nonlinear systems that can be written in the canonical control form and, furthermore, arise when modeling mechanical systems in their generalized Lagrangian coordinates.

As in traditional pinning control problems [8], the goal of the distributed control input $u^{(i)}$ is to drive the trajectory of the system towards the one described by an additional (virtual or physical) node, the *pinner*, whose dynamics are the same as those of the followers, that is,

$$\begin{aligned}\dot{p}_l &= p_{l+1}, \qquad l = 1, \dots, n-1\\ \dot{p}_n &= f(t, p),\end{aligned} \tag{3}$$

with $p_l = \Big[p_l^{[1]}, \dots, p_l^{[d]}\Big]^{\mathrm{T}} \in \mathbb{R}^d$ and $p = \big[p_1^{\mathrm{T}}, \dots, p_n^{\mathrm{T}}\big]^{\mathrm{T}}$.

Pinning control assumes that the control input $u^{(i)}$ injected at node i may only depend on the state of node i itself and of the nodes in its neighborhood $\mathcal{N}_i$. Only a (typically small) subset $\mathcal{P} \subset \mathcal{N}$ of the network nodes also receives information on the state of the pinner. In a graph representation, this means that the pinner is unidirectionally coupled to the so-called pinned nodes in $\mathcal{P}$.

Control Objective: Design an asymptotically vanishing distributed control input $u^{(i)}$ such that global pinning synchronization of the higher order network system (2) is achieved for all possible initial conditions. In formal terms, design $u^{(i)}$ such that, for all $x^i(0) \in \mathbb{R}^{nd}$, $i = 1, \dots, N$, and $p(0) \in \mathbb{R}^{nd}$,

$$\lim_{t\to+\infty} \|e(t)\| = 0, \tag{4a}$$

$$\lim_{t\to+\infty} \|u^{(i)}(t)\| = 0, \quad \forall i = 1, \dots, N, \tag{4b}$$

where $e(t) = x(t) - p(t)$ is the overall *network pinning error*.

From the systems' structure (2)-(3), a coupling is only possible through the nth state component. In view of this, to achieve our control objective we choose for each agent a diffusive protocol where all the state derivatives are leveraged. Denoting with l_{ij} the ijth element of the Laplacian matrix L

associated to $\mathcal{G}$, the resulting protocol is

$$u^{(i)} = -\bar{k}\sum_{h=1}^{n} k_h \sum_{j=1}^{N}\left[l_{ij}\Gamma x_h^{(j)} + g_i(x_h^{(i)} - p_h)\right],\ i\in\mathcal{N}, \quad (5)$$

where the pinning gain g_i is positive if $i \in \mathcal{P}$, while it is 0 otherwise, $\bar{k}$ is the overall coupling gain; and k_h is the control gain associated to the h-th state component. The $d\times d$ inner linking matrix $\Gamma = \Gamma^{\mathrm{T}} > 0$, when different from the identity, can be used to set different coupling intensities for each scalar component [8]. For weak-Lipschitz systems, it is chosen according to Definition 2, while for Lipschitz vector fields it can be freely selected by the control designer as per Lemma 4.

The rationale behind control protocol (5) is to couple homologous state variables. For instance, when applied to kinematic chains of several degrees of freedom described in the position/velocity state-space, in Equation (5) we have $n = 2$ (position and velocity), where $x_1^{(j)}$ represents the position of all the joints in the mechanical system j and $x_2^{(j)}$ the velocities of its joints.

Note that, when $n = 1$, protocol (5) reduces to the standard diffusive pinning control protocol [7], [8]. Therefore, in this paper we directly focus on the case $n \geq 2$. The choice of the diffusive protocol (5) implies that, when (4a) is attained, also (4b) is achieved. Therefore, to prove global pinning controllability, it suffices to show that the error dynamics converge to zero. Combining (2), (3) and (5), the error dynamics can be written as

$$\begin{aligned}\dot{e}_l &= e_{l+1}^{(i)}, \qquad\qquad l = 1,\dots,n-1,\\ \dot{e}_n &= F(t,x) - 1_N\otimes f(t,p) - \bar{k}\sum_{h=1}^{N} k_h(\Omega\otimes\Gamma)e_h,\end{aligned} \quad (6)$$

where $F(t,x) = \left[f^{\mathrm{T}}(t,x^{(1)}),\dots f^{\mathrm{T}}(t,x^{(N)})\right]^{\mathrm{T}}$ and $\Omega = L + G$, with $G = \mathrm{diag}\{g_1,\dots,g_N\}$ being the *pinner matrix*.

IV. Higher order pinning synchronization

Here, we show that the distributed protocol (5) can be effectively used to control the network of systems in companion form (2). Specifically, we illustrate an algorithmic procedure that builds a quadratic Lyapunov function and identifies suitable values for the control gains k_h, $h = 1,\dots,n$, to enforce global pinning synchronization.

The Lyapunov function we consider is the following:

$$V(e) = e^{\mathrm{T}}Pe/2, \quad (7)$$

where $P \in \mathbb{R}^{ndN\times ndN}$ is symmetric and positive definite. Here, we provide an algorithmic procedure to select both matrix P and a positive definite matrix $Q \in \mathbb{R}^{ndN\times ndN}$ that will be instrumental in bounding the time derivative of V.

Specifically, matrices P and Q will be obtained from a backward recursion, which will ensure that, given a proper selection of the control gains, all of its principal minors are positive definite. This will in turn imply the positive definitess of P and Q from the Sylvester's criterion reported in Lemma 2. We start by defining

$$\Xi = \Omega\otimes\Gamma,\ \Xi_0 = 0_{dN}, \Xi_{n+1} = I_{dN}/2,\ \Xi_h = k_h\Xi,\ h = 1,\dots,n,$$

with $\Omega = L + G$ defined as in (6). Next, we consider the following backward recursions to define the size $(q+1)dN \times (q+1)dN$ matrices P_{n-q}, Q_{n-q}, $q = n-1,\dots,0$:

$$\begin{aligned}&P_{n-q} = \begin{bmatrix} P_{\varphi,n-q} & P_{\psi,n-q}\\ P_{\psi,n-q}^{\mathrm{T}} & P_{n-q+1}\end{bmatrix}, Q_{n-q} = \begin{bmatrix} Q_{\varphi,n-q} & Q_{\psi,n-q}\\ Q_{\psi,n-q}^{\mathrm{T}} & Q_{n-q+1}\end{bmatrix},\\ &P_n = \Xi_n, \qquad\qquad Q_n = \Xi_n^2 - \Xi_{n-1},\\ &P_{\varphi,n-q} = 2\Xi_{n-q}\Xi_{n-q+1},\ \ P_{\psi,n-q} = 2\Xi_{n-q}\left[\Xi_{n-q+2},\dots,\Xi_{n+1}\right],\\ &Q_{\varphi,n-q} = \Xi_{n-q}^2 - 2\Xi_{n-q-1}\Xi_{n-q+1},\\ &Q_{\psi,n-q} = -\Xi_{n-q-1}\left[\Xi_{n-q+2},\dots,-\Xi_{n+1}\right].\end{aligned} \quad (8)$$

Given a scalar $\bar{k} > 1$, matrices P_1, Q_1, which are the last step of recursion (8), can be written in block form as

$$P_1 = \begin{bmatrix}\bar{k}P_\vartheta & P_\varsigma\\ P_\varsigma^{\mathrm{T}} & \Xi_n\end{bmatrix},\quad Q_1 = \begin{bmatrix}\bar{k}Q_\vartheta & Q_\varsigma\\ Q_\varsigma^{\mathrm{T}} & \Xi_n^2 - \Xi_{n-1}\end{bmatrix}, \quad (9)$$

from which we define P and Q as

$$P = \begin{bmatrix}P_\vartheta & P_\varsigma\\ P_\varsigma^{\mathrm{T}} & \Xi_n\end{bmatrix},\quad Q = \begin{bmatrix}Q_\vartheta & Q_\varsigma\\ Q_\varsigma^{\mathrm{T}} & \Xi_n^2 - \Xi_{n-1}\end{bmatrix}. \quad (10)$$

Before showing how we can select $k_1,\dots,k_n$ such that $P, Q > 0$, we define a set of auxiliary matrices that are instrumental for the proof.

Auxiliary matrices. Being Ω symmetric and positive definite, it can be written as $\Omega = V_\Omega\Delta_\Omega V_\Omega^{\mathrm{T}}$, where Λ_Q is the diagonal matrix of its positive eigenvalues, and V_Ω is obtained by juxtaposing column-wise its N orthonormal eigenvectors. A similar decomposition (and notation) can be employed for matrix $\Gamma = \Gamma^{\mathrm{T}} > 0$. From the properties of the Kronecker product, we have that $V_\Xi\Delta_\Xi V_\Xi^{\mathrm{T}}$, where $V_\Xi = V_\Omega\otimes V_\Gamma$ and $\Delta_\Xi = \Delta_\Omega\otimes\Delta_\Gamma$. Denoting $\xi^{(1)},\dots,\xi^{(dN)}$ the eigenvalues of Ξ sorted in ascending order, and setting, for all $i = 1,\dots,dN$,

$$\xi_h^{(i)} = k_h\xi^{(i)},\ \ h = 1,\dots,n,\ \ \xi_{n+1}^{(i)} = 1/2, \xi_0^{(i)} = 0,$$

we recursively define matrices $P_{n-q}^{(i)}, Q_{n-q}^{(i)} \in \mathbb{R}^{(q+1)\times(q+1)}$, with $q = n-1,\dots,0$ as

$$\begin{aligned}&P_{n-q}^{(i)} = \begin{bmatrix}P_{\varphi,n-q}^{(i)} & P_{\psi,n-q}^{(i)}\\ P^{(i)}{}_{\psi,n-q}^{T} & P_{n-q+1}^{(i)}\end{bmatrix}, Q_{n-q}^{(i)} = \begin{bmatrix}Q_{\varphi,n-q}^{(i)} & Q_{\psi,n-q}^{(i)}\\ Q^{(i)}{}_{\psi,n-q}^{T} & Q_{n-q+1}^{(i)}\end{bmatrix},\\ &P_n^{(i)} = \xi_n^{(i)}, \qquad\qquad Q_n^{(i)} = \xi_n^{(i)^2} - \xi_{n-1}^{(i)},\\ &P_{\varphi,n-q}^{(i)} = 2\xi_{n-q}^{(i)}\xi_{n-q+1}^{(i)},\quad P_{\psi,n-q}^{(i)} = 2\xi_{n-q}^{(i)}\left[\xi_{n-q+2}^{(i)},\dots,\xi_{n+1}^{(i)}\right],\\ &Q_{\varphi,n-q}^{(i)} = \xi_{n-q}^{(i)^2} - 2\xi_{n-q-1}^{(i)},\ Q_{\psi,n-q}^{(i)} = -\xi_{n-q-1}^{(i)}\left[\xi_{n-q+2}^{(i)},\dots,\xi_{n+1}^{(i)}\right].\end{aligned} \quad (11)$$

and the scalars

$$\delta_{n-q}^{(i)} = \delta_{n-q+1}^{(i)} + 2\xi_{n-q+2}^{(i)}, q = 2,\dots,n-1,\ \ \delta_{n-1}^{(i)} = 1.$$

Finally, by setting

$$A_{n-1}^{(i)} = \begin{bmatrix}2\xi_{n-1}^{(i)}\xi_n^{(i)} & \xi_{n-1}^{(i)}\\ \xi_{n-1}^{(i)} & \xi_n^{(i)}\end{bmatrix},\ \ \begin{aligned}\alpha_{n-1}^{(i)} &= \min\mathrm{eig}\{A_{n-1}^{(i)}\},\\ \beta_{n-1}^{(i)} &= \xi_n^{(i)^2} - \xi_{n-1}^{(i)},\end{aligned}$$

we can then iteratively define, for $q = 2,\dots,n-1$,[1] $\alpha_{n-q}^{(i)} = \min\mathrm{eig}\{A_{n-q}^{(i)}\}$ and $\beta_{n-q}^{(i)} = \min\mathrm{eig}\{B_{n-q}^{(i)}\}$, with

$$A_{n-q}^{(i)} = \begin{bmatrix}2\xi_{n-q}^{(i)}\xi_{n-q+1}^{(i)} & \delta_{n-q}^{(i)}\xi_{n-q}^{(i)}\\ \delta_{n-q}^{(i)}\xi_{n-q}^{(i)} & \alpha_{n-q+1}^{(i)}\end{bmatrix},$$

$$B_{n-q}^{(i)} = \begin{bmatrix}\xi_{n-q+1}^{(i)^2} - 2\xi_{n-q}^{(i)}\xi_{n-q+2}^{(i)} & -\frac{1}{2}\delta_{n-q+1}^{(i)}\xi_{n-q}^{(i)}\\ -\frac{1}{2}\delta_{n-q+1}^{(i)}\xi_{n-q}^{(i)} & \beta_{n-q+1}^{(i)}\end{bmatrix}.$$

Algorithm 1 Gain $\mathcal{K}$ selection for $P_1, Q_1 > 0$

1: Set $k_n \leftarrow 1$
2: Compute $\xi^{\min} = \min \operatorname{eig}\{\Xi\}$
3: Choose $0 < k_{n-1} < \xi^{\min}$
4: Set $\mathcal{K} = \{k_{n-1}, k_n\}$
5: Set $\Xi_{n-1} \leftarrow k_{n-1}\Xi$
6: **for** $q = 2, \ldots, n-1$ **do**
7: Compute $\operatorname{eig}\{\Xi_{n-q+1}\}$
8: **for** $i = 1, \ldots, dN$ **do**
9: Compute $\beta^{(i)}_{n-q+1}$, $\alpha^{(i)}_{n-q+1}$, and $\delta^{(i)}_{n-q}$
10: Set $s^{(i)}_{n-q} \leftarrow \min\{r^{(i)}_{n-q,1}, r^{(i)}_{n-q,2}\}$, with

$$r^{(i)}_{n-q,1} = 2\xi^{(i)}_{n-q+1}\alpha^{(i)}_{n-q+1}/{\delta^{(i)}_{n-q}}^2, \tag{12}$$

$$r^{(i)}_{n-q,2} = \sup_{r\in\mathbb{R}}\Big\{{\delta^{(i)}_{n-q+1}}^2 r^2 + 8\xi^{(i)}_{n-q+2}\beta^{(i)}_{n-q+1} r - 4{\xi^{(i)}_{n-q+1}}^2\beta^{(i)}_{n-q+1} < 0\Big\}. \tag{13}$$

11: Set $\rho^{(i)}_{n-q} \leftarrow \dfrac{s^{(i)}_{n-q}}{\xi^{(i)}_{n-q+1}}$
12: **end for**
13: Choose $0 < \bar{\rho}_{n-q} < \min_{i=1,\ldots,dN} \rho^{(i)}_{n-q}$
14: Set $k_{n-q} \leftarrow \bar{\rho}_{n-q} k_{n-q+1}$, $\mathcal{K} \leftarrow \mathcal{K} \cup \{k_{n-q}\}$, and $\Xi_{n-q} \leftarrow k_{n-q}\Xi$
15: **end for**

Lemma 5: If the gain set $\mathcal{K} = \{k_1, \ldots, k_n\}$ is selected according to Algorithm 1, then $P_1 > 0$ and $Q_1 > 0$.

Proof: Let $\mathfrak{e}_l$ be the lth versor in $\mathbb{R}^n$. The set $S := \{\mathfrak{e}_l \otimes v^{(r)}_\Omega \otimes v^{(s)}_\Gamma | l = 1, \ldots, n;\ r = 1, \ldots, N;\ s = 1, \ldots, d\}$ is a $\mathbb{R}^{ndN}$ basis of orthogonal vectors. Indeed, for all $i \neq j$,

$$(\mathfrak{e}_{l_i} \otimes v^{(r_i)}_\Omega \otimes v^{(s_i)}_\Gamma)^{\mathrm{T}}(\mathfrak{e}_{l_j} \otimes v^{(r_j)}_\Omega \otimes v^{(s_j)}_\Gamma) =$$
$$({\mathfrak{e}_{l_i}}^{\mathrm{T}}\mathfrak{e}_{l_j}) \otimes ({v^{(r_i)}_\Omega}^{\mathrm{T}} v^{(r_j)}_\Omega) \otimes ({v^{(s_i)}_\Gamma}^{\mathrm{T}} v^{(s_j)}_\Gamma) = 0,$$

Noting that S can also be defined as $S = \{\mathfrak{e}_l \otimes v^{(w)}_\Xi | l = 1, \ldots, n;\ w = 1, \ldots, dN\}$, any vector $y \in \mathbb{R}^{ndN}$ can be written as $y = \sum_{i=1}^{Nd} y^{(i)}$, where $y^{(i)} = c^{(i)} \otimes v^{(i)}_\Xi$, with $c^{(i)} = (c^{(i)}_1, \ldots, c^{(i)}_n)^{\mathrm{T}} \in \mathbb{R}^n$ being a vector of coefficients.

Since $v^{(i)}_\Xi$ and $v^{(j)}_\Xi$ are orthogonal, for $i \neq j$ we have that ${y^{(j)}}^{\mathrm{T}} P_1 y^{(i)} = 0$ and ${y^{(j)}}^{\mathrm{T}} Q_1 y^{(i)} = 0$, while from definitions (11) we have ${y^{(i)}}^{\mathrm{T}} P_1 y^{(i)} = {c^{(i)}}^{\mathrm{T}} P^{(i)}_1 c^{(i)}$ and ${y^{(i)}}^{\mathrm{T}} Q_1 y^{(i)} = {c^{(i)}}^{\mathrm{T}} Q^{(i)}_1 c^{(i)}$. Therefore, the following relations hold

$$y^{\mathrm{T}} P_1 y = \sum_{i=1}^{dN} {c^{(i)}}^{\mathrm{T}} P^{(i)}_1 c^{(i)},\ y^{\mathrm{T}} Q_1 y = \sum_{i=1}^{dN} {c^{(i)}}^{\mathrm{T}} Q^{(i)}_1 c^{(i)}. \tag{14}$$

We show that $P^{(i)}_1, Q^{(i)}_1 > 0$ by induction. First, notice that from lines 1–4 in Algorithm 1, we have that the coefficients $\alpha^{(i)}_{n-1}, \beta^{(i)}_{n-1}, \delta^{(i)}_{n-1}$ are strictly positive. Therefore, matrix $P^{(i)}_{n-1}$ $(= A^{(i)}_{n-1})$ is positive defined from Lemmas 2 and 3, since its determinant is positive and $\xi^{(i)}_n > 0$. Also, $\alpha^{(i)}_{n-1} > 0$ and $z^{\mathrm{T}} P^{(i)}_{n-1} z \geq z^{\mathrm{T}} A^{(i)}_{n-1} z \geq \alpha^{(i)}_{n-1} z^{\mathrm{T}} z$ holds or all $z \in \mathbb{R}^2$.

Now, let us suppose that, for a given $q \geq 2$,

$$z^{\mathrm{T}} P^{(i)}_{n-q+1} z \geq z^{\mathrm{T}} A^{(i)}_{n-q+1} z \geq \alpha^{(i)}_{n-q+1} z^{\mathrm{T}} z, \quad \forall z \in \mathbb{R}^q, \tag{15}$$

[1] B_{n-q} and β_{n-q} are defined also for $q = n$.

and $\alpha^{(i)}_{n-q+1}, \delta^{(i)}_{n-q} > 0$. Let $\bar{z}_q \in \mathbb{R}^{q+1}$ be $\bar{z}_q = [z_1, \bar{z}^{\mathrm{T}}_{q-1}]^{\mathrm{T}}$, with $\bar{z}_{q-1} = [z_2, \ldots, z_{q+1}]^{\mathrm{T}}$. We have

$$\begin{aligned}\bar{z}^{\mathrm{T}}_q P^{(i)}_{n-q} \bar{z}_q &= 2\xi^{(i)}_{n-q}\xi^{(i)}_{n-q+1} z_1^2 + \sum_{j=2}^{q} 4\xi^{(i)}_{n-q}\xi^{(i)}_{n-q+j} z_1 z_j \\ &\quad + 2\xi^{(i)}_{n-q} z_1 z_{q+1} + \bar{z}^{\mathrm{T}}_{q-1} P^{(i)}_{n-q+1} \bar{z}_{q-1}.\end{aligned} \tag{16}$$

Defining $h = \arg\min_{j=2,\ldots,q+1} z_1 z_j$, from (15) we obtain

$$\begin{aligned}\bar{z}^{\mathrm{T}}_q P^{(i)}_{n-q} \bar{z}_q &\geq 2\xi^{(i)}_{n-q}\xi^{(i)}_{n-q+1} z_1^2 + 2\Big[1 + \sum_{j=2}^{q} 2\xi^{(i)}_{n-q+j}\Big]\xi^{(i)}_{n-q} z_1 z_h \\ &\quad + \alpha^{(i)}_{n-q+1} z_h^2 = 2\xi^{(i)}_{n-q}\xi^{(i)}_{n-q+1} z_1^2 \\ &\quad + 2\delta^{(i)}_{n-q}\xi^{(i)}_{n-q} z_1 z_h + \alpha^{(i)}_{n-q+1} z_h^2 \\ &= [z_i, z_h] A^{(i)}_{n-q} [z_i, z_h]^{\mathrm{T}}.\end{aligned}$$

Noting that $\alpha^{(i)}_{n-q+1} > 0$, and since the condition (12) in Algorithm 1 ensures that the determinant of $A^{(i)}_{n-q}$ is positive, then $A^{(i)}_{n-q} > 0$ from Lemma 2. By induction, $P^{(i)}_1 > 0$.

Similar steps can be followed to show that $Q_1 > 0$. Indeed, from lines 1–4 of Algorithm 1, we have $Q^{(i)}_n \in \mathbb{R}^{1\times 1} = \beta^{(i)}_{n-1} = {\xi^{(i)}_n}^2 - \xi^{(i)}_{n-1} > 0$. Let us assume, for some $q \geq 1$, that

$$z^{\mathrm{T}} Q^{(i)}_{n-q+1} z \geq \beta_{n-q} z^{\mathrm{T}} z, \quad \forall z \in \mathbb{R}^q. \tag{17}$$

Defining $h = \arg\max_{j=2,\ldots,q+1} z_1 z_j$, and from (17), we have

$$\begin{aligned}\bar{z}^{\mathrm{T}}_q Q^{(i)}_{n-q} \bar{z}_k &\geq \Big[{\xi^{(i)}_{n-q}}^2 - 2\xi^{(i)}_{n-q-1}\xi^{(i)}_{n-q+1}\Big] z_1^2 \\ &\quad - \Big[1 + \sum_{j=2}^{q} 2\xi^{(i)}_{n-q+j}\Big]\xi^{(i)}_{n-q-1} z_1 z_h + \beta^{(i)}_{n-q} z_h^2 \\ &= \Big({\xi^{(i)}_{n-q}}^2 - 2\xi^{(i)}_{n-q-1}\xi^{(i)}_{n-q+1}\Big) z_1^2 - \delta^{(i)}_{n-q} z_1 z_h + \beta^{(i)}_{n-q} z_h^2 \\ &= [z_i, z_h] B^{(i)}_{n-q-1} [z_i, z_h]^{\mathrm{T}}.\end{aligned}$$

Lemma 2 yields $B^{(i)}_{n-q-1} > 0$ since its determinant is positive from (13), and $\beta^{(i)}_{n-q} > 0$. As (17) holds for a given q, by induction it holds for all q, and then $Q^{(i)}_1 > 0$ for all i. As we already proved $P^{(i)}_1 > 0$, the thesis follows. ■

Lemma 6: Matrices P, Q in (10) are positive definite for any $\bar{k} > 1$.

Proof: The quadratic form $y^{\mathrm{T}} P y$ can be rewritten as $y^{\mathrm{T}} P y = y^{\mathrm{T}} P_1 y + (\bar{k} - 1) y^{\mathrm{T}}_\vartheta P_\vartheta y_\vartheta$ and, since P_ϑ is the leading principal minor of P_1, it is positive definite. Therefore $y^{\mathrm{T}} P y > 0$.

Writing $y = \big[y^{\mathrm{T}}_\vartheta, y^{\mathrm{T}}_\varsigma\big]^{\mathrm{T}}$, the quadratic form $y^{\mathrm{T}} Q y$ can be rewritten as $y^{\mathrm{T}} Q y = y^{\mathrm{T}} Q_1 y + (\bar{k} - 1) y^{\mathrm{T}}_\vartheta Q_\vartheta y_\vartheta + (\bar{k} - 1) y^{\mathrm{T}}_\varsigma \Xi^2 y_\varsigma$. Noting that ${y^{(i)}}^{\mathrm{T}} y^{(i)} = \big(c^{(i)} \otimes v^{(i)}_\Xi\big)^{\mathrm{T}} \big(c^{(i)} \otimes v^{(i)}_\Xi\big) = {c^{(i)}}^{\mathrm{T}} c^{(i)}$ and ${y^{(i)}}^{\mathrm{T}} y^{(j)} = 0$ for $i \neq j$, we have

$$\begin{aligned}y^{\mathrm{T}} Q_1 y &= \sum_{i=1}^{dN} {c^{(i)}}^{\mathrm{T}} Q^{(i)}_1 c^{(i)} \geq \sum_{i=1}^{dN} \beta^{(i)}_0 {c^{(i)}}^{\mathrm{T}} c^{(i)} \\ &\geq \bar{\beta} \sum_{i=1}^{dN} {c^{(i)}}^{\mathrm{T}} c^{(i)} \geq \bar{\beta} y^{\mathrm{T}} y,\end{aligned} \tag{18}$$

where $\bar{\beta} = \min_i \beta^{(i)}_0 > 0$. Finally, we obtain

$$y^{\mathrm{T}} Q y \geq \bar{\beta} y^{\mathrm{T}} y + (\bar{k} - 1)\tilde{\beta} y^{\mathrm{T}} y, \tag{19}$$

where $\tilde{\beta} = \min_{i=1,\ldots,dN}\left\{\bar{\beta}, \xi_n^{(i)^2}\right\} > 0$. ■

Remark 1: The reasoning followed in this section shares similarities to the one carried out in [21] in the context of spontaneous synchronization. However, since in [21] a different problem is studied, the derived Lyapunov function has different spectral properties, and key fundamental differences exist in its derivation. For instance, in [21] one could leverage the properties of class $\mathfrak{L}_N$ matrices, while in our case this is not possible due to the presence of the pinner. Additionally, we also have to account for the presence of the inner linking matrix Γ in the coupling protocol, required since each state component has size d, while in [21] $d = 1$ (and $\Gamma = 1$). Note that the possibility of dealing with generic d is very useful in view of applications. For instance, when studying the mechanics of interacting rigid bodies moving on a plane (e.g. in platooning of unmanned ground vehicles), two coordinates need to be considered, and therefore $d = 2$.

Lemma 6 showed that (7) is a valid Lyapunov function candidate with P defined as in (10). Therefore, it can be exploited to provide the main result of the manuscript. In the following theorem, we i) show that the PD distributed protocol (5) can be used to achieve global pinning controllability for network systems in companion form of any order, and ii) provide a feasible choice for the control and coupling gains.

Theorem 1: Let us consider a connected network of N higher-order systems in canonical control form (2). If the set of pinned nodes $\mathcal{P}$ is nonempty, and the function $f(t,z)$ is Γ weak-Lipschitz[2], then there always exists a choice of gains $\mathcal{K} = \{k_1, \ldots, k_n\}$ and $\bar{k} > 1$ such that under the distributed control law (5) the networked system achieves global pinning synchronization (4). Furthermore, a possible choice for $\mathcal{K}$ is given by the selection procedure in Algorithm 1, while $\bar{k}$ can be taken larger than $\max\left\{1, \frac{1}{\tilde{\beta}}\left(\hat{k}w\|\Omega\|^{n+d} + \tilde{\beta} - \bar{\beta}\right)\right\}$.

Proof: Let us compute $\mathcal{K}$ from Algorithm 1, matrices Ξ_h, $h = 1, \ldots, n$ as defined in Section IV, and P, Q as in (10).

The derivative of the candidate Lyapunov function (7) is

$$\dot{V}(e) = e^{\mathrm{T}}P\dot{e} = e^{\mathrm{T}}P\Phi(t,x) + e^{\mathrm{T}}P\Upsilon(e), \tag{20}$$

with $\Phi(t,x) = \left[0^{\mathrm{T}}_{(n-1)dN}, F^{\mathrm{T}}(t,x) - 1^{\mathrm{T}}_N \otimes f^{\mathrm{T}}(t,p)\right]^{\mathrm{T}}$ and $\Upsilon(e) = \left[e_2^{\mathrm{T}}, \ldots, e_n^{\mathrm{T}}, -\left(\bar{k}\sum_{h=1}^{N} k_h(\Omega \otimes \Gamma)e_h\right)^{\mathrm{T}}\right]^{\mathrm{T}}$. Note that

$$e^{\mathrm{T}}P\Phi = M(L) + M(G), \tag{21}$$

where L and G are the Laplacian and pinner matrices, and where given a matrix $J \in \mathbb{R}^{N\times N}$, $M(J) = \sum_{q=1}^{n} k_q e_q^{\mathrm{T}} J \otimes \Gamma\left[F(t,x) - 1_N \otimes f(t,x)\right]$. Since f is Γ weak-Lipschitz, we have that

$$\begin{aligned}
M(J) &= \sum_{q=1}^{n} k_q \frac{1}{2} \sum_{i=}^{N}\sum_{j=1}^{N} J_{ij}(e_q^{(i)} - e_q^{(j)})^{\mathrm{T}}\Gamma\left[f(t,x^{(i)}) - f(t,x^{(j)})\right] \\
&\leq \sum_{q=1}^{n} k_q \frac{1}{2}\sum_{i=}^{N}\sum_{j=1}^{N} J_{ij} w \sum_{h=1}^{n}\left(e_h^{(i)} - e_h^{(j)}\right)^{\mathrm{T}}\left(e_h^{(i)} - e_h^{(j)}\right) \\
&= \hat{k}w\sum_{h=1}^{n} e_h^{\mathrm{T}}(J \otimes I_d)e_h = \hat{k}we^{\mathrm{T}}\left(I_n \otimes J \otimes I_d\right)e,
\end{aligned}$$

[2] In case of pinning synchronization towards an invariant set, this hypothesis can be replaced by requiring $f(t,z)$ locally Lipschitz, see also [26].

for both $J = L$ and $J = G$, where $\hat{k} = \sum_{q=1}^{n} k_q$, thus yielding

$$e^{\mathrm{T}}P\Phi \leq \hat{k}we^{\mathrm{T}}\left(I_n \otimes \Omega \otimes I_d\right)e. \tag{22}$$

We next focus on showing that

$$e^{\mathrm{T}}P\Upsilon(e) = -e^{\mathrm{T}}Qe. \tag{23}$$

Given any $q \in \{0, \ldots, n-1\}$, let us call $\bar{Q}_{n-q} \in \mathbb{R}^{(q+1)\times(q+1)}$ the trailing principal submatrix of Q, and let us denote $\bar{Q}^R_{n-q}$ its first row (which coincides with the first column being Q symmetric), that is,

$$\begin{aligned}
\bar{Q}^R_{n-q} = \Big[&\bar{k}\Xi^2_{n-q} - 2\bar{k}\Xi_{n-q-1}Xi_{n-q+1}, -\bar{k}\Xi_{n-q-1}\Xi_{n-q+2}, \\
&\ldots, -\bar{k}\Xi_{n-q-1}\Xi_n, -\Xi_{n-q-1}/2\Big].
\end{aligned}$$

Denoting q_{ij} the element ij of Q, the sum of the bilinear terms $q_{ij}e_i^{\mathrm{T}}e_j$ of $e^{\mathrm{T}}Qe$ corresponding to the first row and

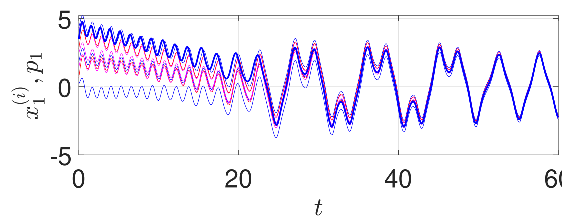

Fig. 1. Time trace of the components $\boldsymbol{x}_1^{(i),[1]}$, $\boldsymbol{i = 1, \ldots, N}$, and $\boldsymbol{p}_1^{[1]}$ (in blue) for the network of chaotic oscillators described by (2), (3), (24) in the presence of the control input (5).

column of $\bar{Q}_{n-q}$ coincides with the sum of the terms of the product $e^{\mathrm{T}}P\Upsilon(e)$ corresponding to row $n-q$ of P truncated of its first $n-q-2$ elements, which can be written as

$$\bar{P}^R_{n-q} = \left[2\bar{k}\Xi_{n-q}[\Xi_{n-q+1}, \Xi_{n-q+1}, \Xi_{n-q+2}, \Xi_n], \Xi_{n-q}\right].$$

Indeed, we have

$$\begin{aligned}
&\sum_{\substack{i=n-q, j=n-q,\ldots,n \\ i=n-q,\ldots,n, j=n-q}} -q_{ij}e_i^{\mathrm{T}}e_j = 2\bar{k}e^{\mathrm{T}}_{n-q}\Xi_{n-q-1}\Xi_{n-q+1}e^{\mathrm{T}}_{n-q} \\
&+ \sum_{j=n-q+1}^{n} 2\bar{k}e^{\mathrm{T}}_{n-q}\Xi_{n-q}\Xi_j e_j - \sum_{j=n-q}^{n}\bar{k}e^{\mathrm{T}}_{n-q}\Xi_{n-q}\Xi_j e_j \\
&- \sum_{i=n-q+1}^{n}\bar{k}e_i^{\mathrm{T}}\Xi_i\Xi_{n-q}e_{n-q} \\
&+ \sum_{i=n-q+1}^{n-1} 2\bar{k}e_i^{\mathrm{T}}\Xi_{n-q-1}\Xi_{i+1}e_{n-q} + e_n^{\mathrm{T}}\Xi_{n-q-1}e_{n-q},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\sum_{\substack{i=n-q, j=n-q,\ldots,n \\ i=n-q,\ldots,n, j=n-q}} -q_{ij}e_i^{\mathrm{T}}e_j \\
&= 2\bar{k}e^{\mathrm{T}}_{n-q}\Xi_{n-q-1}\Xi_{n-q+1}e^{\mathrm{T}}_{n-q} - \bar{k}e^{\mathrm{T}}_{n-q}\Xi^2_{n-q}e_{n-q} \\
&+ \frac{1}{2}\bar{k}\sum_{i=n-q+1}^{n-1}\left[e_i^{\mathrm{T}}\Xi_{n-q-1}\Xi_{i+1}e_{n-q} + e^{\mathrm{T}}_{n-q}\Xi_{n-q-1}\Xi_{i+1}e_i\right] \\
&+ \frac{1}{2}e_n^{\mathrm{T}}\Xi_{n-q-1}e_{n-q} + \frac{1}{2}e^{\mathrm{T}}_{n-q}\Xi_{n-q-1}e_n.
\end{aligned}$$

Iterating for all $q = 0, \ldots, n-1$, we obtain (23). By combining (19) and (20) with (22) and (23), we have

$$\begin{aligned}
\dot{V}(e) &\leq \hat{k}we^{\mathrm{T}}\left(I_n \otimes \Omega \otimes I_d\right)e - \bar{\beta}e^{\mathrm{T}}e - (\bar{k}-1)\tilde{\beta}e^{\mathrm{T}}e, \\
&\leq -\left[(\bar{k}-1)\tilde{\beta} + \bar{\beta} - \hat{k}w\|\Omega\|^{n+d}\right]e^{\mathrm{T}}e.
\end{aligned}$$

Setting $\bar{k} > \max\left\{1, \frac{1}{\tilde{\beta}}\left(\hat{k}w\|\Omega\|^{n+d} + \tilde{\beta} - \bar{\beta}\right)\right\}$, the thesis follows. ■

Remark 2: The result of Theorem 1 can be extended to the case of nonlinear systems of the form $\dot{x}_i = f(x_i) + g(x_i)u_i$ which admit a nonlinear state transformation able to recast them in the canonical control form. If such controllability conditions are satisfied, then Theorem 1 can be applied on the transformed system and a coupling protocol analogous to (5) for the transformed states can be obtained. We refer the reader to [27, Section 6.2] for details on canonical transformations.

V. Numerical validation

To illustrate the effectiveness of our approach, we select as individual dynamics a third order chaotic oscillator, inspired by the well-known Van der Pol second order oscillator. Specifically, in equation (2), we set $n = 3$, $d = 2$, and

$$\begin{aligned} f^{[1]}(t,x) =& -x_2^{[1]} + \mu^{[1]}\left(1 + \left|x_2^{[1]} + \sigma x_2^{[2]}\right|\right)x_3^{[1]} \\ & - x_1^{[1]} + \nu^{[1]}\left(1 + \left|x_1^{[1]} + \sigma x_1^{[2]}\right|\right)x_2^{[1]}, \\ f^{[2]}(t,x) =& -x_2^{[2]} + \mu^{[2]}\left(1 + \left|x_2^{[2]} + \sigma x_2^{[1]}\right|\right)x_3^{[2]} \\ & - x_1^{[2]} + \nu^{[2]}\left(1 + \left|x_1^{[2]} + \sigma x_1^{[1]}\right|\right)x_2^{[2]}, \end{aligned} \tag{24}$$

where we omitted the subscript (i) for brevity, and set $\mu^{[1]} = 2.5, \nu^{[1]} = 8, \mu^{[2]} = 1.25, \nu^{[2]} = 4, \sigma = 0.1$. We consider 10 oscillators (24) coupled on a randomly generated topology, with the first two being pinned, and select Γ as the identity. The initial conditions of the nodes are taken from a uniform distribution in $[0, 5]$ ($[0, 10]$) for the first (second) scalar component of each of the $n = 3$ state components.

In our simulations, the coupling gains are selected according to Theorem 1 as $\bar{k}\mathcal{K} = \{1.9, 44.2, 20\}$. Fig. 1 shows that protocol (5) effectively steers the trajectories of the nodes towards that of the pinner.

VI. Conclusions

In this letter, we provided a solution to the pinning control problem for multi-dimensional higher order nonlinear systems in companion form. For any order of the individual dynamics, we provided a constructive way to demonstrate that a distributed proportional derivative protocol can be used to synchronize the overall network to the pinner's trajectory. This procedure also yields a suitable selection of the coupling and control gains. Different from previous approaches, we did not rely on cancellation, which would require a perfect knowledge of the system's dynamics, but rather preserved the original dynamics of the agents, thus making our protocol more suitable in the presence of disturbances acting on the network system [28].

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