Journal of Algebraic Combinatorics
https://doi.org/10.1007/s10801-020-00970-3

Absolute points of correlations of $PG(3, q^n)$

Giorgio Donati[1] · Nicola Durante[1]

Received: 14 November 2019 / Accepted: 4 August 2020

Abstract
The sets of the absolute points of (possibly degenerate) polarities of a projective space are well known. The sets of the absolute points of (possibly degenerate) correlations, different from polarities, of $\mathrm{PG}(2, q^n)$, have been completely determined by B.C. Kestenband in 11 papers from 1990 to 2014, for non-degenerate correlations and by D'haeseleer and Durante (Electron J Combin 27(2):2–32, 2020) for degenerate correlations. In this paper, we completely determine the sets of the absolute points of degenerate correlations, different from degenerate polarities, of a projective space $\mathrm{PG}(3, q^n)$. As an application we show that, for q even, some of these sets are related to the Segre's $(2^h + 1)$-arc of $\mathrm{PG}(3, 2^n)$ and to the Lüneburg spread of $\mathrm{PG}(3, 2^{2h+1})$.

Keywords Sesquilinear forms · Correlations · Polarities

Mathematics Subject Classification 51E20 · 05B25

1 Introduction and preliminary results

1.1 Sesquilinear forms and correlations

Let V and W be two $\mathbf{F}$-vector spaces, where $\mathbf{F}$ is a field. A map $f : V \longrightarrow W$ is called *semilinear* or *σ-linear* if there exists an automorphism σ of $\mathbf{F}$ such that

$$f(v + v') = f(v) + f(v') \quad \text{and} \quad f(av) = a^\sigma f(v)$$

for all vectors $v \in V$ and all scalars $a \in \mathbf{F}$.
If σ is the identity map, then f is a usual linear map.

✉ Nicola Durante
ndurante@unina.it

Giorgio Donati
giorgio.donati@unina.it

[1] Dipartimento di Matematica e Applicazioni "Caccioppoli", Università di Napoli "Federico II", Complesso di Monte S. Angelo - Edificio T, via Cintia, 80126 Napoli, Italy

Published online: 28 August 2020

Let V be an $\mathbf{F}$-vector space with finite dimension d. A map

$$\langle\ ,\ \rangle : (v, v') \in V \times V \longrightarrow \langle v, v'\rangle \in \mathbf{F}$$

is a *sesquilinear form* or a *semibilinear form* on V if it is a linear map on the first argument and it is a σ-linear map on the second argument, that is:

$$\begin{aligned}
\langle v + v', v''\rangle &= \langle v, v''\rangle + \langle v', v''\rangle, \\
\langle v, v' + v''\rangle &= \langle v, v'\rangle + \langle v, v''\rangle, \\
\langle av, v'\rangle &= a\langle v, v'\rangle, \quad \langle v, av'\rangle = a^{\sigma}\langle v, v'\rangle,
\end{aligned}$$

for all $v, v', v'' \in V, a \in \mathbf{F}$ and σ an automorphism of $\mathbf{F}$. If σ is the identity map, then $\langle\ ,\ \rangle$ is an usual bilinear form.
If $\mathcal{B} = (e_1, e_2, \ldots, e_{d+1})$ is an ordered basis of V, then for $x, y \in V$ we have $\langle x, y\rangle = X^t AY^{\sigma}$, where $A = (\langle e_i, e_j\rangle)$ is the *associated matrix* to the sesquilinear form with respect to the ordered basis $\mathcal{B}$, X and Y are the columns of the coordinates of x, y w.r.t. $\mathcal{B}$. The term *sequi* comes from the Latin, and it means one and a half. For every subspace S of V, put

$$\begin{aligned}
S^{\top} &:= \{y \in V : \langle x, y\rangle = 0 \ \ \forall x \in S\}, \\
S^{\perp} &:= \{y \in V : \langle y, x\rangle = 0 \ \ \forall x \in S\}.
\end{aligned}$$

Both $S^{\top}$ and $S^{\perp}$ are subspaces of V. The subspaces $V^{\top}$ and $V^{\perp}$ are called the *right* and the *left radical* of $\langle\ ,\ \rangle$ and will be also denoted by $Rad_r(V)$ and $Rad_l(V)$, respectively.

Proposition 1.1 *The right and the left radicals of a sesquilinear form of a vector space V have the same dimension.*

Proof Indeed, $\dim V^{\top} = d + 1 - \mathrm{rk}(B)$, where $B = (a_{ij}^{\sigma^{-1}})$ and $\dim V^{\perp} = d + 1 - \mathrm{rk}(A_t)$, where A_t is the transposed of the matrix A. The assertion follows from $\mathrm{rk}(B) = \mathrm{rk}(A) = \mathrm{rk}(A_t)$. $\square$

A *non-degenerate* sesquilinear form $\langle\ ,\ \rangle$ has $V^{\perp} = V^{\top} = \{0\}$.

Definition 1 A σ-sesquilinear form is *reflexive* if $\forall\, u, v \in V$ it is:

$$\langle u, v\rangle = 0 \iff \langle v, u\rangle = 0.$$

Definition 2 Let V be an $\mathbf{F}$-vector space of dimension greater than two. A bijection $g : \mathrm{PG}(V) \longrightarrow \mathrm{PG}(V)$ is a *collineation* if g, together with g^{-1}, maps lines into lines. If V has dimension two, then a *collineation* is a map $\langle v\rangle \in \mathrm{PG}(V) \longrightarrow \langle f(v)\rangle \in \mathrm{PG}(V)$, induced by a bijective semilinear map $f : V \longrightarrow V$.

Theorem 1.2 (Fundamental Theorem) *Let V be an $\mathbf{F}$-vector space. Every collineation of* $\mathrm{PG}(V)$ *is induced by a bijective semilinear map $f : V \longrightarrow V$.*

In the sequel if S is a vector subspace of V, we will denote with the same symbol S the associated projective subspace of $\mathrm{PG}(V)$.

Definition 3 Let $f : V \longrightarrow V$ be a semilinear map, with $Kerf \neq \{0\}$. The map

$$\langle v\rangle \in \mathrm{PG}(V)\backslash Kerf \longrightarrow \langle f(v)\rangle \in \mathrm{PG}(V),$$

will be called a *degenerate collineation* of $\mathrm{PG}(V)$.

Definition 4 A (*degenerate*) *correlation* or (*degenerate*) *duality* of $\mathrm{PG}(d, \mathbf{F})$ is a (degenerate) collineation between $\mathrm{PG}(d, \mathbf{F})$ and its dual space $\mathrm{PG}(d, \mathbf{F})^*$.

Remark 1.3 A correlation of $\mathrm{PG}(d, \mathbf{F})$ can be seen as a bijective map of $\mathrm{PG}(d, \mathbf{F})$ that maps k-dimensional subspaces into $(d-1-k)$-dimensional subspaces reversing inclusion and preserving incidence.

A correlation of $\mathrm{PG}(d, \mathbf{F})$ applied twice gives a collineation of $\mathrm{PG}(d, \mathbf{F})$.

Theorem 1.4 *Any (possibly degenerate) correlation of* $\mathrm{PG}(d, \mathbf{F})$*,* $d > 1$*, is induced by a* σ*-sesquilinear form of the underlying vector space* $\mathbf{F}^{d+1}$*. Conversely, every* σ*-sesquilinear form of* $\mathbf{F}^{d+1}$ *induces two (possibly degenerate) correlations of* $\mathrm{PG}(d, \mathbf{F})$*. The two correlations agree if and only if the form* $\langle\ ,\ \rangle$ *is reflexive.*

Proof A (possibly degenerate) duality of $\mathrm{PG}(d, \mathbf{F})$, $d > 1$, is induced by a σ-linear transformation f of $\mathbf{F}^{d+1}$ into its dual, since it is a (possibly degenerate) collineation. Define a map $\langle\ ,\ \rangle : \mathbf{F}^{d+1} \times \mathbf{F}^{d+1} \longrightarrow \mathbf{F}$ in the following way:

$$\langle u, v\rangle = f(v)(u),$$

that is the result of applying the element $f(v)$ of $(\mathbf{F}^{d+1})^*$ to u. It follows that $\langle\ ,\ \rangle$ is a σ-sesquilinear form. Indeed, it is easy to see that $\langle\ ,\ \rangle$ is linear on the first argument and since

$$\langle u, v_1 + v_1\rangle = f(v_1 + v_2)(u) = f(v_1)(u) + f(v_2)(u) = \langle u, v_1\rangle + \langle u, v_2\rangle,$$

and

$$\langle u, av\rangle = f(av)(u) = a^\sigma f(v)(u) = a^\sigma \langle u, v\rangle,$$

it is semilinear on the second argument. Moreover, $\langle\ ,\ \rangle$ is non-degenerate if and only if f is a bijection.
Conversely, any σ-sesquilinear form on $V = \mathbf{F}^{d+1}$ induces two (possibly degenerate) correlations of $\mathrm{PG}(d, \mathbf{F})$ given by

$$\perp: P = \langle u\rangle \in \mathrm{PG}(d, \mathbf{F})\backslash Rad_l(V) \mapsto P^\perp = \{\langle v\rangle \mid \langle u, v\rangle = 0\} \in \mathrm{PG}(d, \mathbf{F})^*$$

and

$$\top : P = \langle v\rangle \in \mathrm{PG}(d, \mathbf{F})\backslash Rad_r(V) \mapsto P^\top = \{\langle u\rangle \mid \langle u, v\rangle = 0\} \in \mathrm{PG}(d, \mathbf{F})^*.$$

□

Remark 1.5 A (possibly degenerate) correlation induced by a σ-sesquilinear form will be also called a σ*-correlation*. Sometimes we will call *linear* a (degenerate) correlation whose associated form is a bilinear form.

1.2 Reflexive sesquilinear forms and polarities

Definition 5 A (degenerate) *polarity* is a (degenerate) correlation whose square is the identity.

If $\perp$ is a (possibly degenerate) polarity, then for every pair of points P and R the following holds:

$$P \in R^{\perp} \iff R \in P^{\perp}.$$

Proposition 1.6 *A (degenerate) correlation is a (degenerate) polarity if and only if the induced sesquilinear form is reflexive.*

Proof Let $\langle\ ,\ \rangle$ be a reflexive σ-sesquilinear form of $\mathbb{F}_q^{d+1}$. If $u \in \langle v\rangle^{\perp}$, then $v \in \langle u\rangle^{\perp}$. Hence, the map $\langle u\rangle \mapsto \langle u\rangle^{\perp}$ defines a (possibly degenerate) polarity. Conversely, given a (possibly degenerate) polarity $\perp$, if $v \in \langle u\rangle^{\perp}$, then $u \in \langle v\rangle^{\perp}$. So, the induced sesquilinear form is reflexive. □

The non-degenerate, reflexive σ-sesquilinear forms of a $(d+1)$-dimensional $\mathbf{F}$-vector space V have been classified (for a proof see, e.g., Theorem 3.6 in [1] or Theorem 6.3 and Proposition 6.4 in [5]).

Theorem 1.7 *Let $\langle\ ,\ \rangle$ be a non-degenerate, reflexive σ-sesquilinear form of a $(d+1)$-dimensional $\mathbf{F}$-vector space. Then, up to a scalar factor, the form $\langle\ ,\ \rangle$ is one of the following:*

(i) *a* symmetric *form, i.e.,*

$$\forall\ u, v \in V\ \ \langle u, v\rangle = \langle v, u\rangle\ \ (char(\mathbf{F}) = 2 \implies \exists\, v \in V : \langle v, v\rangle \neq 0),$$

(ii) *an* alternating *form, i.e.,*

$$\forall\ v \in V\ \ \langle v, v\rangle = 0 \quad (d \textit{ is necessarily odd}),$$

(iii) *a* Hermitian *form, i.e.,*

$$\forall\ u, v \in V\ \ \langle u, v\rangle = \langle v, u\rangle^{\sigma}\ \ (\sigma^2 = 1, \sigma \neq 1).$$

Let V be a $(d+1)$-dimensional $\mathbf{F}$-vector space and consider the associated projective geometry $\mathrm{PG}(d, \mathbf{F})$. If V is equipped with a sesquilinear from $\langle\ ,\ \rangle$, we may consider in $\mathrm{PG}(d, \mathbf{F})$ the set Γ of *absolute* points of the associated correlations, that is the points

X such that $X \in X^{\perp}$ (or equivalently $X \in X^{\top}$). If A is the associated matrix to the σ-sesquilinear form $\langle\ ,\ \rangle$ w.r.t. an ordered basis of V, then the set Γ has equation $X^t A X^{\sigma} = 0$.

From the previous theorem, the polarities of $\mathrm{PG}(d, \mathbf{F})$ are in one to one correspondence with non-degenerate, reflexive σ-sesquilinear forms of $\mathbf{F}^{d+1}$. Hence, to every polarity of $\mathrm{PG}(d, \mathbf{F})$ there is an associated pair (A, σ), with A a non-singular matrix of order $d + 1$ and σ an automorphism of $\mathbf{F}$. In what follows, we will identify a polarity with a pair (A, σ). From the last theorem of the previous section, we have the following:

Theorem 1.8 *Let (A, σ) be a polarity of* $\mathrm{PG}(d, q)$*, one of the following holds:*

(i) *$\sigma = 1$, A is a symmetric matrix. The polarity is called an* orthogonal *polarity. If q is even, there is a non-absolute point.*
(ii) *$\sigma = 1$, A is a skew-symmetric matrix, d is odd. Every point is an absolute point and the polarity is called a* symplectic *polarity.*
(iii) *$\sigma^2 = 1$, so $\sigma : x \mapsto x^{\sqrt{q}}$, q is a square, A is a Hermitian matrix. The polarity is called a* Hermitian *or* unitary *polarity.*

Recall that a square matrix A is either *symmetric* if $A = A_t$, or *skew-symmetric* if $A = -A_t$ or *Hermitian* if $A = A_t^{\sigma}, \sigma^2 = 1, \sigma \neq 1$. Each of the above polarities of $\mathrm{PG}(d, q)$ determines a set $\Gamma : X^t A X^{\sigma} = 0$, as set of its absolute points. The three types of polarities give rise to the following, well known, subsets of $\mathrm{PG}(d, q)$:

Definition 6 If (A, σ) is a polarity of $\mathrm{PG}(d, q)$, then one of the following holds:

(i) Γ is called a *quadric* of $\mathrm{PG}(d, q)$ (q odd, orthogonal polarity). Γ is a hyperplane of $\mathrm{PG}(d, q)$ (q even, orthogonal polarity).
(ii) Γ is the full pointset of $\mathrm{PG}(d, q)$ (d odd, symplectic polarity) and the geometry determined is a *symplectic polar space*.
(iii) Γ is a *Hermitian* variety of $\mathrm{PG}(d, q)$ (q a square, $\sigma^2 = 1, \sigma \neq 1$, unitary polarity).

Remark 1.9 If q is even, $\sigma = 1$ and (A, σ) is an orthogonal polarity, so A is a symmetric matrix, then the set Γ of its absolute points is a hyperplane. Note that in many books (but not in the book of P. Dembowski [10]) this kind of polarity is called a *pseudo polarity*.

In all the cases of the previous definition, the set Γ is often called also *non-degenerate* since the associated σ-sesquilinear form is non-degenerate. All the above sets have been classified, and for each of them it is possible to give a canonical equation (see, e.g., chapters 22 and 23 in [16]).

Theorem 1.10 *If (A, σ) is a polarity of* $\mathrm{PG}(d, q)$*, then let $\Gamma : X^t A X^{\sigma} = 0$ be the set of its absolute points. The following hold:*

(i) *If Γ is a quadric, so q is odd, then we have:*

(1) *If d is even, then*

$$\Gamma = \mathcal{Q}(d, q) : x_1 x_2 + \cdots + x_{d-1} x_d + x_{d+1}^2 = 0, \quad (\textit{parabolic quadric}).$$

(2) *If d is odd, then either*

$$\Gamma = \mathcal{Q}^-(d,q) : x_1x_2 + \cdots + \alpha x_d^2 + \beta x_d x_{d+1} + \gamma x_{d+1}^2 = 0, \textit{ with}$$

$$\alpha x_d^2 + \beta x_d x_{d+1} + \gamma x_{d+1}^2 \textit{ irreducible polynomial over } \mathbb{F}_q \textit{ (elliptic quadric) or}$$

$$\Gamma = \mathcal{Q}^+(d,q) : x_1x_2 + \cdots + x_d x_{d+1} = 0 \quad \textit{(hyperbolic quadric).}$$

If Γ is a hyperplane, so q is even, then $\Gamma : x_1 = 0$.

(ii) *If Γ is a symplectic polar space, then Γ is the full pointset of* PG(d,q), *d odd. The geometry determined will be denoted by $W(d,q)$. A canonical form for the associated bilinear form is*

$$\langle x, y\rangle = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + \cdots + x_d y_{d+1} - x_{d+1}y_d.$$

(iii) *If Γ is a Hermitian variety, then*

$$\Gamma = \mathcal{H}(d,q) : x_1^{\sqrt{q}+1} + x_2^{\sqrt{q}+1} + \cdots + x_{d+1}^{\sqrt{q}+1} = 0.$$

Remark 1.11 Regarding degenerate, reflexive σ-sesquilinear forms of $V = \mathbb{F}_q^{d+1}$, it is possible to prove that if dim $V^\perp = r+1$, $r \geq 0$, then the set Γ of absolute points in PG(d,q) of the associated degenerate polarity is, in all the possible cases, a cone $\Gamma(\mathcal{V}_r, Q_{d-1-r})$ with vertex a subspace $\mathcal{V}_r$, of dimension r, projecting the set Q_{d-1-r} of absolute points of a polarity in a subspace S_{d-1-r}, of dimension $d-1-r$, skew with $\mathcal{V}_r$. The set Q_{d-1-r} can be either a quadric or a Hermitian variety (if q is a square) or the full pointset of PG$(d-1-r,q)$ (a symplectic geometry). In these cases, we call the set of the absolute points either a *degenerate quadric*,or a *degenerate Hermitian variety* or a *degenerate symplectic geometry*, respectively. The knowledge of the set of the absolute points of a polarity of S_{d-1-r} determines also the knowledge of the set of the absolute points of a degenerate polarity. If Q_{d-1-r} is the full pointset of PG$(d-1-r)$, then d and r must have the same parity and the cone $\Gamma(\mathcal{V}_r, \mathrm{PG}(d-1-r,q))$ is a quotient geometry PG$(d,q)/\mathcal{V}_r$.

Sometimes we will call non-degenerate quadrics, non-degenerate symplectic geometries and non-degenerate Hermitian varieties, the set of the absolute points of a polarity. The following proposition characterizes these sets.

Proposition 1.12 *Let Γ be either a non-degenerate quadric or a non-degenerate symplectic polar space or a non-degenerate Hermitian variety of* PG(d,q) *and denote by $\perp$ the associated polarity. For any point Y of Γ, the set $Y^\perp$ is a hyperplane meeting Γ in a cone $\Gamma(Y, Q_{d-2})$, where Q_{d-2} is either a non-degenerate quadric or a non-degenerate symplectic polar space or a non-degenerate Hermitian variety, respectively and vice versa.*

1.3 Sets of the absolute points of a linear correlation of PG(d, F)

Let Γ be the set of the absolute points of a linear correlation of $\mathrm{PG}(d, \mathbf{F})$ with equation $X^t AX = 0$, $|A| \neq 0$. Assuming that Γ is a proper subset of $\mathrm{PG}(d, \mathbf{F})$, the matrix A cannot be a skew-symmetric matrix and Γ is a (possibly degenerate) quadric and vice versa. Note that the previous holds, independently of the characteristic of the field $\mathbf{F}$. If the characteristic of $\mathbf{F}$ is 2, then there is no relation between $|A|$ and degeneracy or not of the quadric. Observe that, not assuming that A is a symmetric matrix, also for characteristic of $\mathbf{F}$ either odd or 0, this relation is lost and, a bit surprisingly, the following holds:

Proposition 1.13 *If $\Gamma : X^t AX = 0$ is a proper subset of points of $\mathrm{PG}(d, \mathbf{F})$, with $|A| = 0$, then Γ is a (possibly degenerate) quadric. Vice versa, if Γ is a non-empty (possibly degenerate) quadric of $\mathrm{PG}(d, \mathbf{F})$, then there exists a matrix A, with $|A| = 0$, s.t. Γ is projectively equivalent to the set of points with equation $X^t AX = 0$.*

Moreover, since in the finite Desarguesian projective spaces all the quadrics are non-empty (see, e.g., [38]), we have the following:

Proposition 1.14 *If $\Gamma : X^t AX = 0$, with $|A| = 0$, is a set of points of $\mathrm{PG}(d, q^n)$, then Γ is either a degenerate symplectic geometry or a (possibly degenerate) quadric. Vice versa if Γ is either a degenerate symplectic geometry or a (possibly degenerate) quadric of $\mathrm{PG}(d, q^n)$, then there is a matrix A with $|A| = 0$ s.t. Γ is projectively equivalent to the set of points with equation $X^t AX = 0$.*

1.4 $\mathbb{F}_q$-linear sets of PG(d, q^n)

Definition 7 Let $\Omega = \mathrm{PG}(r - 1, q^n)$, $q = p^h$, p a prime. A set L is said an $\mathbb{F}_q$-*linear set* of Ω of rank t if it is defined by the nonzero vectors of an $\mathbb{F}_q$-vector subspace U of $V = \mathbf{F}^r_{q^n}$ of dimension t, that is

$$L = L_U = \{\langle u\rangle_{q^n} : u \in U\setminus\{0\}\}.$$

Let $\Lambda = \mathrm{PG}(W, \mathbb{F}_{q^n})$ be a subspace of Ω of dimension s, we say that Λ has *weight* i with respect to L_U if $\dim_{\mathbb{F}_q}(W \cap U) = i$. An $\mathbb{F}_q$-linear set L_U of Ω of rank t is said to be *scattered* if each point of L_U has weight 1, with respect to L_U.

In [2] it is proved that if L_U is a scattered $\mathbb{F}_q$ linear set of Ω, then $t \leq rn/2$. It is clear, by definition that L_U is a scattered $\mathbb{F}_q$-linear set of rank t if and only if $|L_U| = q^{t-1} + q^{t-2} + \cdots + q + 1$. If L is a scattered linear set of $\mathrm{PG}(r - 1, q^n)$ of rank $rn/2$, it is called a *maximum* scattered linear set.

If $\dim_{\mathbb{F}_q} U = \dim_{\mathbb{F}_{q^n}} V = r$ and $\langle U\rangle_{\mathbb{F}_{q^n}} = V$, then $L_U \cong \mathrm{PG}(U, \mathbb{F}_q)$ is a subgeometry of Ω. In such a case, each point has weight 1, and hence $|L_U| = q^{r-1} + q^{r-2} + \cdots + q + 1$. Let $\Sigma = \mathrm{PG}(t, q)$ be a subgeometry of $\Sigma' = \mathrm{PG}(t, q^n)$ and suppose that there exists a $(t-r)$-dimensional subspace Ω' of Σ' disjoint from Σ. Let $\Omega = \mathrm{PG}(r-1, q^n)$ be an $(r - 1)$-dimensional subspace of Σ' disjoint from Ω', and let Γ be the projection of Σ from Ω' to Ω, i.e., $\Gamma = \{\langle \Omega', x\rangle \cap \Omega : x \in \Sigma\}$. Let $p_{\Omega',\Omega}$ be the map from $\Sigma \setminus \Omega'$ to Ω defined by $x \mapsto \langle \Omega', x\rangle \cap \Omega$. We call Ω' the *center* and Ω the *axis* of $p_{\Omega',\Omega}$. In [31] the following characterization of $\mathbb{F}_q$-linear sets is given:

Theorem 1.15 *If L is a projection of* $\mathrm{PG}(t, q)$ *into* $\Omega = \mathrm{PG}(r-1, q^n)$*, then L is an $\mathbb{F}_q$-linear set of Ω of rank $t+1$ and $\langle L\rangle = \Omega$. Conversely, if L is an $\mathbb{F}_q$-linear set of Ω of rank $t+1$ and $\langle L\rangle = \Omega$, then either L is a subgeometry of Ω or there are a $(t-r)$-dimensional subspace Ω' of $\Sigma' = \mathrm{PG}(t, q^n)$ disjoint from Ω and a subgeometry Σ of Σ' disjoint from Ω' such that $L = p_{\Omega',\Omega}(\Sigma)$.*

A family of maximum scattered linear sets to which a geometric structure, called pseudoregulus, can be associated has been defined in [30].

Definition 8 Let $L = L_U$ be a scattered $\mathbb{F}_q$-linear set of $\Gamma = \mathrm{PG}(2n-1, q^t)$ of rank tn, $t, n \geq 2$. We say that L is of *pseudoregulus type* if:

(i) there exist $m = \frac{q^{nt}-1}{q^t-1}$ pairwise disjoint lines of Γ, say $s_1, s_2, \ldots, s_m$, such that $w_L(s_i) = t$, *i.e.*, $|L \cap s_i| = q^{t-1} + q^{t-2} + \cdots + q + 1 \ \ \forall i = 1, \ldots, m$,

(ii) there exist exactly two $(n-1)$-dimensional subspaces T_1 and T_2 of Γ disjoint from L such that $T_j \cap s_i \neq \emptyset \ \ \forall i = 1, \ldots, m$ and $j = 1, 2$.

We call the set of lines $\mathcal{P}_L = \{s_i : i = 1, \ldots, m\}$ the $\mathbb{F}_q$-*pseudoregulus* (or simply the *pseudoregulus*) of Γ associated with L and we refer to T_1 and T_2 as *transversal spaces* of $\mathcal{P}_L$ (or transversal spaces of L). When $t = n = 2$, these objects already appeared first in [15], where the term pseudoregulus was introduced for the first time. See also [11].

In [12] and in [30] the following class of maximum scattered $\mathbb{F}_q$-linear sets of the projective line $\Gamma = \mathrm{PG}(1, q^t)$ with a structure resembling that of an $\mathbb{F}_q$-linear set of $\mathrm{PG}(2n-1, q^t)$, $n, t \geq 2$, of pseudoregulus type has been studied. Let $P_1 = \langle w\rangle_{q^t}$ and $P_2 = \langle v\rangle_{q^t}$ be two distinct points of Γ and let τ be an automorphism of $\mathbb{F}_{q^t}$ such that $Fix(\tau) = \mathbb{F}_q$. For each $\rho \in \mathbb{F}^*_{q^t}$, the set $W_{\rho,\tau} = \{\lambda w + \rho\lambda^\tau v : \lambda \in \mathbb{F}_{q^t}\}$, is an $\mathbb{F}_q$-vector subspace of $V = \mathbb{F}^2_{q^t}$ of dimension t and $L_{\rho,\tau} = L_{W_{\rho,\tau}}$ is a scattered $\mathbb{F}_q$-linear set of Γ.

Definition 9 In [30] the linear sets $L_{\rho,\tau}$ have been called of *pseudoregulus type* and the points P_1 and P_2 the *transversal points* of $L_{\rho,\tau}$. If $L_{\rho,\tau} \cap L_{\rho',\tau} \neq \emptyset$, then $L_{\rho,\tau} = L_{\rho',\tau}$.

Remark 1.16 (See [12,30]) Note that $L_{\rho,\tau} = L_{\rho',\tau}$ if and only if $N(\rho) = N(\rho')$ (where N denotes the Norm of $\mathbb{F}_{q^t}$ over $\mathbb{F}_q$); so P_1, P_2 and the automorphism τ define a set of $q-1$ mutually disjoint maximum scattered linear sets of pseudoregulus type admitting the same transversal points. Such maximal scattered linear sets, together with P_1 and P_2, cover the pointset of $\mathrm{PG}(1, q^t)$. All the $\mathbb{F}_q$-linear sets of pseudoregulus type in $\Gamma = \mathrm{PG}(1, q^t)$, $t \geq 2$, are equivalent to the linear set L_{1,σ_1} under the action of the collineation group $\mathrm{P\Gamma L}(2, q^t)$.

Remark 1.17 We observe that the pseudoregulus of $\mathrm{PG}(1, q^n)$ is the same as the Norm surface (or sphere) of R.H. Bruck, introduced and studied, in the seventies, by R.H. Bruck (see [3,4]) in the setting of circle geometries.

For more on $\mathbb{F}_q$-linear sets in finite projective spaces, see, e.g., [28,33].

2 The σ-quadrics of PG(d, q^n)

In this section, we will introduce σ-quadrics of PG(d, q^n) and we will determine their intersection with subspaces.

Definition 10 A σ-*quadric* of PG(d, q^n) is the set of the absolute points of a (possibly degenerate) σ-correlation, $\sigma \neq 1$, of PG(d, q^n). A σ-quadric of PG$(2, q^n)$ will be called a σ-*conic*.

Remark 2.1 Let $\Gamma : X^t AX^\sigma = 0$ be the set of the absolute points of a (possibly degenerate) σ-correlation of PG(d, q^2) and let $\sigma : x \mapsto x^q$. In this case, the set Γ is a (possibly degenerate) Hermitian variety of PG(d, q^2) if A is a Hermitian matrix. We have included Hermitian varieties in the definition of a σ-quadric in order to have no exceptions everywhere. Indeed, we will see that Hermitian varieties appear as intersection of σ-quadrics of PG(d, q^2) with subspaces.

Proposition 2.2 *Let* $\Gamma : X^t AX^\sigma = 0$ *be a* σ*-quadric of* PG(d, q^n). *Every subspace* S *intersects* Γ *either in a* σ*-quadric of* S *or it is contained in* Γ.

Proof Let S be an h-dimensional subspace of PG(d, q^n). We may assume, w.l.o.g. that $S : x_{h+2} = 0, \ldots, x_{d+1} = 0$. Let A' be the submatrix of A obtained by deleting the last $d - h$ rows and columns of A. If $A' = 0$, then $S \subset \Gamma$. If $A' \neq 0$, then $S \cap \Gamma$ is a σ-quadric of S. □

For the remaining part of this chapter, we can assume $\sigma \neq 1$. Let $V = \mathbb{F}_{q^n}^{d+1}$, let $\langle\ ,\ \rangle$ be a degenerate σ-sesquilinear form with associated (degenerate) correlations $\perp$, $\top$ and let $\Gamma : X^t AX^\sigma = 0$ be the associated σ-quadric. We will denote by $L = V^\perp$ and $R = V^\top$, the left and right radicals of $\langle\ ,\ \rangle$ respectively, seen as subspaces of PG(d, q^n) that will be called the *vertices* of Γ. Before giving the definition of a non-degenerate σ-quadric, we prove the following:

Proposition 2.3 *Let* $\Gamma : X^t AX^\sigma = 0$ *be a* σ*-quadric of* PG(d, q^n).

- *For every point* $Y \in \Gamma \backslash R$*, the hyperplane* $Y^\top$ *is union of lines through* Y *either contained or 1-secant or 2-secant to* Γ.
- *For every point* $Y \in \Gamma \backslash L$*, the hyperplane* $Y^\perp$ *is union of lines through* Y *either contained or 1-secant or 2-secant to* Γ.

Proof Let $Y \in \Gamma \backslash R$ and let Z be a point of $Y^\top$. The line YZ has equations: $X = \lambda Y + \mu Z$, $(\lambda, \mu) \in$ PG$(1, q^n)$; hence, $Y^\top \cap \Gamma$ is determined by the solutions in (λ, μ) of the following equation:

$$Y^t AY^\sigma \lambda^{\sigma+1} + Y^t AZ^\sigma \lambda\mu^\sigma + Z^t AY^\sigma \lambda^\sigma \mu + Z^t AZ^\sigma \mu^{\sigma+1} = 0. \tag{1}$$

In the previous equation, it is $Y^t AY^\sigma = 0$, since $Y \in \Gamma$ and $Z^t AY^\sigma = 0$, since $Z \in Y^\top$. Hence, Equation (1) becomes $Y^t AZ^\sigma \lambda + Z^t AZ^\sigma \mu = 0$ and four different cases occur

- If $Y^t AZ^\sigma = Z^t AZ^\sigma = 0$, then the line YZ is contained in Γ;

- If $Y^t AZ^\sigma = 0$, $Z^t AZ^\sigma \neq 0$, then the line YZ intersects Γ exactly at the point Y;
- If $Y^t AZ^\sigma \neq 0$, $Z^t AZ^\sigma = 0$, then the line YZ intersects Γ exactly at the point Y;
- If $Y^t AZ^\sigma \neq 0$, $Z^t AZ^\sigma \neq 0$, then the line YZ intersects Γ at Y and at another point.

The second part of the statement follows in a similar way. □

Corollary 2.4 *Let $\Gamma : X^t AX^\sigma = 0$ be a σ-quadric of* $\mathrm{PG}(d, q^n)$ *and let $L = V^\perp$, $R = V^\top$.*

- *For every point $Y \in L$, the set $Y^\top \cap \Gamma$ is union of lines through Y.*
- *For every point $Y \in R$, the set $Y^\perp \cap \Gamma$ is union of lines through Y.*

Remark 2.5 Let $\Gamma : X^t AX^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(d, q^n)$ and let $L = V^\perp$ and $R = V^\top$ be its vertices. For every point $Y \in \Gamma\backslash R$, $Y^\top$ is the tangent hyperplane at Y to the hypersurface with equation $X^t AX^\sigma = 0$ and for every point $Y \in \Gamma\backslash L$, $Y^\perp$ is the tangent hyperplane at Y to the hypersurface with equation $X^t A_t^{\sigma^{-1}} X^{\sigma^{-1}} = 0$. Both these hypersurfaces have as set of $\mathbb{F}_{q^n}$-rational points the set Γ.

Inspired by the characterization of non-degenerate quadrics, Hermitian varieties and symplectic polar spaces of $\mathrm{PG}(d, q^n)$ (see Proposition 1.12), we define a non-degenerate σ-quadric of $\mathrm{PG}(d, q^n)$ by induction on the dimension d of the projective space.

Definition 11 Let Γ be a σ-quadric of $\mathrm{PG}(d, q^n)$, denote by L and R the left and right radicals of the associated sesquilinear form.

(i) Γ is a *non-degenerate* σ-quadric of $\mathrm{PG}(1, q^n)$ if $|\Gamma| \in \{0, 2, q + 1\}$.

(ii) Γ is a *non-degenerate* σ-conic of $\mathrm{PG}(2, q^n)$, if the following hold:

- $L \cap R = \emptyset$,
- the tangent line $L^\top$ to Γ at L intersects Γ exactly at the point L.

(iii) Γ is *non-degenerate* σ-quadric of $\mathrm{PG}(d, q^n)$, $d \geq 3$, if the following hold:

- $L \cap R = \emptyset$,
- $\forall\ Y \in L$, the set $Y^\top \cap \Gamma$ is a cone $\Gamma(Y, Q)$ with vertex Y and base a non-degenerate σ-quadric Q in a $(d-2)$-dimensional subspace S not through Y.

2.1 The σ-quadrics of PG$(1, q^n)$ and of PG$(2, q^n)$

In this subsection, we recall what is known for the σ-quadrics of $\mathrm{PG}(1, q^n)$ and of $\mathrm{PG}(2, q^n)$ with $Fix(\sigma) = \mathbb{F}_q$ (see [9,14]).
In what follows, we will always assume that $\sigma \neq 1$ since if $\sigma = 1$, the set Γ is either a (possibly degenerate) quadric or a (possibly degenerate) symplectic geometry. Moreover, since $Fix(\sigma) = \mathbb{F}_q$ we have that $\sigma : x \mapsto x^{q^m}$, $(m, n) = 1$.

Proposition 2.6 *A σ-quadric of* $\mathrm{PG}(1, q^n)$ *is one of the following:*

- *the empty set, a point, two distinct points,*

- *an* $\mathbb{F}_q$*-subline* $\mathrm{PG}(1, q)$ *of* $\mathrm{PG}(1, q^n)$.

Obviously the following holds:

Proposition 2.7 *Let* Γ *and* Γ' *be a* σ*-quadric and* σ'*-quadric of* $\mathrm{PG}(1, q^n)$*, respectively. The sets* Γ *and* Γ' *are* $\mathrm{P\Gamma L}$*-equivalent if and only if* $|\Gamma| = |\Gamma'|$.

The sets of the absolute points of σ-correlations of $\mathrm{PG}(2, q^n)$ have been completely determined by the huge work of B.C. Kestenband in 11 different papers from 1990 to 2014. There are lots of different classes of such sets. The interested reader can find all of them in 11 different papers from 1990 to 2014 covering 400 pages of mathematics (see [17–27]). In what follows, we will use the following:

Definition 12 A *Kestenband* σ*-conic* of $\mathrm{PG}(2, q^n)$ is the set of absolute points of a σ-correlation, $\sigma \neq 1$, of $\mathrm{PG}(2, q^n)$.

Proposition 2.8 *A* σ*-conic of* $\mathrm{PG}(2, q^n)$ *is one of the following:*

- *a cone with vertex a point* A *projecting a* σ*-quadric of a line* ℓ *not through* A.
- *a degenerate* C_F^m*-set (i.e., the union of a line* ℓ *with a scattered linear set isomorphic to* $\mathrm{PG}(n, q)$*, meeting the line* ℓ *in a maximum scattered* $\mathbb{F}_q$*-linear set of pseudoregulus type),*
- *a* C_F^m*-set (i.e., the union of* $q-1$ *scattered* $\mathbb{F}_q$*-linear sets each of which isomorphic to* $\mathrm{PG}(n-1, q)$ *with two distinct points).*
- *a Kestenband* σ*-conic*

Proposition 2.9 *Let* Γ *be a* σ*-conic of* $\mathrm{PG}(2, q^n)$ *with two different vertices* R *and* L*. Associated to* Γ *there is a* σ*-collineation between the pencils of lines with vertices* R *and* L*. Moreover,* Γ *is also a* σ^{-1} *conic.*

Proof To the σ-conic Γ, there is associated a σ-collineation Φ between the pencils of lines with vertices R and L s.t. Γ is the set of points of intersection of corresponding lines under Φ (resp. Φ') (see [9]). Moreover, the σ-conic Γ with associated the σ-collineation Φ is also a σ^{-1} conic with associated the σ^{-1}-collineation Φ^{-1}. □

Proposition 2.10 *Let* Γ *be a* σ*-conic of* $\mathrm{PG}(2, q^n)$ *with two different vertices* R, L *and let* Γ' *be a* σ'*-conic of* $\mathrm{PG}(2, q^n)$ *with two different vertices* R', L'*. The sets* Γ *and* Γ' *are* $\mathrm{P\Gamma L}$*-equivalent if and only if either* $\sigma' = \sigma$ *or* $\sigma' = \sigma^{-1}$.

Proof Let $\sigma : x \mapsto x^{q^m}$ and let $\sigma' : x \mapsto x^{q^{m'}}$ with $(m, n) = (m', n) = 1$. We may assume that $R' = R$ and $L' = L$ since the group $\mathrm{P\Gamma L}(3, q^n)$ is two-transitive on points. Let Φ (resp. Φ') be the σ-collineation (resp. σ'-collineation) associated with Γ (resp. Γ'). First, assume that Γ is C_F^m-set and that Γ' is a $C_F^{m'}$-set. Let f be a collineation of $\mathrm{PG}(2, q^n)$ mapping Γ into Γ'. Since R and L are the unique points of both Γ and Γ' not incident with $(q+1)$-secant lines, it follows that f stabilizes the set $\{R, L\}$. First, assume that $f(R) = L$. For every line ℓ through the point R, we have that $f(\Phi(\ell)) = (\Phi')^{-1}(f(\ell))$. As Φ and $(\Phi')^{-1}$ are collineations with accompanying automorphism σ and σ'^{-1}, it follows that $\sigma' = \sigma^{-1}$. Next suppose that

$f(R) = R$, $f(L) = L$. For every line ℓ through R, we have that $f(\Phi(\ell)) = \Phi'(f(\ell))$, and hence $\sigma = \sigma'$.
Next assume that Γ is a degenerate C_F^m-set and that Γ' is a degenerate $C_F^{m'}$-set and that $n \geq 3$. Let f be a collineation of $\mathrm{PG}(2, q^n)$ mapping Γ into Γ'. Since the line RL is the unique line contained in both Γ and Γ', the collineation f stabilizes RL. The directions of the sets $\Gamma \backslash RL$ and $\Gamma' \backslash RL$ on the line RL are both $\mathbb{F}_q$-linear sets of pseudoregulus type with transversal points R and L. Since the transversal points of an $\mathbb{F}_q$-linear set of pseudoregulus type of $\mathrm{PG}(1, q^n)$, $n \geq 3$ are uniquely determined (see Proposition 4.3 in [30]), it follows that f stabilizes the set $\{R, L\}$. The assertion follows with the same arguments as in the previous case. □

Finally, the following holds:

Proposition 2.11 *Let Γ be a σ-conic of* $\mathrm{PG}(2, q^n)$ *with vertices $R = L$, and let Γ' be a σ'-conic of* $\mathrm{PG}(2, q^n)$ *with vertices $R' = L'$. The sets Γ and Γ' are* $\mathrm{P\Gamma L}$*-equivalent if and only if $|\Gamma| = |\Gamma'|$.*

Proof We may assume that $R = R'$ since the group $\mathrm{P\Gamma L}(3, q^n)$ is transitive on points. Obviously, if Γ and Γ' are $\mathrm{P\Gamma L}$-equivalent, then $|\Gamma| = |\Gamma'|$. Vice versa if $|\Gamma| = |\Gamma'|$, then both Γ and Γ' are cones with vertex the point R projecting a σ-quadric on a line ℓ not through R. The assertion follows since two σ-quadrics of ℓ are $\mathrm{P\Gamma L}$-equivalent if and only if have the same size (see [9]). □

Remark 2.12 We can divide the σ-conics in two families:

- the degenerate σ-conics: the degenerate C_F^m-sets and the cones with vertex a point V projecting either the empty set or a point or two points or $q + 1$ points on a line ℓ not through the point V;
- the non-degenerate σ-conics: the Kestenband σ-conics and the C_F^m-sets.

In the remaining part of this chapter, we will determine the canonical equations and some properties for the σ-quadrics of $\mathrm{PG}(3, q^n)$ associated to a degenerate correlation of $\mathrm{PG}(3, q^n)$. Let Γ be such a σ-quadric with equation $X^t A X^\sigma = 0$, A a singular matrix. We will consider three separate cases according to $rk(A) \in \{1, 2, 3\}$.

3 Seydewitz's and Steiner's projective generation of quadrics of PG(3, F)

In this section, we recall the projective generation of F. Seydewitz and J. Steiner of quadrics of $\mathrm{PG}(3, \mathbf{F})$, $\mathbf{F}$ a field. We start with Seydewitz's construction. In what follows, if P is a point of $\mathrm{PG}(3, \mathbf{F})$ we will denote with $\mathcal{S}_P$ the star of lines with center P and with $\mathcal{S}_P^*$ the star of planes with center P.

Theorem 3.1 (F. Seydewitz [36]) *Let R and L be two distinct points of* $\mathrm{PG}(3, \mathbf{F})$ *and let $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$ be a projectivity. The set Γ of points of intersection of corresponding elements under Φ is one of the following:*

- *If $\Phi(RL)$ is a plane through the line RL, then Γ is either a quadratic cone or a hyperbolic quadric $\mathcal{Q}^+(3, \mathbf{F})$,*

- *If* $\Phi(RL)$ *is a plane not through the line* RL*, then* Γ *is a non-empty, non-degenerate quadric, i.e., either an elliptic quadric* $\mathcal{Q}^-(3, \mathbf{F})$ *or an hyperbolic quadric* $\mathcal{Q}^+(3, \mathbf{F})$.

Next consider Steiner's construction. In what follows, if s is a line of $\mathrm{PG}(3, \mathbf{F})$ we will denote by $\mathcal{P}_s$ the pencil of planes through s.

Theorem 3.2 (J. Steiner [37]) *Let* r *and* ℓ *be two skew lines of* $\mathrm{PG}(3, \mathbf{F})$. *Let* $\Phi : \mathcal{P}_r \longrightarrow \mathcal{P}_\ell$ *be a projectivity. The set of points of intersection of corresponding planes under* Φ *is a hyperbolic quadric* $\mathcal{Q}^+(3, \mathbf{F})$ *of* $\mathrm{PG}(3, \mathbf{F})$.

Proposition 3.3 *In* $\mathrm{PG}(2, \mathbf{F})$ *the set of the absolute points of a linear (possibly degenerate) correlation satisfies an equation* $X^t AX = 0$*, for some matrix* A*. Therefore, it is one of the followings:*

- *the empty set (e.g.,* $\mathbf{F} = \mathbb{R}$*)*
- *a point, a line, two lines,*
- *a non-degenerate conic,*
- *a degenerate symplectic geometry* $\mathcal{P}_R$*, for some point* R.

Proof It follows immediately from Proposition 1.14. □

In Seydewitz's construction, if we assume that the points R and L coincide, then we get the following:

Proposition 3.4 *Let* R *be a point of* $\mathrm{PG}(3, \mathbf{F})$. *Let* $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_R^*$ *be a projectivity. The set* Γ *of points of intersection of corresponding elements under* Φ *is one of the following:*

- *the point* R *(e.g.,* $\mathbf{F} = \mathbb{R}$*),*
- *a line through* R*, a plane through* R*, two distinct planes through* R*,*
- *a cone with vertex the point* R *and base a non-empty, non-degenerate conic in a plane not through* R*,*
- *a degenerate symplectic geometry* $\mathcal{P}_r$*, for some line* r *through* R.

Proof The set of points of intersection of corresponding elements under Φ is a cone with vertex the point R projecting the set of absolute points of a linear correlation of a plane π not through the point R. Hence, the assertion follows from the previous proposition. □

In Steiner's construction, if the lines r and ℓ either intersect at a point V or coincide, then we get the following:

Proposition 3.5 *Let* r *and* ℓ *be two lines s.t.* $r \cap \ell = \{V\}$. *Let* $\Phi : \mathcal{P}_r \longrightarrow \mathcal{P}_\ell$ *be a projectivity. The set of points of intersection of corresponding planes under* Φ *is one of the following:*

- *a pair of distinct planes,*
- *a cone with vertex the point* V *and base a non-empty, non-degenerate conic in a plane not through* V*,*

Proposition 3.6 *Let r be a line and let $\Phi : \mathcal{P}_r \longrightarrow \mathcal{P}_r$ be a projectivity. The set of points of intersection of corresponding planes under Φ is one of the following:*

- *the line r,*
- *a plane,*
- *a pair of distinct planes,*
- *the degenerate symplectic geometry $\mathcal{P}_r$.*

Remark 3.7 If $\mathbf{F}$ is either an algebraically closed field or a finite field, then Seydewitz's construction in $\mathrm{PG}(3, \mathbf{F})$ gives all the possible quadrics of $\mathrm{PG}(3, \mathbf{F})$ and also the degenerate symplectic geometry $\mathcal{P}_r$, for some line r. If $\mathbf{F} = \mathbb{R}$ is the field of real numbers, then the only missing quadric, up to projectivities, is the quadric with equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, that gives as set of points in $\mathrm{PG}(3, \mathbf{F})$, the empty set. With Steiner's construction in $\mathrm{PG}(3, \mathbf{F})$, we miss also the elliptic quadric.

Note that also the converse holds:

Proposition 3.8 *Let Γ be either a, non-empty, quadric or a degenerate symplectic geometry of* $\mathrm{PG}(3, \mathbf{F})$*,* $\mathbf{F}$ *being a field. There exists two point R and L of Γ and a projectivity $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$ s.t. Γ is the set of points of intersection of corresponding elements under Φ.*

4 σ-quadrics of rank 3 in PG(3, q^n)

Let $\Gamma : X^t A X^\sigma = 0$ be a σ-quadric of $\mathrm{PG}(3, q^n)$ associated to a σ-sesquilinear form $\langle\ ,\ \rangle$. In this section, we assume throughout that $rk(A) = 3$. Therefore, the radicals $V^\perp$ and $V^\top$ are one-dimensional vector subspace spaces of V, so they are points of $\mathrm{PG}(3, q^n)$. We distinguish several cases:
1) $V^\perp \neq V^\top$.
We may assume w.l.o.g. that the point $R = (1, 0, 0, 0)$ is the right radical and the point $L = (0, 0, 0, 1)$ is the left radical. It follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3)x_4^\sigma = 0.$$

The degenerate collineation

$$\top : Y \in \mathrm{PG}(3, q^n)\backslash\{R\} \mapsto X^t A Y^\sigma = 0 \in \mathrm{PG}(3, q^n)^*$$

associated to the sesquilinear form maps points into planes through the point L. Points that are on a common line through R are mapped into the same plane through L. Therefore, $\top$ induces a collineation $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$. Let

$$\mathcal{S}_R = \{\ell_{\alpha,\beta,\gamma} : (\alpha, \beta, \gamma) \in \mathrm{PG}(2, q^n)\}, \text{ where}$$
$$\ell_{\alpha,\beta,\gamma} : \begin{cases} x_1 = \lambda \\ x_2 = \mu\alpha \\ x_3 = \mu\beta \\ x_4 = \mu\gamma \end{cases}, (\lambda, \mu) \in \mathrm{PG}(1, q^n)$$

and

$$\mathcal{S}_L^* = \{\pi_{\alpha',\beta',\gamma'} : (\alpha', \beta', \gamma') \in \mathrm{PG}(2, q^n)\}, \text{ where } \pi_{\alpha',\beta',\gamma'} : \alpha' x_1 + \beta' x_2 + \gamma' x_3 = 0.$$

The collineation Φ is given by $\Phi(\ell_{\alpha,\beta,\gamma}) = \pi_{\alpha',\beta',\gamma'}$, with

$$(\alpha', \beta', \gamma')_t = A'(\alpha, \beta, \gamma)_t^\sigma,$$

where A' is the matrix obtained by A by deleting the last row and the first column. Note that $|A'| \neq 0$ since $rk(A) = 3$. It is easy to see that Γ is the set of points of intersection of corresponding elements under the collineation Φ.
If $Y = (y_1, y_2, y_3, y_4)$ is a point of $\Gamma \setminus \{R\}$, then the tangent plane to Γ at the point Y is the plane $\pi_Y = Y^\top$ with equation $X^t A Y^\sigma = 0$. It follows that for every point Y of $\Gamma \setminus \{R\}$ the plane π_Y contains the point L. The tangent plane $\pi_L = L^\top$ to Γ at the point L is the plane with equation $X^t A L^\sigma = 0$, that is:

$$\pi_L : a_{14}x_1 + a_{24}x_2 + a_{34}x_3 = 0.$$

We again distinguish some cases.

- First, assume that π_L contains the line RL.
  It follows that, w.l.o.g., we may put $\pi_L : x_3 = 0$. Hence, $a_{14} = a_{24} = 0$ and we can put $a_{34} = 1$, obtaining

  $$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)x_3^\sigma + x_3 x_4^\sigma = 0.$$

  With this assumption, the collineation Φ maps the line LR into the plane π_L. Consider now the pencil $\mathcal{P}_{R,\pi_L}$ of lines through R in π_L. We distinguish two cases.

i) Φ maps the lines of $\mathcal{P}_{R,\pi_L}$ into the planes through the line RL.
In this case, we can assume that Φ maps the line $x_3 = x_4 = 0$ into the plane $x_2 = 0$ and the line $x_2 = x_4 = 0$ into the plane $x_1 = 0$ obtaining

$$\Gamma : a x_1 x_3^\sigma + b x_2^{\sigma+1} + x_3 x_4^\sigma = 0.$$

We can assume that Γ contains the points $(0, 1, -1, 1)$ and $(1, 0, 1 - 1)$ obtaining $a = b = 1$ and hence a canonical equation is given by

$$\Gamma : x_1 x_3^\sigma + x_2^{\sigma+1} + x_3 x_4^\sigma = 0.$$

Note that in this case Γ is the set of points studied in [13], where Γ has been called a σ-cone. In this paper, we call this set a degenerate *parabolic σ-quadric* with *collinear vertex points R* and L.
In [13] it has been proved that the following holds:

Theorem 4.1 *Let Γ be a degenerate parabolic σ-quadric Γ of* $\mathrm{PG}(3, q^n)$ *with collinear vertex points R and L. Then $|\Gamma| = q^{2n} + q^n + 1$, RL is the unique line contained in Γ and π_L is the unique plane that meets Γ exactly in RL.*

ii) Φ does not map the lines of $\mathcal{P}_{R,\pi_L}$ into the planes through the line RL.
In this case, there exists a plane π containing RL such that the lines of the pencil $\mathcal{P}_{R,\pi}$ are mapped, under Φ into the planes through RL. Hence, there is a unique line through R (beside RL) contained in Γ. In this case, we may assume that Φ maps the line $x_3 = x_4 = 0$ into the plane $x_1 = 0$ and the line $x_2 = x_4 = 0$ into the plane $x_2 = 0$. Hence:

$$\Gamma : a x_1 x_2^\sigma + b x_2 x_3^\sigma + x_3 x_4^\sigma = 0.$$

Assuming that Γ contains the points $(0, 1, 1, -1)$ and $(1, 1, -1, 0)$, we get $a = b = 1$ and hence a canonical equation is given by

$$\Gamma : x_1 x_2^\sigma + x_2 x_3^\sigma + x_3 x_4^\sigma = 0.$$

We will call this set a *hyperbolic σ-quadric* with *collinear vertex points R* and L.

Theorem 4.2 *Let Γ be a hyperbolic σ-quadric of* $\mathrm{PG}(3, q^n)$ *with collinear vertex points R and L. Then $|\Gamma| = (q^n + 1)^2$ and Γ contains exactly three lines: the line RL, a unique other line through R and a unique other line through L.*

- π_L does not contain the line RL (or equivalently Φ does not map the line RL into a plane through the line RL).
  W.l.o.g. we may put $\pi_L : x_1 = 0$. In this case, there is a plane through R (not containing L), say π_R, such that the pencil of lines through R in π_R is mapped, under Φ, into the pencil of planes through RL. We may assume that $\pi_R : x_4 = 0$. Hence, Φ maps the lines $\ell_{\alpha,\beta,0}$ into the planes $\pi_{0,\beta',\gamma'}$, so we may assume that Φ maps the line $\ell_{1,0,0}$ into the plane $\pi_{0,1,0}$ and the line $\ell_{0,1,0}$ into the plane $\pi_{0,0,1}$. Hence, the points of Γ satisfy the equation

$$a x_2^{\sigma+1} - b x_3^{\sigma+1} + x_1 x_4^\sigma = 0.$$

Assuming, w.l.o.g., that the point $(1, 1, 0, -1)$ belongs to Γ, we obtain $a = 1$. The number of lines through R contained in Γ depends on the number of solutions of the equation $x^{\sigma+1} = b$, and hence, it is either 0, 1, 2 or $q + 1$ depending upon q even or odd and n even or odd. We distinguish several cases:

- If q is even and n is even, then there are either 0 or 1 or $q + 1$ solutions giving either 0 or 1 or $q + 1$ lines through R (and hence through L) contained in Γ.
- If q is even and n is odd, then there is a unique solution of the equation giving one line through R and one line through L contained in Γ.
- If q is odd and n is even, then there are either 0 or $q + 1$ solutions of the equation giving either 0 or $q + 1$ lines through R (and through L) contained in Γ.
- If q is odd and n is odd, then there are either 0 or 2 solutions of the equation giving either 0 or 2 lines through R (and through L) contained in Γ.

In these cases, we will call the set Γ either an *elliptic* or a *parabolic* or a *hyperbolic* or a $(q + 1)$-*hyperbolic* σ-quadric with vertex points R and L according to the number of lines through R contained in Γ is either 0 or 1 or 2 or $q + 1$.
If q is even and n is even, put $r = (q^n - 1, q^m + 1)$.

Theorem 4.3 *If Γ is an elliptic σ-quadric of* $\mathrm{PG}(3, q^n)$ *with vertex points R and L, then Γ has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, with b a non-square if q is odd and $b^{(q^n-1)/r} \neq 1$ if q is even and n is even. Moreover, $|\Gamma| = q^{2n} + 1$ and Γ contains no line.*

Theorem 4.4 *If Γ is a parabolic σ-quadric of* $\mathrm{PG}(3, q^n)$ *with vertex points R and L, then q is even and Γ has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where the equation $x^{\sigma+1} = b$ has a unique solution. Moreover, $|\Gamma| = q^{2n} + q^n + 1$ and Γ contains a unique line through R and a unique line through L.*

Theorem 4.5 *If Γ is a hyperbolic σ-quadric of* $\mathrm{PG}(3, q^n)$ *with vertex points R and L, then both q and n are odd and Γ has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where $x^{\sigma+1} = b$ has exactly two solutions. Moreover, $|\Gamma| = q^{2n} + 2q^n + 1$ and Γ contains exactly two lines through R and exactly two lines through L.*

Theorem 4.6 *If Γ is a $(q + 1)$-hyperbolic σ-quadric of* $\mathrm{PG}(3, q^n)$ *with vertex points R and L, then n is even and Γ has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where $x^{\sigma+1} = b$ has exactly $q + 1$ solutions. Moreover, $|\Gamma| = q^{2n} + (q + 1)q^n + 1$ and Γ contains exactly $q + 1$ lines through R and exactly $q + 1$ lines through L.*

2) $V^{\perp} = V^{\top}$.
We may assume w.l.o.g. that the point $R = L = (1, 0, 0, 0)$ is both the left radical and the right radical. It follows that, in this case, Γ is a cone with vertex the point R. Since the matrix A has rank three with first column and last row equal to 0, by choosing a plane not through the point R, e.g., $\pi : x_1 = 0$, we get that the set $\Gamma \cap \pi$ is a σ-conic of the plane π with associated matrix of rank 3. Hence, it is a Kestenband σ-conic of π. It follows that Γ is a cone with vertex the point R projecting a Kestenband σ-conic in a plane not through R.

5 σ-quadrics of rank at most 2 in PG$(3, q^n)$

In this section, a σ-quadric Γ of PG$(3, q^n)$ will have equation $X^t A X^\sigma = 0$ with $rk(A) \leq 2$.

5.1 σ-quadrics of rank 2

Since $rk(A) = 2$, it follows that the left and right radicals are two lines r and ℓ of PG$(3, q^n)$. We distinguish three cases.
1) $r \cap \ell = \emptyset$.
We may assume w.l.o.g. that $r : x_3 = x_4 = 0$ and $\ell : x_1 = x_2 = 0$. Then:

$$\Gamma : (a_{31}x_3 + a_{23}x_2)x_3^\sigma + (a_{14}x_1 + a_{24}x_2)x_4^\sigma = 0,$$

that is the set of points of PG$(3, q^n)$ of intersection of corresponding planes under a collineation $\Phi : \mathcal{P}_r \longrightarrow \mathcal{P}_\ell$, where

$$\begin{aligned}\mathcal{P}_r &= \{\pi_{a,b} : ax_3 + bx_4 = 0\}_{\{(a,b)\in \mathrm{PG}(1,q^n)\}},\\ \mathcal{P}_\ell &= \{\pi'_{a,b} : ax_1 + bx_2 = 0\}_{\{(a,b)\in \mathrm{PG}(1,q^n)\}}.\end{aligned}$$

The set Γ contains the $q^n + 1$ lines of a pseudoregulus (see [30]) with transversal lines r and ℓ We call this set a *σ-quadric of pseudoregulus type with skew vertex lines* r and ℓ.
Note that if $n = 2$, $\sigma^2 = 1$ a pseudoregulus have been already introduced in [15] and σ-quadrics of pseudoregulus type with skew vertex lines have been introduced in [11] where they were called hyperbolic Q_F-sets.
Let $\Phi : \mathcal{P}_r \longrightarrow \mathcal{P}_s$ be a collineation with accompanying automorphism $\sigma : x \mapsto x^{q^m}$, $(m, n) = 1$.
Assuming that :

$$\Phi(\pi_{1,0}) = \pi'_{1,0}, \Phi(\pi_{0,1}) = \pi'_{0,1}, \Phi(\pi_{1,1}) = \pi'_{1,1},$$

we have that $\Phi(\pi_{a,b}) = \pi'_{a^\sigma,b^\sigma}$.
Hence, the set $\mathcal{Q}$ of points of intersection of corresponding planes under Φ is given by the points whose homogeneous coordinates are the solutions of the linear system

$$\begin{cases} ax_3 + bx_4 = 0 \\ a^\sigma x_1 + b^\sigma x_2 = 0, \end{cases}$$

where $(a, b) \in \mathrm{PG}(1, q^n)$. This system is equivalent to the linear system

$$\begin{cases} ax_3 + bx_4 = 0 \\ ax_1^{\sigma^{-1}} + bx_2^{\sigma^{-1}} = 0. \end{cases}$$

A point $P = (x_1, x_2, x_3, x_4)$ belongs to $\mathcal{Q}$ if and only if the previous linear system, in the unknowns a and b, has non-trivial solutions, hence if and only if $x_1^{\sigma^{-1}} x_4 - x_2^{\sigma^{-1}} x_3 = 0$, that is $x_1 x_4^\sigma - x_2 x_3^\sigma = 0$. This can be seen as a canonical equation of a σ-quadric of pseudoregulus type with skew vertex lines. Let $c \in \mathbb{F}_q^*$ and let $\gamma \in \mathbb{F}_{q^n}^*$ be such that $N(\gamma) = c$. Put

$$\mathcal{Q}_c = \{(\gamma x^\sigma, \gamma y^\sigma, x, y)\}_{(x,y)\in \mathrm{PG}(1,q^n)}.$$

The set $\mathcal{Q}_c$ is a maximum scattered linear set of rank $2n$ of $\mathrm{PG}(3, q^n)$ of pseudoregulus type with transversal lines r and ℓ. Hence, the set $\mathcal{Q}$ is the union of the skew lines r, ℓ and the $q-1$ linear sets of pseudoregulus type $\mathcal{Q}_c$, $c \in \mathbb{F}_{q^n}^*$.
The following holds:

Proposition 5.1 *Let Γ be a σ-quadric of pseudoregulus type of* $\mathrm{PG}(3, q^n)$ *with skew transversal lines r and ℓ. The only lines contained in Γ are r, ℓ and the $q^n + 1$ lines of the pseudoregulus associated to Γ.*

Proposition 5.2 *Let a, b and c be three pairwise skew lines and let ℓ and s be two skew lines meeting a, b and c. There is a unique σ-quadric of pseudoregulus type of* $\mathrm{PG}(3, q^n)$ *with skew vertex lines r and ℓ containing a, b and c.*

2) $r \cap \ell = \{V\}$ is a point.
We may assume w.l.o.g. that $r : x_3 = x_4 = 0$, $\ell : x_2 = x_3 = 0$. In this case, the σ-quadric Γ is a cone with vertex the point V projecting a (degenerate or not) C_F^m-set in a plane not through V. Indeed, let π be a plane not through the point V and let $R = r \cap \pi$, $L = \ell \cap \pi$. It follows that $\Gamma \cap \pi$ is a set of points of π generated by a collineation between the pencils of lines of π with center the points R and L induced by the collineation between the pencil of planes $\mathcal{P}_r$ and $\mathcal{P}_\ell$ that is associated to Γ.
3) $r = \ell$.
We may assume w.l.o.g. that $r = \ell : x_3 = x_4 = 0$. In this case, the σ-quadric is a cone with vertex the line r over a σ-quadric of a line skew with r. That is Γ is either just the line r or a plane through r or a pair of distinct planes through r or $q + 1$ planes through r forming an $\mathbb{F}_q$-subpencil of planes through r.

5.2 σ-quadrics of rank 1

In this subsection, a σ-quadric Γ will have equation $X^t A X^\sigma = 0$ with $rk(A) = 1$. Hence, $\dim V^\perp = \dim V^\top = 3$ so left and right radicals in $\mathrm{PG}(3, q^n)$ are planes. We distinguish two cases:

- $V^\perp \neq V^\top$.
  We may assume that $r : x_4 = 0$ is the right radical and $\ell : x_1 = 0$ is the left radical. Hence, $\Gamma : x_1 x_4^\sigma = 0$, that is the union of two different planes.
- $V^\perp = V^\top$.
  We may assume $r = \ell : x_4 = 0$ is both the left and right radical. Hence, $\Gamma : x_4^{\sigma+1} = 0$, that it is a plane of $\mathrm{PG}(3, q^n)$.

6 σ-quadrics of PG$(3, q^n)$ with $|A| = 0$

In this section, we summarize the results obtained on σ-quadrics.

Proposition 6.1 *Let* $\Gamma : X^t AX^\sigma = 0$ *be a* σ*-quadric of* $\mathrm{PG}(3, q^n)$*, with* $|A| = 0$*. The set* Γ *is one of the following:*

- *a cone with vertex a line* v *projecting a* σ*-quadric of a line* ℓ *skew with* v (*hence either just the line* v *or one, two or* $q+1$ *planes through* v),
- *a cone with vertex a point* V *projecting either a Kestenband* σ*-conic or a (possibly degenerate)* C_F^m*-set of a plane* π*, with* $V \notin \pi$,
- *a degenerate either parabolic or hyperbolic* σ*-quadric with two collinear vertex points,*
- *a non-degenerate either elliptic or parabolic or hyperbolic or* $(q+1)$*-hyperbolic* σ*-quadric with two vertex points,*
- *a non-degenerate* σ*-quadric with two skew vertex lines* (*i.e., a* σ*-quadric of pseudoregulus type*).

Let $\sigma : x \mapsto x^{q^m}$, $\sigma' : x \mapsto x^{q^{m'}}$, $(m, n) = (m', n) = 1$.

Proposition 6.2 *Let* Γ *be a* σ*-quadric and let* Γ' *be a* σ'*-quadric of* $\mathrm{PG}(3, q^n)$*. If* Γ *and* Γ' *are cones, then* Γ *and* Γ' *are* $\mathrm{P\Gamma L}$*-equivalent if and only if the basis of the cones are* $\mathrm{P\Gamma L}$*-equivalent.*

We will say that a σ-quadric and a σ'-quadric of PG$(3, q^n)$ are of the *same type* if they have the same name.

Proposition 6.3 *Let* Γ *be a* σ*-quadric and let* Γ' *be a* σ'*-quadric of* $\mathrm{PG}(3, q^n)$*. If* Γ *and* Γ' *are not cones and have non-empty vertices, then* Γ *and* Γ' *are* $\mathrm{P\Gamma L}$*-equivalent if and only if either* $\sigma' = \sigma$ *or* $\sigma' = \sigma^{-1}$ *and* Γ, Γ' *are of the same type.*

Proof Since Γ and Γ' are not cones, both Γ and Γ' have two distinct vertices. We divide two cases.
Case 1. The vertices of both Γ and Γ' are points.
We may assume that Γ and Γ' have the same vertices R and L. Let Φ (resp. Φ') be the σ-collineation (resp. σ'-collineation) between the star of lines with center R and the star of planes with center L associated to Γ (resp. Γ'). Observe that Γ is also a σ^{-1}-quadric with vertices L and R to which there is associated the σ^{-1}-collineation Φ^{-1}.
Let f be a collineation of PG$(3, q^n)$ mapping Γ into Γ'. Since the number of lines of Γ through either R or L is at most $q+1$, there exists plane π through the line RL s.t. the lines of π through R are mapped under Φ onto planes through a line on L not contained in π. It follows that Φ induces a collineation π between the pencil of lines through R and the pencil of lines through L. This gives that $\Gamma \cap \pi$ is a (possibly degenerate) C_F^m-set of π. It follows that $f(\Gamma \cap \pi)$ is a (possibly degenerate) either C_F^m or C_F^{n-m}-set. Hence, $\sigma' = \sigma$ or $\sigma' = \sigma^{-1}$.
Case 2. The vertices of both $\Gamma$and Γ' are two skew lines.
We may assume that Γ and Γ' have the same vertices r and ℓ. Every plane not through r neither through ℓ meets Γ in a (possibly degenerate) C_F^m set. The assertion follows in the same way as in the previous case. □

7 σ-quadrics of rank 4 of PG(d, q^n)

Opposite to the case of PG$(2, q^n)$, nothing is known for the sets of absolute points of PG(d, q^n) of a nonlinear correlation, different from a polarity, of PG(d, q^n), $d \geq 3$. We recall that took roughly 14 years and 10 different papers to B. Kestenband to classify all σ-conics of PG$(2, q^n)$ with equation $X^t AX^\sigma = 0$, $|A| \neq 0$. In this section, Γ will be a σ-quadric of PG(d, q^n) with equation $X^t AX^\sigma = 0$, $|A| \neq 0$. We determine the dimension of the maximum totally isotropic subspaces contained in Γ.

Proposition 7.1 *If S_h is an h-dimensional subspace of* PG(d, q^n) *contained in a σ-quadric Γ with equation $X^t AX^\sigma = 0$, $|A| \neq 0$, then $h \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Moreover, there exists a σ-quadric with equation $X^t AX^\sigma = 0$, $|A| \neq 0$, containing a subspace with dimension $\left\lfloor \frac{d-1}{2} \right\rfloor$.*

Proof Let S be a subspace with maximum dimension, say h, contained in Γ. We may assume, w.l.o.g., that S has equations $x_{h+2} = x_{h+3} = \ldots x_{d+1} = 0$. It follows that the matrix A has the submatrix formed by the first $h+1$ rows and $h+1$ columns with entries all 0's. This gives that $d+1 = \text{rank}(A) \leq 2(d-h)$, and hence $h \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. If d is odd, then the σ-quadric with equation $x_1 x_{d+1}{}^\sigma + x_2 x_d{}^\sigma + \ldots + x_{d+1} x_1{}^\sigma = 0$ contains the $\frac{d-1}{2}$-dimensional subspace with equations $x_1 = x_2 = \ldots = x_{\frac{d+1}{2}} = 0$. If d is even, then the σ-quadric with equation $x_1 x_{d+1}{}^\sigma + x_2 x_d{}^\sigma + \ldots + x_{\frac{d}{2}+1}^{\sigma+1} + \ldots + x_{d+1} x_1{}^\sigma = 0$ contains the $(\frac{d}{2}-1)$-dimensional subspace with equations $x_1 = x_2 = \ldots = x_{\frac{d}{2}+1} = 0$. □

A similar proof can be given to obtain the following, more general, result

Proposition 7.2 *If S_h is an h-dimensional subspace of* PG(d, q^n) *contained in a σ-quadric Γ with equation $X^t AX^\sigma = 0$, then $h \leq \left\lfloor d - \frac{\text{rank}(A)}{2} \right\rfloor$. Moreover, there exists a σ-quadric with equation $X^t AX^\sigma = 0$, containing a subspace with dimension $\left\lfloor d - \frac{\text{rank}(A)}{2} \right\rfloor$.*

8 Applications of σ-quadrics of PG$(3, 2^n)$

8.1 σ-quadrics of pseudoregulus type and the Segre's $(2^n + 1)$-arc of PG$(3, 2^n)$,

An *arc* in PG$(3, q)$ is a set of points with the property that any 4 of them span the whole space. The maximum size of an arc in PG$(3, q)$ is $q + 1$ as proved by Segre in [34] for q odd and by Casse in [6] for q even. An example of $(q + 1)$-arc of PG$(3, q)$, $q = 2^n$, was given by Segre in [35] and it is a set of points projectively equivalent to the set $\{(1, t, t^\sigma, t^{\sigma+1})\} \cup \{(0, 0, 0, 1)\}$, where $\sigma : x \mapsto x^{2^m}$, $(m, n) = 1$. The $(q + 1)$-arcs of PG$(3, q)$ have been classified by Segre for q odd, $q \geq 5$ [34] and by Casse and Glynn for q even, $q \geq 8$ [7] and are the twisted cubic and, for q even, the Segre's $(q + 1)$-arc.
An *ovoid* of $Q^+(3, q)$ is a set of $q + 1$ points no two collinear on the quadric. A *translation ovoid* is an ovoid $\mathcal{O}$ containing a point P s.t. there is a collineation group

of $Q^+(3, q)$ fixing P linewise and acting sharply transitively on the points of $\mathcal{O}\setminus\{P\}$. An example of translation ovoid of $Q^+(3, q)$ is a non degenerate conic.

Proposition 8.1 *Let* Γ *be a* σ*-quadric of pseudoregulus type of* $\mathrm{PG}(3, q^n)$ *with skew vertex lines* r *and* ℓ*. The set* Γ *is mapped, via the Klein correspondence, into the union of a translation ovoid of a hyperbolic quadric* $\mathcal{Q}^+(3, q^n)$ *with two points* R *and* S*. Moreover, if* $q = 2$*, then this translation ovoid is the Segre's* (2^n+1)*-arc of* $\mathrm{PG}(3, 2^n)$*.*

Proof Let $r : x_3 = x_4 = 0$ and $\ell : x_1 = x_2 = 0$ and let Γ be the σ-quadric with vertex lines ℓ and r with equation $x_1 x_4^\sigma - x_2 x_3^\sigma = 0$. The lines of the associated pseudoregulus are spanned by the points $(0, 0, \alpha, \beta)$ and $(c\alpha^\sigma, c\beta^\sigma, 0, 0)$, $(\alpha, \beta) \in \mathrm{PG}(1, q^n)$, $c \in \mathbb{F}_q^*$. Via the Klein correspondence, the lines of the pseudoregulus are mapped to the set of points $\{(0, \alpha^{\sigma+1}, \alpha^\sigma \beta, \alpha\beta^\sigma, \beta^{\sigma+1}, 0)\}_{\{(\alpha,\beta)\in \mathrm{PG}(1,q^n)\}}$, i.e., the set of points of the ovoid

$$\mathcal{O} = \{(0, \alpha^{\sigma+1}, \alpha^\sigma, \alpha, 1, 0)\} \cup \{P_\infty = (0, 0, 0, 0, 1, 0)\}$$

of the hyperbolic quadric $\mathcal{Q}^+(3, q^n)$ with equations $x_1 = x_6 = 0$, $x_2x_5 - x_3x_4 = 0$ contained in the Klein quadric with equation $x_1x_6 - x_2x_5 + x_3x_4 = 0$. The two vertex lines r and ℓ are mapped, via the Klein correspondence, into the two points $(0, 0, 0, 0, 0, 1)$ and $(1, 0, 0, 0, 0, 0)$ on the line $x_2 = x_3 = x_4 = x_5 = 0$ that is the polar line w.r.t. the three-dimensional subspace with equations $x_1 = x_6 = 0$ containing the ovoid $\mathcal{O}$ under the polarity defined by the Klein quadric.
The ovoid $\mathcal{O}$ is a translation ovoid w.r.t. the point P_∞ of the hyperbolic quadric $\mathcal{Q}^+(3, q^n)$ since the group of projectivities of $\mathrm{PG}(5, q^n)$ induced by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 & 0 \\ 0 & b^\sigma & 0 & 1 & 0 & 0 \\ 0 & b^{\sigma+1} & b^\sigma & b & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$b \in \mathbb{F}_{q^n}$, stabilizes P_∞ and acts sharply transitively on the points of $\mathcal{O}\setminus\{P_\infty\}$. □

8.2 Degenerate σ-quadrics and Lüneburg spread of $\mathrm{PG}(3, 2^n)$.

In 1965, H. Lüneburg proved that if $q^n = 2^{2h+1}$, $h \geq 1$, then the set of absolute lines of a polarity of $W(3, q^n)$ is a symplectic spread, now called the *Lüneburg spread* of $\mathrm{PG}(3, q^n)$ (see [32]).
Let Σ_∞ be a hyperplane of $\mathrm{PG}(4, q^n)$ and let $\mathcal{Q}^+(3, q^n)$ be a hyperbolic quadric of Σ_∞. A set $\mathcal{A}$ of q^{2n} points of $\mathrm{PG}(4, q^n)\setminus\Sigma_\infty$ s.t. the line joining any two of them is disjoint from $\mathcal{Q}^+(3, q^n)$ is called an *affine set* of $\mathrm{PG}(4, q^n)\setminus\Sigma_\infty$. In what follows, we will denote by $\perp$ the polarity of $\mathcal{Q}^+(5, q^n)$. In [29] and also in [31] the following result has been proved.

Theorem 8.2 *Let $\mathcal{O}$ be an ovoid of $\mathcal{Q}^+(5, q^n)$, let P be a point of $\mathcal{O}$ and let Ω be a hyperplane of* $\mathrm{PG}(5, q^n)$ *not containing P. The set $\mathcal{A}_P(\mathcal{O})$ obtained by projecting $\mathcal{O}$ from the point P into Ω is an affine set of $\Omega \backslash P^\perp$. Conversely, if $\mathcal{A}$ is an affine set of $\Omega \backslash P^\perp$, then the set $\mathcal{O} = \{PR \cap \mathcal{Q}^+(5, q^n) : R \in \mathcal{A}\}$ is an ovoid of $\mathcal{Q}^+(5, q^n)$.*

If $\mathcal{S}$ is a spread of $\mathrm{PG}(3, q^n)$ and ℓ is a line of $\mathcal{S}$, then we will denote by $\mathcal{A}_\ell(\mathcal{S})$ the affine set arising from $\mathcal{S}$ with respect to ℓ.
If $\mathcal{S}$ is a symplectic spread, then $\mathcal{A}_\ell(\mathcal{S})$ is a set of q^{2n} points of an affine space $\mathrm{PG}(3, q^n) \backslash \pi_\infty$ such that the line joining any two of them is disjoint from a given non-degenerate conic $\mathcal{C}$ of π_∞.
The affine set arising from the Lüneburg spread has been studied by A. Cossidente, G. Marino and O. Polverino in [8], where the following result has been obtained:

Theorem 8.3 *The affine set $\mathcal{A}$ of the Lüneburg spread of* $\mathrm{PG}(3, 2^{2h+1})$ *is the union of 2^{2h+1}-arcs, and each of them can be completed to a translation hyperoval. The directions of $\mathcal{A}$ on π_∞ are the complement of a regular hyperoval.*

With the following theorem, we observe that the affine set of a Lüneburg spread of $\mathrm{PG}(3, 2^{2h+1})$ is the affine part of a degenerate elliptic σ-quadric of $\mathrm{PG}(3, 2^{2h+1})$ with two vertex points.

Theorem 8.4 *Let Γ be a degenerate parabolic σ-quadric of* $\mathrm{PG}(3, 2^n)$ *with collinear vertex points R and L. The lines joining any two points of $\Gamma \backslash RL$ are disjoint from a translation hyperoval $\mathcal{O}_\infty$ of the plane π_L, projectively equivalent to the set $\{(0, t, t^{\sigma^{-2}}, 1) : t \in \mathbb{F}_{2^n}\} \cup \{R, L\}$.*
If $n = 2h + 1$ and σ is the automorphism of $\mathbb{F}_{2^n}$ given by $\sigma : x \mapsto x^{2^h}$, then $\mathcal{O}_\infty$ is a regular hyperoval. Hence, the set $\Gamma \backslash RL$ is the affine set of the Lüneburg spread of $\mathrm{PG}(3, q^n)$.

Proof Let Γ be a degenerate parabolic σ-quadric with collinear vertex points $R = (0, 0, 1, 0)$ and $L = (0, 1, 0, 0)$ with equation

$$x_4{}^{\sigma+1} + x_1 x_2^\sigma - x_3 x_1{}^\sigma = 0.$$

It follows that the plane π_{RL} has equation $x_1 = 0$. The set $\mathcal{A} = \mathcal{K} \backslash \pi_{RL}$ is given by $\mathcal{A} = \{(1, x, x^\sigma + y^{\sigma+1}, y) : x, y \in \mathbb{F}_{q^n}\}$. Arguing as in Proposition 5.2 in [8], we obtain that the set of directions determined by $\mathcal{A}$ into the plane π_{RL} covers all the points of π_{RL} except to the points of a hyperoval $\mathcal{C}$ given by $\mathcal{C} = \{(0, x, x^{\sigma^{-2}}, 1) : x \in \mathbb{F}_{q^n}^*\}$. Note that if $q = 2^{2h+1}$ and $\sigma : x \mapsto x^{2^h}$, then the hyperoval $\mathcal{C}$ is a hyperconic. The assertion follows (see Sect. 8.2). □

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

References

1. Ball, S.: Finite Geometry and Combinatorial Applications. London Mathematical Society Student Texts, vol. 82. Cambridge University Press, Cambridge (2015)
2. Blokhuis, A., Lavrauw, M.: Scattered spaces with respect to a spread in $\mathrm{PG}(n, q)$. Geom. Dedicata **81**(1–3), 231–243 (2000)
3. Bruck, R.H.: Circle geometry in higher dimensions. In: A Survey of Combinatorial Theory. In Proceedings Internat. Sympos., Colorado State University, Fort Collins, Colo., 1971, pp. 69–77. North-Holland, Amsterdam (1973)
4. Bruck, R.H.: Circle geometry in higher dimensions II. Geom. Dedicata **2**, 133–188 (1973)
5. Cameron, P.J.: Projective and Polar Spaces. University of London. QMW Maths Notes, vol. 13. Queen Mary and Westfield College, School of Mathematical Sciences, London (1992)
6. Casse, L.R.A.: A solution to B. Segre's "Problem $I_{r,q}$" for q even. Atti Accam. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **46**(8), 13–20 (1969)
7. Casse, L.R.A., Glynn, D.: The solution to B. Segre's problem $I_{r,q}$, $r = 3, q = 2^h$. Geom. Dedicata **13**, 157–163 (1982)
8. Cossidente, A., Marino, G., Polverino, O.: Affine sets arising from spreads. Innov. Incidence Geom. **6–7**(1), 127–138 (2007)
9. D'haeseleer, J., Durante, N.: On absolute points of correlations in $\mathrm{PG}(2, q^n)$. Electron. J. Combin. **27**(2), 2–32 (2020)
10. Dembowki, P.: Finite Geometries. Springer, Berlin (1997)
11. Donati, G., Durante, N.: A subset of the Hermitian surface. Innov. Incid. Geom. **3**, 13–23 (2006)
12. Donati, G., Durante, N.: Scattered linear sets generated by collineations between pencils of lines. J. Algebraic Combin. **40**(4), 1121–1134 (2014)
13. Donati, G., Durante, N.: A generalization of the quadratic cone of $\mathrm{PG}(3, q^n)$ and its relation with the affine set of the Lüneburg spread. J. Algebraic Combin. **49**(2), 169–177 (2019)
14. Durante, N.: Geometry of sesquilinear forms. Notes of a course given at the workshop Finite geometry and friends, Bruxelles, (2019)
15. Freeman, J.W.: Reguli and pseudoreguli in $\mathrm{PG}(3, s^2)$. Geom. Dedic. **9**, 267–280 (1980)
16. Hirschfeld, J.W.P., Thas, J.A.: General Galois Geometries. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1991). Oxford Science Publications
17. Kestenband, B.C.: Correlations whose squares are perspectivities. Geom. Dedicata **33**(3), 289–315 (1990)
18. Kestenband, B.C.: The correlations with identity companion automorphism, of finite Desarguesian planes. Linear Algebra Appl. **304**(1–3), 1–31 (2000)
19. Kestenband, B.C.: The correlations of finite Desarguesian planes. I. Generalities. J. Geom. **77**(1–2), 61–101 (2003)
20. Kestenband, B.C.: The correlations of finite Desarguesian planes of even nonsquare order. JP J. Geom. Topol. **5**(1), 1–62 (2005)
21. Kestenband, B.C.: The correlations of finite Desarguesian planes. II. The classification (I). J. Geom. **82**(1–2), 91–134 (2005)
22. Kestenband, B.C.: The correlations of finite Desarguesian planes. III. The classification (II). J. Geom. **83**(1–2), 88–120 (2005)
23. Kestenband, B.C.: The correlations of finite Desarguesian planes. IV. The classification (III). J. Geom. **86**(1–2), 98–139 (2007)
24. Kestenband, B.C.: The correlations of finite Desarguesian planes of square order defined by diagonal matrices. Linear Algebra Appl. **423**(2–3), 366–385 (2007)
25. Kestenband, B.C.: Embedding finite projective geometries into finite projective planes. Int. Electron. J. Geom. **2**(2), 27–33 (2009)

26. Kestenband, B.C.: The correlations of finite Desarguesian planes of odd square order defined by upper triangular matrices with four nonzero entries. JP J. Geom. Topol. **10**(2), 113–170 (2010)
27. Kestenband, B.C.: The correlations of finite Desarguesian planes of even square order defined by upper triangular matrices with four nonzero entries. JP J. Geom. Topol. **16**(2), 127–152 (2014)
28. Lavrauw, M., Van de Voorde, G.: On linear sets on a projective line. Des. Codes Cryptogr. **56**(2–3), 89–104 (2010)
29. Law, M., Penttila, T.: Flocks, ovals and generalised quadrangles (Four Lectures in Napoli, June 2000), preprint n. 40 (2000). Dipartimento di Matematica e Applicazioni R. Caccioppoli Universitá degli Studi di Napoli Federico II
30. Lunardon, G., Marino, G., Polverino, O., Trombetti, R.: Maximum scattered linear sets of pseudoregulus type and the Segre Variety $\mathcal{S}_{n,n}$. J. Algebr. Comb. **39**, 807–831 (2014)
31. Lunardon, G., Polverino, O.: Translation ovoids of orthogonal polar spaces. Forum Math. **16**, 663–669 (2004)
32. Lüneburg, H.: Die Suzukigruppen und ihre Geometrien. Springer, Berlin (1965)
33. Polverino, O.: Linear sets in finite projective spaces. Discrete Math. **310**, 3096–3107 (2010)
34. Segre, B.: Curve razionali normali e k-archi negli spazi finiti. Ann. Mat. Pura Appl. **39**, 357–369 (1955)
35. Segre, B.: Introduction to Galois geometries. Atti Accad. Naz. Lincei Mem. **8**, 133–236 (1967)
36. Seydewitz, F.: Lineäre Konstruktion einer Kurve doppelter Krümmung. Arch. Math. Phys. **10**, 203–214 (1847)
37. Steiner, J.: Systematische Entwichlung der Abhängigkeit Geometrische Gestalten von einander. Reimer, Berlin (1832)
38. Van Maldeghem, H.: On finite conics. https://cage.ugent.be/~hvm/artikels/58.pdf