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Laplacian controllability for graphs obtained by some standard products

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Abstract Let L_G be the Laplacian matrix of a graph G with n vertices, and let $\mathbf{b}$ be a binary vector of length n. The pair $(L_G, \mathbf{b})$ is said to be controllable (and we also say that G is Laplacian controllable for $\mathbf{b}$) if L_G has no eigenvector orthogonal to $\mathbf{b}$. In this paper we study the Laplacian controllability of joins, Cartesian products, tensor products and strong products of two graphs. Besides some theoretical results, we give an iterative construction of infinite families of controllable pairs $(L_G, \mathbf{b})$.

Keywords Laplacian eigenvalues · Controllability · Join · Cartesian product · Tensor product · Strong product

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1 Introduction

We give a very brief introduction, and then go straight to the results. For more details on control systems and their applications, we refer to [7]. The following equation is a standard model for the single-input linear control systems:

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} + \mathbf{b}u. \tag{1}$$

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The scalar $u = u(t)$ is called the *control input*, while M is an $n \times n$ real matrix and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. The system (1) is called *controllable* if for any vector $\mathbf{x}^*$ and time t^*, there always exists a control function $u(t)$, $0 < t < t^*$, such that the solution of (1) gives $\mathbf{x}(t^*) = \mathbf{x}^*$ irrespective of $\mathbf{x}(0)$.

In general, any special structure or property of the matrix M or the vector $\mathbf{b}$ is not prerequisited. However, case in which M is the Laplacian matrix L_G of a graph G and $\mathbf{b}$ a binary vector has received a great deal of attention (see [1,2,9]). In this case, the system (1) is controllable if and only if L_G has no eigenvector orthogonal to $\mathbf{b}$. In fact, this claim holds for a wider class of matrices [2,9]. It is also known that if L_G has a non-simple eigenvalue, then L_G automatically has an associated eigenvector orthogonal to a given vector $\mathbf{b}$ [9]. Thus, a necessary condition for the controllability of (1) is that all eigenvalues of L_G are simple.

We say that the pair $(L_G, \mathbf{b})$ is *controllable* and also that G is *Laplacian controllable for* $\mathbf{b}$ if the corresponding system is controllable. If L_G is not controllable for every binary vector $\mathbf{b}$, then we simply say that G is *Laplacian uncontrollable.*

To simplify terminology, we abbreviate the spectrum, the eigenvalues and the eigenvectors of L_G as the *spectrum*, the *eigenvalues* and the *eigenvectors* of G.

We use $\mathbf{0}$ and $\mathbf{j}$ to denote the all-0 and the all-1 vector, and I and J for the unit and all-1 matrix, respectively. If necessary, the length of the vector or the size of the matrix will be indicated in the subscript. The standard inner product of the vectors $\mathbf{a}, \mathbf{b}$ is denoted by $\langle \mathbf{a}, \mathbf{b} \rangle$. For arbitrary graphs G_1 and G_2, we use $G_1 \cup G_2$ to denote their (disjoint) union and $G_1 \nabla G_2$ to denote the *join* of G_1 and G_2, i.e., the graph obtained by adding an edge between every vertex of G_1 and every vertex of G_2. We use K_n and P_n to denote the complete graph and the path with n vertices, respectively. In particular, the *trivial graph* refers to K_1. Some other notions and the corresponding notation will be introduced in the following sections, upon the corresponding results.

In Sect. 2, we express the Laplacian controllability of the join $G_1 \nabla G_2$ in terms of the Laplacian controllability of G_1 and G_2. We also establish an iterative procedure which gives infinite families of Laplacian controllable graphs and the corresponding binary vectors.

In Sect. 3 we give a sequence of results related to the Laplacian controllability of graphs that are obtained as the Cartesian product of two arbitrary graphs or as the tensor product or the strong product of two arbitrary regular graphs.

2 Join of two graphs

In this section we frequently use the following classical result referred to R. Merris.

Theorem 1 [8] *Let G_1 and G_2 be the graphs with n_1 and n_2 vertices and eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n_1} = 0$ and $\nu_1, \nu_2, \ldots, \nu_{n_2} = 0$, respectively. The eigenvalues of $G_1 \nabla G_2$ are $n_1 + n_2$, $\mu_1 + n_2, \mu_2 + n_2, \ldots, \mu_{n_1-1} + n_2$, $\nu_1 + n_1, \nu_2 + n_1, \ldots, \nu_{n_2-1} + n_1$ and 0.*

If $\mathbf{x}$ is an eigenvector of G_1 orthogonal to $\mathbf{j}$ and associated with an eigenvalue μ, then its extension defined to be zero on each vertex of G_2 is an eigenvector of $G_1 \nabla G_2$ associated with $\mu + n_2$. The eigenvalue $n_1 + n_2$ is associated with the eigenvector whose value is $-n_2$ on each vertex of G_1 and n_1 on each vertex of G_2.

We start with a simple lemma.

Lemma 1 *The join $G_1 \nabla G_2$ is Laplacian uncontrollable whenever*

(i) at least one of G_1, G_2 is a join of two graphs or
(ii) at least one of G_1, G_2 has more than two components.

Proof (i): If, say G_1, is a join of two graphs then its number of vertices appears in its spectrum. But then, by Theorem 1, the number of vertices of $G_1 \nabla G_2$ is among its eigenvalues and has multiplicity at least 2, which leads to the result.

(ii): Similarly, if, say G_1, has at least three components, then the number of vertices of G_2 is an eigenvalue of $G_1 \nabla G_2$ of multiplicity at least 2.

We proceed with Laplacian controllability of $G_1 \nabla G_2$.

Theorem 2 *Given non-trivial graphs G_1 and G_2 with n_1 and n_2 vertices, respectively, let $\mathcal{B}_i$ (for $i \in \{1, 2\}$) denote the set of binary vectors $\mathbf{b_i}$ which are non-orthogonal to any of the eigenvectors of G_i which are orthogonal to $\mathbf{j_{n_i}}$. Then $(L_{G_1 \nabla G_2}, \mathbf{b})$ is controllable if and only if all eigenvalues of $G_1 \nabla G_2$ are simple, $\mathbf{b} = (\mathbf{b_1}^\intercal, \mathbf{b_2}^\intercal)^\intercal$ where $\mathbf{b_i} \in \mathcal{B}_i$, and $n_1 \langle \mathbf{b_2}, \mathbf{j_{n_2}} \rangle \neq n_2 \langle \mathbf{b_1}, \mathbf{j_{n_1}} \rangle$.*

Proof Assume that all eigenvalues of the join are simple and that $\mathbf{b}$ is formed as in the formulation of the theorem. Since the eigenvalues of $G_1 \nabla G_2$ are simple, we have that every its eigenvector is, up to a multiplying constant, determined as in Theorem 1. Let $\mathbf{x}$ be an eigenvector associated with an eigenvalue distinct from $n_1 + n_2$. By virtue of Theorem 1, either $\mathbf{x}$ is a constant vector, which immediately gives $\langle \mathbf{b}, \mathbf{x} \rangle \neq 0$ (as $\mathbf{b} \neq \mathbf{0}$) or $\mathbf{x}$ has the form $(\mathbf{x_1}^\intercal, \mathbf{0}^\intercal)^\intercal$ or $(\mathbf{0}^\intercal, \mathbf{x_2}^\intercal)^\intercal$, where $\mathbf{x_i}$ is an eigenvector of G_i orthogonal to $\mathbf{j_{n_i}}$ (it exists since G_i is non-trivial). It follows that $\langle \mathbf{b}, \mathbf{x} \rangle$ is equal to either $\langle \mathbf{b_1}, \mathbf{x_1} \rangle$ or $\langle \mathbf{b_2}, \mathbf{x_2} \rangle$, which implies $\langle \mathbf{b}, \mathbf{x} \rangle \neq 0$. If $\mathbf{x}$ is associated with $n_1 + n_2$, then it has the form

$$(\underbrace{-n_2, -n_2, \ldots, -n_2}_{n_1}, \underbrace{n_1, n_1, \ldots, n_1}_{n_2})^\intercal,$$

which yields $\langle \mathbf{b}, \mathbf{x} \rangle = -n_2 \langle \mathbf{b_1}, \mathbf{j_{n_1}} \rangle + n_1 \langle \mathbf{b_2}, \mathbf{j_{n_2}} \rangle$, and the proof of one implication is completed.

By assuming that $(L_{G_1 \nabla G_2}, \mathbf{b})$ is controllable, we immediately obtain that the eigenvalues of $G_1 \nabla G_2$ are simple. Assume now, by way of contradiction, that $\mathbf{b}$ is not formed as in the theorem. If $\mathbf{b} = (\mathbf{b_1}^\intercal, \mathbf{b_2}^\intercal)^\intercal$, where $\mathbf{b_i} \in \mathcal{B}_i$, by considering the eigenvector associated with $n_1 + n_2$, we immediately obtain $n_1\langle \mathbf{b_2}, \mathbf{j_{n_2}}\rangle \neq n_2\langle \mathbf{b_1}, \mathbf{j_{n_1}}\rangle$. Further, if $\mathbf{b}$ is a binary vector of the form $(\mathbf{a_1}^\intercal, \mathbf{a_2}^\intercal)^\intercal$, where the length of $\mathbf{a_i}$ is n_i and, say $\mathbf{a_1} \notin \mathcal{B}_1$, then there exists an eigenvector $\mathbf{x_1}$ of G_1 such that $\langle \mathbf{x_1}, \mathbf{j_{n_1}}\rangle = \langle \mathbf{x_1}, \mathbf{a_1}\rangle = 0$. By Theorem 1, $\mathbf{x} = (\mathbf{x_1}^\intercal, \mathbf{0}^\intercal)^\intercal$ is an eigenvector of $G_1\nabla G_2$ which is orthogonal to $\mathbf{b}$, contradicting the controllability of $(L_{G_1\nabla G_2}, \mathbf{b})$.

The remaining situation, in which at least one of G_1 or G_2 is the trivial graph, is settled in the next theorem.

Theorem 3 *Let G be a graph with n vertices, and, when $n \geq 2$, let $\mathcal{B}$ denote the set of binary vectors that are non-orthogonal to any of the eigenvectors of G which are orthogonal to $\mathbf{j}$. We have:*

(i) If $G \cong K_1$, then $(L_{K_1\nabla G}, \mathbf{b})$ is controllable if and only if $\mathbf{b} \in \{(0, 1)^\intercal, (1, 0)^\intercal\}$;
(ii) If $G \ncong K_1$, then $(L_{K_1\nabla G}, \mathbf{b})$ is controllable if and only if all eigenvalues of $K_1\nabla G$ are simple and $\mathbf{b} = (*, \mathbf{b'}^\intercal)^\intercal$ where $\mathbf{b'} \in \mathcal{B}$ and $*$ stands for either 0 or 1.

Proof (i): This follows by direct computation.

(ii): If the eigenvalues of $K_1\nabla G$ are simple, then all its eigenvectors arise from Theorem 1. Let $\mathbf{b}$ be formed as in the theorem, and let $\mathbf{x}$ be an eigenvector of $K_1\nabla G$. If $\mathbf{x}$ is a constant vector or has the form $(-n, \mathbf{j}^\intercal)^\intercal$, then $\langle \mathbf{b}, \mathbf{x}\rangle \neq 0$ (as $\mathbf{b'} \notin \{\mathbf{0}, \mathbf{j}\}$). If $\mathbf{x}$ has the form $(0, \mathbf{x'}^\intercal)^\intercal$ (where $\mathbf{x'}$ is an eigenvector of G orthogonal to $\mathbf{j}$), then $\langle \mathbf{b}, \mathbf{x}\rangle = \langle \mathbf{b'}, \mathbf{x'}\rangle \neq 0$ (as $\mathbf{b'} \in \mathcal{B}$). Thus, $(L_{K_1\nabla G}, \mathbf{b})$ is controllable.

Assume now that $(L_{K_1\nabla G}, \mathbf{b})$ is controllable. Then all eigenvalues of $K_1\nabla G$ are simple. Let further $\mathbf{b} = (*, \mathbf{a'}^\intercal)^\intercal$, where $\mathbf{a'}$ is a binary vector not belonging to $\mathcal{B}$. There is an eigenvector $(0, \mathbf{x'}^\intercal)^\intercal$ of $K_1\nabla G$, such that $\langle \mathbf{x'}, \mathbf{a'}\rangle = 0$. The equality $\langle \mathbf{b}, (0, \mathbf{x'}^\intercal)^\intercal\rangle = 0$ concludes the proof.

Remark 1 In the foregoing theorems, we had an assumption that ‘all eigenvalues of $G_1\nabla G_2$ are simple’. It is not difficult to see when this assumption is satisfied. Namely, it holds if G_1 and G_2 have no repeated non-zero eigenvalues, neither of them is a join nor has more than two components and (with the notation of Theorem 1) $\mu_i + n_2 \neq \nu_j + n_1$ holds for $1 \leq i \leq n_1 - 1$, $1 \leq j \leq n_2 - 1$. Note also that, if G_1 and G_2 are connected, then the set $\mathcal{B}_i$ of Theorem 2 consists of all binary vectors $\mathbf{b_i}$ such that $(L_{G_i}, \mathbf{b_i})$ is controllable; and similarly for the corresponding set of Theorem 3.

Remark 2 In the statement of Theorem 2 (resp. Theorem 3) the set $\mathcal{B}_i$ (resp. $\mathcal{B}$) is formed by taking into account the eigenvectors ‘which are orthogonal to $\mathbf{j_{n_i}}$ (resp. $\mathbf{j}$)’. This assumption is essential as the following example shows. Take $K_1\nabla(K_1 \cup G)$, where G is connected, let $\mathbf{b'}$ be a binary vector such that $(L_G, \mathbf{b'})$ is controllable, and let $\mathcal{C}$ denote the set of binary vectors which are non-orthogonal to all eigenvectors of G. Clearly $\mathcal{C} \subseteq \mathcal{B}$. Then $(1, \mathbf{0}^\intercal)^\intercal$ is an eigenvector associated with zero in $K_1 \cup G$, and so $(0, \mathbf{b'}^\intercal)^\intercal \notin \mathcal{C}$ (since it is orthogonal to $(1, \mathbf{0}^\intercal)^\intercal$), but $K_1\nabla(K_1 \cup G)$ is controllable for $(*, 0, \mathbf{b'}^\intercal)^\intercal$, which can easily be concluded by using Theorem 1.

Take a break with an example.

Example 1 Let $G_1 \cong K_1 \cup K_2$ and $G_2 \cong P_4$.

First, G_1 is Laplacian uncontrollable, as it has a repeated eigenvalue. Nevertheless, since $\mathbf{j}$, $(-1/2, -1/2, 1)^\intercal$ and $(-1, 1, 0)^\intercal$ make a full system of its linearly independent eigenvectors, we obtain

$$\mathcal{B}_1 = \left\{ \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}.$$

Next, since all eigenvalues of G_2 are simple, and the corresponding eigenvectors are $\mathbf{j}, (1, \sqrt{2}-1, 1-\sqrt{2}, -1)^\intercal, (-1, 1, 1-1)^\intercal$ and $(1-\sqrt{2}, 1, -1, \sqrt{2}-1)^\intercal$, we deduce that $(L_{G_2}, \mathbf{b_2})$ is controllable if and only if $\mathbf{b_2}$ is a binary vector of length 4 with exactly one or three 1's.

Since all eigenvalues of $G_1 \nabla G_2$ are simple, it follows that $(L_{G_1 \nabla G_2}, \mathbf{b})$ is controllable if and only if $\mathbf{b} = (\mathbf{b_1}^\intercal, \mathbf{b_2}^\intercal)^\intercal$, for $\mathbf{b_1} \in \mathcal{B}_1$, and $\mathbf{b_2}$ is formed in the described way. There are exactly 32 possibilities for $\mathbf{b}$.

We continue with the following observations. Since $\mathbf{j}$ is associated with zero in the spectrum of any graph, we conclude that $(L_G, \mathbf{j})$ is uncontrollable whenever $G \ncong K_1$. On the other hand, it is obvious that $(L_G, \mathbf{0})$ is also uncontrollable, for any G. Therefore, the extremal cases (for the number of 1's in $\mathbf{b}$) arise when $(L_G, \mathbf{b})$ is controllable and $\langle \mathbf{b}, \mathbf{j} \rangle$ is equal to $n-1$ or 1 (n being the number of vertices of G). Moreover, these extremal cases occur simultaneously for the same graph, since $(L_G, \mathbf{b})$ is controllable if and only if $(L_G, \mathbf{j} - \mathbf{b})$ is controllable. The last follows by $\langle \mathbf{x}, \mathbf{j} - \mathbf{b} \rangle = \langle \mathbf{x}, \mathbf{j} \rangle - \langle \mathbf{x}, \mathbf{b} \rangle = -\langle \mathbf{x}, \mathbf{b} \rangle$, for any eigenvector $\mathbf{x}$ orthogonal to $\mathbf{j}$; for a different proof, see [1]. The path P_4 of Example 1 is a graph for which the extremal cases are attained. In fact, they are attained for any path P_n ($n \geq 2$), as its eigenvalues are simple (see, for example, [5]) and, by the eigenvalue equation, the first coordinate of every eigenvector is non-zero (so, we may take $\mathbf{b} = (1, \mathbf{0}^\intercal)^\intercal$). Moreover, we have a simple lemma.

Lemma 2 *If G is a connected graph with n ($n \geq 2$) vertices, such that $K_1 \nabla G$ admits only simple eigenvalues and $\mathbf{b}$ is a binary vector of length n satisfying $\langle \mathbf{b}, \mathbf{j} \rangle = 1$, then $(L_G, \mathbf{b})$ is controllable if and only if $(L_{K_1 \nabla G}, (0, \mathbf{b}^\intercal)^\intercal)$ is controllable.*

Proof The proof follows from Lemma 1(i) and Theorem 3(ii).

Consequently $K_1 \nabla P_n$ ($n \geq 2$) attains the extremal cases.

We conclude this section by a procedure which produces infinite families of Laplacian controllable graphs and corresponding binary vectors.

Theorem 4 *Let G_0 be an arbitrary non-trivial graph without repeated eigenvalues. Set*

$$G_i = K_1 \nabla (K_1 \cup G_{i-1}), \ \textit{for } i \geq 0.$$

Then, $(L_{G_i}, \mathbf{b})$ *is controllable if and only if*

$$\mathbf{b} = (\underbrace{*, *, \ldots, *}_{2i}, \mathbf{b_0}^\intercal)^\intercal,$$

where $*$ *stands for either* 0 *or* 1 *and* $\mathbf{b_0}$ *is a binary vector such that* $(L_{G_0}, \mathbf{b_0})$ *is controllable.*

Proof If $\mu_1, \mu_2, \ldots, \mu_n = 0$ are the eigenvalues of G_0, using Theorem 1, we get that the eigenvalues of G_i $(i \geq 1)$ are

$$n + 2i, n + 2i - 1, \ldots, n + i + 1, \mu_1 + i, \mu_2 + i, \ldots, \mu_{n-1} + i, i, i - 1, \ldots, 0,$$

and therefore they are distinct, since $\mu_i \leq n$ for $1 \leq i \leq n$.

A full system of linearly independent eigenvectors of G_i is

$$\begin{pmatrix} -(n+2i-1) \\ 1 \\ \mathbf{j_{n+2(i-1)}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathbf{x_1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathbf{x_2} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \mathbf{x_{n+2(i-1)-1}} \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ -(n+2(i-1)) \\ \mathbf{j_{n+2(i-1)}} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \mathbf{j_{n+2(i-1)}} \end{pmatrix},$$

where $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n+2(i-1)-1}}$ are the eigenvectors associated with non-zero eigenvalues of G_{i-1}. We proceed by induction argument.

By setting $i = 1$, we conclude that G_1 has no eigenvector orthogonal to $(*, *, \mathbf{b_0}^\intercal)^\intercal$ (as $\mathbf{b_0} \neq \mathbf{j_n}$, since $G_0 \ncong K_1$). Conversely, if $(L_{G_0}, \mathbf{a_0})$ is uncontrollable, then any binary vector of the form $(*, *, \mathbf{a_0}^\intercal)^\intercal$ is orthogonal to at least one of the eigenvectors of G_1 formed on the basis of $\mathbf{x_k}$'s $(1 \leq k \leq n-1)$.

Assume that the statement holds for G_{i-1} and consider G_i. Then G_i has no eigenvector orthogonal to the vector $\mathbf{b} = (*, *, \ldots, *, \mathbf{b_0}^\intercal)^\intercal$ of length $n+2i$, since, by the induction hypothesis, the vector $(*, *, \ldots, *, \mathbf{b_0}^\intercal)^\intercal$ of length $n + 2(i-1)$ is not orthogonal to any of $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n+2(i-1)-1}}, \mathbf{j_{n+2(i-1)}}$. Every vector distinct from $\mathbf{b}$ is eliminated as in the induction basis.

So, if there are s binary vectors that preserve Laplacian controllability of G_0, then the number of those for G_i is $2^{2i}s$. Other iterative constructions (with some other graphs in the roles of K_1) can be obtained in a similar way.

3 Other products

Let G_1 be a graph with the vertex set $\{u_1, u_2, \ldots, u_{n_1}\}$, eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n_1} = 0$ and associated eigenvectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n_1}}$, and let G_2 be a graph with the vertex set $\{v_1, v_2, \ldots, v_{n_2}\}$, eigenvalues $\nu_1, \nu_2, \ldots, \nu_{n_2} = 0$ and associated eigenvectors $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_{n_2}}$. We assume that both G_1 and G_2 are non-trivial and consider the Cartesian product $G_1 \,\Box\, G_2$, the tensor product $G_1 \times G_2$ and the strong product $G_1 \boxtimes G_2$.

Table 1 Eigenvalues and associated eigenvectors of the Cartesian product, the tensor product and the strong product of G_1 and G_2. For the latter two products, G_1 is regular of degree r and G_2 is regular of degree s. In all cases, we have $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

graph	eigenvalues	eigenvectors
$G_1 \,\Box\, G_2$	$\mu_i + \nu_j$	$\mathbf{x_i} \otimes \mathbf{y_j}$
$G_1 \times G_2$	$s\mu_i + r\nu_j - \mu_i\nu_j$	$\mathbf{x_i} \otimes \mathbf{y_j}$
$G_1 \boxtimes G_2$	$(s+1)\mu_i + (r+1)\nu_j - \mu_i\nu_j$	$\mathbf{x_i} \otimes \mathbf{y_j}$

Recall that the set of vertices of any of these products is the Cartesian product of the sets of vertices of G_1 and G_2. In $G_1 \,\Box\, G_2$ the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i = u_k$ and $v_j \sim v_l$ or $u_i \sim u_k$ and $v_j = v_l$. In $G_1 \times G_2$ the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i \sim u_k$ and $v_j \sim v_l$. In $G_1 \boxtimes G_2$ the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if they are adjacent in any of the previous two products.

The eigenvalues and the eigenvectors of these products are given in Table 1, in which $\otimes$ denotes the standard Kronecker product; for the first product see [4], for the latter two (with an additional assumption that G_1 and G_2 are regular) see [3]. Accordingly, the eigenvectors are the same for any product (which is unsurprising since the corresponding Laplacian matrices are obtained as linear combinations of specified Kronecker products, see [3]).

Let further $*$ stand for any of symbols $\Box$, $\times$ or $\boxtimes$, and assume that in the latter two cases, the corresponding graphs are regular. Here is a straightforward result.

Theorem 5 *If* $\mathbf{c} = (\mathbf{c_1}^\intercal, \mathbf{c_2}^\intercal, \ldots, \mathbf{c_{n_1}}^\intercal)^\intercal$ *is a binary vector, such that the length of each* $\mathbf{c_i}$ *is* n_2*, then* $(L_{G_1 * G_2}, \mathbf{c})$ *is controllable if and only if* $G_1 * G_2$ *has no repeated eigenvalues and* $\sum_{i=1}^{n_1} x_i \langle \mathbf{y}, \mathbf{c_i} \rangle \neq 0$*, for all eigenvectors* $\mathbf{x}$ *of* G_1 *and* $\mathbf{y}$ *of* G_2.

Proof The result follows by observing that $\langle \mathbf{x} \otimes \mathbf{y}, \mathbf{c} \rangle = \sum_{i=1}^{n_1} x_i \langle \mathbf{y}, \mathbf{c_i} \rangle$.

In particular, we have the following corollary.

Corollary 1 *If* $(L_{G_1}, \mathbf{a})$ *and* $(L_{G_2}, \mathbf{b})$ *are controllable and* $G_1 * G_2$ *has no repeated eigenvalues, then* $(L_{G_1 * G_2}, \mathbf{c})$ *is controllable if* $\mathbf{c} = (\mathbf{c_1}^\intercal, \mathbf{c_2}^\intercal, \ldots, \mathbf{c_{n_1}}^\intercal)^\intercal$*, where* $\mathbf{c_i} = a_i \mathbf{b}$ *for* $\mathbf{a} = (a_1, a_2, \ldots, a_{n_1})^\intercal$.

Proof We compute

$$\begin{aligned} \langle \mathbf{x} \otimes \mathbf{y}, \mathbf{c} \rangle &= \langle \mathbf{x} \otimes \mathbf{y},\, (a_1 \mathbf{b}^\intercal, a_2 \mathbf{b}^\intercal, \ldots, a_{n_1} \mathbf{b}^\intercal)^\intercal \rangle \\ &= \sum_{i=1}^{n_1} x_i a_i \langle \mathbf{y}, \mathbf{b} \rangle = \langle \mathbf{x}, \mathbf{a} \rangle \langle \mathbf{y}, \mathbf{b} \rangle \neq 0, \end{aligned}$$

and we are done.

Observe that if at least one of G_1 or G_2 has a repeated eigenvalue, then $G_1 * G_2$ also have a repeated eigenvalue, so it is Laplacian uncontrollable.

According to Table 1, orthogonality of a binary vector to the eigenvectors of $G_1 * G_2$ does not depend on $* \in \{\Box, \times, \boxtimes\}$. Here we obtain some similar situations. We use $\overline{G}$ to denote the complement of G, and we also recall that the *bipartite complement* $\overline{\overline{G}}$ of a bipartite graph G is a bipartite graph with the same colour classes having the edge between them exactly where G does not.

Theorem 6 *Let $* \in \{\Box, \times, \boxtimes\}$. For a graph G_1 and a connected graph G_2, we have:*

(i) Every eigenvector of $G_1 * G_2$ is an eigenvector of $G_1 * \overline{G}_2$;
(ii) If G_2 is bipartite and regular, then every eigenvector of $G_1 * G_2$ is an eigenvector of $G_1 * \overline{\overline{G}}_2$.

Proof The proofs of both claims rely on the fact that, under the given assumptions, the eigenvectors of G_2 are the eigenvectors of $\overline{G}_2$ (resp. $\overline{\overline{G}}_2$) for (i) (resp. for (ii)). Indeed, then the result follows by Table 1.

The mentioned fact for G_2 and $\overline{G}_2$ is known from literature, and the reader can consult [6]. Here we prove the latter one. After an appropriate vertex permutation, the Laplacian matrix of G_2 assumes the form

$$L_{G_2} = sI - \begin{pmatrix} O & N \\ N^\intercal & O \end{pmatrix}.$$

Then,

$$L_{\overline{\overline{G}}_2} = (n-s)I - \begin{pmatrix} O & J-N \\ J-N^\intercal & O \end{pmatrix} = nI - L_{G_2} - \begin{pmatrix} O & J \\ J & O \end{pmatrix}.$$

If $\mathbf{x}$ is an eigenvector of G_2, then

$$L_{\overline{\overline{G}}_2}\mathbf{x} = nI\mathbf{x} - L_{G_2}\mathbf{x} - \begin{pmatrix} O & J \\ J & O \end{pmatrix}\mathbf{x}.$$

Now, for $\mathbf{x} = \mathbf{j_{n_2}}$ and $\mathbf{x} = (\mathbf{j_n}^\intercal, -\mathbf{j_n}^\intercal)^\intercal$, we get $L_{\overline{\overline{G}}_2}\mathbf{x} = 0\mathbf{x}$ and $L_{\overline{\overline{G}}_2}\mathbf{x} = 2(n-s)\mathbf{x}$, so the claim follows. If $\mathbf{x}$ is some of the remaining eigenvectors (associated with an eigenvalue ν), then we have $L_{\overline{\overline{G}}_2}\mathbf{x} = (n-\nu)\mathbf{x}$, and we are done.

We now investigate the tensor product of some peculiar graphs. It follows from definition that $G_1 \times G_2$ of non-trivial graphs is connected if G_1, G_2 are connected and at least one of them is non-bipartite. Otherwise, it is disconnected, and so Laplacian uncontrollable. The tensor product $K_2 \times G$ is called a *bipartite double* of G. Clearly, it is always bipartite. We denote it by $\mathrm{bd}(G)$. By Table 1, if G is regular of degree r with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, then the eigenvalues of $\mathrm{bd}(G)$ are $\mu_1, \mu_2, \ldots, \mu_n, 2r-\mu_1, 2r-\mu_2, \ldots, 2r-\mu_n$.

Theorem 7 *For a regular graph G, the pair $(L_{\mathrm{bd}(G)}, \mathbf{c})$ is controllable if and only if $\mathrm{bd}(G)$ has no repeated eigenvalues and $\mathbf{c} = (\mathbf{c_1}^\intercal, \mathbf{c_2}^\intercal)^\intercal$ is a binary vector such that $\mathbf{c_1}$ and $\mathbf{c_2}$ are equal in length and $\langle \mathbf{c_1} \pm \mathbf{c_2}, \mathbf{x}\rangle \neq 0$, for every eigenvector $\mathbf{x}$ of G.*

Proof Since $(1,1)^\intercal$ and $(1,-1)^\intercal$ make a full set of linearly independent eigenvectors of K_2, we get that $(\mathbf{x}^\intercal, \mathbf{x}^\intercal)^\intercal$ and $(\mathbf{x}^\intercal, -\mathbf{x}^\intercal)^\intercal$ are the eigenvectors of $\mathrm{bd}(G)$, where $\mathbf{x}$ is an eigenvector of G. Now, we easily conclude that $\mathrm{bd}(G)$ is controllable if and only if it has no repeated eigenvalues and $\mathbf{c}$ is formed as in the theorem.

Observe that

$$L_{\mathrm{bd}(G)} = rI_{2n} + (I_2 - J_2) \otimes A_G, \tag{2}$$

where A_G is the standard adjacency matrix of G.

Consider now a related product. If the vertices of K_2 are denoted by u_1 and u_2, then the *extended bipartite double* $\mathrm{ebd}(G)$ of G is obtained from its bipartite double by inserting and edge between the vertices (u_1, v) and (u_2, v), for all vertices v of G. The graph $\mathrm{ebd}(G)$ is connected if and only if G is connected. If, as before, G is regular, then

$$L_{\mathrm{ebd}(G)} = (r+1)I_{2n} + (I_2 - J_2) \otimes (A_G + I_n). \tag{3}$$

Considering the latter identity, we conclude that the eigenvalues of $\mathrm{ebd}(G)$ are $\mu_1, \mu_2, \ldots, \mu_n, 2(r+1)-\mu_1, 2(r+1)-\mu_2, \ldots, 2(r+1)-\mu_n$. Moreover, we have the following result.

Theorem 8 *If G is a regular graph, then* $\mathrm{bd}(G)$ *and* $\mathrm{ebd}(G)$ *share the same eigenvectors.*

Proof From (2) and (3), both $L_{\mathrm{bd}(G)}$ and $L_{\mathrm{ebd}(G)}$ share the same eigenvectors with $(I_2 - J_2) \otimes A_G$.

We conclude the section by eliminating a possibility for Laplacian controllable products. A graph is said to be *Laplacian integral* if its spectrum consists entirely of integers.

Theorem 9 *Let $*$ be a fixed element of $\{\square, \times, \boxtimes\}$ and assume that at least one of G_1, G_2 is non-trivial. If $G_1 * G_2$ is a Laplacian integral product described in Table* 1*, then $G_1 * G_2$ is Laplacian uncontrollable.*

Proof If $G_1 * G_2$ is Laplacian integral, then since zero is an eigenvalue of both G_1 and G_2, we conclude that they are Laplacian integral, as well. Since they cannot have repeated eigenvalues, we have that the eigenvalues of G_i are the n_i distinct numbers which belong to $\{0, 1, \ldots, n_i\}$. Now, it is a matter of routine to verify that $G_1 * G_2$ must have a repeated eigenvalue.

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Conflict of interest

The authors declare that they have no conflict of interest.

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